Inverse Kinematics

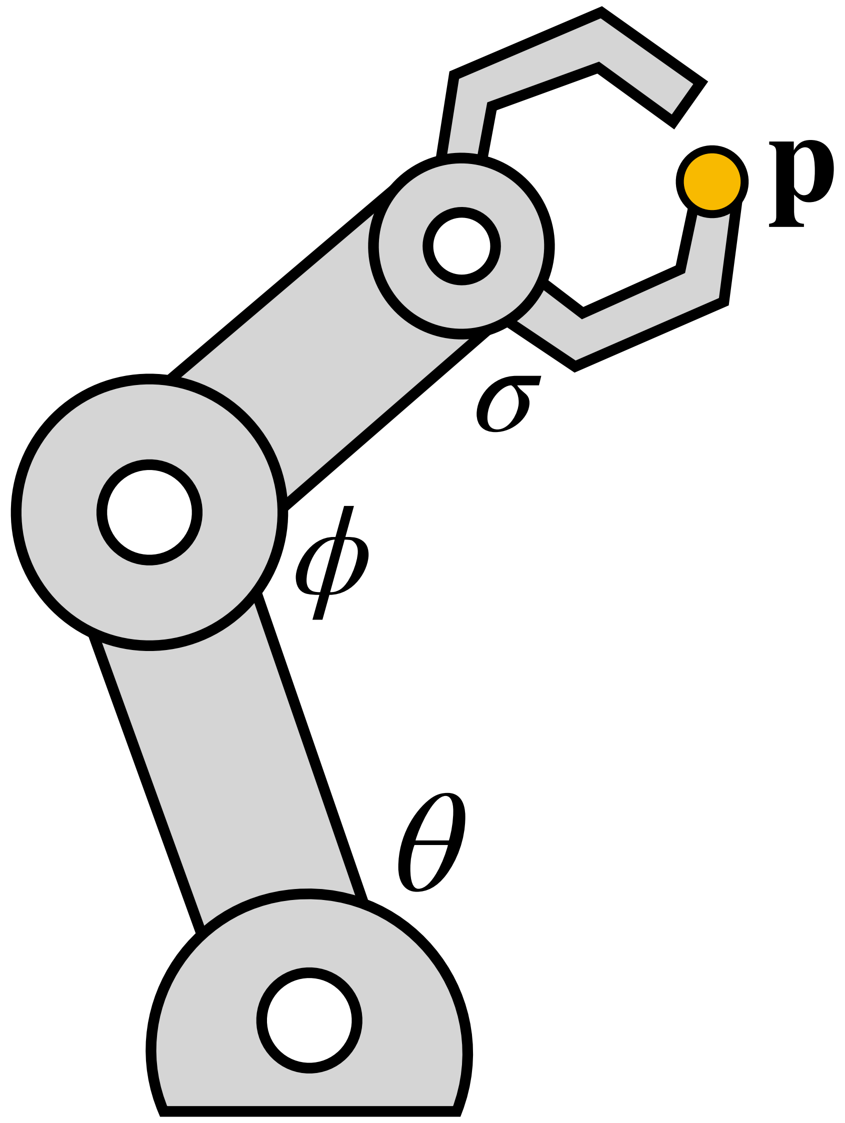
C. Karen Liu

Kinematics

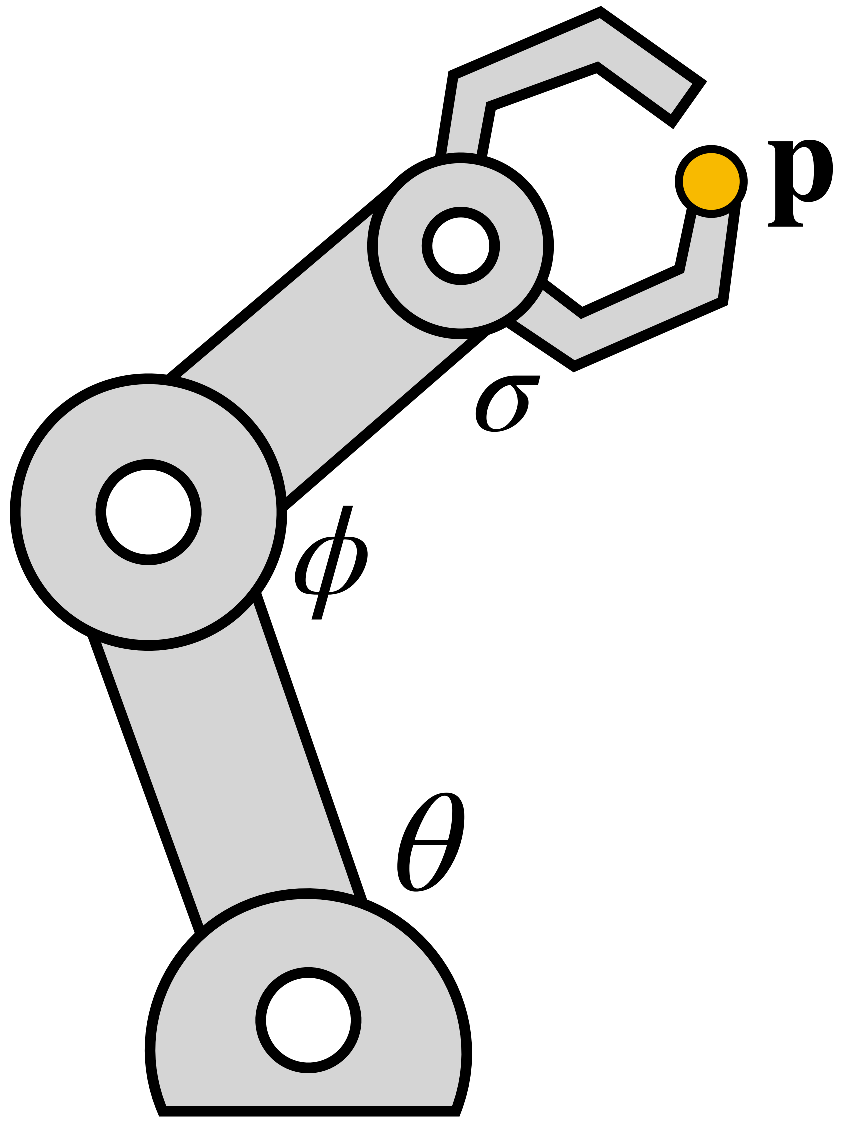
- Forward kinematics
  - Given a pose of a structure, what is the 3D position of a point on the structure?
- Inverse kinematics
  - Given a target position for a point on the structure, what is the pose such that the point reaches the target position?

Quiz

Which one is solving Inverse Kinematics?



$$[\theta, \phi, \sigma] = f(\mathbf{p})$$



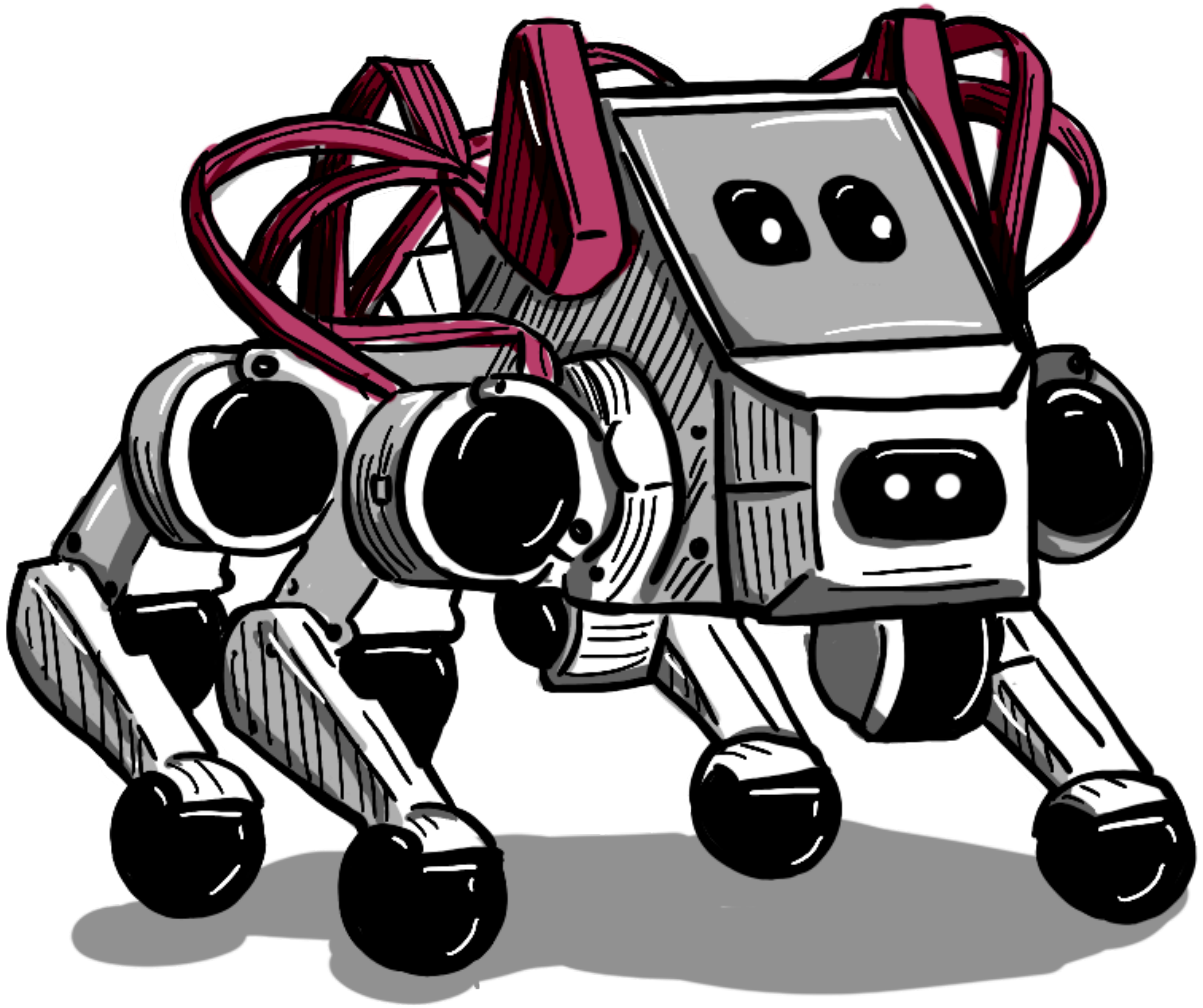
$$\mathbf{p} = f(\theta, \phi, \sigma)$$

Why inverse?

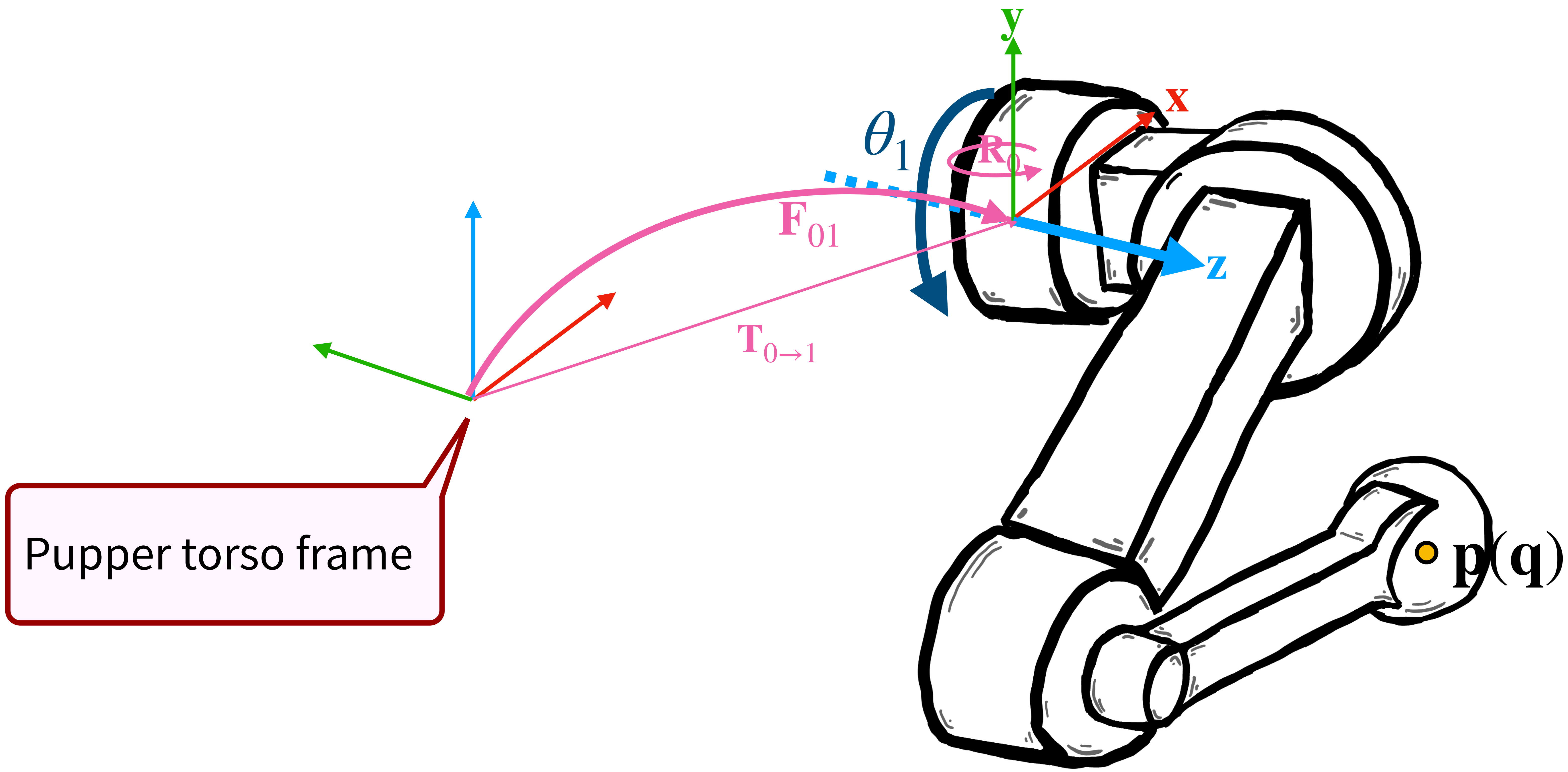
- The world is described in the Cartesian space but the movement is described in the pose space.
  - IK provides more intuitive control.
  - IK maintains environment constraints.

Define a pose

$$\mathbf{q} \equiv \{x, y, z, r, p, y, \theta_1, \cdots, \theta_{12}\}$$

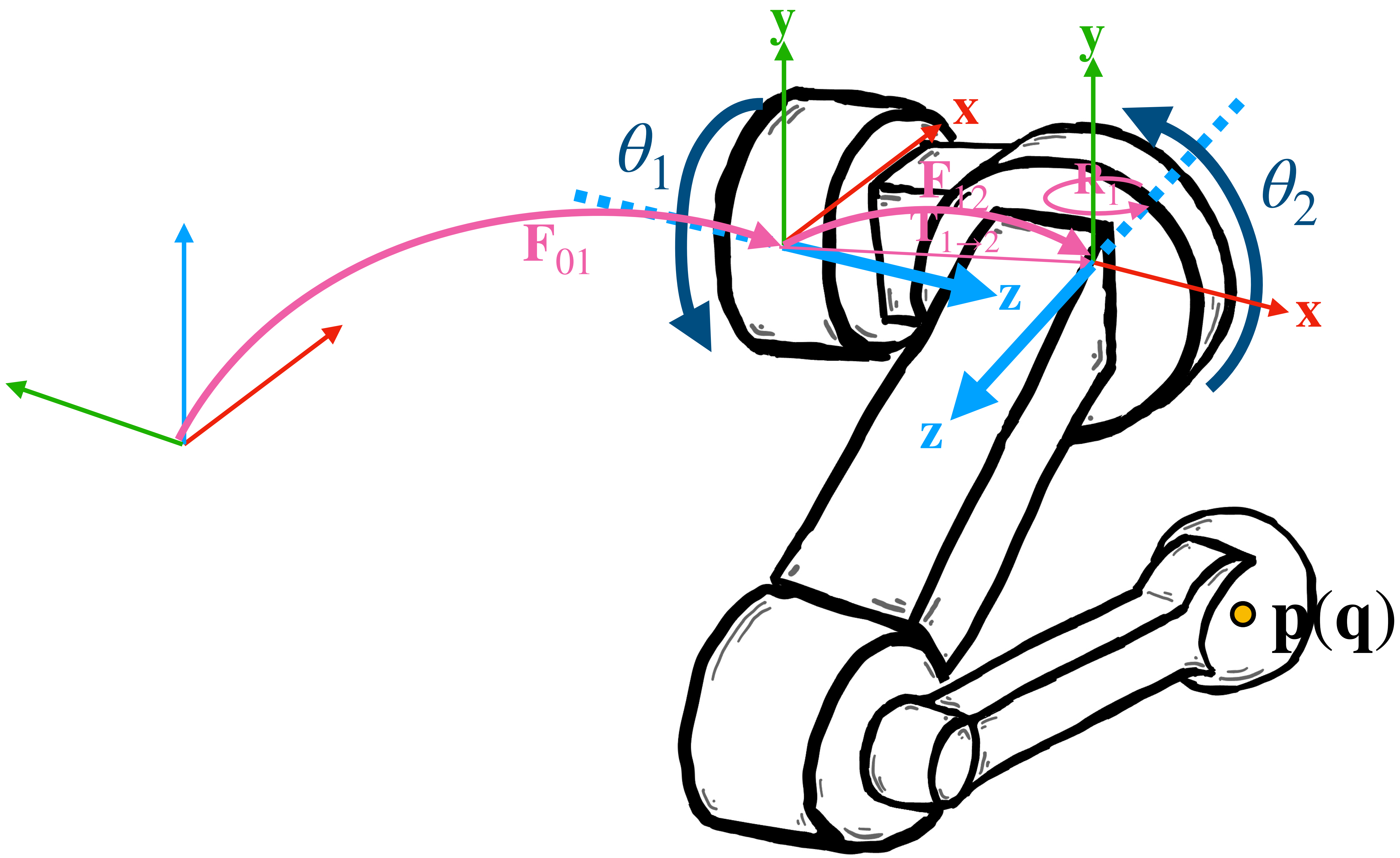


Transformation chain



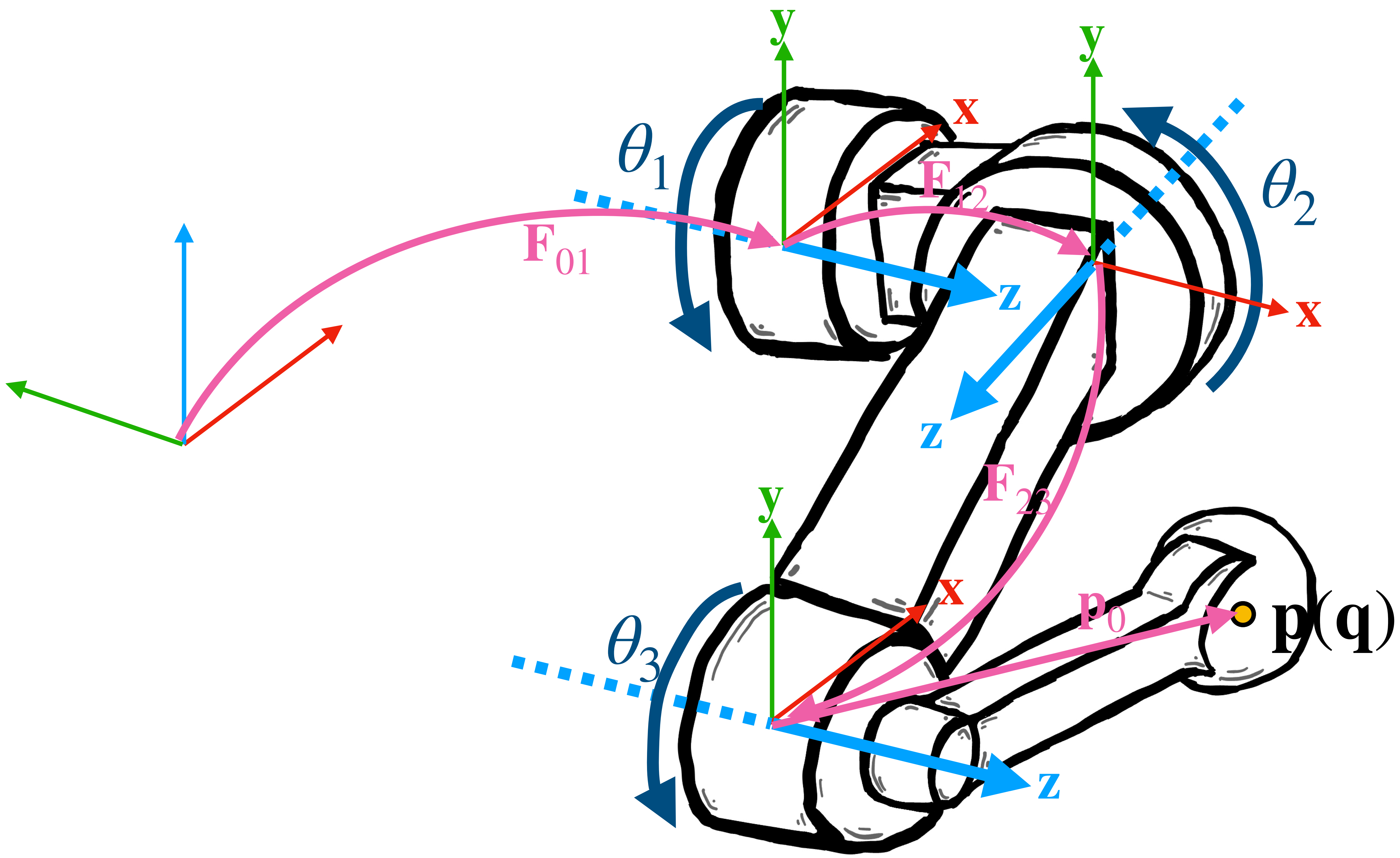
$$\mathbf{p}(\mathbf{q}) = \overset{\mathbf{F}_{01}}{\cancel{\mathbf{T}_{0\to 1}\,\mathbf{R}_0}}\ \mathbf{R}_z(\theta_1)$$

Transformation chain



$$\mathbf{p}(\mathbf{q}) = \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\overset{\mathbf{F}_{12}}{\cancel{\mathbf{T}_{1\to 2}\mathbf{R}_1}}\mathbf{R}_z(\theta_2)$$

Transformation chain



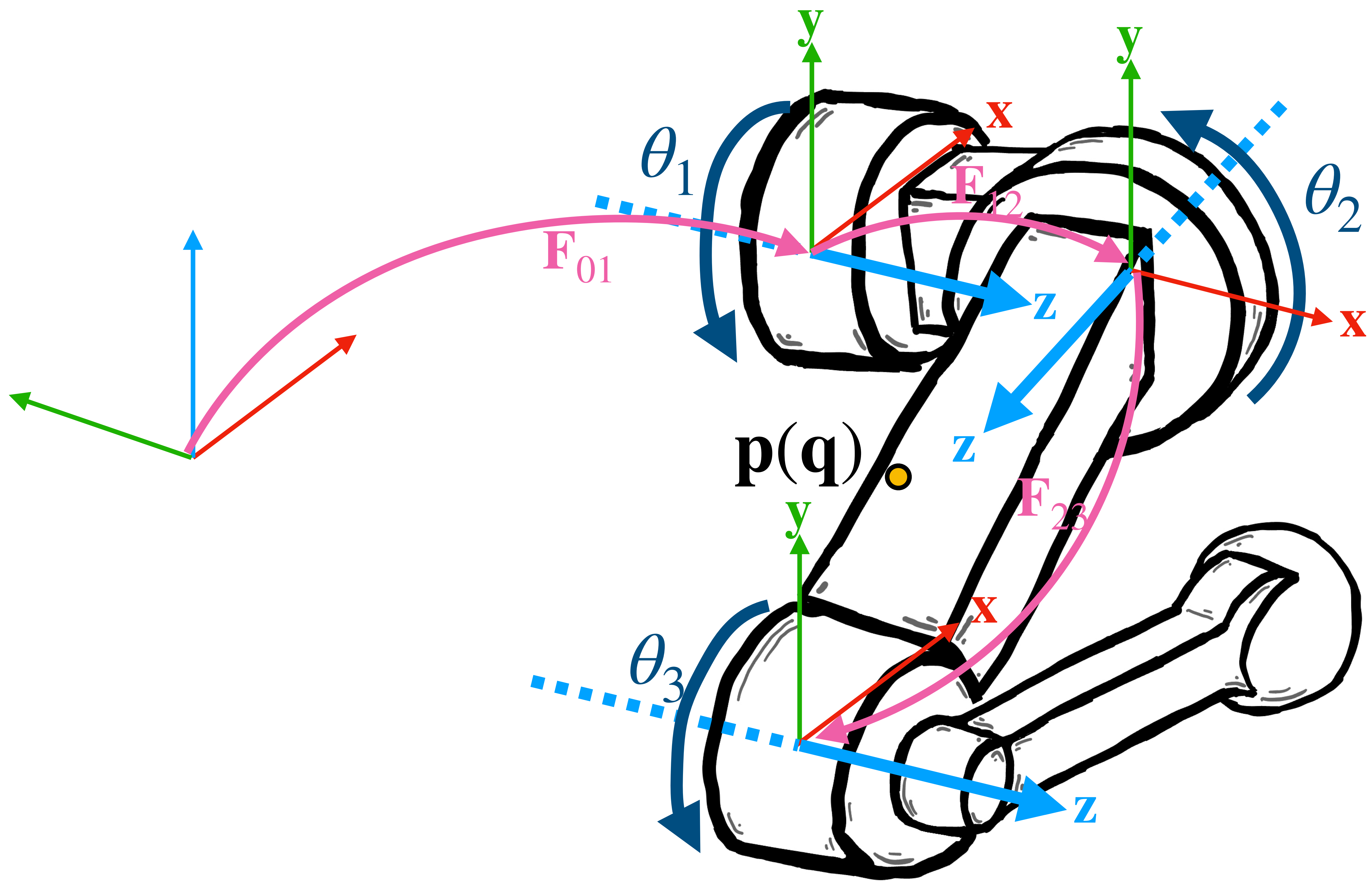
A is a transformation matrix that maps a point from the foot coord frame to the torso frame

$$\mathbf{p}(\mathbf{q}) = \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\,\mathbf{p}_0$$
$$= A(\theta_1, \theta_2, \theta_3)\,\mathbf{p}_0 = A(\mathbf{q})\,\mathbf{p}_0$$

$\mathbf{p}_0$ is the local coordinates of the end-effector in foot frame. In our example, $\mathbf{p}_0 = [0.062, -0.062, 0.018]$

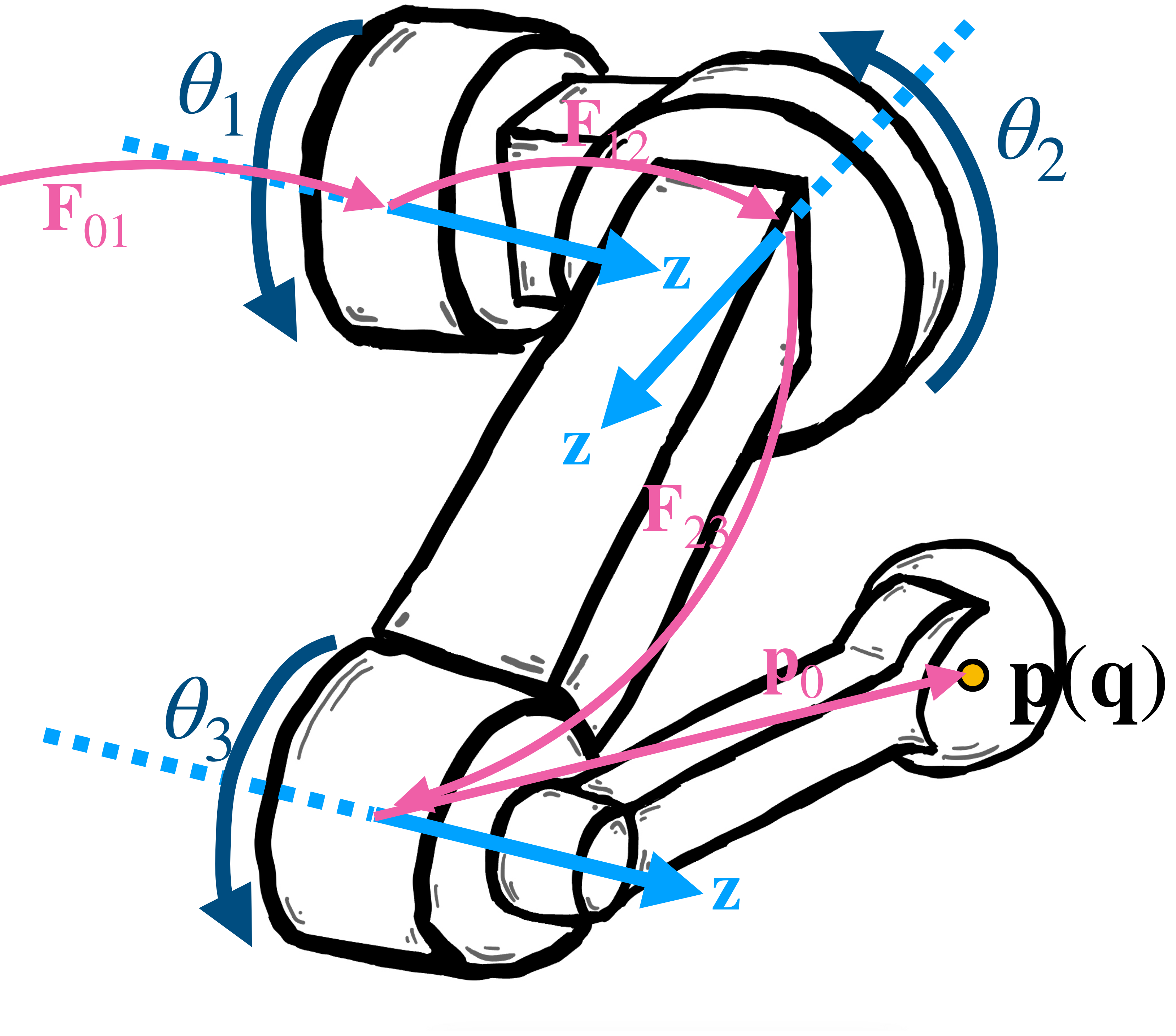
Quiz

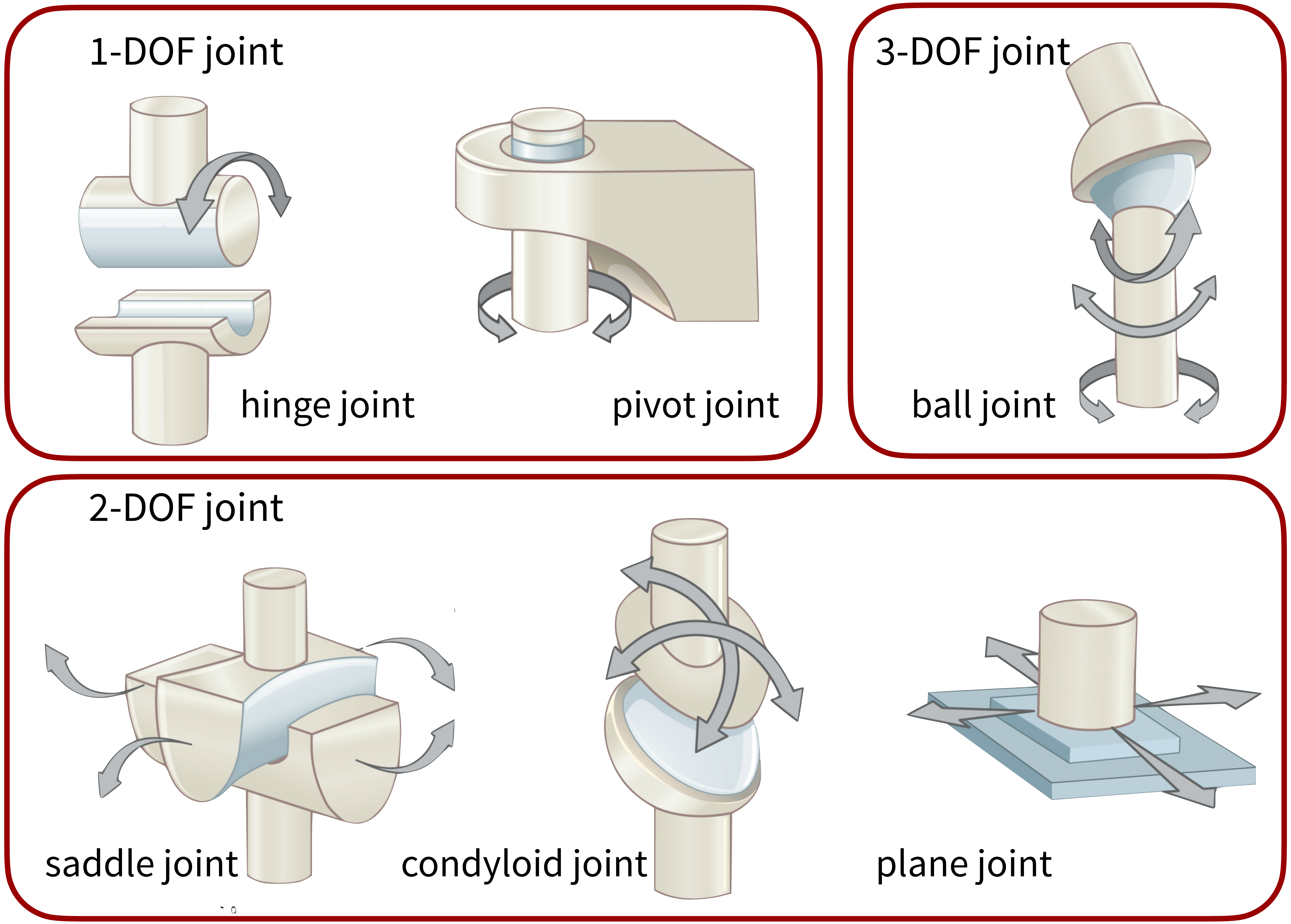
What if the point of interest is on the “shin” of Pupper? What is the transformation chain?



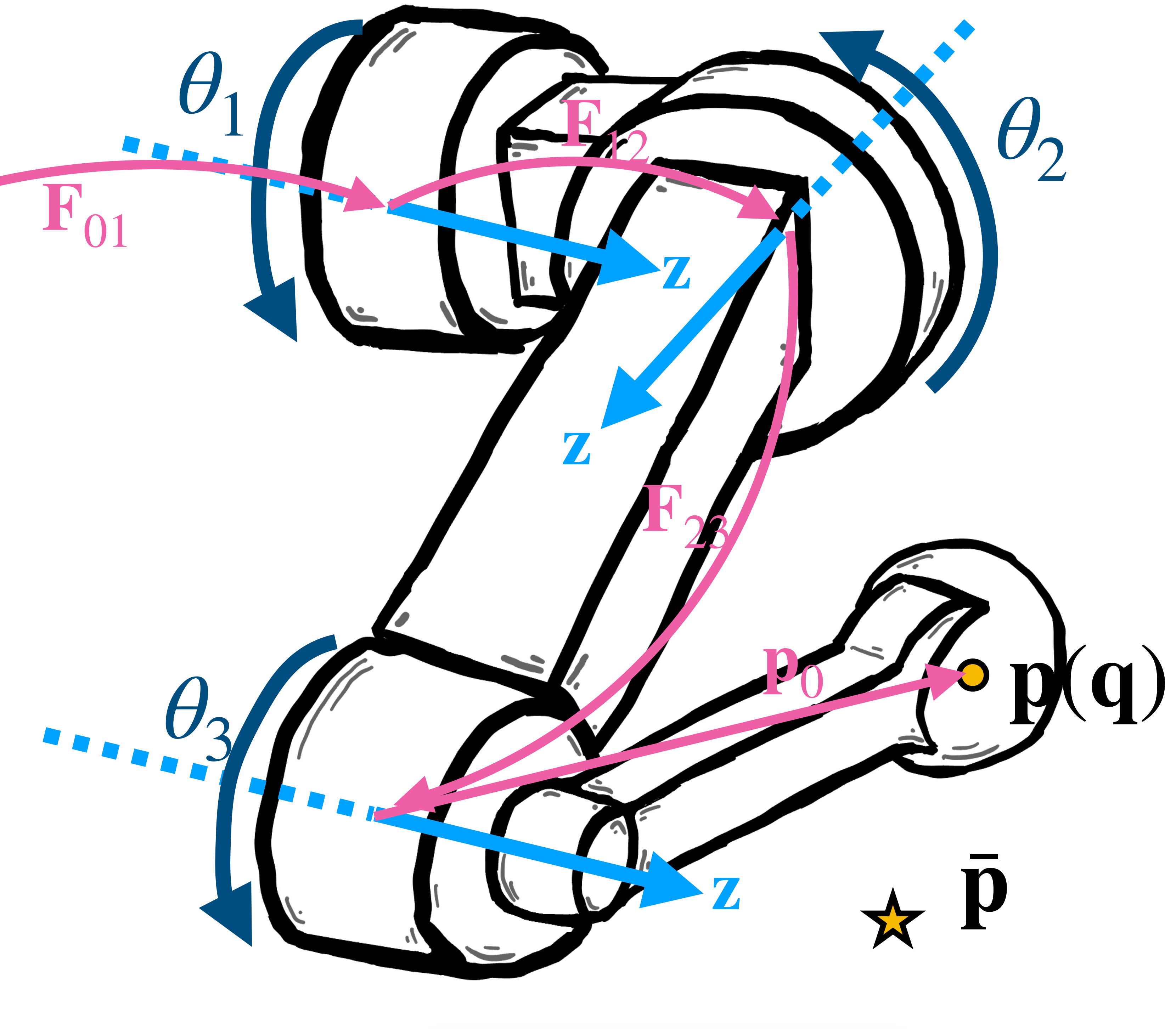
Answer: $\mathbf{p}(\mathbf{q}) = \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\,\mathbf{p}_0$

Joints


θ_1
θ_2
θ_3
$\mathbf{F}_{01}$
$\mathbf{F}_{12}$
$\mathbf{F}_{23}$
z
z
z
$\mathbf{p}_0$
$\mathbf{p}(\mathbf{q})$


1-DOF joint
hinge joint
pivot joint
3-DOF joint
ball joint
2-DOF joint
saddle joint
condyloid joint
plane joint

Formulate IK problem



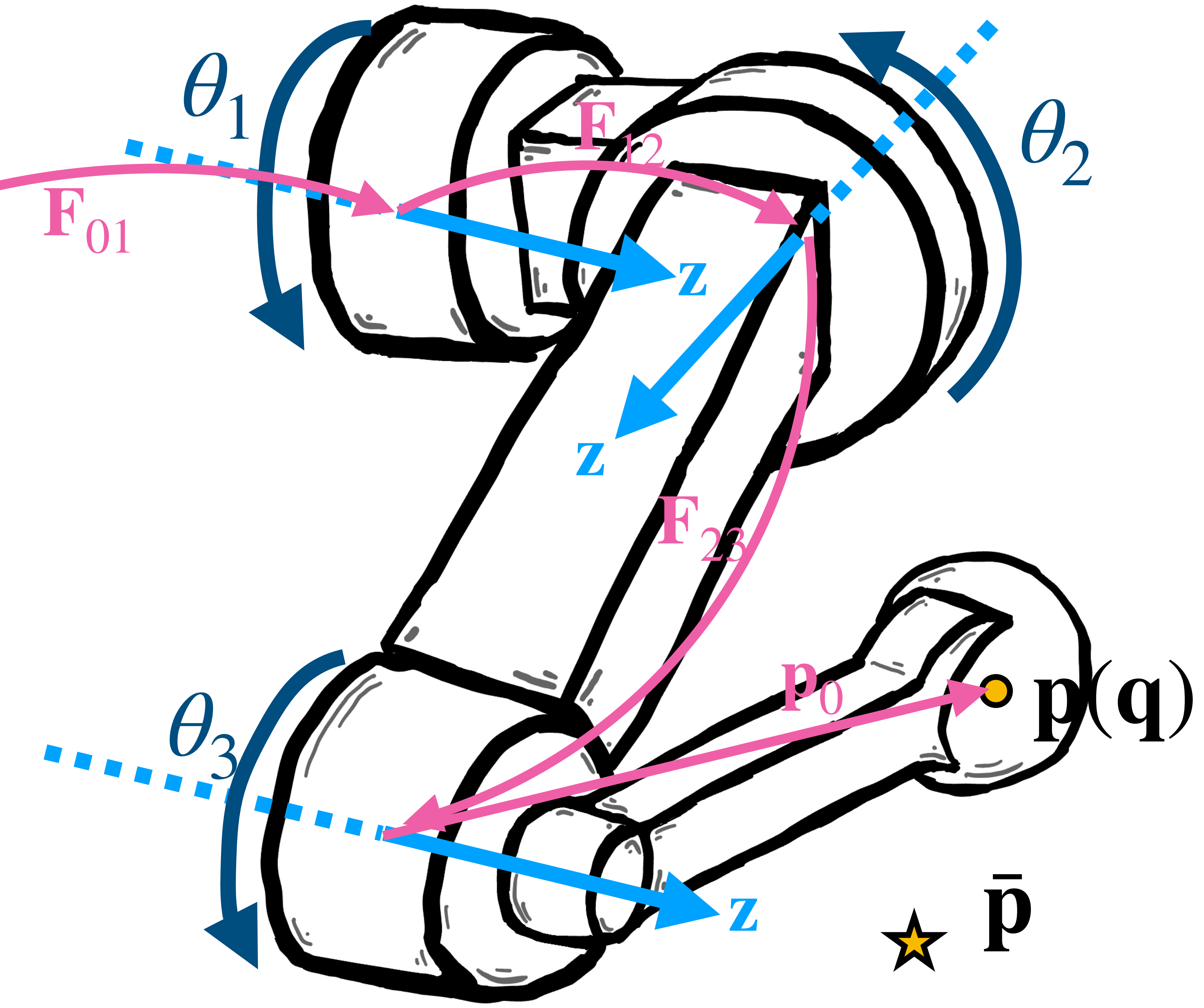
What is the condition for a pose $\mathbf{q}$ such that the toe reaches point $\bar{\mathbf{p}}$?

$$\mathbf{C}(\mathbf{q}) = \mathbf{p}(\mathbf{q}) - \bar{\mathbf{p}} = \mathbf{0}$$

This is an *inverse* kinematic question, but we need to evaluate toe position, $\mathbf{p}(\mathbf{q})$, given a pose $\mathbf{q}$, and *that* is a *forward* kinematics question.

What is the coordinates of the toe in the torso frame?

Formulate IK problem



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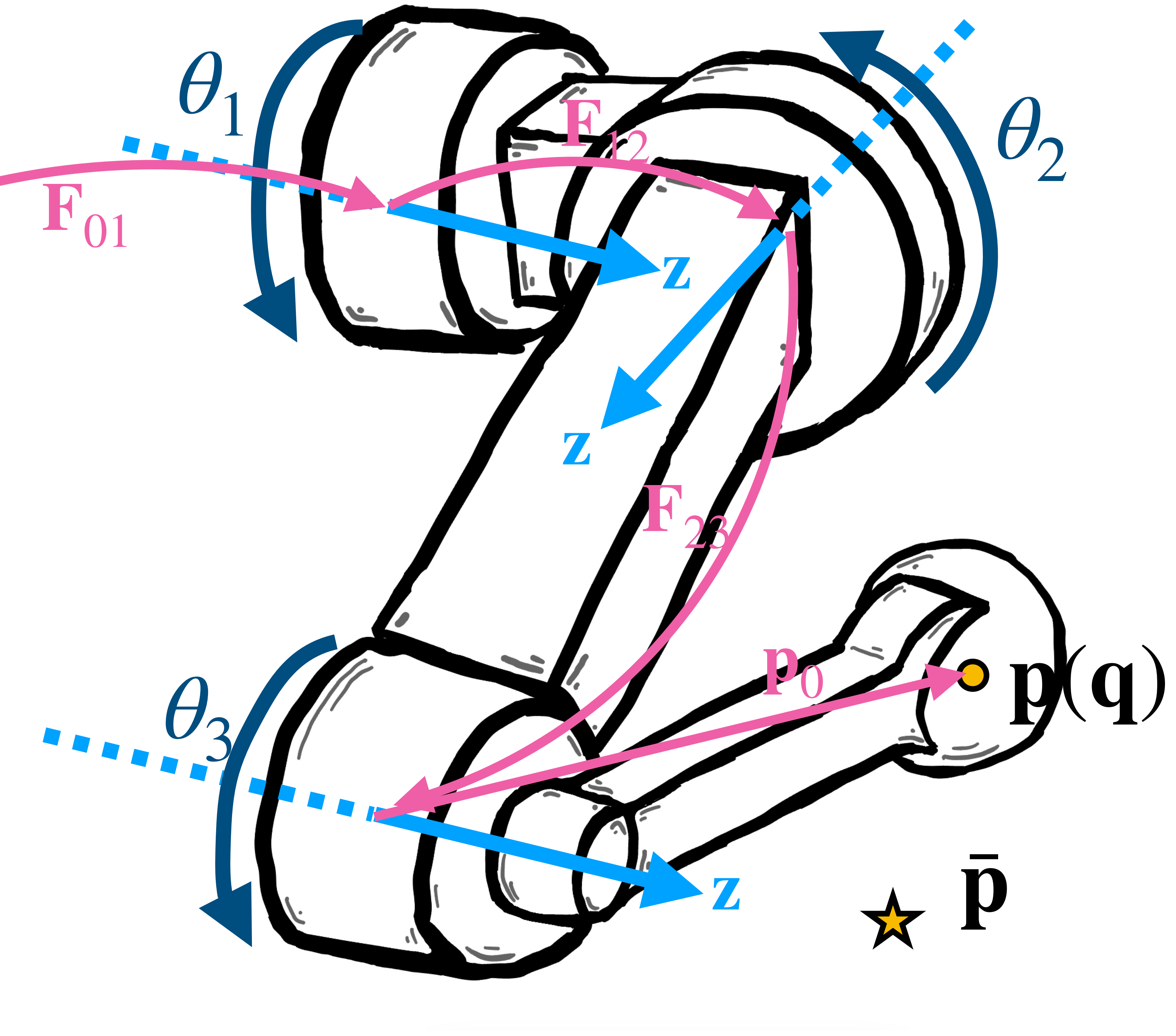
What is the coordinates of the toe in the torso frame?

$$\mathbf{p}(\mathbf{q}) = \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\mathbf{p}_0$$

the toe in torso frame

the toe in local frame

Formulate IK problem



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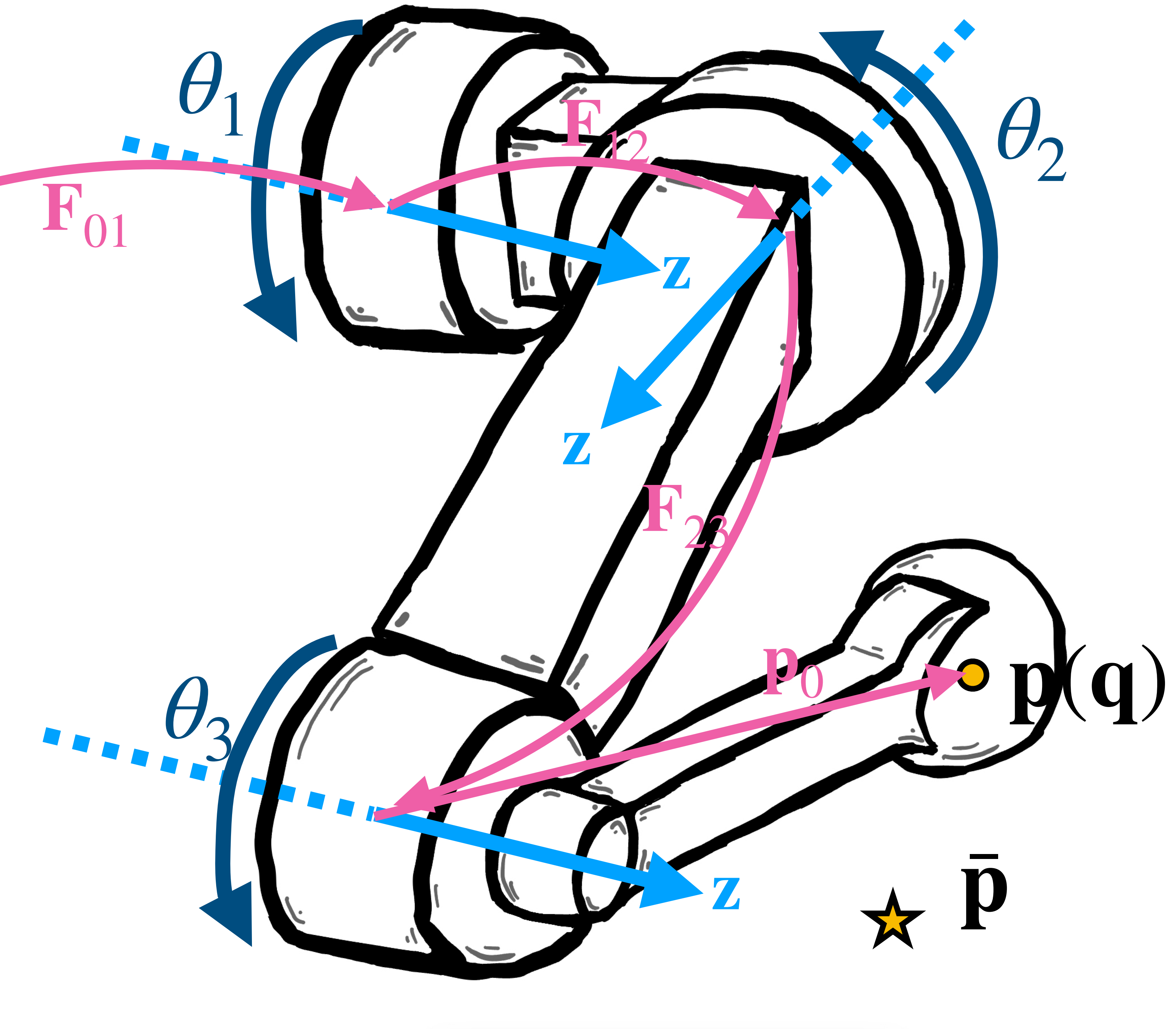
This is an *inverse* kinematic question, but we need to evaluate toe position, $\mathbf{p}(\mathbf{q})$, given a pose $\mathbf{q}$, and *that* is a *forward* kinematics question.

What is the coordinates of the toe in the torso frame?

$$\mathbf{p}(\mathbf{q}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix}$$

In practice, just drop the fourth element, “1” at the end of FK calculation

Formulate IK problem



What is the condition for a pose $\mathbf{q}$ such that the toe reaches point $\bar{\mathbf{p}}$?

$$\mathbf{C}(\mathbf{q}) = \mathbf{p}(\mathbf{q}) - \bar{\mathbf{p}} = \mathbf{0}$$

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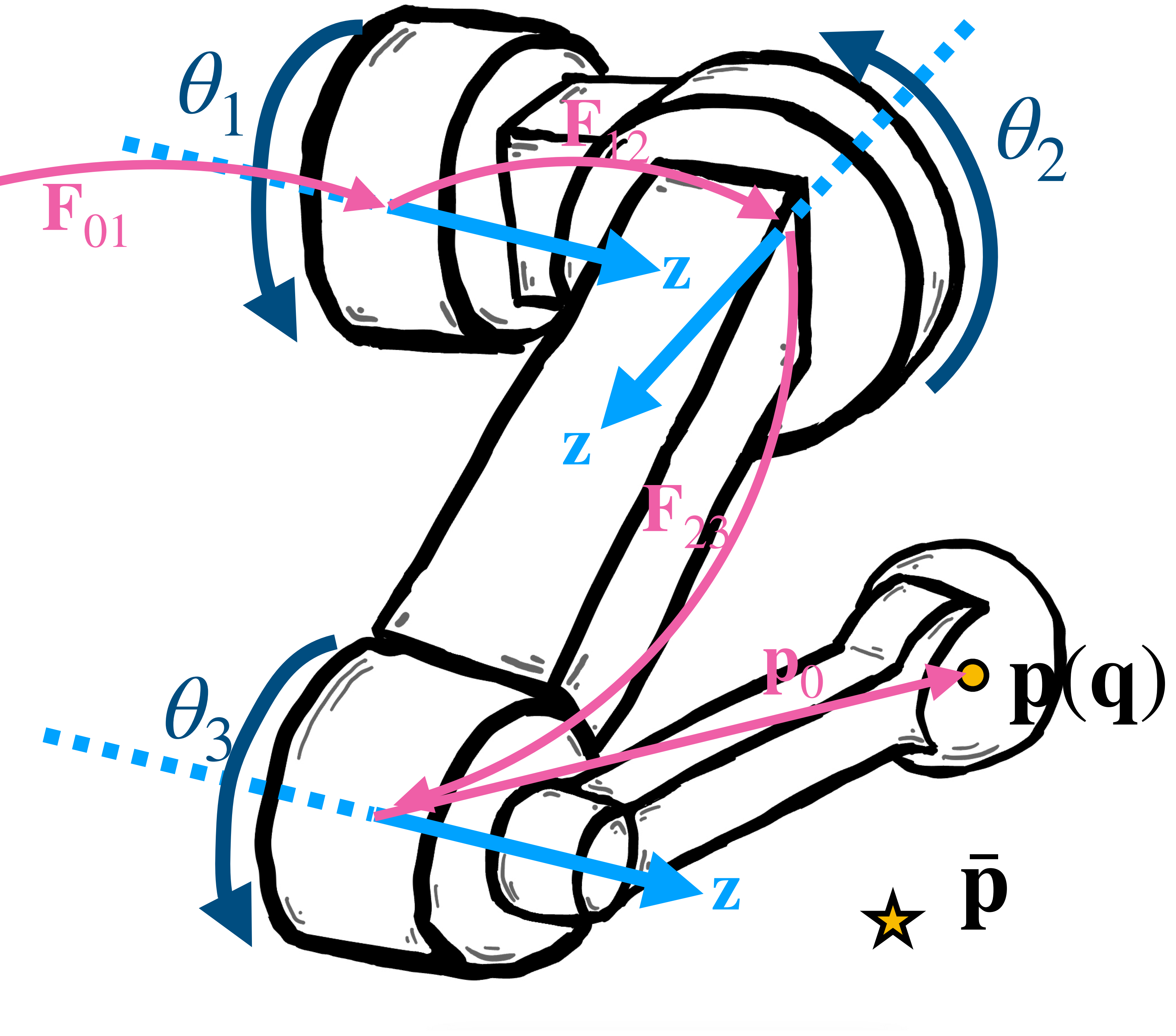
$\mathbf{C}(\mathbf{q}) = \mathbf{0}$ is a nonlinear root finding problem.

What is the coordinates of the toe in the torso frame?

$$\mathbf{p}(\mathbf{q}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3) \begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix}$$

Solve for the constraint



What is the condition for a pose $\mathbf{q}$ such that the toe reaches point $\bar{\mathbf{p}}$?

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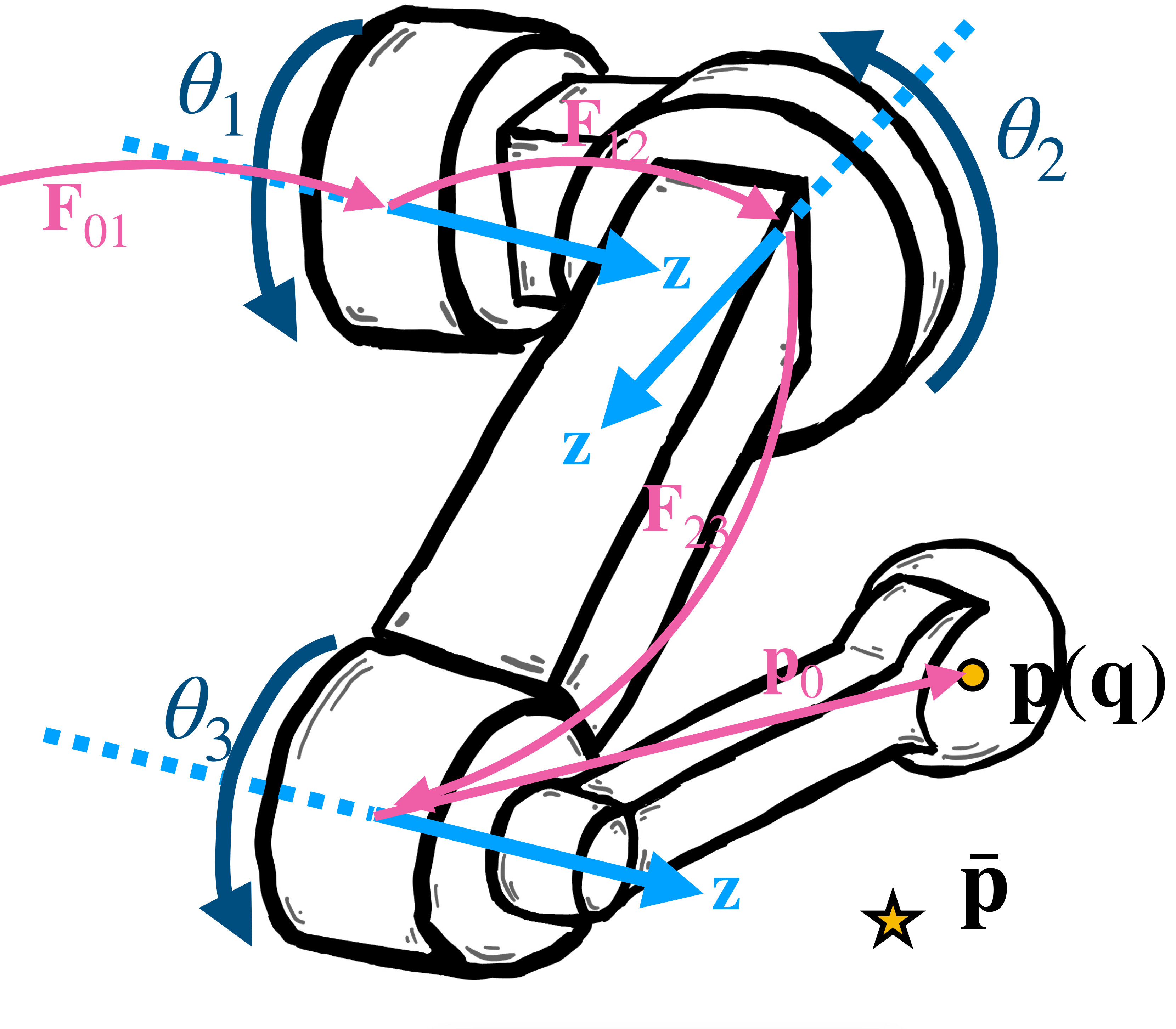
$$\mathbf{C}(\mathbf{q}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix} - \bar{\mathbf{p}} = \mathbf{0}$$

- When $\mathbf{C}(\mathbf{q})$ is nonlinear in $\mathbf{q}$, it's difficult to solve analytically
- There could be
  - A single solution
  - No solution
  - Multiple solutions

Solve it numerically, as an optimization problem

Solve IK as optimization problem



$\mathbf{C}(\mathbf{q}) = \mathbf{p}(\mathbf{q}) - \bar{\mathbf{p}}$

$$\min_{\mathbf{q}} \quad \|\mathbf{C}(\mathbf{q})\|^2$$

Among all possible $\mathbf{q}$'s, find one that minimizes the objective function: $\|\mathbf{C}(\mathbf{q})\|^2$

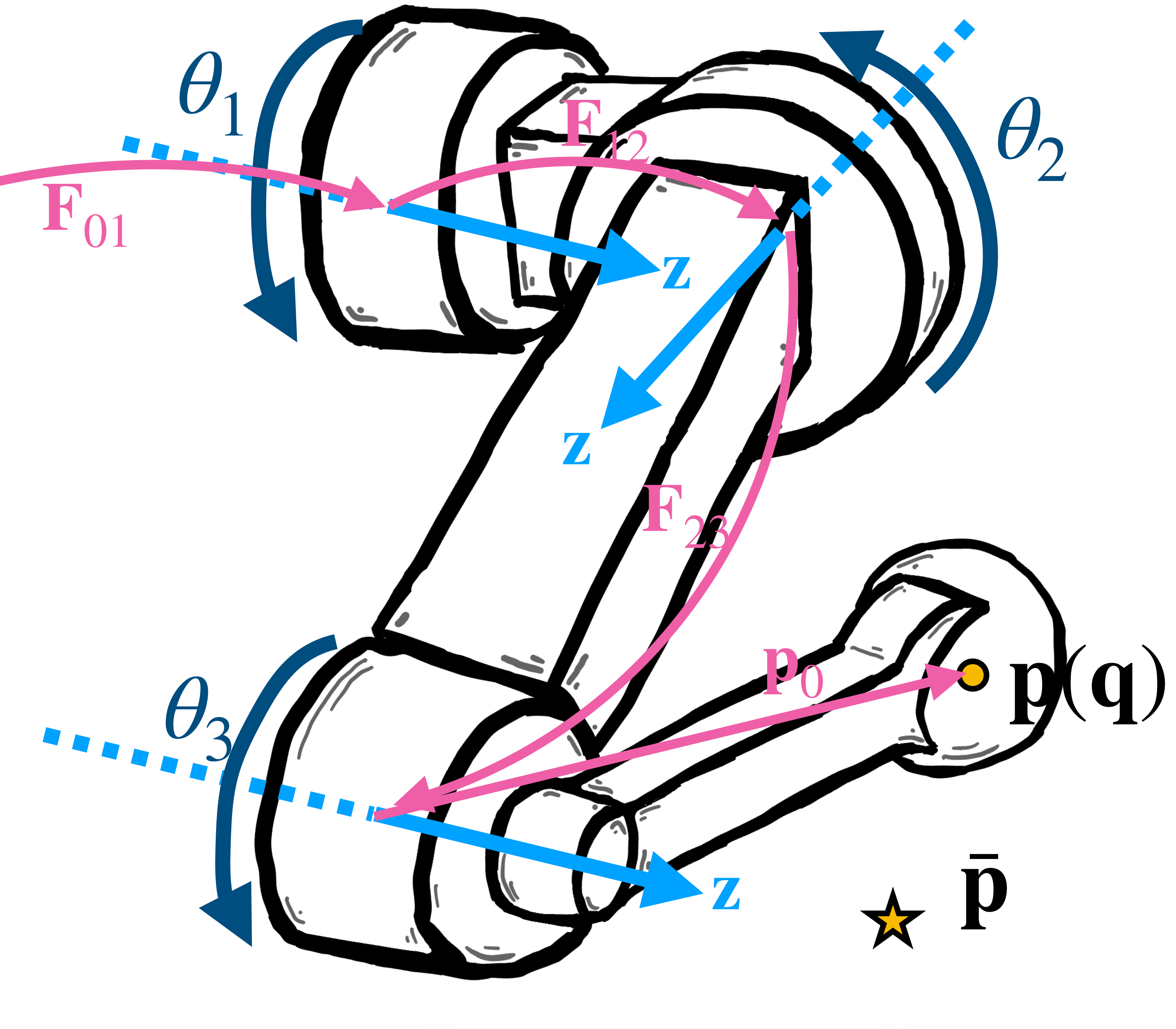
What is the smallest possible value for $\|\mathbf{C}(\mathbf{q})\|^2$?

Zero!

How to find $\mathbf{q}$ such that $\|\mathbf{C}(\mathbf{q})\|^2$ is minimized?

Gradient Descent Method

Gradient descent



$$\min_{\mathbf{q}} \quad \|\mathbf{C}(\mathbf{q})\|^2$$

// while not converged

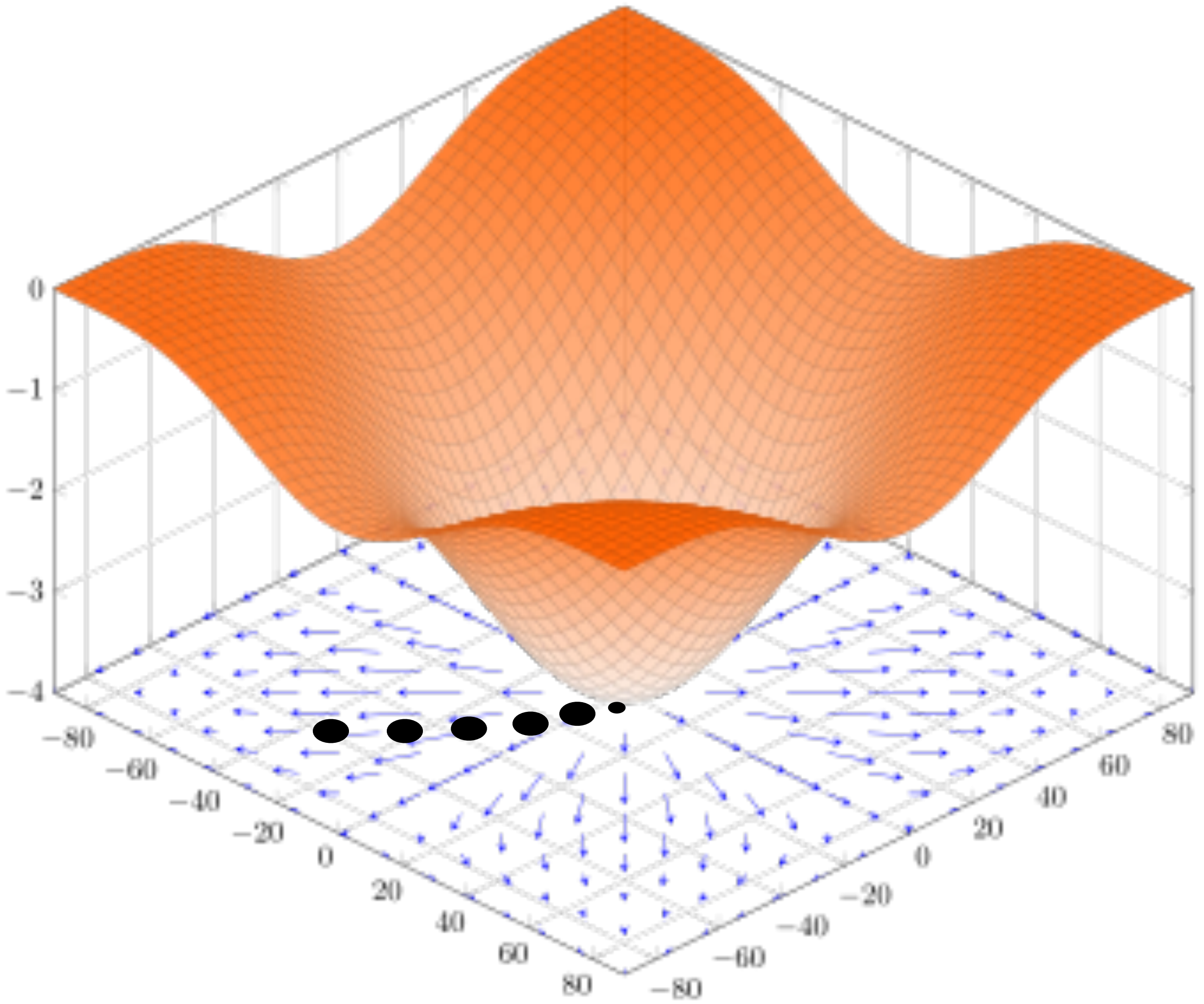
// compute gradient of objective function at current $\mathbf{q}$

// update $\mathbf{q}$ by moving along the negative gradient direction by a small step

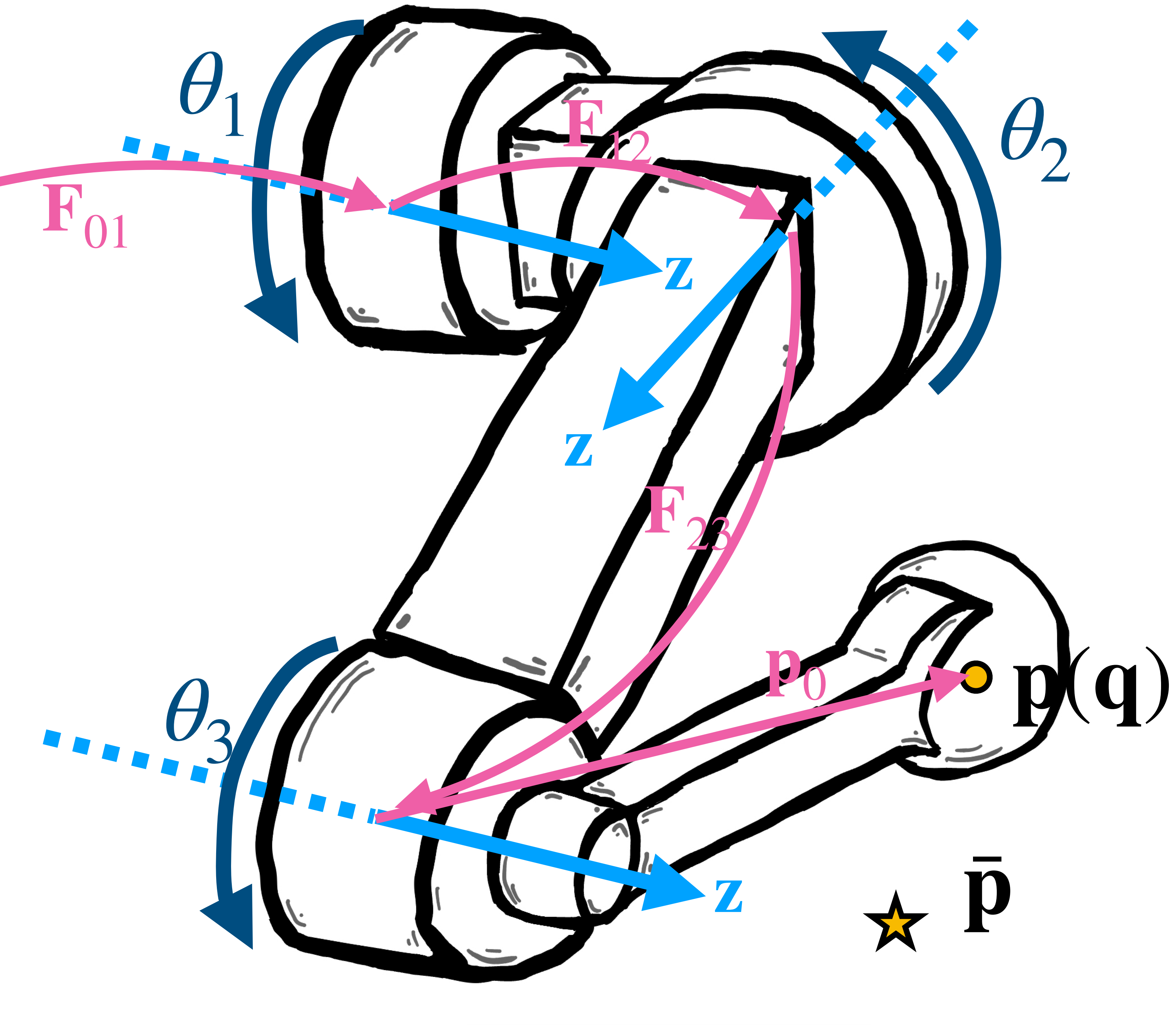
$$\textbf{while} \;\; \|\mathbf{C}(\mathbf{q})\|^2 > \epsilon$$

$$\mathbf{d} = 2\frac{\partial \mathbf{C}(\mathbf{q})}{\partial \mathbf{q}}^T \mathbf{C}(\mathbf{q})$$

$$\mathbf{q} = \mathbf{q} - \alpha \mathbf{d}$$



Compute gradient



$$\mathbf{d} = 2\boxed{\frac{\partial \mathbf{C}(\mathbf{q})}{\partial \mathbf{q}}^T}\boxed{\mathbf{C}(\mathbf{q})}$$

To compute gradient in each optimization iteration,

we need to evaluate the constraint:

$$\mathbf{C}(\mathbf{q}) = \mathbf{p}(\mathbf{q}) - \bar{\mathbf{p}}$$

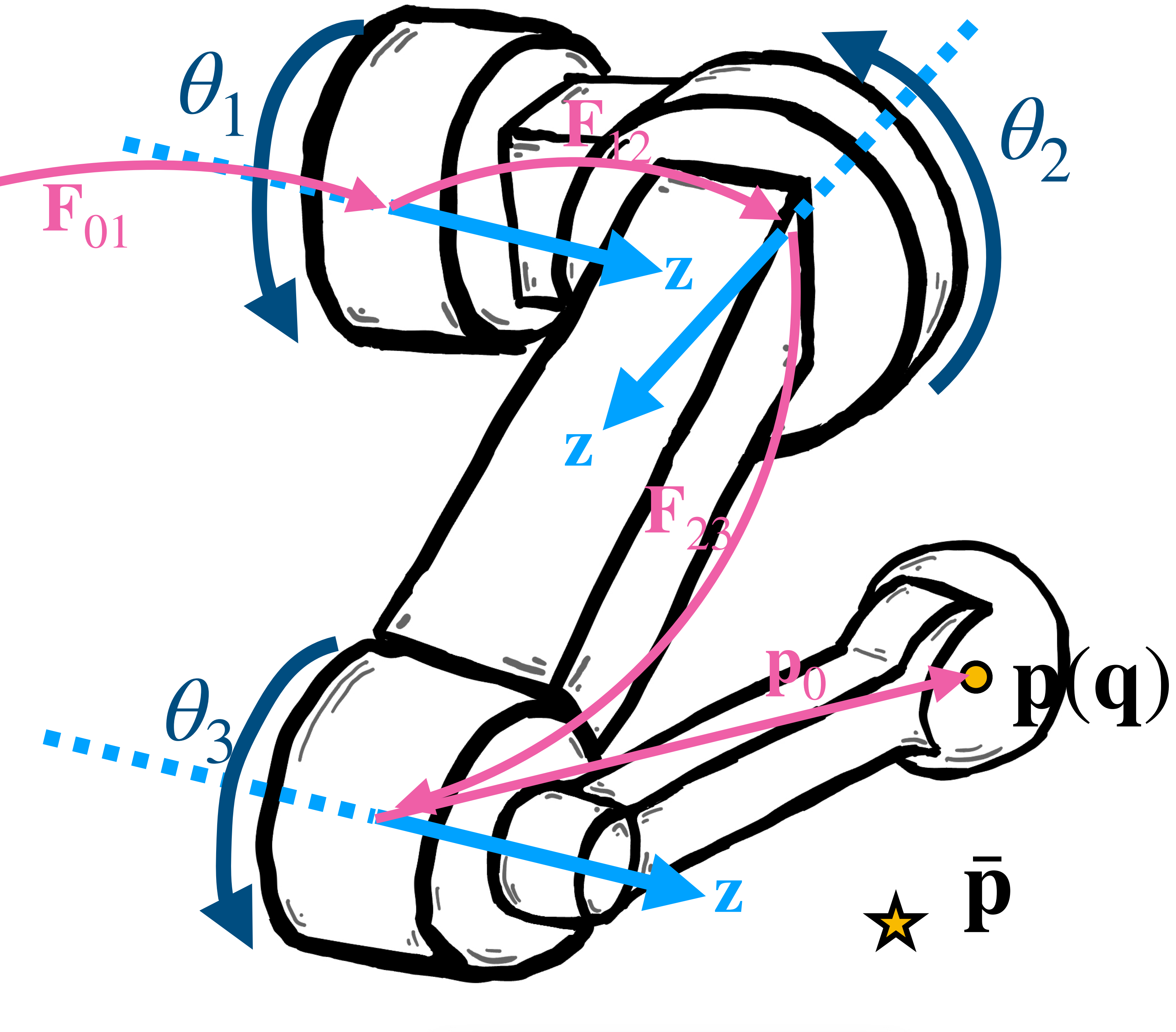
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix} - \bar{\mathbf{p}}$$

we need to compute the partial derivatives of $\mathbf{C}$

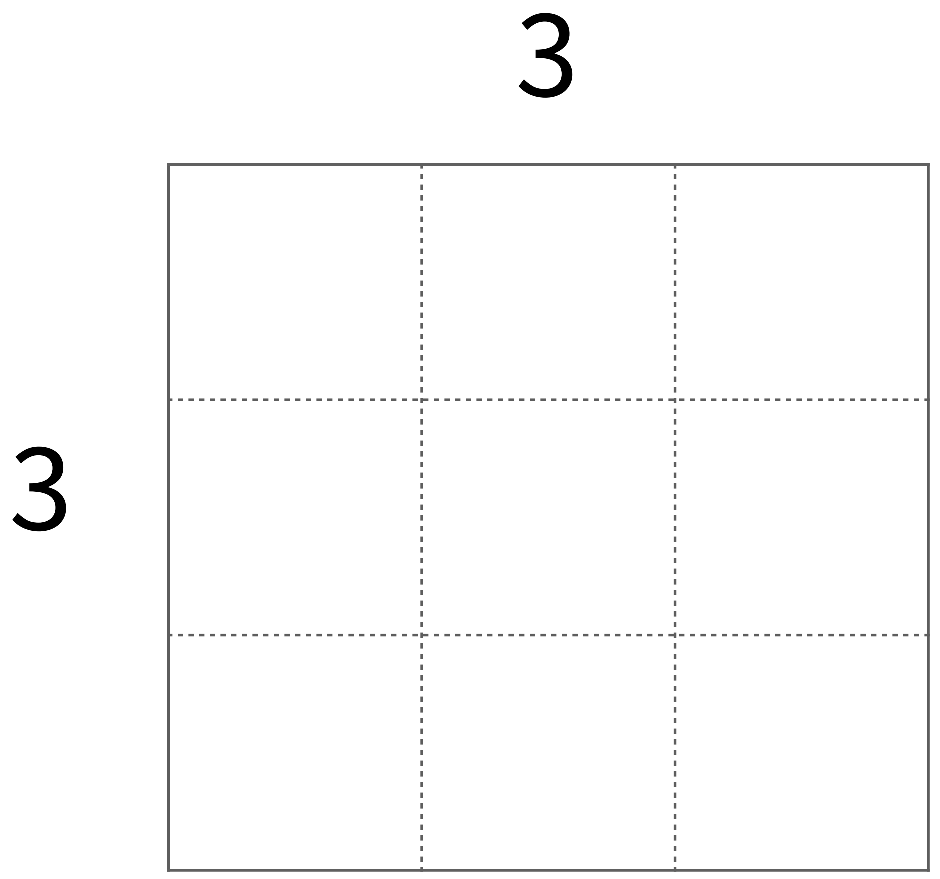
$$\frac{\partial \mathbf{C}(\mathbf{q})}{\partial \mathbf{q}} = \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}} = ?$$

This is called the Jacobian Matrix

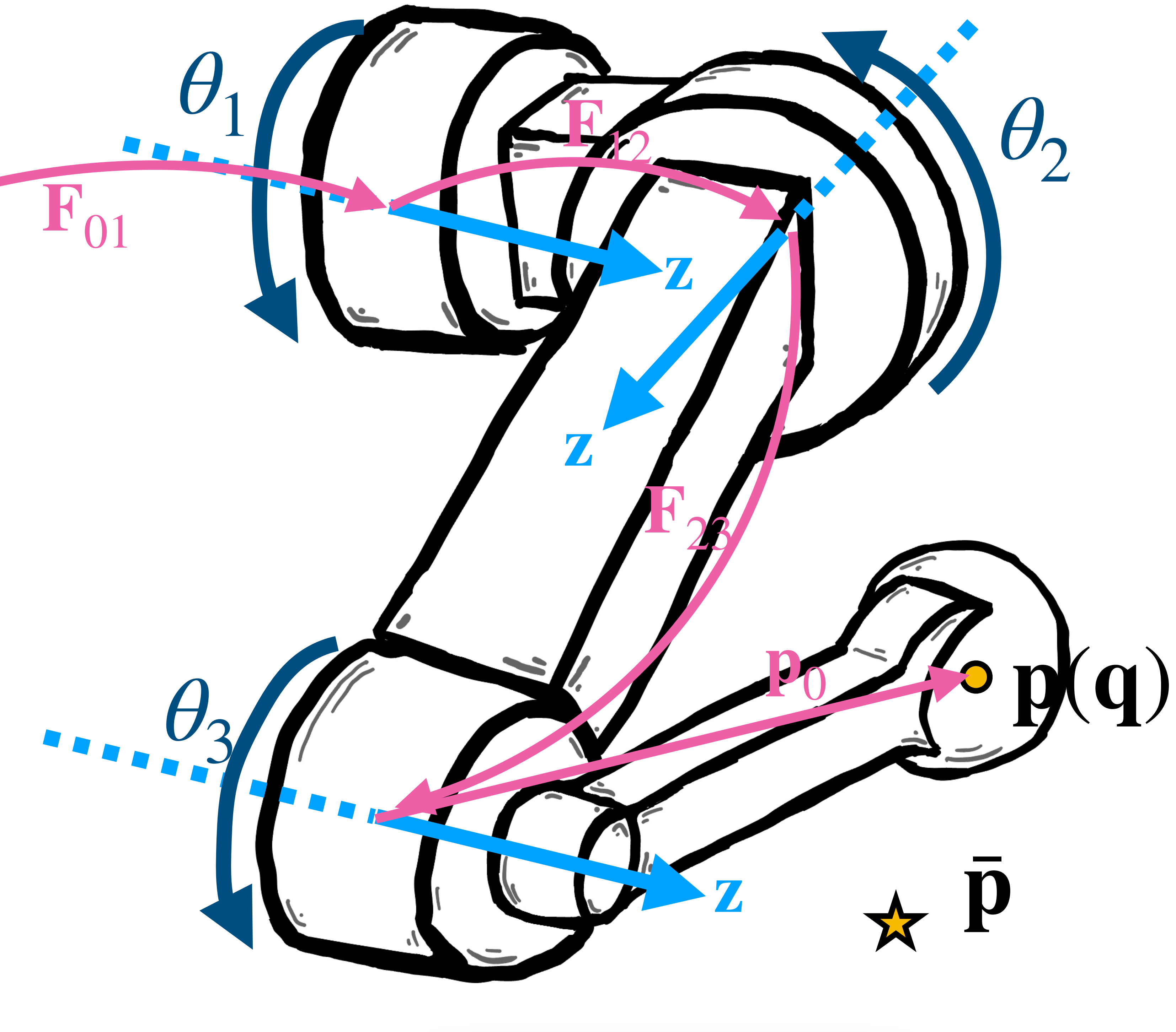
Jacobian matrix



What's the dimension of Jacobian $\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}}$?



Jacobian matrix

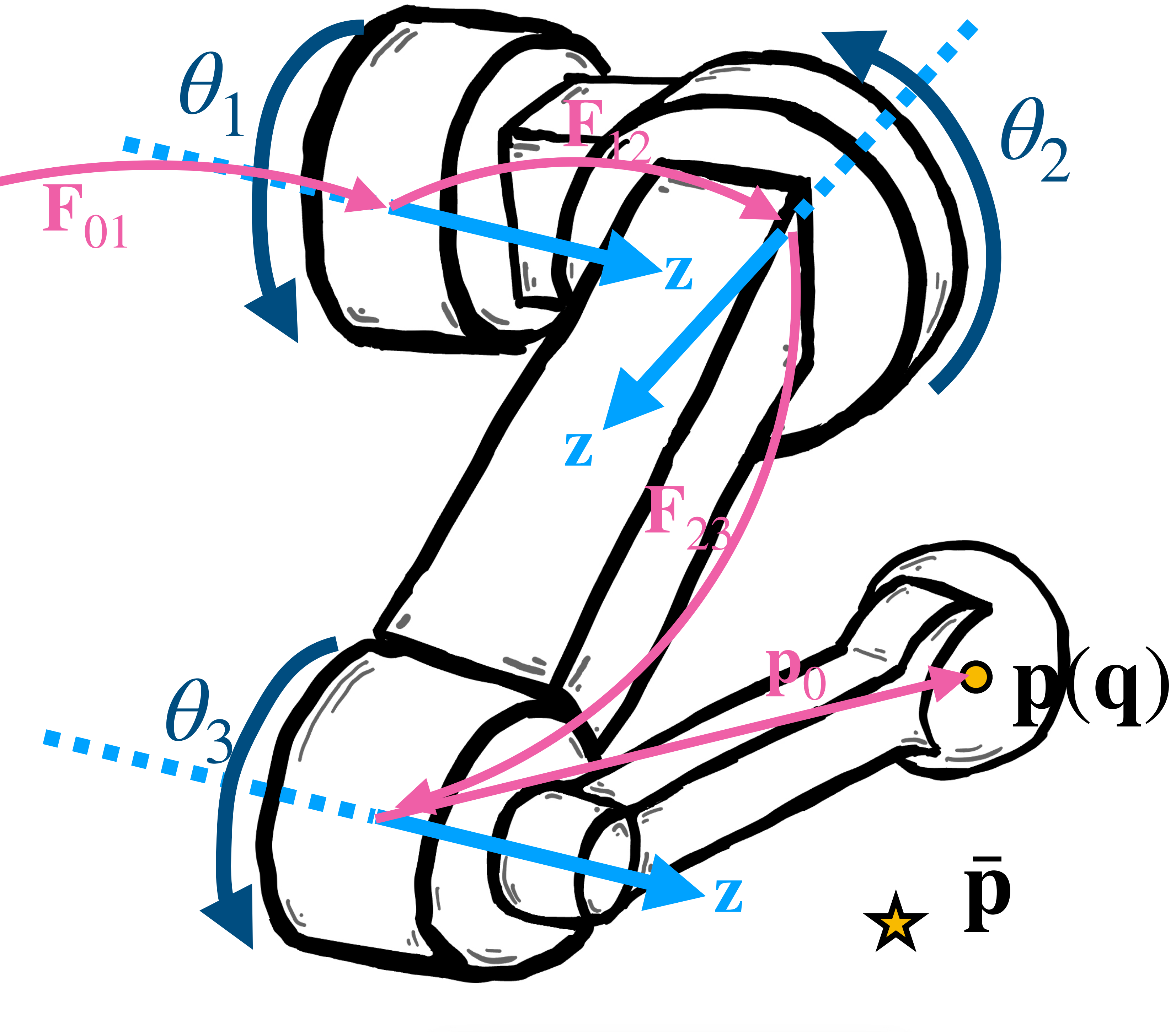


$$\mathbf{p}(\mathbf{q}) = \mathrm{Proj}\left(\mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix}\mathbf{p}_0 \\ 1\end{bmatrix}\right)$$

$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix}$$

"Proj" takes the top 3 elements of a 4x1 vector

Jacobian matrix



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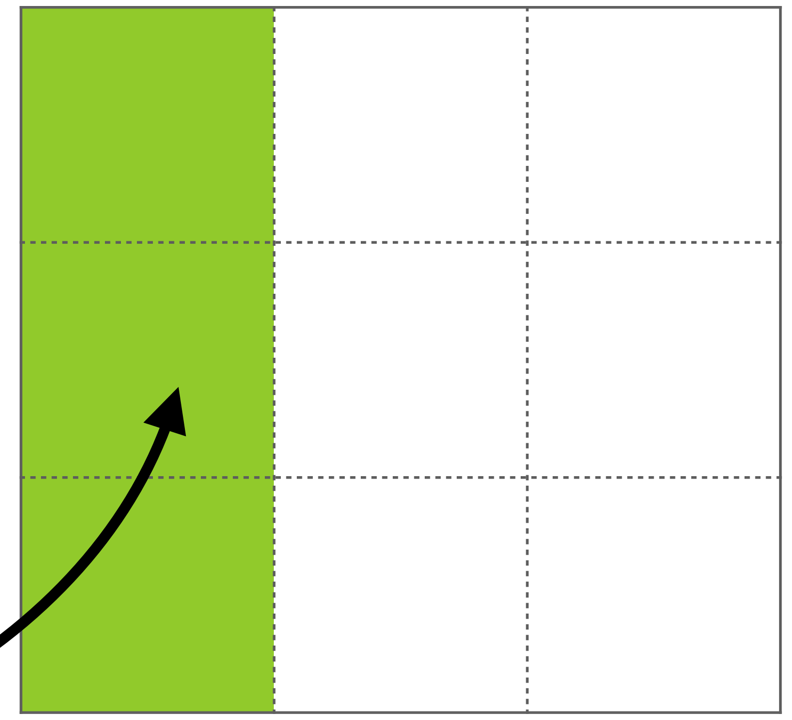
$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \theta_1} = \mathrm{Proj}\left(\mathbf{F}_{01}\frac{\partial \mathbf{R}_z(\theta_1)}{\partial \theta_1}\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix}\mathbf{p}_0\\1\end{bmatrix}\right)$$

$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}} =$$



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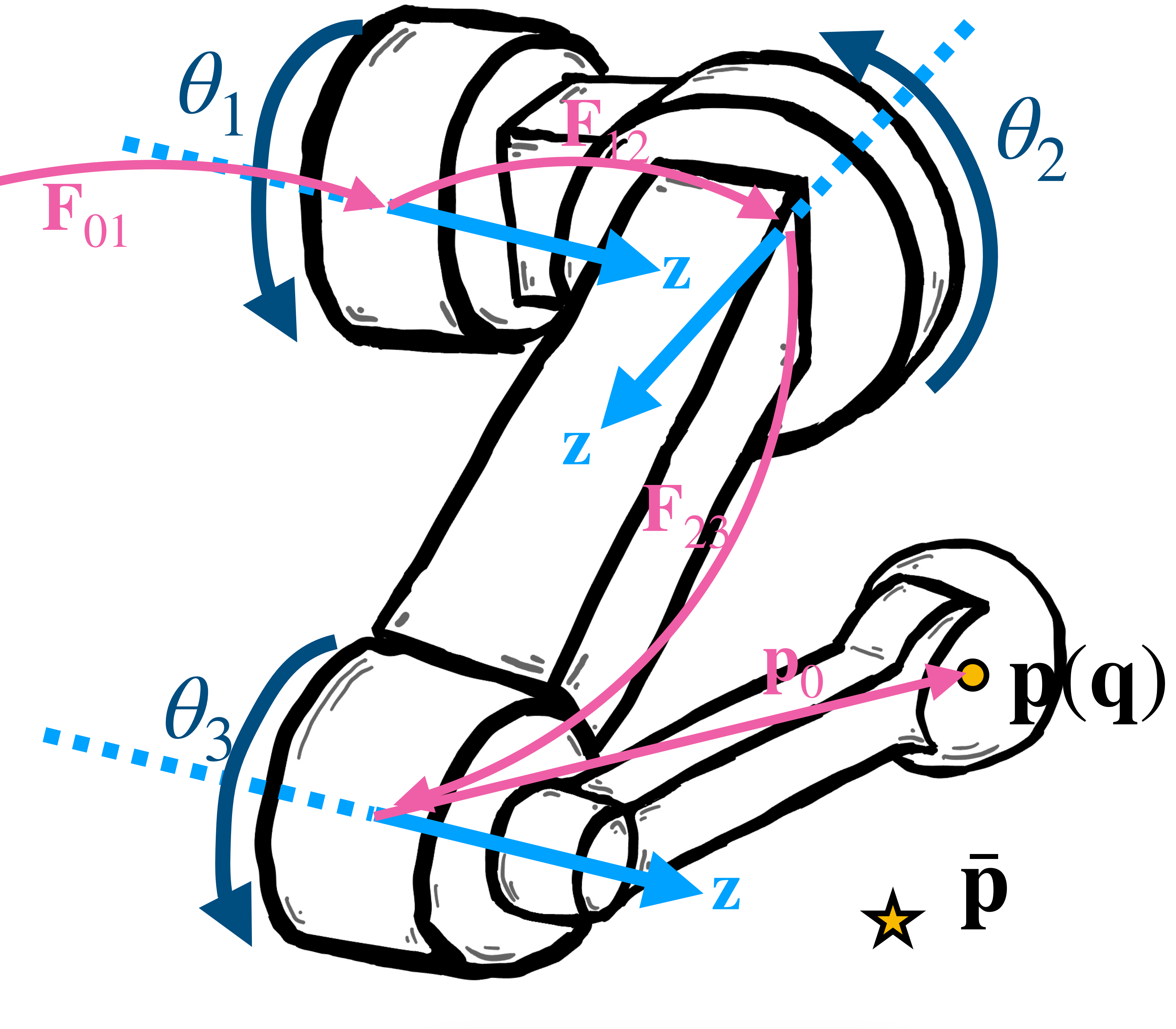

$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \theta_1} = \mathrm{Proj}\left(\mathbf{F}_{01}\frac{\partial \mathbf{R}_z(\theta_1)}{\partial \theta_1}\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix}\mathbf{p}_0\\1\end{bmatrix}\right)$$

$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \theta_2} = \mathrm{Proj}\left(\mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\frac{\partial \mathbf{R}_z(\theta_2)}{\partial \theta_2}\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix}\mathbf{p}_0\\1\end{bmatrix}\right)$$

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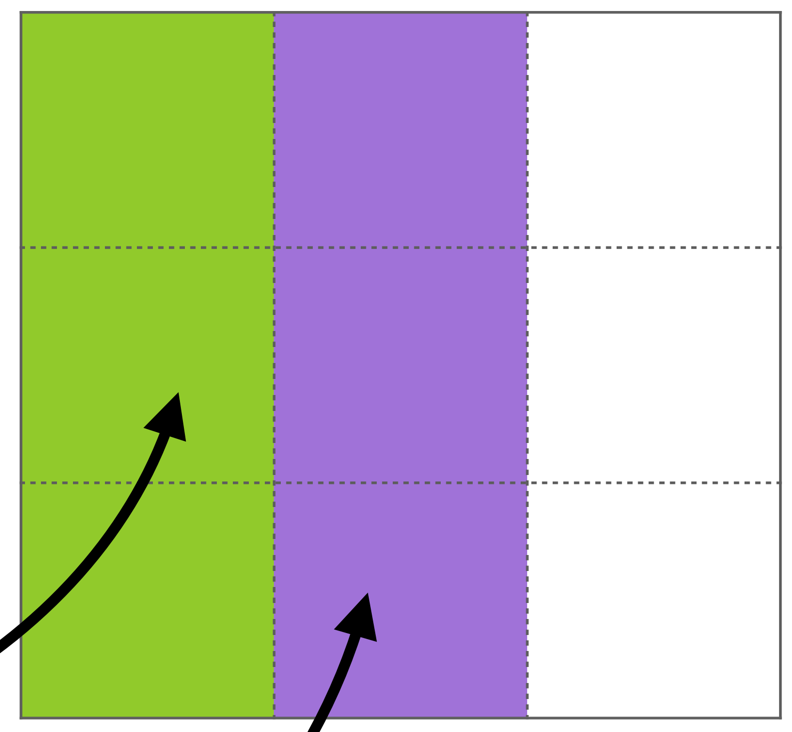

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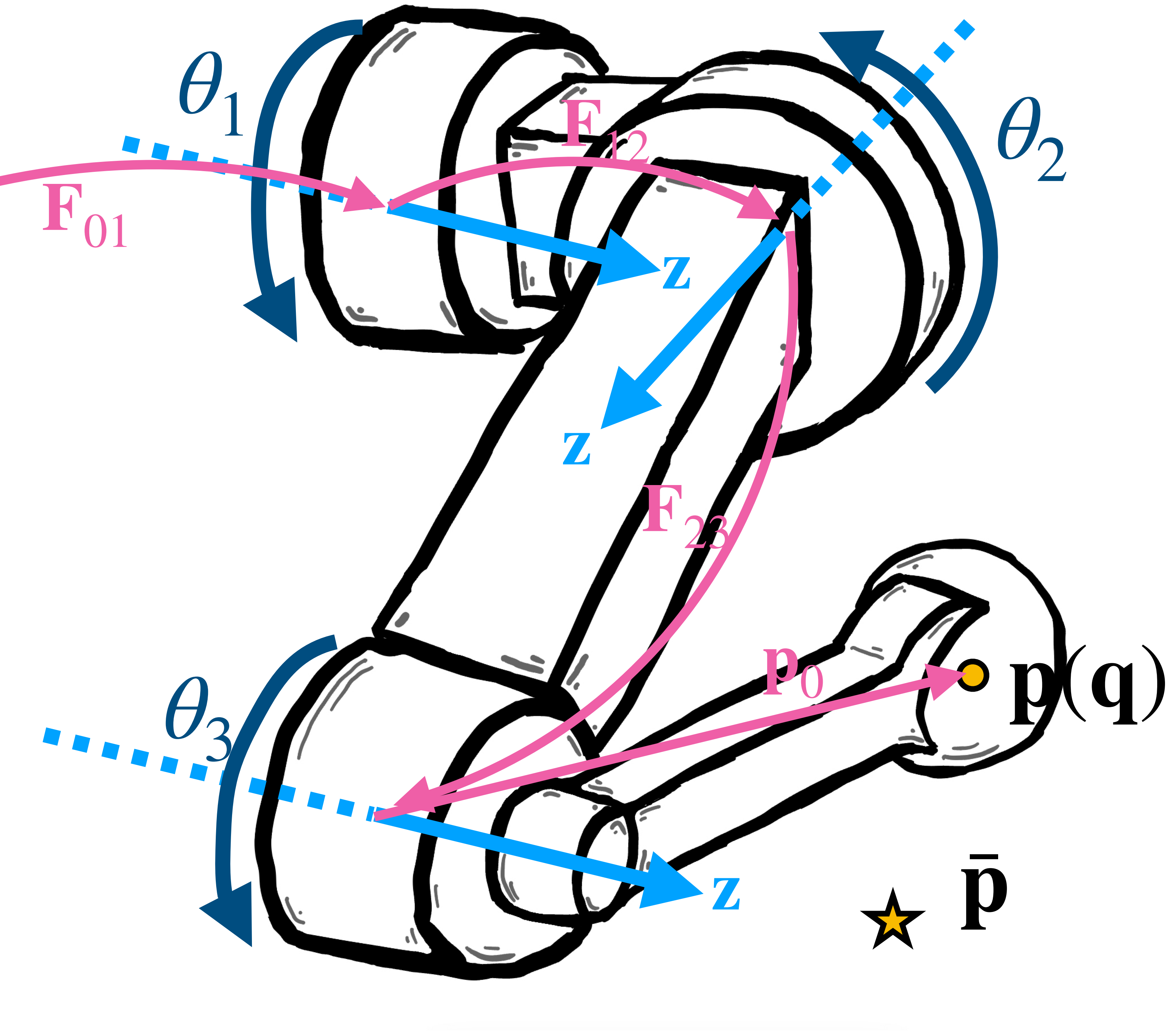

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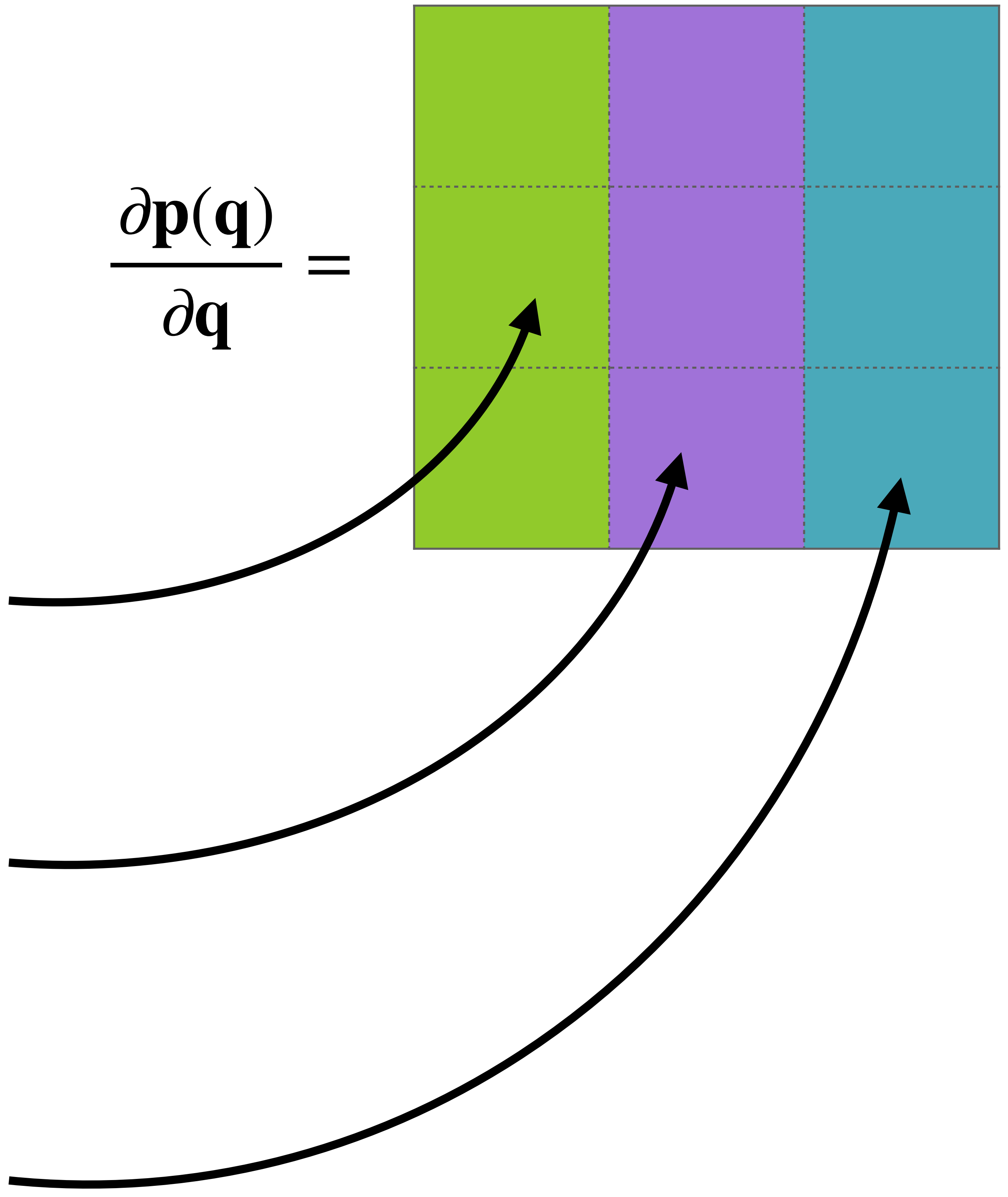

$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \theta_1} = \mathrm{Proj}\left(\mathbf{F}_{01}\boxed{\frac{\partial \mathbf{R}_z(\theta_1)}{\partial \theta_1}}\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix}\mathbf{p}_0\\1\end{bmatrix}\right)$$

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Jacobian matrix

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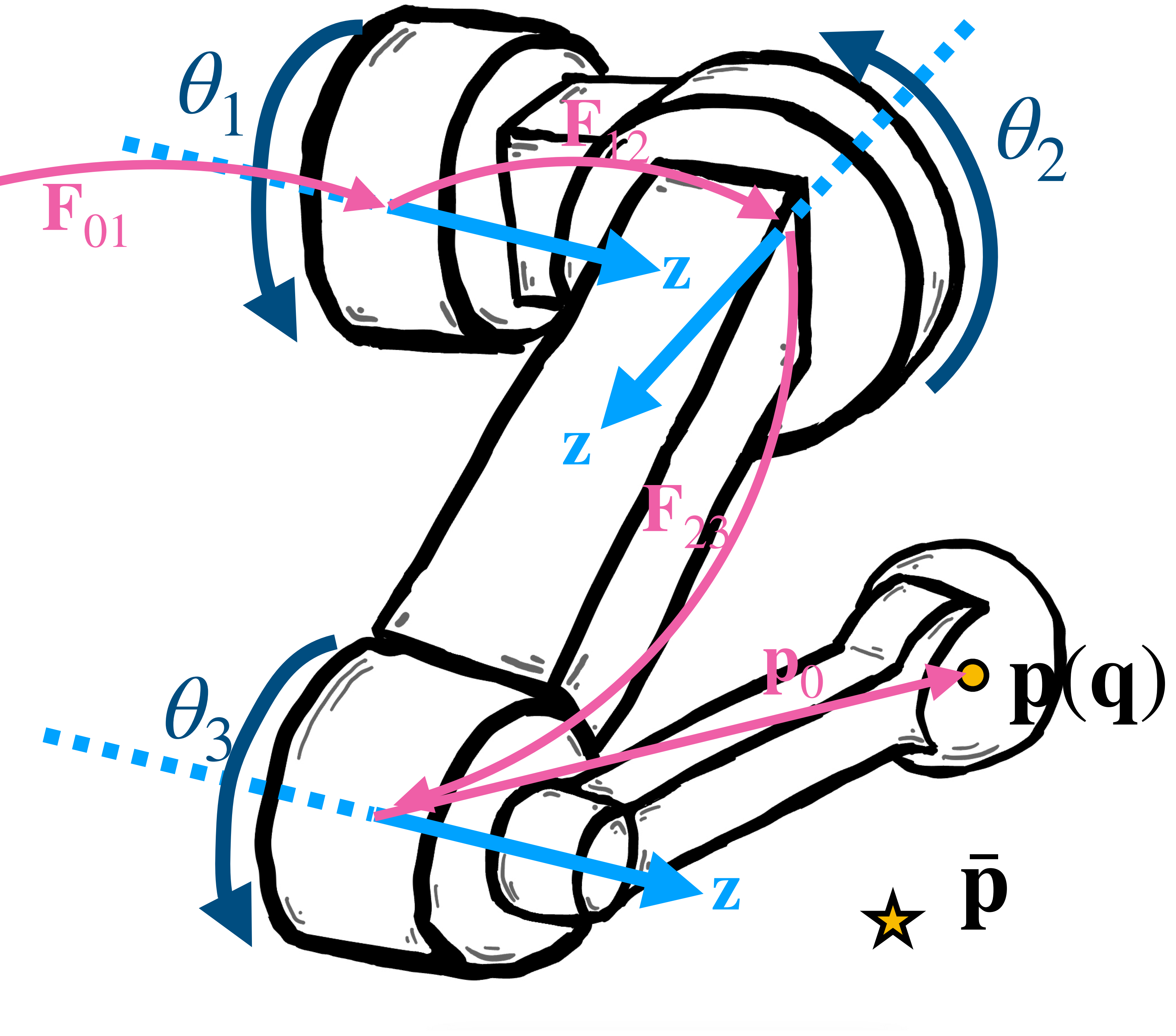

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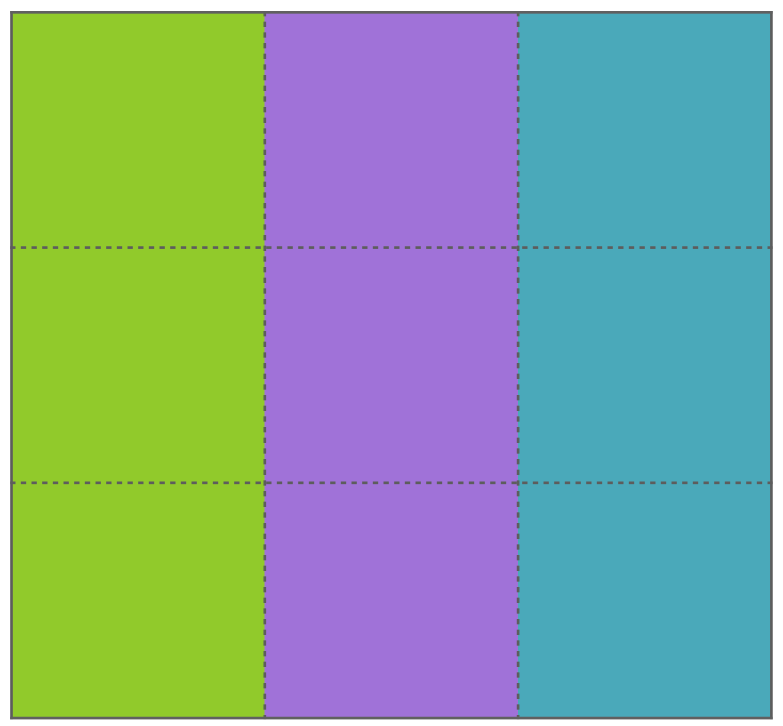


$$\frac{\partial \mathbf{R}_z(\theta_1)}{\partial \theta_1} = \begin{bmatrix} -\sin\theta_1 & -\cos\theta_1 & 0 & 0 \\ \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_z(\theta_2)}{\partial \theta_2} = \begin{bmatrix} -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_z(\theta_3)}{\partial \theta_3} = \begin{bmatrix} -\sin\theta_3 & -\cos\theta_3 & 0 & 0 \\ \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

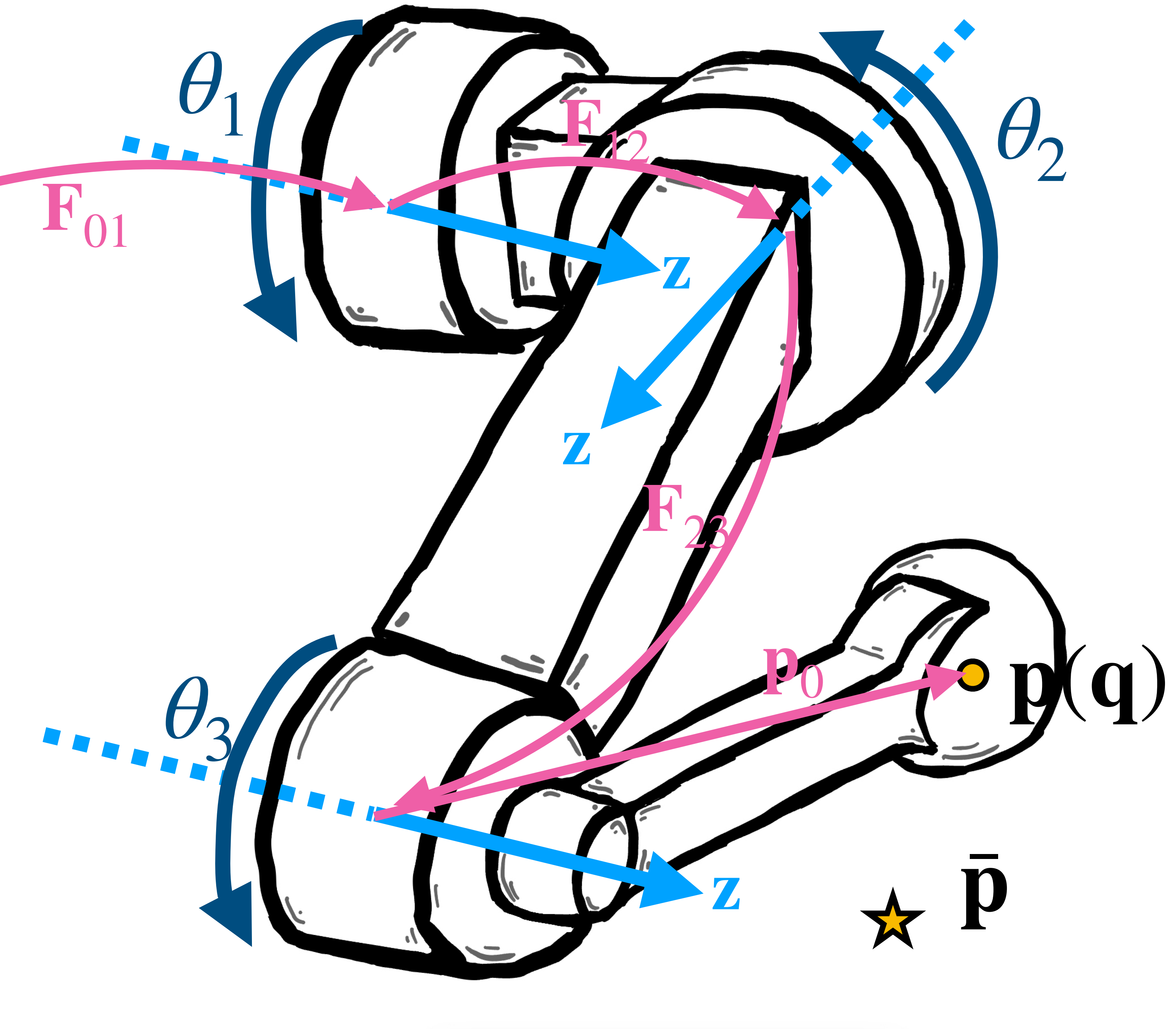
recall affine transformations

$$\mathrm{Rot}_x(\theta_x) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathrm{Rot}_y(\theta_y) = \left(\begin{array}{ccc|c} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

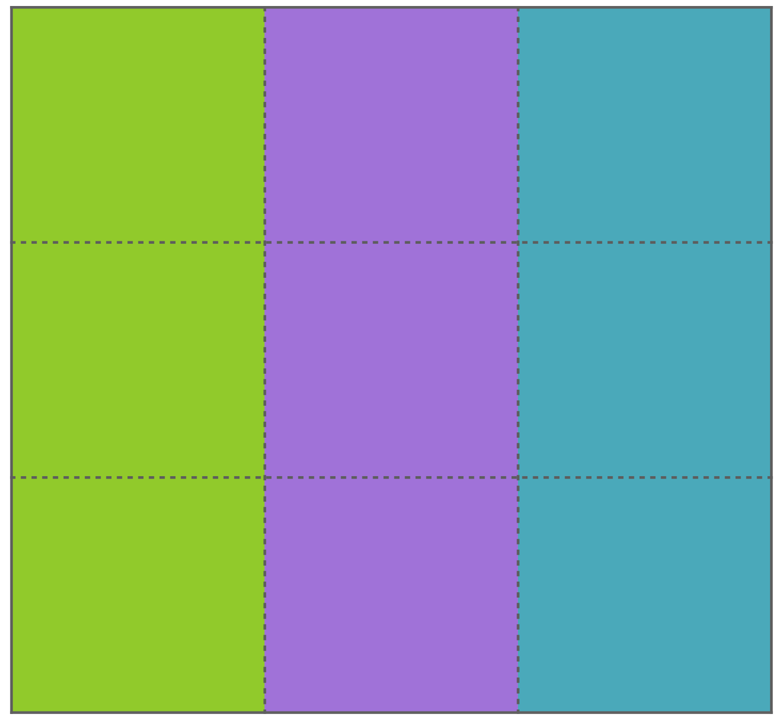
$$\mathrm{Rot}_z(\theta_z) = \left(\begin{array}{ccc|c} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Jacobian matrix



$$\mathbf{p}(\mathbf{q}) = \mathrm{Proj}\left(\mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix}\mathbf{p}_0\\1\end{bmatrix}\right)$$

$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}} =$$

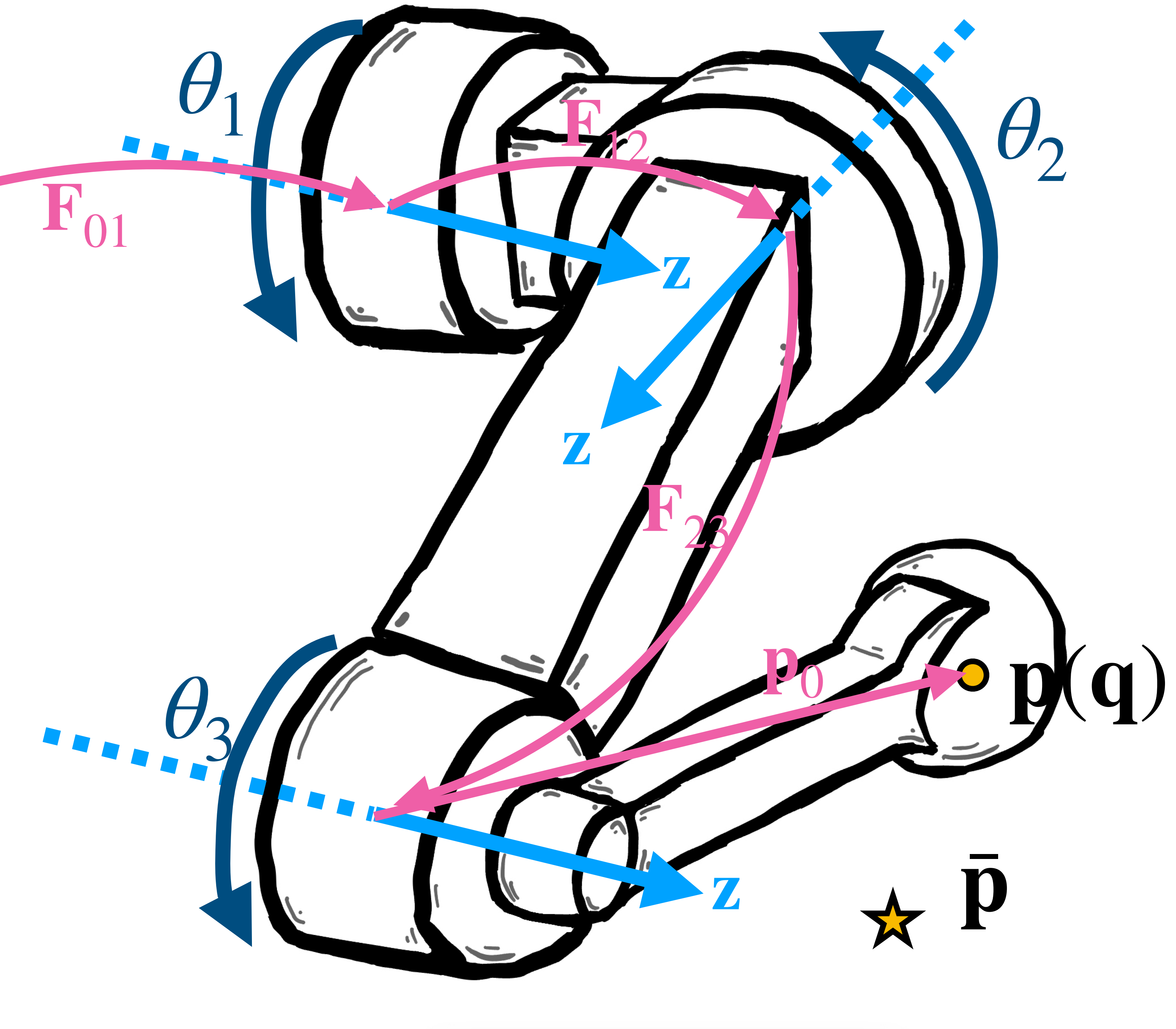


$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \theta_1} = \mathrm{Proj}\left(\mathbf{F}_{01}\begin{bmatrix}-\sin\theta_1 & -\cos\theta_1 & 0 & 0\\ \cos\theta_1 & -\sin\theta_1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix}\mathbf{p}_0\\1\end{bmatrix}\right)$$

$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \theta_2} = \mathrm{Proj}\left(\mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\begin{bmatrix}-\sin\theta_2 & -\cos\theta_2 & 0 & 0\\ \cos\theta_2 & -\sin\theta_2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix}\mathbf{p}_0\\1\end{bmatrix}\right)$$

$$\frac{\partial \mathbf{p}(\mathbf{q})}{\partial \theta_3} = \mathrm{Proj}\left(\mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\begin{bmatrix}-\sin\theta_3 & -\cos\theta_3 & 0 & 0\\ \cos\theta_3 & -\sin\theta_3 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{p}_0\\1\end{bmatrix}\right)$$

Compute gradient



$$\mathbf{d} = 2\frac{\partial \mathbf{C}(\mathbf{q})}{\partial \mathbf{q}}^T \mathbf{C}(\mathbf{q})$$

To compute gradient in each optimization iteration,

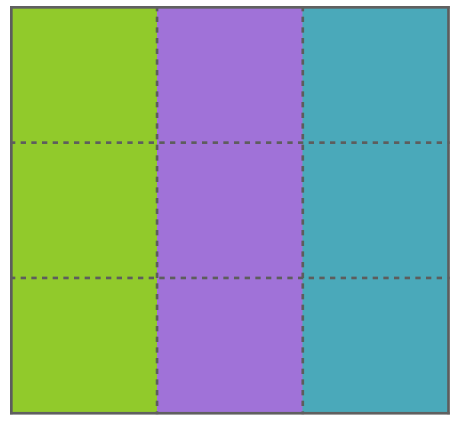
we need to evaluate the constraint:

$$\mathbf{C}(\mathbf{q}) = \mathbf{p}(\mathbf{q}) - \bar{\mathbf{p}}$$

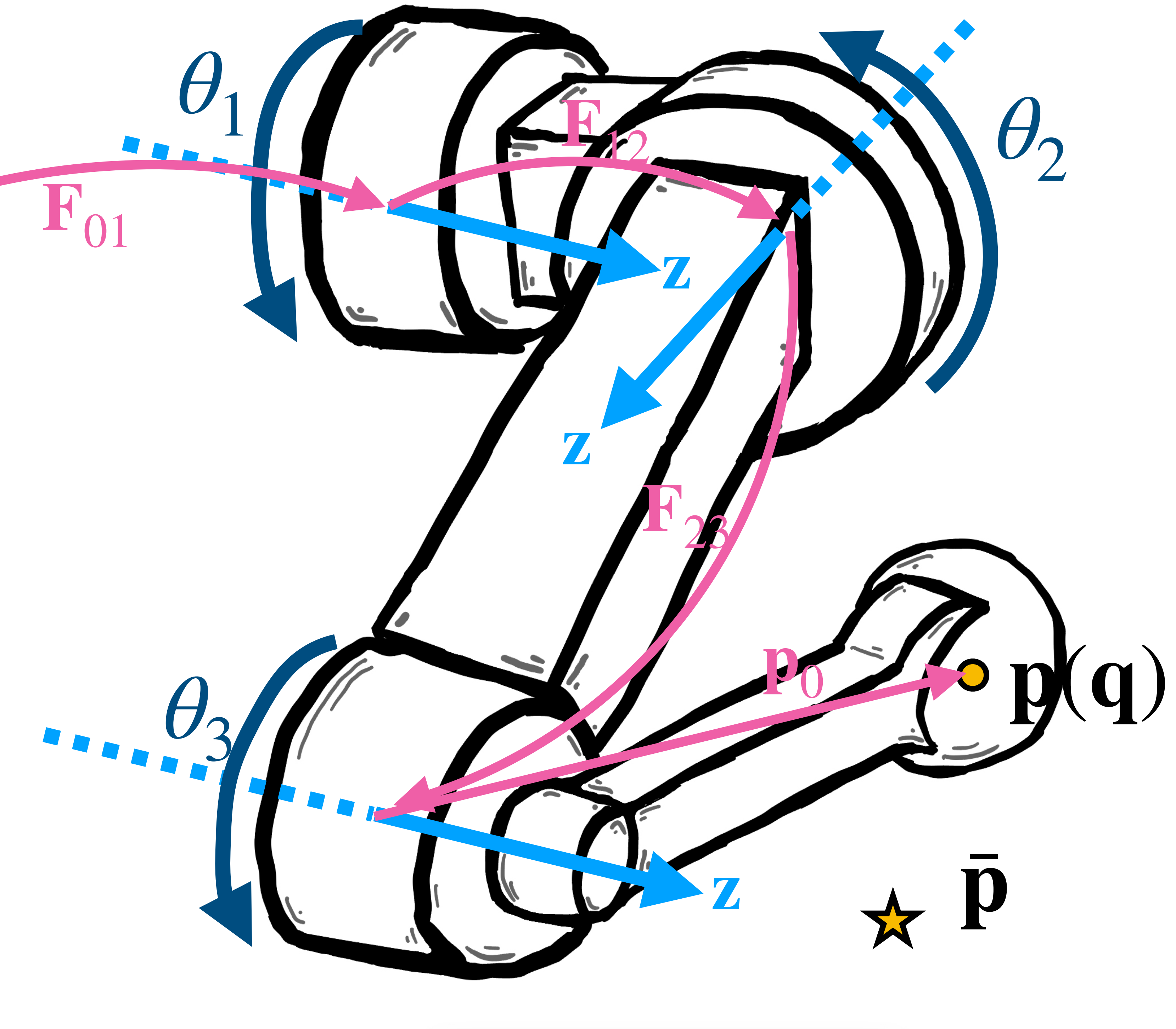
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{F}_{01}\mathbf{R}_z(\theta_1)\mathbf{F}_{12}\mathbf{R}_z(\theta_2)\mathbf{F}_{23}\mathbf{R}_z(\theta_3)\begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix} - \bar{\mathbf{p}}$$

we need to compute the partial derivatives of $\mathbf{C}$

$$\frac{\partial \mathbf{C}(\mathbf{q})}{\partial \mathbf{q}} =$$



Gradient descent



$$\min_{\mathbf{q}} \ \|\mathbf{C}(\mathbf{q})\|^2$$

// while not converged

// compute gradient of objective function at current $\mathbf{q}$

// update $\mathbf{q}$ by moving along the negative gradient direction by a small step

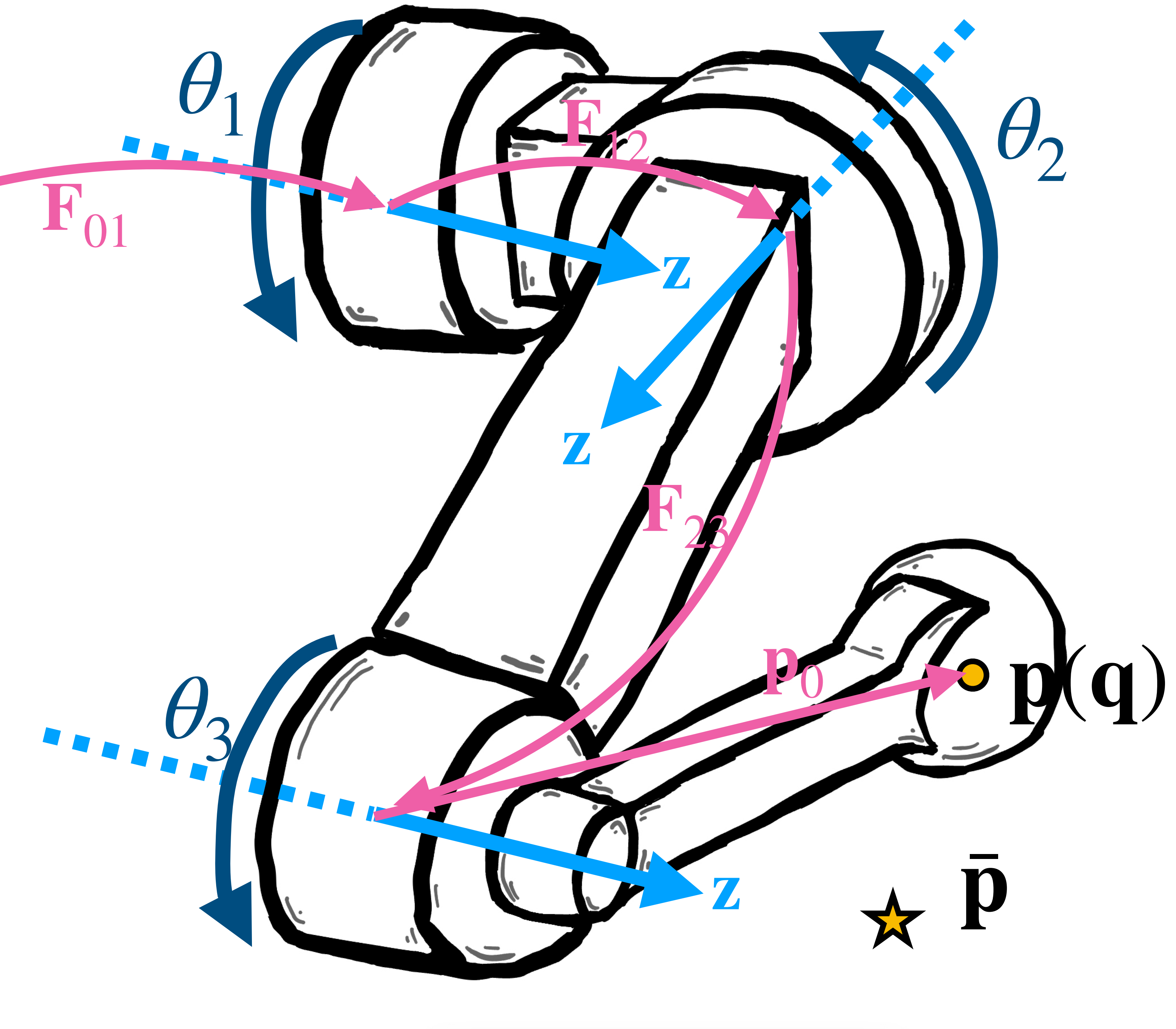
$$\textbf{while} \ \|\mathbf{C}(\mathbf{q})\|^2 > \epsilon$$

$$\mathbf{d} = 2\frac{\partial \mathbf{C}(\mathbf{q})}{\partial \mathbf{q}}^T \mathbf{C}(\mathbf{q})$$

$$\mathbf{q} = \mathbf{q} - \alpha \mathbf{d}$$

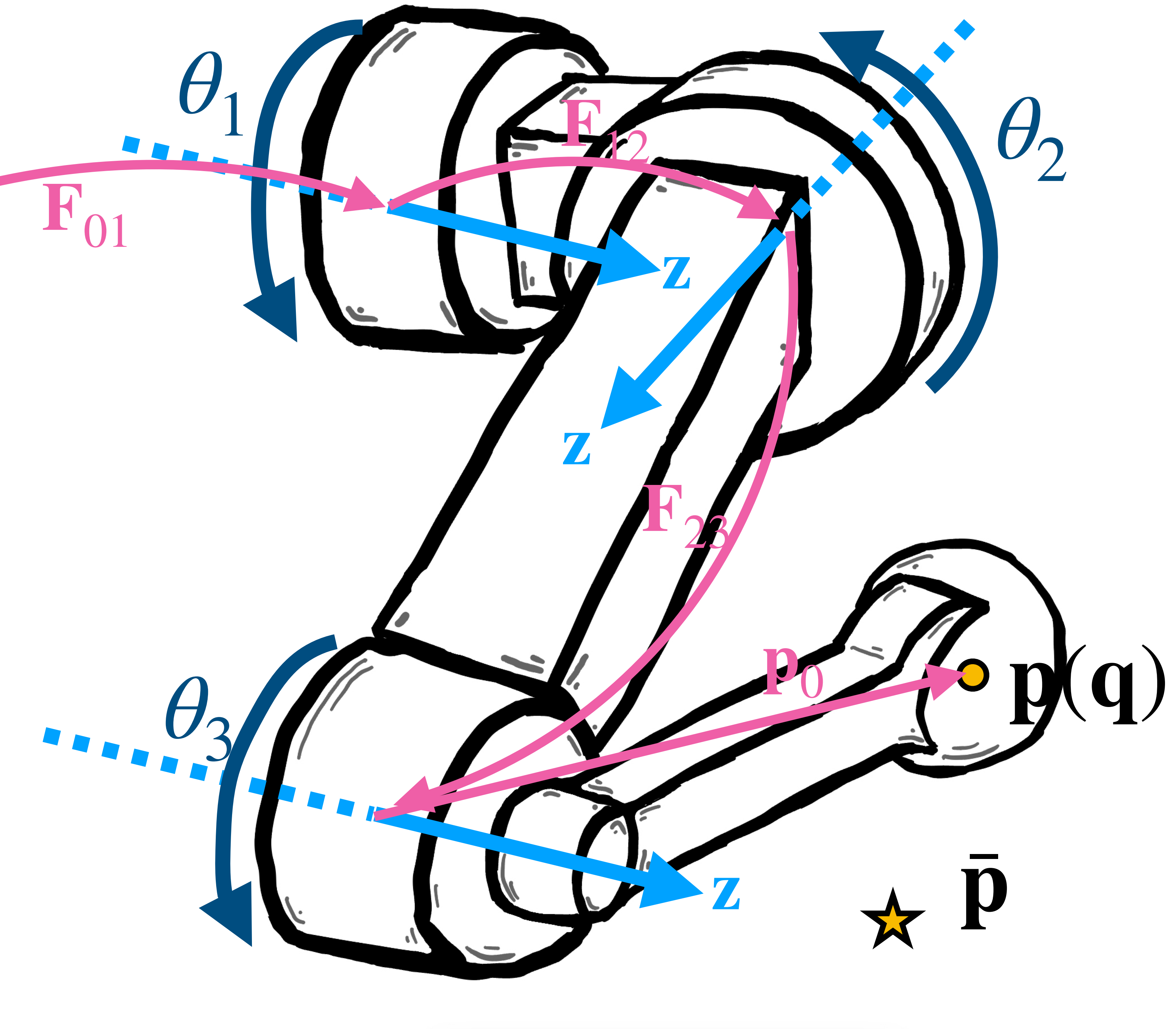
An easier way to compute the gradient is called Finite Differencing. This is what we are going to implement in Lab 3.

Gradient descent



$$\min_{\mathbf{q}} \quad \|\mathbf{C}(\mathbf{q})\|^2$$

Finite differenting



$$\min_{\mathbf{q}} \quad \|\mathbf{C}(\mathbf{q})\|^2$$

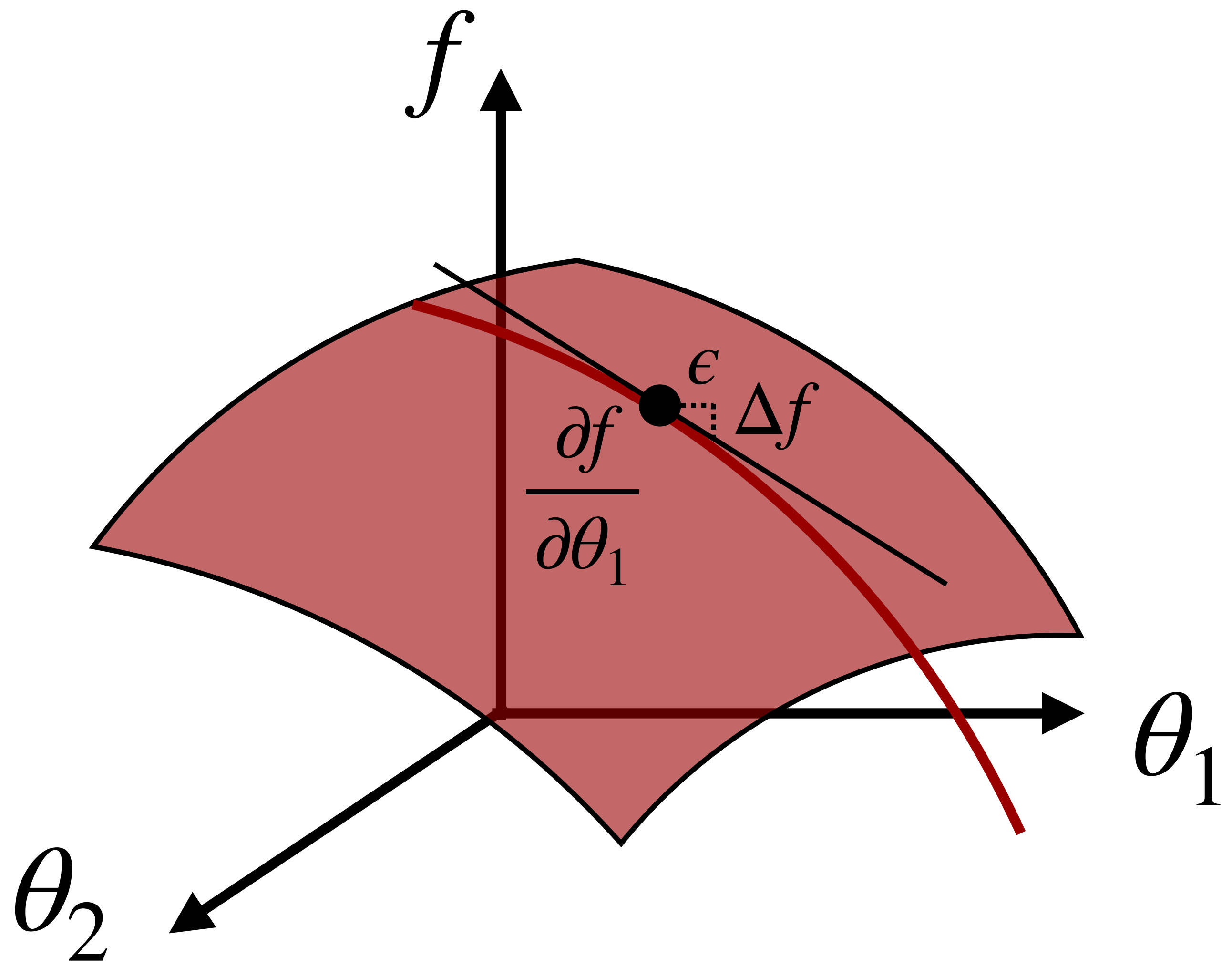
where $\quad f(\mathbf{q}) \equiv \|\mathbf{C}(\mathbf{q})\|^2 = \|\mathbf{p}(\mathbf{q}) - \bar{\mathbf{p}}\|^2$

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} = \left[\frac{\partial f(\mathbf{q})}{\partial \theta_1}, \frac{\partial f(\mathbf{q})}{\partial \theta_2}, \frac{\partial f(\mathbf{q})}{\partial \theta_3}\right]$$

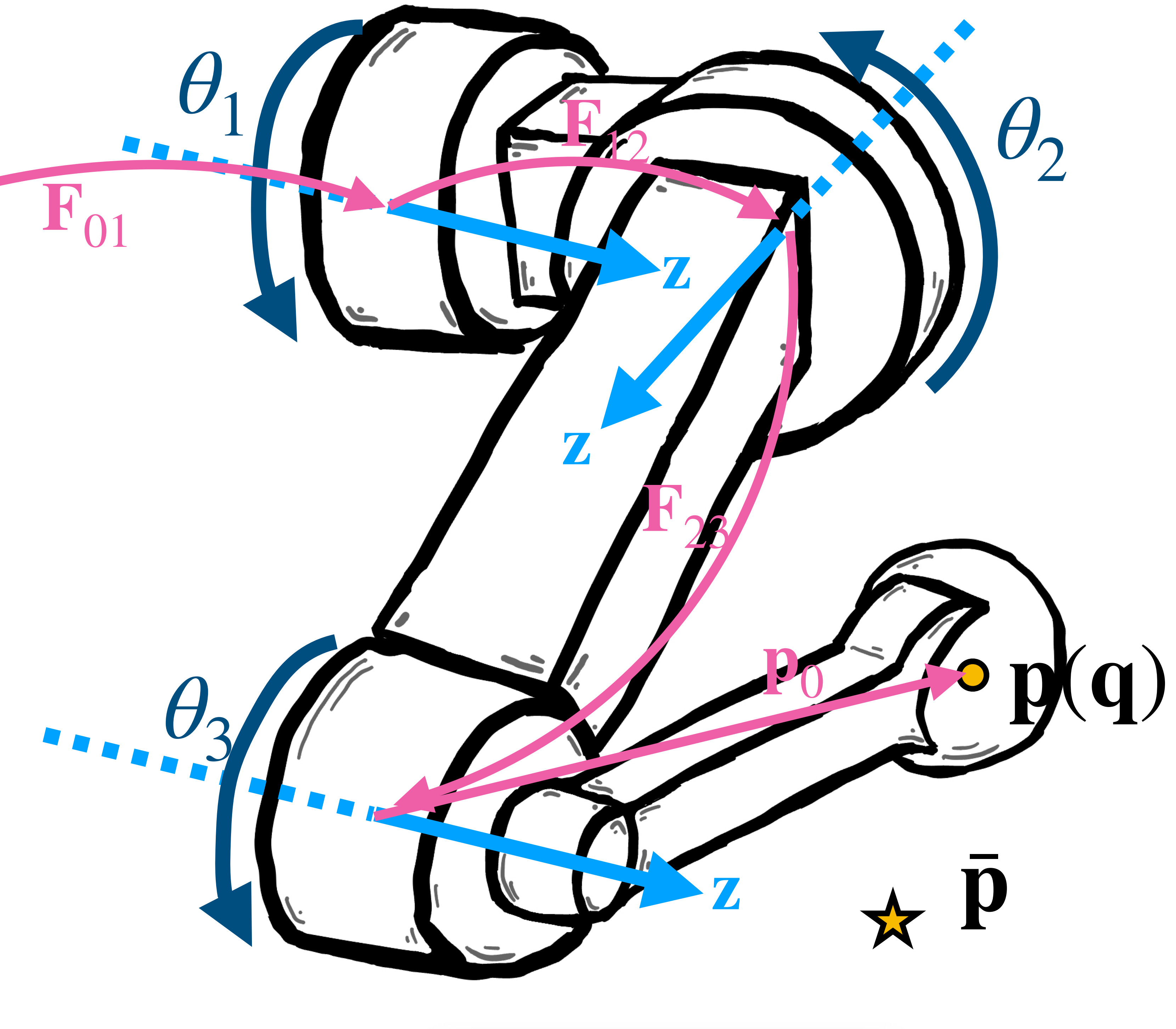
$$\frac{\partial f(\mathbf{q})}{\partial \theta_1} \approx \frac{\Delta f}{\Delta \theta_1} = \frac{f(\mathbf{q} + \begin{bmatrix}\epsilon \\ 0 \\ 0\end{bmatrix}) - f(\mathbf{q})}{\epsilon}$$

$$\frac{\partial f(\mathbf{q})}{\partial \theta_2} \approx ?$$

$$\frac{\partial f(\mathbf{q})}{\partial \theta_3} \approx ?$$



Gradient descent



$$\min_{\mathbf{q}} \quad \|\mathbf{C}(\mathbf{q})\|^2$$

// while not converged

// compute gradient of objective function at current $\mathbf{q}$

// update $\mathbf{q}$ by moving along the negative gradient direction by a small step

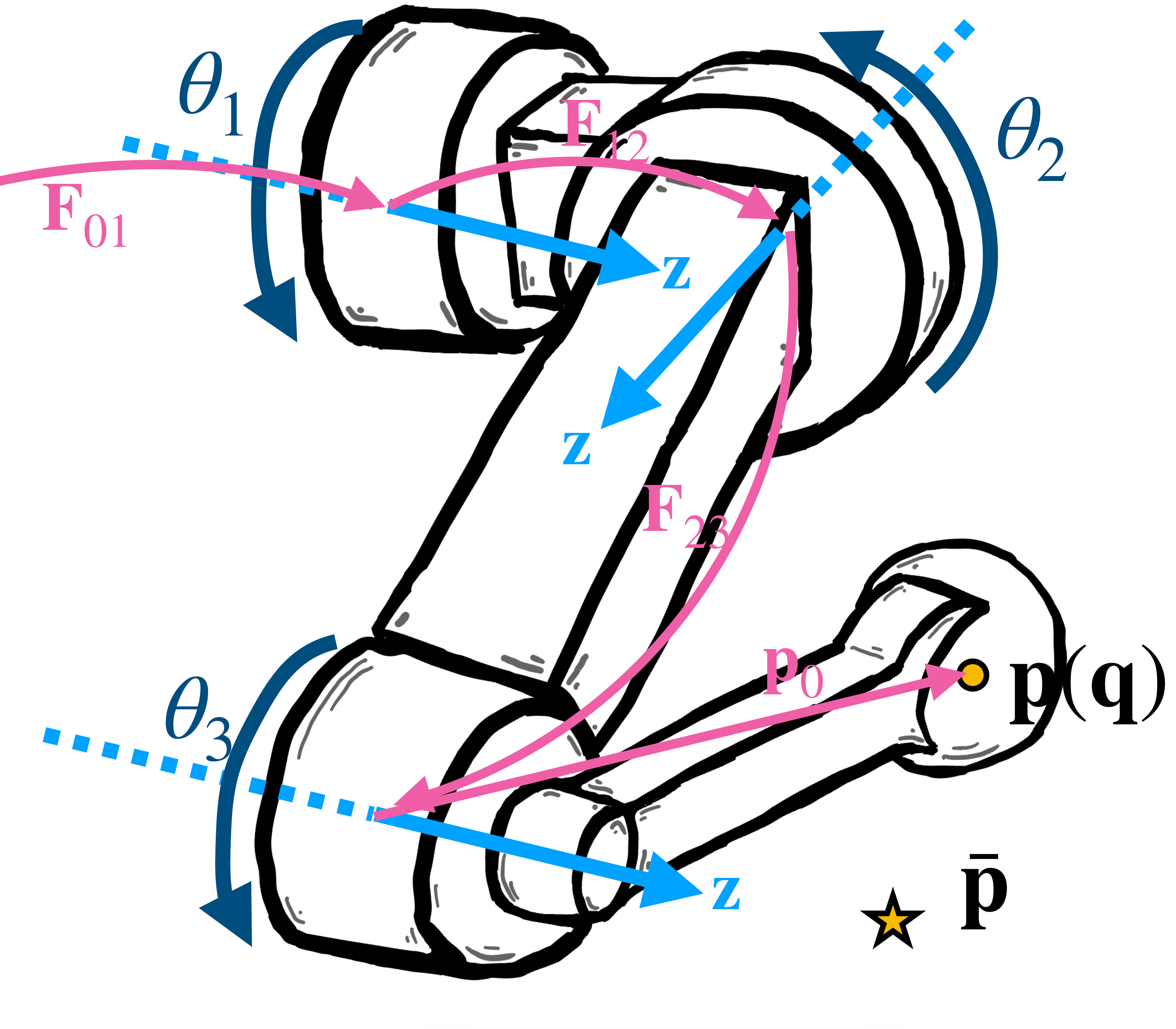
while $f(\mathbf{q}) > \epsilon$

$$\mathbf{d} = \boxed{\text{finiteDiff}(f, \mathbf{q})}$$

Implement this for Lab 3

$$\mathbf{q} = \mathbf{q} - \alpha \mathbf{d}$$

Gradient descent



$$\min_{\mathbf{q}} \quad \|\mathbf{C}(\mathbf{q})\|^2$$

// while not converged

// compute gradient of objective function at current $\mathbf{q}$

// update $\mathbf{q}$ by moving along the negative gradient direction by a small step

while $f(\mathbf{q}) > \epsilon$

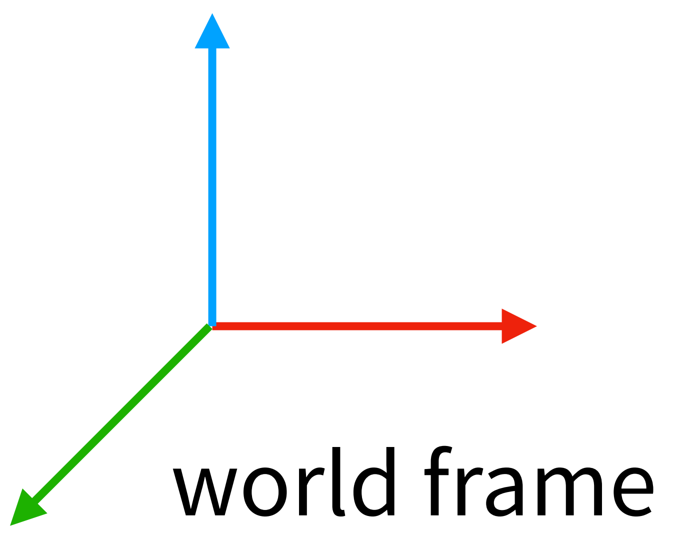
$\mathbf{d} = \text{finiteDiff}(f, \mathbf{q})$

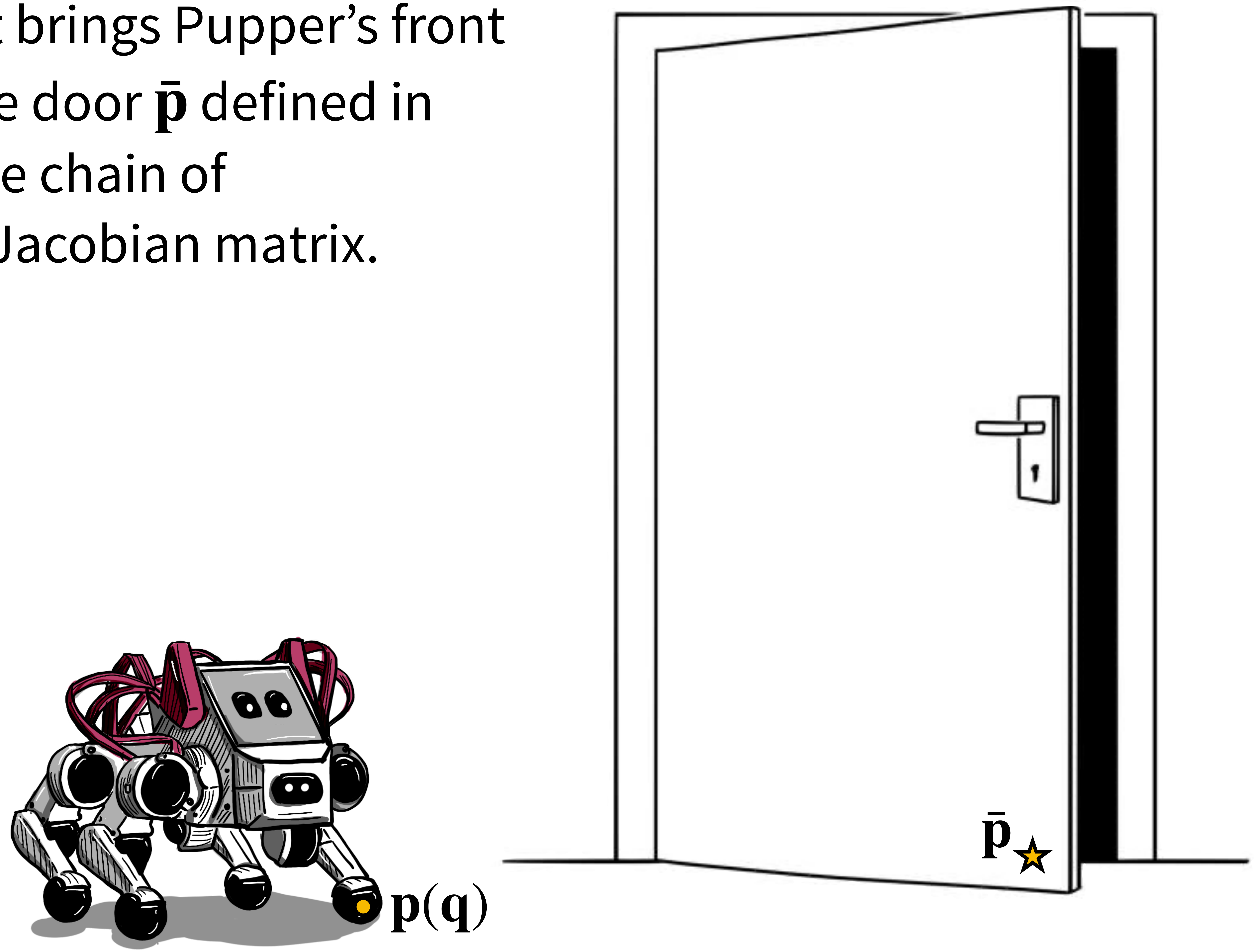
$\mathbf{q} = \mathbf{q} - \alpha\mathbf{d}$

How to determine α?
Experiment it by yourself

Challenge: Close the door, Pupper!

Formulate an IK constraint that brings Pupper's front left toe $\mathbf{p}(\mathbf{q})$ to the corner of the door $\bar{\mathbf{p}}$ defined in the world frame. Write down the chain of transformation and derive the Jacobian matrix.





Questions?