Computer Vision: Lecture 5

2023-11-13

Today's Lecture

Two-view geometry

- Relative orientation of two cameras
- The epipolar constraints
- The uncalibrated case: The Fundamental Matrix
- The 8-point algorithm

Camera Matrix Cheat Sheet

- Camera (projection) matrix $\mathtt{P} = \mathtt{K}(\mathtt{R} \mid \mathbf{T}) = (\mathtt{P}_{3,3} \mid \mathtt{P}_4)$
  - $\mathtt{R}$ is rotation matrix, $\mathtt{K} = \begin{pmatrix} \gamma f & sf & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix}$ is upper triangular calibration matrix
  - *Focal length* f and *principal point* (x_0, y_0)
  - Advice: always scale $\mathtt{P}$ such that $\|\mathtt{P}_{3,1:3}\| = 1$ and $\det(\mathtt{P}_{3,3}) > 0$ (why?)
- Image point $\mathbf{x} \sim \mathtt{P}\mathbf{X} \iff \lambda\mathbf{x} = \mathtt{P}\mathbf{X}$ for some $\lambda \neq 0$ $\quad (\mathbf{X} \in \mathbb{P}^3)$
  - Transform to camera coordinate system (CCS): $\mathbf{X}' = \mathtt{R}\mathbf{X} + \mathbf{T}$ $\quad (\mathbf{X} \in \mathbb{R}^3)$
  - Plus $\mathtt{K}$ and π: $\lambda\mathbf{x} = \mathtt{K}(\mathtt{R}\mathbf{X} + \mathbf{T}) = \mathtt{KR}(\mathbf{X} - \mathbf{C})$ $\quad (\mathbf{X} \in \mathbb{R}^3)$
  - Camera center $\mathbf{C}$ is origin in CCS: defining relation $\mathbf{0} = \mathtt{R}\mathbf{C} + \mathbf{T}$
  - How can we compute camera center given $\mathtt{P}$? Hint: what is $\mathtt{K}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?
- Transform 3D direction $\mathbf{d}$ to CCS: $\mathbf{d}' = \mathtt{R}\mathbf{d}$
  - Principal (or optical) axis $\mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (z-axis) in CCS
  - Principal axis $\mathbf{a}$ in WCS: defining relation $\mathbf{e}_z = \mathtt{R}\,\mathbf{a}$
  - How can we compute principal axis given $\mathtt{P}$? Hint: what is $\mathtt{P}_{3,1:3}$?
  - Principal point: image of the principical axis $\begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \sim \mathtt{P}_{3,3}\,\mathbf{a}$
- 3D line in WCS generated by image point $\mathbf{x}$: $\mathbf{X}(\lambda) = \lambda\mathtt{P}_{3,3}^{-1}\mathbf{x} - \mathtt{P}_{3,3}^{-1}\mathtt{P}_4$
- 2D points on line $\mathbf{l}$ are projections of 3D points on plane $\Pi = \mathtt{P}^\top\mathbf{l}$

Recap Taxonomy

	3D points	Camera matrices	Image points
Camera resectioning & pose estimation	Known	Unknown	Known
Triangulation	Unknown	Known	Known
Homography	Unknown but planar	Unknown	Known
Epipolar geometry, relative pose & SfM	**Unknown**	**Unknown**	**Known**

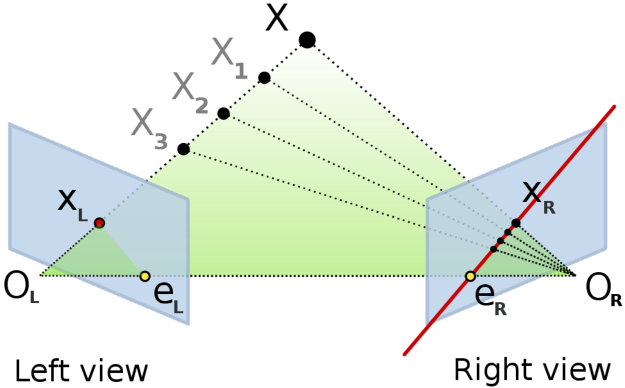
Epipolar Geometry: Introduction

Questions

Imagine two cameras are observing a 3D scene. You are camera 1.

- How does camera 1 see lines corresponding to image points in camera 2?
- Do these meet? If yes, where? If no, why not?

Epipolar Geometry: Introduction



An image is worth more than 1000 words.

Epipolar Geometry

- We simplify the setup
  - Calibrated cameras, and we work with normalized image points
  - $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$, $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$
    You are the center of the world
- 3D ray corresponding to image point $\mathbf{y}$ in camera 2 (in world coordinates)

$$\mathbf{X}_\lambda = \mathbf{C} + \lambda \mathtt{R}^\top \mathbf{y}$$

- The image of $\mathbf{X}_\lambda$ in camera 1

$$\mathbf{x}_\lambda \sim \lambda \mathtt{R}^\top \mathbf{y} + \mathbf{C}$$

Geometric intuition from last slide: $\{\mathbf{x}_\lambda : \lambda \in \mathbb{R}\}$ is a line

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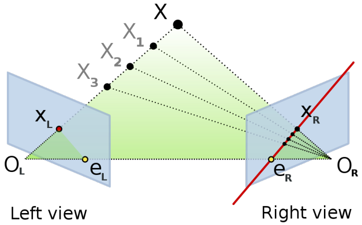


- X_λ lies on the plane spanned by $\mathbf{0}$, $\mathbf{C}$, $\mathbf{X} = \mathbf{X}_1$ $\qquad (\lambda = 1)$

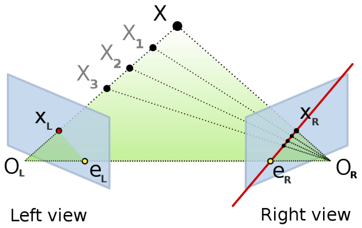
$$\Pi = \begin{pmatrix} \mathbf{X} \times \mathbf{C} \\ 0 \end{pmatrix}$$

- Remember: line-plane relation $\Pi = \mathtt{P}^\top \mathbf{l}$

$$\Pi = \begin{pmatrix} \mathbf{X} \times \mathbf{C} \\ 0 \end{pmatrix} = \mathtt{P}_1^\top \mathbf{l}_1 = \begin{pmatrix} \mathtt{I} \\ \mathbf{0}^\top \end{pmatrix} \mathbf{l}_1$$

Therefore $\mathbf{l}_1 = \mathbf{X} \times \mathbf{C}$

Epipolar Geometry



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Epipolar Geometry

- Recall $\mathbf{l}_1 = \mathbf{X} \times \mathbf{C}$
- Expand $\mathbf{C}$ and $\mathbf{X}$ and apply properties of $\times$

$$\begin{aligned}
\mathbf{l}_1 &= (\mathbf{C} + \mathtt{R}^\top \mathbf{y}) \times \mathbf{C} && \text{Distributive property} \\
&= \underbrace{\mathbf{C} \times \mathbf{C}}_{=\mathbf{0}} + (\mathtt{R}^\top \mathbf{y}) \times \mathbf{C} && \text{expand } \mathbf{C} \\
&= (\mathtt{R}^\top \mathbf{y}) \times \mathbf{C} = -(\mathtt{R}^\top \mathbf{y}) \times (\mathtt{R}^\top \mathbf{T}) && \text{Rotation equivariance} \\
&= -\mathtt{R}^\top (\mathbf{y} \times \mathbf{T}) && \text{anticommutative} \\
&= \mathtt{R}^\top (\mathbf{T} \times \mathbf{y}) = \mathtt{R}^\top [\mathbf{T}]_\times \mathbf{y} && \text{skew-symm. matrix}
\end{aligned}$$

- Skew-symmetric matrix

$$[\mathbf{a}]_\times := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \qquad [\mathbf{a}]_\times + [\mathbf{a}]_\times^\top = 0$$

Rank 2 for $\mathbf{a} \neq \mathbf{0}$

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- $\mathbf{l}_1 = \mathtt{R}^\top [\mathbf{T}]_\times \mathbf{y}$
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Epipolar geometry (normalized image points)

Given camera matrices $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$. Corresponding image points $\mathbf{x}$ (in camera 1) and $\mathbf{y}$ (in camera 2) satisfy the relation

$$\mathbf{y}^\top \mathtt{E}\,\mathbf{x} = 0,$$

where $\mathtt{E} := [\mathbf{T}]_\times \mathtt{R}$ is called the *essential matrix*.
$\mathtt{E}^\top \mathbf{y}$ is the *epipolar line* in camera 1, and $\mathtt{E}\,\mathbf{x}$ is the epipolar line in camera 2.

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Epipolar Geometry: A Direct Derivation in a Single Slide

- As before we assume $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$
- 3D line generated by $\mathbf{x}$ in camera 1 is just $\lambda_1 \mathbf{x}$ with $\lambda_1 \in \mathbb{R}$
- Project into 2nd camera, where it should align with image point $\mathbf{y}$:

$$\lambda_2 \mathbf{y} = \lambda_1 \mathtt{R}\mathbf{x} + \mathbf{T} \implies \mathbf{y} \times (\lambda_1 \mathtt{R}\mathbf{x} + \mathbf{T}) = \mathbf{0} \iff \lambda_1 \mathbf{y} \times (\mathtt{R}\mathbf{x}) = -\mathbf{y} \times \mathbf{T}$$

- Apply cross-product trick another time to eliminate λ_1:

$$(\mathbf{y} \times \mathbf{T}) \times (\mathbf{y} \times (\mathtt{R}\mathbf{x})) = \mathbf{0}$$

- A useful identity: $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) = (\mathbf{a}^\top (\mathbf{b} \times \mathbf{c}))\mathbf{a}$:

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- Since $\mathbf{y} \neq \mathbf{0}$ it must hold that

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Epipolar Geometry

- What if $\mathbf{x}$ and $\mathbf{y}$ are given in sensor coordinates?

$$\mathbf{x} = \mathtt{K}_1 \tilde{\mathbf{x}} \qquad \mathbf{y} = \mathtt{K}_2 \tilde{\mathbf{y}}$$

$$\tilde{\mathbf{x}} = \mathtt{K}_1^{-1} \mathbf{x} \qquad \tilde{\mathbf{y}} = \mathtt{K}_2^{-1} \mathbf{y}$$

- Epipolar relation between corresponding normalized image points

$$\tilde{\mathbf{y}}^\top \mathbf{E} \tilde{\mathbf{x}} = 0 \iff \mathbf{y}^\top \underbrace{\mathtt{K}_2^{-\top} \mathtt{E} \mathtt{K}_1^{-1}}_{=:\mathtt{F}} \mathbf{x} = 0$$

Epipolar geometry (general case)

Given camera matrices $\mathtt{P}_1 = \mathtt{K}_1(\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = \mathtt{K}_2(\mathtt{R} \mid \mathbf{T})$. Corresponding image points $\mathbf{x}$ (in camera 1) and $\mathbf{y}$ (in camera 2) satisfy the relation

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Epipolar Geometry

Properties of $\mathtt{F}$

- $\mathtt{F} \in \mathbb{R}^{3\times 3}$
- Can $\mathtt{F}$ have full rank? Why? Why not?
- What is $\mathrm{null}(\mathtt{F})$ and $\mathrm{null}(\mathtt{F}^\top)$?
- How many d.o.f. does $\mathtt{F}$ have?

Epipolar Geometry

Properties of $\mathtt{F}$

- $\mathtt{F} \in \mathbb{R}^{3\times 3}$
- $\mathtt{F}$ has rank 2
- Left epipole $\mathbf{e}_1$
  - Image of camera center 2 in camera 1
  - $\mathbf{e}_1 \sim \mathrm{null}(\mathtt{F})$
- Right epipole $\mathbf{e}_2$
  - Image of camera center 1 in camera 2
  - $\mathbf{e}_2 \sim \mathrm{null}(\mathtt{F}^\top)$
- $\mathtt{F}$ has 7 d.o.f.

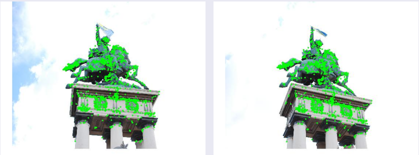
Epipolar Geometry

Epipolar Geometry

The projected lines should all meet in a point. The so called **epipole** is the projection of the camera center of the other camera.

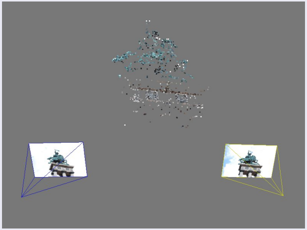
Relative Orientation: Problem Formulation

Given



Two images and corresponding points.

Compute



The structure (3D-points) and the motion (camera matrices).

Relative Orientation: Problem Formulation

Mathematical Formulation

Given two sets of corresponding points $\{\mathbf{x}_i\}$ and $\{\bar{\mathbf{x}}_i\}$, compute camera matrices $\mathtt{P}_1$, $\mathtt{P}_2$ and 3D-points $\{\mathbf{X}_i\}$ such that

$$\lambda_i \mathbf{x}_i = \mathtt{P}_1 \mathbf{X}_i$$

and

$$\bar{\lambda}_i \bar{\mathbf{x}}_i = \mathtt{P}_2 \mathbf{X}_i.$$

Estimating the Fundamental Matrix

Estimating $\mathtt{F}$

If $\mathbf{x}_i$ and $\bar{\mathbf{x}}_i$ corresponding points

$$\bar{\mathbf{x}}_i^T \mathtt{F} \mathbf{x}_i = 0.$$

If $\mathbf{x}_i = (x_i, y_i, z_i)^\top$ and $\bar{\mathbf{x}}_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i)^\top$ then

$$\begin{aligned}\bar{\mathbf{x}}_i^T \mathtt{F} \mathbf{x}_i &= F_{11}\bar{x}_i x_i + F_{12}\bar{x}_i y_i + F_{13}\bar{x}_i z_i \\ &+ F_{21}\bar{y}_i x_i + F_{22}\bar{y}_i y_i + F_{23}\bar{y}_i z_i \\ &+ F_{31}\bar{z}_i x_i + F_{32}\bar{z}_i y_i + F_{33}\bar{z}_i z_i\end{aligned}$$

The Fundamental Matrix

Estimating $\mathtt{F}$

In matrix form (one row for each correspondence):

$$\underbrace{\begin{bmatrix} \bar{x}_1 x_1 & \bar{x}_1 y_1 & \bar{x}_1 z_1 & \dots & \bar{z}_1 z_1 \\ \bar{x}_2 x_2 & \bar{x}_2 y_2 & \bar{x}_2 z_2 & \dots & \bar{z}_2 z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{x}_n x_n & \bar{x}_n y_n & \bar{x}_n z_n & \dots & \bar{z}_n z_n \end{bmatrix}}_{\mathtt{M}} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ \vdots \\ F_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- More concise: each row of $\mathtt{M}$ is

$$\operatorname{vec}(\bar{\mathbf{x}}_i \mathbf{x}_i^\top)^\top$$

- $\operatorname{vec}(\mathtt{A}) \in \mathbb{R}^{nm}$ stacks columns of $\mathtt{A} \in \mathbb{R}^{n \times m}$

The Fundamental Matrix

Estimating F

In matrix form (one row for each correspondence):

$$\underbrace{\begin{bmatrix} \bar{x}_1 x_1 & \bar{x}_1 y_1 & \bar{x}_1 z_1 & \dots & \bar{z}_1 z_1 \\ \bar{x}_2 x_2 & \bar{x}_2 y_2 & \bar{x}_2 z_2 & \dots & \bar{z}_2 z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{x}_n x_n & \bar{x}_n y_n & \bar{x}_n z_n & \dots & \bar{z}_n z_n \end{bmatrix}}_{\mathtt{M}} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ \vdots \\ F_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Solve using homogeneous least squares (SVD).
$\mathtt{F}$ has 9 entries (but the scale is arbitrary). Need at least 8 equations (point correspondences).

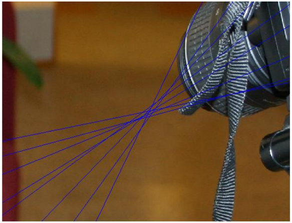
The Fundamental Matrix

Issues

Resulting F may not have $\det(F) = 0$.
Pick the closest matrix A with $\det(A) = 0$.

Can be solved using SVD, in Matlab:

$$\begin{aligned}&[\mathtt{U}, \mathtt{S}, \mathtt{V}] = \mathtt{svd}(\mathtt{F});\\&\mathtt{S}(3,3) = 0;\\&\mathtt{A} = \mathtt{U} * \mathtt{S} * \mathtt{V}';\end{aligned}$$

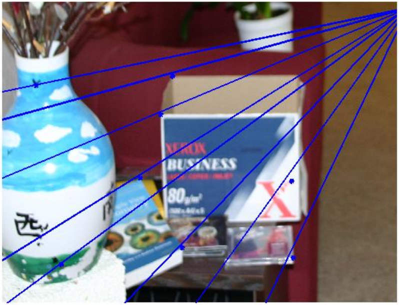


The Fundamental Matrix

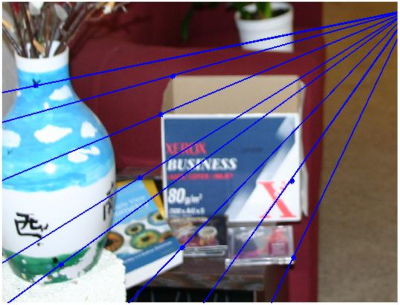
Issues

Normalization needed (see DLT).
If x_1 and $\bar{x}_1 \approx 1000$ pixels, the coefficients $z_1\bar{z}_1 = 1$, $x_1\bar{z}_1 = 1000$ and $x_1\bar{x}_1 = 1000000$. May give poor numerics.

Not normalized:



Normalized:



The Fundamental Matrix

The 8-point algorithm

- Extract at least 8 point correspondences.
- Normalize the coordinates (see DLT).
- Form $\mathtt{M}$ and solve
$$\min_{\|\mathbf{v}\|^2=1} \|\mathtt{M}\mathbf{v}\|^2$$
using SVD.
- Form the matrix $\mathtt{F}$ (ensure that $\det(\mathtt{F}) = 0$).
- Transform back to the original coordinates.
- Compute a pair of cameras from $\mathtt{F}$ (next lecture).
- Compute the scene points (Triangulation).

To do

- Lab session after this lecture: E-D2480, ES61, ES62 & ES63
- Next time: Calibrated epipolar geometry.
- Continue to work on Assignment 2.
- Search for "The Fundamental Matrix Song" on youtube.

More reading:

- Szeliski, Section 11.2 on "Pose Estimation" (including DLT & triangulation)
- Szeliski, Section 11.3 on "Two-Frame Structure from Motion"
- Hartley & Zisserman, Chapter 9 "Epipolar Geometry and the Fundamental Matrix" is free
  `http://www.robots.ox.ac.uk/~vgg/hzbook/hzbook2/HZepipolar.pdf`