Computer Vision: Lecture 8

2023-11-23

Planned contents

Week 1	Intro, camera model	Projective geometry
Week 2	Camera calibration, DLT I	DLT II, feature matching
Week 3	Two-view geometry I	Two-view geometry II
Week 4	Robust estimation	Minimal solvers & degeneracies
Week 5	MLE & Non-Linear opt.	Factorization project pres.
Week 6	Non-rigid SfM (guest)	Bundle adjustment I, Uncertainty
Week 7	Bundle adjustment II	Dense reconstruction

Today's Lecture

Minimal Solvers

- 7-point method
- Root finding for univariate polynomials
- Action matrix method
- ~~Gröbner bases~~
- 5-point method

Degeneracies

- Scenes dominated by a plane

Model Fitting: Minimal Solvers

- Recall RANSAC formula

$$T \geq \left\lceil \frac{\log(1-\alpha)}{\log(1-\varepsilon^s)} \right\rceil$$

- Fewer iterations when s is smaller
- Strong incentive to work with smallest $s = \text{d.o.f.}$
  - 8-point method for $\mathtt{F}$?
  - 8-point method for $\mathtt{E}$?
  - 6-point method for $\mathtt{P} = (\mathtt{R} \mid \mathbf{T})$?
  - These are all non-minimal
- Minimal solvers
  - Constraints coming from data are often linear
  - Internal constraints are usually polynomials
  - $\mathtt{F}$ has rank ≤ 2: $\det(\mathtt{F}) = 0$
  - $\mathtt{E}$ is essential matrix: $\det(\mathtt{E}) = 0$ and $2\mathtt{E}\mathtt{E}^\top\mathtt{E} = \mathrm{trace}(\mathtt{E}\mathtt{E}^\top)\mathtt{E}$
  - $\mathtt{R}$ is rotation matrix: e.g. $\mathtt{R}$ is induced by unit quaternion $\mathbf{q}$
- We need to solve system of polynomial equations (fast)

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7-Point Algorithm

- 7 point correspondences needed to estimate $\mathtt{F}$

$$\bar{\mathbf{x}}_i^\top \mathtt{F} \mathbf{x}_i = 0 \qquad i = 1, \dots, 7$$

- Reshape $\mathtt{F} \in \mathbb{R}^{3\times 3}$ to $\mathbf{f} \in \mathbb{R}^9$

$$\mathtt{M}\mathbf{f} = 0 \qquad \mathtt{M} \in \mathbb{R}^{7\times 9}$$

- $\mathbf{f}$ has to be in null-space of $\mathtt{M}$: $\mathbf{f} = t\mathbf{u}_1 + r\mathbf{u}_2$

$$\mathbf{u}_1, \mathbf{u}_2 \in \text{null}(\mathtt{M}) \qquad \mathbf{u}_1 \perp \mathbf{u}_2$$

- Fix scale of $\mathtt{F}$: $t + r = 1 \implies \mathbf{f} = t\mathbf{u}_1 + (1-t)\mathbf{u}_2 = \mathbf{u}_2 + t(\mathbf{u}_1 - \mathbf{u}_2)$
- $\det(\mathtt{F}) = 0$ yields cubic polynomial in t
  - 1 or 3 real solutions
- We solve a single univariate (cubic) polynomial

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7-Point Algorithm

- How to calulate $\det(\mathtt{F}) = \det(\mathtt{U}_2 + t(\mathtt{U}_2 - \mathtt{U}_1))$?
  - Coefficients c_i of polynomial $c_0 + c_1 t + c_2 t^2 + c_3 t^3$ from $\mathbf{u}_1$ & $\mathbf{u}_2$
- Use a CAS (Maple, Mathematica or Maxima)
- Maxima

```
/* t*N1 + (1-t)*N2 = N2 + t*(N1-N2) = N + t*D */
M: matrix([t*D(1,1)+N(1,1),t*D(1,2)+N(1,2),t*D(1,3)+N(1,3)], \
          [t*D(2,1)+N(2,1),t*D(2,2)+N(2,2),t*D(2,3)+N(2,3)], \
          [t*D(3,1)+N(3,1),t*D(3,2)+N(3,2),t*D(3,3)+N(3,3)]);

g: expand(determinant(M));

/* Disable "pretty" formatting */
display2d:false$

c3: factor(coeff(g, t^3));
c2: factor(coeff(g, t^2));
c1: factor(coeff(g, t));
c0: factor(g - c3*t^3 - c2*t^2 - c1*t);
```

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5-Point Algorithm

- 5 (normalized) point correspondences needed to estimate E

$$\bar{\mathbf{x}}_i \mathbf{E} \mathbf{x}_i = 0 \qquad i = 1, \dots, 7$$

- Reshape $\mathtt{E} \in \mathbb{R}^{3\times 3}$ to $\mathbf{e} \in \mathbb{R}^9$

$$\mathtt{M}\mathbf{e} = 0 \qquad \mathtt{M} \in \mathbb{R}^{5\times 9}$$

- $\mathbf{e}$ has to be in null-space of M: $\mathbf{e} = x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3 + w\mathbf{u}_4$

$$\mathbf{u}_1, \dots, \mathbf{u}_4 \in \mathrm{null}(\mathtt{M}) \qquad \mathbf{u}_i \perp \mathbf{u}_j$$

- Fix scale of E: $w = 1$ and $\mathbf{e} = x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3 + \mathbf{u}_4$
- $\det(\mathtt{E}) = 0$ and $2\mathtt{E}\mathtt{E}^\top \mathtt{E} = \mathrm{trace}(\mathtt{E}\mathtt{E}^\top)\mathtt{E}$ yield cubic polynomials in x, y and z
- What can we do?

5-Point Algorithm

- 5 (normalized) point correspondences needed to estimate $\mathtt{E}$

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Minimal Solvers — Polynomial Systems of Equations

Quick outlook

- Most minimal model fitting tasks in 3D computer vision require solving a polynomial system
- Resultant: determinant of the Sylvester matrix: coefficients $\in \mathbb{R}[y]$

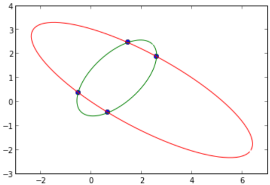
$$\det \begin{pmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_2 & b_1 & b_0 \end{pmatrix} = 0$$

  Yields univariate polynomial
- Action matrix method: non-symmetric eigenvalue problem

$$x\mathbf{m}(x,y) = \mathtt{M}_x^\top \mathbf{m}(x,y)$$

Solving Polynomial Systems of Equations

- Bezout's theorem: n polynomial equations each with degree d_i can have up to $\prod_{i=1}^{n} d_i$ (complex) solutions
  - Intersection of two ellipses: 2 polynomials of degree 2 $\implies$ up to four points



- 5-point method: ≥ 3 cubic polynomials, $3^3 = 27$ solutions?
- Luckily, worst case bound is almost never reached in 3D computer vision
- Solving polynomal systems is nevertheless very hard
  - Often requires clever reformulation/reparametrization of the problem

Solving Polynomial Systems of Equations

Methods to solve polynomial systems

- Numerical optimization / Newton's method for finding roots

$$\min_{x,y,z} \sum_i \|p_i(x,y,z)\|^2$$

  Can be slow, finds only one (or no) solution
- Resultant method
  - Reduces problem to univariate polynomial
  - Generally useful only for systems of 2 polynomials
- Action matrix method
  - Reduces problem to an eigenvalue problem
- Gröbner/standard bases
  - Reduces problem to univariate polynomial

Root Finding for Univariate Polynomials

Finding roots

Given $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{d-1} x^{d-1} + x^d$, determine solutions $x \in \mathbb{R}$ such that $p(x) = 0$.

- Up to d real solutions: we need to find all real roots
- Closed form expressions for $d \leq 4$
- Eigenvalues of the non-symmetric *companion matrix*

$$\mathtt{C}(p) = \begin{pmatrix} \mathbf{0} & \mathtt{I} \\ -a_0 & -\mathbf{a}_{1:d-1}^\top \end{pmatrix}$$

- Bisection method: requires $x^-, x^+ \in \mathbb{R}$ such that $p(x^-)p(x^+) \leq 0$
- Isolating real roots
  - Finding disjoint intervals $[x_k^-, x_k^+]$ such that $p(x_k^-)p(x_k^+) \leq 0$
  - Sturm chains
- *Jenkins-Traub algorithm*

Resultant Method

- Let

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$
$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

- Coefficients $a_i, b_j \in \mathbb{R}[y]$ (polynomial field)
- Sylvester matrix $\mathtt{S}_{p,q} \in \mathbb{R}[y]^{(m+n)\times(m+n)}$ $(m = 4,\, n = 3)$

$$\begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix}$$

- $\det(\mathtt{S}_{p,q}) = 0 \iff p(x)$ and $q(x)$ have a common root
  - $\det(\mathtt{S}_{p,q})$ is a polynomial in y

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Resultant Method: Example

- $p(x) = x^2 - y - 3$ and $q(x) = x(y-1)$
- Sylvester matrix $\mathtt{S}_{p,q}$

$$\mathtt{S}_{p,q} = \begin{pmatrix} 1 & 0 & -(y+3) \\ y-1 & 0 & 0 \\ 0 & y-1 & 0 \end{pmatrix}$$

- $\det(\mathtt{S}_{p,q}) = 0$ using Sarrus' scheme

$$\mathtt{S}_{p,q} = \begin{vmatrix} 1 & 0 & -(y+3) \\ y-1 & 0 & 0 \\ 0 & y-1 & 0 \end{vmatrix} = -(y+3)(y-1)^2$$

- We solve univariate cubic polynomial $-(y+3)(y-1)^2$: $y = -3$ or $y = 1$
- Given y, solve for x
  - $y = -3$: $x = 0$
  - $y = 1$: $x^2 = 4$, i.e. $x = \pm 2$

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Action Matrix Method

Algorithm

- Select a monomial basis.
- Apply the mapping $\mathcal{T}_x$ to the monomial basis and reduce the expressions until the result consists of monomials from the basis.
- Construct the action matrix $\mathtt{M}_x^\top$.
- Compute eigenvalues and eigenvectors of $\mathtt{M}_x^\top$.
- Extract the solutions from the eigenvectors.

Note: The theory guarentees that the solutions will be among the eigenvectors, but not all eigenvectors are solutions. Need to test the solutions.

Action Matrix Method

- Running example: $p(x,y) = x^2 - y - 3$ and $q(x,y) = x(y-1) = xy - x$
- What monomial basis should we choose?
- All monomials of degree 2: 1, x, y, x^2, xy and y^2
- Monomial basis

$$\mathbf{m}(x,y) = \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix}$$

- Every polynomial $r(x,y)$ (in our case of degree-2) can be written as

$$\mathbf{c}_r^\top \mathbf{m}(x,y)$$

- E.g.

$$p(x,y) = \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}^\top \mathbf{m}(x,y)$$

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Action Matrix Method

- We introduce operator $\mathcal{T}_x$ that multiplies a polynomial with x

$$\mathcal{T}_x p(x,y) = x(x^2 - y - 3) = x^3 - xy - 3x$$

  Analogously we define $\mathcal{T}_y$
- Creates new monomials outside our basis
- Idea: express new monomials in terms of our basis by using properties of a solution (x_0, y_0) (i.e. $p(x_0, y_0) = q(x_0, y_0) = 0$)

$$\mathcal{T}_x \mathbf{m}(x,y) = x\,\mathbf{m}(x,y) = \mathtt{M}_x^\top \mathbf{m}(x,y) \qquad \text{at } (x,y) = (x_0, y_0)$$

- x_0 is an eigenvalue of $\mathtt{M}_x^\top$ (and $\mathbf{m}(x_0, y_0)$ is a corresponding eigenvector)
- Necessary, but not sufficient condition
  - We need to verify which eigenvalue satisfies our polynomial system

Action Matrix Method

- Running example: $p(x, y) = x^2 - y - 3$ and $q(x, y) = x(y - 1) = xy - x$
- $\mathcal{T}_x \mathbf{m}(x, y) = (x, x^2, xy, x^3, x^2y, xy^2)^\top$
- Assume (x_0, y_0) satisfies $p(x_0, y_0) = 0$ and $q(x_0, y_0) = 0$
- New monomials x^3, x^2y and xy^2 can be rewritten at a root (x_0, y_0)

$$\begin{aligned}
1 &\mapsto x_0 \\
x_0 &\mapsto x_0^2 = y_0 + 3 \\
y_0 &\mapsto x_0 y_0 = x_0 \\
x_0^2 &\mapsto x_0^3 = x_0(y_0 + 3) = x_0 y_0 + 3x_0 = 4x_0 \\
x_0 y_0 &\mapsto x_0^2 y_0 = x_0^2 = y_0 + 3 \\
y_0^2 &\mapsto x_0 y_0^2 = x_0 y_0 = x_0
\end{aligned}$$

Action Matrix Method

- Running example: $p(x,y) = x^2 - y - 3$ and $q(x,y) = x(y-1) = xy - x$
- For a solution (x_0, y_0) we have

$$x_0 \mathbf{m}(x_0,y_0) = \begin{pmatrix} x_0 \\ x_0^2 \\ x_0 y_0 \\ x_0^3 \\ x_0^2 y_0 \\ x_0 y_0^2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0+3 \\ x_0 \\ 4x_0 \\ y_0+3 \\ x_0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}}_{=:\mathtt{M}_x^\top} \begin{pmatrix} 1 \\ x_0 \\ y_0 \\ x_0^2 \\ x_0 y_0 \\ y_0^2 \end{pmatrix}$$

- $\mathtt{M}_x$ is called the action matrix
- Analogous construction for $\mathtt{M}_y$

Action Matrix Method

- Running example: $p(x,y) = x^2 - y - 3$ and $q(x,y) = x(y-1) = xy - x$
- Action matrix

$$\mathtt{M}_x^\top = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $\mathtt{M}_x$ (and $\mathtt{M}_x^\top$) has eigenvalues -2, 2 and 0 (with multiplicity 4)

Action Matrix Method

- Running example: $p(x,y) = x^2 - y - 3$ and $q(x,y) = x(y-1) = xy - x$
- Note: we can use a smaller monomial basis

$$\mathbf{m}'(x,y) = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

Since monomials appearing in p and q are linear or linear multiplied with x

- Therefore

$$x_0 \mathbf{m}'(x_0,y_0) = \begin{pmatrix} x_0 \\ x_0^2 \\ x_0 y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 + 3 \\ x_0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{(\mathtt{M}'_x)^\top} \mathbf{m}'(x_0,y_0)$$

- $\mathtt{M}'_x$ (and $(\mathtt{M}'_x)^\top$) has eigenvalues -2, 2 and 0
- $\mathbf{m}'(x,y) = (1,x,y)^\top$ is not sufficient for $\mathcal{T}_y/\mathtt{M}'_y$

Action Matrix Method

Can we always use the action matrix method?

Unfortunately no. Often preprocessing necessary ($\rightsquigarrow$Gröbner basis). Other issues:

- There are $\binom{n+d}{d}$ monomials of degree $\leq d$ in n variables
  - 5-point method: $n = 3$, $d = 3$: 20 monomials
  - Can grow quickly for less "structured" problems
- Complex eigenvalue decomposition required
  - For general, non-symmetric real matrices

- Coefficients of polynomials depend on data
  - And therefore the entries of the action matrix $\mathtt{M}_x^\top$
  - But elimination steps will be the same
- Sometimes it can be benefical to use non-minimal solvers
  - Trade one more data points $s \rightsquigarrow s + k$ for a much simpler (and faster) solver
  - E.g. 7 or 8-point method instead of 5-point method
  - But: minimal solver often has fewer degeneracies

The 5-Point Method by Nistér

Finding an essential matrix can be done with five correspondences by solving

$$\begin{aligned} \bar{\mathbf{x}}_i^T \mathtt{E} \mathbf{x}_i &= 0 \qquad\qquad i = 1, \dots, 5 \\ \det(\mathtt{E}) &= 0 \\ 2\mathtt{E}\mathtt{E}^\top \mathtt{E} - \operatorname{trace}(\mathtt{E}\mathtt{E}^\top)\mathtt{E} &= 0 \end{aligned}$$

Form the system matrix $\mathtt{M}$ from the five correspondences. $\mathtt{M}$ has a 4 dimensional null-space:

$$\mathtt{M}(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4) = 0$$

Reshaping into matrices

$$\bar{\mathbf{x}}_i^T (\alpha_1 \mathtt{E}_1 + \alpha_2 \mathtt{E}_2 + \alpha_3 \mathtt{E}_3 + \alpha_4 \mathtt{E}_4) \mathbf{x}_i = 0 \quad i = 1, \dots, 5$$

The 5-Point Method

- Use the remaining equations to determine $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.
- Trace constraint, 9 equations

$$2\mathtt{E}\mathtt{E}^\top\mathtt{E} - \mathrm{trace}(\mathtt{E}\mathtt{E}^\top)\mathtt{E} = \sum_{i=1}^4\sum_{j=1}^4\sum_{k=1}^4 \alpha_i\alpha_j\alpha_k \left(2\mathtt{E}_i\mathtt{E}_j^\top\mathtt{E}_k - \mathsf{trace}(\mathtt{E}_i\mathtt{E}_j^\top)\mathtt{E}_k\right) \overset{!}{=} 0$$

One equation:

$$\det(\mathtt{E}) = \sum_{i=1}^4\sum_{j=1}^4\sum_{k=1}^4 \alpha_i\alpha_j\alpha_k \big(e_{11}^i e_{22}^j e_{33}^k + e_{12}^i e_{23}^j e_{31}^k + e_{13}^i e_{21}^j e_{32}^k$$

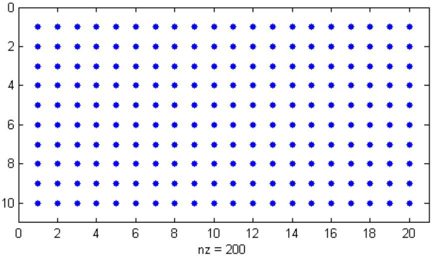
$$-e_{11}^i e_{23}^j e_{32}^k - e_{12}^i e_{21}^j e_{33}^k - e_{13}^i e_{22}^j e_{31}^k\big) \overset{!}{=} 0$$

The 5-Point Method

- α_1 can be assumed to be one (scale ambiguity)
- Lex order for monomials

$$\begin{aligned}(\alpha_4^3, \alpha_3\alpha_4^2, \alpha_3^2\alpha_4, \alpha_3^3, \alpha_2\alpha_4^2, \alpha_2\alpha_3\alpha_4, \alpha_2\alpha_3^2, \alpha_2^2\alpha_4, \alpha_2^2\alpha_3, \alpha_2^3,\\ \alpha_4^2, \alpha_3\alpha_4, \alpha_3^2, \alpha_2\alpha_4, \alpha_2\alpha_3, \alpha_2^2, \alpha_4, \alpha_3, \alpha_2, 1)\end{aligned}$$

- Gaussian elimination gives reductions for all third order terms
  - Lucky coincidence: 10 degree-3 monomials and 10 equations



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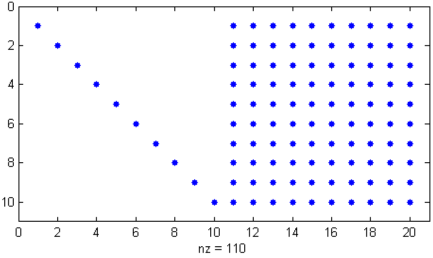
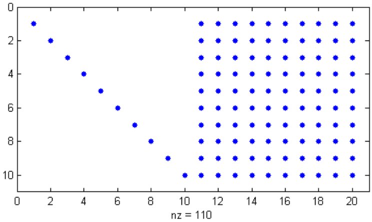


The 5-Point Method

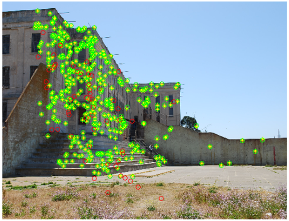
- We can express all degree-3 polynomials via degree-≤ 2 ones
- Action matrix $\mathtt{M}_{\alpha_i}$ is 10×10 matrix
- $\Longrightarrow$ up to 10 real solutions for $\mathtt{E}$



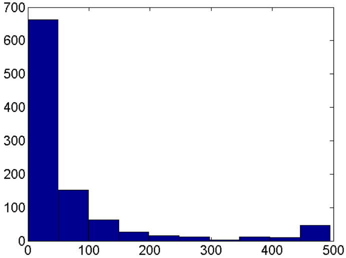
Remarks

- This is a simplified variant of the 5-point method
- Nistér's method smartly combines these polynomials to obtain a univariate one of degree 10
- Real roots determined via Sturm chains and bisection method
- Solver takes μs on moderate PC in 2003

The 5-Point Method

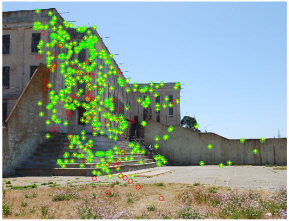


Histogram over the size of the consensus set in each iteration of RANSAC (1000 iterations), using 5 points, 8 points and 10 points respectively.



10 points (non-minimal)

The 5-Point Method



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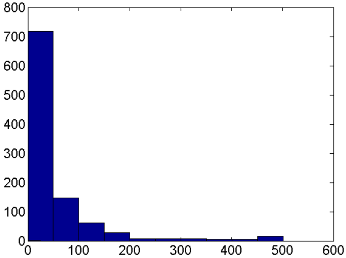
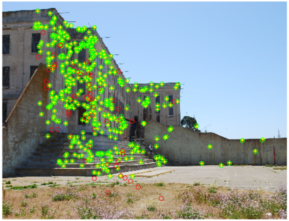


8 points (non-minimal)

The 5-Point Method



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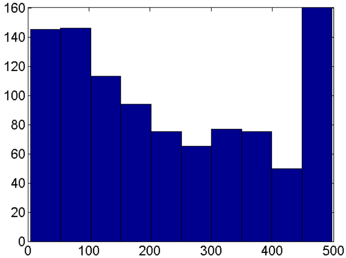


5 points

Degeneracies

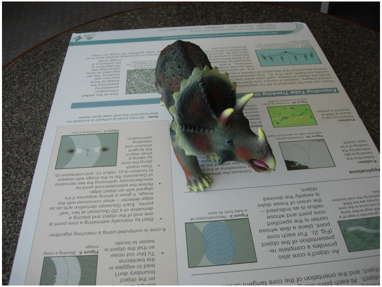
Discussion

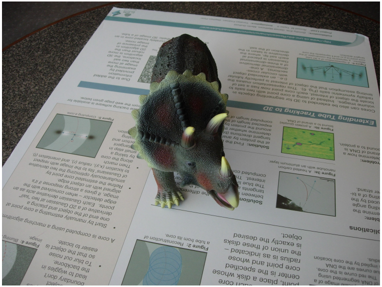
When will estimation of $\mathtt{F}/\mathtt{E}$ or a homography not work?
How can one detect such situations?

Degenerate Configurations: Planar Scenes

Question

Can we estimate F when the scene is planar (or the minimal sample can be described by a homography)?





- The answer is No
- The 7- and 8-point method will return unstable / non-unique estimates
- Man-made environments: scenes are often dominated by planes!

Degenerate Configurations: Planar Scenes

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Can we estimate F when the scene is planar (or the minimal sample can be described by a homography)?

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Degenerate Configurations: Planar Scenes

- Let $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
- There is a homography $\mathtt{H}$ such that $\mathbf{y}_i \sim \mathtt{H}\mathbf{x}_i$
- Epipolar constraint for any $\mathbf{x} \leftrightarrow \mathbf{y}$ with $\mathbf{y} \sim \mathtt{H}\mathbf{x}$

$$0 = \mathbf{y}^\top \mathtt{F}\mathbf{x} \sim (\mathtt{H}\mathbf{x})^\top \mathtt{F}\mathbf{x} = \mathbf{x}^\top \mathtt{H}^\top \mathtt{F}\mathbf{x} \qquad \forall \mathbf{x}$$

- Quadratic form $\mathbf{x}^\top \mathtt{Q}\mathbf{x}$, only symmetrized matrix matters

$$\text{Condition: } \mathbf{x}^\top (\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H})\mathbf{x} = 0 \qquad \forall \mathbf{x}$$

- Sufficient (and neceesary) condition: $\mathtt{H}^\top \mathtt{F} + \mathtt{F}^\top \mathtt{H} = 0$
  - $\mathtt{H}$ *compatible* with $\mathtt{F}$
- If all $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ come from a single 3D plane
  - 2-dimensional space of solutions for $\mathtt{F}$
  - 9 entries of $\mathtt{F}$ minus 6 skew-symmetric constraints minus scale freedom
- Also problematic if $\mathbf{x}_i \leftrightarrow \mathbf{y}_i$ are almost related by a homography

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Degenerate Configurations: Planar Scenes

What can we do?

Calibrated setup

- 5-point method to estimate $\mathtt{E}$ has only few issues with planar scenes
- **Find $\mathtt{H}$ and decompose into $\mathtt{R}$ and $\mathbf{T}$**

Uncalibrated setup (or ignoring calibration)

- Detect degeneracy and use a different method to estimate $\mathtt{F}$
  - Discard $\mathtt{F}$ if estimated from ≥ 5 planar correspondences
    - Might require many iterations
  - Better: DGENSAC

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Draw sample set S of 7 (8) correspondences
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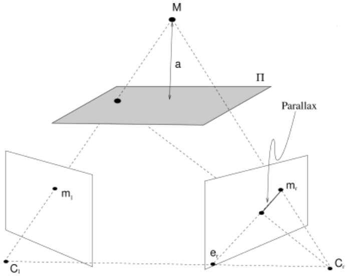
Plane + Parallax Method

- We have $\mathtt{H}$, what do we need to estimate $\mathtt{F}$?
  - 4 correspondences on the plane: $\{\mathbf{x}_i \leftrightarrow \mathbf{y}_i\}_{i=1,\dots,4}$
  - 2 correspondences off the plane: $\{\mathbf{x}_i \leftrightarrow \mathbf{y}_i\}_{i=5,6}$
- Epipolar constraint satisfied for

$$\mathbf{x}_5 \leftrightarrow \mathbf{y}_5 \qquad \mathbf{x}_5 \leftrightarrow \mathtt{H}\mathbf{x}_5 \qquad \mathbf{x}_6 \leftrightarrow \mathbf{y}_6 \qquad \mathbf{x}_6 \leftrightarrow \mathtt{H}\mathbf{x}_6$$

  - Epipole $\mathbf{e}_2$ is intersection of $\overline{(\mathbf{y}_5, \mathtt{H}\mathbf{x}_5)}$ and $\overline{(\mathbf{y}_6, \mathtt{H}\mathbf{x}_6)}$
- Fundamental matrix $\mathtt{F} = [\mathbf{e}_2]_\times \mathtt{H}$
  - Q: is $[\mathbf{e}_2]_\times \mathtt{H}$ an essential matrix in the calibrated setup?



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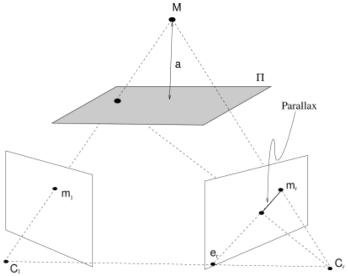

Discussion

We do not know in advance whether there is a dominant plane in the scene. What can we do?

- Search for a fundamental matrix $\mathtt{F}$ and discard degenerate samples?
- Search for a homography $\mathtt{H}$?
  - Any downsides of only looking for either $\mathtt{F}$ or $\mathtt{H}$?
- Can we search for $\mathtt{F}$ and $\mathtt{H}$ in parallel?
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To do

- Finish assignment 2, work on assignment 3

Lab sessions today: IDE-D2505-7, MTI2, MTI3, MTI4