Assignment 7

1) Mean, Variance, and Kurtosis of Mixed Normal

$$X = \sigma Z$$

1) $\sigma\ (rv) = \begin{cases} 1 & Pr(0.5) \\ 4 & Pr(0.5) \end{cases}$

2) $Z \sim N(0,1)$

3) $\sigma \perp Z$

Mean:
$$\begin{aligned} E(X) &= E(\sigma Z) \\ &= E(\sigma)E(Z) \qquad \sigma \perp Z \\ &= E(\sigma) \times 0 \\ &= 0 \end{aligned}$$

Variance:
$$\begin{aligned} Var(X) &= Var(\sigma Z) \\ &= Var(Z\sigma) \\ &= Var[E(Z\sigma \mid Z)] + E[Var(Z\sigma \mid Z)] \\ &= Var[Z E(\sigma \mid Z)] + E[Z^2 Var(\sigma \mid Z)] \\ &= Var[Z E(\sigma)] + E[Z^2 Var(\sigma)] \\ &= E(\sigma)^2 \underbrace{Var(Z)}_{1} + Var(\sigma)\underbrace{E(Z^2)}_{1} \\ &= E(\sigma)^2 + Var(\sigma) \\ &= 2.5^2 + \\ &= 6.25 + 2.25 = 8.5 \end{aligned}$$

$$E(\sigma) = 1 \times 0.5 + 4 \times 0.5 = 2.5$$

$$E(\sigma^2) = 1^2 \times 0.5 + 4^2 \times 0.5 = 8.5$$

$$\begin{aligned} Var(\sigma) &= E(\sigma^2) - E(\sigma)^2 \\ &= 8.5 - 2.5^2 \\ &= 8.5 - 6.25 = 2.25 \end{aligned}$$

$$\begin{aligned} Var(\sigma^2) &= E(\sigma^4) - E(\sigma^2)^2 \\ &= 1^4 \times 0.5 + 4^4 \times 0.5 - 8.5^2 \\ &= 56.25 \end{aligned}$$

Kurtosis:
$$\begin{aligned} K(x) &= 3\left(1 + \frac{Var(\sigma^2)}{(E(\sigma^2))^2}\right) \\ &= 3\left(1 + \frac{56.25}{(8.5)^2}\right) \\ &= 5.336 \end{aligned}$$

2) VaR Calculations

1) Conditional VaR (2.5%)

VaR_normal ← -qnorm(0.025, 0, $\sqrt{0.5}$)

= 1.385904

2) Unconditional VaR (2.5%)

$$\hat{\sigma}^2_{t+1} = 0.2 + 0.09\, x_t^2 + 0.9\, \sigma_t^2$$
$$= 0.2 + 0.09\,(1.5)^2 + 0.9\,(0.5)$$
$$= 0.8525$$
$$E(x_t) = 0 = \mu$$

VaR_normal ← -qnorm(0.025, 0, $\sqrt{0.8525}$)

= 1.809653

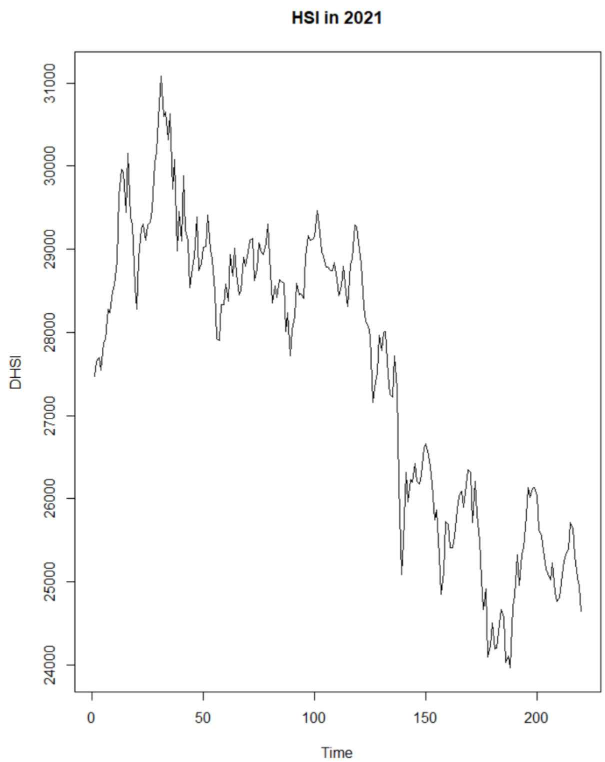
3) Conditional VaR underestimates (or overestimates) the risk associated, while unconditional VaR is more accurate. Due to heteroskedasticity, the variance is conditional on time; hence, should be determined by previous month variance and return. Since the time series is a GARCH(1,1) model, heteroskedasticity is more reasonable than homoskedasticity. The current situation is that conditional VaR is underestimated, this may be due to having a lower variance in the current period.

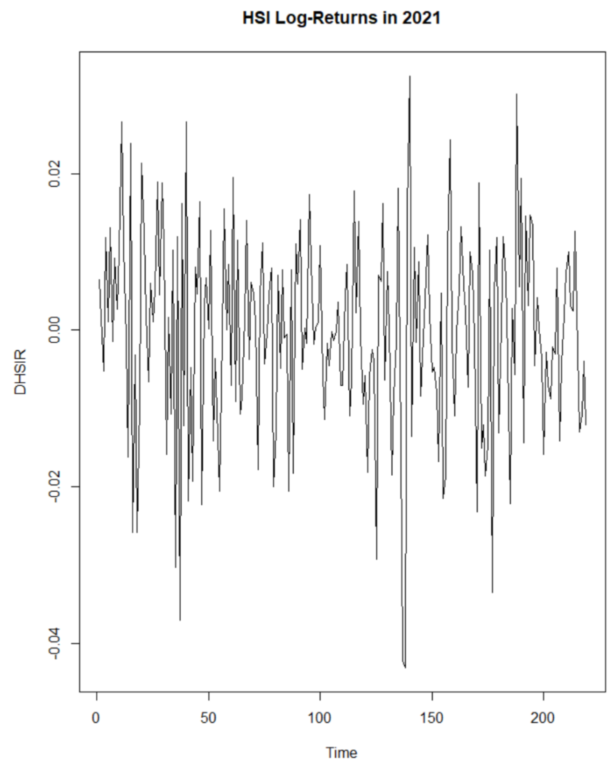
3) Hang Seng Index Analysis

3.1) Time Series Plot of Index and Return

```
DHSI <- HSI$Close
DHSIR <- diff(log(DHSI))

ts.plot(DHSI,main="HSI in 2021")
ts.plot(DHSIR,main="HSI Log-Returns in 2021")
```

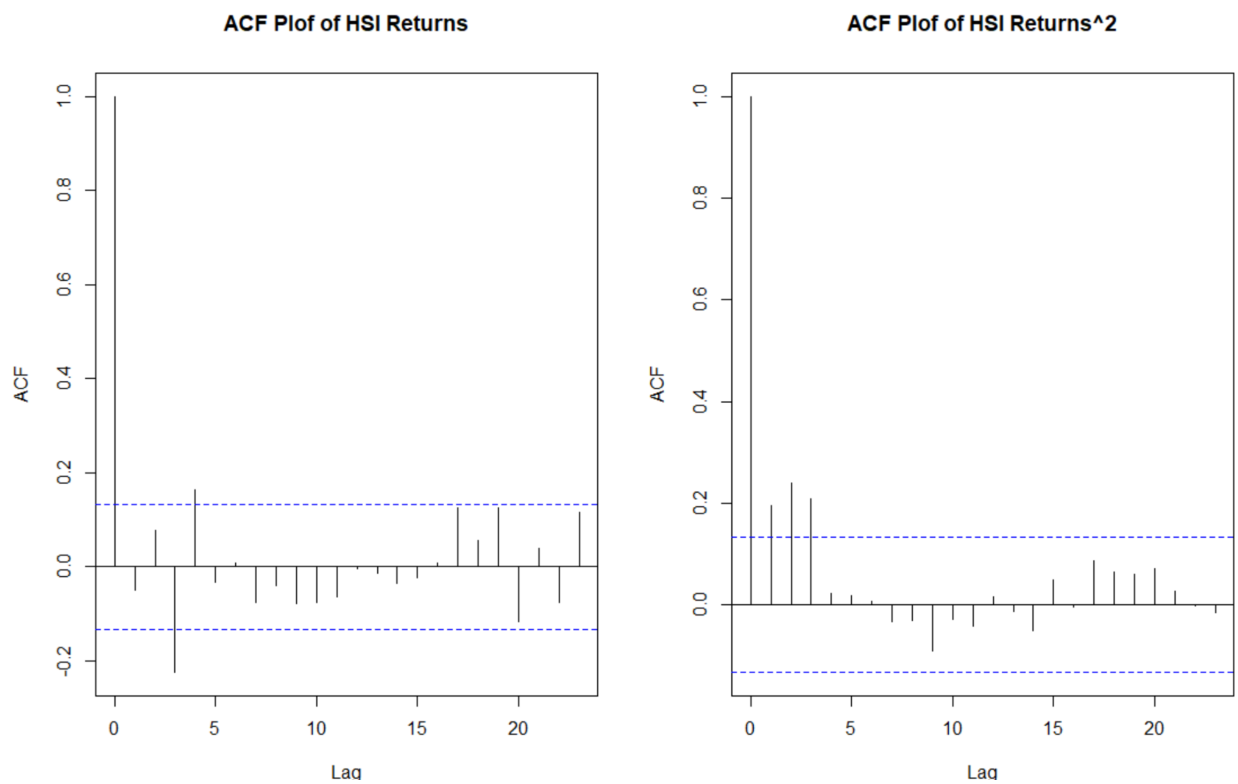




The time series has a decreasing trend with daily fluctuations. The log returns seem to be stationary with mean reversion. The volatility is clustered, which means that there are periods where high variance is followed by high variance, while lower variance is followed by low variance. This would mean the time series is under a heteroskedastic assumption because the variance is conditional on time.

3.2) ACF Returns and Squared Returns

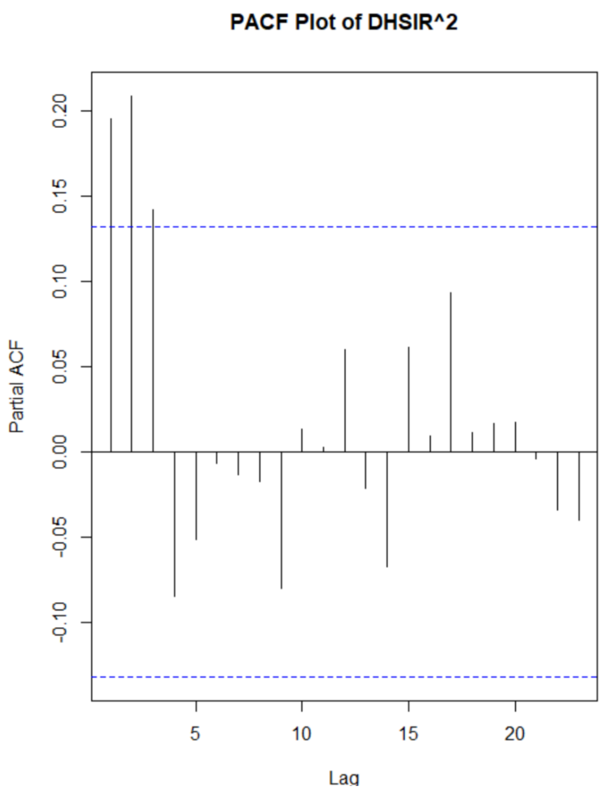
```
acf(DHSIR, main = "ACF Plof of HSI Returns")
acf((DHSIR)^2, main = "ACF Plof of HSI Returns^2")
```

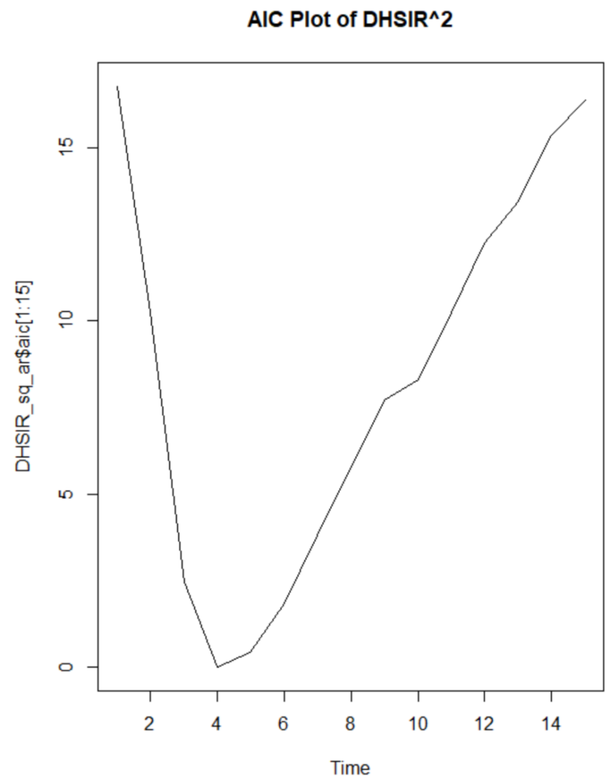


The ACF shows that the squared returns of HSI are more significantly and clearly correlated than the returns of HSI. This would mean DHSIR is not a Gaussian time series because there is correlation in the squared returns, which means the time series is not independent. The time series could possibly be heavy-tailed.

3.3) ARCH Order

```
pacf((DHSIR)^2,main="PACF Plot of DHSIR^2")
DHSIR_sq_ar <- ar((DHSIR)^2)
ts.plot(DHSIR_sq_ar$aic[1:15],main="AIC Plot of DHSIR^2")
DHSIR_sq_ar$order
```





Based on PACF and AIC, the order should be 3.

3.4) GARCH(1,1) Fit

```
library(fGarch)

DHSIR_training = DHSIR[1:length(DHSIR)-5]
DHSIR_GARCH <- garchFit( ~ garch(1,1), data = DHSIR_training, trace = FALSE)
summary(DHSIR_GARCH)
```

```
Coefficient(s):
         mu        omega       alpha1        beta1
-8.6486e-05   3.9168e-05   1.5655e-01   6.0683e-01

Std. Errors:
 based on Hessian

Error Analysis:
         Estimate  Std. Error  t value Pr(>|t|)
mu     -8.649e-05   8.307e-04   -0.104 0.917079
omega   3.917e-05   2.258e-05    1.735 0.082777 .
alpha1  1.565e-01   6.898e-02    2.269 0.023248 *
beta1   6.068e-01   1.643e-01    3.692 0.000222 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

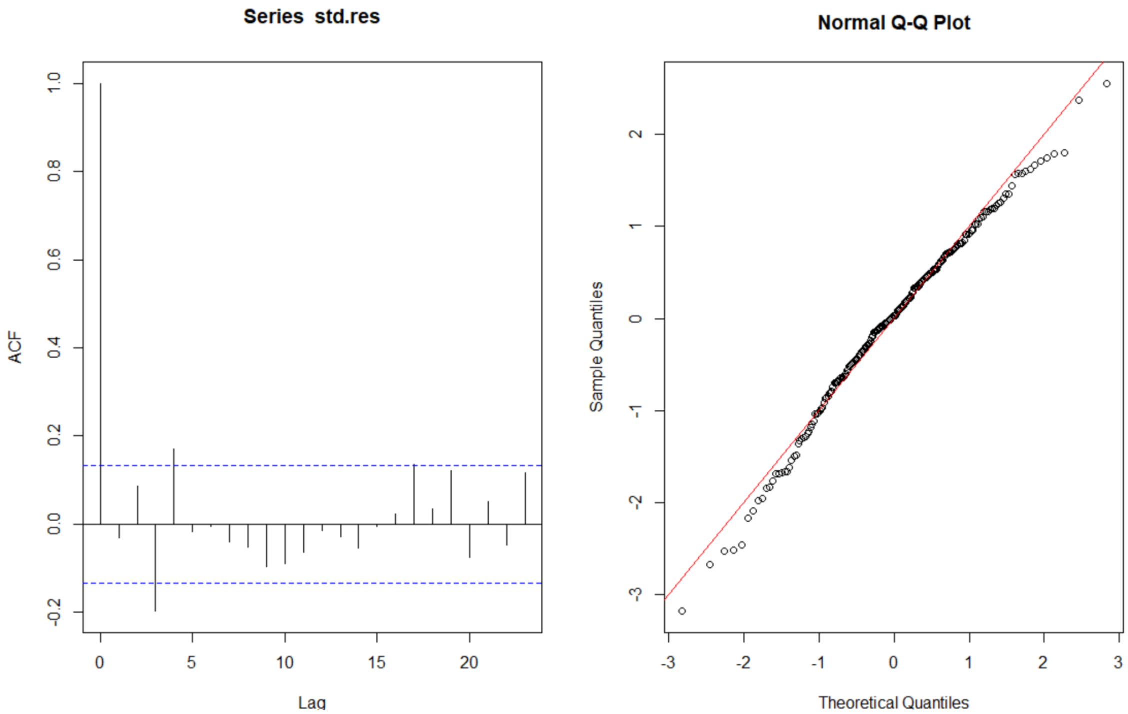
μ = -8.649e-05 with p-value 0.917079 > 0.05; therefore, not significant at 5% level.

3.5) GARCH(1,1) without Mean ~ Model Diagnostics

```
DHSIR_GARCH <- garchFit( ~ garch(1,1), data = DHSIR_training, trace = FALSE, include.mean = FALSE)

  # Model Diagnostics
summary(DHSIR_GARCH)

std.res <- (DHSIR_training)/DHSIR_GARCH@sigma.t
acf(std.res)
qqnorm(std.res)
abline(0,1,col="red")
```



The standardized residuals seem to show marginal correlations, which could indicate that the modelling is not complete because the residuals are not independent. Also the QQ plot shows that the residuals are marginally normal, with slight heavy-tailedness / light-tailedness.

```
Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  5.702193  0.05778093
 Shapiro-Wilk Test  R    W      0.9886259 0.08739031
 Ljung-Box Test     R    Q(10)  21.67269  0.01686167
 Ljung-Box Test     R    Q(15)  23.51191  0.07385934
 Ljung-Box Test     R    Q(20)  33.17965  0.03224014
 Ljung-Box Test     R^2  Q(10)  6.09017   0.8076315
 Ljung-Box Test     R^2  Q(15)  10.57039  0.7824121
 Ljung-Box Test     R^2  Q(20)  13.03305  0.8759614
 LM Arch Test       R    TR^2   8.823933  0.7178908
```

The tests above for the model diagnostics will determine the normality and uncorrelatedness of the residuals. When the p-value is lower than 5% the the null hypothesis is rejected and it either means normality or uncorrelatedness is not evident.

The Jarque-Bera and Shapiro-Wilk Test shows that the residuals are indeed normal. The Ljung-Box Test shows at Q(10) and Q(20) the residuals are correlated, while at Q(15) the residuals are uncorrelated. For the squared residuals, the Ljung-Box Test shows that the squared residuals are uncorrelated. Lastly, the LM Arch Test shows that there are no more ARCH components in the time series.

Overall, the residuals under a normality and squared residuals uncorrelatedness assumption is okay, but the non-squared residuals may be correlated.

3.6) Volatility Forecast (T+1)

```
DHSIR_GARCH@fit$coef

omega <- DHSIR_GARCH@fit$coef[1]
alpha <- DHSIR_GARCH@fit$coef[2]
beta <- DHSIR_GARCH@fit$coef[3]

n <- length(DHSIR_training)
volatility_forecast_1 <- omega + alpha* DHSIR[n]^2+ beta*DHSIR_GARCH@sigma.t[n]^2

prediction <- predict(DHSIR_GARCH, n.ahead = 1)
sd_forecast <- prediction$standardDeviation
volatility_forecast_2 <- sd_forecast^2
c(volatility_forecast_1, volatility_forecast_2)
```

Volatility Forecast: 0.0001320311

3.7) 1%-VaR and 1%-Expected Shortfall (T+1)

```
mu <- 0

  # 1%-VaR forecast under normal for the next day
VaR_normal <- -qnorm(0.01, mu, sd_forecast)

  # 1%-ES forecast under normal
N<-100000
X<-rnorm(N,mu,sd_forecast)
ES_normal <- mean( - X[- X > VaR_normal])

c(VaR_normal,ES_normal)
```

1%-VaR: 0.02673085
1%-Expected Shortfall: 0.03074560