Computer Vision: Lecture 2

Christopher Zach

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Today's Lecture

Projective Geometry

- Homogeneous coordinates, projective space
- Lines, planes and conics
- Projective transformations

Relevant chapters in Szeliski's book:

- 2.1: Geometric primitives and transformations
- 2.1.1: 2D transformations
- 2.1.4: 3D to 2D projections

Homogeneous coordinates and $\mathbb{P}^2$

Projective space $\mathbb{P}^2$

If there exists a $\lambda \neq 0$ such that $\mathbf{x} = \lambda \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then we write $\mathbf{x} \sim \mathbf{y}$. An equivalence relation $\sim$ induces equivalence classes

$$[\mathbf{x}] := \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} \sim \mathbf{x}\}$$

$\mathbb{P}^2$ is the quotient space of $\mathbb{R}^3$ by $\sim$: $\mathbb{P}^2 = \{[\mathbf{x}] : \mathbf{x} \in \mathbb{R}^3\} = \mathbb{R}^3 / \sim$

- Straightforward to extend to $\mathbb{P}^k = \mathbb{R}^{k+1} / \sim$
- If $\mathbf{x} = (x_1, x_2, x_3)^\top$ with $x_3 \neq 0$, then $\begin{pmatrix} x_1/x_3 \\ x_2/x_3 \\ 1 \end{pmatrix}$ is a representative for $[\mathbf{x}]$
- Bijection between $(x_1, x_2, 1)^\top \in [\mathbf{x}] \in \mathbb{P}^2$ and $(x_1, x_2)^\top \in \mathbb{R}^2$
- We can move between $\mathbb{R}^2$ and $\mathbb{P}^2$ as needed (if $x_3 \neq 0$)!

$(x_1, x_2, x_3)^\top$ are homogeneous coordinates corresponding to $(x_1/x_3, x_2/x_3)^\top$

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Homogeneous coordinates and $\mathbb{P}^2$

- Converting to homogeneous coordinates ("homogenizing")

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \in \left[\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \right] \in \mathbb{P}^2$$

  Representative $(x_1, x_2, 1)^\top$ is element of $\mathbb{R}^3$
- Converting to Cartesian coordinates ("de-homogenizing")

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{P}^2 \rightsquigarrow \begin{pmatrix} x_1/x_3 \\ x_2/x_3 \end{pmatrix} \in \mathbb{R}^2$$

  if $x_3 \neq 0$

Homogeneous coordinates and $\mathbb{P}^k$

Why are homogeneous coordinates useful?

- Recall Euclidean transformation

$$\mathbf{X}' = \mathtt{R}\mathbf{X} + \mathbf{T} \qquad\qquad \mathbf{X}, \mathbf{X}' \in \mathbb{R}^3$$

  Affine expression (linear + offset)

- Interpret $\mathbf{X} \in \mathbb{P}^3$:

$$\mathbf{X}' = \underbrace{\begin{pmatrix} \mathtt{R} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{pmatrix}}_{\in \mathbb{R}^{4x4}} \mathbf{X} \qquad\qquad \mathbf{X}, \mathbf{X}' \in \mathbb{P}^3$$

  Purely linear expression

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Homogeneous coordinates and $\mathbb{P}^k$

Why are homogeneous coordinates useful?

- Recall projection onto the image plane, followed by calibration matrix ($s = 0$)

$$\mathbf{p} = \mathtt{K} \cdot \pi(\mathbf{X}) = \begin{pmatrix} \gamma f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X_1/X_3 \\ X_2/X_3 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma f X_1/X_3 + x_0 \\ f X_2/X_3 + y_0 \\ 1 \end{pmatrix} \in \mathbb{P}^2$$

- Observe that

$$\pi(\mathtt{K}\mathbf{X}) = \frac{1}{X_3} \begin{pmatrix} \gamma f X_1 + x_0 X_3 \\ f X_2 + y_0 X_3 \\ X_3 \end{pmatrix} = \mathtt{K} \cdot \pi(\mathbf{X}) \in [\mathtt{K}\mathbf{X}]$$

- We can apply the perspective division as final step

$$\underbrace{[\mathtt{K}(\mathtt{R}\ \mathbf{T})\mathbf{X}]}_{\mathbf{p} \in \mathbb{P}^2} = \pi(\underbrace{\mathtt{K}(\mathtt{R}\mathbf{X} + \mathbf{T})}_{=\mathtt{P}\mathbf{X}})$$

Projection matrix $\mathtt{P} = \mathtt{K}(\mathtt{R} \mid \mathbf{T}) \in \mathbb{R}^{3\times 4}$

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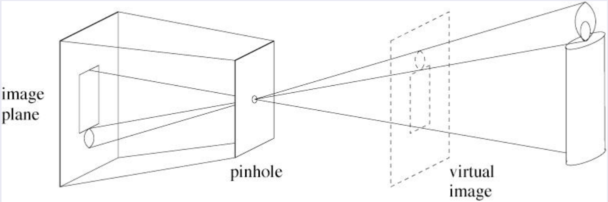
Homogeneous coordinates and $\mathbb{P}^k$

Why are homogeneous coordinates useful?

- Allowing to work with quantities such as $[(x_1, x_2, 0)^\top]$
- Points, lines and planes at infinity
- More on that later

Recap Camera Model

The Pinhole Camera



$$\mathbf{x} = \pi(\mathtt{K}(\mathtt{R}\mathbf{X} + \mathbf{T})) = \pi(\mathtt{K}\,\mathtt{R}(\mathbf{X} - \mathbf{C}))$$

- Camera center $\mathbf{C}$ or translation vector $\mathbf{T} = -\mathtt{R}\mathbf{C}$
- Rotation matrix $\mathtt{R}$
- Camera calibration matrix (intrinsics) $\mathtt{K}$
- Perspective projection π

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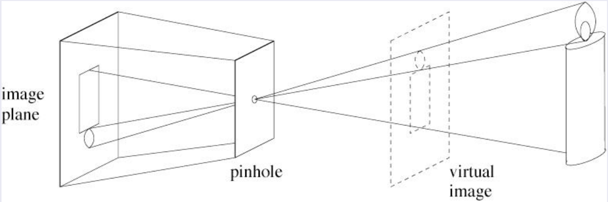

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$$\mathbf{x} = \pi(\mathtt{P}\mathbf{X}) \iff \exists\lambda \neq 0 : \lambda\mathbf{x} = \mathtt{P}\mathbf{X}$$

λ ... "projective depth"

Discussion

- Pros and cons with pinhole model? Limitations?
- What is the position (camera centre) of the camera $\mathtt{P} = \mathtt{K}(\mathtt{R} \mid \mathbf{T})$?
- What is the projection of the camera centre for camera matrix $\mathtt{P}$?
- How can one determine if a 3D point $\mathbf{X}$ is in front of camera?

Discussion

Unprojecting an image point

Which 3D points project to image point $\mathbf{x}$ for given camera matrix $\mathtt{P}$?
What is the viewing ray for image point $\mathbf{x}$ of camera $\mathtt{P}$?

- We are interested in $\mathbf{X}(\lambda)$ in world coordinates
  - $\lambda \dots$ projective depth / parameter to move on the line/ray
- In world coordindates: $\mathbf{X}(\lambda)$ satisfies

$$\lambda \mathbf{x} = \mathtt{P} \underbrace{\mathbf{X}(\lambda)}_{\mathbf{X} \in \mathbb{P}^3} = \mathtt{P}_{3,3} \underbrace{\mathbf{X}(\lambda)}_{\mathbf{X} \in \mathbb{R}^3} + \mathtt{P}_4 \qquad \mathtt{P} = (\mathtt{P}_{3,3} \mid \mathtt{P}_4)$$

- Solve for $\mathbf{X}(\lambda)$

$$\mathbf{X}(\lambda) = \mathtt{P}_{3,3}^{-1}(\lambda \mathbf{x} - \mathtt{P}_4) = \lambda \mathtt{P}_{3,3}^{-1} \mathbf{x} - \mathtt{P}_{3,3}^{-1} \mathtt{P}_4$$

- Calibrated camera $\mathtt{P} = \mathtt{K}(\mathtt{R} \mid \mathbf{T})$

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The Pinhole Camera Model

Attention

Different conventions used in the literature/software

- Computer vision community mostly uses
  - Camera is looking in z direction
  - World coordinates to camera coordinates: $\mathbf{X}' = \mathtt{R}\mathbf{X} + \mathbf{T}$
- Computer graphics (OpenGL)
  - Camera is looking in $-z$ direction
- "Position-centric" parametrization
  - World coordinates to camera coordinates: $\mathbf{X}' = \mathtt{R}(\mathbf{X} \pm \mathbf{C})$
  - $\pm\mathbf{C}$ might be called "translation"
- Inverse parametrization
  - World coordinates to camera coordinates: $\mathbf{X}' = \mathtt{R}^\top(\mathbf{X} \pm \mathbf{C})$
  - "Look at" parametrization
- Lots of ambiguities about image coordinates (mm) and sensor coordinates (pixels)

At least the right-handed coordinate system convention is generally accepted.

The Pinhole Camera Model: Summary

Summary

- Model for a pinhole camera

$$\mathbf{x} \sim \underbrace{\mathtt{K}(\mathtt{R} \mid \mathbf{T})}_{=\mathtt{P}} \mathbf{X} = \mathtt{P}\mathbf{X} \qquad \mathbf{X} \in \mathbb{P}^3$$

- Camera represented by its projection matrix (or camera matrix) $\mathtt{P} \in \mathbb{R}^{3\times 4}$
  - $\mathtt{P}$ has 11 d.o.f.
  - If $\mathtt{K}$ is known, then the camera is said to be calibrated
- Camera center
  - in camera coordinates: $\mathbf{0}$
  - in world coordinates: $-\mathtt{R}^\top \mathbf{T}$

Recap Homogeneous Coordinates

- Projective space $\mathbb{P}^k$

$$\mathbf{x} \sim \mathbf{y} :\iff \exists \lambda \neq 0 : \mathbf{x} = \lambda \mathbf{y} \qquad\qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^{k+1}$$
$$[\mathbf{x}] := \{\mathbf{y} \in \mathbb{R}^{k+1} : \mathbf{y} \sim \mathbf{x}\}$$
$$\mathbb{P}^k := \{[\mathbf{x}] : \mathbf{x} \in \mathbb{R}^{k+1}\} = \mathbb{R}^{k+1}/\sim$$

- We identify $\mathbf{x}/x_{k+1}$ as a point in $\mathbb{R}^k$ if $x_{k+1} \neq 0$
- Advantages
  - Represent quantities that are defined up to non-zero scale
  - Treat expressions in a purely linear way
  - Gracefully handle (some) infinities

Discussion

- Which of the following statements is correct?
  1. The set of points in $\mathbb{R}^2$ is a subset of points in $\mathbb{P}^2$
  2. The set of points in $\mathbb{R}^2$ is equal to the set of points in $\mathbb{P}^2$
  3. The set of points in $\mathbb{P}^2$ is a subset of points in $\mathbb{R}^2$

Lines in $\mathbb{P}^2$

- Line in $\mathbb{R}^2$: $(a, b) \neq (0, 0)$

$$ax + by + c = 0 \iff \mathbf{l}^\top \mathbf{x} = 0 \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \mathbf{l} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- Line is defined up to scale

$$\forall \lambda \neq 0 : \mathbf{l}^\top \mathbf{x} = 0 \iff \lambda \mathbf{l}^\top \mathbf{x} = 0$$

- Therefore $\mathbf{l}, \lambda \mathbf{l} \in [\mathbf{l}] \in \mathbb{P}^2$

Lines in $\mathbb{P}^2$

Unique intersection

Two lines $\mathbf{l}_1$, $\mathbf{l}_2 \in \mathbb{P}^2$ with $\mathbf{l}_1 \neq \mathbf{l}_2$ have a unique intersection in $\mathbb{P}^2$.

Intersection point $\mathbf{x}$ is given as solution of

$$\mathbf{l}_1^\top \mathbf{x} = 0 \wedge \mathbf{l}_2^\top \mathbf{x} = 0 \iff \begin{pmatrix} \mathbf{l}_1^\top \\ \mathbf{l}_2^\top \end{pmatrix} \mathbf{x} = \mathbf{0}.$$

Since $\mathbf{l}_1 \neq \mathbf{l}_2$ the rank of $\begin{pmatrix} \mathbf{l}_1^\top \\ \mathbf{l}_2^\top \end{pmatrix} = 2$ and $\mathbf{x} \neq \mathbf{0}$ lies in a 1-D nullspace.

- Example: $\mathbf{l}_1 = (-1, 0, 1)^\top$, $\mathbf{l}_2 = (1, 0, 1)^\top$ (2 parallel vertical lines).
- Intersection $(0, \lambda, 0)^\top \in [(0, 1, 0)^\top]$: point at infinity in $\mathbb{R}^2$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lim_{\varepsilon \to \pm 0} \begin{pmatrix} 0 \\ 1 \\ \varepsilon \end{pmatrix} \equiv \lim_{\varepsilon \to \pm 0} \begin{pmatrix} 0/\varepsilon \\ 1/\varepsilon \end{pmatrix}$$

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$$\mathbf{l}_1^\top \mathbf{x} = 0 \wedge \mathbf{l}_2^\top \mathbf{x} = 0 \iff \begin{pmatrix} \mathbf{l}_1^\top \\ \mathbf{l}_2^\top \end{pmatrix} \mathbf{x} = \mathbf{0}.$$

Since $\mathbf{l}_1 \neq \mathbf{l}_2$ the rank of $\binom{\mathbf{l}_1^\top}{\mathbf{l}_2^\top} = 2$ and $\mathbf{x} \neq \mathbf{0}$ lies in a 1-D nullspace.

- Example: $\mathbf{l}_1 = (-1, 0, 1)^\top$, $\mathbf{l}_2 = (1, 0, 1)^\top$ (2 parallel vertical lines).
- Intersection $(0, \lambda, 0)^\top \in [(0, 1, 0)^\top]$: point at infinity in $\mathbb{R}^2$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lim_{\varepsilon \to \pm 0} \begin{pmatrix} 0 \\ 1 \\ \varepsilon \end{pmatrix} \equiv \lim_{\varepsilon \to \pm 0} \begin{pmatrix} 0/\varepsilon \\ 1/\varepsilon \end{pmatrix}$$

Lines in $\mathbb{P}^2$

- Duality has many meanings
  - Duality in optimization: Langrange and Fenchel duality
    - Dual cones
  - Dual vector space: all linear functionals $\mathbb{V}^*$ acting on vector space $\mathbb{V}$
  - Duality in projective geometry: incidence-preserving operations
- Lines can be interpreted as linear mapping $\langle \mathbf{l}, \cdot \rangle \in (\mathbb{R}^3)^*$
- Lines and points are dual when it comes to incidence relations

  Two points uniquely determine a line, and two lines uniquely determine a point.

Discussion

Unprojecting an image line

Given a line $\mathbf{l}$ in the image and a camera matrix $\mathtt{P}$, can we "unproject" the line?

- Image points $\mathbf{x}$ on the line satisfy $\mathbf{l}^\top \mathbf{x} = 0$
- Image points satisfy $\mathbf{x} \sim \mathtt{P}\mathbf{X}$
- Therefore $0 = \mathbf{l}^\top \mathtt{P}\mathbf{X} = (\mathtt{P}^\top \mathbf{l})^\top \mathbf{X}$

$\Pi = \mathtt{P}^\top \mathbf{l}$ is 3D-plane projecting to line $\mathbf{l}$ in camera $\mathtt{P}$.

3D plane

$\Pi \in \mathbb{P}^3$ with $(\Pi_1, \Pi_2, \Pi_3)^\top \neq \mathbf{0}$ defines a 3D plane, $\{\mathbf{X} \in \mathbb{P}^3 : \Pi^\top \mathbf{X} = 0\}$.

Point-plane duality

Three points uniquely determine a plane, and 3 planes uniquely determine a point.

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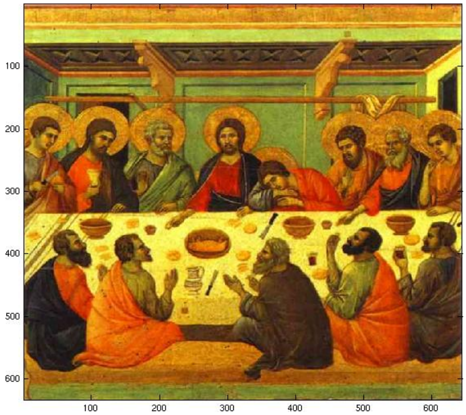
X-at-infinity

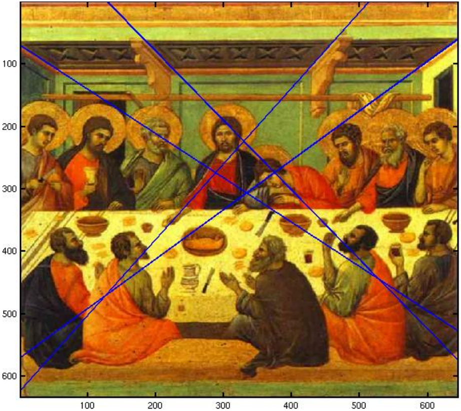
- Extend incindence properties to all configuration (including parallel ones)
- Line at infinity: $\mathbf{l} \sim (0, 0, 1)^\top \in \mathbb{P}^2$
  - Points on the line at infinity: intersection of parallel 2D lines
- Plane at infinity: $\Pi \sim (0, 0, 0, 1)^\top \in \mathbb{P}^3$
  - Points on the plane at infinity: intersection of parallel 3D lines
  - Lines on the plane at infinity: intersection of parallel 3D planes

Vanishing points and lines



The last supper by Duccio (around 1310).

Vanishing points and lines



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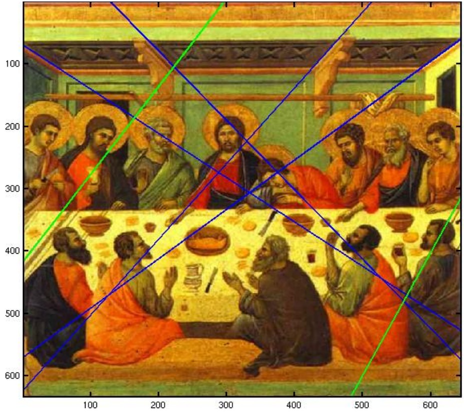

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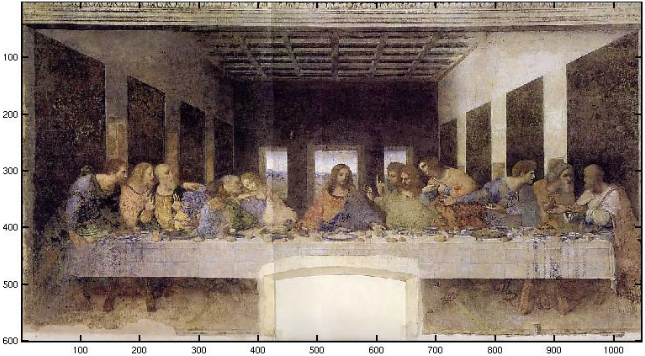

The last supper by da Vinci (1499).

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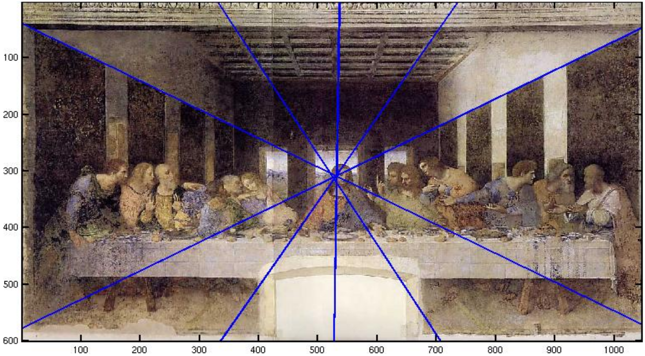

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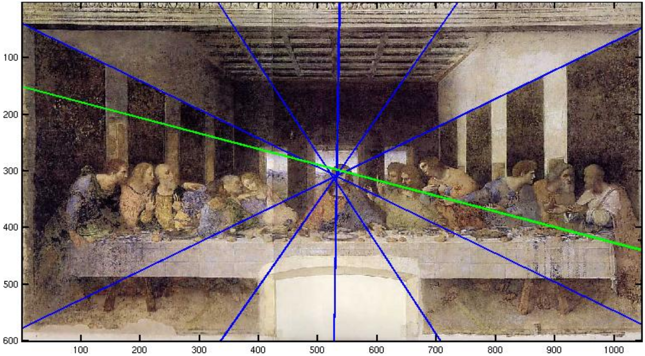

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Vanishing points

- Where do two parallel lines in $\mathbb{R}^3$ with direction $\mathbf{d}$ intersect?

$$\mathbf{X}_1(t) = \mathbf{X}_1 + t\mathbf{d} \qquad \mathbf{X}_2(t) = \mathbf{X}_2 + t\mathbf{d} \qquad t \in \mathbb{R}$$

- Interpret as quantities in $\mathbb{P}^3$: $s = 1/t$

$$\begin{pmatrix} \mathbf{X}_i(t) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_i \\ 1 \end{pmatrix} + t \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} \sim \begin{pmatrix} s\mathbf{X}_i \\ s \end{pmatrix} + \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} \qquad i = 1, 2$$

- Lines intersect at $\binom{\mathbf{d}}{0} \in \mathbb{P}^3$
  - More principled: unique intersection of 3 planes (two of them are parallel)
- Projection into image

$$\mathtt{K}(\mathtt{R} \mid \mathbf{T}) \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} = \mathtt{K}\,\mathtt{R}\,\mathbf{d}$$

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Conics in $\mathbb{P}^2$

Conic

Let $\mathtt{C} \in \mathbb{R}^{3\times 3}$ be a symmetric matrix. A conic $\mathcal{C}$ is the set of points $\mathbf{x}$ from $\mathbb{P}^2$ that satisfy $\mathbf{x}^\top \mathtt{C}\mathbf{x} = 0$,

$$\mathcal{C} := \{\mathbf{x} : \mathbf{x}^\top \mathtt{C}\mathbf{x} = 0\}.$$

- Example: circle at the origin with radius 1

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = x^2 + y^2 - 1 = 0$$

- Do we need to assume that $\mathtt{C}$ is symmetric?
- Tangent line at $\mathbf{x}$: $\mathbf{l} = \mathtt{C}\mathbf{x}$ satisfies $\mathbf{l}^\top \mathbf{x} = 0$
- The set of all tangent lines

$$\{\mathbf{l} = \mathtt{C}\mathbf{x} : \exists \mathbf{x} : \mathbf{x}^\top \mathtt{C}\mathbf{x} = 0\} \overset{\mathbf{x}=\mathtt{C}^{-1}\mathbf{l}}{=} \{\mathbf{l} : (\mathtt{C}^{-1}\mathbf{l})^\top \mathtt{C}\mathtt{C}^{-1}\mathbf{l} = 0\} = \{\mathbf{l} : \mathbf{l}^\top \mathtt{C}^{-1}\mathbf{l} = 0\}$$

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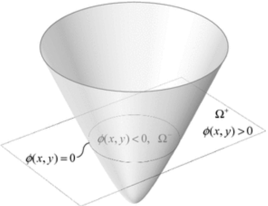

Extra slide: why is $\mathtt{C}\mathbf{x}$ a tangent line?

- Introduce $f(\mathbf{x}) := \mathbf{x}^\top \mathtt{C}\mathbf{x}$, $\mathbf{x} \in \mathbb{P}^2$
- Gradient $\nabla f(\mathbf{x})$:

$$\nabla f(\mathbf{x}) = (\mathtt{C} + \mathtt{C}^\top)\mathbf{x}$$

  Automatically symmetrizes $\mathtt{C}$!
- Therefore $f(\mathbf{x})$ is constant in direction orthogonal to $\nabla f(\mathbf{x}) = (\mathtt{C} + \mathtt{C}^\top)\mathbf{x}$
- Line $\mathbf{l} = (\mathtt{C} + \mathtt{C}^\top)\mathbf{x}$ is tangent line to level set $\{\mathbf{y} : f(\mathbf{y}) = f(\mathbf{x})\}$
- $\mathtt{C}$ symmetric: $\mathbf{l} \sim \mathtt{C}\mathbf{x}$

Quadrics

Quadric

Let $\mathtt{Q} \in \mathbb{R}^{4\times 4}$ be a symmetric matrix. A quadric $\mathcal{Q}$ is the set of points $\mathbf{X}$ from $\mathbb{P}^3$ that satisfy $\mathbf{X}^\top \mathtt{Q}\mathbf{X} = 0$,

$$\mathcal{Q} := \{\mathbf{X} : \mathbf{X}^\top \mathtt{Q}\mathbf{X} = 0\}.$$

- Example: sphere at the origin with radius 1

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = x^2 + y^2 + z^2 - 1 = 0$$

- Tangent plane at $\mathbf{X}$: $\Pi = \mathtt{Q}\mathbf{X}$ satisfies $\Pi^\top \mathbf{X} = 0$
- The set of all tangent planes

$$\{\Pi : \Pi^\top \mathtt{Q}^{-1} \Pi = 0\}$$

$\mathtt{Q}^{-1}$ is called the dual quadric

Discussion

Projection of a quadric

How does a quadric $\mathtt{Q}$ look in an image with camera matrix $\mathtt{P}$?

- We consider the silhouette of the quadric in the image
- Silhouette is generated by lines in the image tangential to $\mathtt{Q}$
  - Lines $\mathbf{l}$ correspond to 3D planes $\Pi = \mathtt{P}^\top \mathbf{l}$
  - Planes Π are tangental to $\mathtt{Q}$, i.e. in the dual cone

$$0 = \Pi^\top \mathtt{Q}^{-1} \Pi = (\mathtt{P}^\top \mathbf{l})^\top \mathtt{Q}^{-1} \mathtt{P}^\top \mathbf{l} = \mathbf{l}^\top \mathtt{P}\mathtt{Q}^{-1}\mathtt{P}^\top \mathbf{l}$$

  - $\mathtt{P}\mathtt{Q}^{-1}\mathtt{P}^\top$ is the dual conic in the image
  - $(\mathtt{P}\mathtt{Q}^{-1}\mathtt{P}^\top)^{-1}$ is the image conic of the quadric $\mathtt{Q}$

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Projective Transformations

Projective transformation

A projective transformation $\mathtt{H}$ is an invertible linear mapping from $\mathbb{P}^k \to \mathbb{P}^k$,

$$\mathbf{y} = \mathtt{H}\mathbf{x}.$$

It can be represented by an invertible matrix $\mathtt{H} \in \mathbb{R}^{(k+1)\times(k+1)}$.

- Projective transformations are defined (non-zero) up-to-scale

$$\mathtt{H}\mathbf{x} \sim \lambda \mathtt{H}\mathbf{x}$$

- *Homographies* are projective transformations in $\mathbb{P}^2$, $\mathtt{H} \in \mathbb{R}^{3\times 3}$
  - 8 degrees of freedom (why?)
  - 4 correspondences $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ to estimate $\mathtt{H}$

Homographies

Plane transfer between images

If $\mathbf{X}$ is on a 3D plane Π, then there exists a homography $\mathtt{H}$ such that

$$\mathbf{y} = \mathtt{P}_2\mathbf{X} = \mathtt{H}\mathtt{P}_1\mathbf{X} = \mathtt{H}\mathbf{x}$$

- Assume $\Pi = (0, 0, 1, 0)^\top$ ($z = 0$ plane). Then $\mathbf{X} = (X_1, X_2, 0, \mu)^\top \in \mathbb{P}^3$.
- Write $\mathtt{P}_i = (\mathtt{A}_i \mid \mathbf{b}_i)$ for $i = 1, 2$

$$\mathbf{x} = \mathtt{A}_1 \begin{pmatrix} X_1 \\ X_2 \\ 0 \end{pmatrix} + \mu\mathbf{b}_1 = \underbrace{\left(\mathtt{A}_1(1:3, 1:2) \mid \mathbf{b}_1\right)}_{=:\mathtt{C}} \begin{pmatrix} X_1 \\ X_2 \\ \mu \end{pmatrix}$$

- Solve for $(X_1, X_2, \mu) = \mathtt{C}^{-1}\mathbf{x}$
- Insert into 2nd camera

$$\mathbf{y} \sim \mathtt{P}_2 \begin{pmatrix} X_1 \\ X_2 \\ 0 \\ \mu \end{pmatrix} = \underbrace{\left(\mathtt{A}_2(1:3, 1:2) \mid \mathbf{b}_2\right)\mathtt{C}^{-1}}_{=\mathtt{H}}\mathbf{x}$$

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- Solve for $(X_1, X_2, \mu) = \mathtt{C}^{-1}\mathbf{x}$
- Insert into 2nd camera

$$\mathbf{y} \sim \mathtt{P}_2 \begin{pmatrix} X_1 \\ X_2 \\ 0 \\ \mu \end{pmatrix} = \underbrace{\left(\mathtt{A}_2(1:3,1:2) \mid \mathbf{b}_2\right)\mathtt{C}^{-1}}_{=\mathtt{H}}\mathbf{x}$$

Homographies

Plane transfer between images

If $\mathbf{X}$ is on a 3D plane Π, then there exists a homography $\mathtt{H}$ such that

$$\mathbf{y} = \mathtt{P}_2\mathbf{X} = \mathtt{H}\mathtt{P}_1\mathbf{X} = \mathtt{H}\mathbf{x}$$

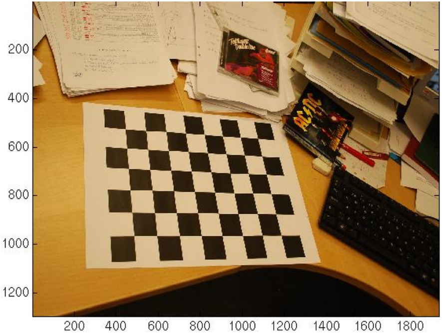
- Assume $\Pi = (0,0,1,0)^\top$ ($z = 0$ plane). Then $\mathbf{X} = (X_1, X_2, 0, \mu)^\top \in \mathbb{P}^3$.
- Write $\mathtt{P}_i = (\mathtt{A}_i \mid \mathbf{b}_i)$ for $i = 1, 2$

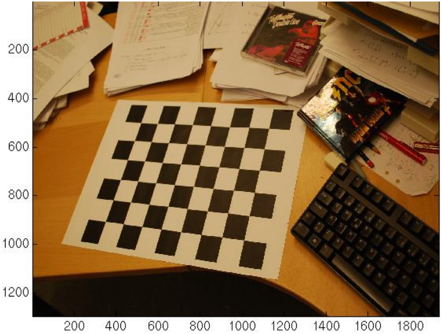
$$\mathbf{x} = \mathtt{A}_1 \begin{pmatrix} X_1 \\ X_2 \\ 0 \end{pmatrix} + \mu \mathbf{b}_1 = \underbrace{\left(\mathtt{A}_1(1:3, 1:2) \mid \mathbf{b}_1\right)}_{=:\mathtt{C}} \begin{pmatrix} X_1 \\ X_2 \\ \mu \end{pmatrix}$$

- Solve for $(X_1, X_2, \mu) = \mathtt{C}^{-1}\mathbf{x}$
- Insert into 2nd camera

$$\mathbf{y} \sim \mathtt{P}_2 \begin{pmatrix} X_1 \\ X_2 \\ 0 \\ \mu \end{pmatrix} = \underbrace{\left(\mathtt{A}_2(1:3, 1:2) \mid \mathbf{b}_2\right)\mathtt{C}^{-1}}_{=\mathtt{H}} \mathbf{x}$$

Example: Point Transfer via a Plane.

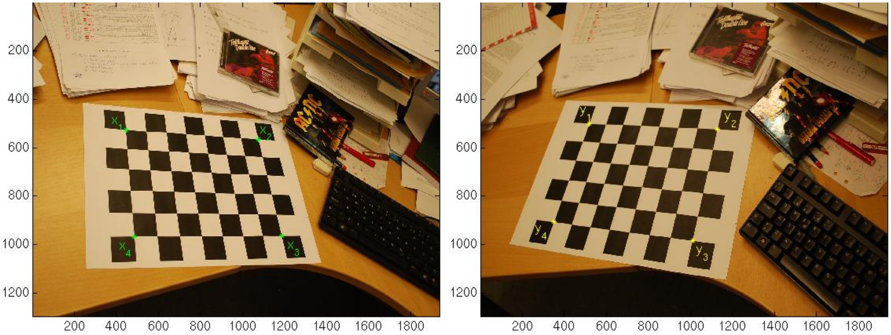




If a set of points $\mathbf{X}_i$ lying on the same plane is projected into two cameras $\mathbf{x}_i \sim \mathtt{P}_1\mathbf{X}_i$, $\mathbf{y}_i \sim \mathtt{P}_2\mathbf{X}_i$, then there is a homography such that $\mathbf{x}_i \sim \mathtt{H}\mathbf{y}_i$.

Projective Transformations

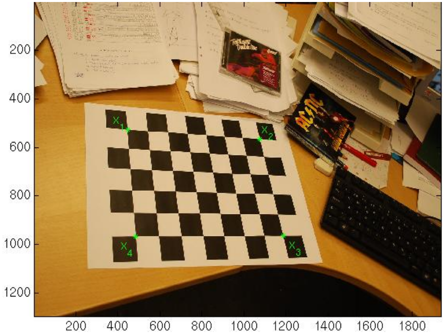
Example: Point Transfer via a Plane.

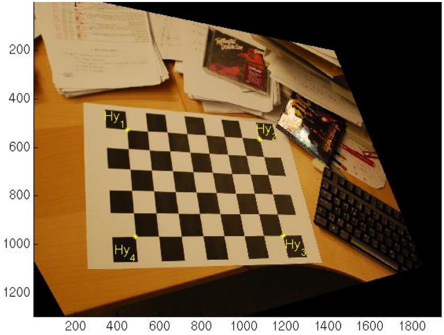


Compute the homography by selecting (at least) 4 points, and solving $\lambda_i \mathbf{x}_i = \mathtt{H}\mathbf{y}_i$.

Projective Transformations

Example: Point Transfer via a Plane.

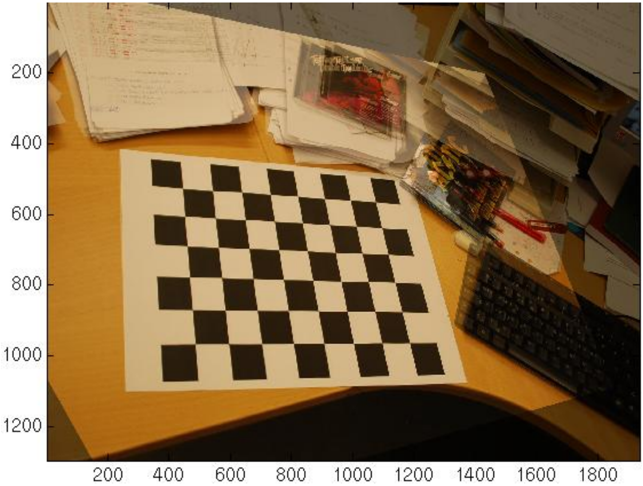




Apply transformation to right image.
(Uses matlabs imtransform.)

Projective Transformations

Example: Point Transfer via a Plane.



Mean value of the two images. Points on the plane seem to agree.

Projective transformations: Special Cases

Affine Transformations ($\mathbb{P}^n \to \mathbb{P}^n$)

$$\mathtt{H} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix},$$

where $\mathtt{A} \in \mathbb{R}^{n\times n}$ (invertible) and $\mathbf{t} \in \mathbb{R}^n$.

- Parallel lines are mapped to parallel lines.
- Preserves the line at infinity (points at infinity are mapped to points at infinity, and regular points are mapped to regular points).
- Can be written $\mathbf{y} = \mathtt{A}\mathbf{x} + \mathbf{t}$ for points in $\mathbb{R}^n$.

Projective transformations: Special Cases

Similarity Transformations ($\mathbb{P}^n \to \mathbb{P}^n$)

$$\mathtt{H} = \begin{bmatrix} s\mathrm{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix},$$

where $\mathtt{R} \in \mathbb{R}^{n\times n}$ rotation, $\mathbf{t} \in \mathbb{R}^n$, $s > 0$

- Special case of affine transformation.
- Preserves angles between lines.

Projective transformations: Special Cases

Euclidian Transformations (Rigid body motion $\mathbb{P}^n \to \mathbb{P}^n$)

$$\mathtt{H} = \begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix},$$

where $\mathtt{R} \in \mathbb{R}^{n\times n}$ rotation, $\mathbf{t} \in \mathbb{R}^n$.

- Special case of similarity.
- Preserves distances.

Projective transformations: Special Cases

	Projective	Affine	Similarity	Euclidean
Maps lines to lines	Y	Y	Y	Y
Preserves parallel lines	N	Y	Y	Y
Preserves angles	N	N	Y	Y
Preserves distances	N	N	N	Y

Exercises

- Given camera matrix $\mathtt{P} = (\mathtt{R} \mid \mathbf{t})$, what is the mapping between stars $(\mathbf{y}_i, 0)$ and their images $\mathbf{x}_i$, where $\mathbf{y}_i, \mathbf{x}_i \in \mathbb{P}^2$?

To do

- Work on assignment 1
- Lab session after this lecture: MTI1–MTI4