MATH 2043
All-in-one Summary

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1 Completeness of Ordered Field

- **Ordered Field**: A field K together with order.
- **Completeness of $\mathbb{R}$**: There exists a unique ordered field satisfying LUB, i.e. $\mathbb{R}$ is the unique complete ordered field.

Let R be any ordered field.

- **Supremum**: For any $S \subset R$, $\lambda = \sup S$ if
  - λ is an upper bound: For every $s \in S$, $s \leq \lambda$.
  - λ is the smallest: If μ is another upper bound, $\lambda \leq \mu$. Equivalently, for every $\varepsilon > 0$, there exists $s \in S$, such that $\lambda - \varepsilon < s \leq \lambda$.
- **Least Upper Bound Property**: R has least upper bound property if every nonempty bounded above subset of R has a least upper bound in R.
- **Archimedean Property**: $\mathbb{N} \subset R$ is bounded.
  - For every $x \in R$, there exists $n \in \mathbb{N}$, such that $n > x$.
  - For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $\frac{1}{n} < \frac{1}{N} < \varepsilon$, for every $n > N$.
  - $\lim_{n\to\infty} \frac{1}{n} = 0$.
- LUB$\Rightarrow$AP.
- $S \subset R$ is nonempty and bounded above.
  - If λ is an upper bound and there exists (x_n) in S such that $\lim_{n\to\infty} x_n = \lambda$, then $\lambda = \sup S$.
  - The opposite direction holds if R satisfies AP.
- LUB$\Leftrightarrow$MCT.
- MCT$\Rightarrow$AP.
- **Dedekind Cut**: $\underline{A} \subset \mathbb{Q}$ is a cut If

  - $\underline{A} \neq \mathbb{Q}$ and $\underline{A} \neq \emptyset$.
  - If $x \in \underline{A}$, $y \in \mathbb{Q}$ and $y < x$, then $y \in \underline{A}$.
  - $\underline{A}$ has no maximum: For every $x \in \underline{A}$, there exists $z \in \underline{A}$, such that $x < z$.
- The collection of all cuts $\underline{\mathbb{R}}$ satisfies LUB.
- If K is another complete ordered field, then there exists an ordered field isomorphism $\varphi : \underline{\mathbb{R}} \to K$, such that
  - φ is bijective.
  - If $a < b$, then $\varphi(a) < \varphi(b)$.
  - $\varphi(a+b) = \varphi(a) + \varphi(b)$.
  - $\varphi(ab) = \varphi(a)\varphi(b)$.

2 Limit Superior and Limit Inferior

From now on, we work with $\mathbb{R}$.

- **Limit Set**: $\mathrm{LIM}(x_n) :=$ set of all limits of convergent subsequences in $\hat{\mathbb{R}}$.
  - $L \in \mathrm{LIM}(x_n)$ iff for every $\varepsilon > 0$, there exist infinitely many x_n with $|x_n - L| < \varepsilon$.
- **Limit Superior**: $\limsup_{n\to\infty}(x_n) := \sup \mathrm{LIM}(x_n)$.
- Let (x_n) be bounded and $L \in \mathbb{R}$.
  - If $L > \limsup_{n\to\infty} x_n$, then there exist finitely many $x_n > L$.
  - If $L < \limsup_{n\to\infty} x_n$, then there exist infinitely many $x_n > L$.
- $\limsup_{n\to\infty} x_n \in \mathrm{LIM}(x_n)$.
- (x_n) converges iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.
- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} M_n$, where $M_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$.
- **Ratio Test**:
  - If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely.
  - If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
- **Root Test**:
  - If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely.
  - If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
- Root Test$\Rightarrow$Ratio Test.

3 Topology on $\mathbb{R}$

- **Open Set**: $S \subset \mathbb{R}$ is open if for every $x \in S$, there exists $\varepsilon > 0$, such that $(x-\varepsilon, x+\varepsilon) \subset S$.
- **Closed Set**: S is closed if $S^{\complement}$ is open.
  - $S \subset \mathbb{R}$ is closed iff if (x_n) in S converges to $L \in \mathbb{R}$, then $L \in S$.
  - If nonempty $S \subset \mathbb{R}$ is closed and bounded, then $\sup S = \max S$.

  - Union of open sets is open.
  - Intersection of closed sets is closed.
  - Finite intersection of open sets is open.
  - Finite union of closed sets is closed.
- **Continuity**: $f(x)$ is continuous at $x = a$ iff
  - $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$.

  or

  - for every $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(a)| < \varepsilon$ if $|x - a| < \delta$.
- $f(x)$ is continuous on $\mathbb{R}$ iff for every open $U \subset \mathbb{R}$, $f^{-1}(U)$ is open.
- **Continuity on $D \subset \mathbb{R}$**: $f(x)$ is continuous on $D \subset \mathbb{R}$ if for every $\varepsilon > 0$, there exists $\delta_x > 0$, such that if $|x - y| < \delta_x$, then $|f(x) - f(y)| < \varepsilon$.
- **Uniform Continuity on $D \subset \mathbb{R}$**: $f(x)$ is uniformly continuous on $D \subset \mathbb{R}$ if for every $\varepsilon > 0$, there exists $\delta > 0$, such that for every $x, y \in D$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.
- If f is continuous on compact set D, then f is uniformly continuous on D.
- **Lipschitz Continuity on $D \subset \mathbb{R}$**: $f(x)$ is Lipschitz continuous on $D \subset \mathbb{R}$ if there exists $\alpha > 0$, such that $|f(x) - f(y)| \leq \alpha |x - y|$, for every $x, y \in D$.
- Lipschitz Continuity$\Rightarrow$Uniform Continuity.
- If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $|f'(x)| \leq M$ for some M, then f is uniformly continuous.
- **Compact Set**: $X \subset \mathbb{R}$ is compact if any open cover has a finite subcover, i.e. for every collection of open sets $\bigcup_\alpha U_\alpha$, such that $X \subset \bigcup_\alpha U_\alpha$, there exists $U_{\alpha_1}, \ldots, U_{\alpha_N}$, such that $X \subset \bigcup_{i=1}^{N} U_{\alpha_i}$.
- **Heine-Borel Theorem**: $X \subset \mathbb{R}$ is compact iff X is closed and bounded.
- HBT$\Leftrightarrow$AC.

4 Metric Spaces

- **Metric Space**: (X,d) is a metric space if $d : X \times X \to \mathbb{R}$ such that
  - $d(x,y) \geq 0$, for every $x,y \in X$ and $d(x,y) = 0$ iff $x = y$.
  - $d(x,y) = d(y,x)$, for every $x,y \in X$.
  - $d(x,y) \leq d(x,z) + d(z,y)$, for every $x,y,z \in X$.
- **Limit in Metric Space:** A sequence (x_n) in X converges to some $L \in X$ iff for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for every $n > N$, $d(x_n, L) < \varepsilon$.
- **Normed Vector Space**: $(V, \|\cdot\|)$ is a normed vector space if $\|\cdot\| : V \to \mathbb{R}$ such that
  - $\|\mathbf{x}\| \geq 0$, for every $\mathbf{x} \in V$ and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$.
  - $\|c\mathbf{x}\| = |c|\,\|\mathbf{x}\|$, for every $c \in \mathbb{R}$ and $\mathbf{x} \in V$.
  - $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, for every $\mathbf{x}, \mathbf{y} \in V$.
- A normed vector space is a metric space with $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.
- $(\mathbf{x}_n)$ in $\mathbb{R}^N$ converges to some $\mathbf{x}$ in $\|\cdot\|_2$ iff (x_n^i) converges to $x^i \in \mathbb{R}$, for every i, where x_n^i is the i-th component of $\mathbf{x}_n$.
- **Bolzano-Weierstrass Theorem in $\mathbb{R}^N$**: If $\|\mathbf{x}_n\| \leq B$ for some $B \in \mathbb{R}$ for every $n \in \mathbb{N}$, then there exists subsequence $\left(\mathbf{x}_{n_k}\right)$ converging to some $\mathbf{x}$.
- **Complete Space (Banach Space)**: $(V, \|\cdot\|)$ is complete if every Cauchy sequence in V converges in V.

5 Topology on Metric Spaces

With respect to metric d or norm $\|\cdot\|$ in X,

- **Open Ball**: $B_d(x,r) = \{y \in X : d(x,y) < r\}$.
- **Closed Ball**: $\overline{B}(x,r) = \{y \in X : d(x,y) \leq r\}$.
- **Sphere**: $S(x,r) = \{y \in X : d(x,y) = r\}$.

Let $E \subset X$.

- **Interior Point**: $p \in E$ is an interior point if there exists $\varepsilon > 0$, such that $B(p, \varepsilon) \subset E$. The set of all interior points of E is denoted by E°.
- **Open Set**: E is open in X if every $p \in E$ is an interior point.
- E° is open.
- **Closed Set**: E is closed in X iff $E^\complement$ is open.

- **Limit Point**: $p \in X$ is a limit point iff for every $\varepsilon > 0$, there exists $q \in E$ with $q \neq p$, such that $q \in B(p, \varepsilon)$. The set of all limit points of E is denote by E'.
- **Isolated Point**: If $p \in E$ is not a limit point, then it is an isolated point.
- **Perfect Set**: E is perfect if it is closed and has no isolated points.
- E is closed iff every limit point of E is in E, or equivalently every converging sequence in E has limit in E.
- **Boundness**: E is bounded if there exists $M > 0$, such that $d(p,q) < M$, for every $p, q \in E$.
- The properties of intersection and union of open and closed sets follow from the topology on $\mathbb{R}$.
- **Closure**: The closure of E is defined by $\overline{E} = E \cup E'$.
- $\overline{X \setminus E} = X \setminus E^\circ$.
- $\overline{E}$ is closed and is the smallest closed set containing E.
- E is closed iff $E = \overline{E}$.
- **Dense Set**: E is dense in $X \Leftrightarrow \overline{E} = X \Leftrightarrow E^{\complement}$ has empty interior $\Leftrightarrow$ for every $p \in X$, either $p \in E$ or $p \in E' \Leftrightarrow$ for every $p \in X$ and every $\varepsilon > 0$, there exists $q \in E$, such that $q \in B(p, \varepsilon)$.
- **Boundary**: The boundary of E is defined by ∂E, which is the set of all $x \in X$, such that for every $\varepsilon > 0$, $B(x.\varepsilon)$ contains some point in E and some point not in E.
  - $\overline{E} = E \cup \partial E$.
  - $E^\circ = E \setminus \partial E = \overline{E} \setminus \partial E$.
  - ∂E is closed.
- **Compact Set**: $K \subset X$ is compact if every open cover of K has finite subcover.
- **Sequentially Compact Set**: $K \subset X$ is sequentially compact if every sequence in K has a converging subsequence with limit in K, or equivalently every infinite subset contains a limit point.
- Compact $\Leftrightarrow$ Sequentially compact.
- **Lebesgue Number Theorem**: Let K be a sequentially compact set in K covered by $\bigcup_\alpha U_\alpha$. Then, there exists $\delta > 0$, such that for every $x \in K$, $B(x, \delta) \subset U_\alpha$ for some α.
- **Complete Set**: $K \subset X$ is complete if every Cauchy sequence in K converges in K.
  - If K is complete then K is closed.
- **Totally Bounded Set**: $K \subset X$ is totally bounded if for every $\varepsilon > 0$, there exists some finite cover by open balls of radius ε.
  - If K is totally bounded then K is bounded.

- **Heine-Borel Theorem in a metric space**: K is compact $\Leftrightarrow$ K is sequentially compact $\Leftrightarrow$ K is complete and totally bounded.
  - In $(\mathbb{R}^n, \|\cdot\|_2)$, K is compact iff K is closed and bounded, K is complete iff K is closed, and K is totally bounded iff K is bounded.
- Let $Y \subset X$ with induced metrix $d|_Y$. $K \subset X$ is compact in X iff $K \subset Y$ is compact in Y.
- **Continuity**: $f : X \to Y$ is continuous at $a \in X$ iff for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $d_X(x,a) < \delta$, then $d_Y(f(x), f(a)) < \varepsilon$.
- f is continuous on $X \Leftrightarrow$ for every open $U \subset Y$, $f^{-1}(U)$ is open in X. $\Leftrightarrow$ for every closed $V \subset Y$, $f^{-1}(V)$ is closed in X.
- f is continuous on $X \Leftrightarrow \overline{f^{-1}(E)} \subset f^{-1}\left(\overline{E}\right)$, for every $E \subset Y \Leftrightarrow f\left(\overline{D}\right) \subset \overline{f(D)}$, for every $D \subset X$.
- All norms are continuous.
- If f is continuous and K is compact, then $f(K)$ is compact.
- **Extreme Value Theorem**: Let $f : K \to \mathbb{R}$ be continuous, where K is compact. Then, f attains maximum and minimum on K, i.e. there exists $x_{max} \in K$, such that $f(x_{\max}) = \sup_{x \in K} f(x)$.
- Continuity$\Rightarrow$Uniform Continuity.
- **Equivalent Norms**: Two norms $\|\cdot\|, \|\cdot\|'$ on V are equivalent iff there exists $c > 0$, such that $\frac{1}{c}\|\mathbf{x}\| \leq \|\mathbf{x}\|' \leq c\|\mathbf{x}\|$, for every $\mathbf{x} \in V$.
  - $\mathbf{x}_n \to \mathbf{x}$ in $\|\cdot\| \iff \mathbf{x}_n \to \mathbf{x}$ in $\|\cdot\|'$.
  - If $\|\cdot\|$, $\|\cdot\|'$ are equivalent, then the collection of open sets in $(V, \|\cdot\|), (V, \|\cdot\|')$ is the same.
  - If $f : (V, \|\cdot\|) \to X$ is continuous, then $f : \left(V, \|cdot\|'\right) \to X$ is continuous.
- **Equivalence of norms in $\mathbb{R}^n$**: Any norm $\|\cdot\|_*$ in $\mathbb{R}^n$ is equiavalent to $\|\cdot\|_2$.
  - $\|\cdot\|_*$ is continuous with respect to $\|\cdot\|_2$.
- **Disconnected Set**: $S \subset X$ is disconnected if there exists $A, B \subset X$, such that $S = A \sqcup B$, $A, B \neq \emptyset$ and $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. If S is not disconnected, then S is connected.
  - $S \subset \mathbb{R} \iff$ If $x, y \in S$ and $x < y$, then $[x,y] \subset S$.
- Let $f : (X, d_X) \to (Y, d_Y)$ be continuous. Then if $S \subset X$ is connected, $f(S) \subset Y$ is connected.
  - **Intermediate Value Theorem**.
- If S is connected, then $\overline{S}$ is connected.
- **Path Connectedness**: $S \subset X$ is path-connected if for every $x, y \in S$, there exists continuous path $\gamma : [0,1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

- If $S \subset X$ is path-connected, then it is connected.

- **Nowhere Dense Set**: $A \subset X$ is nowhere dense if $\overline{A}$ has empty interior or equivalently $A^{\complement}$ contains an open dense subset.
  - Note that if S is dense, then $\overline{S} = X$, i.e. for every $p \in X$ and every $\varepsilon > 0$, $B(p,\varepsilon) \cap S \neq \emptyset$, or equivalently, $S^{\complement}$ has empty interior.

- **First Category Set**: $A \subset X$ is first category if it is a countable union of nowhere dense subsets. Otherwise, it is second category.

- **Baire's Category Theorem**: If X is complete metric space, then
  - countable intersection of open dense set is dense.
  - first category set has empty interior.
  - complement of first category set is dense.

- **Weak BCT**(dense$\Rightarrow$nonempty): If X is complete metric space (nonempty),
  - countable intersection of open dense is nonempty.
  - X is second category.
  - If X is countable union of closed set, then at least one of the closed set has nonempty interior.

- The set of discontinuity of a function $f : \mathbb{R} \to \mathbb{R}$ is a countable union of closed set.
  - $\mathbb{R} \setminus \mathbb{Q}$ cannot be the set of discontinuity of a function $f : \mathbb{R} \to \mathbb{R}$.

6 Multivariable Calculus

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$. Let $\mathbf{a} \in \mathbb{R}^n$.

- **Continuity**: $\mathbf{F}$ is continuous at $\mathbf{a}$ if for every $\varepsilon > 0$, there exists $\delta > 0$, such that for every $\|\mathbf{x} - \mathbf{a}\| < \delta$, $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{a})\| < \varepsilon$.

- **Differentiability ($\mathbb{R}^n \to \mathbb{R}$)**: $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$, if there exists a linear approximation $L(\mathbf{x})$ at $\mathbf{a}$, such that $F(\mathbf{x}) = F(\mathbf{a}) + L(\mathbf{x} - \mathbf{a}) + o(\|\mathbf{x} - \mathbf{a}\|)$.
  - If F is differentiable at $\mathbf{a}$, then $\frac{\partial F}{\partial x_i}(\mathbf{a})$ exists for every $1 \le i \le n$ and $L(\mathbf{x} - \mathbf{a}) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\mathbf{a})(x_i - a_i) = \nabla F_{\mathbf{a}} \cdot (\mathbf{x} - \mathbf{a})$.
  - The converse is not true, e.g. $f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is not differentiable at $(0,0)$.

- **Differentiability ($\mathbb{R}^n \to \mathbb{R}^m$)**: $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$ if each component F^i is differentiable at $\mathbf{a}$, i.e. $\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{a}) + D\mathbf{F}_{\mathbf{a}} \cdot (\mathbf{x} - \mathbf{a}) + o(\|\mathbf{x} - \mathbf{a}\|)$, where $D\mathbf{F}_{\mathbf{a}}$ is the $m \times n$ Jacobian matrix $\left(\frac{\partial F^i}{\partial x_j}\right)$.

- If $F:\mathbb{R}^n \to \mathbb{R}$, $DF_{\mathbf{a}} = \nabla F_{\mathbf{a}} = \left(\frac{\partial F}{\partial x_i}\right)_{1\times n}$.

- $\mathscr{C}^1$: $\mathbf{F}$ is $\mathscr{C}^1$ on an open set $U \subset \mathbb{R}^n$ if all partial derivatives $\frac{\partial F^i}{\partial x_j}$ exist and are continuous on U.
  - If $\mathbf{F}$ is $\mathscr{C}^1$ on U, then $\mathbf{F}$ is differentiable on U.

- **Chain Rule**: If $\mathbf{G}:\mathbb{R}^k \to \mathbb{R}^n$ is differentiable at $\mathbf{a}$ and $\mathbf{F}:\mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{G}(\mathbf{a})$, then $\mathbf{F}\circ\mathbf{G}:\mathbb{R}^k \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$ with $D(\mathbf{F}\circ\mathbf{G})_{\mathbf{a}} = D\mathbf{F}_{\mathbf{G}(\mathbf{a})} D\mathbf{G}_{\mathbf{a}}$.
  - $\mathbf{F}:\mathbb{R}^n \to \mathbb{R}^n$ is differentiable at $\mathbf{a}$ and its inverse function $\mathbf{F}^{-1}:\mathbb{R}^n \to \mathbb{R}^n$ exists and is differentiable at $\mathbf{F}(\mathbf{a})$, then $(D\mathbf{F}_{\mathbf{a}})^{-1} = D\left(\mathbf{F}^{-1}\right)_{\mathbf{F}(\mathbf{a})}$.

- **Inverse Function Theorem**: If $\mathbf{F}:\mathbb{R}^n \to \mathbb{R}^m$ is $\mathscr{C}^1$ and $D\mathbf{F}_{\mathbf{a}}$ is invertible, then there exists an open $U \subset \mathbb{R}^n$ such that $\tilde{\mathbf{F}} := \mathbf{F}|_U : U \to \mathbf{F}(U)$ is invertible and $\tilde{\mathbf{F}}^{-1}$ is $\mathscr{C}^1$.

- **Banach Contraction Mapping Theorem**: Let X be a complete metric space and $F:X\to X$. If there exists $\alpha \in [0,1)$ such that $d(F(x),F(y)) \le \alpha d(x,y)$ for every $x,y \in X$, then there exists a unique fixed point $x_* \in X$.

- **Vector-Valued Mean Value Theorem**: Let $\mathbf{F}:\mathbb{R}^n \to \mathbb{R}^m$ be $\mathscr{C}^1$ and $V \subset \mathbb{R}^n$ be a convex subset. If $\|D\mathbf{F}\| \le M$ is bounded by some $M>0$ on V, then $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \le M\|\mathbf{x}-\mathbf{y}\|$ for every $\mathbf{x},\mathbf{y} \in V$.

- **Implicit Function Theorem**: Let $\mathbf{F}:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be $\mathscr{C}^1$ and $\mathbf{F}(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{0}$. If $D_{\mathbf{y}}\mathbf{F}_{(\mathbf{x}_0,\mathbf{y}_0)}$ is invertible, then there exists an open $U \subset \mathbb{R}^n$ with $\mathbf{x}_0 \in U$ and exists a unique $\mathbf{G}:U \to \mathbb{R}^m$ such that
  - $\mathbf{G}(\mathbf{x}_0) = \mathbf{y}_0$,
  - $\mathbf{F}(\mathbf{x},\mathbf{G}(\mathbf{x})) = \mathbf{0}$,

  and $\mathbf{G}$ is $\mathscr{C}^1$ near $\mathbf{x}_0$ with $D\mathbf{G} = -(D_{\mathbf{y}}\mathbf{F})^{-1} D_{\mathbf{x}}\mathbf{F}$.

- **Higher Order Differentiability**: $F:\mathbb{R}^N \to \mathbb{R}$ is k-th order differentiable at $\mathbf{a} \in \mathbb{R}^N$ if there exists a degree k polynomial $P_k(\mathbf{x})$ such that $F(\mathbf{x}) = P_k(\mathbf{x}) + o\left(\|\mathbf{x}-\mathbf{a}\|^k\right)$.
  - If $\frac{\partial F}{\partial x_i}$ exists near $\mathbf{a}$ and $\frac{\partial F}{\partial x_i}$ is differentiable at $\mathbf{a}$, then F is 2^{nd} order differentiable at $\mathbf{a}$.
  - If F is $\mathscr{C}^2$ on $U \subset \mathbb{R}^N$, it is 2^{nd} order differentiable.
  - If $F:\mathbb{R}^N \to \mathbb{R}$ has $(k-1)$-th partial derivatives near $\mathbf{a}$ and differentiable at $\mathbf{a}$, then F is k-th order differentiable at $\mathbf{a}$.
  - If F is $\mathscr{C}^k$, then F is k-th differentiable.

- **Mixed Partial Theorem**: Let $f:\mathbb{R}^N \to \mathbb{R}$ be differentiable at $\mathbf{a}$. If f_x, f_y exist near $\mathbf{a}$ and are differentiable at $\mathbf{a}$, then $f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a})$. If f is $\mathscr{C}^2$, then the conclusion also holds.

- **Hessian Matrix**: Let $F:\mathbb{R}^N \to \mathbb{R}$ be $\mathscr{C}^2$. $\mathbf{H}_F = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{i,j=1}^N$.

- **Second Order Derivative Test**: Assume that $F:\mathbb{R}^N \to \mathbb{R} \in \mathscr{C}^2$. If F has a critical point at $\mathbf{a}$, i.e. $\nabla F_{\mathbf{a}} = \mathbf{0}$, then $\mathbf{a}$ is a

- local min if $\mathbf{H}_F$ is positive definite at $\mathbf{a}$,
- local max if $\mathbf{H}_F$ is negative definite at $\mathbf{a}$,
- saddle point if $\mathbf{H}_F$ is indefinite at $\mathbf{a}$.

7 Riemann Integration

Let $f : \mathbb{R} \to \mathbb{R}$ be defined on $[a,b]$.

- **Notation**:
  - $P : a = x_0 < x_1 < \ldots < x_n = b$
  - $U(P, f) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x)(x_i - x_{i-1})$
  - $L(P, f) = \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)(x_i - x_{i-1})$
  - $S(P^*, f) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$
- **Equivalence of Riemann Integral and Darboux Integral**: Assume that f is bounded on $[a,b]$. The following are equivalent:
  - **Darboux Integral**: For every $\varepsilon > 0$, there exists P, such that $U(P, f) - L(P, f) < \varepsilon$.
  - **Riemann Integral**: There exists $I \in \mathbb{R}$, such that for every $\varepsilon > 0$, there exists $\delta > 0$, such that for every $\|P\| < \delta$, $I - \varepsilon < L(P, f) \le S(P^*, f) \le U(P, f) < I + \varepsilon$, i.e. $|S(P^*, f) - I| < \varepsilon$.
- **Oscillation**: The oscillation of a bounded f on D is defined by $\omega_D(f) := \sup_{x,y\in D} |f(x) - f(y)| = \sup_{x\in D} f(x) - \inf_{x\in D} f(x)$.

  $\omega(P, f) := \sum_{i=1}^{n} \omega_{[x_{i-1},x_i]}(f)(x_i - x_{i-1})$. Hence, we can rewrite the Riemann and Darboux Integral:
  - For every $\varepsilon > 0$, there exists P, such that $\omega(P, f) < \varepsilon$.
  - For every $\varepsilon > 0$, there exists $\delta > 0$, such that for every $\|P\| < \delta$, $\omega(P, f) < \varepsilon$.

  $\omega_f(x) = \inf_{\delta>0} \omega_{(x-\delta,x+\delta)}(f)$.
- **Properties of Riemann Integrable Functions**:
  - Riemann integrable functions are bounded.
  - If f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$, and hence $f \in \mathscr{R}[a,b]$.
  - If f is bounded and has finitely many discontinuities, then $f \in \mathscr{R}[a,b]$.
  - **Mean Value Theorem for Integrals**: If $f \in \mathscr{R}[a,b]$ is continuous on $[a,b]$, then there exists $c \in [a,b]$, such that $\frac{1}{b-a} \int_a^b f(x)\mathrm{d}x = f(c)$.
- **Lebesgue Integrability Criterion**: f is Riemann integrable on $[a,b]$ if and only if f is bounded and $\mathscr{S}_f$ has Lebesgue measure zero, where $\mathscr{S}_f$ is the set of discontinuities of f on $[a,b]$.

- If f is Riemann integrable on $[a,b]$ and g is continuous on $f[a,b]$ and is bounded, then $g \circ f \in \mathscr{R}[a,b]$.
- If f is Riemann integrable on $[a,b]$ and g differs from f at finitely many points on $[a,b]$, then $g \in \mathscr{R}[a,b]$.

- **Lebesgue Measure Zero**: $S \subset \mathbb{R}$ has Lebesgue measure zero if for every $\varepsilon > 0$, there exists a countable collection of (α_n, β_n), such that $S \subset \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$ and $\sum_{n=1}^{\infty} (\beta_n - \alpha_n) \leq \varepsilon$.
  - **a.e.**: If a property is true everywhere except at a set of measure zero, then the property is true almost everywhere.

- **Properties of Lebesgue Measure Zero**:
  - A countable set has measure zero.
  - If B has measure 0 and $A \subset B$, then A has measure zero.
  - If A_k has measure zero, then $\bigcup_{k=1}^{\infty} A_k$ has measure zero.
  - The Cantor set is uncountable but has measure zero.
  - For every $S \subset \mathbb{R}$, if S contains an open interval, then S does not have measure zero.

- **Thomae's Function**:
$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \text{ in the reduced form} \\ 0, & \text{otherwise.} \end{cases}$$
  f is discontinuous at $\mathbb{Q}$ only and f is bounded. Hence, f is Riemann integrable on $[0,1]$.

- **Dirichlet Function**: $\mathbf{1}_{\mathbb{Q}}(x)$ is discontinuous everywhere and hence is not Riemann integrable on any $[a,b]$.

8 Uniform Convergence

Let $(f_n) : D \to \mathbb{R}$ be a sequence of functions on any nonempty set D.

- **Uniform Convergence**: f_n uniformly converges to f on D if for every $\varepsilon > 0$, there exists $N > 0$, such that for every $n > N$, $|f_n(x) - f(x)| < \varepsilon$ for every $x \in D$, i.e. $\lim_{n\to\infty} \sup_{x\in D} |f_n(x) - f(x)| = 0$.

  $\sum_{n=1}^{\infty} f_n(x)$ uniformly converges on D if $\sum_{n=1}^{N} f_n(x)$ uniformly converges on D to $F(x) = \sum_{n=1}^{\infty} f_n(x)$ as $N \to \infty$.

- **Cauchy's Criterion**: f_n converges to f on D if and only if f_n is a Cauchy sequence under $\|\cdot\|_\infty$, i.e. for every $\varepsilon > 0$, there exists $N > 0$, such that for every $m,n > N$, $|f_n(x) - f_m(x)| < \varepsilon$, for every $x \in D$.

  $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D if and only if for every $\varepsilon > 0$, there exists $N > 0$, such that for every $n > N$, $\left|\sum_{k=n+1}^{n+p} f_k(x)\right| < \varepsilon$, for every $p \in \mathbb{N}$ and every $x \in D$.

- **Comparison Test**: Let $(u_n), (v_n) : D \to \mathbb{R}$ be two sequences of functions on any nonempty set D. If $|u_n(x)| \leq v_n(x)$ for every $x \in D$ and every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} v_n(x)$ converges uniformly on D, then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on D.

- **Weierstrass M-Test**: If $|f_n(x)| \leq M_n$, for every $x \in D$ and every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D.

- **Dirichlet Test**: Let $(u_n), (v_n) : D \to \mathbb{R}$ be two sequences of functions on any nonempty set D. If
  - $u_n(x) \geq u_{n+1}(x)$, for every $n \in \mathbb{N}$ and every $x \in D$,
  - $u_n(x)$ converges to 0 uniformly on D.
  - $\left|\sum_{n=1}^{N} v_n(x)\right| \leq B$, for every $x \in D$ and every $N \in \mathbb{N}$.

- **Abel Test**: Let $(u_n), (v_n) : D \to \mathbb{R}$ be two sequences of functions on any nonempty set D. If
  - $u_n(x) \geq u_{n+1}(x)$, for every $n \in \mathbb{N}$ and every $x \in D$,
  - $|u_n(x)| \leq B$, for every $x \in D$ and for every $n \in \mathbb{N}$,
  - $\sum_{n=1}^{\infty} v_n(x)$ converges uniformly on D,

  then $\sum_{n=1}^{\infty} u_n(x)v_n(x)$ converges uniformly on D.

- **Uniform Convergence Theorem**: Let $(f_n) : D \to X$, where X is a metric space. If f_n converges to f uniformly on D and f_n is continuous at $c \in D$ for every $n \in \mathbb{N}$, then f is continuous at c, i.e.

$$\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to c} f_n(x) = f(c).$$

  If f_n is continuous at $x = c$ and $s(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D, then $s(x)$ is continuous at $x = c$.

Let $(f_n) : [a,b] \to \mathbb{R}$ be a sequence of real-valued functions.

- **Interchange of Limit and Integration**: Assume that $f_n \in \mathscr{R}[a,b]$ converges to f uniformly on $[a,b]$. Let $g_n(x) = \int_a^x f_n(x)\mathrm{d}t$, for every $x \in [a,b]$. Then,
  - $f \in \mathscr{R}[a,b]$,
  - g_n converges to some g uniformly on $[a,b]$ and $g(x) = \int_a^x f(t)\mathrm{d}t$.

  If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,b]$, then it is Riemann integrable on $[a,b]$ and can be integrated term by term.

- **Interchange of Limit and Differentiation**: Let f_n be differentiable on (a,b) for every $n \in \mathbb{N}$. If
  - $f_n(x_0)$ converges for some point $x_0 \in (a,b)$,
  - there exists g such that f_n' converges uniformly to g on (a,b),

  then

- there exists f such that f_n converges to f uniformly on (a,b),
- $f'(x) = g(x)$, for every $x \in (a,b)$.

If

- f_n is differentiable on (a,b),
- $\sum_{n=1}^{\infty} f_n(x_0)$ converges for some $x_0 \in (a,b)$,
- there exists $g(x) = \sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on (a,b),

then

- $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on (a,b),
- $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$.

- **Uniform Convergence of Power Series**: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series. The radius of convergence is given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

$\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$ and diverges for $|x| > R$.

If $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$, then it converges uniformly on $[-r,r] \subset (-R,R)$.

  - Hence, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R,R)$.
  - We can integrate and differentiate f term-by-term on $(-R,R)$.
  - f is smooth on $(-R,R)$.

- **Abel's Limit Theorem**: If $f(x) = \sum_{n=0}^{\infty} a_n(x_n)$ converges for $x \in (-r,r]$, then $f(x)$ is left-continuous at $x = r$.

9 Arzelà-Ascoli Theorem

Let $(f_n) : X \to \mathbb{R}$ be a sequence of real-valued functions on a metric space (X,d).

- **Pointwise Boundness**: f_n is pointwise bounded on X if for every $x \in X$, there exists $B_x > 0$, such that $|f_n(x)| \leq B_x$ for every $n \in \mathbb{N}$.

- **Uniform Boundness**: f_n is uniformly bounded on X if there exists $B > 0$, such that for every $x \in X$ and every $n \in \mathbb{N}$, $|f_n(x)| \leq B$.

- **Equicontinuity**: f_n is equicontinuous at $x \in X$ if for every $\varepsilon > 0$, there eixsts $\delta_x > 0$ such that for every y such that $d(x,y) < \delta$ and every $n \in \mathbb{N}$, $|f_n(x) - f_n(y)| < \varepsilon$.

- **Uniform Equicontinuity**: f_n is uniformly equicontinuous on X if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for every $x,y \in X$ and every $n \in \mathbb{N}$, if $d(x,y) < \delta$, then $|f_n(x) - f_n(y)| < \varepsilon$.

– In general, if $|f_n'(x)| \leq M$, for every $n \in \mathbb{N}$ and every $x \in X$, then f_n is uniformly equicontinuous on X.

– If $K \subset X$ is compact and $(f_n) : K \to \mathbb{R}$ is equicontinuous at any $x \in K$, then (f_n) is pointwise bounded if and only if it is uniformly bounded.

– Assume that K is compact and $(f_n) : K \to \mathbb{R}$ is uniformly equicontinuous on K. If f_n converges pointwise on a dense subset $E \subset K$, then f_n converges uniformly on K.

– Let X be a countable set and $(f_n) : X \to \mathbb{R}$ be pointwise bounded. Then, (f_n) has a subsequence that pointwise converges.

- **Arzelà-Ascoli Theorem**: Let $K \subset X$ be a compact set and $(f_n) : K \to \mathbb{R}$. If

  – (f_n) is pointwise bounded (or uniformly bounded),

  – (f_n) is uniformly equicontinuous (or pointwise equicontinuous),

  then (f_n) has a uniformly convergent subsequence.

10 Stone-Weierstrass Theorem

- **Weierstrass Approximation Theorem**: If $f : [a,b] \to \mathbb{C}$ is continuous, then there exists a sequence of polynomials (P_n), such that $P_n \to f$ on $[a,b]$.

  If $f : [a,b] \to \mathbb{R}$, (P_n) can be taken to be real.

- **Algebra**: A set $\mathscr{A}$ of real or complex functions on a metric space X is an algebra if for every $f, g \in \mathscr{A}$ and every $c \in \mathbb{R}$ or $\mathbb{C}$,

  – $f + g \in \mathscr{A}$,

  – $cf \in \mathscr{A}$,

  – $f \cdot g \in \mathscr{A}$.

- **Separating points and Vanishing at no points**: Assume that $\mathscr{A}$ is an algebra on a metric space X.

  – $\mathscr{A}$ separates points on X if for every $x_1, x_2 \in X$ such that $x_1 \neq x_2$, there exists $f \in \mathscr{A}$, such that $f(x_1) \neq f(x_2)$.

  – $\mathscr{A}$ vanishes at no points of X if for every $x \in X$, there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$.

- **Interpolation**: If $\mathscr{A}$ separates points and vanishes at no point of X, then for every $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and every $c_1, c_2 \in \mathbb{R}$ or $\mathbb{C}$, there exists $f \in \mathscr{A}$, such that $f(x_1) = c_1$ and $f(x_2) = c_2$.

Assume that $\mathscr{A} \subset \mathscr{C}(X, \mathbb{R})$ is an algebra.

- **Uniform Closure**:

  – $\mathscr{A}$ is uniformly closed if it is closed in $\mathscr{C}(X, \mathbb{R})$ under metric space topology, i.e. if $f_n \in \mathscr{A}$ and f_n converges to some f uniformly, then $f \in \mathscr{A}$.

    - The uniform closure of $\overline{\mathscr{A}}$ of $\mathscr{A}$ is its closure in $\mathscr{C}(X,\mathbb{R})$ under metric space topology, i.e. it consists of all the uniform limit from $\mathscr{A}$.

  If $\mathscr{A} \subset \mathscr{C}(X,\mathbb{R})$ is an algebra, then $\overline{\mathscr{A}}$ is a uniformly closed algebra.

- **Stone-Weierstrass Theorem in $\mathbb{R}$**: Let X be a compact metric space. Let $\mathscr{A} \subset \mathscr{C}(X,\mathbb{R})$ be an algebra, such that

    - $\mathscr{A}$ separates points on X,
    - $\mathscr{A}$ vanishes at no points of X.

  Then, $\mathscr{A}$ is dense in $\mathscr{C}(X,\mathbb{R})$,

    - i.e. $\overline{\mathscr{A}} = \mathscr{C}(X,\mathbb{R})$,
    - i.e. any $f \in \mathscr{C}(X,\mathbb{R})$ can be uniformly approximated by functions in $\mathscr{A}$.
    - i.e. for every $f \in \mathscr{C}(X,\mathbb{R})$, there exists $f_n \in \mathscr{A}$ such that f_n converges uniformly to f on X.

  If $X \subset \mathbb{R}^n$ is compact and $\mathscr{A} = \text{Poly}(X)$, then $\overline{\mathscr{A}} = \mathscr{C}(X,\mathbb{R})$, i.e. every continuous function can be uniformly approximated by polynomials on X.

Assume that $\mathscr{A} \subset \mathscr{C}(X,\mathbb{C})$ is an algebra.

- **Self-Adjoint**: $\mathscr{A}$ is self-adjoint if for every $f \in \mathscr{A}$, $\overline{f} \in \mathscr{A}$. $\overline{f}(x) = \overline{f(x)}$, for every $x \in X$.

- **Stone-Weierstrass Theorem in $\mathbb{C}$**: Let X be a compact metric space. Let $\mathscr{A} \subset \mathscr{C}(X,\mathbb{C})$ be an algebra, such that

    - $\mathscr{A}$ separates points on X,
    - $\mathscr{A}$ vanishes at no points of X,
    - $\mathscr{A}$ is self-adjoint.

  Then, $\mathscr{A}$ is dense in $\mathscr{C}(X,\mathbb{C})$.