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# Network models

## Network Science

### Lecture 4

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## Empirical network features:

- Power-law (heavy-tailed) degree distribution
- Small average distance (graph diameter)
- Large clustering coefficient (transitivity)

## Generative models:

- Random graph model (Erdos & Renyi, 1959)
- Preferential attachment model (Barabasi & Albert, 1999)
- Small world model (Watts & Strogatz, 1998)

- Citation networks
- Collaboration networks
- Web
- Social networks

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Most of the networks we study are dynamic, they evolve over time, expanding by adding new nodes and edges

Barabasi and Albert, 1999

Dynamic growth: start at  $t = 0$  with  $n_0$  nodes and  $m_0 \geq n_0$  edges

## 1. Growth

At each time step add a new node with  $m$  edges ( $m \leq n_0$ ), connecting to  $m$  nodes already in network  $k_i(i) = m$

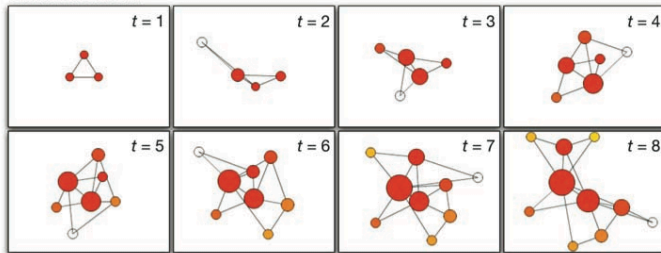
## 2. Preferential attachment

The probability of linking to existing node  $i$  is proportional to the node degree  $k_i$

$$\Pi(k_i) = \frac{k_i}{\sum_i k_i}$$

after  $t$  timesteps:  $t + n_0$  nodes,  $mt + m_0$  edges

Scale-Free Model



Barabasi, 1999

Continues approximation: continues time, real variable node degree  $\langle k_i(t) \rangle$  - expected value over multiple realizations

Time-dependent degree of a single node:

$$k_i(t + \delta t) = k_i(t) + m\Pi(k_i)\delta t$$

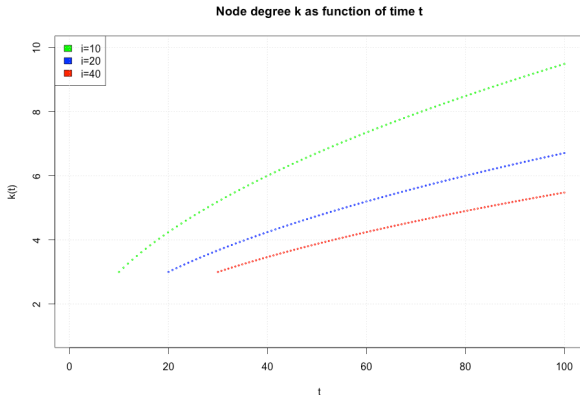
$$\frac{dk_i(t)}{dt} = m\Pi(k_i) = m \frac{k_i}{\sum_i k_i} = \frac{mk_i}{2mt} = \frac{k_i(t)}{2t}$$

node  $i$  is added at time  $t_i$ :  $k_i(t_i) = m$

$$\int_m^{k_i(t)} \frac{dk_i}{k_i} = \int_{t_i}^t \frac{dt}{2t}$$

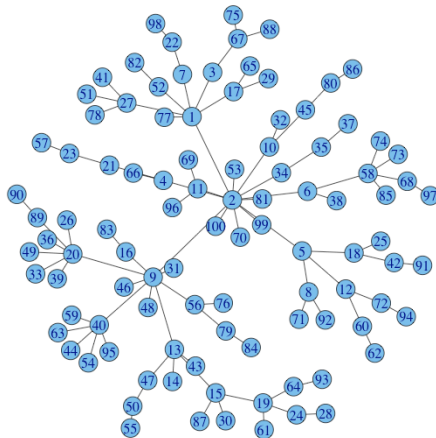
Solution:

$$k_i(t) = m \left( \frac{t}{t_i} \right)^{1/2}$$



$$k_i(t) = m \left( \frac{t}{t_i} \right)^{1/2}; \quad \frac{dk_i(t)}{dt} = \frac{m}{2} \frac{1}{\sqrt{tt_i}}$$





Time evolution of a node degree

$$k_i(t) = m \left( \frac{t}{t_i} \right)^{1/2}$$

Find probability  $P(k' \leq k)$  of a randomly selected node to have  $k' \leq k$  at time  $t$  (fraction of nodes with  $k' \leq k$ ). Nodes with  $k_i(t) \leq k$ :

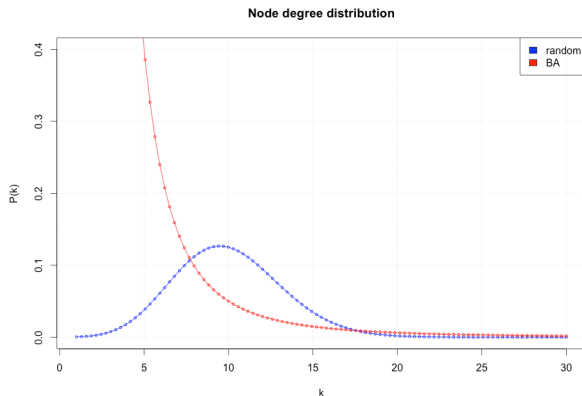
$$m \left( \frac{t}{t_i} \right)^{1/2} \leq k \quad \Rightarrow \quad t_i \geq \frac{m^2}{k^2} t$$

Cumulative function:

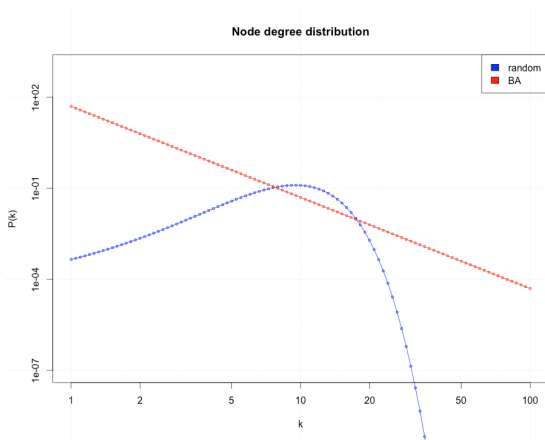
$$F(k) = P(k' \leq k) = \frac{n_0 + t - m^2 t / k^2}{n_0 + t} \approx 1 - \frac{m^2}{k^2}$$

Distribution function:

$$P(k) = \frac{d}{dk} F(k) = \frac{2m^2}{k^3}$$

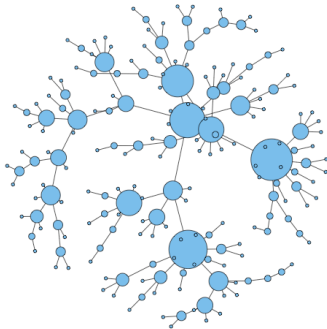
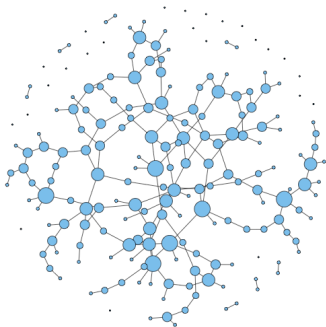


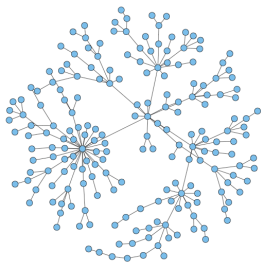
$$BA : P(k) = \frac{2m^2}{k^3}, \quad ER : P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}, \quad \langle k \rangle = pn$$



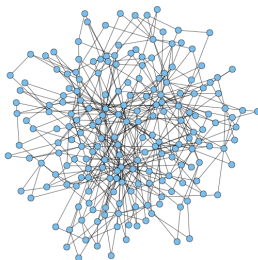
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# Preferential attachment vs random graph

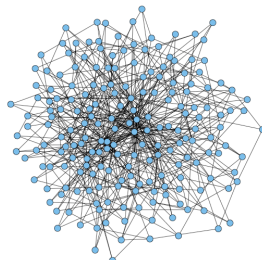




$m = 1$



$m = 2$



$m = 3$

## 1. Growth

At each time step add a new node with  $m$  edges ( $m \leq n_0$ ), connecting to  $m$  nodes already in network  $k_i(i) = m$

## 2. Attachment uniformly at random

The probability of linking to existing node  $i$  is

$$\Pi(k_i) = \frac{1}{n_0 + t - 1}$$

Node degree growth:

$$k_i(t) = m \left( 1 + \log \left( \frac{t}{i} \right) \right)$$

Node degree distribution function:

$$P(k) = \frac{e}{m} \exp \left( -\frac{k}{m} \right)$$

- Power law distribution function:

$$P(k) = \frac{2m^2}{k^3}$$

- Average path length (analytical result) :

$$\langle L \rangle \sim \log(N) / \log(\log(N))$$

- Clustering coefficient (numerical result):

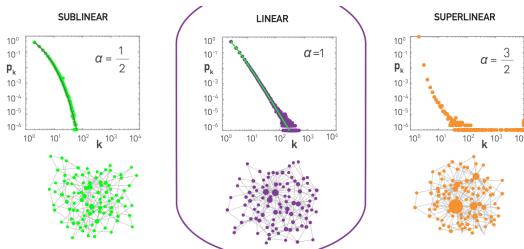
$$C \sim N^{-0.75}$$



- Non-linear preferential attachment models:

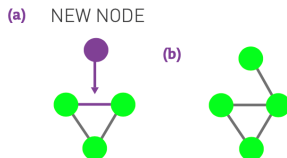
$$\Pi(k) \sim k^\alpha$$

- $\alpha = 0$ , no hubs, exponential distribution
- $0 < \alpha < 1$ , sublinear, smaller hubs, stretched exponential
- $\alpha = 1$ , scale-free, hubs, power law
- $\alpha > 1$ , superlinear, super hubs, hubs-and-spoke



Local growth mechanism:

- Growth: at each time step add a new node
- Link selection: select link at random and connect to one of two nodes at the ends

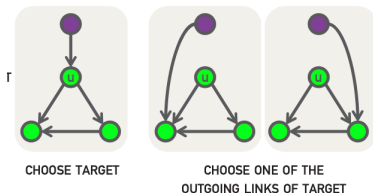


Probability to connect to a node with degree  $k$ :

$$q_k = \frac{kp_k}{\langle k \rangle}$$

Local growth mechanism:

- Random connection: with probability  $p$  connect to a random node  $u$
- Copying: with probability  $1 - p$  randomly choose an outgoing link from  $u$  and connect to its target



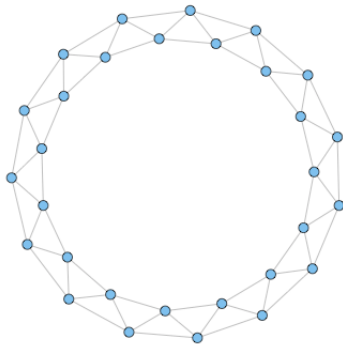
Probability to connect to a node with degree  $k$  :

$$\Pi(k) = \frac{p}{n} + (1 - p) \frac{k}{2m}$$

- Polya urn model, George Polya, 1923
- Yule process, Udny Yule, 1925
- Distribution of wealth, Herbert Simon, 1955
- Evolution of citation networks, cumulative advantage, Derek de Solla Price, 1976
- Preferential attachment network model, Barabasi and Albert, 1999

Local random models vs global optimization models

Motivation: keep high clustering, get small diameter



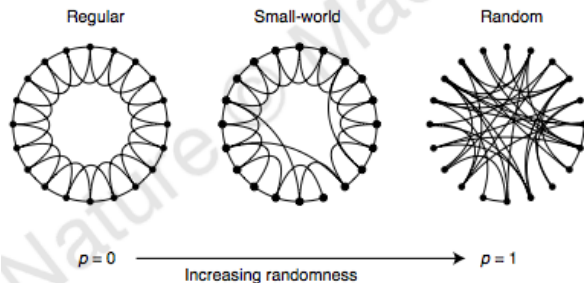
Clustering coefficient  $C = 1/2$

Graph diameter  $d = 8$

Watts and Strogatz, 1998

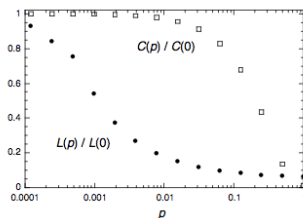
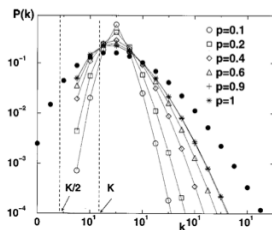
Single parameter model, interpolation between regular lattice and random graph

- start with regular lattice with  $n$  nodes,  $k$  edges per vertex (node degree),  $k \ll n$
- randomly connect with other nodes with probability  $p$ , forms  $pnk/2$  "long distance" connections from total of  $nk/2$  edges
- $p = 0$  regular lattice,  $p = 1$  random graph

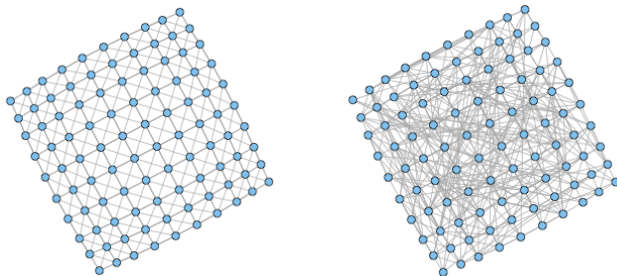


Watts, 1998

- Node degree distribution:  
Poisson like
- Ave. path length  $\langle L(p) \rangle$  :  
 $p \rightarrow 0$ , ring lattice,  $\langle L(0) \rangle = n/2k$   
 $p \rightarrow 1$ , random graph,  $\langle L(1) \rangle = \log(n)/\log(k)$
- Clustering coefficient  $C(p)$  :  
 $p \rightarrow 0$ , ring lattice,  $C(0) = 3/4 = \text{const}$   
 $p \rightarrow 1$ , random graph,  $C(1) = k/n$







20% rewiring:

ave. path length = 3.58  $\rightarrow$  ave. path length = 2.32

clust. coeff = 0.49  $\rightarrow$  clust. coeff = 0.19

	Random	BA model	WS model	Empirical networks
$P(k)$	$\frac{\lambda^k e^{-\lambda}}{k!}$	$k^{-3}$	poisson like	power law
$C$	$\langle k \rangle / N$	$N^{-0.75}$	const	large
$\langle L \rangle$	$\frac{\log(N)}{\log(\langle k \rangle)}$	$\frac{\log(N)}{\log \log(N)}$	$\log(N)$	small

- Emergence of Scaling in Random Networks, A.L. Barabasi and R. Albert, Science 286, 509-512, 1999
- Collective dynamics of small-world networks. Duncan J. Watts and Steven H. Strogatz. Nature 393 (6684): 440-442, 1998