Computer Vision: Lecture 1

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2023-10-30

Outline

- General course information
- What is computer vision & 3D reconstruction from images??
- Modelling cameras
- Homogeneous coordinates

General Course Information

- Lectures: $14 \times 2h$
  - Check TimeEdit for lecture hall
  - If I am late Thursday mornings, blame Hisingsbron...
- Lab sessions: $14 \times 2h$
- Assignments: 4 (*mandatory* and *optional* parts)
- Project (*mandatory* and *optional* parts)

Course requirements

- Passing the course: Completed assignments and project in time
  - You need 50% of mandatory theoretical exercises points for *each* assignment
  - You need to complete the mandatory computer exercises
- Grade 4: Same as above and at least 50% of all achievable *optional* points
- Grade 5: Same as above and oral exam

There is no written exam in the course

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Assignments and Exercises

- 4 assignments, which contain both theoretical and computer exercises
- You have the opportunity to work with the assignments and ask questions during the exercise sessions
- After completing an assignment, you submit a report in pdf together with your code in Canvas
- You have about 2 weeks for each assignment (and 1 month for the project) Exact dates are written in the assignments, alternatively check Canvas
- You may collaborate with another students for the computer exercises. Groups larger than two students are not allowed.
  - State clearly in the report who you collaborated with
- The reports should be done individually. (You should understand and be able to explain the solution you hand in.)

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Assignments and Exercises

Requirements to pass

- You need to obtain 50% of points for mandatory theory questions in each assignment
- You need to complete the mandatory computer exercises
  - Most of them will be used in the final project
- If you don't complete an assignment:
  - You may have the chance to submit a revised solution up to 1 week after getting feedback from the TA
  - Only applicable if your solutions require relatively minor corrections
- Late submission
  - You can use 5 late days in total
  - Downside: no optional points (for that assignment) is graded
- Use exercise sessions (but use them wisely)

Project

Requirements to pass

- Project: mandatory and optional parts
- Detailed description will be presented in late November / early December
  - Project outline: you will create software for 3D reconstruction from images
  - Details on the project, pitfalls and how to get optional points
- Deadline is early January

Cheating / Plagiarism

- Don't cheat
- Don't use repositories offering solutions from previous years
- Don't copy code from another student group
- Behave like a trustworthy 5th year student
- Ultimately passing this course should be easy

Low risk but high penalty!

Code development

- Matlab is the default programming language
  - TAs will focus on Matlab-related questions
  - Code provided by us will be in Matlab
  - Octave should work as well
- You may submit Python code
  - Restrict yourself to well-known libraries
    - NumPy for linear algebra
    - OpenCV for image loading and feature extracting & matching
    - Matplotlib / Open3D for visualization
- You cannot simply call a library function when you should implement an algorithm!
  - Exception: functions explicitly allowed or corresponding to supplied routines

Matlab	Python
`a = [1 0; 0 1]`	`a = np.array([[1, 0], [0, 1]])`
`b = [4 1; 2 2]`	`b = np.array([[4, 1], [2, 2]])`
`a*b`	`np.matmul(a, b)`

Literature

Literature

- Lecture and lecture slides are self-contained
- Optional: lecture notes
  - `https://www.maths.lth.se/matematiklth/personal/calle/datorseende19/index.html`
  - May be helpful
  - *Are neither necessary nor sufficient for this course*
- Some additional scientific papers

Additional Reading

- Hartley, Zisserman, Multiple View Geometry, 2004.
- Szeliski, Computer Vision Algorithms and Applications, 2010. Availible online; http://szeliski.org/Book/

Planned contents

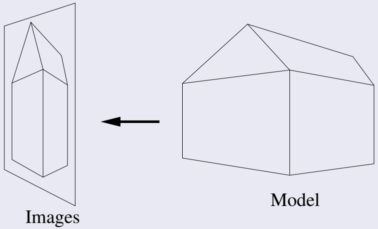
Week 1	Intro, camera model	Projective geometry
Week 2	Camera calibration, DLT I	DLT II, feature matching
Week 3	Two-view geometry I	Two-view geometry II
Week 4	Robust estimation	Minimal solvers
Week 5	MLE & Non-Linear opt., Factorization	Non-Seq. SfM, project pres.
Week 6	Bundle adjustment I, Uncertainty	Bundle adjustment II
Week 7	Non-rigid SfM (guest)	Dense reconstruction

Slides based on the ones by Carl Olsson & Fredrik Kahl
Lecture notes: Carl Olsson & Fredrik Kahl

What is Computer Vision?

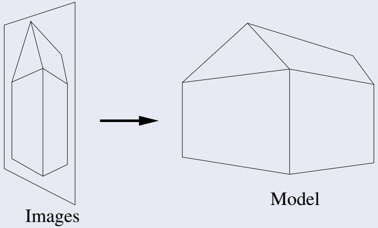
- 1970–2015: enabling software to understand images in 3D
- 2015–20xx: umbrella term for image analysis, comp. photography, pattern recognition etc.

Computer Graphics



Generate images from a 3D model.

Computer Vision



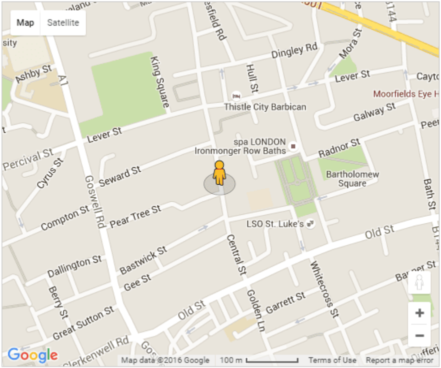
The inverse problem: Generate 3D model from images.

What can we do with 3D models?

Aerial photogrammetry: city planning, flood/disaster simulation

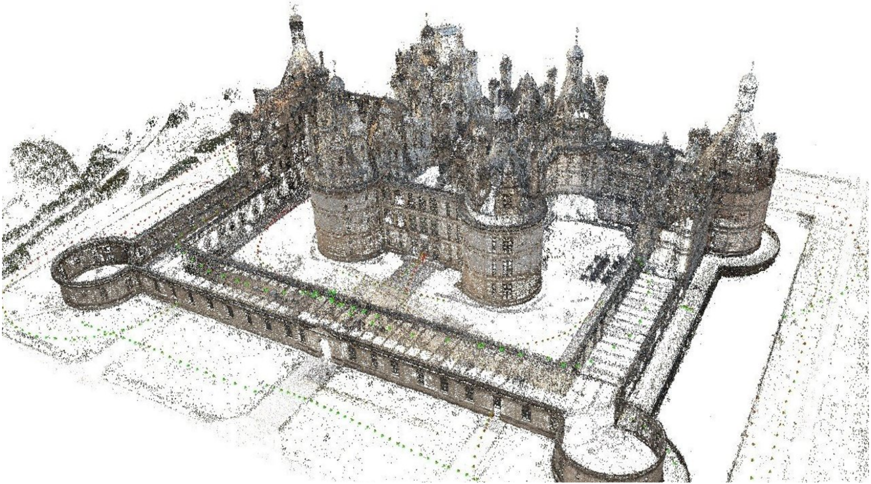
What can we do with 3D models?

Aerial + terrestrial photogrammetry: street view



What can we do with 3D models?

Cultural heritage preservation



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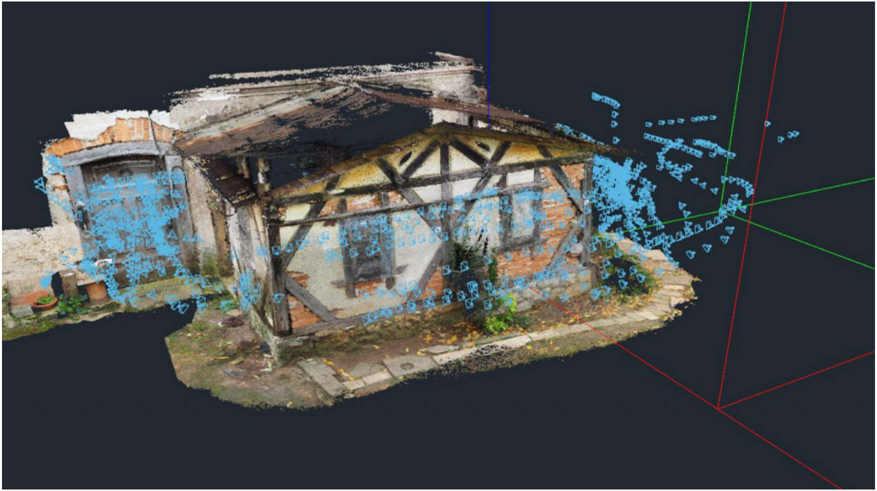
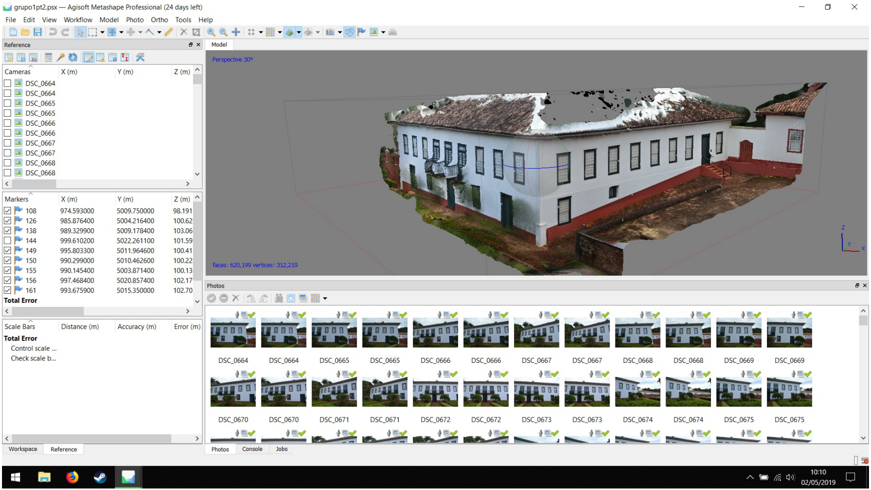


What can we do with 3D models?

Architectural planning



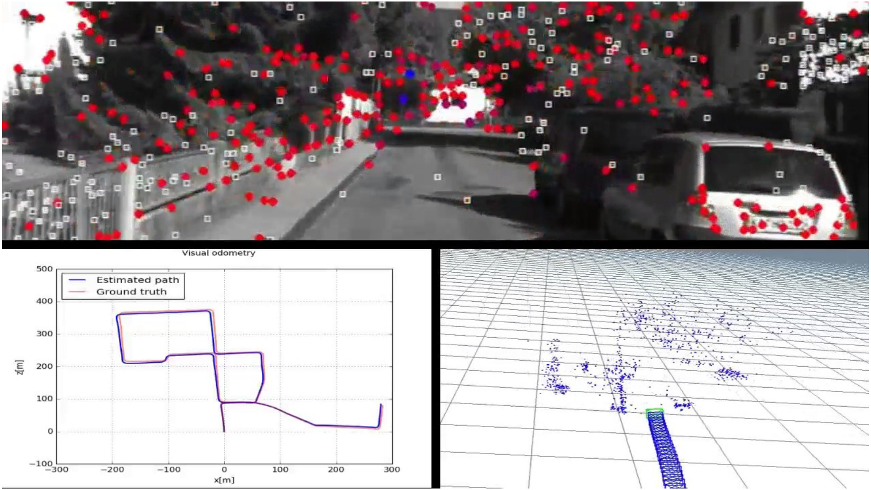
By NELAC - N.ELAC

What can we do with 3D models?

Augmented reality

What can we do with 3D models?

Autonomous vehicles & robotics

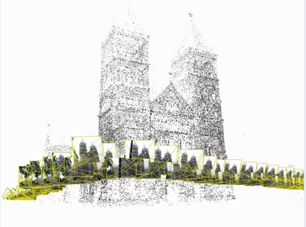


Main Goal of the Course: Multiview Reconstruction

Given Images

4 images out of a sequence with 435 images.

Compute 3D Model



Multiview Reconstruction Pipeline

Point Detection and Matching

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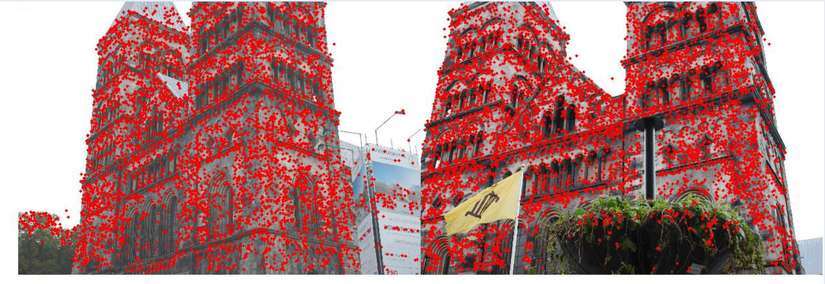
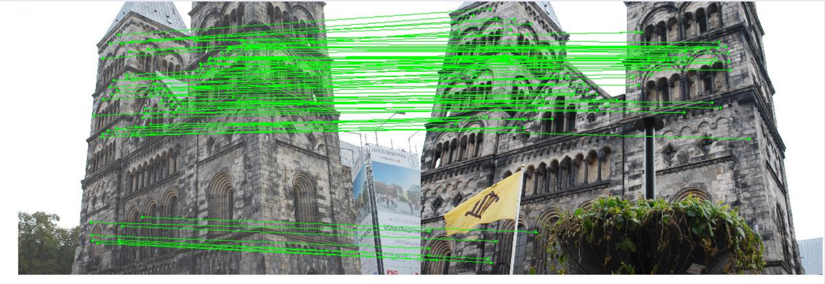


Detect interesting (descriptive) points in all images.

Multiview Reconstruction Pipeline

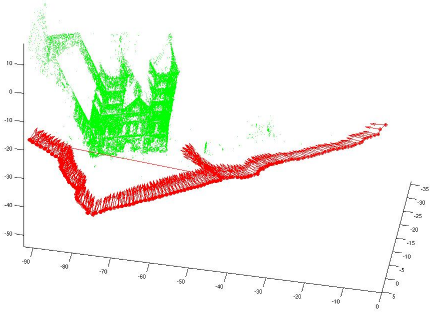
Point Detection and Matching



Match points between images.

Multiview Reconstruction Pipeline

Geometric Computations (main part of this course!)



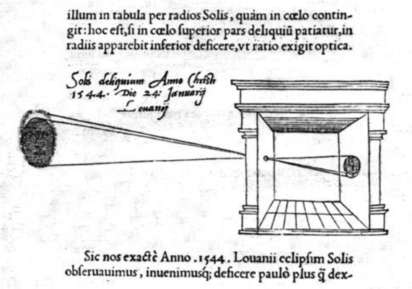
Compute 3D-positions of the matched points, position and orientation of the cameras.

Video

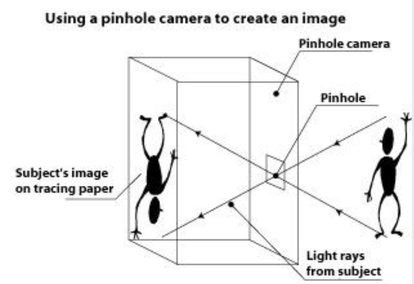
Live demo

Camera Model

The Pinhole Camera



Reinerus Gemma-Frisus camera obscura from 1544.



Camera Model

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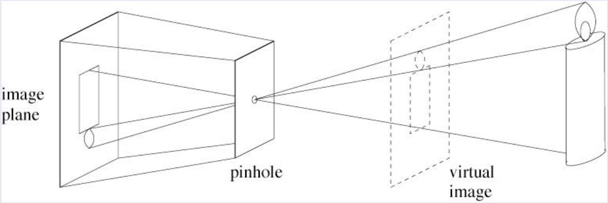
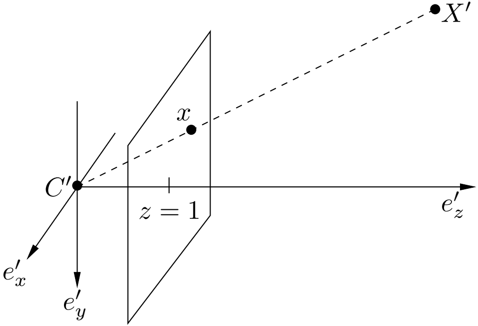


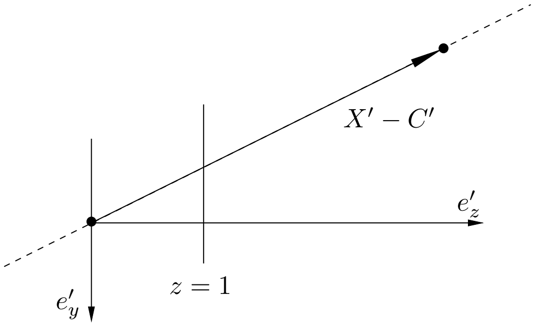
- Idealized model of how a real camera with a lens system works
- Approximately correct for modern consumer cameras

The Pinhole Camera Model



The mathematical model for a pinhole camera

The Pinhole Camera Model



The model viewed from the side. (The vector e'_x points out of the figure.)

The Pinhole Camera Model

Going from 3D points to image points involves 3 steps

- Transforming from world coordinates to camera coordinates

$$\mathbf{X}' = \mathtt{R}(\mathbf{X} - \mathbf{C}) = \mathtt{R}\mathbf{X} + \mathbf{T}$$

- Perspective projection: camera coordinates to image coordinates

$$\mathbf{x} = \frac{1}{X_3'} \begin{pmatrix} X_1' \\ X_2' \\ X_3' \end{pmatrix} = \begin{pmatrix} X_1'/X_3' \\ X_2'/X_3' \\ 1 \end{pmatrix}$$

- Calibration matrix: image coordinates to sensor coordinates

$$\mathbf{p} = \mathtt{K}\mathbf{x} = \begin{pmatrix} \gamma f & s & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}$$

If $\mathtt{K}$ is given, then the camera is *calibrated*, otherwise *uncalibrated*. Modern cameras: $\gamma = 1$ and $s = 0$. More about that in lecture 3.

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Transforming from world coordinates to camera coordinates

- Euclidean transformation

$$\mathbf{X}' = \mathtt{R}(\mathbf{X} - \mathbf{C}) = \mathtt{R}\mathbf{X} + \mathbf{T}$$

- Rotates and shifts points (position vectors) in 3D
- Matrix $\mathtt{R} \in \mathbb{R}^{3\times 3}$ is a rotation matrix

$$\mathtt{R}^\top \mathtt{R} = \mathtt{I} \qquad \det(\mathtt{R}) = 1$$

- Camera center $\mathbf{C} \in \mathbb{R}^3$ and translation vector $\mathbf{T} \in \mathbb{R}^3$

$$\mathbf{C}' = \mathbf{0} = \mathtt{R}\mathbf{C} + \mathbf{T} \iff \mathbf{C} = -\mathtt{R}^\top \mathbf{T} \iff \mathbf{T} = -\mathtt{R}\mathbf{C}$$

The Pinhole Camera Model

Transforming from world coordinates to camera coordinates

- How do direction vectors transform?

$$\mathbf{d}' = \mathtt{R}\mathbf{d}$$

- How do planes transform?

$$\mathbf{X}' = \mathtt{R}\mathbf{X} + \mathbf{T} \qquad \begin{pmatrix} X_1' \\ X_2' \\ X_3' \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \mathtt{R} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{pmatrix}}_{=:\mathtt{M}} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{pmatrix}$$

Plane equation $0 = \Pi^\top \binom{\mathbf{X}}{1}$

$$0 = \Pi^\top \begin{pmatrix} \mathbf{X} \\ 1 \end{pmatrix} = \Pi^\top \mathtt{M}^{-1}\mathtt{M} \begin{pmatrix} \mathbf{X} \\ 1 \end{pmatrix} = \left(\mathtt{M}^{-\top}\Pi\right)^\top \begin{pmatrix} \mathbf{X}' \\ 1 \end{pmatrix}$$

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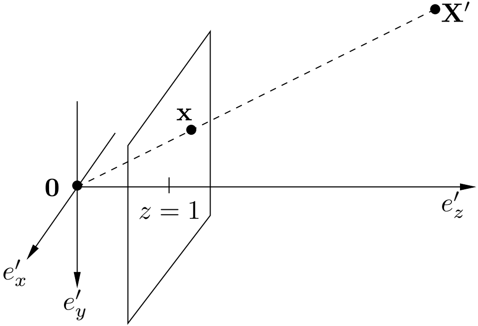

The mathematical model for a pinhole camera

The Pinhole Camera Model

Perspective projection: from camera coordinates to image coordinates

- Non-linear mapping $\pi : \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathbf{x} = \pi(\mathbf{X}') = \frac{1}{X_3'} \begin{pmatrix} X_1' \\ X_2' \\ X_3' \end{pmatrix} = \begin{pmatrix} X_1'/X_3' \\ X_2'/X_3' \\ 1 \end{pmatrix}$$

- $\pi(s\mathbf{X}') = \pi(\mathbf{X}')$ for all $s \neq 0$

$$\pi^{-1}(\mathbf{x}) = \{s\mathbf{x} : s \neq 0\}$$

- X_3' is the *depth* of $\mathbf{X}'$: point-plane distance!

Discussion

- What does it mean if the depth $X_3' < 0$? If $X_3' < 1$?
- What happens if $X_3' \to \infty$? If $X_3' = 0$?
- Isn't $\mathbf{x} \in \mathbb{R}^3$, but image/sensor coordinates should be 2D?

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Homogeneous coordinates and $\mathbb{P}^2$

Projective space $\mathbb{P}^2$

If there exists a $\lambda \neq 0$ such that $\mathbf{x} = \lambda \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then we write $\mathbf{x} \sim \mathbf{y}$. An equivalence relation $\sim$ induces equivalence classes

$$[\mathbf{x}] := \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} \sim \mathbf{x}\}$$

$\mathbb{P}^2$ is the quotient space of $\mathbb{R}^3$ by $\sim$: $\mathbb{P}^2 = \{[\mathbf{x}] : \mathbf{x} \in \mathbb{R}^3\} = \mathbb{R}^3 / \sim$
By definition, we exclude the origin $\mathbf{0} = (0,0,0)^\top$ from $\mathbb{P}^2$

- Straightforward to extend to $\mathbb{P}^k = \mathbb{R}^{k+1} / \sim$
- If $\mathbf{x} = (x_1, x_2, x_3)^\top$ with $x_3 \neq 0$, then $\begin{pmatrix} x_1/x_3 \\ x_2/x_3 \\ 1 \end{pmatrix}$ is a representative for $[\mathbf{x}]$
- Bijection between $(x_1, x_2, 1)^\top \in [\mathbf{x}] \in \mathbb{P}^2$ and $(x_1, x_2)^\top \in \mathbb{R}^2$
- We can move between $\mathbb{R}^2$ and $\mathbb{P}^2$ as needed (if $x_3 \neq 0$)!

$(x_1, x_2, x_3)^\top$ are homogeneous coordinates corresponding to $(x_1/x_3, x_2/x_3)^\top$

Homogeneous coordinates and $\mathbb{P}^2$

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$(x_1, x_2, x_3)^\top$ are homogeneous coordinates corresponding to $(x_1/x_3, x_2/x_3)^\top$

Homogeneous coordinates and $\mathbb{P}^2$

Projective space $\mathbb{P}^2$

If there exists a $\lambda \neq 0$ such that $\mathbf{x} = \lambda \mathbf{y}$ for $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$, then we write $\mathbf{x} \sim \mathbf{y}$. An equivalence relation $\sim$ induces equivalence classes

$$[\mathbf{x}] := \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} \sim \mathbf{x}\}$$

$\mathbb{P}^2$ is the quotient space of $\mathbb{R}^3$ by $\sim$: $\mathbb{P}^2 = \{[\mathbf{x}] : \mathbf{x} \in \mathbb{R}^3\} = \mathbb{R}^3 / \sim$
By definition, we exclude the origin $\mathbf{0} = (0, 0, 0)^\top$ from $\mathbb{P}^2$

- Straightforward to extend to $\mathbb{P}^k = \mathbb{R}^{k+1} / \sim$
- If $\mathbf{x} = (x_1, x_2, x_3)^\top$ with $x_3 \neq 0$, then $\begin{pmatrix} x_1/x_3 \\ x_2/x_3 \\ 1 \end{pmatrix}$ is a representative for $[\mathbf{x}]$
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Homogeneous coordinates and $\mathbb{P}^2$

- Converting to homogeneous coordinates ("homogenizing")

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \in \left[\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \right] \in \mathbb{P}^2$$

  Representative $(x_1, x_2, 1)^\top$ is element of $\mathbb{R}^3$

- Converting to Cartesian coordinates ("de-homogenizing")

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{P}^2 \rightsquigarrow \begin{pmatrix} x_1/x_3 \\ x_2/x_3 \end{pmatrix} \in \mathbb{R}^2$$

  if $x_3 \neq 0$

Homogeneous coordinates and $\mathbb{P}^k$

Why are homogeneous coordinates useful?

- Recall Euclidean transformation

$$\mathbf{X}' = \mathtt{R}\mathbf{X} + \mathbf{T} \qquad\qquad \mathbf{X}, \mathbf{X}' \in \mathbb{R}^3$$

  Affine expression (linear + offset)

- Interpret $\mathbf{X} \in \mathbb{P}^3$:

$$\mathbf{X}' = \underbrace{\begin{pmatrix} \mathtt{R} & \mathbf{T} \\ \mathbf{0}^\top & 1 \end{pmatrix}}_{\in \mathbb{R}^{4x4}} \mathbf{X} \qquad\qquad \mathbf{X}, \mathbf{X}' \in \mathbb{P}^3$$

  Purely linear expression

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  Purely linear expression

Homogeneous coordinates and $\mathbb{P}^k$

Why are homogeneous coordinates useful?

- Recall projection onto the image plane, followed by calibration matrix ($s = 0$)

$$\mathbf{p} = \mathtt{K} \cdot \pi(\mathbf{X}) = \begin{pmatrix} \gamma f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X_1/X_3 \\ X_2/X_3 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma f X_1/X_3 + x_0 \\ f X_2/X_3 + x_1 \\ 1 \end{pmatrix} \in \mathbb{P}^2$$

- Observe that

$$\pi(\mathtt{K}\mathbf{X}) = \frac{1}{X_3} \begin{pmatrix} \gamma f X_1 + x_0 X_3 \\ f X_2 + x_1 X_3 \\ X_3 \end{pmatrix} = \mathtt{K} \cdot \pi(\mathbf{X}) \in [\mathtt{K}\mathbf{X}]$$

- We can apply the perspective division as final step

$$\underbrace{[\mathtt{K}(\mathtt{R}\ \mathbf{T})\mathbf{X}]}_{\mathbf{X} \in \mathbb{P}^3} = \pi(\underbrace{\mathtt{K}(\mathtt{R}\mathbf{X} + \mathbf{T})}_{= \mathtt{P}\mathbf{X}})$$

Projection matrix $\mathtt{P} = \mathtt{K}(\mathtt{R} \mid \mathbf{T}) \in \mathbb{R}^{3 \times 4}$

Homogeneous coordinates and $\mathbb{P}^k$

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- We can apply the perspective division as final step

$$\underbrace{[\mathtt{K}(\mathtt{R}\ \mathbf{T})\mathbf{X}]}_{\mathbf{X} \in \mathbb{P}^3} \equiv \pi(\underbrace{\mathtt{K}(\mathtt{R}\mathbf{X} + \mathbf{T})}_{= \mathtt{P}\mathbf{X}})$$

Projection matrix $\mathtt{P} = \mathtt{K}(\mathtt{R} \mid \mathbf{T}) \in \mathbb{R}^{3\times 4}$

Homogeneous coordinates and $\mathbb{P}^k$

Why are homogeneous coordinates useful?

- Allowing to work with quantities such as $[(x_1, x_2, 0)^\top]$
- Points, lines and planes at infinity
- More on that in the next lecture

The Pinhole Camera Model

Attention

Different conventions used in the literature/software

- Computer vision community mostly uses
  - Camera is looking in z direction
  - World coordinates to camera coordinates: $\mathbf{X}' = \mathtt{R}\mathbf{X} + \mathbf{T}$
- Computer graphics (OpenGL)
  - Camera is looking in $-z$ direction
- "Position-centric" parametrization
  - World coordinates to camera coordinates: $\mathbf{X}' = \mathtt{R}(\mathbf{X} \pm \mathbf{C})$
  - $\pm\mathbf{C}$ might be called "translation"
- Inverse parametrization
  - World coordinates to camera coordinates: $\mathbf{X}' = \mathtt{R}^\top(\mathbf{X} \pm \mathbf{C})$
  - "Look at" parametrization
- Lots of ambiguities about image coordinates (mm) and sensor coordinates (pixels)

At least the right-handed coordinate system convention is generally accepted.

The Pinhole Camera Model: Summary

Summary

- Model for a pinhole camera

$$\mathbf{x} \sim \underbrace{\mathtt{K}(\mathtt{R} \mid \mathbf{T})}_{=\mathtt{P}} \mathbf{X} = \mathtt{P}\mathbf{X} \qquad \mathbf{X} \in \mathbb{P}^3$$

- Camera represented by its projection matrix (or camera matrix) $\mathtt{P} \in \mathbb{R}^{3\times 4}$
  - $\mathtt{P}$ has 11 d.o.f.
  - If $\mathtt{K}$ is known, then the camera is said to be calibrated
- Camera center
  - in camera coordinates: $\mathbf{0}$
  - in world coordinates: $-\mathtt{R}^\top \mathbf{T}$

Next Time

- Lab session today in ES61, ES62, ES63 and via Zoom

   

Projective Geometry

- Projective space
- Projective transformations