Harvey Mudd College Math Tutorial:

The Multivariable Chain Rule

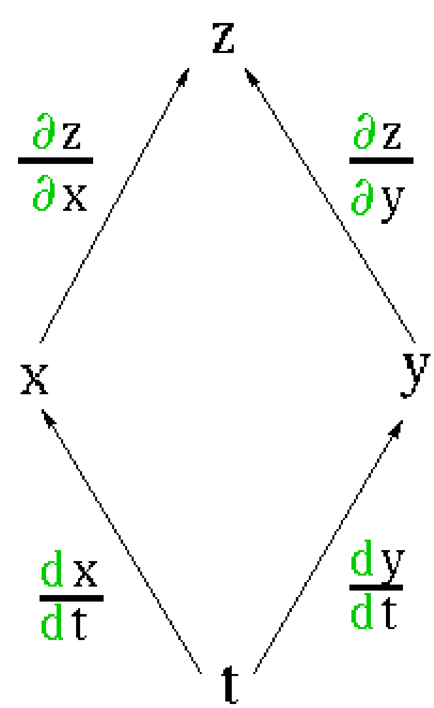
Suppose that $z = f(x, y)$, where x and y themselves depend on one or more variables. Multivariable Chain Rules allow us to differentiate z with respect to any of the variables involved:

Let $x = x(t)$ and $y = y(t)$ be differentiable at t and suppose that $z = f(x, y)$ is differentiable at the point $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Proof

Although the formal proof is not trivial, the variable-dependence diagram shown here provides a simple way to remember this Chain Rule. Simply add up the two paths starting at z and ending at t, multiplying derivatives along each path.



Example

Let $z = x^2y - y^2$ where x and y are parametrized as $x = t^2$ and $y = 2t$.
Then

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2xy)(2t) + (x^2 - 2y)(2) \\
&= (2t^2 \cdot 2t)(2t) + \Big((t^2)^2 - 2(2t)\Big)(2) \\
&= 8t^4 + 2t^4 - 8t \\
&= 10t^4 - 8t.
\end{aligned}$$

Alternate Solution

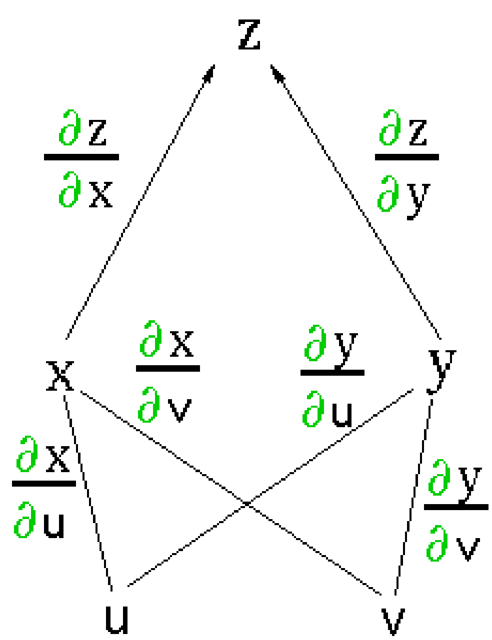
We now suppose that x and y are both multivariable functions.

Let $x = x(u,v)$ and $y = y(u,v)$ have first-order partial derivatives at the point (u,v) and suppose that $z = f(x,y)$ is differentiable at the point $(x(u,v), y(u,v))$. Then $f(x(u,v), y(u,v))$ has first-order partial derivatives at (u,v) given by

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.
\end{aligned}$$

Proof

Again, the variable-dependence diagram shown here indicates this Chain Rule by summing paths for z either to u or to v.



Example

Let $z = e^{x^2 y}$, where $x(u,v) = \sqrt{uv}$ and $y(u,v) = 1/v$. Then

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \left(2xye^{x^2 y}\right)\left(\frac{\sqrt{v}}{2\sqrt{u}}\right) + \left(x^2 e^{x^2 y}\right)(0) \\
&= 2\sqrt{uv}\cdot\frac{1}{v}e^{(\sqrt{uv})^2\cdot\frac{1}{v}}\cdot\frac{\sqrt{v}}{2\sqrt{u}} + (\sqrt{uv})^2\cdot e^{(\sqrt{uv})^2\cdot\frac{1}{v}}\cdot(0) \\
&= e^u + 0 \\
&= e^u \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \left(2xye^{x^2 y}\right)\left(\frac{\sqrt{u}}{2\sqrt{v}}\right) + \left(x^2 e^{x^2 y}\right)\left(-\frac{1}{v^2}\right) \\
&= 2\sqrt{uv}\cdot\frac{1}{v}e^{(\sqrt{uv})^2\cdot\frac{1}{v}}\cdot\frac{\sqrt{u}}{2\sqrt{v}} + (\sqrt{uv})^2 e^{(\sqrt{uv})^2\cdot\frac{1}{v}}\cdot\left(-\frac{1}{v^2}\right) \\
&= \frac{u}{v}e^u - \frac{u}{v}e^u \\
&= 0.
\end{aligned}$$

Alternate Solution

These Chain Rules generalize to functions of three or more variables in a straight forward manner.

Key Concepts

- Let $x = x(t)$ and $y = y(t)$ be differentiable at t and suppose that $z = f(x, y)$ is differentiable at the point $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

- Let $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v) and suppose that $z = f(x, y)$ is differentiable at the point $(x(u, v), y(u, v))$. Then $f(x(u, v), y(u, v))$ has first-order partial derivatives at (u, v) given by

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.\end{aligned}$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]