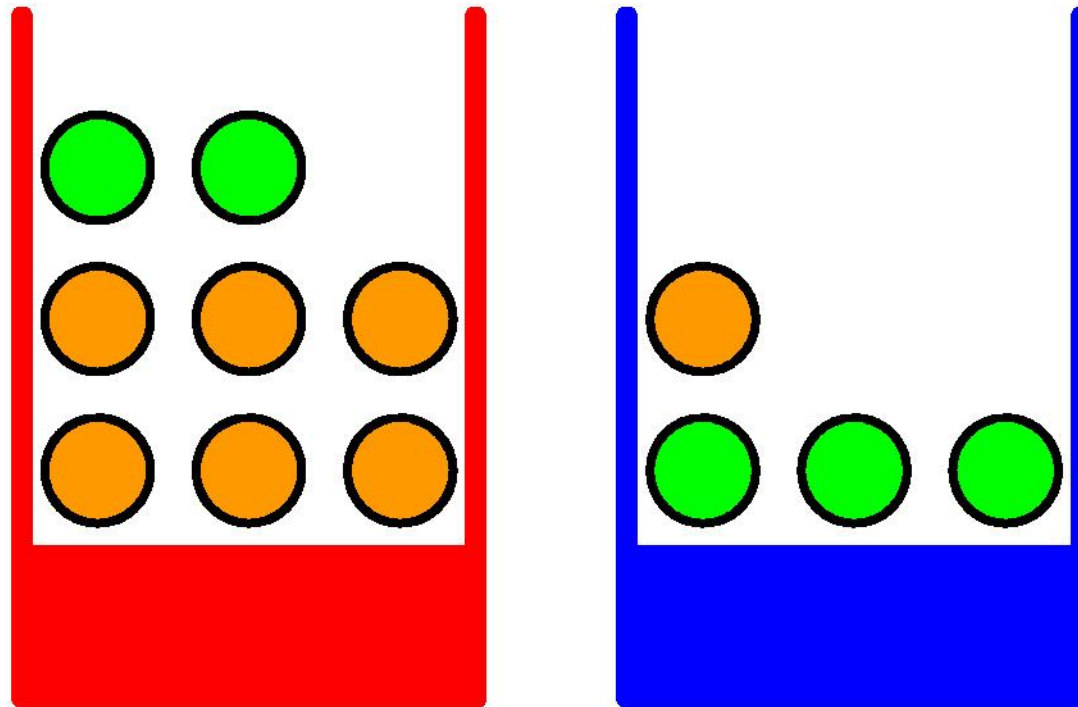


# Probability Theory

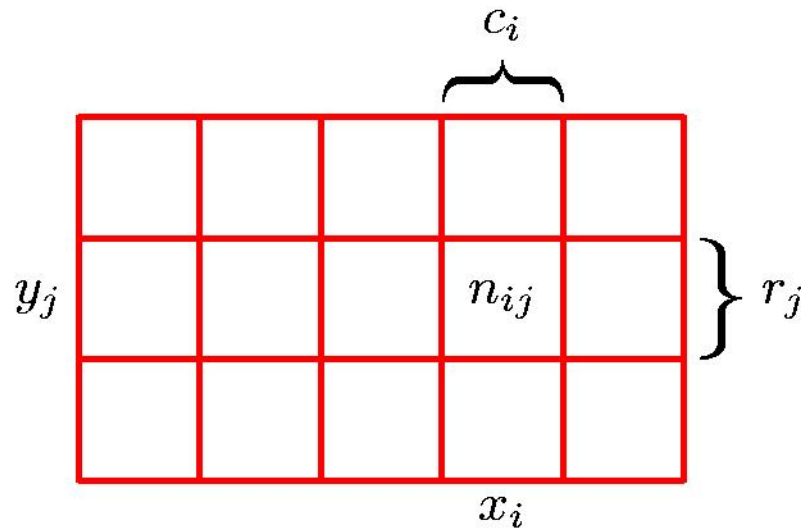
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Apples and Oranges



# Probability Theory

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A 3x5 grid of cells outlined in red. The top row is grouped by a horizontal curly brace labeled  $c_i$ . The rightmost column is grouped by a vertical curly brace labeled  $r_j$ . The cell at the intersection of the second row and fourth column is labeled  $n_{ij}$ . The label  $y_j$  is to the left of the second row, and  $x_i$  is below the fourth column.

			$n_{ij}$	

Marginal Probability

$$p(X = x_i) = \frac{c_i}{N}.$$

Joint Probability

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

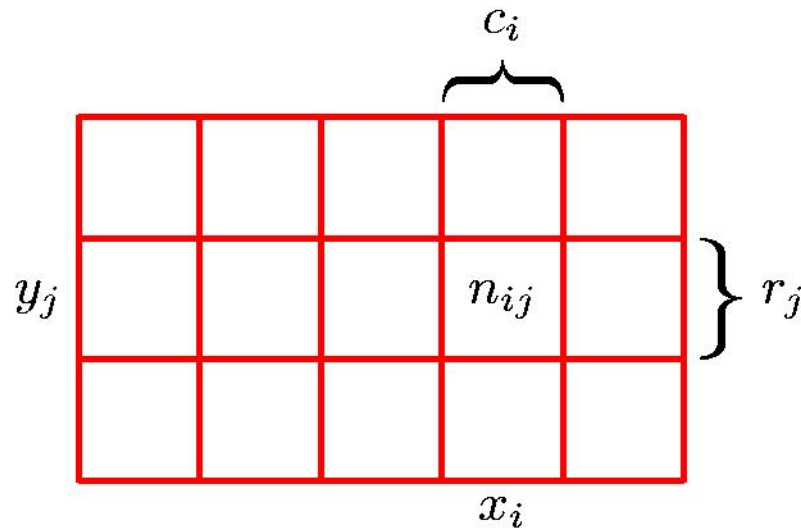
Conditional Probability

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$

---

# Probability Theory

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Sum Rule

$$\begin{aligned} p(X = x_i) &= \frac{c_i}{N} = \frac{1}{N} \sum_{j=1}^L n_{ij} \\ &= \sum_{j=1}^L p(X = x_i, Y = y_j) \end{aligned}$$

Product Rule

$$\begin{aligned} p(X = x_i, Y = y_j) &= \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N} \\ &= p(Y = y_j | X = x_i) p(X = x_i) \end{aligned}$$

---

# The Rules of Probability

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Sum Rule

$$p(X) = \sum_Y p(X, Y)$$

Product Rule

$$p(X, Y) = p(Y|X)p(X)$$

---

# Bayes' Theorem

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$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}$$

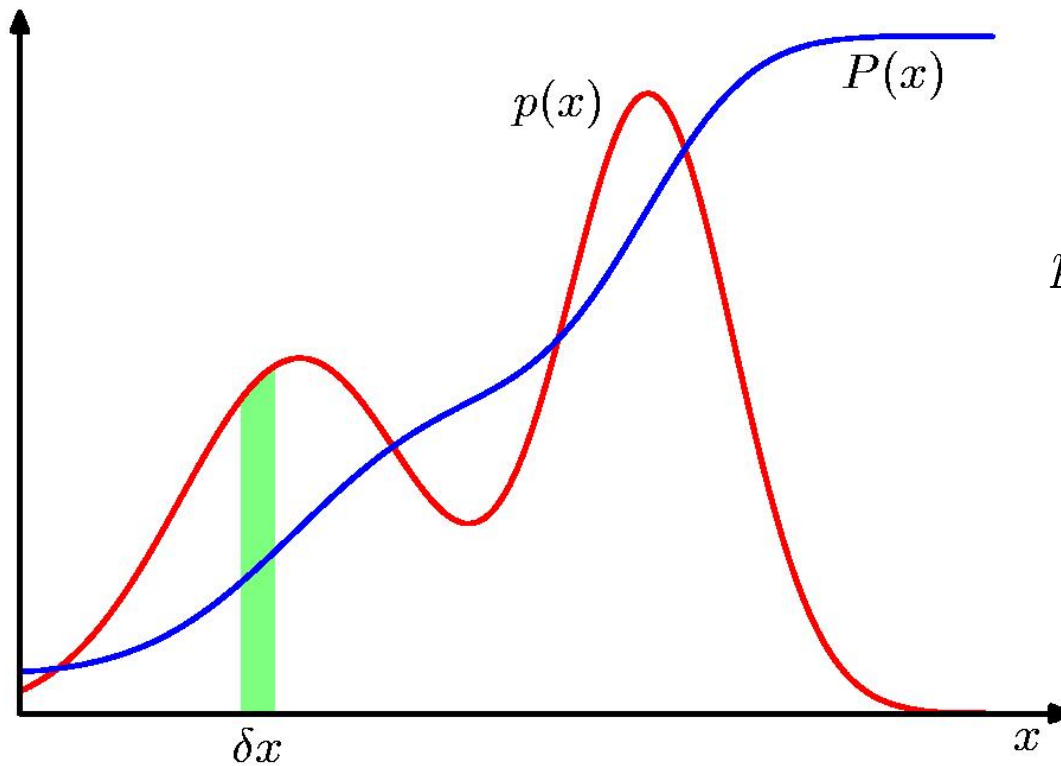
$$p(X) = \sum_Y p(X|Y)p(Y)$$

posterior  $\propto$  likelihood  $\times$  prior

---

# Probability Densities

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$$p(x \in (a, b)) = \int_a^b p(x) dx$$

$$P(z) = \int_{-\infty}^z p(x) dx$$

$$p(x) \geq 0$$

$$\int_{-\infty}^{\infty} p(x) dx = 1$$


---

# Expectations

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$$\mathbb{E}[f] = \sum_x p(x) f(x)$$

$$\mathbb{E}[f] = \int p(x) f(x) \, dx$$

$$\mathbb{E}_x[f|y] = \sum_x p(x|y) f(x)$$


Conditional Expectation  
(discrete)

$$\mathbb{E}[f] \simeq \frac{1}{N} \sum_{n=1}^N f(x_n)$$

Approximate Expectation  
(discrete and continuous)

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# Variances and Covariances

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$$\text{var}[f] = \mathbb{E} \left[ (f(x) - \mathbb{E}[f(x)])^2 \right] = \mathbb{E}[f(x)^2] - \mathbb{E}[f(x)]^2$$

$$\begin{aligned} \text{cov}[x, y] &= \mathbb{E}_{x,y} [\{x - \mathbb{E}[x]\} \{y - \mathbb{E}[y]\}] \\ &= \mathbb{E}_{x,y}[xy] - \mathbb{E}[x]\mathbb{E}[y] \end{aligned}$$

$$\begin{aligned} \text{cov}[\mathbf{x}, \mathbf{y}] &= \mathbb{E}_{\mathbf{x}, \mathbf{y}} [\{\mathbf{x} - \mathbb{E}[\mathbf{x}]\} \{\mathbf{y}^T - \mathbb{E}[\mathbf{y}^T]\}] \\ &= \mathbb{E}_{\mathbf{x}, \mathbf{y}}[\mathbf{x}\mathbf{y}^T] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{y}^T] \end{aligned}$$

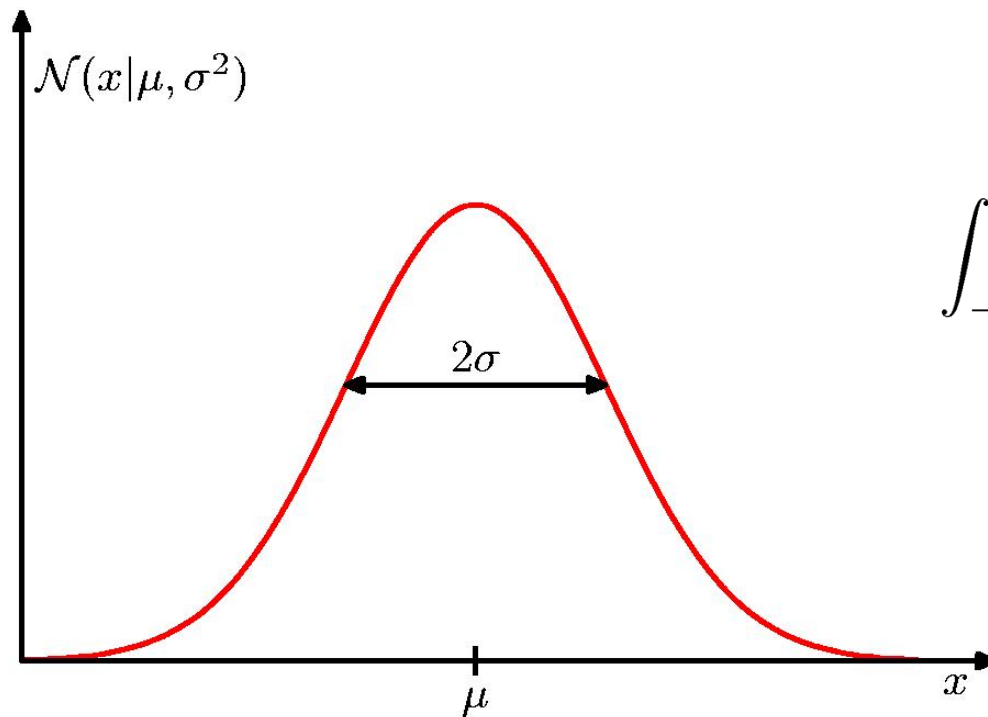
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# The Gaussian Distribution

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$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp \left\{ -\frac{1}{2\sigma^2} (x - \mu)^2 \right\}$$



$$\mathcal{N}(x|\mu, \sigma^2) > 0$$

$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) \, dx = 1$$

# Gaussian Mean and Variance

---

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) x \, dx = \mu$$

$$\mathbb{E}[x^2] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) x^2 \, dx = \mu^2 + \sigma^2$$

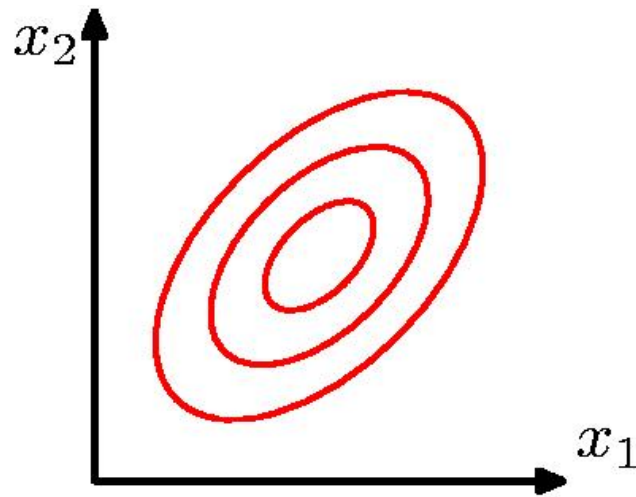
$$\text{var}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2$$

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# The Multivariate Gaussian

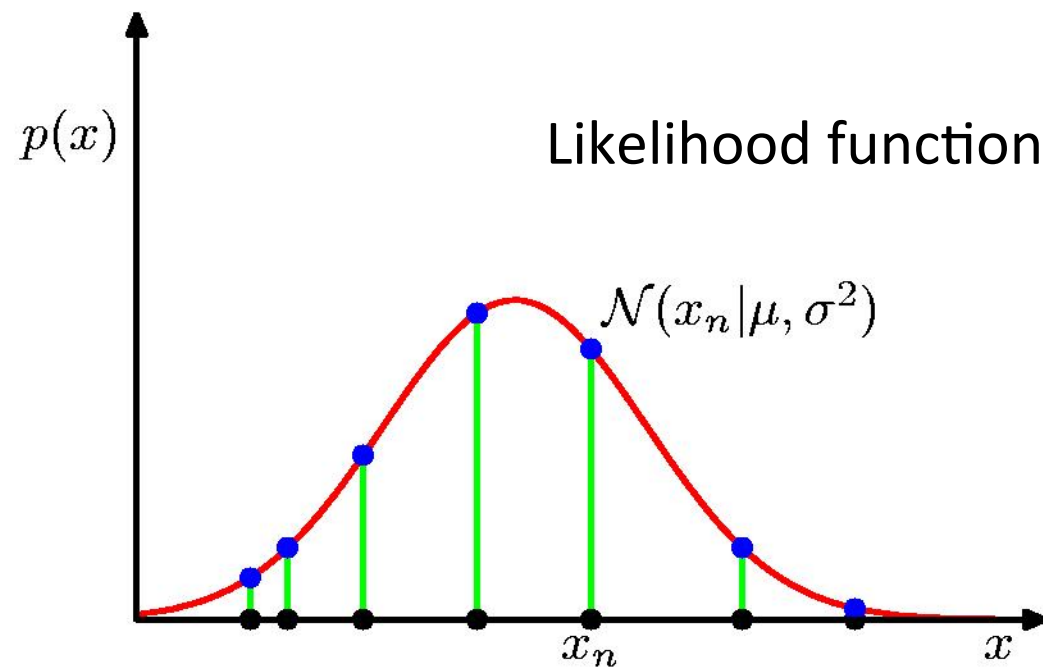
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$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp \left\{ -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \right\}$$



# Gaussian Parameter Estimation

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$$p(\mathbf{x} | \mu, \sigma^2) = \prod_{n=1}^N \mathcal{N}(x_n | \mu, \sigma^2)$$

---

# Maximum (Log) Likelihood

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$$\ln p(\mathbf{x}|\mu, \sigma^2) = -\frac{1}{2\sigma^2} \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2} \ln \sigma^2 - \frac{N}{2} \ln(2\pi)$$

$$\mu_{\text{ML}} = \frac{1}{N} \sum_{n=1}^N x_n \qquad \sigma_{\text{ML}}^2 = \frac{1}{N} \sum_{n=1}^N (x_n - \mu_{\text{ML}})^2$$

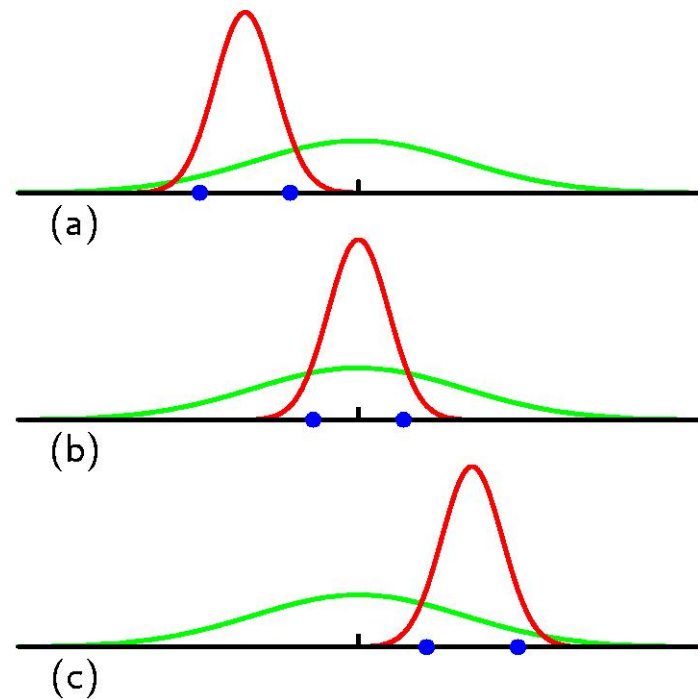
# Properties of $\mu_{\text{ML}}$ and $\sigma_{\text{ML}}^2$

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$$\mathbb{E}[\mu_{\text{ML}}] = \mu$$

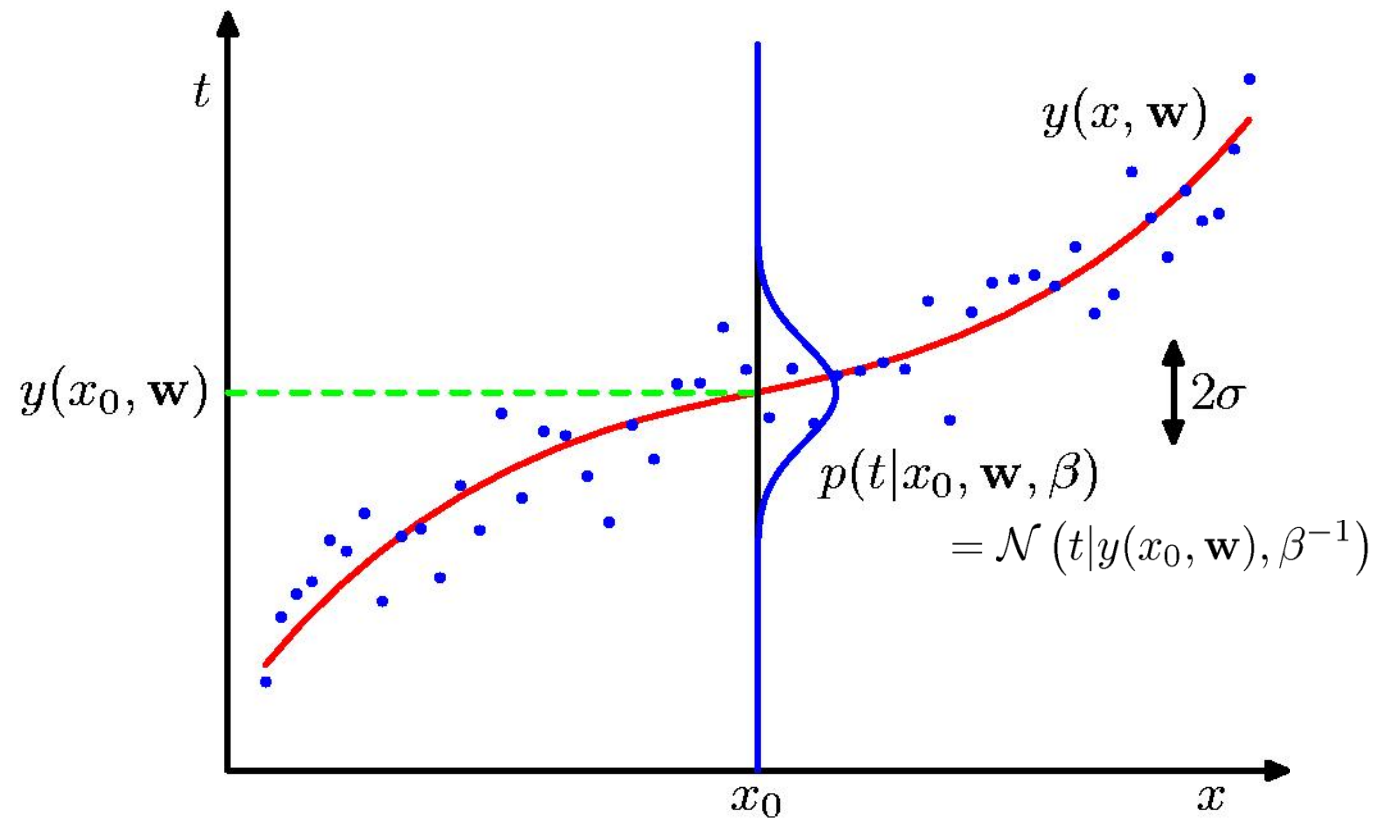
$$\mathbb{E}[\sigma_{\text{ML}}^2] = \left(\frac{N-1}{N}\right) \sigma^2$$

$$\begin{aligned} \tilde{\sigma}^2 &= \frac{N}{N-1} \sigma_{\text{ML}}^2 \\ &= \frac{1}{N-1} \sum_{n=1}^N (x_n - \mu_{\text{ML}})^2 \end{aligned}$$



# Curve Fitting Re-visited

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# Maximum Likelihood

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$$p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta) = \prod_{n=1}^N \mathcal{N}(t_n | y(x_n, \mathbf{w}), \beta^{-1})$$

$$\ln p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta) = - \underbrace{\frac{\beta}{2} \sum_{n=1}^N \{y(x_n, \mathbf{w}) - t_n\}^2}_{\beta E(\mathbf{w})} + \frac{N}{2} \ln \beta - \frac{N}{2} \ln(2\pi)$$

Determine  $\mathbf{w}_{\text{ML}}$  by minimizing sum-of-squares error,  $E(\mathbf{w})$ .

$$\frac{1}{\beta_{\text{ML}}} = \frac{1}{N} \sum_{n=1}^N \{y(x_n, \mathbf{w}_{\text{ML}}) - t_n\}^2$$

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# MAP: A Step towards Bayes

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$$p(\mathbf{w}|\alpha) = \mathcal{N}(\mathbf{w}|\mathbf{0}, \alpha^{-1}\mathbf{I}) = \left(\frac{\alpha}{2\pi}\right)^{(M+1)/2} \exp\left\{-\frac{\alpha}{2}\mathbf{w}^T\mathbf{w}\right\}$$

$$p(\mathbf{w}|\mathbf{x}, \mathbf{t}, \alpha, \beta) \propto p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \beta)p(\mathbf{w}|\alpha)$$

$$\beta\tilde{E}(\mathbf{w}) = \frac{\beta}{2} \sum_{n=1}^N \{y(x_n, \mathbf{w}) - t_n\}^2 + \frac{\alpha}{2}\mathbf{w}^T\mathbf{w}$$

Determine  $\mathbf{w}_{\text{MAP}}$  by minimizing regularized sum-of-squares error,  $\tilde{E}(\mathbf{w})$ .

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