1.4 $\begin{cases} \dot{x} = (\sigma+1)x + 3y \\ \dot{y} = -2x + (\sigma-1)y \end{cases}$,

$$\dot{\mathbf{x}} = \begin{bmatrix} (\sigma+1) & 3 \\ -2 & (\sigma-1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

a) $\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{2\sigma \pm \sqrt{(2\sigma)^2 - 4((\sigma+1)(\sigma-1) - (-2\cdot 3))}}{2}$

$= \frac{2\sigma \pm \sqrt{4\sigma^2 - 4(\sigma^2 - 1 + 6}}{2} = \frac{2\sigma \pm \sqrt{4\sigma^2 - 4\sigma^2 - 20}}{2} =$

$= \frac{2\sigma \pm \sqrt{20}\, i}{2} = \sigma \pm \frac{\sqrt{20}}{\sqrt{4}} i = \sigma \pm \sqrt{\frac{20}{4}}\, i \Rightarrow$ Ans:

$\lambda_{1,2} = \sigma \pm \sqrt{5}\, i$

---

b) Find eigenvector: $\left(\begin{array}{cc|c} (\sigma+1) - (\sigma + \sqrt{5}i) & 3 & 0 \\ -2 & (\sigma-1) - (\sigma + \sqrt{5}i) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1-\sqrt{5}i & 3 & 0 \\ -2 & -1-\sqrt{5}i & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & (1+\sqrt{5}i)/2 & 0 \\ -2 & -1-\sqrt{5}i & 0 \end{array}\right)$

$R_1 \to R_1 \cdot \left(\frac{1+\sqrt{5}i}{6}\right)$ $\quad R_2 \to R_2 + 2R_1$

$= \left(\begin{array}{cc|c} 1 & (1+\sqrt{5}i)/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow V_1 = -\frac{(1+\sqrt{5}i)}{2} V_2 \Rightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -(1+\sqrt{5}i)/2 \\ 1 \end{bmatrix}$

$\Rightarrow \lambda = \sigma + \sqrt{5}i, \quad \underline{V} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + i\begin{bmatrix} -\sqrt{5}/2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = c_1 e^{\sigma t}\left(\cos(\sqrt{5}t)\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} - \sin(\sqrt{5}t)\begin{bmatrix} -\sqrt{5}/2 \\ 0 \end{bmatrix}\right) +$$
$$+ c_2 e^{\sigma t}\left(\cos(\sqrt{5}t)\begin{bmatrix} -\sqrt{5}/2 \\ 0 \end{bmatrix} + \sin(\sqrt{5}t)\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right),$$

$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = c_1 e^0\left(\cos(0)\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} - \sin(0)\begin{bmatrix} -\sqrt{5}/2 \\ 0 \end{bmatrix}\right) +$

$+ c_2 e^0\left(\cos(0)\begin{bmatrix} -\sqrt{5}/2 \\ 0 \end{bmatrix} + \sin(0)\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right) = c_1\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -\sqrt{5}/2 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} u = -\frac{c_1 - c_2\sqrt{5}}{2} \\ v = c_1 \end{cases} \Rightarrow u = \frac{-v - c_2\sqrt{5}}{2} \Rightarrow c_2 = -\frac{(2u+v)}{\sqrt{5}}$

Ans: $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = v e^{\sigma t}\left(\cos(\sqrt{5}t)\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} - \sin(\sqrt{5}t)\begin{bmatrix} -\sqrt{5}/2 \\ 0 \end{bmatrix}\right) - \frac{(2u+v)}{\sqrt{5}} e^{\sigma t}\left(\cos(\sqrt{5}t)\begin{bmatrix} -\sqrt{5}/2 \\ 0 \end{bmatrix} + \right.$
$\left. + \sin(\sqrt{5}t)\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right)$

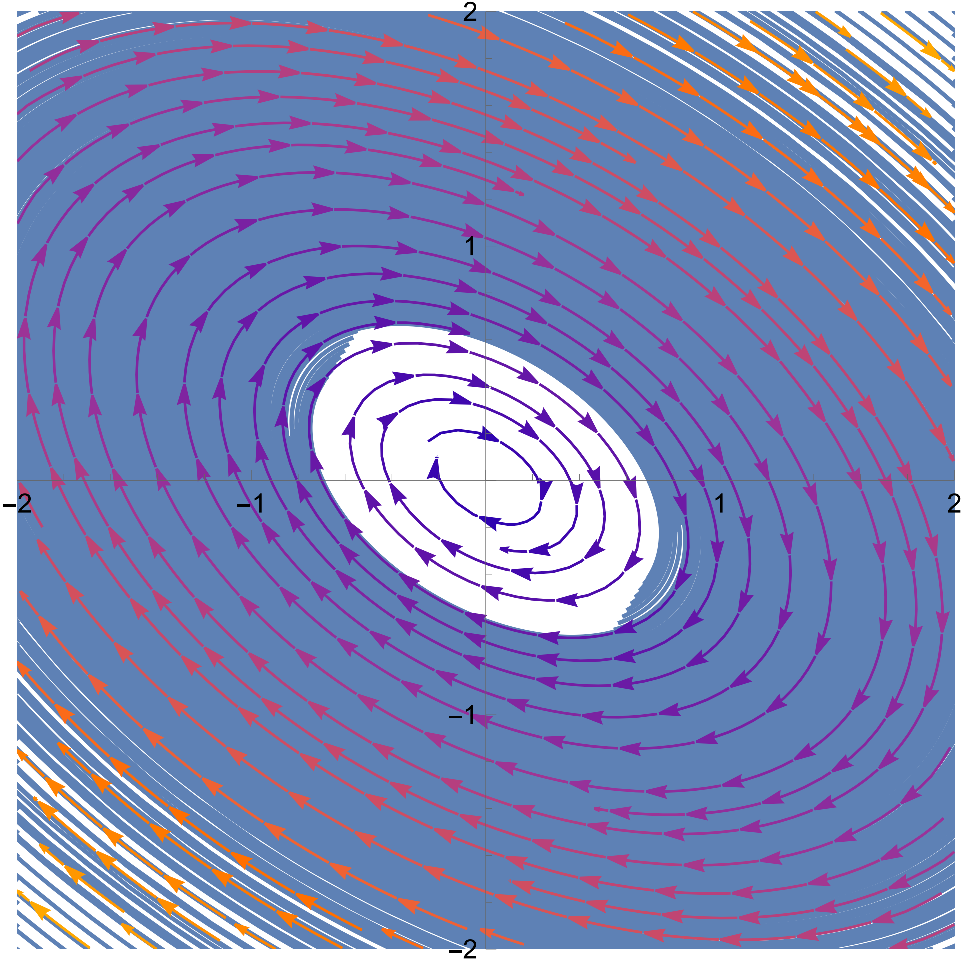
d)

$$\cos\left(\frac{t}{T}\right) = \cos\left(\frac{\sqrt{5}\,t}{2\pi}\right) \Rightarrow \frac{1}{T} = \frac{\sqrt{5}}{2\pi} \Rightarrow T = \frac{2\pi}{\sqrt{5}}$$

Ans: The period of the ellipse is

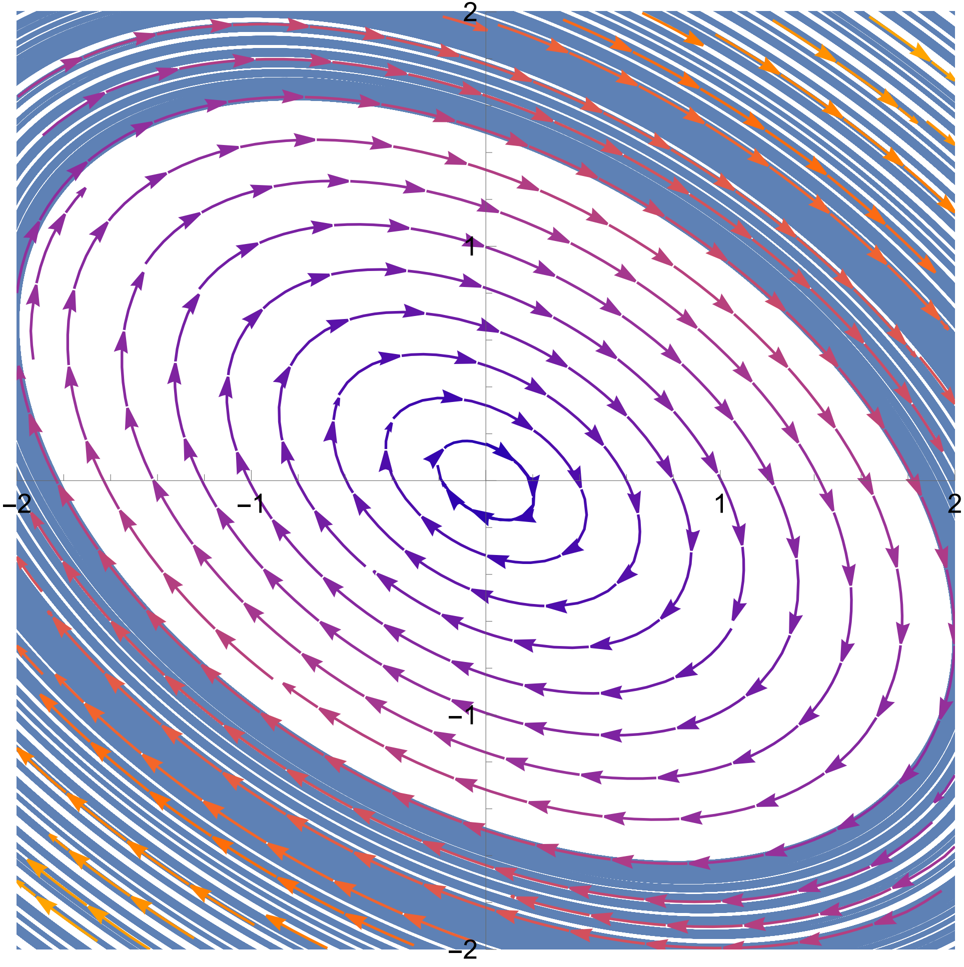
$$T = \frac{2\pi}{\sqrt{5}}$$

Out[●]=



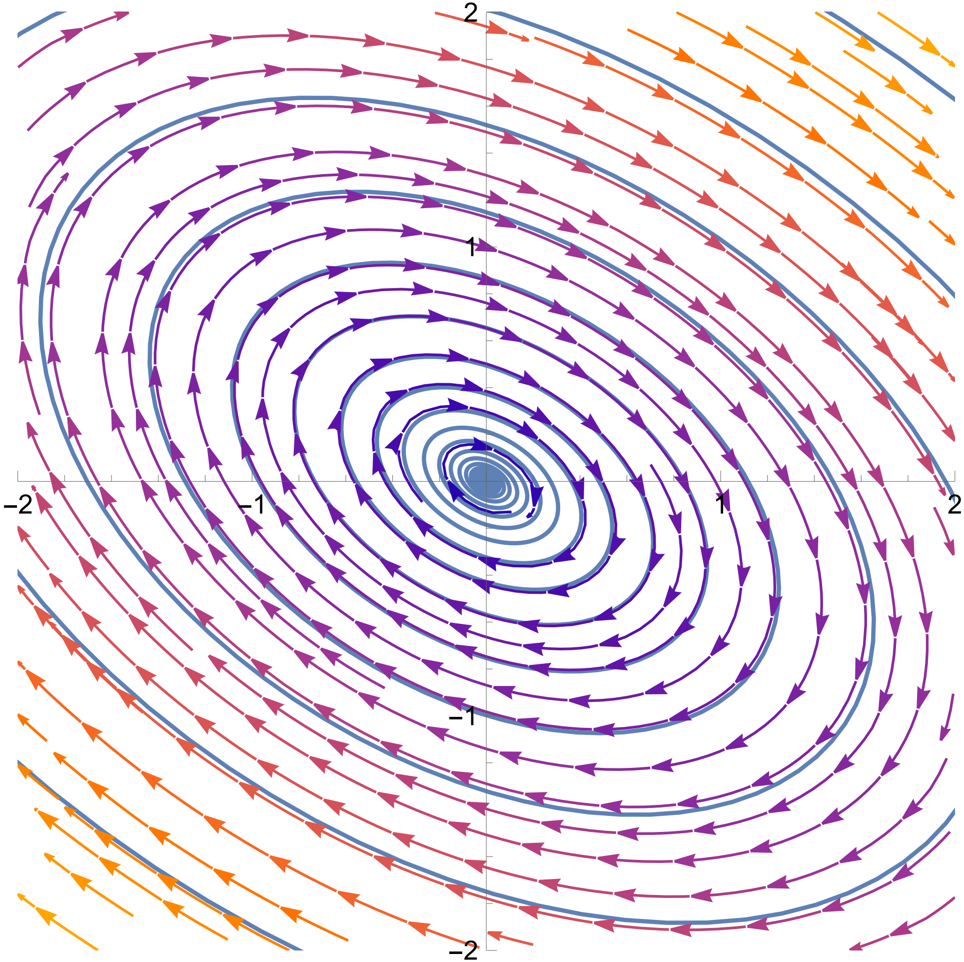
Sigma = −1/10

Out[ ]=



Sigma = 0

Out[ ]=



Sigma = 1/10

```
In[14]:= ClearAll["Global`*"]
Clear[minx, miny, maxx, maxy]
minx = -2;
miny = -2;
maxx = 2;
maxy = 2;
```

```
In[62]:= Clear[sol, x, y, t]
sol[x0_, y0_] := NDSolve[
  {x'[t] == (0 + 1) * x[t] + 3 * y[t],
   y'[t] == -2 * x[t] + (0 - 1) * y[t],
   x[0] == x0, y[0] == y0},
  {x, y}, {t, -100, 10}]
```

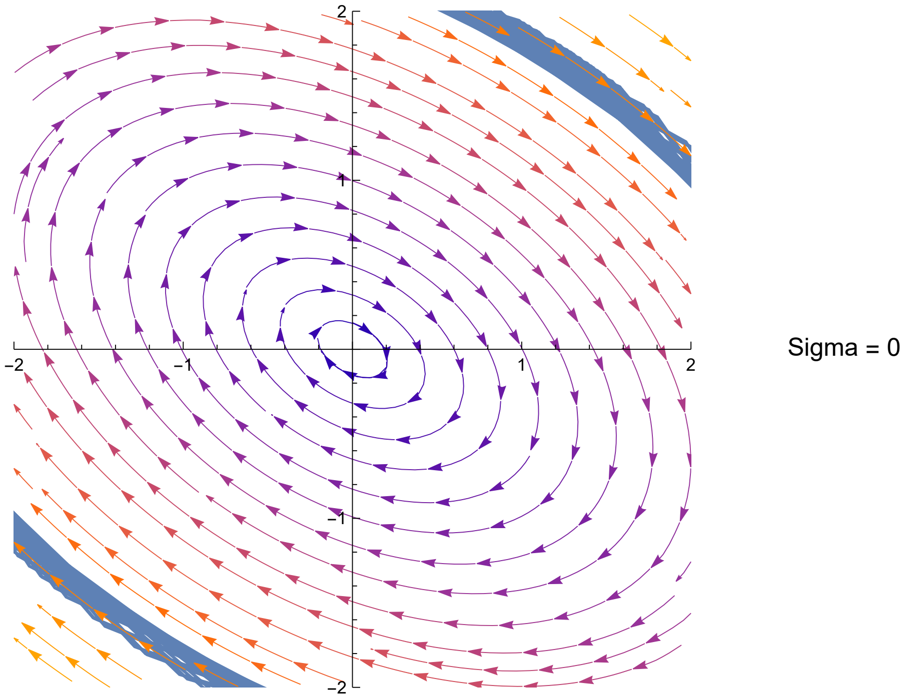
```
In[22]:= initialCond = Join[
   Table[{x, maxy}, {x, minx , maxx, 0.1}],
   Table[{minx , y } , {y, miny , maxy, 0.1}],
   Table[{x , miny}, {x , minx , maxx, 0.1}],
   Table[{maxx, y} , {y, miny , maxy, 0.1}]
  ];
```

In[64]:=
```
Show[ParametricPlot[
   Evaluate[{x[t], y[t]} /. sol[initialCond[[50, 1]], initialCond[[50, 2]]]],
   {t, -1000, 10}, PlotRange → {{minx, maxx}, {miny, maxy}}],
  StreamPlot[{(0 + 1) * x + 3 * y, -2 * x + (0 - 1) * y}, {x, -2, 2}, {y, -2, 2}],
  ListPlot[{{0, 0}}, PlotStyle → {PointSize[0.03], Red},
   PlotMarkers → {"", Large}, PlotLegends → {"Sigma = 0"}]
]
```

InterpolatingFunction: Input value {−999.979} lies outside the range of data in the interpolating function. Extrapolation will be used.

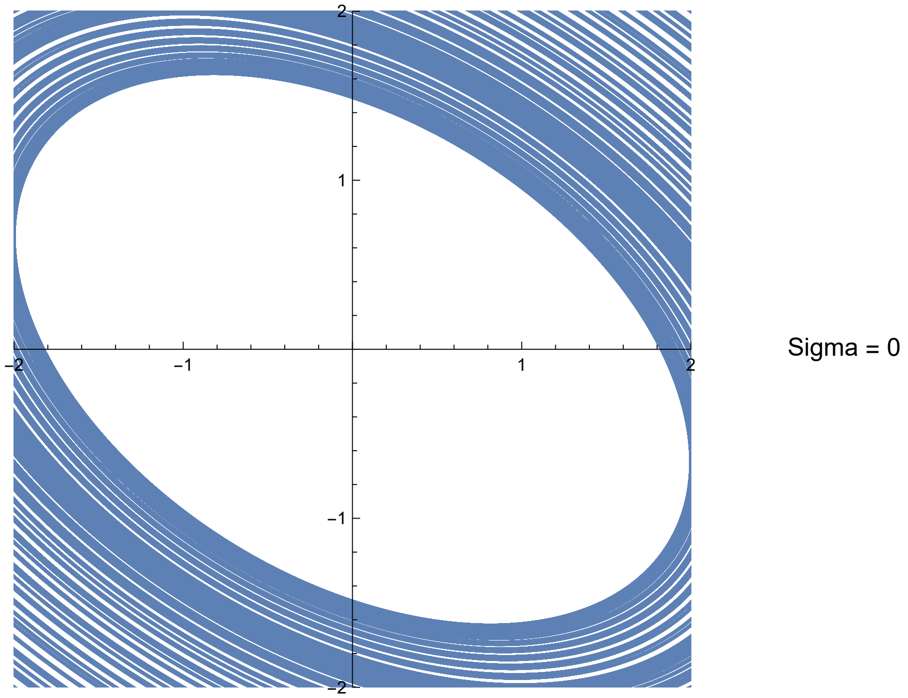
InterpolatingFunction: Input value {−999.979} lies outside the range of data in the interpolating function. Extrapolation will be used.

Out[64]=



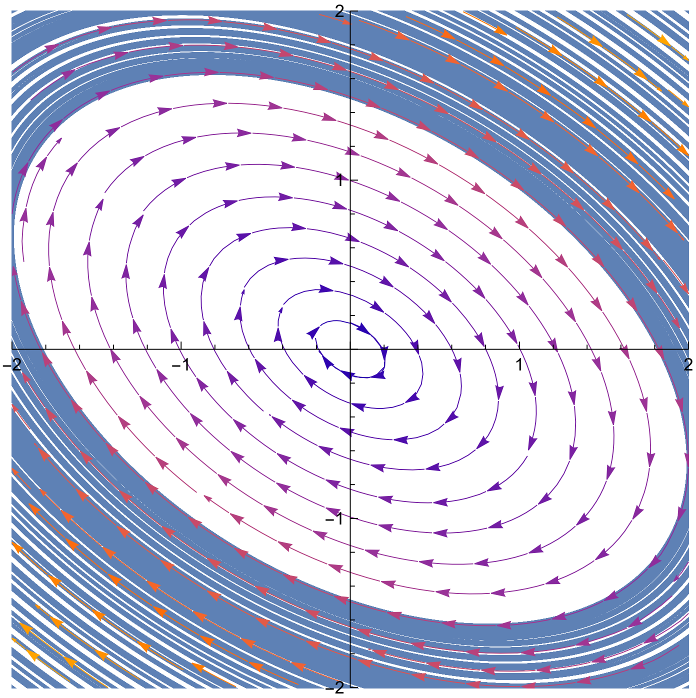
In[65]:=
```
p2 = Show[
  Table[
   ParametricPlot[
    Evaluate[{x[t], y[t]} /. sol[initialCond〚i, 1〛, initialCond〚i, 2
        〛]], {t, -100, 10}, PlotRange → {{minx, maxx}, {miny, maxy}}],
   {i, 1, Length[initialCond]}],
  ListPlot[{{0, 0}}, PlotStyle → {PointSize[0.03], Red},
   PlotMarkers → {"", Large}, PlotLegends → {"Sigma = 0"}]
 ]
```

Out[65]=



In[66]:=
```
Show[p2, StreamPlot[{(0 + 1) * x + 3 * y, -2 * x + (0 - 1) * y}, {x, -2, 2}, {y, -2, 2}],
 PlotRange → {{minx, maxx}, {miny, maxy}}]
```

Out[66]=



Sigma = 0

```
(*Computation for 1.4 b, e, f.*)

In[ ]:= Clear[xsol, ysol, x, y, u, v, t]
{xsol, ysol} = DSolveValue[
        {x'[t] == (p + 1) * x[t] + 3 * y[t],
        y'[t] == -2 * x[t] + (p - 1) * y[t],
        x[0] == u, y[0] == v},
        {x, y}, {t, 0, 20}];
xsol;
ysol;
```

Function$\left[\{t\}, \frac{1}{5} e^{p t} \left(v \left(\frac{-1}{2}\right) \text{Cos}\left[\sqrt{5}\ t\right] - \sqrt{5}\ (2 u + v)\ \text{Sin}\left[\sqrt{5}\ t\right]\right)\right]$;

In[ ]:= $\left\{\left\{x \to \text{Function}\left[\{t\}, \frac{1}{5} e^{p t} \left(5 u \text{Cos}\left[\sqrt{5}\ t\right] + \sqrt{5}\ u \text{Sin}\left[\sqrt{5}\ t\right] + 3 \sqrt{5}\ v \text{Sin}\left[\sqrt{5}\ t\right]\right)\right],\right.\right.$
$\left.\left. y \to \text{Function}\left[\{t\}, -\frac{1}{5} e^{p t} \left(-5 v \text{Cos}\left[\sqrt{5}\ t\right] + 2 \sqrt{5}\ u \text{Sin}\left[\sqrt{5}\ t\right] + \sqrt{5}\ v \text{Sin}\left[\sqrt{5}\ t\right]\right)\right]\right\}\right\}$;

```
In[41]:= Clear[u, v, p, r]
u = 1;
v = 1;
p = 0;
```

$x = v * e^{p*t} * \left(\left(\frac{-1}{2}\right) * \text{Cos}\left[\sqrt{5}\ t\right] - \left(\frac{-\sqrt{5}}{2}\right) * \text{Sin}\left[\sqrt{5}\ t\right]\right) - \frac{(2 * u + v)}{\sqrt{5}} * e^{p*t} * \left(\left(\frac{-\sqrt{5}}{2}\right) * \text{Cos}\left[\sqrt{5}\ t\right] + \left(\frac{-1}{2}\right) * \text{Sin}\left[\sqrt{5}\ t\right]\right)$;

$y = v * e^{p*t} * \text{Cos}\left[\sqrt{5}\ t\right] - \frac{(2 * u + v)}{\sqrt{5}} * e^{p*t} * \text{Sin}\left[\sqrt{5}\ t\right]$;

```
r = Sqrt[x^2 + y^2] // FullSimplify;

In[48]:= Clear[min, max, ratio]
min = MinValue[r, t] // FullSimplify;
max = MaxValue[r, t] // FullSimplify;
ratio = max / min
```

Out[51]=

$$\sqrt{\frac{5 + \sqrt{5}}{5 - \sqrt{5}}}$$

```
In[52]:= tMax = ArgMax[r, t] // FullSimplify;
```

In[53]:=
```
Clear[u, v, p, r]
u = 1;
v = 1;
p = 0;
```

$$X[t\_] = v * e^{p*t} * \left(\left(\frac{-1}{2}\right) * \mathrm{Cos}\left[\sqrt{5}\ t\right] - \left(\frac{-\sqrt{5}}{2}\right) * \mathrm{Sin}\left[\sqrt{5}\ t\right]\right) - \frac{(2*u+v)}{\sqrt{5}} * e^{p*t} * \left(\left(\frac{-\sqrt{5}}{2}\right) * \mathrm{Cos}\left[\sqrt{5}\ t\right] + \left(\frac{-1}{2}\right) * \mathrm{Sin}\left[\sqrt{5}\ t\right]\right);$$

$$Y[t\_] = v * e^{p*t} * \mathrm{Cos}\left[\sqrt{5}\ t\right] - \frac{(2*u+v)}{\sqrt{5}} * e^{p*t} * \mathrm{Sin}\left[\sqrt{5}\ t\right];$$

In[68]:=
```
Clear[v]
v = {X[tMax], Y[tMax]} // FullSimplify
```

$$\mathrm{norm} = \mathrm{Sqrt}\left[(X[tMax] - 0)^2 + (Y[tMax] - 0)^2\right] \text{ // FullSimplify}$$

Out[69]=

$$\left\{\sqrt{\frac{7}{10}\left(3+\sqrt{5}\right)}, -\sqrt{\frac{7}{5}}\right\}$$

Out[70]=

$$\sqrt{\frac{7}{10}\left(5+\sqrt{5}\right)}$$