Computer Vision: Lecture 6

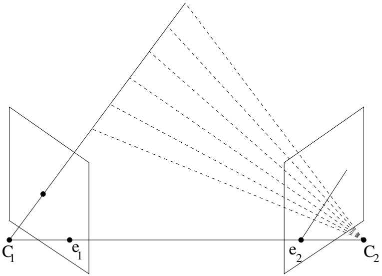
2023-11-16

Recap

If $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = (\mathtt{A} \mid \mathbf{t})$:

$$\bar{\mathbf{x}}^T \underbrace{[\mathbf{t}]_\times \mathtt{A}}_{:=\mathtt{F}} \mathbf{x} = 0$$

$\mathtt{F}\mathbf{x}$ — epipolar line in image 2, $\mathtt{F}^\top \mathbf{e}_2 = \mathbf{0}$
$\mathtt{F}^\top \bar{\mathbf{x}}$ — epipolar line in image 1, $\mathtt{F}\mathbf{e}_1 = \mathbf{0}$



Uncalibrated Structure from Motion with 2 cameras:

- Solve for $\mathtt{F}$ using 8-point solver
- Extract cameras from $\mathtt{F}$ (Today's Lecture)
- Triangulate 3D points

Today's Lecture

Two view geometry

- Computing cameras from F
- The calibrated case: The Essential Matrix
- The 8-point algorithm (again)
- Computing the cameras from E.

Camera matrices from $\mathtt{F}$

- We assume $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$
  - Projective ambiguity
- Can you determine $\mathtt{P}_2 = (\mathtt{A} \mid \mathbf{t})$ such that $\mathtt{F} = [\mathbf{t}]_\times \mathtt{A}$?
- Geometric intuition does not help...
- One choice is given by

$$\mathtt{P}_2 = \left([\mathbf{e}_2]_\times \mathtt{F} \mid \mathbf{e}_2 \right)$$

  $\mathbf{e}_2 \in \mathrm{null}(\mathtt{F}^\top)$ is the 2nd epipole (image of camera 1 in camera 2)
- Recall skew-symmetric matrix from last lecture

$$[\mathbf{a}]_\times := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \qquad [\mathbf{a}]_\times + [\mathbf{a}]_\times^\top = 0$$

- How can we prove that?

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Camera matrices from $\mathtt{F}$

- Claim: for given $\mathtt{F}$ we can choose $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = ([\mathbf{e}_2]_\times \mathtt{F} \mid \mathbf{e}_2)$
- Let $\binom{\mathbf{X}}{\mu} \in \mathbb{P}^3$ $\quad\quad \mathbf{X} \in \mathbb{R}^3$

$$\mathbf{x} \sim \mathtt{P}_1 \begin{pmatrix} \mathbf{X} \\ \mu \end{pmatrix} = (\mathtt{I} \mid \mathbf{0}) \begin{pmatrix} \mathbf{X} \\ \mu \end{pmatrix} = \mathbf{X}$$

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- Now we check $\bar{\mathbf{x}}^\top \mathtt{F}\mathbf{x}$:

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Question

Camera center 2

What is the camera center of $\mathtt{P}_2 = ([\mathbf{e}_2]_\times \mathtt{F} \mid \mathbf{e}_2)$?
Hint: recall that $\mathtt{F}\mathbf{e}_1 = \mathbf{0}$.

Solution:

- Camera center $\binom{\mathbf{C}_2}{\mu} \in \mathbb{P}^3$:

$$\mathtt{P}_2 \begin{pmatrix} \mathbf{C}_2 \\ \mu \end{pmatrix} = \mathbf{0} \iff [\mathbf{e}_2]_\times \mathtt{F}\mathbf{C}_2 + \mu \mathbf{e}_2 = \mathbf{0}$$
$$\implies \mathbf{C}_2 = \mathbf{e}_1 \wedge \mu = 0$$

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Camera matrices from $\mathtt{F}$

General solution

For given $\mathtt{F}$ we can choose

$$\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0}) \qquad \mathtt{P}_2 = \left([\mathbf{e}_2]_\times \mathtt{F} + \mathbf{e}_2 \mathbf{v}^\top \mid \lambda \mathbf{e}_2\right)$$

for any $\mathbf{v} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$.

- For given $\mathtt{F}$ we have a 4-parameter family of solutions $\mathtt{P}_2$ even after fixing $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$
- $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = (\mathtt{A} \mid \mathbf{t})$ ... "canonical camera pair"
- We may apply a projective transformation $\mathtt{H} \in \mathbb{R}^{4\times 4}$

$$\mathtt{P}_1' = \mathtt{P}_1 \mathtt{H} \qquad \mathtt{P}_2' = \mathtt{P}_2 \mathtt{H}$$

Will not affect $\mathtt{F}$

Relative Orientation: The Calibrated Case

Problem Formulation

Given two sets of corresponding (normalized) points $\{\mathbf{x}_i\}$ and $\{\bar{\mathbf{x}}_i\}$, compute camera matrices $\mathtt{P}_1 = (\mathtt{R}_1 \mid \mathbf{T}_1)$, $\mathtt{P}_2 = (\mathtt{R}_2 \mid \mathbf{T}_2)$ and 3D-points $\{\mathbf{X}_i\}$ such that

$$\lambda_i \mathbf{x}_i = \mathtt{P}_1 \mathbf{X}_i$$

and

$$\bar{\lambda}_i \bar{\mathbf{x}}_i = \mathtt{P}_2 \mathbf{X}_i.$$

Relative Orientation: Problem Formulation

Simplification

If $\mathtt{P}_1 = (\mathtt{R}_1 \mid \mathbf{T}_1)$, $\mathtt{P}_2 = (\mathtt{R}_2 \mid \mathbf{T}_2)$, apply the (Euclidean) transformation

$$\mathtt{H} = \begin{pmatrix} \mathtt{R}_1^\top & -\mathtt{R}_1^\top \mathbf{T}_1 \\ \mathbf{0}^\top & 1 \end{pmatrix}.$$

Then

$$\mathtt{P}_1 \mathtt{H} = \begin{pmatrix} \mathtt{R}_1 & \mathbf{T}_1 \end{pmatrix} \begin{pmatrix} \mathtt{R}_1^\top & -\mathtt{R}_1^\top \mathbf{T}_1 \\ \mathbf{0}^\top & 1 \end{pmatrix} = (\mathtt{I} \mid \mathbf{0})$$

Hence, we may assume that the cameras are

$$\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0}) \qquad\qquad \mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T}).$$

The Essential Matrix

The Essential Matrix

The camera pair $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$ has the fundamental matrix

$$\mathtt{E} = [\mathbf{T}]_\times \mathtt{R}$$

$\mathtt{E}$ is called the essential matrix.

- $\mathtt{R}$ has 3 dof, $\mathbf{T}$ 3 dof, but the scale is arbitrary, therefore $\mathtt{E}$ has 5 d.o.f.
- $\mathtt{E}$ has $\det(\mathtt{E}) = 0$
- $\mathtt{E}$ has two nonzero equal singular values.

The Essential Matrix

The 8-point algorithm (again)

- Extract at least 8 point correspondences.
- Normalize the coordinates (multiply with $\mathtt{K}^{-1}$, $\mathtt{K}$ inner parameters).
- Form $\mathtt{M}$ and solve

$$\min_{\mathbf{v}:\|\mathbf{v}\|=1} \left\|\mathtt{M}\mathbf{v}\right\|^2$$

  using SVD.
- Reshape $\mathbf{v} \in \mathbb{R}^9$ to $\mathtt{E} \in \mathbb{R}^{3\times 3}$
- Enforce constraints on $\mathtt{E}$
  - Ensure that $\det(\mathtt{E}) = 0$
  - $\mathtt{E}$ has two nonzero equal singular values
- Compute a pair of cameras from $\mathtt{E}$.
- Compute the scene points (triangulation).

The Essential Matrix

Issues

Resulting E may not have $\det(\mathtt{E}) = 0$ and two nonzero equal singular values. Pick the closest essential matrix.

Can be solved using svd, in matlab:

```
[U,S,V] = svd(E);
s = (S(1,1)+S(2,2))/2;
S = diag([s s 0]);
E = U*S*V';
```

Note: Since the scale of the essential matrix is arbitrary we may assume that $s = 1$. We can therefore always use `S = diag([1 1 0])`.

Computing the cameras from $\mathtt{E}$

Goal

Find $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$ such that $\mathtt{E} = [\mathbf{T}]_\times \mathtt{R}$.

Outline:

- Ensure that $\mathtt{E}$ has the SVD
$$\mathtt{E} = \mathtt{U}\Sigma\mathtt{V}^\top = \mathtt{U} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathtt{V}^\top$$
where $\det(\mathtt{U}\mathtt{V}^\top) = 1$
- Compute a factorization $\mathtt{E} = \mathtt{S}\mathtt{R}$ where $\mathtt{S}$ is skew symmetric and $\mathtt{R}$ a rotation
- Compute a $\mathbf{T}$ such that $[\mathbf{T}]_\times = \mathtt{S}$
- Form the camera $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$

Decomposing $\mathtt{E}$

- First decompose $\Sigma = \underbrace{\mathtt{Z}}_{\text{skew sym.}} \underbrace{\mathtt{W}}_{\text{orthogonal}} \iff \Sigma \mathtt{W}^\top = \mathtt{Z}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{pmatrix} = \begin{pmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ 0 & 0 & 0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$

$$\implies \begin{cases} w_{11} = w_{22} = 0 \\ w_{31} = z_2 = 0 \\ w_{32} = -z_1 = 0 \\ w_{12} = z_3 = -w_{21} \end{cases} \iff \begin{pmatrix} 0 & w_{21} & 0 \\ w_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -z_3 & 0 \\ z_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Two solutions $\Sigma = \mathtt{ZW} = \mathtt{Z}^\top \mathtt{W}^\top$, where

$$\mathtt{W} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathtt{Z} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Decomposing $\mathtt{E}$

- First decompose $\Sigma = \underbrace{\mathtt{Z}}_{\text{skew sym.}} \underbrace{\mathtt{W}}_{\text{orthogonal}} \iff \Sigma\mathtt{W}^\top = \mathtt{Z}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{pmatrix} = \begin{pmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ 0 & 0 & 0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$

$$\implies \begin{cases} w_{11} = w_{22} = 0 \\ w_{31} = z_2 = 0 \\ w_{32} = -z_1 = 0 \\ w_{12} = z_3 = -w_{21} \end{cases} \iff \begin{pmatrix} 0 & w_{21} & 0 \\ w_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -z_3 & 0 \\ z_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Two solutions $\Sigma = \mathtt{Z}\mathtt{W} = \mathtt{Z}^\top\mathtt{W}^\top$, where

$$\mathtt{W} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathtt{Z} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Decomposing $\mathtt{E}$

- Two solutions for $\mathtt{E}$

$$\mathtt{E} = \mathtt{U\Sigma V}^\top = \mathtt{UZWV}^\top = \overbrace{\mathtt{UZU}^\top}^{\mathtt{S}_1}\overbrace{\mathtt{UWV}^\top}^{\mathtt{R}_1}$$
$$\mathtt{E} = \mathtt{U\Sigma V}^\top = \mathtt{UZ}^\top\mathtt{W}^\top\mathtt{V}^\top = \underbrace{\mathtt{UZ}^\top\mathtt{U}^\top}_{\mathtt{S}_2}\underbrace{\mathtt{UW}^\top\mathtt{V}^\top}_{\mathtt{R}_2}$$

- A "twisted pair"
- Ex: show that $\mathtt{S}_1$ and $\mathtt{S}_2$ are skew symmetric and that $\mathtt{R}_1$ and $\mathtt{R}_2$ are rotations

The Twisted Pair

- $\mathtt{S}_1 = -\mathtt{S}_2$ (i.e. are the same up to scale)
- Next step: find $\mathbf{T}$ such that $[\mathbf{T}]_\times = \mathtt{S}_1$
- $[\mathbf{T}]_\times \mathbf{T} = 0 \implies \mathbf{T} \in \mathrm{null}(\mathtt{S}_1) = \mathrm{null}(\mathtt{UZU}^\top)$

$$\mathbf{T} = \mathtt{U}(:, 3)$$

- Or read it directly from $\mathtt{S}_1$

$$\mathtt{S}_1 = [\mathbf{T}]_\times := \begin{pmatrix} 0 & -T_3 & T_2 \\ T_3 & 0 & -T_1 \\ -T_2 & T_1 & 0 \end{pmatrix}$$

- Overall: $\mathtt{P}_1(\mathtt{I} \mid \mathbf{0})$ and 4 possible solutions:

$$\mathtt{P}_2 = \left(\mathtt{UWV}^\top \mid \pm\mathbf{T}\right) \qquad \text{or} \qquad \mathtt{P}_2 = \left(\mathtt{UW}^\top\mathtt{V}^\top \mid \pm\mathbf{T}\right)$$

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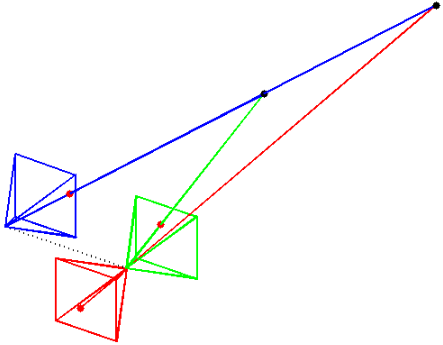
The Twisted Pair: Example

$\mathtt{P}_2 = (\mathtt{I} \mid \mathbf{T})$ or $\mathtt{P}_2 = (\mathtt{R}_2 \mid \mathbf{T})$:

$$\mathtt{R}_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$



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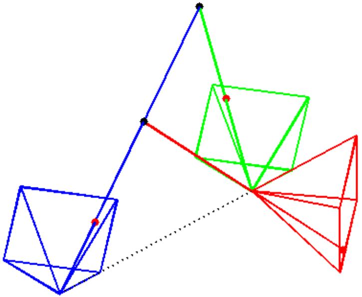
$$\mathbf{T} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$



Scale Ambiguity

- Scale is arbitrary $\lambda \mathtt{E}$ is also a valid essential matrix

$$\lambda \mathtt{E} = [\lambda \mathbf{T}]_\times \mathtt{R}_1 = [\lambda \mathbf{T}]_\times \mathtt{R}_2$$

- Gives two solutions $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and

$$\mathtt{P}_2 = \left(\mathtt{UWV}^\top \mid \lambda \mathbf{T}\right) \qquad \text{or} \qquad \mathtt{P}_2 = \left(\mathtt{UW}^\top \mathtt{V}^\top \mid \lambda \mathbf{T}\right)$$

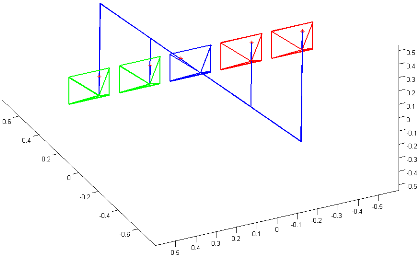
- Scales *baseline* between camera centers and rescales scene

Scale Ambiguity: Example

$$\mathtt{P}_2 = (\mathtt{I} \mid \lambda \mathbf{T})$$

$$\mathbf{T} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Green — $\lambda > 0$
Red — $\lambda < 0$.



4 Solutions

4 possible 3D reconstructions

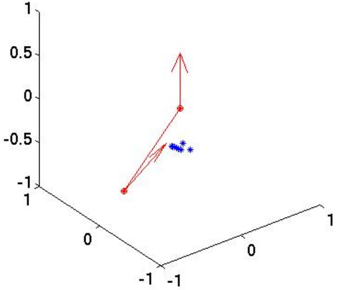
Conclusion: One of the 4 solutions

$$P_2 = \left(\mathtt{UWV}^\top \mid \mathbf{T}\right) \qquad P_2 = \left(\mathtt{UWV}^\top \mid -\mathbf{T}\right)$$
$$P_2 = \left(\mathtt{UW}^\top\mathtt{V}^\top \mid \mathbf{T}\right) \qquad P_2 = \left(\mathtt{UW}^\top\mathtt{V}^\top \mid -\mathbf{T}\right)$$

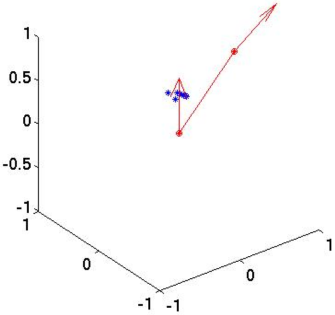
has points in front of both cameras.

- "Cheirality constraint": 3D points have to lie in front of both cameras
- If not:
  - Wrong configuration
  - Noise in image points and numerical issues
  - Corresponding points are outliers, i.e. false positives

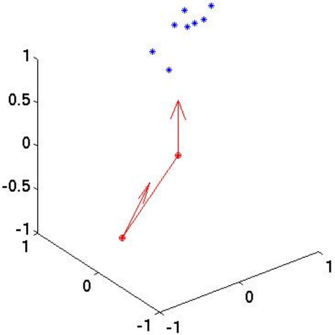
4 Solutions: Example



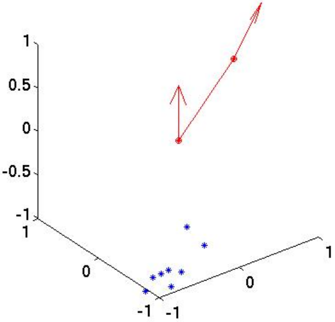
4 Solutions: Example



4 Solutions: Example

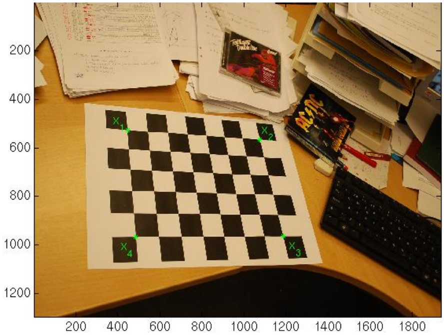


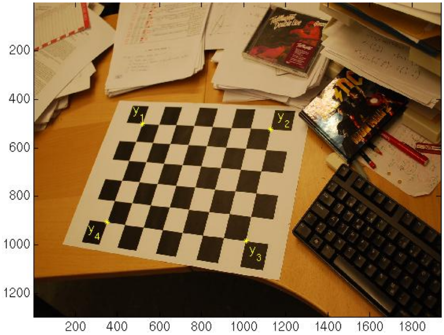
4 Solutions: Example



Homographies: Calibrated Setup

Remember homographies?





Homographies: Calibrated Setup

- Let $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$, $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$ and $\Pi = (\mathbf{n}, d)$
- Points $\mathbf{X} \in \mathbb{R}^3$ on the plane satisfy $\mathbf{n}^\top \mathbf{X} + d = 0$
- Image point $\mathbf{x}$ in camera 1: $\mathbf{X} = \lambda \mathbf{x}$
- Now $\mathbf{X} \in \Pi$:
$$\lambda \mathbf{n}^\top \mathbf{x} + d = 0 \implies \lambda = -\frac{d}{\mathbf{n}^\top \mathbf{x}} \wedge \mathbf{X} = -\frac{d}{\mathbf{n}^\top \mathbf{x}} \mathbf{x}$$
- Corresponding image point im camera 2:
$$\mathtt{P}_2 \begin{pmatrix} \mathbf{X} \\ 1 \end{pmatrix} = \mathtt{R}\mathbf{X} + \mathbf{T} = -\mathtt{R}\frac{d}{\mathbf{n}^\top \mathbf{x}} \mathbf{x} + \mathbf{T} \sim \mathtt{R}\mathbf{x} - \frac{\mathbf{T}\mathbf{n}^\top \mathbf{x}}{d} = \underbrace{\left(\mathtt{R} - \frac{\mathbf{T}\mathbf{n}^\top}{d} \right)}_{=\mathtt{H}} \mathbf{x}$$

Homography in the calibrated setup

For a canonical camera pair, $\mathtt{P}_1 = (\mathtt{I} \mid \mathbf{0})$ and $\mathtt{P}_2 = (\mathtt{R} \mid \mathbf{T})$, and 3D plane $\Pi = (\mathbf{n}, d)$ the induced homography is given by $\mathtt{H} = \mathtt{R} - \frac{1}{d}\mathbf{T}\mathbf{n}^\top$.

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Homographies: Calibrated Setup

- Comparison of d.o.f.
  - Homography $\mathtt{H}$ has 8 d.o.f.
  - $\mathtt{R}$, $\mathbf{T}$ and Π together have 3+(3-1)+(4-1) = 8 d.o.f.
- Given $\mathtt{R}$, $\mathbf{T}$ and Π there is a unique (up to scale) $\mathtt{H} = \mathtt{R} - \frac{1}{d}\mathbf{T}\mathbf{n}^\top$
- Can we extract $\mathtt{R}$, $\mathbf{T}$ and Π from a given $\mathtt{H}$?

Homography decomposition

A homography $\mathtt{H}$ estimated from *normalized* image points has two solutions for $\mathtt{R}$, $\mathbf{T}$ and Π such that $\mathtt{H} = \mathtt{R}_i - \frac{1}{d_i}\mathbf{T}_i\mathbf{n}_i^\top$ for $i = 1, 2$.

- Matlab routine will be provided
- Python
  - Convert Matlab routine, or
  - `num, Rs, Ts, Ns = cv2.decomposeHomographyMat(H, K)` (with $\mathtt{K} = \mathtt{I}$)
- Finding out which solution is the correct one is not always straightforward
  - More at the project presentation

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To do

Lab sessions today: MTI2, MTI4, SB-D020, SB-D409

- Next time: Robust Estimation.
- Work on Assignment 2.

More reading:

- Szeliski, Section 11.3 on Two-View Structure from Motion.