

1.2 b) $\dot{x} = rx + 4x^3 - 9x^5$, fixed points at $\dot{x} = 0$
 $\Rightarrow \dot{x} = x(r + 4x^2 - 9x^4) = 0$

Solutions for x :
$$\begin{cases} x = 0 \\ r + 4x^2 - 9x^4 = 0 \quad (1) \end{cases}$$

(1) $\Rightarrow x^4 - \frac{4}{9}x^2 - \frac{r}{9} = 0$

$$\begin{aligned} \Rightarrow x^2 &= -\frac{\left(\frac{-4}{9}\right)}{2} \pm \sqrt{\left(\frac{\left(\frac{-4}{9}\right)}{2}\right)^2 - \left(-\frac{r}{9}\right)} = \frac{2}{9} \pm \sqrt{\left(\frac{2}{9}\right)^2 + \frac{r}{9}} = \\ &= \frac{2}{9} \pm \sqrt{\frac{4}{81} + \frac{9r}{81}} = \frac{2}{9} \pm \sqrt{\frac{4+9r}{9}} \end{aligned}$$

$$\Rightarrow x^2 = \frac{2 \pm \sqrt{4+9r}}{9}$$

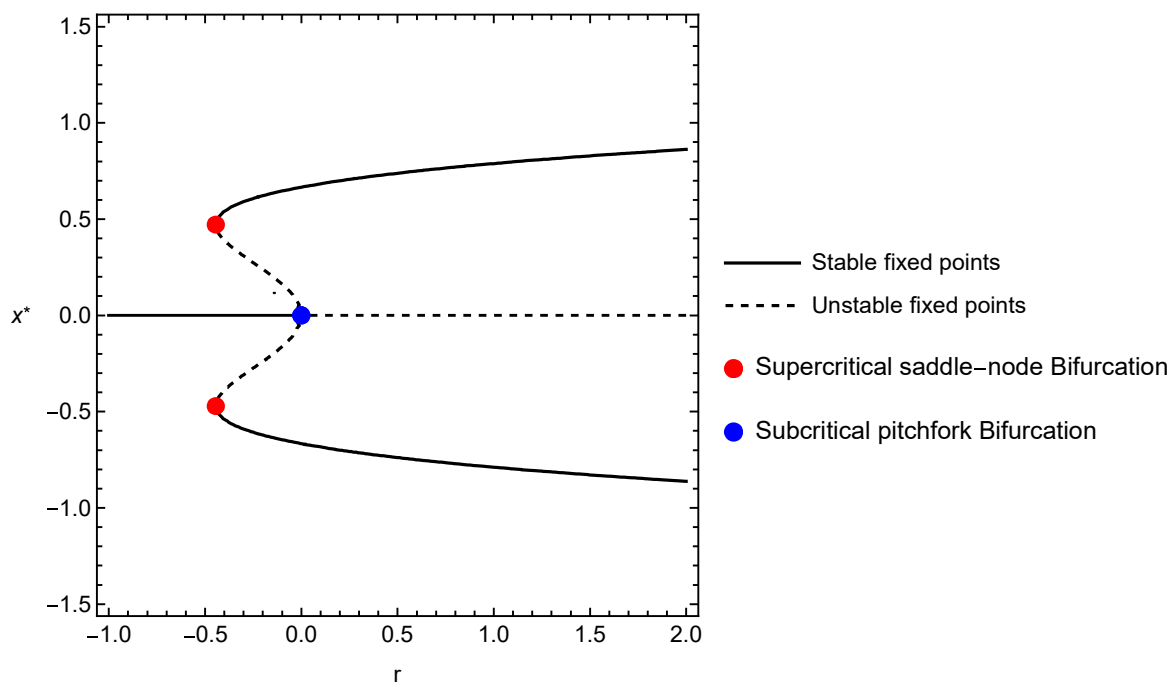
- x^2 is complex if $r < -\frac{4}{9}$
- x^2 is real and positive if $-\frac{4}{9} < r < 0$
- x^2 is real, positive and negative if $0 < r$

Also accounting to $x=0$, it follows that:

- 1 equilibria if $r < -\frac{4}{9}$
- 5 equilibria if $-\frac{4}{9} < r < 0$
- 3 equilibria if $0 < r$

We can see that as r increases, the number of equilibria increases from 1 to 5 at $r = -\frac{4}{9}$. Also that the number of equilibria decreases from 5 to 3 at $r = 0$. Thus, we have bifurcation points at $r_{c,1} = -\frac{4}{9}$ and $r_{c,2} = 0$.

In the diagram from (a) we can see that the 2 saddle-node bifurcations happen at $r_c = -\frac{4}{9}$



1.2 Subcritical pitchfork

a)

```
In[ ]:= Clear["Global`*"]

ptsSaddle = {{-4/9, 2/3}, {-4/9, -2/3}};

(*Bifurcation points are calculated in (b)*)
ptsPitch = {{0, 0}};
λ = D[r x + 4 x^3 - 9 x^5, x];

(*Stable fixed point if λ < 0 and unstable fixed point if λ > 0*)
Show[
  ContourPlot[
    ConditionalExpression[r x + 4 x^3 - 9 x^5, λ < 0] == 0,
    ConditionalExpression[r x + 4 x^3 - 9 x^5, λ > 0] == 0,
    {r, -1, 2}, {x, -1.5, 1.5},
    ContourStyle -> {{Black}, {Black, Dashed}},
    FrameLabel -> {Style["r", Black, Medium], Style["x", Black, Medium]},
    RotateLabel -> False,
    LabelStyle -> (FontSize -> 12),
    PlotLegends -> {"Stable fixed points", "Unstable fixed points"}],
  ListPlot[ptsSaddle,
    PlotLegends -> {"Supercritical saddle-node Bifurcation"},
    PlotStyle -> {PointSize[0.03], Red}],
  ListPlot[ptsPitch,
    PlotLegends -> {"Subcritical pitchfork Bifurcation"},
    PlotStyle -> {PointSize[0.03], Blue}]
]
```

$Out[] =$ 