Computer Vision, Assignment 3
Epipolar Geometry

1 Instructions

In this assignment you will study epipolar geometry. You will use the fundamental matrix and the essential matrix for simultaneously reconstructing the 3D structure and the camera motion from two images.

Please see Canvas for detailed instructions on what is expected for a passing/higher grade. All exercises not marked **OPTIONAL** are “mandatory” in the sense described on Canvas.

The maximum amount of core for the theoretical exercises in Assignment 3 is 18. 50% of 18 is 9.

2 The Fundamental Matrix

Theoretical Exercise 1 (3 points).

If $P_1 = [I\ 0]$ and

$$P_2 = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}$$

Compute the fundamental matrix.

Suppose the point $x = (0, 2)$ is the projection of a 3D-point $\mathbf{X}$ into P_1. Compute the epipolar line in the second image generated from x.

Which of the points $(1, 0)$, $(3, 2)$ and $(1, 1)$ could be a projection of the same point $\mathbf{X}$ into P_2?

For the report: Answers are enough.

Theoretical Exercise 2 (3.5 points).

If $P_1 = [I\ 0]$ and

$$P_2 = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 3 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}. \tag{2}$$

Compute the epipoles, by projecting the camera centers.

Compute the fundamental matrix, its determinant and verify that $e_2^T F = 0$ and $Fe_1 = 0$.

For the report: Complete solution.

Theoretical Exercise 3 (**OPTIONAL**, 12 optional points).

For a general camera pair $P_1 = [I\ 0]$ and $P_2 = [A\ t]$. Compute the epipoles, by projecting the camera centers. (You may assume that A is invertible.)

Verify that for the fundamental matrix $F = [t]_\times A$ the epipoles will always fulfill $e_2^T F = 0$ and $Fe_1 = 0$.

Given the above result explain why the fundamental matrix has to have determinant 0.

For the report: Complete solution.

Theoretical Exercise 4 (1 point). When computing the fundamental matrix F using the 8-point algorithm it is recommended to use normalization. Suppose the image points have been normalized using

$$\tilde{\mathbf{x}}_1 \sim N_1 \mathbf{x}_1 \text{ and } \tilde{\mathbf{x}}_2 \sim N_2 \mathbf{x}_2. \tag{3}$$

If $\tilde{F}$ fulfills $\tilde{\mathbf{x}}_2^T \tilde{F} \tilde{\mathbf{x}}_1 = 0$ what is the fundamental matrix F that fulfills $\mathbf{x}_2^T F \mathbf{x}_1 = 0$ for the original (un-normalized) points?

For the report: Answer is enough.

Computer Exercise 1.

In this exercise you will compute the fundamental matrix for the two images in Figure 1 of a part of the fort Kronan in Gothenburg.

The file `compEx1data.mat` contains a cell `x` with matched points for the two images.

Figure 1: kronan1.jpg and kronan2.jpg.

Part I. Compute normalization matrices N_1 and N_2. These matrices should subtract the mean and re-scale using the standard deviation, as in assignment 2. Normalize the image points of the two images with N_1 and N_2 respectively.

Set up the matrix M in the eight point algorithm (use all the points), and solve the homogeneous least squares system using SVD. Check that the minimum singular value and $||Mv||$ are both small. Construct the normalized fundamental matrix $\tilde{F}$ from the solution v. Write the code as a function `estimate_F_DLT(x1s, x2s)` which would be useful later.

Don't forget to make sure that $\det(\tilde{F}) = 0$ for your solution. Having a separate function `enforce_fundamental(F_approx)` that does that will be convenient for you. Check that the epipolar constraints $\tilde{\mathbf{x}}_2^T \tilde{F} \tilde{\mathbf{x}}_1 = 0$ are roughly fulfilled.

Compute the un-normalized fundamental matrix F (using the formula from exercise 4) and the epipolar lines $l = F\mathbf{x}_1$. Pick 20 points in the second image at random and plot these in the same figure as the image. Also plot the corresponding epipolar lines in the same image using the function `rital.m`. Verify that the points are close to the corresponding epipolar lines.

Compute the distance between all the points and their corresponding epipolar lines and plot these in a histogram with 100 bins. What is the mean distance? You should create a function `compute_epipolar_errors(F, x1s, x2s)` as this would be useful later.

Part II. See what happens if we do everything without normalization (that is, set $N_1 = N_2 = I$ and run the code again). What is the mean distance in this case?

```
Useful matlab commands:

xx = x2n(:,i)*x1n(:,i)'; %Computes a 3x3 matrix containing all multiplications
%of coordinates from x1n(:,i) and x2n(:,i).
M(i,:) = xx(:)'; %Reshapes the matrix above and adds to the M matrix

Fn = reshape(v,[3 3]);
%Forms an F-matrix from the solution v of the leat squares problem

plot(diag(x2n'*Fn*x1n));
%Computes and plots all the epipolar constraints (should be roughly 0)

l = F*x{1};  %Computes the epipolar lines
l = l./sqrt(repmat(l(1,:).^2 + l(2,:).^2,[3 1]));
%Makes sure that the line has a unit normal
%(makes the distance formula easier)
hist(abs(sum(l.*x{2})),100);
%Computes all the the distances between the points
%and there corresponding lines, and plots in a histogram
```

For the report: Submit the m-file, the fundamental matrix for the original (un-normalized) points (where $F(3,3) = 1$, use $F = F./F(3,3)$), the histogram and the plot of the epipolar lines. Briefly answer the questions in the text.

Theoretical Exercise 5 (3.5 points).

Consider the fundamental matrix

$$F = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}.$$

Verify that the projection of the scene points $(0, 3, 1)$ and $(-1, 2, 0)$ in the cameras $P_1 = [I\ 0]$ and $P_2 = [[e_2]_\times F\ e_2]$, fulfill the epipolar constraint ($x_2^T F x_1 = 0$). What is the camera center of P_2?

For the report: Complete solution.

3 The Essential Matrix

Theoretical Exercise 6 (**OPTIONAL**, 13 optional points).

The goal of this exercise is to show that an essential matrix has two nonzero identical singular values.

Suppose the 3×3 skew symmetric matrix $[t]_\times$ has a singular value decomposition

$$[t]_\times = USV^T, \tag{4}$$

where U, V are orthogonal and S diagonal with non-negative elements. Show that the eigenvalues of $[t]_\times^T [t]_\times$ are the squared singular values. (Hint: Show that $S^T S = S^2$ diagonalizes $[t]_\times^T [t]_\times$, see your linear algebra book.)

Verify that if w satisfies

$$-t \times (t \times w) = \lambda w. \tag{5}$$

for some λ, then w is an eigenvector of $[t]_\times^T [t]_\times$ with eigenvalue λ.

In "Linjär Algebra" by Sparr (on page 96) we find the formula

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w. \tag{6}$$

Show that $w = t$ is an eigenvector to $[t]_\times^T [t]_\times$ with eigenvalue 0 and that any w that is perpendicular to t is an eigenvector with eigenvalue $||t||^2$. Are these all of the eigenvectors?

Show that the singular values of $[t]_\times$ are $0, ||t||$ and $||t||$.

If $E = [t]_\times R$ and $[t]_\times$ has the SVD in (4), state an SVD of E. What are the singular values of E?

For the report: A complete solution.

Computer Exercise 2.

The file `compEx2data.mat` contains the calibration matrix K for the two images in Computer Exercise 1. Normalize the image points using the inverse of K.

Set up the matrix M in the eight point algorithm, and solve the homogeneous least squares system using SVD. Check that the minimum singular value and Mv are both small. You can of course re-use function `estimate_F_DLT`.

Construct the essential matrix from the solution v. Don't forget to make sure that E has two equal singular values and the third one zero. As the essential matrix is a homogeneous entity, scale does not matter, but for grading purposes, please also make sure that the two non-zero singular values are both equal to 1. Check that the epipolar constraints $\tilde{\mathbf{x}}_2^T E \tilde{\mathbf{x}}_1 = 0$ are roughly fulfilled.

Compute the fundamental matrix for the un-normalized coordinate system from the essential matrix and compute the epipolar lines $l = F\mathbf{x}_1$. Pick 20 of the detected points in the second image at random and plot these in the same figure as the image. Also plot the corresponding epipolar lines in the same figure using the function `rital.m`.

Compute the distance between the points and their corresponding epipolar lines and plot these in a histogram with 100 bins. How does this result compare to the corresponding result in Computer Exercise 1?

```
Useful matlab commands:

[U,S,V] = svd(Eapprox);
if det(U*V') < 0
    V = -V;
end
E = U*diag([1 1 0])*V';
% Creates a valid essential matrix from an approximate solution.
% Note: Computing svd on E may still give U and V that does not fulfill
% det(U*V') = 1 since the svd is not unique.
% So don't recompute the svd after this step.
```

Write the following functions that might be useful later for the project:

- `enforce_essential(E_approx)` for the code given above.
- `convert_E_to_F(E,K1,K2)` that gives you a fundamental matrix from an essential matrix and the two calibration matrices corresponding to the two images (note that in our case $K1 = K2 = K$).

For the report: Submit the m-file, the essential matrix (scaled such that the non-zero singular values are 1), the histogram and the plot of the epipolar lines. Report the average point-to-line distance in pixels, and a comparison with the result of Computer Exercise 1.

Theoretical Exercise 7 (7 points).

Suppose that an essential matrix E has the singular value decomposition

$$E = U\text{diag}([1\ 1\ 0])V^T \tag{7}$$

where

$$U = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{8}$$

Verify that $\det(UV^T) = 1$.

Compute the essential matrix and verify that $x_1 = (2, 0)$ (in camera 1) and $x_2 = (1, -3)$ (in camera 2) is a plausible correspondence.

If x_1 is the projection of $\mathbf{X}$ in $P_1 = [I\ 0]$ show that $\mathbf{X}$ must be one of the points

$$\mathbf{X}(s) = \begin{pmatrix} 2 \\ 0 \\ 1 \\ s \end{pmatrix}. \tag{9}$$

For each of the solutions

$$P_2 = [UWV^T \ u_3] \text{ or } [UWV^T \ -u_3] \text{ or } [UW^TV^T \ u_3] \text{ or } [UW^TV^T \ -u_3], \tag{10}$$

where

$$W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

and u_3 is the third column of U, compute s such that $\mathbf{X}(s)$ projects to x_2.

For what choice of P_2 is the camera pair (P_1, P_2) valid, such that the 3D point $\mathbf{X}(s)$ is in front of both cameras? HINT: For a calibrated camera $P = [R, t]$, unscaled such that $\det R = 1$, and for a finite 3D point $\mathbf{X}$ in homogeneous coordinates, unscaled such that $\mathbf{X}_4 = 1$, $P\mathbf{X}$ yields the cartesian coordinates of the 3D point in the camera coordinate frame.

For the report: Complete solution.

Computer Exercise 3.

For the essential matrix obtained in Computer Exercise 2 compute the four camera solutions in (10) (make sure that `det(U*V') > 0`; otherwise set `V = -V`). Write a function `extract_P_from_E(E)` that does this for you, it will also be useful in the future. The return type of `P` would be a cell of 4 arrays from (10).

Triangulate the points using DLT for each of the four camera solutions, and determine for which of the solutions the points are in front of the cameras. (Since there is noise involved it might not be possible to find a solution with all points in front of the cameras. In that case select the one with the highest number of points in front of the cameras.)

Compute the corresponding camera matrices for the original (un-normalized) coordinate system and plot the image points and the projected 3D-points in the same figure. Do the errors look small? Verify that the projections are reasonably well aligned with the image points.

Plot the 3D points and camera centers and principal axes in a 3D plot. Does it look like you expected it to ?

For the report: Submit the m-file and the plots. Comment on how reasonable the reconstruction is.