Computer Vision: Lecture 7

2023-11-20

Today's Lecture

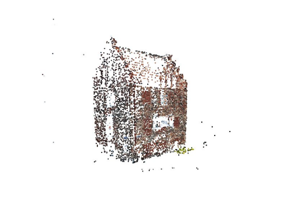
Outline

- Repetition
- A First Reconstruction System
- Outliers and RANSAC
- The 7-point method
- Degeneracies

Repetition: Computing the Camera Matrix

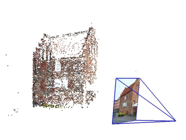
Known

Image points $\mathbf{x}_i$.



Known scene points $\mathbf{X}_i$.

Estimate



A camera matrix P such that

$$\lambda_i \mathbf{x}_i = P\mathbf{X}_i$$

Solved using DLT in Lecture 3.

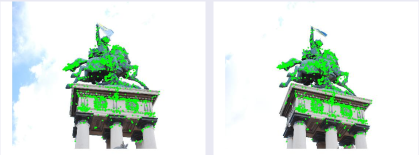
Discussion on Camera Pose

You want to estimate P. How many 2D-3D corresponding do you need?

- Suppose P is uncalibrated. How many?
- Suppose P is calibrated. How many?

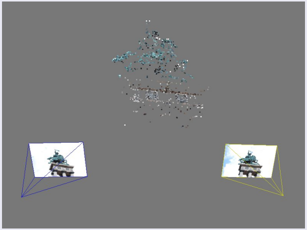
Repetition: Relative Orientation

Known:



Two corresponding point sets $\{\bar{\mathbf{x}}_i\}$ and $\{\mathbf{x}_i\}$.

Sought:



Scene points $\{\mathbf{X}_i\}$ and cameras $\mathtt{P}_1$, $\mathtt{P}_2$, such that

$$\lambda_i \mathbf{x}_i = \mathtt{P}_1 \mathbf{X}_i$$
$$\bar{\lambda}_i \bar{\mathbf{x}}_i = \mathtt{P}_2 \mathbf{X}_i$$

Repetition: Relative Orientation

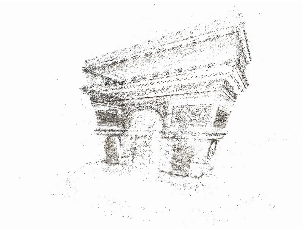
The Fundamental Matrix (see Lecture 5 & 6)

For cameras $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = (\mathtt{A} \mid \mathbf{t})$. The corresponding image points $\mathbf{x}_i$ and $\bar{\mathbf{x}}_i$ fulfill

$$\bar{\mathbf{x}}_i^T \mathbf{F} \mathbf{x}_i = 0,$$

where $\mathtt{F} = [\mathbf{t}]_\times \mathtt{A}$.

- The scene points $\mathbf{X}_i$ have been eliminated.
- Solve $\mathtt{F}$ using 8-point alg, compute cameras (Lectures 5 & 6).



Problem: Projective ambiguity

Repetition: Relative Orientation

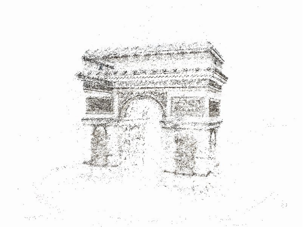
The Essential Matrix (see Lecture 5 & 6)

For cameras $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$. The corresponding image points $\mathbf{x}_i$ and $\bar{\mathbf{x}}_i$ fulfill

$$\bar{\mathbf{x}}_i^T \mathbf{E} \mathbf{x}_i = 0,$$

where $\mathtt{E} = [\mathbf{T}]_\times \mathtt{R}$.

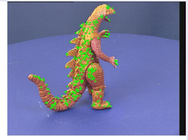
- The scene point $\mathbf{X}_i$ has been eliminated.
- Solve $\mathtt{E}$ using modified 8-point alg, compute cameras (Lecture 6).



No projective ambiguity

Repetition: Triangulation

Known



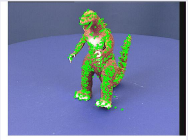
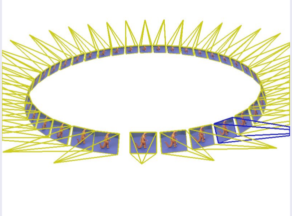


Image points $\{\mathbf{x}_{ij}\}$.



Camera matrices $\mathtt{P}_j$

Sought

3D points $\mathbf{X}_i$, such that

$$\lambda_{ij}\mathbf{x}_{ij} = \mathtt{P}_j \mathbf{X}_i$$

See slides of lecture 4

A First Reconstruction System

Sequential Reconstruction

Given lots of images (and matching image points)

    ...

How do we compute the entire reconstruction?

1. For an initial pair of images, compute the cameras and visible scene points, using 8-point alg.
2. For a new image viewing some of the previously reconstructed scene points, find the camera matrix, using DLT.
3. Compute new scene points using triangulation.
4. If there are more cameras goto step 2.

A First Reconstruction System

Issues

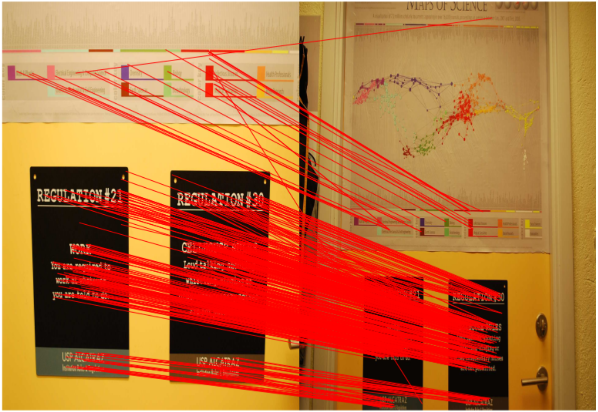
- **Outliers**
- **Noise sensitivity**
- Unreliable 3D points
- (How to select initial pair)

We get back to theses issues later in the course, but we will focus on *non-sequential* 3D reconstruction.

The Outlier Problem

What is an outlier?

An outlier is a measurement that does not fulfill the small-scale noise assumption. Main source in 3D computer vision for outliers are mismatches.



RANdom SAmpling Consensus - RANSAC

Idea

If the proportion of outliers is small, pick a random subset of measurements. With a certain probability this set will be outlier free.
Estimate a model (line, `F`, `H` etc.) from this random subset.

Algorithm

Repeat until a stopping criterion is reached

1. Randomly select a small number of the measurements, and fit the model to these.
2. Evaluate the estimated model with respect to the remaining measurements. The consensus set is are the measurements with an error less than some predefined threshold.

Return the model with the largest consensus set.

RANdom SAmpling Consensus - RANSAC

Idea

If the proportion of outliers is small, pick a random subset of measurements. With a certain probability this set will be outlier free.
Estimate a model (line, `F`, `H` etc.) from this random subset.

Algorithm

Repeat until a stopping criterion is reached

1. Randomly select a small number of the measurements, and fit the model to these.
2. Evaluate the estimated model with respect to the remaining measurements. The consensus set is are the measurements with an error less than some predefined threshold.

Return the model with the largest consensus set.

RANdom SAmpling Consensus - RANSAC

Idea

If the proportion of outliers is small, pick a random subset of measurements. With a certain probability this set will be outlier free.
Estimate a model (line, `F`, `H` etc.) from this random subset.

Algorithm

Repeat until a stopping criterion is reached

1. Randomly select a small number of the measurements, and fit the model to these.
2. Evaluate the estimated model with respect to the remaining measurements. The consensus set is are the measurements with an error less than some predefined threshold.

Return the model with the largest consensus set.

RANdom SAmpling Consensus - RANSAC

Idea

If the proportion of outliers is small, pick a random subset of measurements. With a certain probability this set will be outlier free.
Estimate a model (line, F, H etc.) from this random subset.

Algorithm

Repeat until a stopping criterion is reached

1. Randomly select a small number of the measurements, and fit the model to these.
2. Evaluate the estimated model with respect to the remaining measurements. The consensus set is are the measurements with an error less than some predefined threshold.

Return the model with the largest consensus set.

RANdom SAmpling Consensus - RANSAC

Idea

If the proportion of outliers is small, pick a random subset of measurements. With a certain probability this set will be outlier free.
Estimate a model (line, $\mathtt{F}$, $\mathtt{H}$ etc.) from this random subset.

Algorithm

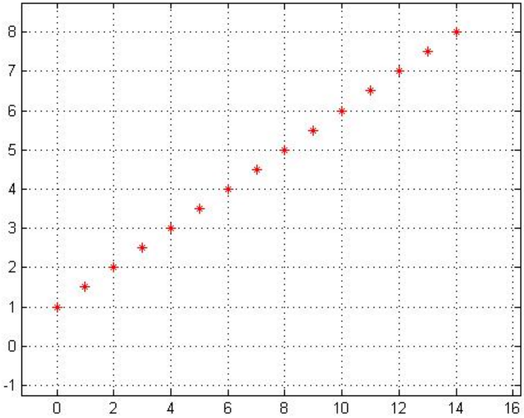
Repeat until a stopping criterion is reached

1. Randomly select a small number of the measurements, and fit the model to these.
2. Evaluate the estimated model with respect to the remaining measurements. The consensus set is are the measurements with an error less than some predefined threshold.

Return the model with the largest consensus set.

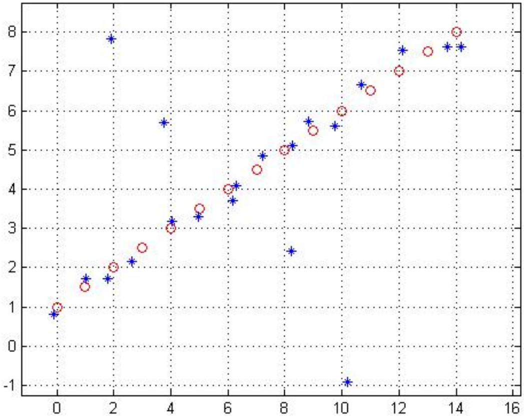
Line Fitting Example

Ideal data exactly lying on a line: noise-free data



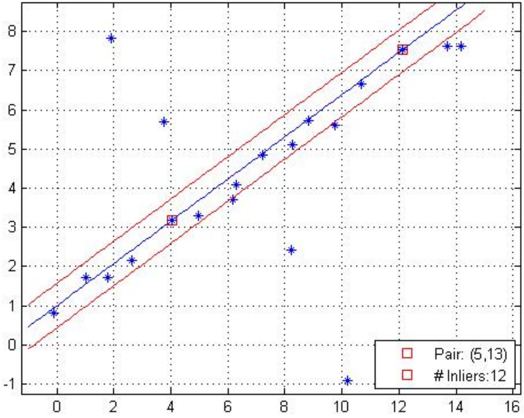
Line Fitting Example

Real data given to you: noisy measurements and outliers



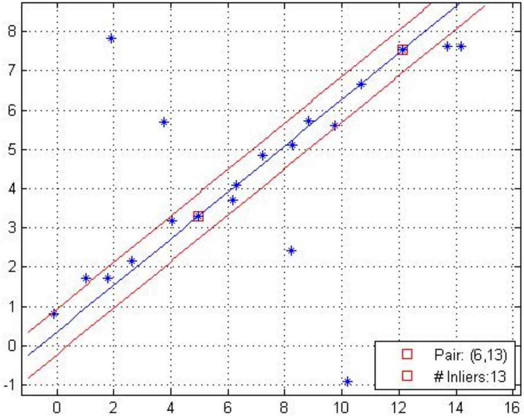
Line Fitting Example

Select 2 points and fit a line.
First iteration:



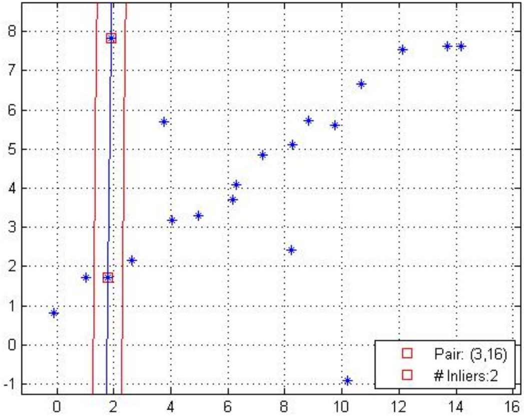
Line Fitting Example

Select 2 points and fit a line.
More iterations:



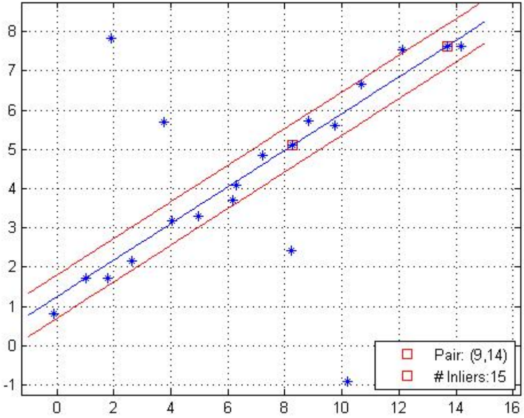
Line Fitting Example

Select 2 points and fit a line.
More iterations:



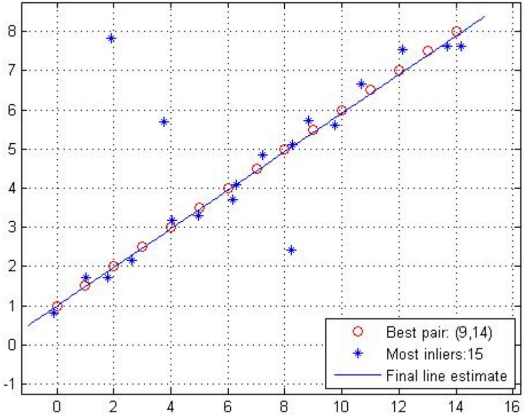
Line Fitting Example

Select 2 points and fit a line.
More iterations:



Line Fitting Example

Final result:



RANSAC

Question

How many iterations do we need?

RANSAC

Question

How many iterations do we need?

- Given N data points $\mathcal{X} = \{\mathbf{x}_i : i = 1, \dots, N\}$
- Let ε be the fraction of inliers

$$\mathrm{P}(\mathbf{x}_i \text{ is inlier}) = \varepsilon$$

- We need s (e.g. $s = 2$) data points to estimate a model
- $\mathcal{S}$ is a set of size s uniformly sampled from $\mathcal{X}$

$$\mathrm{P}(\text{all data points in } \mathcal{S} \text{ are inliers}) = \varepsilon^s$$

- Probability of $\mathcal{S}$ containing at least one outlier: $1 - \varepsilon^s$
- We sample T independent sets $\mathcal{S}$.
  Probability that all of them are corrupted by ≥ 1 outliers

$$\mathrm{P}(\text{all sampled sets } \mathcal{S} \text{ contain one or more outliers}) = (1 - \varepsilon^s)^T$$

RANSAC

Question

How many iterations do we need?

- Given N data points $\mathcal{X} = \{\mathbf{x}_i : i = 1, \dots, N\}$
- Let ε be the fraction of inliers

$$\mathrm{P}(\mathbf{x}_i \text{ is inlier}) = \varepsilon$$

- We need s (e.g. $s = 2$) data points to estimate a model
- $\mathcal{S}$ is a set of size s uniformly sampled from $\mathcal{X}$

$$\mathrm{P}(\text{all data points in } \mathcal{S} \text{ are inliers}) = \varepsilon^s$$

- Probability of $\mathcal{S}$ containing at least one outlier: $1 - \varepsilon^s$
- We sample T independent sets $\mathcal{S}$.
  Probability that all of them are corrupted by ≥ 1 outliers

$$\mathrm{P}(\text{all sampled sets } \mathcal{S} \text{ contain one or more outliers}) = (1 - \varepsilon^s)^T$$

RANSAC

Question

How many iterations do we need?

- Given N data points $\mathcal{X} = \{\mathbf{x}_i : i = 1, \dots, N\}$
- Let ε be the fraction of inliers

$$\mathrm{P}(\mathbf{x}_i \text{ is inlier}) = \varepsilon$$

- We need s (e.g. $s = 2$) data points to estimate a model
- $\mathcal{S}$ is a set of size s uniformly sampled from $\mathcal{X}$

$$\mathrm{P}(\text{all data points in } \mathcal{S} \text{ are inliers}) = \varepsilon^s$$

- Probability of $\mathcal{S}$ containing at least one outlier: $1 - \varepsilon^s$
- We sample T independent sets $\mathcal{S}$.
  Probability that all of them are corrupted by ≥ 1 outliers

$$\mathrm{P}(\text{all sampled sets } \mathcal{S} \text{ contain one or more outliers}) = (1 - \varepsilon^s)^T$$

RANSAC

Question

How many iterations do we need?

- Previous slide: probability that all sets $\mathcal{S}$ are corrupted by ≥ 1 outliers

$$\mathrm{P}(\text{all sampled sets } \mathcal{S} \text{ contain one or more outliers}) = (1 - \varepsilon^s)^T$$

- Assuming ε we want T large enough to guarantee an all-inlier set with probability α (e.g. $\alpha = 95\%$)

$$1 - (1 - \varepsilon^s)^T \geq \alpha \qquad \text{or} \qquad (1 - \varepsilon^s)^T \leq 1 - \alpha$$

- Solve for T

$$T \geq \left\lceil \frac{\log(1-\alpha)}{\log(1-\varepsilon^s)} \right\rceil$$

- This is called the "RANSAC formula"

RANSAC

Question

How many iterations do we need?

- Previous slide: probability that all sets $\mathcal{S}$ are corrupted by ≥ 1 outliers

$$\mathrm{P}(\text{all sampled sets } \mathcal{S} \text{ contain one or more outliers}) = (1 - \varepsilon^s)^T$$

- Assuming ε we want T large enough to guarantee an all-inlier set with probability α (e.g. $\alpha = 95\%$)

$$1 - (1 - \varepsilon^s)^T \geq \alpha \qquad \text{or} \qquad (1 - \varepsilon^s)^T \leq 1 - \alpha$$

- Solve for T

$$T \geq \left\lceil \frac{\log(1-\alpha)}{\log(1-\varepsilon^s)} \right\rceil$$

- This is called the "RANSAC formula"

RANSAC

Question

How many iterations do we *really* need?

- Recall the *RANSAC formula*

$$T \geq \left\lceil \frac{\log(1-\alpha)}{\log(1-\varepsilon^s)} \right\rceil$$

- We do not know ε?
  - Start with a lower bound for ε and update ε and T during the iterations
  - E.g. initial $\varepsilon = 10\%$
- Big assumption of the RANSAC formula
  - Every all-inlier set $\mathcal{S}$ will contain all true inliers in the consensus set
  - Not the case in general
  - Actual number of ransac iterations about 3-4$\times$ higher

RANSAC

Question

How many iterations do we *really* need?

- Recall the *RANSAC formula*

$$T \geq \left\lceil \frac{\log(1-\alpha)}{\log(1-\varepsilon^s)} \right\rceil$$

- We do not know ε?
  - Start with a lower bound for ε and update ε and T during the iterations
  - E.g. initial $\varepsilon = 10\%$
- Big assumption of the RANSAC formula
  - Every all-inlier set $\mathcal{S}$ will contain all true inliers in the consensus set
  - Not the case in general
  - Actual number of ransac iterations about 3-4$\times$ higher

RANSAC

Question

How many iterations do we *really* need?

- Recall the *RANSAC formula*

$$T \geq \left\lceil \frac{\log(1-\alpha)}{\log(1-\varepsilon^s)} \right\rceil$$

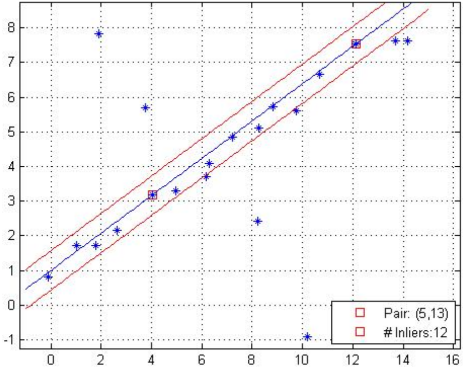
- We do not know ε?
  - Start with a lower bound for ε and update ε and T during the iterations
  - E.g. initial $\varepsilon = 10\%$
- Big assumption of the RANSAC formula
  - Every all-inlier set $\mathcal{S}$ will contain all true inliers in the consensus set
  - Not the case in general
  - Actual number of ransac iterations about 3-4$\times$ higher

RANSAC

Question

How many iterations do we *really* need?

- $N = 20$, $\alpha = 95\%$, $\varepsilon_0 = 10\%$, $T = 299$
- $t = 1$: consensus set of size 12 $\implies$ $\varepsilon = 60\%$, $T = 7$

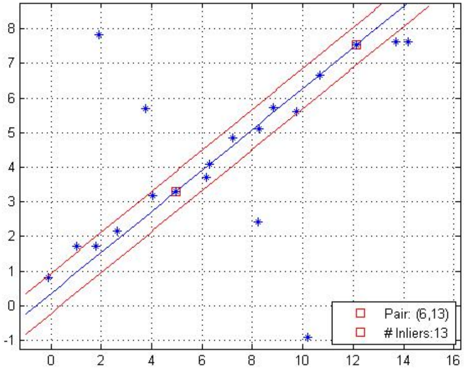


RANSAC

Question

How many iterations do we *really* need?

- $N = 20$, $\alpha = 95\%$, $\varepsilon_0 = 10\%$, $T = 299$
- $t = 2$: consensus set of size 13 $\implies$ $\varepsilon = 65\%$, $T = 6$

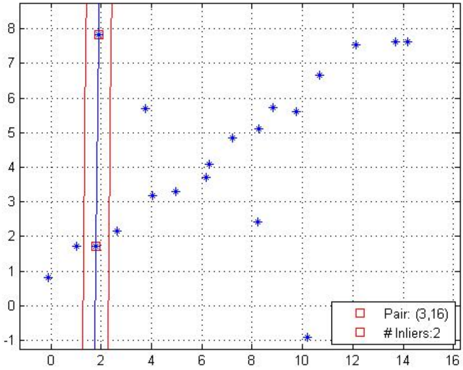


RANSAC

Question

How many iterations do we *really* need?

- $N = 20$, $\alpha = 95\%$, $\varepsilon_0 = 10\%$, $T = 299$
- $t = 3$: consensus set of size 2. Poor model corrupted by outlier.

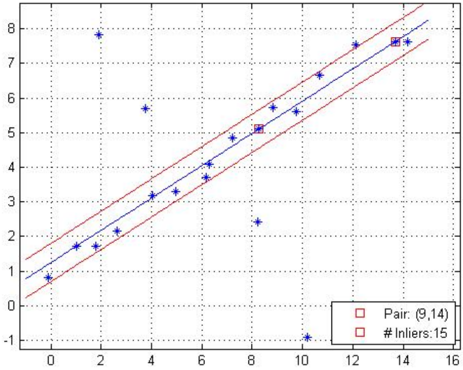


RANSAC

Question

How many iterations do we *really* need?

- $N = 20$, $\alpha = 95\%$, $\varepsilon_0 = 10\%$, $T = 299$
- $t = 4$: consensus set of size 15 $\implies$ $\varepsilon = 75\%$, $T = 4$. Done!



RANSAC

Question

RANSAC takes $O(TN)$ runtime. Can we improve on this?

- Better sampling strategy using additional information
  - Aims to reduce T by picking more probable inlier sets
  - Guided MLESAC: non-uniform sampling of data points
  - ProSAC: start with most likely inlier set and progress systematically
- Early bailout for bad-looking models
  - E.g. $N = 1000$, $\varepsilon = 50\%$ and after 100 data points you have 5 inliers
  - WaldSAC: uses sequential analysis from statistics
  - "HoeffdingSAC": makes use of Hoeffding's inequality

$$\mathrm{P}\big(\bar{Z} \le \mathbb{E}(\bar{Z}) - t\big) \le e^{-2nt^2} \qquad \bar{Z} = \frac{1}{n} Z_i \qquad Z_i \sim \mathrm{Ber}(p)$$

  - Require random shuffling of data points

RANSAC

Question

RANSAC takes $O(TN)$ runtime. Can we improve on this?

- Better sampling strategy using additional information
  - Aims to reduce T by picking more probable inlier sets
  - Guided MLESAC: non-uniform sampling of data points
  - ProSAC: start with most likely inlier set and progress systematically
- Early bailout for bad-looking models
  - E.g. $N = 1000$, $\varepsilon = 50\%$ and after 100 data points you have 5 inliers
  - WaldSAC: uses sequential analysis from statistics
  - "HoeffdingSAC": makes use of Hoeffding's inequality

$$\mathrm{P}\big(\bar{Z} \le \mathbb{E}(\bar{Z}) - t\big) \le e^{-2nt^2} \qquad \bar{Z} = \frac{1}{n} Z_i \qquad Z_i \sim \mathrm{Ber}(p)$$

  - Require random shuffling of data points

RANSAC

Question

RANSAC takes $O(TN)$ runtime. Can we improve on this?

- Better sampling strategy using additional information
  - Aims to reduce T by picking more probable inlier sets
  - Guided MLESAC: non-uniform sampling of data points
  - ProSAC: start with most likely inlier set and progress systematically
- Early bailout for bad-looking models
  - E.g. $N = 1000$, $\varepsilon = 50\%$ and after 100 data points you have 5 inliers
  - WaldSAC: uses sequential analysis from statistics
  - "HoeffdingSAC": makes use of Hoeffding's inequality

$$\mathrm{P}\big(\bar{Z} \leq \mathbb{E}(\bar{Z}) - t\big) \leq e^{-2nt^2} \qquad \bar{Z} = \frac{1}{n} Z_i \qquad Z_i \sim \mathrm{Ber}(p)$$

  - Require random shuffling of data points

Model Fitting: Minimal Solvers

- Recall RANSAC formula

$$T \geq \left\lceil \frac{\log(1-\alpha)}{\log(1-\varepsilon^s)} \right\rceil$$

- Fewer iterations when s is smaller
- Strong incentive to work with smallest $s =$ d.o.f.
  - 8-point method for $\mathtt{F}$?
  - 8-point method for $\mathtt{E}$?
  - 6-point method for $\mathtt{P} = (\mathtt{R} \mid \mathbf{T})$?
  - These are all non-minimal
- Minimal solvers
  - Constraints coming from data are often linear
  - Internal constraints are usually polynomial
  - $\mathtt{F}$ has rank ≤ 2: $\det(\mathtt{F}) = 0$
  - $\mathtt{E}$ is essential matrix: $\det(\mathtt{E}) = 0$ and $2\mathtt{E}\mathtt{E}^\top\mathtt{E} = \mathrm{trace}(\mathtt{E}\mathtt{E}^\top)\mathtt{E}$
  - $\mathtt{R}$ is rotation matrix: e.g. $\mathtt{R}$ is induced by unit quaternion $\mathbf{q}$
- We need to solve system of polynomial equations (fast)

Model Fitting: Minimal Solvers

- Recall RANSAC formula

$$T \geq \left\lceil \frac{\log(1-\alpha)}{\log(1-\varepsilon^s)} \right\rceil$$

- Fewer iterations when s is smaller
- Strong incentive to work with smallest $s =$ d.o.f.
  - 8-point method for $\mathtt{F}$?
  - 8-point method for $\mathtt{E}$?
  - 6-point method for $\mathtt{P} = (\mathtt{R} \mid \mathbf{T})$?
  - These are all non-minimal
- Minimal solvers
  - Constraints coming from data are often linear
  - Internal constraints are usually polynomial
  - $\mathtt{F}$ has rank ≤ 2: $\det(\mathtt{F}) = 0$
  - $\mathtt{E}$ is essential matrix: $\det(\mathtt{E}) = 0$ and $2\mathtt{E}\mathtt{E}^\top\mathtt{E} = \mathrm{trace}(\mathtt{E}\mathtt{E}^\top)\mathtt{E}$
  - $\mathtt{R}$ is rotation matrix: e.g. $\mathtt{R}$ is induced by unit quaternion $\mathbf{q}$
- We need to solve system of polynomial equations (fast)

7-Point Algorithm

- 7 point correspondences needed to estimate $\mathtt{F}$

$$\bar{\mathbf{x}}_i^\top \mathtt{F} \mathbf{x}_i = 0 \qquad i = 1, \dots, 7$$

- Reshape $\mathtt{F} \in \mathbb{R}^{3\times 3}$ to $\mathbf{f} \in \mathbb{R}^9$

$$\mathtt{M}\mathbf{f} = 0 \qquad \mathtt{M} \in \mathbb{R}^{7\times 9}$$

- $\mathbf{f}$ has to be in null-space of $\mathtt{M}$: $\mathbf{f} = t\mathbf{u}_1 + r\mathbf{u}_2$

$$\mathbf{u}_1, \mathbf{u}_2 \in \text{null}(\mathtt{M}) \qquad \mathbf{u}_1 \perp \mathbf{u}_2$$

- Fix scale of $\mathtt{F}$: $t + r = 1 \implies \mathbf{f} = t\mathbf{u}_1 + (1-t)\mathbf{u}_2 = \mathbf{u}_2 + t(\mathbf{u}_1 - \mathbf{u}_2)$
- $\det(\mathtt{F}) = 0$ yields cubic polynomial in t
  - 1 or 3 real solutions
- We solve a single univariate (cubic) polynomial

7-Point Algorithm

- 7 point correspondences needed to estimate $\mathtt{F}$

$$\bar{\mathbf{x}}_i^\top \mathtt{F} \mathbf{x}_i = 0 \qquad i = 1, \dots, 7$$

- Reshape $\mathtt{F} \in \mathbb{R}^{3\times 3}$ to $\mathbf{f} \in \mathbb{R}^9$

$$\mathtt{M}\mathbf{f} = 0 \qquad \mathtt{M} \in \mathbb{R}^{7\times 9}$$

- $\mathbf{f}$ has to be in null-space of $\mathtt{M}$: $\mathbf{f} = t\mathbf{u}_1 + r\mathbf{u}_2$

$$\mathbf{u}_1, \mathbf{u}_2 \in \operatorname{null}(\mathtt{M}) \qquad \mathbf{u}_1 \perp \mathbf{u}_2$$

- Fix scale of $\mathtt{F}$: $t + r = 1 \implies \mathbf{f} = t\mathbf{u}_1 + (1-t)\mathbf{u}_2 = \mathbf{u}_2 + t(\mathbf{u}_1 - \mathbf{u}_2)$
- $\det(\mathtt{F}) = 0$ yields cubic polynomial in t
  - 1 or 3 real solutions
- We solve a single univariate (cubic) polynomial

7-Point Algorithm

- 7 point correspondences needed to estimate $\mathtt{F}$

$$\bar{\mathbf{x}}_i^\top \mathtt{F} \mathbf{x}_i = 0 \qquad i = 1, \dots, 7$$

- Reshape $\mathtt{F} \in \mathbb{R}^{3\times 3}$ to $\mathbf{f} \in \mathbb{R}^9$

$$\mathtt{M}\mathbf{f} = 0 \qquad \mathtt{M} \in \mathbb{R}^{7\times 9}$$

- $\mathbf{f}$ has to be in null-space of $\mathtt{M}$: $\mathbf{f} = t\mathbf{u}_1 + r\mathbf{u}_2$

$$\mathbf{u}_1, \mathbf{u}_2 \in \operatorname{null}(\mathtt{M}) \qquad \mathbf{u}_1 \perp \mathbf{u}_2$$

- Fix scale of $\mathtt{F}$: $t + r = 1 \implies \mathbf{f} = t\mathbf{u}_1 + (1-t)\mathbf{u}_2 = \mathbf{u}_2 + t(\mathbf{u}_1 - \mathbf{u}_2)$
- $\det(\mathtt{F}) = 0$ yields cubic polynomial in t
  - 1 or 3 real solutions
- We solve a single univariate (cubic) polynomial

7-Point Algorithm

- 7 point correspondences needed to estimate $\mathtt{F}$

$$\bar{\mathbf{x}}_i^\top \mathtt{F} \mathbf{x}_i = 0 \qquad i = 1, \dots, 7$$

- Reshape $\mathtt{F} \in \mathbb{R}^{3\times 3}$ to $\mathbf{f} \in \mathbb{R}^9$

$$\mathtt{M}\mathbf{f} = 0 \qquad \mathtt{M} \in \mathbb{R}^{7\times 9}$$

- $\mathbf{f}$ has to be in null-space of $\mathtt{M}$: $\mathbf{f} = t\mathbf{u}_1 + r\mathbf{u}_2$

$$\mathbf{u}_1, \mathbf{u}_2 \in \operatorname{null}(\mathtt{M}) \qquad \mathbf{u}_1 \perp \mathbf{u}_2$$

- Fix scale of $\mathtt{F}$: $t + r = 1 \implies \mathbf{f} = t\mathbf{u}_1 + (1-t)\mathbf{u}_2 = \mathbf{u}_2 + t(\mathbf{u}_1 - \mathbf{u}_2)$
- $\det(\mathtt{F}) = 0$ yields cubic polynomial in t
  - 1 or 3 real solutions
- We solve a single univariate (cubic) polynomial

7-Point Algorithm

- How to calulate $\det(\mathtt{F}) = \det(\mathtt{U}_2 + t(\mathtt{U}_2 - \mathtt{U}_1))$?
  - Coefficients c_i of polynomial $c_0 + c_1 t + c_2 t^2 + c_3 t^3$ from $\mathbf{u}_1$ & $\mathbf{u}_2$
- Use a CAS (Maple, Mathematica or Maxima)
- Maxima

```
/* t*N1 + (1-t)*N2 = N2 + t*(N1-N2) = N + t*D */
M: matrix([t*D(1,1)+N(1,1),t*D(1,2)+N(1,2),t*D(1,3)+N(1,3)], \
          [t*D(2,1)+N(2,1),t*D(2,2)+N(2,2),t*D(2,3)+N(2,3)], \
          [t*D(3,1)+N(3,1),t*D(3,2)+N(3,2),t*D(3,3)+N(3,3)]);

g: expand(determinant(M));

/* Disable "pretty" formatting */
display2d:false$

c3: factor(coeff(g, t^3));
c2: factor(coeff(g, t^2));
c1: factor(coeff(g, t));
c0: factor(g - c3*t^3 - c2*t^2 - c1*t);
```

Degeneracies

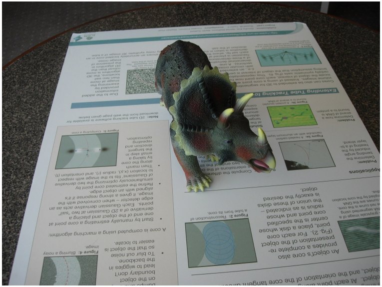
Discussion

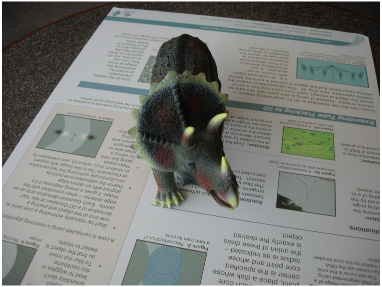
When will estimation of $\mathtt{F}/\mathtt{E}$ or a homography not work?
How can one detect such situations?

Degenerate Configurations: Planar Scenes

Question

Can we estimate F when the scene is planar (or the minimal sample can be described by a homography)?





- The answer is No
- The 7- and 8-point method will return unstable / non-unique estimates
- Man-made environments: scenes are often dominated by planes!

Degenerate Configurations: Planar Scenes

Question

Can we estimate $\mathtt{F}$ when the scene is planar (or the minimal sample can be described by a homography)?

- The answer is No
- The 7- and 8-point method will return unstable / non-unique estimates
- Man-made environments: scenes are often dominated by planes!

Degenerate Configurations: Planar Scenes

- Let $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
- There is a homography $\mathtt{H}$ such that $\mathbf{y}_i \sim \mathtt{H}\mathbf{x}_i$
- Epipolar constraint for any $\mathbf{x} \leftrightarrow \mathbf{y}$ with $\mathbf{y} \sim \mathtt{H}\mathbf{x}$

$$0 = \mathbf{y}^\top \mathtt{F}\mathbf{x} \sim (\mathtt{H}\mathbf{x})^\top \mathtt{F}\mathbf{x} = \mathbf{x}^\top \mathtt{H}^\top \mathtt{F}\mathbf{x} \qquad \forall \mathbf{x}$$

- Quadratic form $\mathbf{x}^\top \mathtt{Q}\mathbf{x}$, only symmetrized matrix matters

$$\text{Condition: } \mathbf{x}^\top (\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H})\mathbf{x} = 0 \qquad \forall \mathbf{x}$$

- Sufficient (and neceesary) condition: $\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H} = 0$
  - $\mathtt{H}$ *compatible* with $\mathtt{F}$
- If all $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
  - 2-dimensional space of solutions for $\mathtt{F}$
  - 9 entries of $\mathtt{F}$ minus 6 skew-symmetric constraints minus scale freedom
- Also problematic if $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ are almost related by a homography

Degenerate Configurations: Planar Scenes

- Let $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
- There is a homography $\mathtt{H}$ such that $\mathbf{y}_i \sim \mathtt{H}\mathbf{x}_i$
- Epipolar constraint for any $\mathbf{x} \leftrightarrow \mathbf{y}$ with $\mathbf{y} \sim \mathtt{H}\mathbf{x}$

$$0 = \mathbf{y}^\top \mathbf{F}\mathbf{x} \sim (\mathtt{H}\mathbf{x})^\top \mathbf{F}\mathbf{x} = \mathbf{x}^\top \mathtt{H}^\top \mathbf{F}\mathbf{x} \qquad \forall \mathbf{x}$$

- Quadratic form $\mathbf{x}^\top \mathtt{Q}\mathbf{x}$, only symmetrized matrix matters

$$\text{Condition: } \mathbf{x}^\top (\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H})\mathbf{x} = 0 \qquad \forall \mathbf{x}$$

- Sufficient (and neceesary) condition: $\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H} = 0$
  - $\mathtt{H}$ *compatible* with $\mathtt{F}$
- If all $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
  - 2-dimensional space of solutions for $\mathtt{F}$
  - 9 entries of $\mathtt{F}$ minus 6 skew-symmetric constraints minus scale freedom
- Also problematic if $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ are almost related by a homography

Degenerate Configurations: Planar Scenes

- Let $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
- There is a homography $\mathtt{H}$ such that $\mathbf{y}_i \sim \mathtt{H}\mathbf{x}_i$
- Epipolar constraint for any $\mathbf{x} \leftrightarrow \mathbf{y}$ with $\mathbf{y} \sim \mathtt{H}\mathbf{x}$

$$0 = \mathbf{y}^\top \mathtt{F}\mathbf{x} \sim (\mathtt{H}\mathbf{x})^\top \mathtt{F}\mathbf{x} = \mathbf{x}^\top \mathtt{H}^\top \mathtt{F}\mathbf{x} \qquad \forall \mathbf{x}$$

- Quadratic form $\mathbf{x}^\top \mathtt{Q}\mathbf{x}$, only symmetrized matrix matters

$$\text{Condition: } \mathbf{x}^\top (\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H})\mathbf{x} = 0 \qquad \forall \mathbf{x}$$

- Sufficient (and neceesary) condition: $\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H} = 0$
  - $\mathtt{H}$ *compatible* with $\mathtt{F}$
- If all $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
  - 2-dimensional space of solutions for $\mathtt{F}$
  - 9 entries of $\mathtt{F}$ minus 6 skew-symmetric constraints minus scale freedom
- Also problematic if $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ are almost related by a homography

Degenerate Configurations: Planar Scenes

- Let $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
- There is a homography $\mathtt{H}$ such that $\mathbf{y}_i \sim \mathtt{H}\mathbf{x}_i$
- Epipolar constraint for any $\mathbf{x} \leftrightarrow \mathbf{y}$ with $\mathbf{y} \sim \mathtt{H}\mathbf{x}$

$$0 = \mathbf{y}^\top \mathtt{F}\mathbf{x} \sim (\mathtt{H}\mathbf{x})^\top \mathtt{F}\mathbf{x} = \mathbf{x}^\top \mathtt{H}^\top \mathtt{F}\mathbf{x} \qquad \forall \mathbf{x}$$

- Quadratic form $\mathbf{x}^\top \mathtt{Q}\mathbf{x}$, only symmetrized matrix matters

$$\text{Condition: } \mathbf{x}^\top (\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H})\mathbf{x} = 0 \qquad \forall \mathbf{x}$$

- Sufficient (and neceesary) condition: $\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H} = \mathtt{0}$
  - $\mathtt{H}$ *compatible* with $\mathtt{F}$
- If all $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
  - 2-dimensional space of solutions for $\mathtt{F}$
  - 9 entries of $\mathtt{F}$ minus 6 skew-symmetric constraints minus scale freedom
- Also problematic if $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ are almost related by a homography

Degenerate Configurations: Planar Scenes

- Let $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
- There is a homography $\mathtt{H}$ such that $\mathbf{y}_i \sim \mathtt{H}\mathbf{x}_i$
- Epipolar constraint for any $\mathbf{x} \leftrightarrow \mathbf{y}$ with $\mathbf{y} \sim \mathtt{H}\mathbf{x}$

$$0 = \mathbf{y}^\top \mathtt{F}\mathbf{x} \sim (\mathtt{H}\mathbf{x})^\top \mathtt{F}\mathbf{x} = \mathbf{x}^\top \mathtt{H}^\top \mathtt{F}\mathbf{x} \qquad \forall \mathbf{x}$$

- Quadratic form $\mathbf{x}^\top \mathtt{Q}\mathbf{x}$, only symmetrized matrix matters

$$\text{Condition: } \mathbf{x}^\top (\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H})\mathbf{x} = 0 \qquad \forall \mathbf{x}$$

- Sufficient (and neceesary) condition: $\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H} = 0$
  - $\mathtt{H}$ *compatible* with $\mathtt{F}$
- If all $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
  - 2-dimensional space of solutions for $\mathtt{F}$
  - 9 entries of $\mathtt{F}$ minus 6 skew-symmetric constraints minus scale freedom
- Also problematic if $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ are almost related by a homography

Degenerate Configurations: Planar Scenes

What can we do?

Calibrated setup

- 5-point method to estimate $\mathtt{E}$ has only few issues with planar scenes
- **Find $\mathtt{H}$ and decompose into $\mathtt{R}$ and $\mathbf{T}$**

Uncalibrated setup (or ignoring calibration)

- Detect degeneracy and use a different method to estimate $\mathtt{F}$
  - Discard $\mathtt{F}$ if estimated from ≥ 5 planar correspondences
    - Might require many iterations
  - Better: DGENSAC

```
Draw sample set S of 7 (8) correspondences
Estimate F via the 7-point (8-point) method
If F is best model so far and
   >= 5 matches in S are related by a homography H
   Re-estimate F via the plane+parallax method
end if
```

Degenerate Configurations: Planar Scenes

What can we do?

Calibrated setup

- 5-point method to estimate $\mathtt{E}$ has only few issues with planar scenes
- **Find $\mathtt{H}$ and decompose into $\mathtt{R}$ and $\mathbf{T}$**

Uncalibrated setup (or ignoring calibration)

- Detect degeneracy and use a different method to estimate $\mathtt{F}$
  - Discard $\mathtt{F}$ if estimated from ≥ 5 planar correspondences
    - Might require many iterations
  - Better: DGENSAC

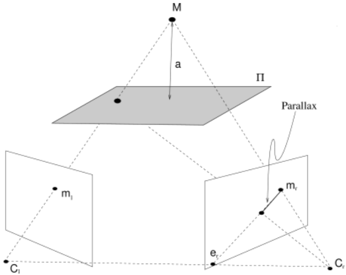
```
Draw sample set S of 7 (8) correspondences
Estimate F via the 7-point (8-point) method
If F is best model so far and
   >= 5 matches in S are related by a homography H
   Re-estimate F via the plane+parallax method
end if
```

Plane + Parallax Method

- We have $\mathtt{H}$, what do we need to estimate $\mathtt{F}$?
  - 4 correspondences on the plane: $\{\mathbf{x}_i \leftrightarrow \mathbf{y}_i\}_{i=1,\dots,4}$
  - 2 correspondences off the plane: $\{\mathbf{x}_i \leftrightarrow \mathbf{y}_i\}_{i=5,6}$
- Epipolar constraint satisfied for

$$\mathbf{x}_5 \leftrightarrow \mathbf{y}_5 \qquad \mathbf{x}_5 \leftrightarrow \mathtt{H}\mathbf{x}_5 \qquad \mathbf{x}_6 \leftrightarrow \mathbf{y}_6 \qquad \mathbf{x}_6 \leftrightarrow \mathtt{H}\mathbf{x}_6$$

  - Epipole $\mathbf{e}_2$ is intersection of $\overline{(\mathbf{y}_5, \mathtt{H}\mathbf{x}_5)}$ and $\overline{(\mathbf{y}_6, \mathtt{H}\mathbf{x}_6)}$
- Fundamental matrix $\mathtt{F} = [\mathbf{e}_2]_\times \mathtt{H}$
  - Q: is $[\mathbf{e}_2]_\times \mathtt{H}$ an essential matrix in the calibrated setup?

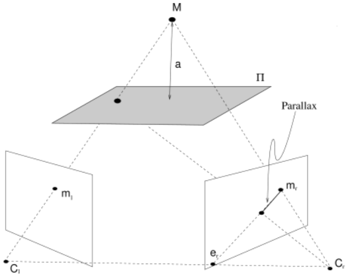


Plane + Parallax Method

- We have $\mathtt{H}$, what do we need to estimate $\mathtt{F}$?
  - 4 correspondences on the plane: $\{\mathbf{x}_i \leftrightarrow \mathbf{y}_i\}_{i=1,\dots,4}$
  - 2 correspondences off the plane: $\{\mathbf{x}_i \leftrightarrow \mathbf{y}_i\}_{i=5,6}$
- Epipolar constraint satisfied for

$$\mathbf{x}_5 \leftrightarrow \mathbf{y}_5 \qquad \mathbf{x}_5 \leftrightarrow \mathtt{H}\mathbf{x}_5 \qquad \mathbf{x}_6 \leftrightarrow \mathbf{y}_6 \qquad \mathbf{x}_6 \leftrightarrow \mathtt{H}\mathbf{x}_6$$

  - Epipole $\mathbf{e}_2$ is intersection of $\overline{(\mathbf{y}_5, \mathtt{H}\mathbf{x}_5)}$ and $\overline{(\mathbf{y}_6, \mathtt{H}\mathbf{x}_6)}$
- Fundamental matrix $\mathtt{F} = [\mathbf{e}_2]_\times \mathtt{H}$
  - Q: is $[\mathbf{e}_2]_\times \mathtt{H}$ an essential matrix in the calibrated setup?



Discussion

We do not know in advance whether there is a dominant plane in the scene. What can we do?

- Search for a fundamental matrix $\mathtt{F}$ and discard degenerate samples?
- Search for a homography $\mathtt{H}$?
  - Any downsides of only looking for either $\mathtt{F}$ or $\mathtt{H}$?
- Can we search for $\mathtt{F}$ and $\mathtt{H}$ in parallel?
  - If yes, how do we modify the RANSAC procedure?

Discussion

We do not know in advance whether there is a dominant plane in the scene. What can we do?

- Search for a fundamental matrix $\mathtt{F}$ and discard degenerate samples?
- Search for a homography $\mathtt{H}$?
  - Any downsides of only looking for either $\mathtt{F}$ or $\mathtt{H}$?
- Can we search for $\mathtt{F}$ and $\mathtt{H}$ in parallel?
  - If yes, how do we modify the RANSAC procedure?

To do

Lab sessions today: E-D2480, ES62, ES63 & KD1

- Next time: minimal solvers, 5-point method and maximum likelihood
- Finish assignment 2 and start with assignment 3.

More reading:

- Szeliski 8.1.4 on Robust Least Squares and RANSAC.