Godunov Solver for 2D Euler Equations with MUSCL and Roe Flux

The Godunov solver for the 2D Euler equations with MUSCL reconstruction and Roe flux solves the hyperbolic conservation law system:

2D Euler Equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0,$$

where $\mathbf{U} = (\rho, \rho u, \rho v, E)^T$ is the conserved state vector, and the flux vectors are:

$$\mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix},$$

with ρ (density), u, v (velocities), p (pressure), $E = \rho\left(e + \frac{u^2+v^2}{2}\right)$ (total energy), and e (internal energy). The equation of state is:

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho(u^2+v^2)\right),$$

where γ is the adiabatic index.

Godunov Solver Framework

The Godunov method uses a finite volume approach, updating cell averages via interface fluxes. For cell (i, j) at time t^n:

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^{n} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j}\right) - \frac{\Delta t}{\Delta y}\left(\mathbf{G}_{i,j+1/2} - \mathbf{G}_{i,j-1/2}\right).$$

MUSCL Reconstruction

MUSCL reconstructs left and right states at interfaces for second-order accuracy.

1. **Primitive Variables**: Reconstruct $\mathbf{W} = (\rho, u, v, p)^T$.
2. **Slope Computation**: Compute limited slopes. For cell (i, j), in x-direction:

$$\Delta \mathbf{W}_{i,j}^{x} = \text{minmod}\left(\mathbf{W}_{i+1,j} - \mathbf{W}_{i,j}, \mathbf{W}_{i,j} - \mathbf{W}_{i-1,j}\right),$$

in y-direction:

$$\Delta \mathbf{W}_{i,j}^{y} = \text{minmod}\left(\mathbf{W}_{i,j+1} - \mathbf{W}_{i,j}, \mathbf{W}_{i,j} - \mathbf{W}_{i,j-1}\right),$$

where:

$$\text{minmod}(a, b) = \begin{cases} \text{sign}(a)\min(|a|, |b|) & \text{if } ab > 0, \\ 0 & \text{otherwise}. \end{cases}$$

3. **Interface States**: For x-interface at $(i+1/2, j)$:

$$\mathbf{W}^L_{i+1/2,j} = \mathbf{W}_{i,j} + \frac{1}{2}\Delta\mathbf{W}^x_{i,j}, \quad \mathbf{W}^R_{i+1/2,j} = \mathbf{W}_{i+1,j} - \frac{1}{2}\Delta\mathbf{W}^x_{i+1,j}.$$

   For y-interface at $(i, j+1/2)$:

$$\mathbf{W}^L_{i,j+1/2} = \mathbf{W}_{i,j} + \frac{1}{2}\Delta\mathbf{W}^y_{i,j}, \quad \mathbf{W}^R_{i,j+1/2} = \mathbf{W}_{i,j+1} - \frac{1}{2}\Delta\mathbf{W}^y_{i,j+1}.$$

4. **Conserved States**: Convert to conserved variables:

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho\left(e + \frac{u^2+v^2}{2}\right) \end{pmatrix}, \quad e = \frac{p}{(\gamma-1)\rho}.$$

Roe Flux

For x-interface at $(i+1/2, j)$, the Roe flux is:

$$\mathbf{F}_{i+1/2,j} = \frac{1}{2}\left(\mathbf{F}(\mathbf{U}^L) + \mathbf{F}(\mathbf{U}^R) - |\mathbf{A}_{\text{Roe}}|\left(\mathbf{U}^R - \mathbf{U}^L\right)\right).$$

Roe Averaging

For states $\mathbf{U}^L = (\rho^L, \rho^L u^L, \rho^L v^L, E^L)^T$, $\mathbf{U}^R = (\rho^R, \rho^R u^R, \rho^R v^R, E^R)^T$:

$$\sqrt{\rho} = \sqrt{\rho^L} + \sqrt{\rho^R}, \quad w = \frac{\sqrt{\rho^L}}{\sqrt{\rho}}, \quad 1 - w = \frac{\sqrt{\rho^R}}{\sqrt{\rho}},$$

$$u = wu^L + (1-w)u^R, \quad v = wv^L + (1-w)v^R,$$

$$H = wH^L + (1-w)H^R, \quad H = \frac{E+p}{\rho}, \quad a = \sqrt{(\gamma-1)\left(H - \frac{u^2+v^2}{2}\right)}.$$

Roe Matrix

The Jacobian $\mathbf{A}_{\text{Roe}} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$ has eigenvalues:

$$\lambda_1 = u - a, \quad \lambda_2 = u, \quad \lambda_3 = u, \quad \lambda_4 = u + a.$$

The flux contribution is:

$$|\mathbf{A}_{\text{Roe}}|\left(\mathbf{U}^R - \mathbf{U}^L\right) = \sum_k |\lambda_k| \alpha_k \mathbf{e}_k,$$

where α_k are wave strengths from projecting $\mathbf{U}^R - \mathbf{U}^L$ onto eigenvectors $\mathbf{e}_k$. For y-direction, use $\frac{\partial \mathbf{G}}{\partial \mathbf{U}}$, swapping u and v.

Time Stepping

Choose Δt via CFL condition:

$$\Delta t \leq \text{CFL} \cdot \min\left(\frac{\Delta x}{\max|u \pm a|}, \frac{\Delta y}{\max|v \pm a|}\right), \quad \text{CFL} < 1.$$

This describes the Godunov solver with MUSCL reconstruction and Roe flux for the 2D Euler equations.