4.2 a) $D_q = \frac{1}{1-q} \lim_{\varepsilon \to 0} \frac{\ln I_q(\varepsilon)}{\ln\left(\frac{1}{\varepsilon}\right)}$

$$I_q(\varepsilon) = \sum_{j=0}^{N_{box}} P_j^q(\varepsilon) = \sum_{k}^{n} \binom{n}{k}\left(p^k(1-p)^{n-k}\right)^q =$$

$$= \sum_{k}^{n} \binom{n}{k} p^{qk}(1-p)^{q(n-k)} = \sum_{k}^{n} \binom{n}{k}(p^q)^k\left((1-p)^q\right)^{n-k} =$$

$$= \left((1-p)^q + p^q\right)^n$$

$$\Rightarrow D_q = \frac{1}{1-q} \lim_{\varepsilon \to 0} -\frac{\ln\left(\left((1-p)^q + p^q\right)^n\right)}{\ln(\varepsilon)} = \frac{1}{1-q} \lim_{\varepsilon \to 0} -\frac{n \ln\left((1-p)^q + p^q\right)}{\ln(\varepsilon)}$$

$$\varepsilon = \left(\frac{1}{3}\right)^n \Rightarrow \ln(\varepsilon) = n \ln\left(\frac{1}{3}\right) \Rightarrow n = -\frac{\ln \varepsilon}{\ln 3}$$

$$\Rightarrow D_q = \frac{1}{1-q} \lim_{\varepsilon \to 0} \frac{-(-\ln \varepsilon) \ln\left((1-p)^q + p^q\right)}{\ln 3 \cdot \ln \varepsilon} =$$

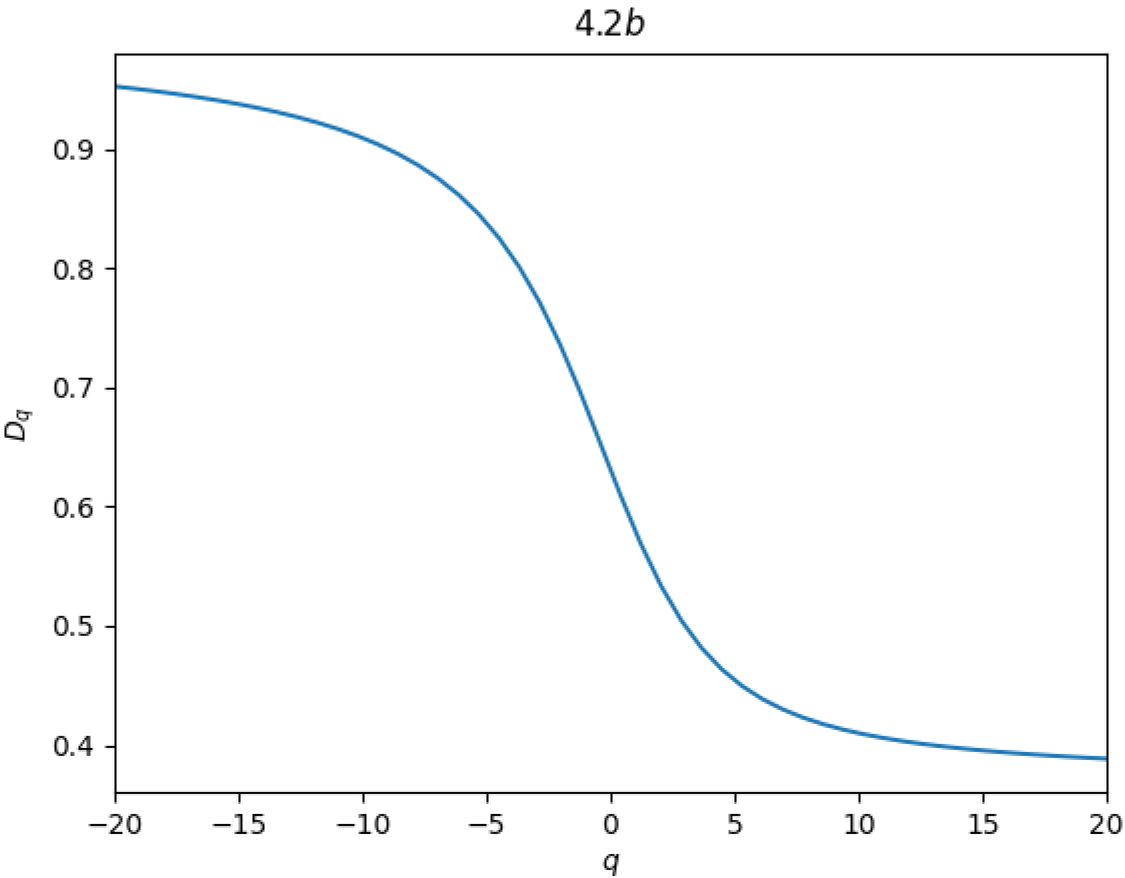
$$= \frac{\ln\left((1-p)^q + p^q\right)}{(1-q)\ln(3)} = \left\{p = \frac{1}{3}\right\} = \frac{\ln\left(\left(\frac{2}{3}\right)^q + \left(\frac{1}{3}\right)^q\right)}{(1-q)\ln(3)} =$$

$$= \frac{\ln\left(\frac{2^q}{3^q} + \frac{1^q}{3^q}\right)}{(1-q)\ln 3} = \frac{\ln\left(\frac{2^q+1}{3^q}\right)}{(1-q)\ln 3} = \frac{\ln(2^q+1) - \ln(3^q)}{(1-q)\ln 3} =$$

$$= \frac{\ln(2^q+1) - q\ln 3}{(1-q)\ln 3}$$

If $q=0$: $\frac{\ln(2^0+1) - 0 \cdot \ln 3}{(1-0)\ln 3} = \frac{\ln 2}{\ln 3}$

$$\boxed{\text{Ans: } D_q = \frac{\ln(2^q+1) - q\ln 3}{(1-q)\ln 3}}$$


4.2b
0.9
0.8
0.7
0.6
0.5
0.4
D_q
−20
−15
−10
−5
0
5
10
15
20
q

4.2 c) $D_{q\to 1} = \lim_{q\to 1} \frac{\ln(2^q+1) - q\ln 3}{(1-q)\ln 3} = \left\{\begin{matrix}\text{L'Hospitals} \\ \text{rule}\end{matrix}\right\} =$

$= \lim_{q\to 1} \frac{\left(\frac{2^q \ln 2}{2^q+1} - \ln 3\right)}{-\ln 3} = \frac{\frac{2\ln 2}{3} - \frac{3\ln 3}{3}}{-\ln 3} = \frac{2\ln 2 - 3\ln 3}{-3\ln 3} =$

$= 1 - \frac{2\ln 2}{3\ln 3}$

$D_{q=2} = \frac{\ln(2^2+1) - 2\ln 3}{(1-2)\ln 3} = \frac{\ln 5 - 2\ln 3}{-\ln 3} = \frac{\ln 5 - \ln(3^2)}{-\ln 3} =$

$= \frac{\ln\left(\frac{5}{9}\right)}{-\ln 3}$

$\boxed{\underline{\text{Ans}}: \quad D_1 = 1 - \frac{2\ln 2}{3\ln 3}, \quad D_2 = \frac{\ln\left(\frac{5}{9}\right)}{-\ln 3}}$

4.3 d) $D_{q\to -\infty} = \lim_{q\to -\infty} \frac{\ln(2^q+1) - q\ln 3}{(1-q)\ln 3} \approx \lim_{q\to -\infty} \frac{-q\ln 3}{(1-q)\ln 3} =$

$= \lim_{q\to -\infty} \frac{-q}{(1-q)} \approx 1$

$D_{q\to\infty} = \lim_{q\to\infty} \frac{\ln(2^q+1) - q\ln 3}{(1-q)\ln 3} = \left\{\begin{matrix}\text{L'Hospitals} \\ \text{rule}\end{matrix}\right\} = \lim_{q\to\infty} \frac{\left(\frac{\cancel{2^q}\ln 2}{\cancel{2^q+1}} - \ln 3\right)}{-\ln 3} \approx$

$\approx \frac{\ln 2 - \ln 3}{-\ln 3} = 1 - \frac{\ln 2}{\ln 3}$

$\boxed{\underline{\text{Ans}}: \quad D_{-\infty} \approx 1, \quad D_{\infty} \approx 1 - \frac{\ln 2}{\ln 3}}$

```python
import numpy as np
import matplotlib.pyplot as plt
import sys

qmin = -20
qmax = 20
q = np.linspace(qmin,qmax,50)
Dq = (np.log(2**q+1)-q*np.log(3))/((1-q)*np.log(3))

plt.figure()
plt.plot(q,Dq)
plt.title('$4.2b$')
plt.xlabel('$q$')
plt.ylabel('$D_q$')
plt.xlim(qmin,qmax)
plt.savefig('Dynamical Systems/DS HW4/4.2/4.2b2.png')
plt.show()
```