These notes are an excerpt of H. Strogatz, Physica D 143 (2000)

7. Synchronisation of coupled oscillators

7.1. Introduction

Collective synchronisation: network of oscillators locks into a common mode despite the fact that the frequencies of the individual oscillators are all slightly different.

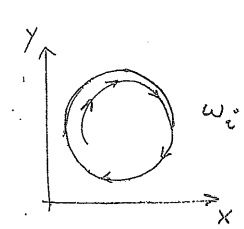
This mechanism is thought to operate in a wide variety of biological and engineering systems

- networks of pacemaker cells in (human) heart
- network of circadian pacemaker cells in (human) brain
- metabolic synchronisation of yeast-cell suspension
- synchronously flashing fireflies
- synchronously chirping crickets

- synchronous motion of ant colonies → E.Ott
- synchronisation of laser arrays
- synchronisation of metronomes on a flexible shelf → wwtw.
- Millennium bridge (reinforcement vs. damping) → E.Ott

7.2. Formulation of the problem

Consider ensemble of limit-cycle oscillators i (see for example p. 175) with nearly identical frequencies ω_i and weak interactions.



Weak interactions, small detuning → separation of time scales: the oscillators rapidly relax to their limit cycles and can therefore be characterised by their phases θ_i. On a much longer time scale the phase dynamics is influenced by the interactions.

Numerical observation: phase transition

detuning large w.r.t. coupling	incoherent phase: each oscillator moves at its own frequency
detuning small w.r.t. coupling	clusters of oscillators freeze into synchrony

Hypothesis: individual phases θ_i are "pulled towards mean-field phase ψ" (and not to the phase of any other oscillator). The magnitude of this effect is proportional to the extent of coherence, i.e., to the fraction of oscillators frozen into synchrony.

positive-feedback loop

Problem: derive mathematical model supporting this hypothesis.

Two tasks

① derive model from realistic microscopic dynamics

② analyse model (mean-field approach → lecture notes on neural networks

and stability analysis)

In the following concentrate on ②.

7.3. Kuramoto's model

Ensemble of nearly identical, weakly interacting limit-cycle oscillators.

Approximate equation for phase dynamics at large times

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \qquad i = 1, \dots, N$$

(Kuramoto 1984).

Nobody has yet succeeded in solving or analysing these equations in their most general form (as quoted above):

– $\Gamma_{ij}(\theta) = \sum_k a_k^{(ij)} e^{ik\theta}$ arbitrary

– network connectivity

- fully connected
- nearest-neighbour coupling on square lattice
- random partial connectivity
- scale-free networks

Kuramoto simplified the problem further by assuming $\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N} \sin(\theta_j - \theta_i)$

- retain just one Fourier mode
- weight all couplings equally, K.
- network fully coupled (each oscillator is connected to all others)

Kuramoto's model

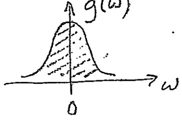
$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \qquad (*)$$

$K \geq 0$ ...... proper normalisation when $N \to \infty$

– frequency distribution $\text{Prob}(\omega_i = \omega) = g(\omega)$

mean frequency $\Omega = \frac{1}{N}\sum_{j=1}^{N} \omega_j$. Strogatz p. 3

– assume that $g(\omega)$ is unimodal and symmetric, $g(\Omega + \omega) = g(\Omega - \omega)$.



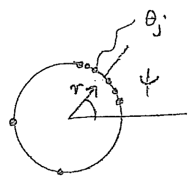
go to frame rotating at frequency Ω: $\theta_i \to \theta_i + \Omega t$.

Eq. (*) is invariant and $g(\omega) = g(-\omega)$.

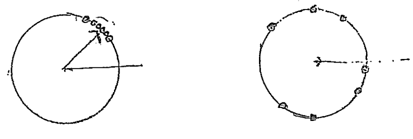
7.4. Order parameter (→ Neural networks p. 45)

Order parameter:

$$r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \qquad (**)$$



$r(t)$ measures phase-coherence, $\psi(t)$ is average phase.



$r \approx 1$ — population in collective rythm

$r \approx 0$ — individual oscillations add incoherently in (**) no macroscopic rythm

Mean-field form of eq. (*)

Multiply eq. (**) with $e^{-i\theta_i}$

$$r e^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}$$

Take imaginary part

$$r\sin(\psi-\theta_i) = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i)$$

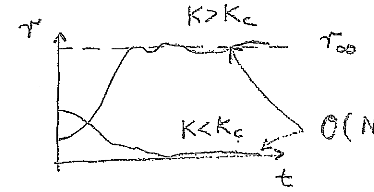
Insert into eq. (*) to obtain

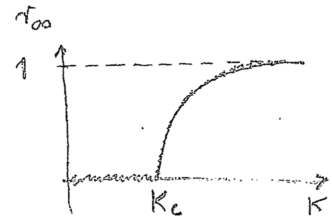
$$\dot\theta_i = \omega_i + Kr\sin(\psi-\theta_i)$$

Coupling only through mean fields r and ψ. This form appears to be consistent with hypothesis on p.40.

7.5. Results of computer simulations for Kuramoto's model (large N)

$g(\omega) = \frac{\gamma}{\pi}\frac{1}{\gamma^2+\omega^2}$ Lorentzian





Questions

- compute K_c and $r_\infty(K)$
- compute apparent stability of $r_\infty=0$-branch below K_c and the bifurcating branch above K_c
- finite N results, convergence as $N\to\infty$?

7.6. Mean-field analysis

$$\dot\theta_i = \omega_i + Kr\sin(\psi-\theta_i)$$

$$r(t)e^{i\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}$$

Consider limit of $N\to\infty$ and seek steady-state solution

$r(t) = \text{const.} \equiv r$

$\psi(t) = \text{const.}$ (Can take $\psi=0$; $\psi(t)$ rotates at frequency Ω, go to rotating frame).

Equation of motion

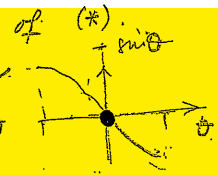
$$\dot\theta_i = \omega_i - Kr\sin(\theta_i) \qquad (*)$$

Self-consistency condition: compute θ_i and determine $r(t)e^{i\psi(t)}$. Must find that $r(t)=\text{const}\equiv r$, and $\psi=0$.

Linear stability analysis of (*).

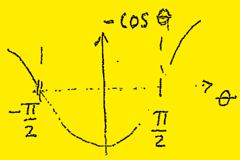
Steady-state condition

$$\omega_i - Kr\sin\theta_i = 0$$



Steady states for $|\omega_i|\le Kr$, since $|\sin\theta_i|\le 1$.

Steady states are stable



$$\frac{d}{d\theta}(-\sin\theta) < 0 \quad \text{for } |\theta_i| < \frac{\pi}{2}$$

For such oscillators $\theta_i^* = \text{const.}$ They are called phase-locked because they are rotating rigidly at frequency Ω in the original frame.

The remaining oscillators are unstable. They rotate in a non-uniform manner, and are called drifting.

locked oscillators ≙ centre of g

drifting oscillators ≙ tails of g

Assume that drifting oscillators are described by stationary distribution

$\rho(\omega,\theta)\,d\theta =$ fraction of oscillators with frequency ω with phase between θ and $\theta+d\theta$

Stationarity requires that $\rho(\omega,\theta)$ must be inversely proportional to speed $|\dot\theta|$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\theta}(v\rho)=0$$

$$\rho(\omega,\theta)=\frac{C}{|\omega-Kr\sin\theta|}$$

where C is determined so that $\int_0^{2\pi} d\theta\,\rho(\omega,\theta)=$

Find

$$C=\frac{\sqrt{\omega^2-K^2r^2}}{2\pi}.$$

Now check for self-consistency

$$\langle e^{i\theta}\rangle=\langle e^{i\theta}\rangle_{drift}+\langle e^{i\theta}\rangle_{lock}$$

$$\langle e^{i\theta}\rangle=re^{i\psi}=r \quad \text{since } \psi=0 \text{ was assumed}$$

Since $\psi=0$ was assumed, $\langle e^{i\theta}\rangle=r$ (remember $\langle e^{i\theta}\rangle=re^{i\psi}$).

Evaluate the two contributions separately.

First locked contribution

$$\sin\theta^*=\frac{\omega}{Kr} \qquad |\omega|\le Kr$$

In the limit of $N\to\infty$ there are just as many oscillators at θ^* as there are at $-\theta^*$ (since $g(\omega)$ is symmetric)

So

average over oscillators

$$\langle\sin\theta\rangle_{lock}=0.$$

Thus

$$\langle e^{i\theta}\rangle_{lock}=\langle\cos\theta\rangle_{lock}$$

$$=\int_{-Kr}^{Kr} d\omega\, g(\omega)\cos[\theta(\omega)]$$

Change variables from ω to θ

$$\langle e^{i\theta}\rangle_{lock}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\,\underbrace{Kr\cos\theta}_{\frac{d\omega}{d\theta}}\,g(Kr\sin\theta)\cos\theta$$

$$=Kr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\cos^2\theta\, g(Kr\sin\theta)$$

Now consider the contribution from the drifting oscillators.

$$\langle e^{i\theta}\rangle_{drift}=\int_{-\pi}^{\pi}d\theta\int_{|\omega|>Kr}d\omega\, g(\omega)\,e^{i\theta}\rho(\theta,\omega)$$

Make use of symmetry (see p. 320)

$$g(\omega)=g(-\omega)$$

$$\rho(\theta+\pi,-\omega)=\rho(\theta,\omega)$$

to show that $\langle e^{i\theta}\rangle_{drift}=0.$

Self-consistency condition

$$r=Kr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\cos^2\theta\, g(Kr\sin\theta).$$

Trivial solution $r=0$: uncoherent state with $\rho(\theta,\omega)=(2\pi)^{-1}$.

Second solution, partially synchronized state given by

$$1=K\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\cos^2\theta\, g(Kr\sin\theta)$$

Determine K_c (p. 317) by letting $r\to 0^+$

$$K_c=\frac{1}{g(0)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cos^2\theta}=\frac{2}{\pi g(0)}.$$

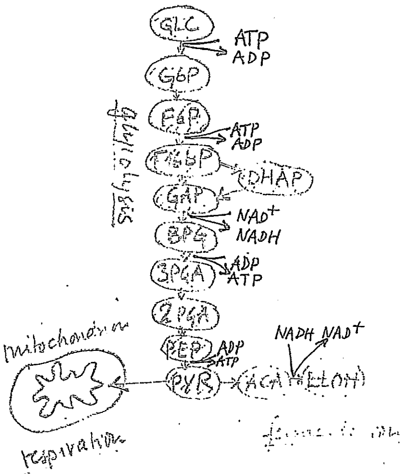
What is the functional form of $r(K)$ for $K>K_c$? Expand self-consistency condition in powers of r.

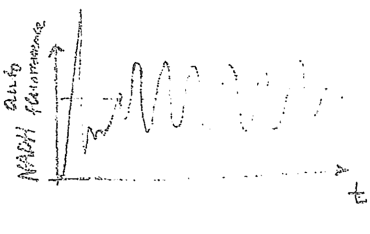
$$1=\frac{K}{K_c}+\frac{K(Kr)^2}{2}g''(0)\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cos^2\theta\sin^2\theta}_{=\frac{\pi}{8}}+\dots$$

First order vanishes since $g'(0)=0$.

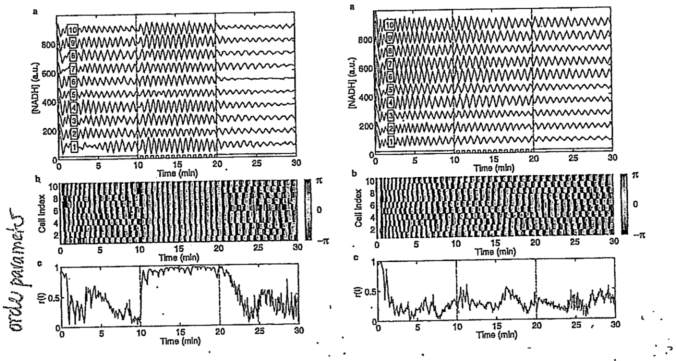
7.7. glycolytic oscillations in yeast cells

glycolysis in yeast cells. Metabolic oscillations.





Entrainment of NADH oscillations by cyanide concentration oscillations



entrainment　　　　no entrainment

A-K Gustavsson et al. Sci. Rep. 5 (2015)