Homework 1

Deadline: November 6, 23:59

Exercise 1

Consider two boxes with white and black balls. Box 1 contains three black and five white balls and box 2 contains two black and five white balls. First a box is chosen at random with a prior probability $p(\text{box} = 1) = p(\text{box} = 2) = 0.5$, secondly a ball picked at random from that box. This ball turns out to be black. What is the posterior probability that this black ball came from box 1?

Exercise 2

The weather in Gothenburg can be summarized as: if it rains or snows one day there is a 60% chance it will also rain or snow the following day; if it does not rain or snow one day there is an 80% chance it will not rain or snow the following day either.

(i) Assuming that the prior probability it rained or snowed yesterday is 50%, what is the probability that it was raining or snowing yesterday given that it does not rain or snow today?

(ii) If the weather follows the same pattern as above, day after day, what is the probability that it will rain or snow on any day (based on an effectively infinite number of days of observing the weather)?

(iii) Use the result from part 2 above as a new prior probability of rain/snow yesterday and recompute the probability that it was raining/snowing yesterday given that it's does not rain or snow today.

Exercise 3

Prove that the Beta distribution

$$\text{Beta}(\mu; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}, \quad \text{where} \quad \mu \in [0,1], \quad \Gamma(a) = \int_0^\infty e^{-x}x^{a-1}dx$$

is correctly normalized, i.e.,

$$\int_0^1 \text{Beta}(\mu; a, b)d\mu = 1 \iff \int_0^1 \mu^{a-1}(1-\mu)^{b-1}d\mu = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Exercise 4

Consider a Beta distributed random variable $\mu \in [0, 1]$ with

$$\mu \sim \mathrm{Beta}(\mu; a, b)$$

show that

(i) the mean is μ is $\frac{a}{a+b}$. Hint: $\Gamma(a + 1) = a\Gamma(a)$

(ii) the variance is μ is $\frac{ab}{(a+b)^2(a+b+1)}$.

Exercise 5

Consider a variable $x \in 0, 1$ and $p(x = 1) = \mu$ representing flipping of a coin. With 50% probability we think the coin is fair, i.e., $\mu = 0.5$, and with 50% probability we think that is unfair, i.e., $\mu \neq 0.5$. We encode this prior belief with the following prior

$$p(\mu) = \frac{1}{2}\mathrm{Beta}(\mu; 1, 1) + \frac{1}{2}\delta(\mu - 0.5)$$

(i) Assume that we get one observation $x_1 = 1$. What is the posterior $p(\mu|x_1)$? In particular, how does the belief of the fairness of the coin change under this observation?

(ii) Assume that we get one additional observation $x_2 = 1$. What is the posterior $p(\mu|x_1, x_2)$? In particular, how does the belief of the fairness of the coin change under this observation?

(iii) Compute the probability of the coin being fair by defining an event *fair* with the prior probability $p(fair) = 0.5$. Compute $p(fair|x_1, x_2)$ using Bayes' theorem based on the observations $x_1 = 1, x_2 = 1$.