Project Part 1

Eric Zou

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1 Question 1

We want to find the change from $S_{t+1} to S_t$, First trace the lines of each in the model and add together the in arrows and subtract the out arrows.

$$S_{t+1} - S_t = -\frac{\beta S_t I_t}{N} - \mu S_t + \mu S_t + \mu I_t + \mu R_t$$

Moving things around and simplifying, we get

$$S_{t+1} = S_t - \frac{\beta * S_t * I_t}{N} + u(I_t + R_t)$$

It's the same process for the other two models.

$$I_{t+1} - I_t = \frac{\beta S_t I_t}{N} - \mu I_t - \gamma I_t$$

Simplify

$$I_{t+1} = I_t(1 - \mu - \gamma) + \frac{\beta * S_t * I_t}{N}$$

Now for R

$$R_{t+1} - R_t = \gamma I_t - \mu R_t$$

simplify

$$R_{t+1} = R_t(1 - \mu) + \beta * I_t$$

So the three models are:

$$S_{t+1} = S_t - \frac{\beta * S_t * I_t}{N} + u(I_t + R_t)$$

$$I_{t+1} = I_t(1 - \mu - \gamma) + \frac{\beta * S_t * I_t}{N}$$

$$R_{t+1} = R_t(1 - \mu) + \beta * I_t$$

2 Question 2

$$S_t - \frac{\beta * S_t * I_t}{N} + uI_t + uR_t + I_t - I_t\mu - I_t\gamma + R_t - R_t\mu + \beta * I_t$$

$$S_t + I_t - I_t\gamma + I_t\beta - R_t$$

We notice all the the u cancels out of the equation

3 Question 3

We want to express the model in terms of only S_t and I_t.

I_t stays the same since there is no R

We use the initial condition of $S_0 + I_0 + R_0$ as N will not change

$$R_t = N - S_t - I_t$$

$$S_{t+1} = S_t - \frac{\beta * S_t * I_t}{N} + u(N - S_t)$$

$$I_{t+1} = I_t(1 - \mu - \gamma) + \frac{\beta * S_t * I_t}{N}$$

4 Question 4

$$S^* = S^* - \frac{\beta S^* I_t^*}{N} + u(N - S^*)$$

$$I^* = I^*(1 - \mu - \gamma) + \frac{\beta S^* I^*}{N}$$

To solve this we will set I_t and I_{t+1} to I^*, same with S. Starting with the equation for I, we distribute the I

$$I^* = I^* - \mu I^* - \gamma I^* + \frac{\beta S^* I^*}{N}$$

Cancel out the I^*

$$0 = -\mu I^* - \gamma I^* + \frac{\beta S^* I^*}{N}$$

Factor out I^*

$$0 = I^*(-\mu - \gamma + \frac{\beta S^*}{N})$$

For this to work, either I^* or $-\mu - \gamma + \frac{\beta S^*}{N}$ is equal to zero. Startin with $I^* = 0$ plugged into the other equation to find S^*:

$$S^* = S^* - 0 + u(N - S^*)$$

$$0 = u(N - S^*)$$

$$S^* = N$$

First(Disease free) stable point: $\begin{bmatrix} N \\ 0 \end{bmatrix}$

To find second(Endemic) fixed point, find when $-\mu - \gamma + \frac{\beta S^*}{N} = 0$ and plug into first equation

$$S^* = \frac{N(\gamma + \mu)}{\beta}$$

Plugging in

$$S^* = S^* - \frac{\beta \frac{N(\gamma+\mu)}{\beta} I^*}{N} + u(N - S^*)$$

Cancel S^* and simplify fraction

$$0 = -I(\gamma + \mu) + u(N - \frac{N(\gamma + \mu)}{\beta})$$

$$I = \frac{u(N - \frac{N(\gamma+\mu)}{\beta})}{(\gamma + \mu)}$$

Simplify a bit more

$$I = \frac{u(\beta - (\gamma + \mu))}{\beta(\gamma + \mu)}$$

So second(Endemic) equilibrium is $\begin{bmatrix} \frac{N(\gamma+\mu)}{\beta} \\ \frac{u(\beta-(\gamma+\mu))}{\beta(\gamma+\mu)} \end{bmatrix}$

5 Question 5

To find jacobian, first find partial derivatives

$$\frac{dS}{dS} = 1 - \frac{\beta I_t}{N} - \mu$$

$$\frac{dI}{dS} = -\frac{\beta S_t}{N}$$

$$\frac{dS}{dI} = \frac{\beta I_t}{N}$$

$$\frac{dI}{dI} = (1 - \mu - \gamma) + \frac{\beta S_t}{N}$$

$$\begin{bmatrix} 1 - \frac{\beta I_t}{N} - u & -\frac{\beta S_t}{N} \\ \frac{\beta I_t}{N} & (1 - \mu - \gamma) + \frac{\beta S_t}{N} \end{bmatrix}$$

Evaluating the Jacobian at the disease-free equilibrium

$$\begin{bmatrix} 1 - \frac{\beta(0)}{N} - u & -\frac{\beta(N)}{N} \\ \frac{\beta(0)}{N} & (1 - \mu - \gamma) + \frac{\beta(N)}{N} \end{bmatrix}$$

Simplify

$$\begin{bmatrix} 1-u & -\beta \\ 0 & 1-\mu-\gamma+\beta \end{bmatrix}$$

6 Question 6

Start by subtracting the diagonals from λ

$$\begin{bmatrix} \lambda-(1-u) & -\beta \\ 0 & \lambda-(1-\mu-\gamma+\beta) \end{bmatrix}$$

$$(\lambda-1+\mu)(\lambda-1+\mu+\gamma-\beta)$$

$$\lambda_1 = 1-u$$

$$\lambda_2 = 1-\mu-\gamma+\beta$$

To find stability we evaluate when the eigenvalues will be less than/greater than

$$1 > 1-\mu$$

So we know μ has to be positive

$$1 > 1-\mu-\gamma+\beta$$

$$0 > -\mu-\gamma+\beta$$

$$\mu+\gamma > \beta$$

$$1 > \frac{\beta}{\mu+\gamma}$$

$$R_0 = \frac{\beta}{\mu+\gamma}$$

7 Question 7

First, find the jacobian at the endemic equilibrium

$$\begin{bmatrix} 1-\frac{\beta I_t}{N}-u & -\frac{\beta S_t}{N} \\ \frac{\beta I_t}{N} & (1-\mu-\gamma)+\frac{\beta S_t}{N} \end{bmatrix}$$

Plugging in $\begin{bmatrix} \frac{N(\gamma+\mu)}{\beta} \\ \frac{u(\beta-(\gamma+\mu))}{\beta(\gamma+\mu)} \end{bmatrix}$

$$\begin{bmatrix} 1-\frac{u(\beta-\gamma-\mu)}{N(\gamma+\mu)}-u & -(\gamma+\mu) \\ \frac{u(\beta-\gamma-\mu)}{N(\gamma+\mu)} & (1-\mu-\gamma)+\gamma+\mu \end{bmatrix}$$

Simplifying down and replacing μ, γ, β with R_0 we obtained in question 6

$$\begin{bmatrix} 1-\mu R_0 & -\frac{\beta}{R_0} \\ \mu(R_0-1) & 1 \end{bmatrix}$$

Now We use the formula $|Tr(J)| < 1 + Det(J) < 2$ to see what values we need for stability

$$Tr(J) = (1)(1-\mu R_0)$$

$$Det(J) = (1-\mu R_0) + \frac{\beta\mu(R_0-1)}{R_0}$$

So it is LAS as long as $\frac{\beta\mu(R_0-1)}{R_0}$ is positive and $(1-\mu R_0) + \frac{\beta\mu(R_0-1)}{R_0} < 2$.

8 Question 8

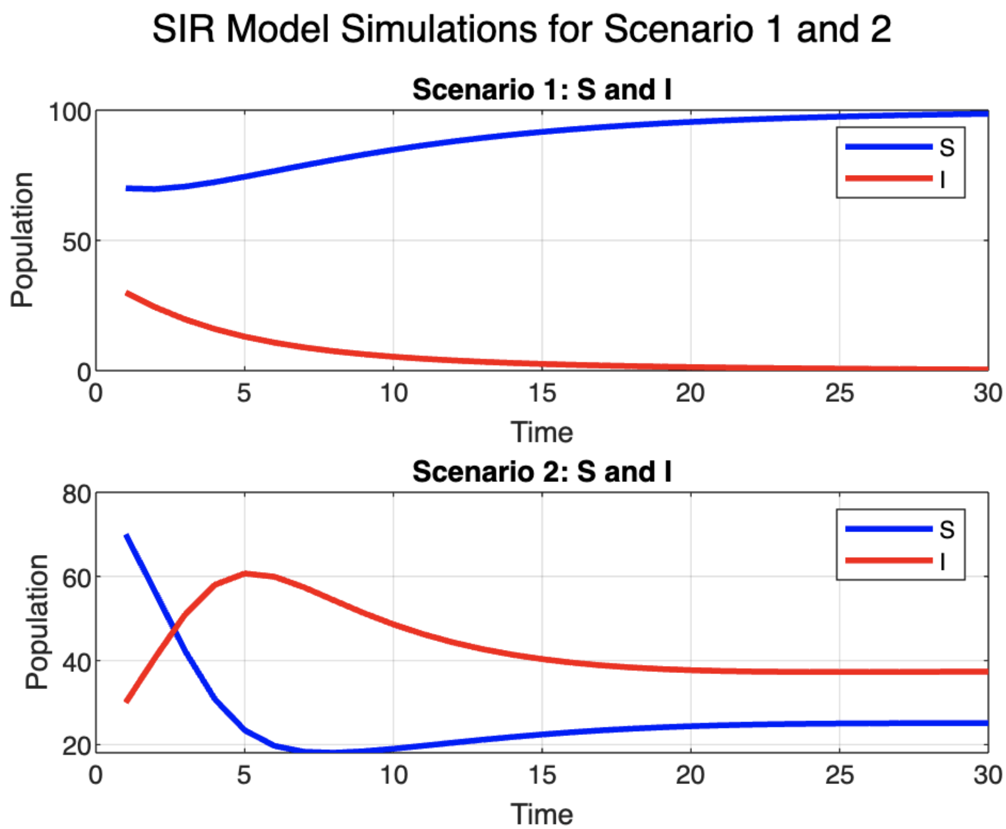
$$R_0 = \frac{\mu+\gamma}{\beta}$$

We plug in the given values to find R_0 for each example

$$R_0 1 = \frac{0.3}{0.2+0.2} = 3/4$$

$$R_0 2 = \frac{0.8}{0.1+0.1} = 4$$

9 Question 9



Yes, this matches what we see in the numerical analysis. For the first example, R_0 is 3/4, which is below 1. So it follows that the disease will not have enough secondary infections to survive and will phase out of the population. This is confirmed by the model. Susceptible is approaching 100 and Infected is approaching 0.

For the second model, R_0 is 4, so the model will persist and cause an outbreak. This is confirmed by the model, S and I are not converging to 100 or 0, and are instead staying around 40 and 30 respectively.