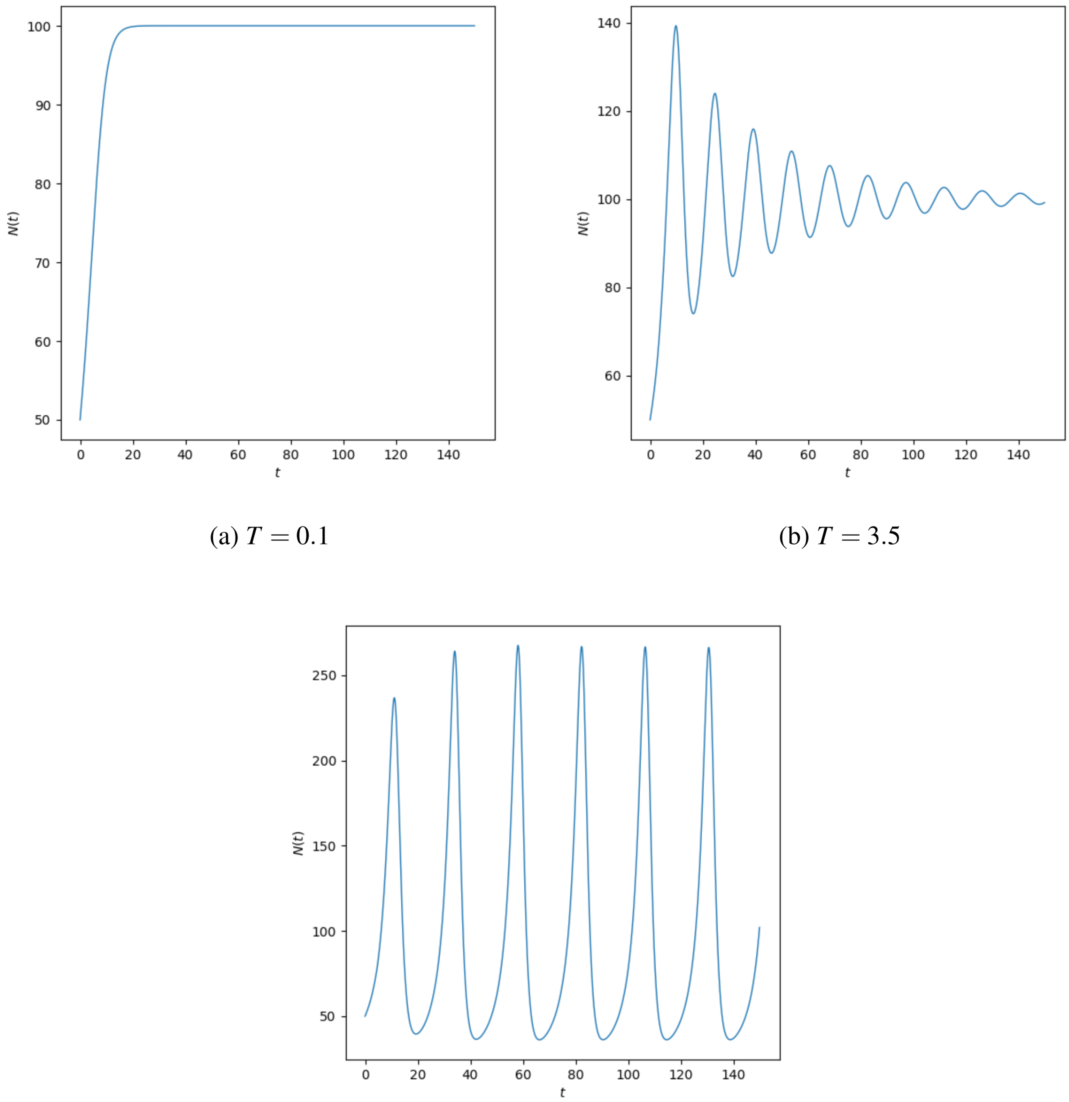
Problem 1

Collaborators: Erik Norlin & Hannes Nilsson

a)



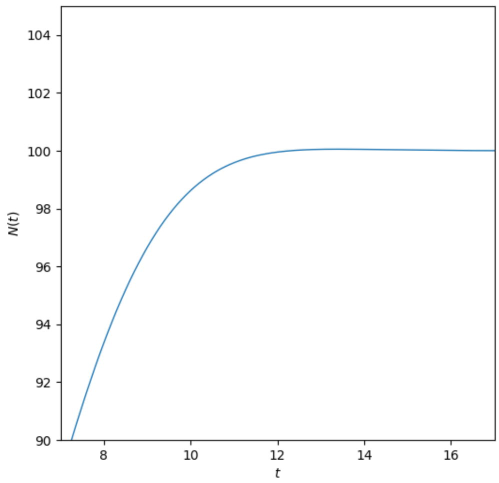
(a) $T = 0.1$

(b) $T = 3.5$

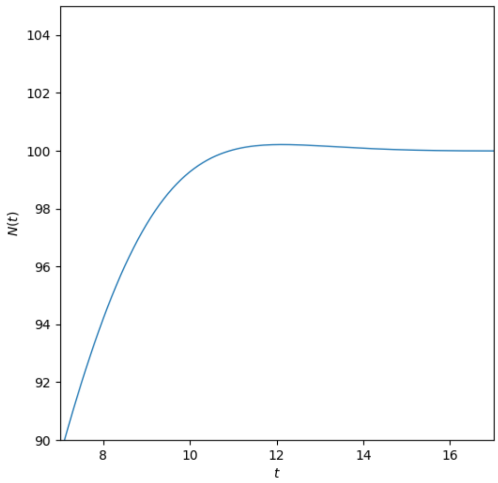
(c) $T = 5.0$

Figur 1: Examples of no oscillations (a), damped oscillations (b) and stable oscillations (c).

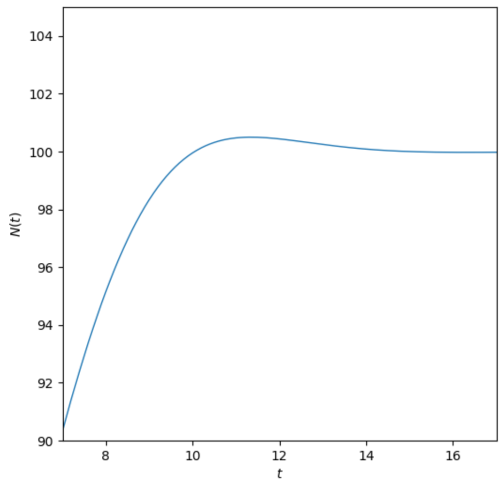
b)



(a) $T = 1.1$



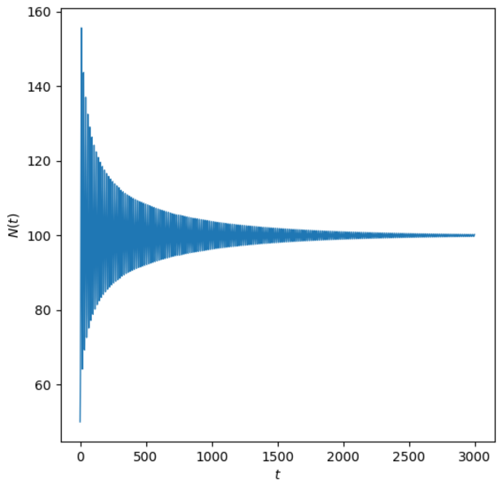
(b) $T = 1.2$



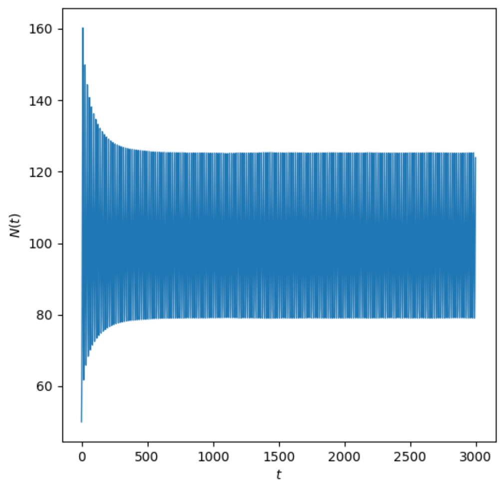
(c) $T = 1.3$

Figur 2: No oscillations can be seen in (a) when $T = 1.1$. A small overshoot can be vaguely seen in (b) when $T = 1.2$. Overshooting is clearly visible in (c) when $T = 1.3$. According to the theory for damped oscillations, if there's an overshoot, there's also an undershoot with a smaller amplitude which indicates oscillation.

c)



(a) $T = 3.9$



(b) $T = 4.0$

Figur 3: The dynamics show damped oscillations for a long time when $T = 3.9$ (a) which indicates that there's only a spiral in the system, while the dynamics show stable oscillations when $T = 4.0$ (b) i.e. the dynamics instead converge to a limit cycle. We can say that the dynamics undergoes a Hopf bifurcation between 3.9 and 4.0 for the value of T.

d)

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-T)}{K}\right)\left(\frac{N(t)}{A} - 1\right).$$

Linearizing around fixed point $N^* = K \implies N(t) \approx K + \eta(t) \quad \& \quad \dot{N}(t) \approx \dot{\eta}(t)$:

$$\begin{aligned}
\dot{\eta}(t) &= r\big(K + \eta(t)\big)\left(1 - \frac{K + \eta(t-T)}{K}\right)\left(\frac{K + \eta(t)}{A} - 1\right) \\
&= \big(rK + r\eta(t)\big)\left(1 - \frac{K}{K} + \frac{\eta(t-T)}{K}\right)\left(\frac{K}{A} + \frac{\eta(t)}{A} - 1\right) \\
&= \frac{\eta(t-T)}{K}\big(rK + r\eta(t)\big)\left(\frac{K-A}{A} + \frac{\eta(t)}{A}\right).
\end{aligned}$$

Since $\eta << 1$, we consider only the linear term, and plug in the values $A = 20$, $K = 100$, $r = 0.1$:

$$\dot{\eta}(t) \approx rK\frac{K-A}{A}\frac{\eta(t-T)}{K} = \frac{2}{5}\eta(t-T).$$

Now we make the ansatz $\eta(t) = Be^{\lambda t}$:

$$\lambda Be^{\lambda t} = \frac{2}{5}\eta(t-T) = \frac{2B}{5}e^{\lambda(t-T)}.$$

Dividing by $Be^{\lambda t}$ on both sides gives:

$$\lambda = \frac{2}{5}e^{-\lambda T}.$$

Assume λ is complex and rewrite as:

$$\begin{aligned}
\lambda' + i\lambda'' &= \frac{2}{5}e^{(-\lambda' - i\lambda'')T} \\
&= \frac{2}{5}e^{-\lambda' T}e^{-i\lambda'' T} \\
&= \frac{2}{5}e^{-\lambda' T}\big(\cos(-\lambda'' T) + i\sin(-\lambda'' T)\big) \\
&= \frac{2}{5}e^{-\lambda' T}\big(\cos(\lambda'' T) - i\sin(\lambda'' T)\big)
\end{aligned}$$

From here, we can deduce the real and imaginary parts of λ as:

$$\lambda' = \frac{2}{5}e^{-\lambda' T}\cos(\lambda'' T).$$

$$\lambda'' = \frac{2}{5}e^{-\lambda' T}\sin(\lambda'' T).$$

At a Hopf bifurcation, we have $Re(\lambda) = 0$. This gives us:

$$\lambda'' T_H = \frac{\pi}{2} \implies \lambda'' = \frac{2}{5}e^{0}\sin(\pi/2) = \frac{2}{5}.$$

We can now find T_H as:

$$T_H = \frac{\pi}{2\lambda''} = \frac{5\pi}{4} \approx 3.93.$$

This analysis is in agreement with our experimental results, where we found the Hopf bifurcation to take place between $T = 3.9$ and $T = 4.0$.

Appendix

Code for simulations (Python)

```
import numpy as np
import matplotlib.pyplot as plt
from ddeint import ddeint
import sys


# Numerical solver
T = 5
T_end = T*5
dt = 0.1
t = np.linspace(0,T_end,int(T_end/dt)+1)
T_i = np.linspace(0,T,int(T/dt)+1)
N0 = lambda t: 50
```

```
def model(Y, t, d):
    N = Y(t)
    Nd = Y(t-d)
    r = 0.1
    K = 100
    A = 20
    dNdt_solver = r*N*(1-Nd/K)*(N/A-1)
    return dNdt_solver

# Hopf bif. between T = 3.9 and 4.0
d = 1.1
N_solver = ddeint(model, N0, t, fargs=(d,))

# for d in T_i:
#     # 1.1b) damped oscillations starts at around d = 4.4
    # N_solver = ddeint(model, N0, t, fargs=(d,))
    # ax.plot(t, N_solver, linewidth=1, label='delay = %.01f'%d)
    # break

# Plotting solver
fig, ax = plt.subplots(figsize=(6,6))
ax.plot(t, N_solver, linewidth=1, label='delay = %.01f'%d)
ax.set_xlabel('$t$')
ax.set_ylabel('$N(t)$')
ax.set_xlim([7,17])
ax.set_ylim([90,105])
ax.set_box_aspect(1)

# plt.legend(loc="upper left")
title = '/1.1b Start of damped oscillations, T={}'.format(d)
location = r'C:\Users\erikn\OneDrive - Chalmers\Computational Biology\CB HW 1'
plt.savefig(location+title+'.png')
plt.show()
```