Problem Set 2

Erik Norlin
Karl Lundgren

February, 2023

Problem 2.1

a) Here we rewrite the dynamics to dimensionless parameters, neglect diffusion and calculating the spatially homogeneous steady states.

$$\frac{\partial n}{\partial t} = rn\left(1 - \frac{n}{k}\right) - \frac{An}{1 + n/B} + D\frac{\partial^2 n}{\partial x^2} \tag{1}$$

$$t = \frac{\tau}{A},\ x = \frac{\xi}{\sqrt{A/D}},\ n = uB \tag{2}$$

$$\Rightarrow \frac{\partial(uB)}{\partial(\tau/A)} = ruB\left(1 - \frac{uB}{k}\right) - \frac{AuB}{1 + uB/B} + D\frac{\partial^2(uB)}{\partial(\frac{\xi}{\sqrt{(A/D)}})^2} \tag{3}$$

$$\Rightarrow AB\frac{\partial u}{\partial \tau} = ruB\left(1 - \frac{uB}{k}\right) - \frac{AuB}{1 + u} + \frac{DAB}{D}\frac{\partial^2 u}{\partial \xi^2} \tag{4}$$

$$\Rightarrow \frac{\partial u}{\partial \tau} = \frac{ru}{A}\left(1 - \frac{uB}{k}\right) - \frac{u}{1 + u} + \frac{\partial^2 u}{\partial \xi^2} \tag{5}$$

$$\rho = \frac{r}{A},\ q = \frac{k}{B} \tag{6}$$

$$\Rightarrow \frac{\partial u}{\partial \tau} = \rho u\left(1 - \frac{u}{q}\right) - \frac{u}{1 + u} + \frac{\partial^2 u}{\partial \xi^2} \tag{7}$$

Neglecting diffusion for deriving the spatially homogenous steady states as following.

$$\Rightarrow \frac{\partial u}{\partial \tau} = \rho u\left(1 - \frac{u}{q}\right) - \frac{u}{1 + u} = 0 \tag{8}$$

$$\Rightarrow u\left(\rho\left(1 - \frac{u}{q}\right) - \frac{1}{1 + u}\right) = 0 \tag{9}$$

$$u_1^* = 0 \tag{10}$$

$$\Rightarrow \rho(1+u)\left(1-\frac{u}{q}\right)-1=0 \tag{11}$$

$$\Rightarrow \rho\left(1+u-\frac{u}{q}-\frac{u^2}{q}\right)-1=0 \tag{12}$$

$$\Rightarrow 1+u-\frac{u}{q}-\frac{u^2}{q}-\frac{1}{\rho}=0 \tag{13}$$

$$\Rightarrow u^2+u-qu-q+\frac{q}{\rho}=0 \tag{14}$$

$$\Rightarrow u^2+u(1-q)+q\left(\frac{1}{\rho}-1\right)=0 \tag{15}$$

$$\Rightarrow u^*_{2,3}=\frac{q-1}{2}\pm\sqrt{\frac{(1-q)^2}{4}-q\left(\frac{1}{\rho}-1\right)} \tag{16}$$

Ans: The spatially homogeneous steady states of the dimensionless system without diffusion in terms of ρ and q are

$$u^*_1=0,\; u^*_{2,3}=\frac{q-1}{2}\pm\sqrt{\frac{(1-q)^2}{4}-q\left(\frac{1}{\rho}-1\right)} \tag{17}$$

b)

$$\rho=0.5,\; q=8 \tag{18}$$

$$\Rightarrow u^*_1=0,\; u^*_2=1.43845,\; u^*_3=5.56155 \tag{19}$$

i) $\xi_0=20$ and $u_0=u^*_3=5.56155$

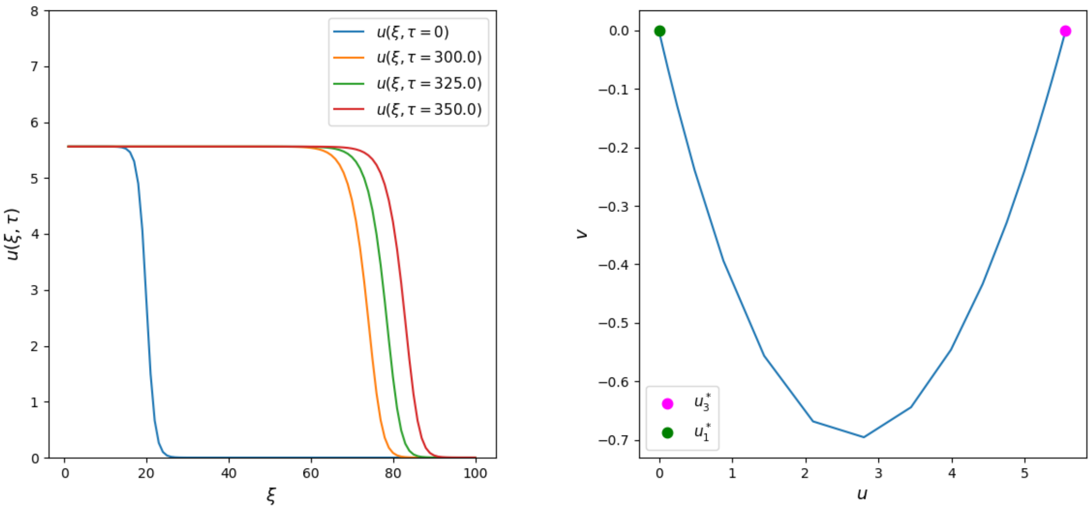

Figure 1: For case (i), in the plot to the left we can see that a travelling wave is formed and is travelling with a positive velocity because ξ is increasing as τ is increasing. The population is diffusing and is converging around the fixed point u_3^*. In the phase plane in the plot to the right ($\tau = 325$), the travelling wave connects two fixed points, u_1^* and u_3^*.

In figure 1 we can see that a travelling wave connects the two fixed points, u_1^* and u_3^*, in phase space. We compute the velocity c of the wave by choosing two points with different values of ξ and τ and the same value of u. The velocity can then be easily computed with $c = \frac{\Delta \xi}{\Delta \tau}$. We then do the same thing again for another time interval to make sure that we have a fully evolved travelling wave with a constant velocity (see figure 1). For case (i) the velocity of the wave is $c \approx 0.18$

To classify the fixed points that the travelling wave connect in phase space we first make a change of variables from two to one. To do this we make the ansatz that

$$u(\xi, \tau) = U(\xi - c\tau) = U(z) \tag{20}$$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial \tau} = -c\dfrac{dU}{dz} & (21) \\ \dfrac{\partial u}{\partial \xi} = \dfrac{dU}{dz} & (22) \end{cases}$$

Eq. 7 becomes

$$\Rightarrow -c\frac{\partial U}{\partial z} = \rho U \left(1 - \frac{U}{q}\right) - \frac{U}{1+U} + \frac{\partial^2 U}{\partial z^2} \tag{23}$$

We introduce a help variable $v = \frac{dU}{dz}$ to get a coupled dynamical system with first order differential equations.

$$\Rightarrow \begin{cases} \dfrac{\partial U}{\partial z} = v & (24) \\ \dfrac{\partial v}{\partial z} = \dfrac{U}{1+U} - cv - \rho U\left(1 - \dfrac{U}{q}\right) & (25) \end{cases}$$

The fixed points turns out to be the same as before.

$$\Rightarrow U_1^* = 0,\, U_2^* = 1.43845,\, U_3^* = 5.56155 \tag{26}$$

We use linear stability analysis to classify the fixed points that connect in the phase space, i.e. $U_3^* = 5.56155$ and $U_1^* = 0$. The jacobian of the dynamical system becomes

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ \frac{1}{1+U} - \frac{U}{(1+U)^2} - \rho + \frac{2\rho U}{q} & -c \end{bmatrix}$$

For case (i) with $U_3^* = 5.56155$ and $c = 0.18$ the jacobian becomes

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 0.21842 & -0.18 \end{bmatrix}$$

Where TrJ = -0.18 and $\Delta = -0.21842$. This implies that $U_3^* = u_3^*$ is a saddle point ($\Delta < 0$). With $U_1^* = 0$ the jacobian becomes

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.18 \end{bmatrix}$$

Where TrJ = -0.18 and $\Delta = -0.5$. This implies that $U_1^* = u_1^*$ is a saddle point ($\Delta < 0$).

Ans: For the case of (i) we have a travelling wave travelling with a constant velocity of 0.18, meaning that the population grows and spreads with time. In figure 1 we can see that the travelling wave connects the two fixed points, U_1^* and U_3^*, in phase space. These two fixed points are both classified as saddle points.

ii) $\xi_0 = 50$ and $u_0 = u_2^* = 1.43845$

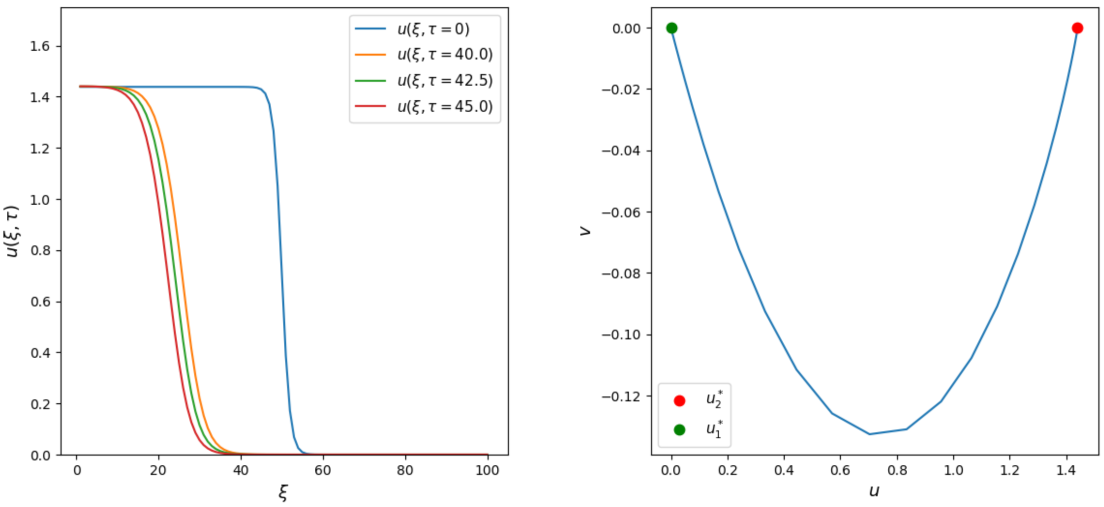

Figure 2: For case (ii), in the left plot we can see that a travelling wave is formed and is travelling with a negative velocity because ξ is decreasing as τ is increasing. The population is diffusing and is converging around u_1^*. In the phase plane in the plot to the right ($\tau = 42.5$), the travelling wave connects two fixed points, u_1^* and u_2^*.

In figure 2 we can see that a travelling wave connects the two fixed points, u_1^* and u_2^*, in phase space. We compute the velocity c of the wave the same way as before. For case (ii) the velocity is $c \approx -0.7$. We also assume the same ansatz as before so we have the same system as (23) and (24). The jacobian at the fixed point U_2^* with $c = -0.7$ becomes

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -0.152015 & 0.7 \end{bmatrix}$$

Where $\text{TrJ} = 0.7$ and $\Delta = 0.152015$. Furthermore,

$$\frac{\text{TrJ}^2}{4} = \frac{0.7^2}{4} = 0.1225 < \Delta \tag{27}$$

This implies that $U_2^* = u_2^*$ is an unstable spiral ($\text{TrJ} > 0$ and $\Delta > \frac{\text{TrJ}^2}{4}$). With $U_1^* = 0$ the jacobian becomes

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.7 \end{bmatrix}$$

Where $\text{TrJ} = 0.7$ and $\Delta = -0.5$. This implies that $U_1^* = u_1^*$ is a saddle point ($\Delta < 0$), just as before.

Ans: For the case of (ii) we have a travelling wave travelling with a constant velocity of -0.7, meaning that the population decays with time. In figure 2 we can see that the travelling wave connects the two fixed points, U_1^* and U_2^*, in phase space, classified as a saddle point and an unstable spiral respectively.

iii) $\xi_0 = 50$ and $u_0 = 1.1 * u_2^* = 1.1 * 1.43845$

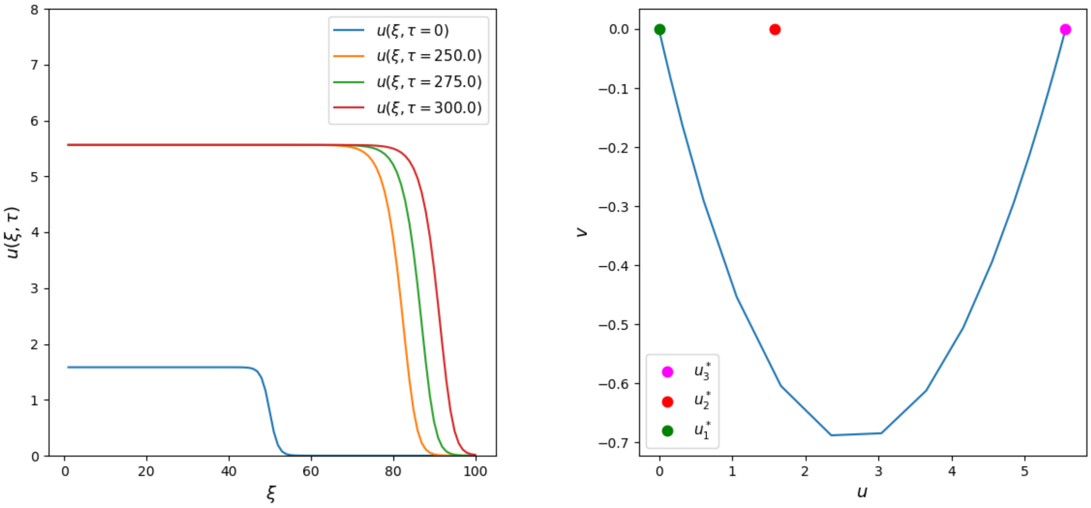

Figure 3: For case (iii), in the left plot we can see that a travelling wave is formed and is travelling with a positive velocity because ξ is increasing as τ is increasing. The population is diffusing and is converging around u_3^*. The "jump" in the dynamics is due to u_2^* turning out to be an unstable fixed point, causing the dynamics to converge to u_3^*. In the phase plane in the plot to the right ($\tau = 275$), the travelling wave connects two fixed points, u_1^* and u_3^*.

In figure 3 we can see that a travelling wave connects the two fixed points, u_1^* and u_3^*, in phase space. We compute the velocity c the same way as before. For case (iii) the velocity is $c \approx 0.18$. We also assume the same ansatz as before so we have the same system as (23) and (24). The jacobian at the fixed point U_3^* with $c = 0.18$ becomes

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 0.21842 & -0.18 \end{bmatrix}$$

Where TrJ = -0.18 and $\Delta = -0.21842$. This implies that $U_3^* = u_3^*$ is a saddle point ($\Delta < 0$). With $U_1^* = 0$ the jacobian becomes

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.18 \end{bmatrix}$$

Where TrJ = -0.18 and $\Delta = -0.5$. This implies that $U_1^* = u_1^*$ is a saddle point ($\Delta < 0$).

Ans: For the case of (iii) we have a travelling wave travelling with a constant velocity of 0.18, meaning that the population grows and expands with time. In figure 3 we can see that the travelling wave connects the two fixed points, U_1^* and U_3^*, in phase space, both classified as a saddle points.

c) For the follwing cases we get

i) $\xi_0 = 50$ and $u_0 = u_3^* = 5.56155$

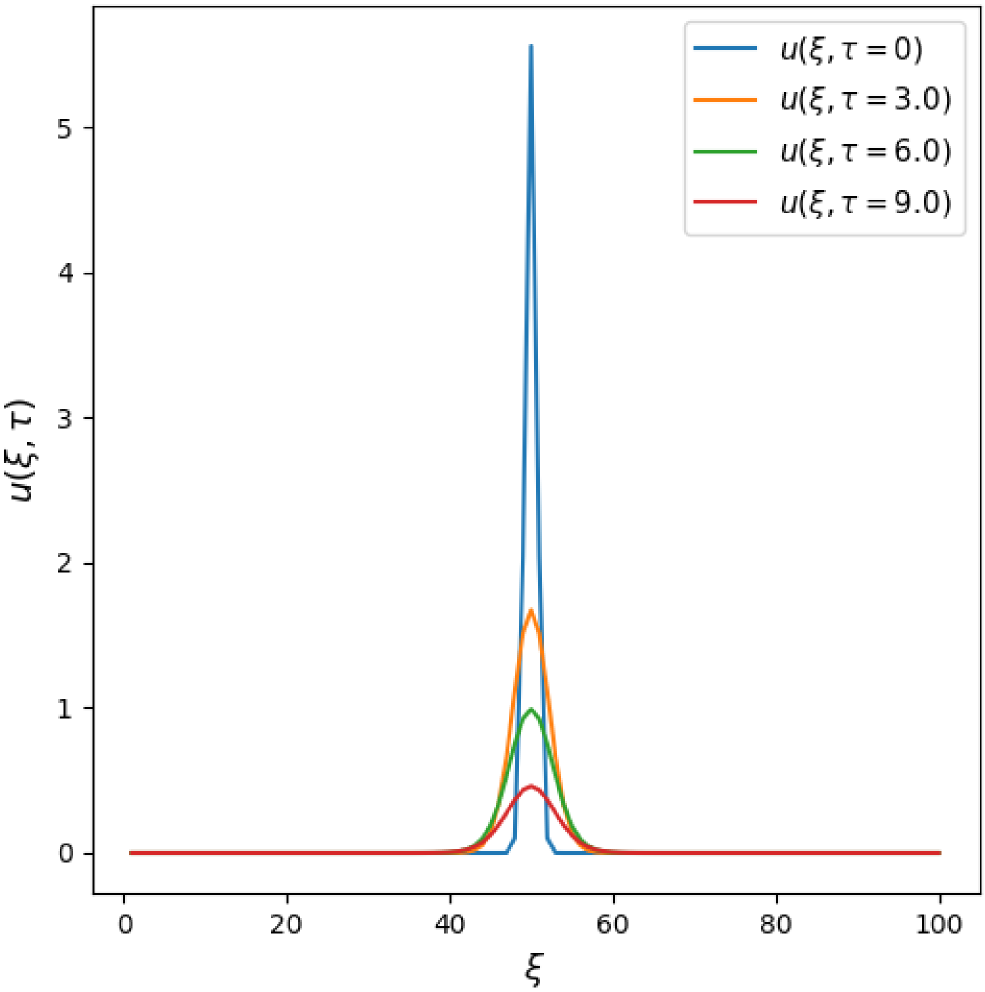

Figure 4: Wave profile for case $u_0 = u_3^* = 5.56155$. For this case we get no travelling wave as can be seen here, because the dynamics converges to $u_1^* = 0$ as the time increases, giving the dynamics not a chance to evolve into a travelling wave.

ii) $\xi_0 = 50$ and $u_0 = 3 * u_3^* = 3 * 5.56155$

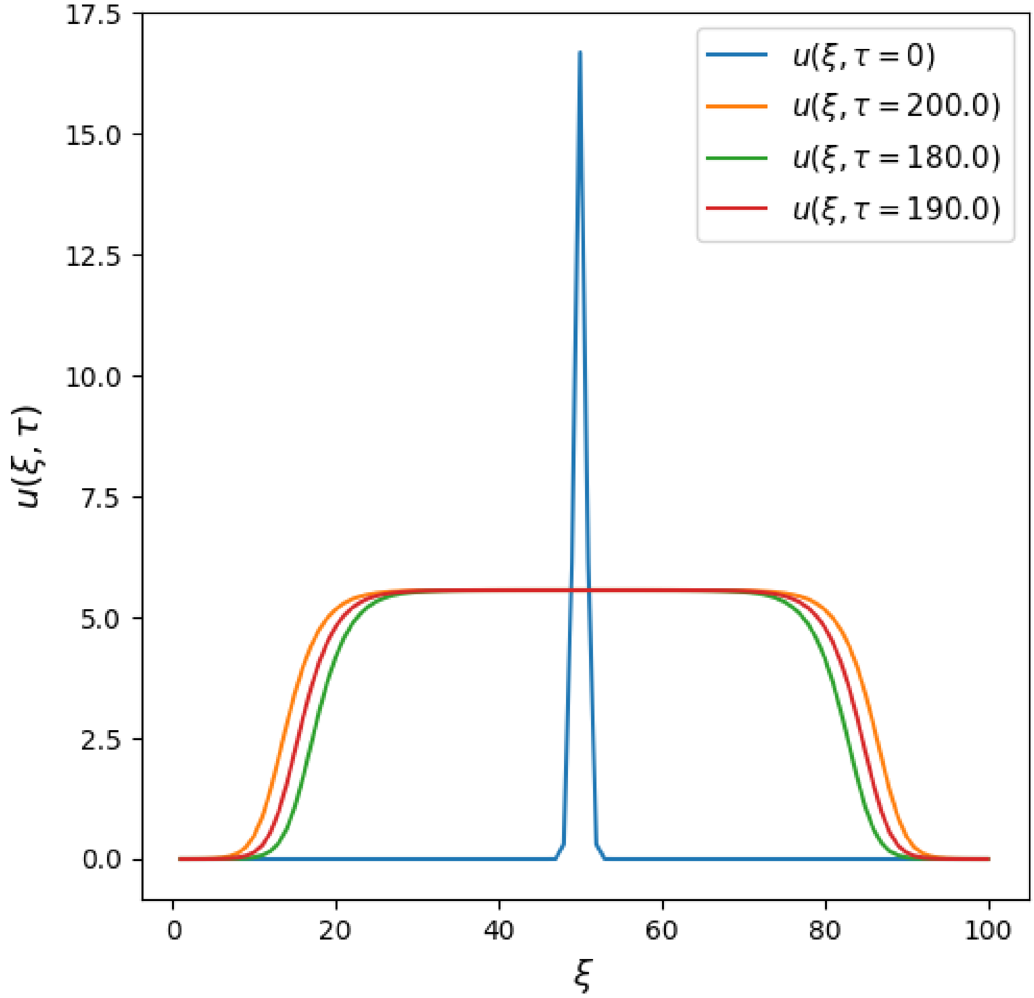

Figure 5: Wave profile for case $u_0 = 3 * u_3^* = 3 * 5.56155$. For this case we get travelling waves in two directions as can be seen here. The travelling waves have constant positive velocities of approximately 0.18 in each direction, causing the population to diffuse and expand, and finally converges to u_3^*.

Ans: For case (i) we get no travelling waves, as can be seen in figure 4. For case (ii) we get two travelling waves that causes the population to diffuse and expand as can be seen in figure 5.

Appendix

Code for simulations (Python), 1b

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
# import matplotlib.backends.backend_qt5agg
import PyQt5
import matplotlib as mpl
```

```
# import latex
mpl.use('Qt5Agg')
# plt.rcParams['text.usetex'] = True

location = r'C:\Users\erikn\OneDrive - Chalmers\Computational Biology\CB HW 2'
task = 3

rho = 0.5
q = 8

L = 100
xi_list = np.linspace(1,L,L)

xi_0 = 20
u0_1 = 5.56155

xi_0 = 50
u0 = 1.43845

xi_0 = 50
u0 = 1.1 * 1.43845

T = 400
dt = 0.1
t_list = np.linspace(0, T, int(T/dt))

def u0_wave_func(u0):

    u_list = np.zeros((len(t_list),len(xi_list)))
    # u = u0

    for i in range(L):

        xi = xi_list[i]
        u = u0 / (1+np.exp(xi-xi_0))
        u_list[0,i] = u

    return u_list

u_list = u0_wave_func(u0)

for t in range(len(t_list)-1):
    for j in range(len(xi_list)):

        xi = xi_list[j]

        if xi == 1:
            diffusion = (u_list[t, j+1] - u_list[t, 0]) / 1**2
        elif xi == L:
            diffusion = -(u_list[t, j] - u_list[t, j-1]) / 1**2
        else:
```

```
            diffusion = (u_list[t, j+1] + u_list[t, j-1] - 2*u_list[t, j]) /
                ↪ 1**2

        u = u_list[t,j]
        u_list[t+1,j] = u + (rho*u*(1-u/q) - u/(1+u) + diffusion)*dt

tau_fixed_early = 2500
tau_fixed_mid = 2750
tau_fixed_late = 3000

fig1, ax = plt.subplots( figsize=(6,6))
ax.plot(xi_list, u_list[0,:], label='$u(\\xi, \\tau=0)$')
ax.plot(xi_list, u_list[tau_fixed_early,:], label='$u(\\xi, \\tau={})$'.format(
    ↪ tau_fixed_early*dt))
ax.plot(xi_list, u_list[tau_fixed_mid,:], label='$u(\\xi, \\tau={})$'.format(
    ↪ tau_fixed_mid*dt))
ax.plot(xi_list, u_list[tau_fixed_late,:], label='$u(\\xi, \\tau={})$'.format(
    ↪ tau_fixed_late*dt))
ax.set_xlabel('$\\xi$', fontsize=13)
ax.set_ylabel('$u(\\xi, \\tau)$', fontsize=13)
ax.set_ylim((0,8))
ax.set_box_aspect(1)
plt.legend(loc="upper right", prop={'size': 11})

location = r'C:\Users\erikn\OneDrive - Chalmers\Computational Biology\CB HW 2'
title = '/2.1b_wave_{}'.format(task)
plt.savefig(location+title+'.png')


v_list = np.zeros_like(u_list[0,:])

for i in range(len(v_list)-1):
    v = (u_list[tau_fixed_mid,i+1] - u_list[tau_fixed_mid,i]) / 1
    v_list[i] = v

v_last = len(v_list)-1
v_list[v_last] = v_list[v_last-1]

fig2, ax = plt.subplots(figsize=(6,6))
ax.plot(u_list[tau_fixed_mid,:], v_list)
ax.plot(u0_1,0,'.', markersize=15, color='magenta', label='$u_3^*$')
ax.plot(u0,0,'.', markersize=15, color='red', label='$u_2^*$')
ax.plot(0,0,'.', markersize=15, color='green', label='$u_1^*$')
ax.set_xlabel('$u$', fontsize=13)
ax.set_ylabel('$v$', fontsize=13)
plt.legend(loc="lower left", prop={'size': 11})
ax.set_box_aspect(1)

title = '/2.1b_phase_{}'.format(task)
plt.savefig(location+title+'.png')
```

```
plt.show()
```

Code for simulations (Python), 1c

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
# import matplotlib.backends.backend_qt5agg
import PyQt5
import matplotlib as mpl
# import latex
mpl.use('Qt5Agg')
# plt.rcParams['text.usetex'] = True

location = r'C:\Users\erikn\OneDrive - Chalmers\Computational Biology\CB HW 2'
task = 2

rho = 0.5
q = 8

L = 100
xi_list = np.linspace(1,L,L)

xi_0 = 50
u0 = 5.56155 * 3
# u0 = 5.56155

T = 400
dt = 0.1
t_list = np.linspace(0, T, int(T/dt))

def u0_wave_func(u0):

    u_list = np.zeros((len(t_list),len(xi_list)))
    # u = u0

    for i in range(L):

        xi = xi_list[i]
        u = u0*np.exp(-(xi-xi_0)**2)
        u_list[0,i] = u

    return u_list

u_list = u0_wave_func(u0)

for t in range(len(t_list)-1):
    for j in range(len(xi_list)):

        xi = xi_list[j]
```

```
        if xi == 1:
            diffusion = (u_list[t, j+1] - u_list[t, 0]) / 1**2
        elif xi == L:
            diffusion = -(u_list[t, j] - u_list[t, j-1]) / 1**2
        else:
            diffusion = (u_list[t, j+1] + u_list[t, j-1] - 2*u_list[t, j]) /
                ↪ 1**2

        u = u_list[t,j]
        u_list[t+1,j] = u + (rho*u*(1-u/q) - u/(1+u) + diffusion)*dt

# next wave 0.7 or -0.7, last 0.18

tau_fixed_early = 2000
tau_fixed_mid = 1800
tau_fixed_late = 1900

fig1, ax = plt.subplots( figsize=(6,6))
ax.plot(xi_list, u_list[0,:], label='$u(\\xi, \\tau=0)$')
ax.plot(xi_list, u_list[tau_fixed_early,:], label='$u(\\xi, \\tau={})$'.format(
    ↪ tau_fixed_early*dt))
ax.plot(xi_list, u_list[tau_fixed_mid,:], label='$u(\\xi, \\tau={})$'.format(
    ↪ tau_fixed_mid*dt))
ax.plot(xi_list, u_list[tau_fixed_late,:], label='$u(\\xi, \\tau={})$'.format(
    ↪ tau_fixed_late*dt))
ax.set_xlabel('$\\xi$', fontsize=13)
ax.set_ylabel('$u(\\xi, \\tau)$', fontsize=13)
# ax.set_ylim((0,7.5))
ax.set_box_aspect(1)
plt.legend(loc="upper right", prop={'size': 11})

location = r'C:\Users\erikn\OneDrive - Chalmers\Computational Biology\CB HW 2'
title = '/2.1c_wave_{}'.format(task)
plt.savefig(location+title+'.png')


# v_list = np.zeros_like(u_list[0,:])

# for i in range(len(v_list)-1):
# v = (u_list[tau_fixed_mid,i+1] - u_list[tau_fixed_mid,i]) / 1
# v_list[i] = v

# v_last = len(v_list)-1
# v_list[v_last] = v_list[v_last-1]

# fig2, ax = plt.subplots(figsize=(6,6))
# ax.plot(u_list[tau_fixed_mid,:], v_list)
# ax.plot(u0,0,'.', markersize=15, color='black', label='Unstable spiral')
# ax.plot(0,0,'.', markersize=15, color='red', label='Saddle point')
# ax.set_xlabel('$u$', fontsize=13)
```

```
# ax.set_ylabel('$v$', fontsize=13)
# plt.legend(loc="lower left", prop={'size': 11})
# ax.set_box_aspect(1)

# title = '/2.1c_phase'
# plt.savefig(location+title+'.png')

plt.show()
```