Assignment 1 EEN020 Computer Vision

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Points in Homogeneous Coordinates

Theoretical Exercise 1

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{P}^2 \Rightarrow x = \begin{pmatrix} x/z \\ y/z \end{pmatrix} \in \mathbb{R}^2$$

$$x_1 = \begin{pmatrix} 4 \\ -16 \\ 2 \end{pmatrix} \Rightarrow x_1 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} -3 \\ 7 \\ -1 \end{pmatrix} \Rightarrow x_2 = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

$$x_3 = \begin{pmatrix} 9\lambda \\ -3\lambda \\ 6\lambda \end{pmatrix} \Rightarrow x_3 = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$$

The interpretation of

$$x_4 = \begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix}$$

is that it is a point at infinity because dividing the vector by 0 is not possible so if we instead divide by ϵ and taking the limit of ϵ to 0 we get infinity.

$$x_5 = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$$

is not the same point as x_4 in $\mathbb{P}^2$ but the same point in inhomogeneous coordinates since they are both points at infinity.

Computer Exercise 1

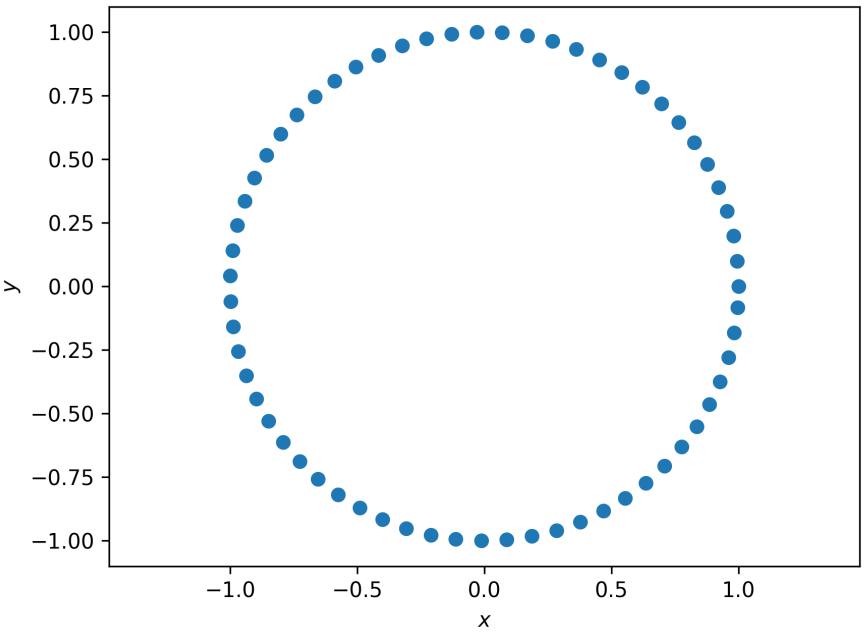

Figure 1: 2D plot of the points in `compEx1.mat`.

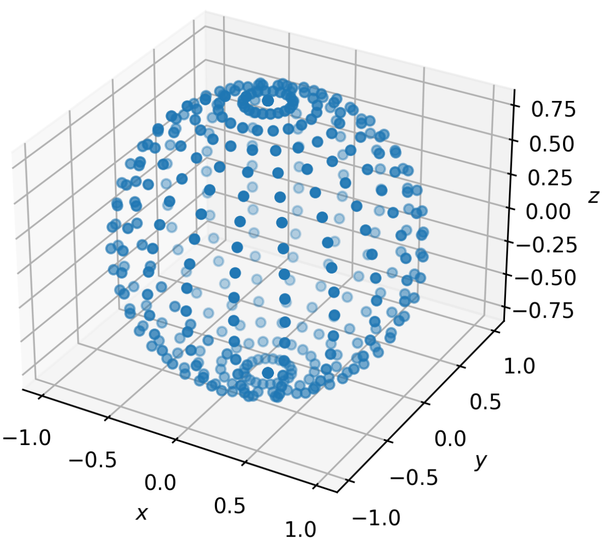

Figure 2: 3D plot of the points in `compEx1.mat`.

Lines

Theoretical Exercise 2

To compute the intersection of two (unparallel) lines we use the cross product because there exists only one point where there is an orthogonal vector to both lines, and this where the lines intersect. Given two lines l and l' we have an orthogonal vector $x = l \times l'$, because of the triple scalar product identity $l^T(l \times l') = l'^T(l \times l') = 0$ we have that $l^T x = l'^T x = 0$. So if x is a point then this point is unique and represents the intersection of the two lines.

The intersection of

$$l_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ and } l_2 = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$$

is

$$x = l_1 \times l_2 = \begin{pmatrix} -2 \\ 7 \\ -9 \end{pmatrix} \in \mathbb{P}^2 \Rightarrow x = \begin{pmatrix} 2/9 \\ -7/9 \end{pmatrix} \in \mathbb{R}^2$$

The intersection of

$$l_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ and } l_4 = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$$

is

$$x = l_3 \times l_4 = \begin{pmatrix} 0 \\ 17 \\ 0 \end{pmatrix} \in \mathbb{P}^2$$

This is a point at infinity in inhomogeneous coordinates and does therefore not belong to $\mathbb{R}^2$. The lines l_3 and l_4 are parallel in $\mathbb{R}^2$ because they do not intersect at a finite point.

By the same argument of calculating the intersection of two lines we can calculate the line intersecting two points. The line l_5 that goes through

$$x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \mathbb{R}^2 \Rightarrow x_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{P}^2 \text{ and } x_2 = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \in \mathbb{R}^2 \Rightarrow x_2 = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} \in \mathbb{P}^2$$

is

$$l_5 = x_1 \times x_2 = \begin{pmatrix} -2 \\ 7 \\ -9 \end{pmatrix}$$

Theoretical Exercise 3

We define

$$M = \begin{pmatrix} 6 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

The nullspace of M is $\mathcal{N}(M) = \{x \in \mathbb{P}^2; Mx = 0\}$. The intersection of l_1 and l_2 is $x = (-2,7,-9)^T$ as previously calculated. To see if $x \in \mathcal{N}(M)$ we calculate

$$Mx = \begin{pmatrix} 6 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 7 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which shows that x lies in the nullspace of M.

The intersection of the point in homogeneous coordinates of l_1 and l_2 is in the nullspace of M because for a point x to be on both lines we have that

$$0 = \begin{pmatrix} l_1^T x \\ l_2^T x \end{pmatrix} = \begin{pmatrix} l_1^T \\ l_2^T \end{pmatrix} x = Mx$$

Technically, there are other vectors that lies in the nullspace of M because scaling $x = (-2,7,-9)^T$ still satisfies $Mx = 0$. However, the direction of x is still unique because no other vectors in any other direction satisfies $Mx = 0$.

Computer Exercise 2

In figure 3 we can see that the lines are not parallel in the plot but seem to be parallel in 3D because they align along straight contours of the building that are presumably parallel in reality.

The distance between $l_2 \cap l_3$ and l_1 is ≈ 8.27 which is relatively small compared to the size of the image, which we can see in figure 3. A reason why the distance should be zero is because parallel lines in $\mathbb{R}^2$ vanish at a single point in $\mathbb{P}^2$. However, the contours of the building might not be perfectly parallel in reality. It could also be that the pair of points are not perfectly aligned on the contours of the building. Other factors may also play a role. But in principle all three lines should intersect at a single point in $\mathbb{P}^2$ if they are truly parallel in reality.

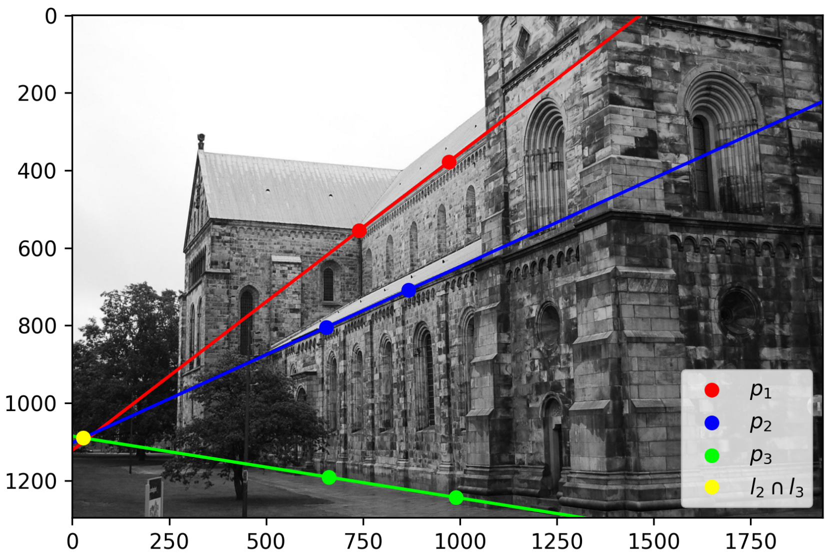

Figure 3: Plot of the three pair of points in `compEx2.mat`, the computed lines going through each pair of points, and the point of intersection between the lines going through p_2 and p_3.

Projective Transformations

Theoretical Exercise 4

Let

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

be the projective transformation,

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

be two points, and y_1 and y_2 be two points such that $y_1 \sim Hx_1$ and $y_2 \sim Hx_2$.

$$y_1 \sim Hx_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = y_1 \in \mathbb{P}^2$$

$$y_2 \sim Hx_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \tilde{y}_2 \in \mathbb{P}^2$$

where y_2 is a point at infinity.

$$l_1 = x_1 \times x_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$l_2 = y_1 \times \tilde{y}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$(H^{-1})^T l_1 = \frac{1}{2} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \sim l_2$$

Theoretical Exercise 5

$$0 = l_1^T x = l_1^T H^{-1} H x = [(H^{-1})^T l_1]^T H x \sim l_2^T y = 0$$

Thus, every projective transformation H preserves lines.

Theoretical Exercise 6

$$H_1 = \begin{pmatrix} \sqrt{3} & 0 & 1 \\ 1 & -\sqrt{3} & -1 \\ 0 & 0 & 1 \end{pmatrix}, \; H_2 = \begin{pmatrix} 1 & -\sqrt{3} & 1 \\ \sqrt{3} & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$
$$H_3 = \begin{pmatrix} \sqrt{5} & 1 & 1 \\ 1 & \sqrt{5} & 1 \\ 1/2 & 1/4 & 1 \end{pmatrix}, \; H_4 = \begin{pmatrix} 1 & -5 & 2 \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 2\sqrt{3} \end{pmatrix}$$

a)

H_1, H_2, and H_3 are projective transformations because they are invertable.

b)

H_1, and H_2 are affine transformations because they are invertable and their last row is $(0^T, 1)$ (if we scale H_2 with $1/2$ which is allowed).

c)

H_2 is a similarity transformation because it is invertable, its last row is $(0^T, 1)$ (if we scale it), and the upper left sub-matrix, sR, is a rotational matrix, i.e., it is invertable and $\det(sR)/s = 1$.

d)

H_2 is a Euclidean transformation because it is invertable, its last row is $(0^T, 1)$ (if we scale it), and the upper left sub-matrix, R, is an unscaled rotational matrix, i.e., it is invertable and $\det(R) = 1$.

e)

H_2 preserve lengths between points because it is a Euclidean transformation.

f)

H_1, H_2, and H_3 map lines to lines because they are projective transformations.

g)

H_1, and H_2 map parallel lines to parallel lines because they are affine transformations.

The Pinhole Camera

Theoretical Exercise 7

Let

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \text{ and } X_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

be homogeneous 3D coordinates in the camera and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

be the camera matrix. We have the following transformations

$$y_1 \sim PX_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \in \mathbb{P}^2$$

$$y_2 \sim PX_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \in \mathbb{P}^2$$

$$y_3 \sim PX_3 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \in \mathbb{P}^2$$

The projection of X_1 is a point at infinity in inhomogeneous coordinates.

We compute the camera center of P where $M = P_{1:3,1:3}$ as

$$C = -M^{-1}P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We compute the principal axis, a, where $M = P_{1:3,1:3}$ and $m = M_{3,1:3}$ as

$$a = \det(M)m = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Computer Exercise 3

Bename camera center as C_i and normalised principal axis as a_i for image $i \in \{1,2\}$.

$$C_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ with } a_1 \approx \begin{pmatrix} 0.31 \\ 0.95 \\ 0.84 \end{pmatrix}$$

$$C_2 \approx \begin{pmatrix} 6.64 \\ 14.85 \\ -15.07 \end{pmatrix}, \text{ with } a_2 \approx \begin{pmatrix} 0.03 \\ 0.34 \\ 0.94 \end{pmatrix}$$

In figure 4 we can see that the camera centers are aligned with how the images (see figure 5 and 6) are taken. The principal axes are directed towards the middle of the object from the camera centers, which we can observe in the images that the cameras are directed in similar directions towards the object.

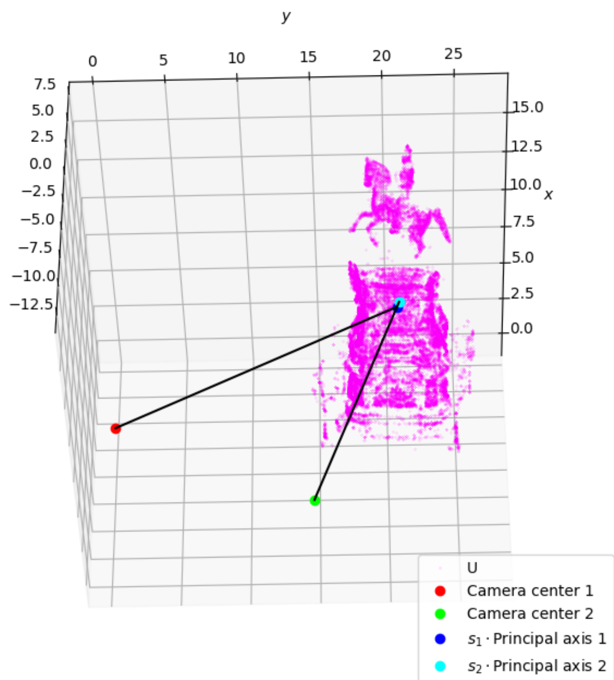

Figure 4: Plot of the 3D points together with the two different camera centers and their respective scaled principal axis.

In figure 5 and 6 we can see that the projected 3D points onto the images aligns accurately on the contours which indicates that the transformations are reasonable.

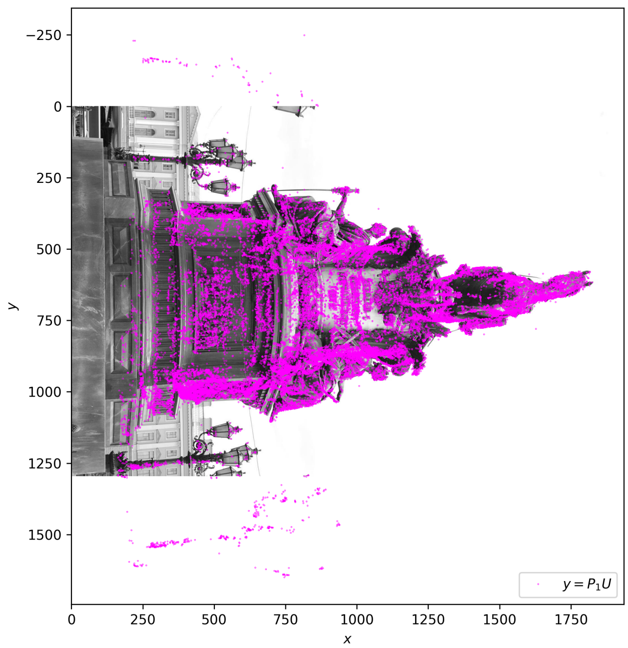

Figure 5: The pink points are the 3D points projected onto image 1 using the camera matrix P_1.

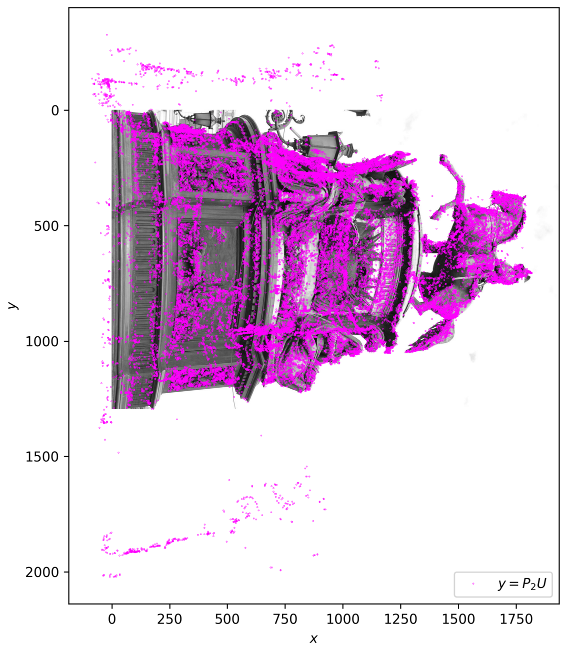

Figure 6: The pink points are the 3D points projected onto image 2 using the camera matrix P_2.

OPTIONAL

Theoretical Exercise 8

Given the camera matrix and a point

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \in \mathbb{P}^2$$

let

$$U(s) = \begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \\ s \end{pmatrix} \in \mathbb{P}^3$$

and

$$x \sim P_1 U(s) = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = x$$

This means that for any value of s, $U(s)$ will always project to x. It is not possible to determine s only using P_1 because P_1 eliminates s in the transformation.

To compute s we assume that U belongs to the plane

$$\Pi \begin{pmatrix} \pi \\ 1 \end{pmatrix}$$

where $\pi \in \mathbb{R}^3$, $U(s) = (x^T, s)$, and $\Pi^T U(s) = 0$, we have that

$$\Pi^T U(s) = 0 \Rightarrow \begin{pmatrix} \pi^T & 1 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = \pi^T x + s = 0 \Rightarrow s = -\pi^T x$$

Now we show that $(R - t\pi^T)$ is a homography H that maps x to y.

$$y \sim P_2 U(s) = \begin{pmatrix} R & t \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = Rx + ts = Rx - t\pi^T x = (R - t\pi^T)x = Hx$$

Computer Exercise 4

We can see in figure 7 that the origin of the coordinate system is in the upper left corner of the image.

Figure 7: Plot of original image, and corner points of the poster.

We can see in figure 8 that the origin of coordinate system in this case is in the left center of the normalised poster.

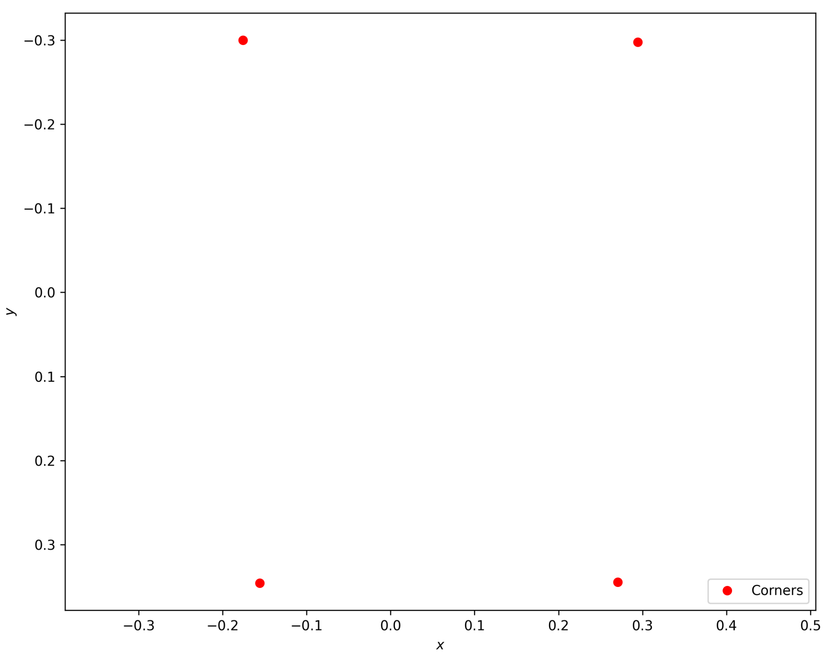

Figure 8: Normalised corner points from the original image.

We can see in figure 9, 10 and figure 11 from three different angles in 3D that the 3D points looks reasonable. Cameras are pointing towards the center of the poster, and also a little bit from above which we can also see in figure 7.

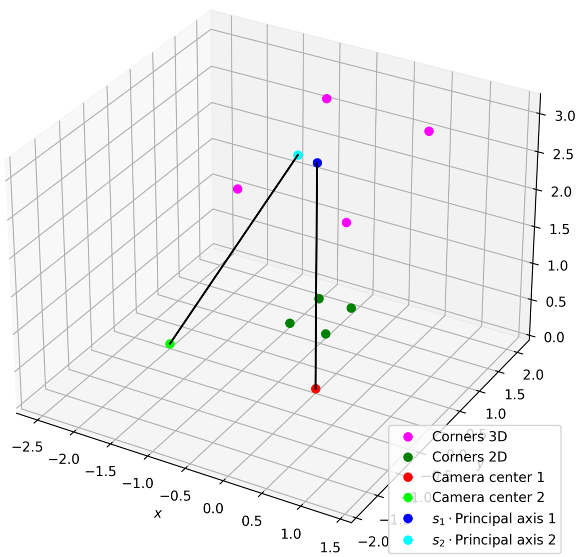

Figure 9: 3D plot of normalised corner points, projected corner points to 3D points, camera center 1 and 2, principal axes of camera 1 and 2.

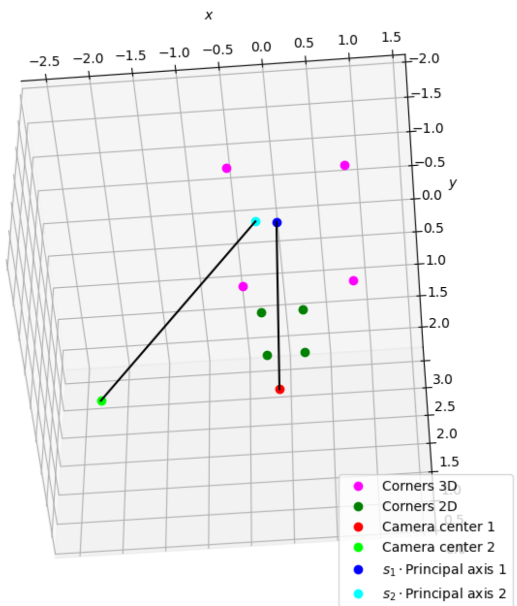

Figure 10: 3D plot of normalised corner points, projected corner points to 3D points, camera center 1 and 2, principal axes of camera 1 and 2.

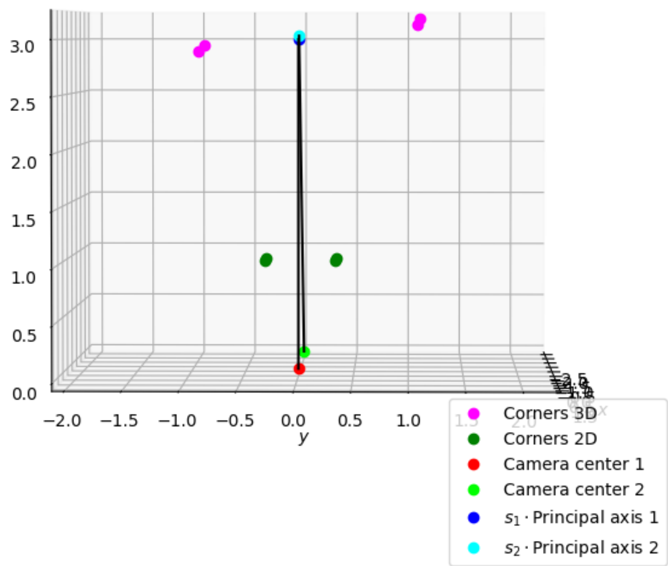

Figure 11: 3D plot of normalised corner points, projected corner points to 3D points, camera center 1 and 2, principal axes of camera 1 and 2. We can see here that the directions of the cameras come a little bit from above the poster as expected.

We can see in figure 12 that the camera is pointing towards the center of the poster from the left and a little bit from above as expected. The projected points from 3D to camera 2, as well as using homography transformation from camera 1 to camera 2 results in the same points.

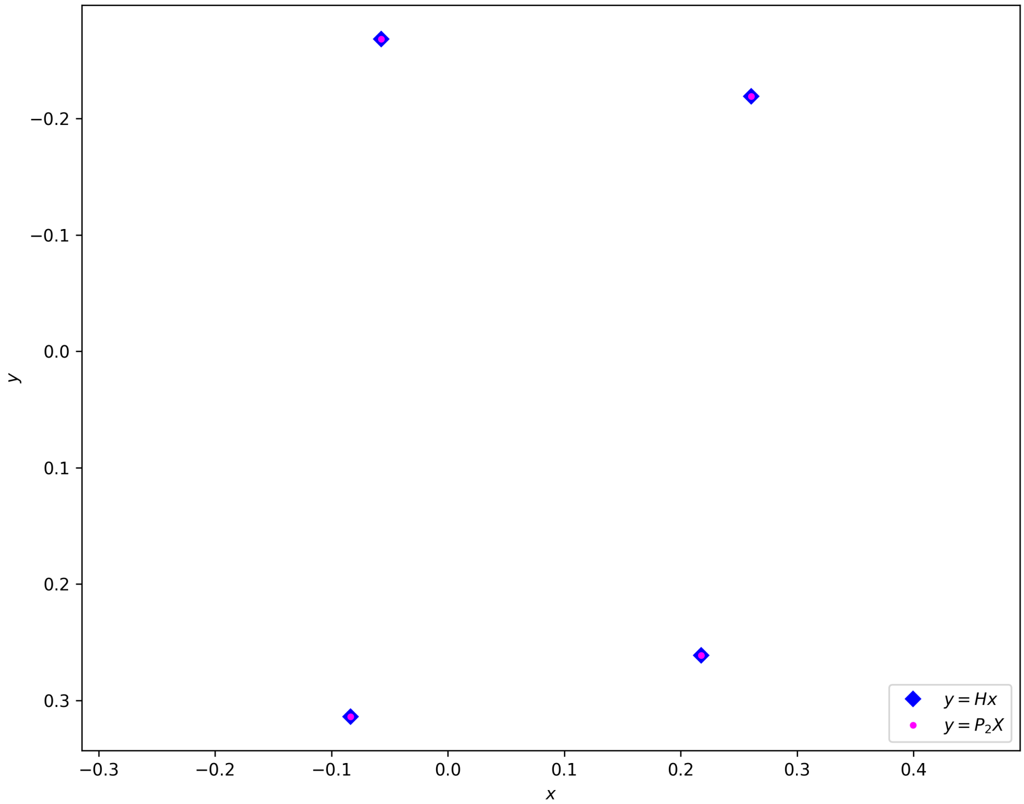

Figure 12: Transformed normalised corner points to camera 2 using homography, and projected corner points from 3D to camera 2. Pink points (projected 3D points) lies precisely on the blue points (transformed points using homography).

Figure 13 shows the original image transformed from camera 1 to camera 2 using the computed unnormalised homography, which shows the image from the left as expected.

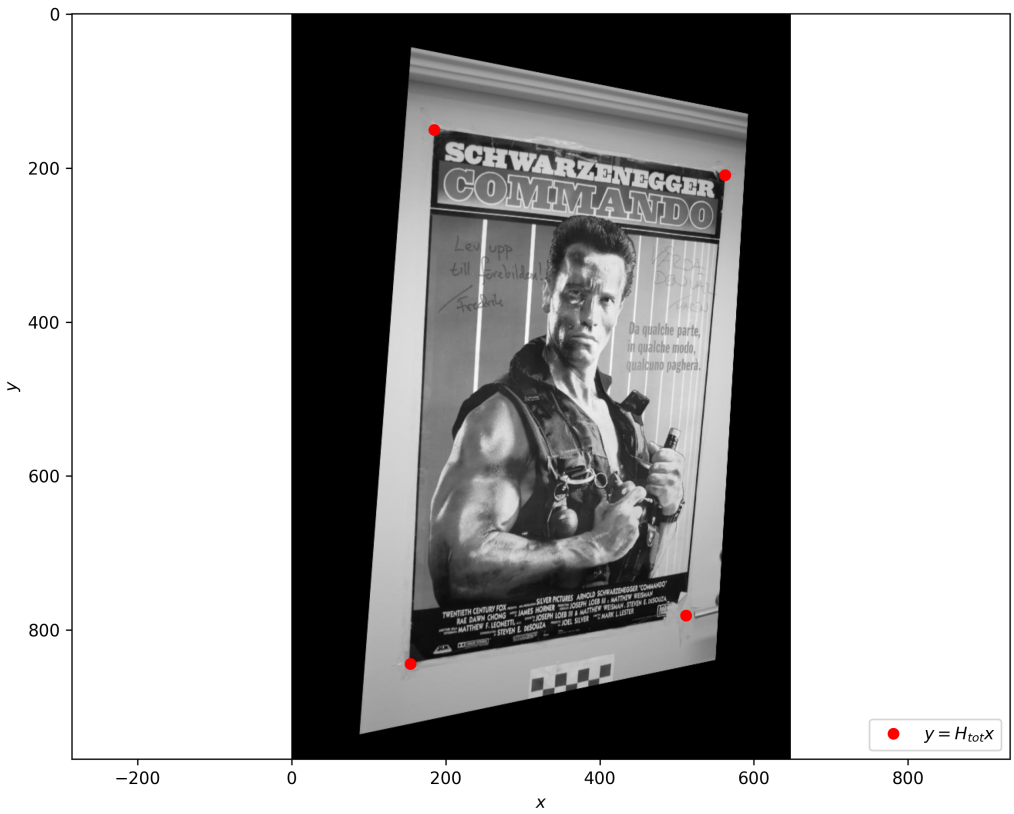

Figure 13: Transformed original image, and transformed unnormalised corner points using homography.