Assignment 1 SSY316

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Exercise 1

To find the posterior probability that the black ball came from Box 1, we can use Bayes' theorem. Let B represent the event that a black ball is chosen and B_1 and B_2 represent the events that the black ball came from Box 1 and Box 2, respectively. Also, let B_1 represent the event of selecting Box 1.

According to Bayes' theorem, the posterior probability $P(B_1|B)$ is given by:

$$P(B_1|B) = \frac{P(B|B_1) \cdot P(B_1)}{P(B)}$$

We can compute the terms in the formula as follows:

1. $P(B|B_1)$ is the probability of drawing a black ball from Box 1, which is the ratio of the number of black balls in Box 1 to the total number of balls in Box 1: $P(B|B_1) = \frac{3}{8}$. For Box 2 this is $P(B_2) = \frac{2}{7}$

2. $P(B_1)$ is the prior probability of selecting Box 1, which is 0.5 in this case.

3. $P(B)$ is the total probability of drawing a black ball, which can be computed as

$$P(B) = P(B|B_1)P(B_1) + P(B|B_2)P(B_2) = \frac{1}{2}\left(\frac{3}{8} + \frac{2}{7}\right) = \frac{1}{2} \times \frac{37}{56}$$

Now, we can find the posterior probability $P(B_1|B)$:

$$P(B_1|B) = \frac{P(B|B_1) \cdot P(B_1)}{P(B)} = \frac{\frac{3}{8} \times \frac{1}{2}}{\frac{1}{2} \times \frac{37}{56}} = \frac{21}{37}$$

Exercise 2

i)

To find the probability that it was raining or snowing yesterday given that it does not rain or snow today, we can use Bayes' theorem. Let X_{t-1} be the event that it rained/snowed yesterday, X_t be the event that it rains/snows today,

Y_{t-1} be the even that it did not rain/snow yesterday, and Y_t that it does not rain/snow today. We are given the following probabilities:

$$\begin{cases} P(X_t|X_{t-1}) = 0.6 \\ P(X_t|Y_{t-1}) = 0.2 \\ P(Y_t|X_{t-1}) = 0.4 \\ P(Y_t|Y_{t-1}) = 0.8 \end{cases}$$

For this subtask we are given $P(X_{t-1}) = 0.5$ which implies that $P(Y_{t-1}) = 0.5$, and we want to find $P(X_{t-1}|Y_t)$, i.e., the probability that it was raining or snowing yesterday given that it does not rain or snow today.

Using Bayes' theorem, we have:

$$P(X_{t-1}|Y_t) = \frac{P(Y_t|X_{t-1}) \cdot P(X_{t-1})}{P(Y_t)}$$

where

$$\begin{cases} P(Y_t|X_{t-1}) = 0.4 \\ P(X_{t-1}) = 0.5 \\ P(Y_t) = P(Y_t|Y_{t-1}) \cdot P(Y_{t-1}) + P(Y_t|X_{t-1}) \cdot P(X_{t-1}) = 0.8 \cdot 0.5 + 0.4 \cdot 0.5 \end{cases}$$

Hence,

$$P(X_{t-1}|Y_t) = \frac{0.4 \cdot 0.5}{(0.8 + 0.4) \cdot 0.5} = \frac{1}{3}$$

ii)

If the weather follows the same pattern as above, day after day, we want to calculate $P(X_n)$, i.e., the probability that it will rain or snow on any day (based on an effectively infinite number of days of observing the weather).

The posterior probability should converge to the prior probability as the number of days goes to infinity because the prior belief becomes more accurate the more data we gather. Thus, $P(X_n) = P(X_{n-1})$ and $P(Y_n) = P(Y_{n-1})$ as $n \to \infty$

We define a system of equations

$$\begin{cases} P(X_n) = 0.6P(X_n) + 0.2P(Y_n) \\ 1 = P(X_n) + P(Y_n) \end{cases}$$

The first equation gives us that $P(Y_n) = 2P(X_n)$, putting this into the second equation gives us that $P(X_n) = \frac{1}{3}$. Thus, the probability that it will rain on any given day is $\frac{1}{3}$.

iii)

Now we use the result from (ii) above as a new prior probability of rain/snow yesterday, i.e., $P(X_{t-1}) = \frac{1}{3}$ and $P(Y_{t-1}) = \frac{2}{3}$ and recompute $P(X_{t-1}|Y_t)$ i.e., the probability that it was raining/snowing yesterday given that it's does not rain or snow today. We get that

$$P(X_{t-1}|Y_t) = \frac{0.4 \cdot \frac{1}{3}}{0.8 \cdot \frac{2}{3} + 0.4 \cdot \frac{1}{3}} = \frac{1}{5}$$

Exercise 3

To prove that the Beta distribution is correctly normalized, we need to show that the integral of the Beta distribution function from 0 to 1 is equal to 1. Let's proceed with the calculation step by step.

The Beta distribution function is given by:

$$Beta(\mu; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}$$

To show that the Beta distribution is normalized, we need to show that:

$$\int_0^1 Beta(\mu; a, b) d\mu = 1 \Leftrightarrow \int_0^1 \mu^{a-1}(1-\mu)^{b-1} d\mu = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$\Rightarrow$:

Let's start by substituting the expression for $Beta(\mu; a, b)$ into the integral on the left hand side:

$$\int_0^1 Beta(\mu; a, b) d\mu = \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1} d\mu = 1$$

$$\Rightarrow \int_0^1 \mu^{a-1}(1-\mu)^{b-1}\, d\mu = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$\Leftarrow$:

Move the expression on the right hand side to the left hand side:

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mu^{a-1}(1-\mu)^{b-1}\, d\mu = 1$$

$$\Rightarrow \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}\, d\mu = 1$$

$$\Rightarrow \int_0^1 Beta(\mu; a, b) d\mu = 1$$

Exercise 4

Here we will utilize the fact that

$$\int_0^1 \mu^{a-1}(1-\mu)^{b-1}d\mu = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

and also that

$$\Gamma(a+1) = a\Gamma(a)$$

i)

The mean is :

$$\begin{aligned}
E[\mu] &= \int_0^1 \mu Beta(\mu; a, b)d\mu \\
&= \int_0^1 \mu \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}d\mu \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mu^{a+1-1}(1-\mu)^{b-1}d\mu \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+1+b)} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)} \\
&= \frac{a}{a+b}
\end{aligned}$$

ii)

The variance is $\text{Var}[\mu] = E[\mu^2] - E[\mu]^2$ where we have already derived $E[\mu]$, hence $E[\mu]^2 = (\frac{a}{a+b})^2$ so we proceed to derive $E[\mu^2]$

$$\begin{aligned}
E[\mu^2] &= \int_0^1 \mu^2 Beta(\mu; a, b)d\mu \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mu^{a+1}(1-\mu)^{b-1}d\mu \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+2+b)} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)} \frac{(a+1)\Gamma(a+1)}{(a+1+b)\Gamma(a+1+b)} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)} \frac{(a+1)a\Gamma(a)}{(a+1+b)(a+b)\Gamma(a+b)} \\
&= \frac{a(a+1)}{(a+1+b)(a+b)}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}[\mu] &= \frac{a(a+1)}{(a+1+b)(a+b)} - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{(a^2+a)(a+b)}{(a+1+b)(a+b)^2} - \frac{a^2(a+1+b)}{(a+b)^2(a+1+b)} \\
&= \frac{a^3+a^2b+a^2+ab-a^3-a^2-a^2b}{(a+b)^2(a+1+b)} \\
&= \frac{ab}{(a+b)^2(a+1+b)}
\end{aligned}
$$

Exercise 5

We consider a variable $x \in \{0,1\}$ and $P(x=1)=\mu$ representing flipping of a coin. With 50% probability we think the coin is fair, i.e., μ = 0.5, and with 50% probability we think that is unfair, i.e., $\mu \neq 0.5$. We encode this prior belief with the following prior $P(\mu) = \frac{1}{2}\mathrm{Beta}(\mu;1,1) + \frac{1}{2}\delta(\mu-0.5)$.

In this exercise we will at some points utilize the fact that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a),$$

also that

$$f(x)\delta(x-a) = f(a)\delta(x-a),$$

and as well that

$$\Gamma(n) = (n-1)!$$

i)

We assume that we get one observation $x_1 = 1$ and we want to calculate the posterior $P(\mu|x_1)$. In particular, we want to see how the belief of the fairness of the coin change under this observation. We use Bayes' theorem

$$P(\mu|x) = \frac{P(x|\mu)\cdot P(\mu)}{P(x)}$$

where

$$
\begin{cases}
P(x_1|\mu) = \mu \\
P(\mu) = \frac{1}{2}\mathrm{Beta}(\mu;1,1) + \frac{1}{2}\delta(\mu-0.5) = \\
P(x_1) = \int_0^1 P(x_1|\mu)P(\mu)
\end{cases}
$$

and

$$
\begin{aligned}
P(x_1) &= \int_0^1 \frac{1}{2}\mu\text{Beta}(\mu;1,1) + \frac{1}{2}\mu\delta(\mu-0.5)d\mu \\
&= \int_0^1 \frac{1}{2}\frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\mu^{1-1+1}(1-\mu)^{1-1}d\mu + \int_0^1 \frac{1}{2}\mu\delta(\mu-0.5)d\mu \\
&= \left[\frac{\mu^2}{4}\right]_0^1 + \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}
$$

It follows that the posterior probability is

$$
\begin{aligned}
P(\mu|x_1) &= \frac{\frac{1}{2}\mu\text{Beta}(\mu;1,1) + \frac{1}{2}\mu\delta(\mu-0.5)}{\frac{1}{2}} \\
&= 2\frac{1}{2}\frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\mu^{1-1+1}(1-\mu)^{1-1} + 2\frac{1}{2\cdot 2}\delta(\mu-0.5) \\
&= \frac{1}{2}2\mu + \frac{1}{2}\delta(\mu-0.5) \\
&= \frac{1}{2}\frac{\Gamma(2+1)}{\Gamma(2)\Gamma(1)}\mu^{2-1}(1-\mu)^{1-1} + \frac{1}{2}\delta(\mu-0.5) \\
&= \frac{1}{2}\text{Beta}(\mu;2,1) + \frac{1}{2}\delta(\mu-0.5)
\end{aligned}
$$

The probability that the coin is fair given $x_1 = 1$ is

$$
\begin{aligned}
P(\mu = 0.5|x_1) &= \frac{1}{2}\frac{\Gamma(2+1)}{\Gamma(2)\Gamma(1)}0.5^{2-1}(1-0.5)^{1-1} \\
&= 0.5
\end{aligned}
$$

Thus, our belief of fairness of the coin does not change which makes sense because flipping the coin only once reveals too little information about how fair the coin is.

ii)

We now assume that we get one additional observation $x_2 = 1$, and we want to calculate the posterior $p(\mu|x_1, x_2)$. In particular, we want to see how the belief of the fairness of the coin change under this observation.

We update our prior probability to the posterior probability that we computed earlier, we have therefore that

$$
\begin{cases}
P(x_2|\mu) = \mu \\
P(\mu) = P(\mu|x_1) = \frac{1}{2}\text{Beta}(\mu;2,1) + \frac{1}{2}\delta(\mu-0.5) \\
P(x_2) = \int_0^1 P(x_2|\mu)P(\mu)
\end{cases}
$$

where

$$
\begin{aligned}
P(x_2) &= \int_0^1 \frac{1}{2}\mu \text{Beta}(\mu; 2, 1) + \frac{1}{2}\mu\delta(\mu - 0.5) d\mu \\
&= \int_0^1 \frac{1}{2} \frac{\Gamma(2+1)}{\Gamma(2)\Gamma(1)} \mu^{2-1+1}(1-\mu)^{1-1} d\mu + \int_0^1 \frac{1}{2}\mu\delta(\mu - 0.5) d\mu \\
&= \left[\frac{\mu^3}{3}\right]_0^1 + \frac{1}{4} \\
&= \frac{7}{12}
\end{aligned}
$$

It follows that the posterior probability is

$$
\begin{aligned}
P(\mu|x_1, x_2) &= \frac{\frac{1}{2}\mu \text{Beta}(\mu; 2, 1) + \frac{1}{2}\mu\delta(\mu - 0.5)}{\frac{7}{12}} \\
&= \frac{12}{7} \frac{1}{2} \frac{\Gamma(2+1)}{\Gamma(2)\Gamma(1)} \mu^{2-1+1}(1-\mu)^{1-1} + \frac{12}{7} \frac{1}{2 \cdot 2} \delta(\mu - 0.5) \\
&= \frac{12}{7}\mu^2 + \frac{3}{7}\delta(\mu - 0.5) \\
&= \frac{12}{7 \cdot 3} \frac{\Gamma(3+1)}{\Gamma(3)\Gamma(1)} \mu^{3-1}(1-\mu)^{1-1} + \frac{3}{7}\delta(\mu - 0.5) \\
&= \frac{4}{7} \text{Beta}(\mu; 3, 1) + \frac{3}{7}\delta(\mu - 0.5)
\end{aligned}
$$

The probability that the coin is fair given $x_1 = 1$ and $x_2 = 1$ is

$$
\begin{aligned}
P(\mu = 0.5|x_1, x_2) &= \frac{4}{7} \frac{\Gamma(3+1)}{\Gamma(3)\Gamma(1)} 0.5^{3-1}(1 - 0.5)^{1-1} \\
&= \frac{4}{7} \cdot \frac{6}{2} \cdot \frac{1}{4} \\
&= \frac{3}{7}
\end{aligned}
$$

Thus, our belief of fairness of the coin has changed to being less than 50% which makes sense because getting the same outcome every time indicates that it is more probable that the coin is unfair.

iii)

Now we want to compute the probability of the coin being fair by defining an event fair with the prior probability $p(fair) = 0.5$. We compute $p(fair|x_1, x_2)$ using Bayes' theorem based on the observations $x_1 = 1$ and $x_2 = 1$.

$$p(fair|x_1, x_2) = \frac{p(x_1, x_2|fair) \cdot p(fair)}{p(x_1, x_2)}$$

where

$$\begin{cases} P(x_1, x_2|fair) = 0.5^2 = 0.25 \\ P(fair) = P(unfair) = 0.5 \\ P(x_1, x_2) = P(x_1, x_2|fair) \cdot P(fair) + P(x_1, x_2|unfair) \cdot P(unfair) \end{cases}$$

and

$$P(x_1, x_2|unfair) = \int_0^1 P(x_1, x_2|\mu, unfair) \cdot P(\mu|unfair) d\mu$$

We set $P(x_1, x_2|\mu, unfair) = \mathrm{Bin}(2, 2, \mu)$ because this represents the probability of getting two successes in two coin flips i.e., $x_1 = 1$ and $x_2 = 1$.

$$P(x_1, x_2|unfair) = \int_0^1 \mathrm{Bin}(2, 2, \mu) \cdot P(\mu|unfair) d\mu$$

It follows that the posterior probability is

$$\begin{aligned} p(fair|x_1, x_2) &= \frac{0.25 \cdot 0.5}{0.25 \cdot 0.5 + P(x_1, x_2|unfair) \cdot 0.5} \\ &= \frac{1}{1 + 4P(x_1, x_2|unfair)} \end{aligned}$$