ARMA Modeling

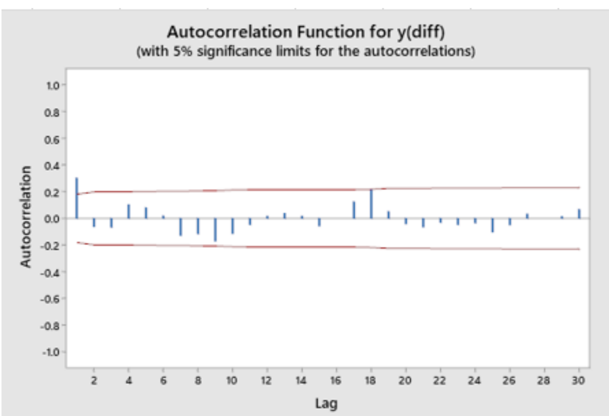
1) Please derive the autocorrelation function (ACF) for the MA(2) model.

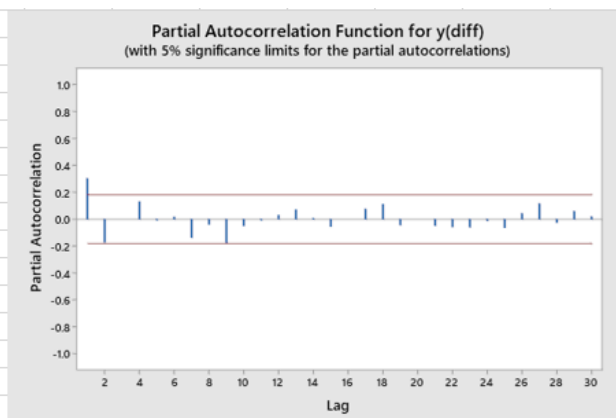
Derivations are at the last few pages.

2) A company that makes absorbent paper towels, would like to develop a forecasting model for weekly sales over and above 100,000 rolls, in units of 10,000 rolls. The time series of actual sales for the last 120 weeks is in a separate data file. Based on the time series plots, SAC, and SPAC, please answer the following questions:

a) Identify a tentative model for the first differences of the said time series and explain your answer.

After differencing the time series, an Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) for y(diff) is computed on Minitab:





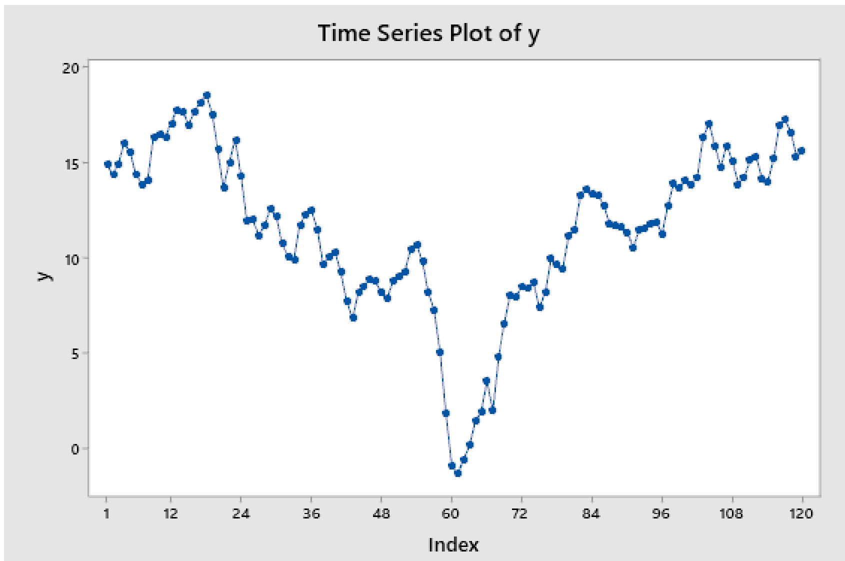
The notable observations for the graphs:

- ACF tapers to zero but seems to have a more extreme cutoff at lag 2
    - However at lag 18 the values are near the red boundaries
- PACF also tapers to zero but have not as extreme cutoff at lag 2 onwards
    - lag 2,4,7... values are near the red boundaries

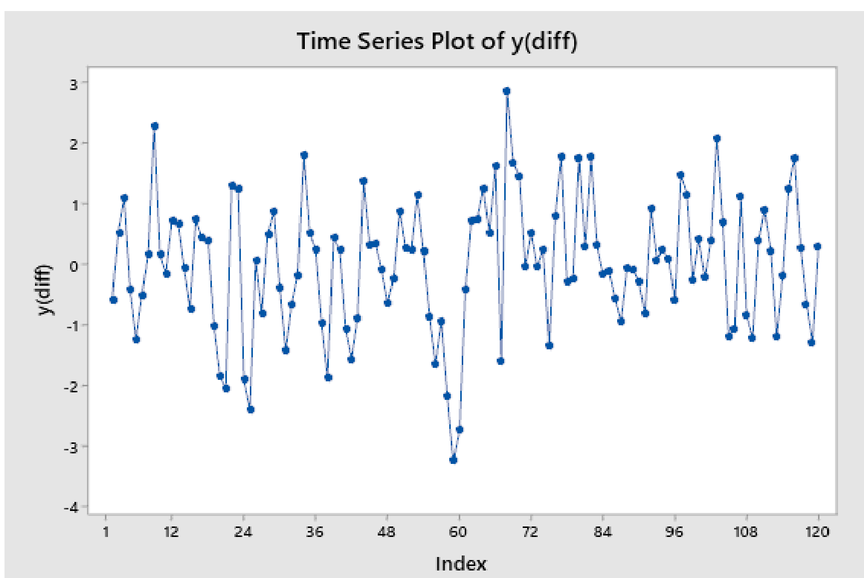
Based on the observations the **tentative models are MA(1) and AR(1)**, respectively. However, only by checking the AIC and BIC of the models, will the most optimal one be chosen. But it can be deduced that the best model may be MA(1) because the ACF graph shows a more extreme cut off, while PACF shows a gradual decrease.

b) Why do we consider the first differences instead of the original time series?

The reason we consider the first difference instead of the original time series is because the original time series does not have a constant mean level and exhibits drifting and wandering behavior. Only a stationary series - with constant mean level - can be modelled with AR(p) or MA(q). It is clearly not a stationary series, as shown in the figure below:



After taking successive differences of the time series, we obtain the first difference series, which looks like it has a constant mean level.



MA(2)

$$\gamma_0 = E(\overset{a}{a_t} - \overset{x}{\theta_1}\overset{b}{a_{t-1}} - \overset{y}{\theta_2}\overset{c}{a_{t-2}})^2$$
$$= E(-2\theta_1 a_t a_{t-1} + \theta_1^2 a_{t-1}^2 + 2\theta_1\theta_2 a_{t-1}a_{t-2} + a_t^2 + 2\theta_2 a_t a_{t-2} + \theta_2^2 a_{t-2}^2)$$
$$= E(-2\theta_1 \cancel{a_t} a_{t-1} + \theta_1^2 a_{t-1}^2 + 2\theta_1\theta_2 \cancel{a_{t-1}} a_{t-2} + a_t^2 + 2\theta_2 \cancel{a_t} a_{t-2} + \theta_2^2 a_{t-2}^2)$$
$$= E(\theta_1^2 a_{t-1}^2 + a_t^2 + \theta_2^2 a_{t-2}^2)$$
$$= \sigma_a^2(1 + \theta_1^2 + \theta_2^2)$$

$$\gamma_1 = E[(\overset{a}{a_t} - \overset{x}{\theta_1}\overset{b}{a_{t-1}} - \overset{y}{\theta_2}\overset{c}{a_{t-2}})(\overset{b}{a_{t-1}} - \overset{x}{\theta_1}\overset{c}{a_{t-2}} - \overset{y}{\theta_2}\overset{d}{a_{t-3}})]$$
$$= E(a_t a_{t-1} - \theta_1 a_t a_{t-2} - \theta_2 a_t a_{t-3} - \theta_1 a_{t-1}^2 + \theta_1^2 a_{t-1}a_{t-2} + \theta_1\theta_2 a_{t-1}a_{t-3} - \theta_2 a_{t-1}a_{t-2} + \theta_1\theta_2 a_{t-2}^2 + \theta_2^2 a_{t-2}a_{t-3})$$
$$= E(\cancel{a_t} a_{t-1} - \theta_1 \cancel{a_t} a_{t-2} - \theta_2 \cancel{a_t} a_{t-3} - \theta_1 a_{t-1}^2 + \theta_1^2 \cancel{a_{t-1}} a_{t-2} + \theta_1\theta_2 \cancel{a_{t-1}} a_{t-3} - \theta_2 \cancel{a_{t-1}} a_{t-2} + \theta_1\theta_2 a_{t-2}^2 + \theta_2^2 \cancel{a_{t-2}} a_{t-3})$$
$$= E(-\theta_1 a_{t-1}^2 + \theta_1\theta_2 a_{t-2}^2)$$
$$= \sigma_a^2(\theta_1\theta_2 - \theta_1)$$

$$\gamma_2 = E[(\overset{a}{a_t} - \overset{x}{\theta_1}\overset{b}{a_{t-1}} - \overset{y}{\theta_2}\overset{c}{a_{t-2}})(\overset{c}{a_{t-2}} - \overset{x}{\theta_1}\overset{d}{a_{t-3}} - \overset{y}{\theta_2}\overset{e}{a_{t-4}})]$$
$$= E(a_t a_{t-2} - \theta_1 a_t a_{t-3} - \theta_2 a_t a_{t-4} - \theta_1 a_{t-1}a_{t-2} + \theta_1^2 a_{t-1}a_{t-3} + \theta_1\theta_2 a_{t-1}a_{t-4} - \theta_2 a_{t-2}^2 + \theta_1\theta_2 a_{t-2}a_{t-3} + \theta_2^2 a_{t-2}a_{t-4})$$
$$= E(\cancel{a_t} a_{t-2} - \theta_1 \cancel{a_t} a_{t-3} - \theta_2 \cancel{a_t} a_{t-4} - \theta_1 \cancel{a_{t-1}} a_{t-2} + \theta_1^2 \cancel{a_{t-1}} a_{t-3} + \theta_1\theta_2 \cancel{a_{t-1}} a_{t-4} - \theta_2 a_{t-2}^2 + \theta_1\theta_2 \cancel{a_{t-2}} a_{t-3} + \theta_2^2 \cancel{a_{t-2}} a_{t-4})$$
$$= E(-\theta_2 a_{t-2}^2)$$
$$= \sigma^2(-\theta_2)$$

$$\gamma_3 = E[(\overset{a}{a_t} - \overset{x}{\theta_1}\overset{b}{a_{t-1}} - \overset{y}{\theta_2}\overset{c}{a_{t-2}})(\overset{d}{a_{t-3}} - \overset{x}{\theta_1}\overset{e}{a_{t-4}} - \overset{y}{\theta_2}\overset{f}{a_{t-5}})]$$
$$= E(a_t a_{t-3} - \theta_1 a_t a_{t-4} - \theta_2 a_t a_{t-5} - \theta_1 a_{t-1}a_{t-3} + \theta_1^2 a_{t-1}a_{t-4} + \theta_1\theta_2 a_{t-1}a_{t-5} - \theta_2 a_{t-2}a_{t-3} + \theta_1\theta_2 a_{t-2}a_{t-4} + \theta^2 a_{t-2}a_{t-5})$$
$$= E(\cancel{a_t} a_{t-3} - \theta_1 \cancel{a_t} a_{t-4} - \theta_2 \cancel{a_t} a_{t-5} - \theta_1 \cancel{a_{t-1}} a_{t-3} + \theta_1^2 \cancel{a_{t-1}} a_{t-4} + \theta_1\theta_2 \cancel{a_{t-1}} a_{t-5} - \theta_2 \cancel{a_{t-2}} a_{t-3} + \theta_1\theta_2 \cancel{a_{t-2}} a_{t-4} + \theta^2 \cancel{a_{t-2}} a_{t-5})$$
$$= E(0) = 0$$

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\sigma_a^2(\theta_1\theta_2 - \theta_1)}{\sigma_a^2(1+\theta_1^2+\theta_2^2)}$$

$$= \frac{(\theta_1\theta_2 - \theta_1)}{(1+\theta_1^2+\theta_2^2)}$$

$$= \frac{-\theta_1(1-\theta_2)}{(1+\theta_1^2+\theta_2^2)}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\sigma_a^2(-\theta_2)}{\sigma_a^2(1+\theta_1^2+\theta_2^2)}$$

$$= \frac{-\theta_2}{(1+\theta_1^2+\theta_2^2)}$$

$$\rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{0}{\sigma_a^2(1+\theta_1^2+\theta_2^2)}$$

$$= 0$$

$$\rho_k = 0 \qquad k = 3, 4 \ldots$$