Problem 3

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a)

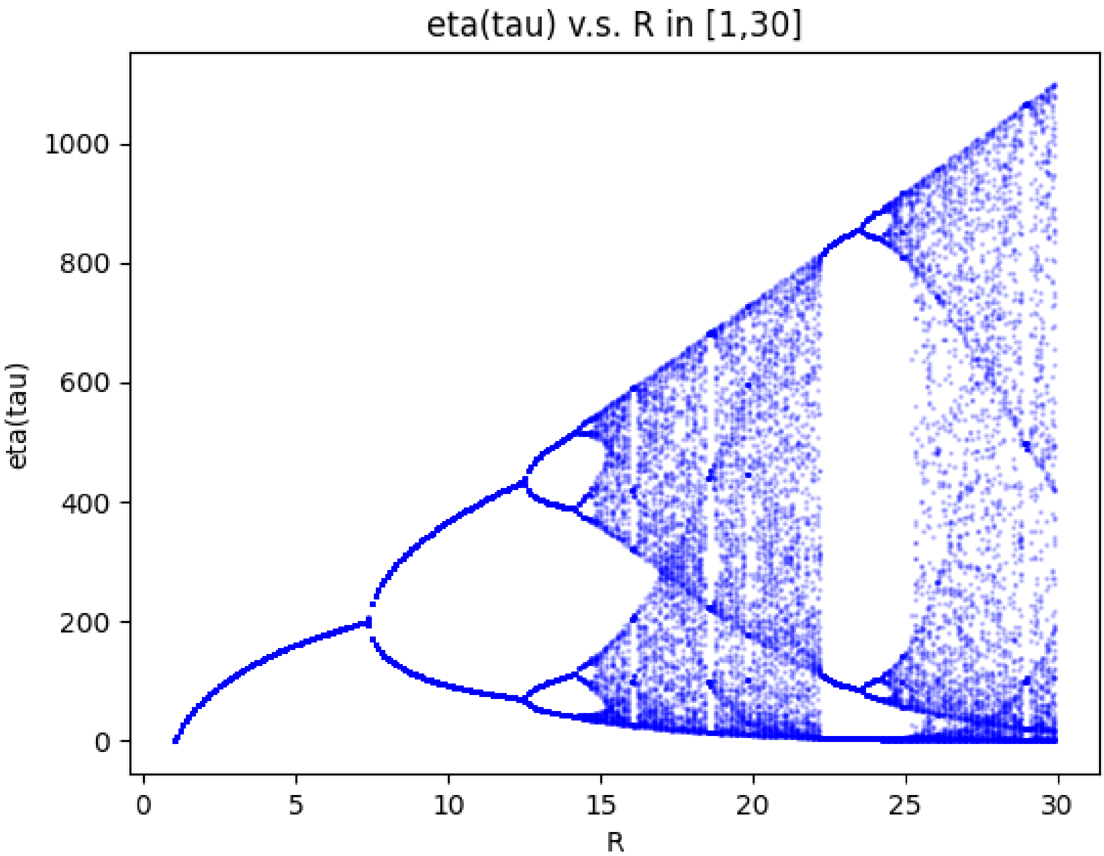


Figure 1: $\eta(\tau)$ vs R

As anticipated, we observe period doubling bifurcations leading up to chaotic periods of infinite-period orbits as we increase R. In between certain chaotic regions, we also see some more regular behavior in the dynamics.

b)

As per the bifurcation diagram in part a) (Figure 1), we should have a stable fixed point, 2-point, 3-point, and 4-point cycles at the values 5, 10, 23, and 13 respectively. This is exactly what we observe by plotting the corresponding orbits.

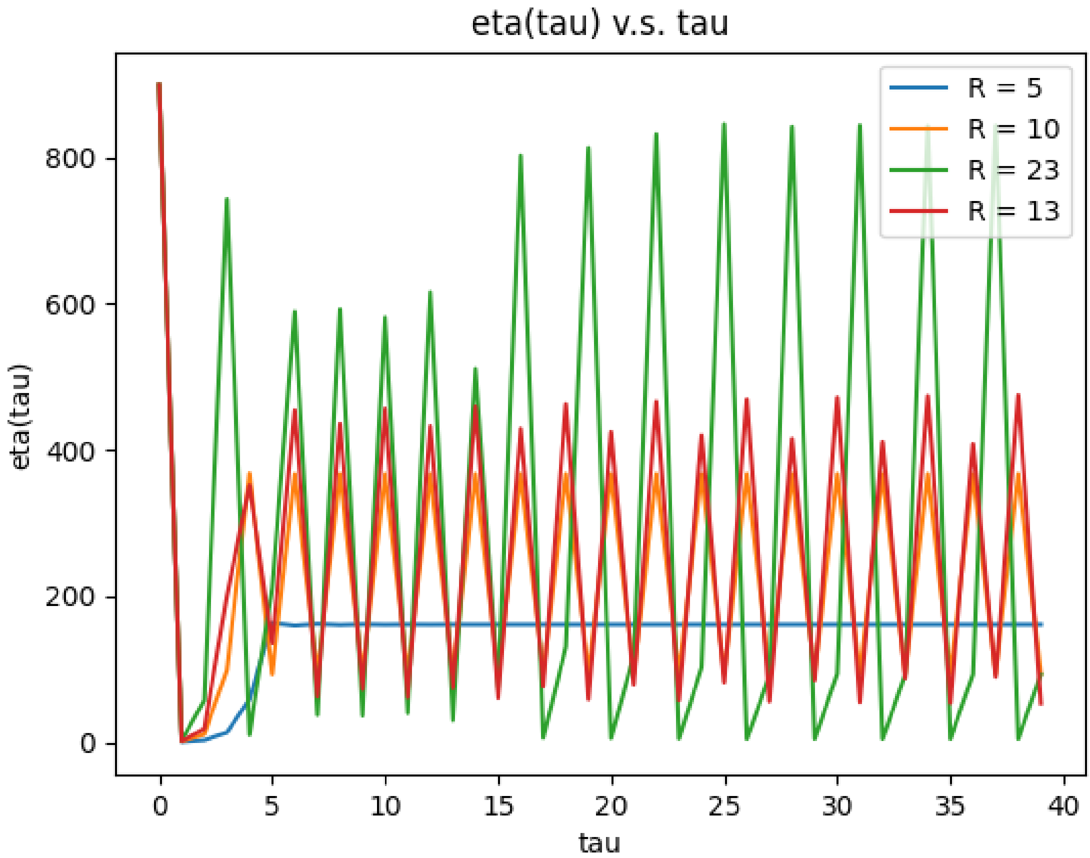
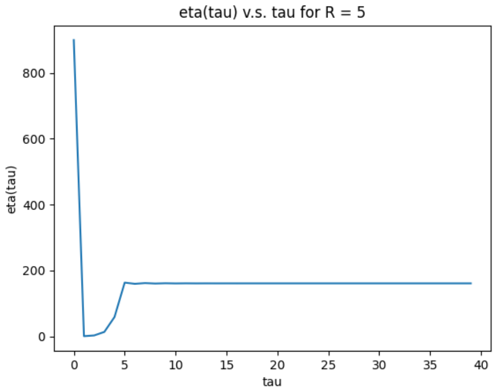
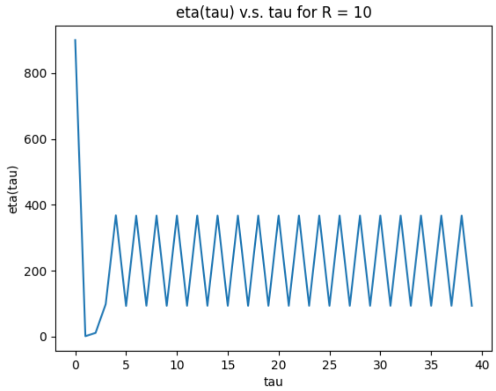


Figure 2: $\eta(\tau)$ vs τ for 4 different values of R

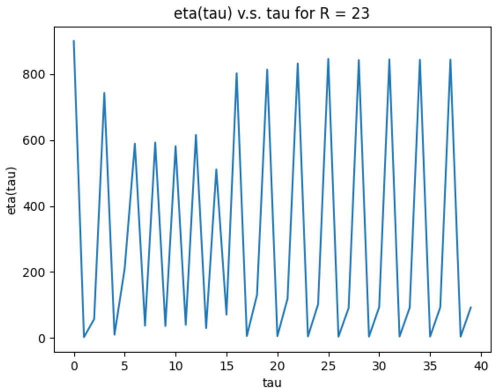
Plotting the four cases in separate plots make it easier to count track the orbits, and we can confirm our prediction in all of the cases.



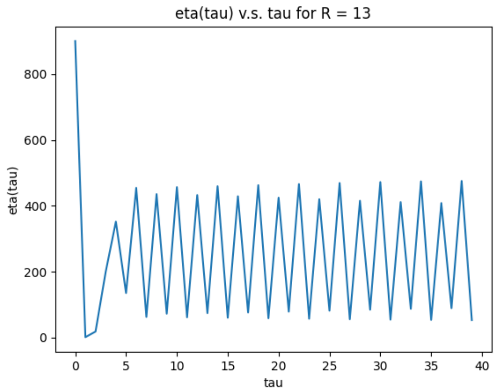
(a) $R = 5$



(b) $R = 10$



(c) $R = 23$



(d) $R = 13$

Figure 3: Caption

c)

By zooming in very closely, we can observe that the first bifurcation, where the population dynamics go from a stable fixed point to a 2-point cycle happens at approximately $R \approx 7.1$. This, we see from the orbits start to distribute over two distinct points rather than one as is the case before the bifurcation has taken place.

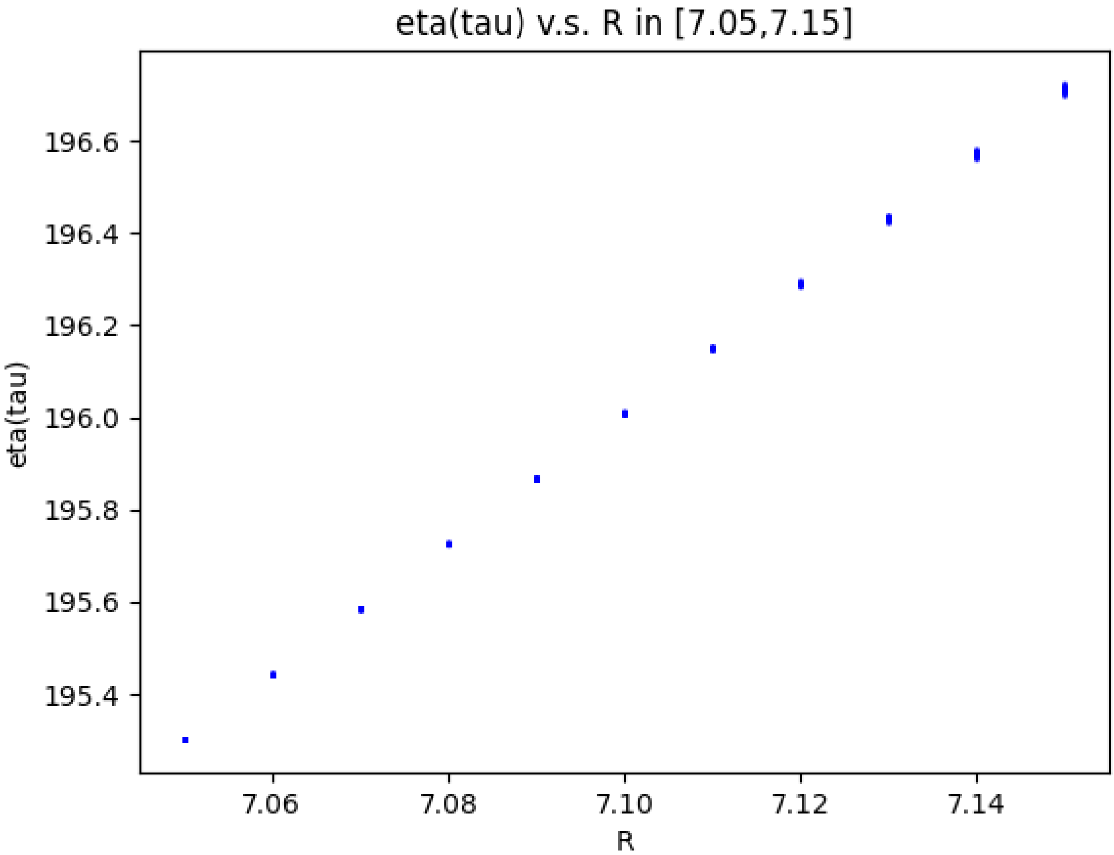


Figure 4: $\eta(\tau)$ vs R at first bifurcation.

Using the same procedure as for the first bifurcation, we can determine that the second one, where the period doubles to make a 4-point cycle occur at $R \approx 12.4$. It is a bit tricky to notice right where this happens, but at $R = 12.44$, two distinct points have definitely emerged in both branches, although they look look melted together due to the size of the markers in the plot.

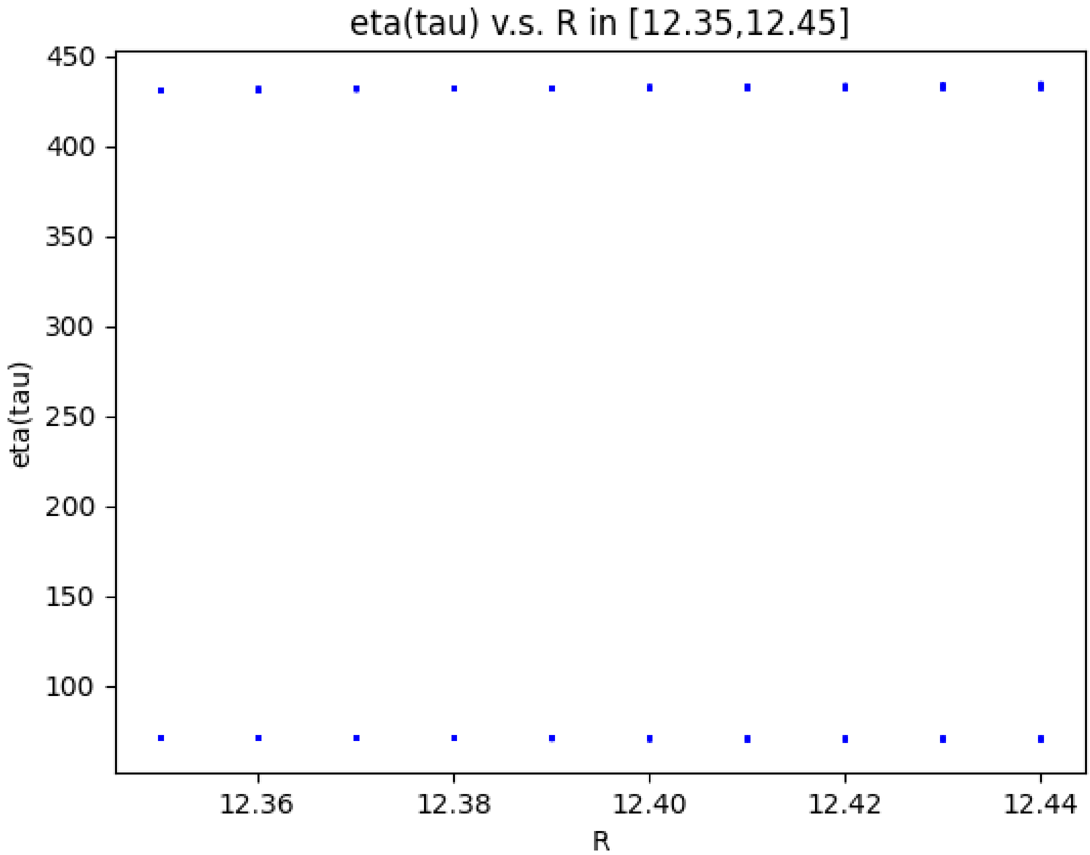


Figure 5: $\eta(\tau)$ vs R at second bifurcation.

d)

By studying the bifurcation diagram in part a) (Figure 1), we see that R_∞ seem to occur after $R = 14$, fairly close to $R = 15$. By refining the intervals in R at which we iterate the dynamics to steps of 0.01, and zooming in on this region, the two last period doubling bifurcations we are able to observe are estimated to occur at $R = 14.64$, and $R = 14.74$ respectively. Using these values, we can estimate R_∞ using Feigenbaum's constant $F \approx 4.669$ and an infinite sum:

$$R_\infty = R_k + \sum_{n=0}^{\infty} \frac{R_{k+1} - R_k}{F^n} \approx 14.64 + \sum_{n=0}^{\infty} \frac{14.74 - 14.64}{4.669^n} \approx 14.8.$$

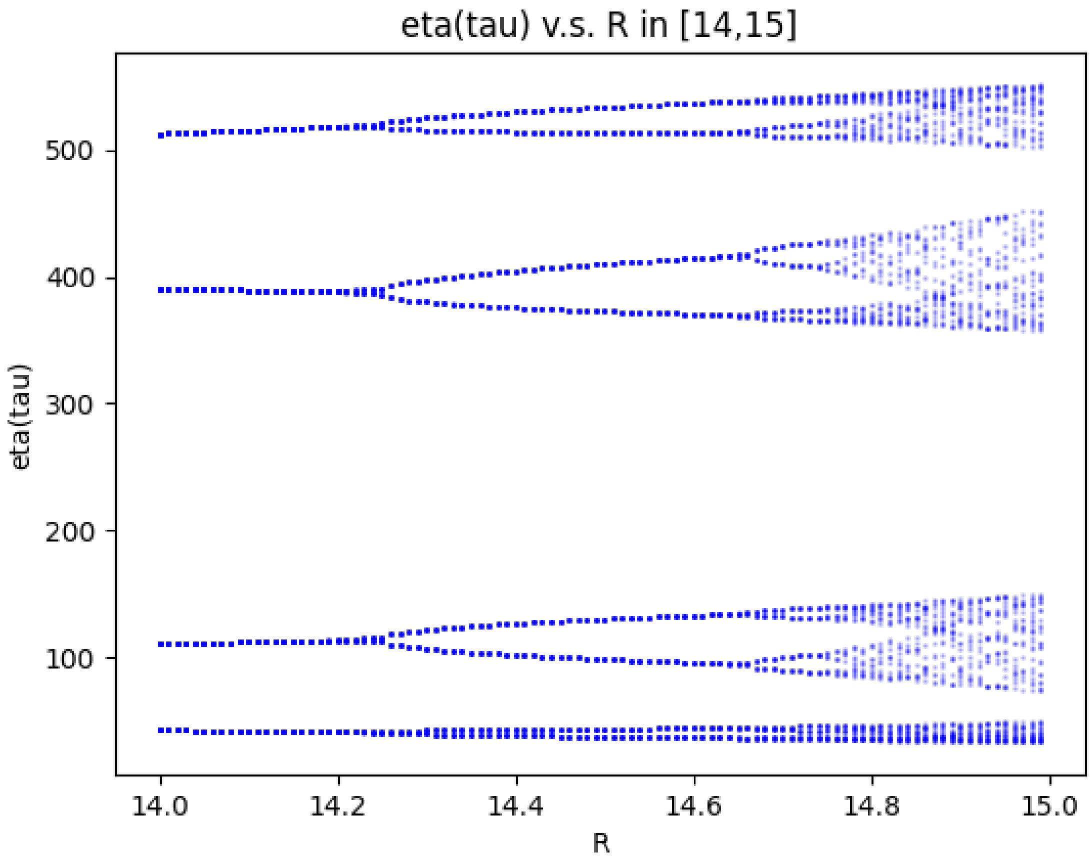


Figure 6: $\eta(\tau)$ vs R at second bifurcation.

Appendix

Code for simulations (Python)

```
import numpy as np
import matplotlib.pyplot as plt


def etaNext(eta,R,alpha,eta0):

    return R*eta*np.exp(-alpha*eta)

alpha = 0.01
eta0 = 900
```

```
#part a)
R = np.arange(1,30,0.1)
orbits = np.ones_like(R) * eta0
etaList = [orbits]

for t in range(300):
    etaList.append(etaNext(np.array(etaList[-1]),R,alpha,eta0))

etaNp = np.array(etaList[-101:-1])
print(etaNp.shape, R.shape)

for i in range(100):
    plt.plot(R,etaNp[i],linestyle='none',marker='o',markersize=1,color='blue',alph
plt.title('eta(tau) v.s. R in [1,30]')
plt.xlabel('R')
plt.ylabel('eta(tau)')
plt.savefig('hw1_3a.png')
plt.show()

#part b)
R = [5,10,23,13]
orbits =  np.ones_like(R) * eta0
etaList = [orbits]

for t in range(1,40):
    etaList.append(etaNext(np.array(etaList[-1]),R,alpha,eta0))

etaNp = np.array(etaList)
print(etaNp.shape)
legends = []
for i in range(len(R)):
    plt.plot(np.arange(40),etaNp[:,i])
    plt.title('eta(tau) v.s. tau for R = ' + str(R[i]))
    plt.xlabel('tau')
    plt.ylabel('eta(tau)')
    plt.savefig('hw1_3b2_R' + str(R[i]))
    plt.show()
    legends.append('R = ' + str(R[i]))
#plt.title('eta(tau) v.s. tau')
#plt.xlabel('tau')
```

```
#plt.ylabel('eta(tau)')
#plt.legend(legends)
#plt.savefig('hw1_3b.png')
#plt.show()


#part c)
R = np.arange(7.05,7.15,0.01)
orbits = np.ones_like(R) * eta0
etaList = [orbits]

for t in range(300):
    etaList.append(etaNext(np.array(etaList[-1]),R,alpha,eta0))

etaNp = np.array(etaList[-101:-1])
print(etaNp.shape, R.shape)

for i in range(100):
    plt.plot(R,etaNp[i],linestyle='none',marker='o',markersize=1,color='blue',alph
plt.title('eta(tau) v.s. R in [7.05,7.15]')
plt.xlabel('R')
plt.ylabel('eta(tau)')
plt.savefig('hw1_3c1')
plt.show()


R = np.arange(12.35,12.45,0.01)
orbits = np.ones_like(R) * eta0
etaList = [orbits]

for t in range(300):
    etaList.append(etaNext(np.array(etaList[-1]),R,alpha,eta0))

etaNp = np.array(etaList[-101:-1])
print(etaNp.shape, R.shape)

for i in range(100):
    plt.plot(R,etaNp[i],linestyle='none',marker='o',markersize=1,color='blue',alph
plt.title('eta(tau) v.s. R in [12.35,12.45]')
plt.xlabel('R')
```

```
plt.ylabel('eta(tau)')
plt.savefig('hw1_3c2')
plt.show()


#part d)
R = np.arange(14,15,0.01)
orbits = np.ones_like(R) * eta0
etaList = [orbits]

for t in range(300):
    etaList.append(etaNext(np.array(etaList[-1]),R,alpha,eta0))

etaNp = np.array(etaList[-101:-1])
print(etaNp.shape, R.shape)

for i in range(100):
    plt.plot(R,etaNp[i],linestyle='none',marker='o',markersize=1,color='blue',alph
plt.title('eta(tau) v.s. R in [14,15]')
plt.xlabel('R')
plt.ylabel('eta(tau)')
plt.savefig('hw1_3dR14_15.png')
plt.show()


R = np.arange(14.6,14.8,0.001)
orbits = np.ones_like(R) * eta0
etaList = [orbits]

for t in range(300):
    etaList.append(etaNext(np.array(etaList[-1]),R,alpha,eta0))

etaNp = np.array(etaList[-101:-1])
print(etaNp.shape, R.shape)

for i in range(100):
    plt.plot(R,etaNp[i],linestyle='none',marker='o',markersize=1,color='blue',alph
plt.title('eta(tau) v.s. R in [14.6,14.8]')
plt.xlabel('R')
plt.ylabel('eta(tau)')
```

```
plt.savefig('hw1_3dR146_148.png')
plt.show()
```