2D Shallow Water Equations for Finite Volume Solver

Governing Equations

The 2D shallow water equations in conservative form are:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S}$$

Where:

- Conserved variables: $\mathbf{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}$
  - h: water depth
  - hu: x-momentum
  - hv: y-momentum
- Fluxes:
  - $\mathbf{F} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}$
  - $\mathbf{G} = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}$
- Source term: $\mathbf{S} = \begin{bmatrix} 0 \\ -gh\frac{\partial z}{\partial x} \\ -gh\frac{\partial z}{\partial y} \end{bmatrix}$
  - g: gravitational acceleration
  - z: bed elevation

Riemann Problem

For the Riemann problem, define a discontinuity across a cell interface:

- Left state: $\mathbf{U}_L = \begin{bmatrix} h_L \\ (hu)_L \\ (hv)_L \end{bmatrix}$
- Right state: $\mathbf{U}_R = \begin{bmatrix} h_R \\ (hu)_R \\ (hv)_R \end{bmatrix}$
- Initial condition: Specify different h, hu, hv on either side of a line (e.g., $x = 0$).

Roe Flux

The Roe flux for the interface between left and right states is:

$$\mathbf{F}_{Roe} = \frac{1}{2}\left(\mathbf{F}(\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R)\right) - \frac{1}{2}\sum_{k=1}^{3} |\tilde{\lambda}_k| \tilde{\mathbf{K}}_k \tilde{\alpha}_k$$

Where:

- $\tilde{\lambda}_k$: eigenvalues of the Roe-averaged Jacobian
- $\tilde{\mathbf{K}}_k$: eigenvectors
- $\tilde{\alpha}_k$: wave strengths
- Roe averages:
  - $\tilde{u} = \frac{\sqrt{h_L}u_L + \sqrt{h_R}u_R}{\sqrt{h_L} + \sqrt{h_R}}$
  - $\tilde{v} = \frac{\sqrt{h_L}v_L + \sqrt{h_R}v_R}{\sqrt{h_L} + \sqrt{h_R}}$
  - $\tilde{h} = \sqrt{h_L h_R}$
- Eigenvalues: $\tilde{\lambda}_1 = \tilde{u} - \sqrt{g\tilde{h}}$, $\tilde{\lambda}_2 = \tilde{u}$, $\tilde{\lambda}_3 = \tilde{u} + \sqrt{g\tilde{h}}$
- Eigenvectors: $\tilde{\mathbf{K}}_1 = \begin{bmatrix} 1 \\ \tilde{u} - \sqrt{g\tilde{h}} \\ \tilde{v} \end{bmatrix}$, $\tilde{\mathbf{K}}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\tilde{\mathbf{K}}_3 = \begin{bmatrix} 1 \\ \tilde{u} + \sqrt{g\tilde{h}} \\ \tilde{v} \end{bmatrix}$
- Wave strengths: Solve $\mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{3} \tilde{\alpha}_k \tilde{\mathbf{K}}_k$

MUSCL Reconstruction

For second-order accuracy, use MUSCL (Monotonic Upstream-centered Scheme for Conservation Laws):

- Reconstruct $\mathbf{U}$ at cell interfaces using linear interpolation.
- Slope limiter (e.g., minmod, superbee) to ensure monotonicity:

  $$\mathbf{U}_{i+1/2,L} = \mathbf{U}_i + \frac{1}{2}\phi(\mathbf{r})(\mathbf{U}_i - \mathbf{U}_{i-1})$$

  $$\mathbf{U}_{i+1/2,R} = \mathbf{U}_{i+1} - \frac{1}{2}\phi(\mathbf{r})(\mathbf{U}_{i+2} - \mathbf{U}_{i+1})$$

  Where $\phi(\mathbf{r})$ is the limiter function, and $\mathbf{r}$ is the ratio of consecutive gradients.

Mesh Handling

- Import unstructured mesh (quadrilateral/triangular) from a standard format (e.g., Gmsh .msh).
- Store cell connectivity, face normals, and areas.

- Compute fluxes across each face using Roe solver and MUSCL-reconstructed states.