3.2)

a) $\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + \nu r^2 \end{cases}$

The radius (r_0) for the limit cycle is when $\dot{r} = 0$

$\Rightarrow \dot{r} = 0 = \mu r - r_0^3 \Rightarrow r_0^3 = \mu r \Rightarrow r_0^2 = \mu \Rightarrow r_0 = \sqrt{\mu}$

$\Rightarrow \dot{\theta} = \omega + \nu r_0^2 = \omega + \nu \sqrt{\mu}^2 = \omega + \nu\mu$ — one lap

$\Rightarrow \frac{d\theta}{dt} = \omega + \nu\mu \Rightarrow dt = \frac{1}{(\omega + \nu\mu)} d\theta \Rightarrow T = \int_0^{2\pi} \frac{1}{\omega + \nu\mu} d\theta =$

$= \left[\frac{\theta}{\omega + \nu\mu} \right]_0^{2\pi} = \frac{2\pi}{\omega + \nu\mu}$

<u>Ans:</u> $[r_0, T] = \left[\sqrt{\mu}, \frac{2\pi}{\omega + \nu\mu}\right]$

WHITELINES

3.2) $\begin{cases} \dot{x}_1 = F_1(X) = \frac{1}{10}x_1 - x_2^3 - x_1x_2^2 - x_1^2x_2 - x_2 - x_1^3 \\ \dot{x}_2 = F_2(X) = x_1 + \frac{1}{10}x_2 + x_1x_2^2 + x_1^3 - x_2^3 - x_1^2x_2 \end{cases}$

b, c) $\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$ (1), $\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = w + \nu r^2 \end{cases}$ (2)

(1) and (2):

$$\Rightarrow \dot{x}_1 = \dot{r}\cos\theta - \dot{\theta}r\sin\theta = (\mu r - r^3)\cos\theta - (w + \nu r^2)r\sin\theta =$$

$$= (\mu - r^2)r\cos\theta - (w + \nu r^2)r\sin\theta =$$

$$= (\mu - (x_1^2 + x_2^2))(x_1^2 + x_2^2)^{1/2}\cos(\tan^{-1}(\tfrac{x_2}{x_1})) - (w + \nu(x_1^2 + x_2^2))(x_1^2 + x_2^2)^{1/2}\sin(\tan^{-1}(\tfrac{x_2}{x_1}))$$

$$\dot{x}_2 = \dot{r}\sin\theta + \dot{\theta}r\cos\theta = (\mu r - r^3)\sin\theta + (w + \nu r^2)r\cos\theta =$$

$$= (\mu - r^2)r\sin\theta + (w + \nu r^2)r\cos\theta =$$

$$= (\mu - (x_1^2 + x_2^2))(x_1^2 + x_2^2)^{1/2}\sin(\tan^{-1}(\tfrac{x_2}{x_1})) + (w + \nu(x_1^2 + x_2^2))(x_1^2 + x_2^2)^{1/2}\cos(\tan^{-1}(\tfrac{x_2}{x_1}))$$

$$\dot{x}_1 = \frac{d(r\cos\theta)}{dt} = \frac{1}{10}r\cos\theta - r^3\sin^3\theta - r^3\sin^2\theta\cos\theta - r^3\cos^2\theta\sin\theta - r\sin\theta -$$

$$- r^3\cos^3\theta =$$

$$= r(\tfrac{1}{10}\cos\theta - r^2\sin^3\theta - r^2\sin^2\theta\cos\theta - r^2\cos^2\theta\sin\theta - \sin\theta - r^2\cos^3\theta) =$$

$$= r(\tfrac{1}{10}\cos\theta - r^2(\sin^3\theta + \sin^2\theta\cos\theta + \cos^2\theta\sin\theta + \cos^3\theta) - \sin\theta) =$$

$$= r(\tfrac{1}{10}\cos\theta - \sin\theta - r^2(\sin\theta(\underbrace{\sin^2\theta + \cos^2\theta}_{1} + \sin\theta\cos\theta) + \cos^3)) =$$

$$= r(\tfrac{1}{10}\cos\theta - \sin\theta - r^2(\sin^2\theta\cos\theta + \sin\theta + \cos^3\theta)) =$$

$$= r(\tfrac{1}{10}\cos\theta - \sin\theta - r^2(\cos\theta(\underbrace{\sin^2\theta + \cos^2\theta}_{1}) + \sin\theta)) =$$

$$= r(\tfrac{1}{10}\cos\theta - \sin\theta - r^2(\cos\theta + \sin\theta)) =$$

$$= r(\tfrac{1}{10}\cos\theta - \sin\theta - r^2\cos\theta - r^2\sin\theta) =$$

$$= r(\cos\theta(\tfrac{1}{10} - r^2) - \sin\theta(1 + r^2)) =$$

$$= (\tfrac{1}{10} - r^2)r\cos\theta - (1 + r^2)r\sin\theta$$ compare with:

$$(\mu - r^2)r\cos\theta - (w + \nu r^2)r\sin\theta$$

$$\mu = \tfrac{1}{10}, \quad w = 1, \quad \nu = 1$$

$$\dot{x}_2 = \frac{d(r\sin\theta)}{dt} = r\cos\theta + \frac{1}{10} r\sin\theta + r^3\sin^2\theta\cos\theta + r^3\cos^3\theta -$$

$$-r^3\sin^3\theta - r^3\cos^2\theta\sin\theta =$$

$$= r(\cos\theta + \tfrac{1}{10}\sin\theta + r^2(\sin^2\theta\cos\theta + \cos^3\theta - \sin^3\theta - \cos^2\theta\sin\theta))) =$$

$$= r(\cos\theta + \tfrac{1}{10}\sin\theta + r^2(\sin\theta(\sin\theta\cos\theta \underbrace{- \sin^2\theta - \cos^2\theta}_{-1}) + \cos^3\theta)) =$$

$$= r(\cos\theta + \tfrac{1}{10}\sin\theta + r^2(\sin^2\cos\theta - \sin\theta + \cos^3\theta)) =$$

$$= r(\cos\theta + \tfrac{1}{10}\sin\theta + r^2(\cos\theta\underbrace{(\sin^2\theta + \cos^2\theta)}_{1} - \sin\theta)) =$$

$$= r(\cos\theta + \tfrac{1}{10}\sin\theta + r^2\cos\theta - r^2\sin\theta =$$

$$= r(\cos\theta(1+r^2) + \sin\theta(\tfrac{1}{10} - r^2) =$$

$$= (\tfrac{1}{10} - r^2)r\sin\theta + (1+r^2)r\cos\theta$$

compare with:

$$(\mu - r^2)r\sin\theta + (\omega + \nu r^2)r\cos\theta$$

$$\Rightarrow \quad \mu = \frac{1}{10}, \quad \omega = 1, \quad \nu = 1$$

<u>Ans</u>: $\mu = \frac{1}{10}$, $\omega = 1$, $\nu = 1$

3.2 d) $\dot{\mathbb{M}} = \mathbb{J}\,\mathbb{M}$

$$\mathbb{J} = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{10} - x_2^2 - 2x_1x_2 - 3x_1^2\right) & \left(-3x_2^2 - 2x_1x_2 - x_1^2 - 1\right) \\ \left(1 + x_2^2 + 3x_1^2 - 2x_1x_2\right) & \left(\frac{1}{10} + 2x_1x_2 - 3x_2^2 - x_1^2\right) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{M}_{11} & \dot{M}_{12} \\ \dot{M}_{21} & \dot{M}_{22} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \overbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}^{\mathbb{1} \text{ at } t=0} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \dot{M}_{11} = J_{11} M_{11} + J_{12} M_{21} \\ \dot{M}_{12} = J_{11} M_{12} + J_{12} M_{22} \\ \dot{M}_{21} = J_{21} M_{11} + J_{22} M_{21} \\ \dot{M}_{22} = J_{21} M_{12} + J_{22} M_{22} \end{cases}$$

3.2 g) $\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + \nu r^2 \end{cases}$ $\begin{cases} \mu = \frac{1}{10} \\ \nu = 1 \\ \omega = 1 \end{cases}$

$$\mathbb{J}_{polar} = \begin{pmatrix} \mu - 3r^2 & 0 \\ 2\nu r & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} - 3r^2 & 0 \\ 2r & 0 \end{pmatrix}$$

$$\mathbb{M}_{polar} = e^{\mathbb{J}_{polar} \cdot t}, \quad T = \frac{2\pi}{\omega + \nu\mu} = \frac{2\pi}{1 + \frac{1}{10}} = \frac{20\pi}{11}$$

$$\mathbb{J}_{polar} \cdot T = \begin{pmatrix} \left(\frac{2\pi}{11} - \frac{60\pi r^2}{11}\right) & 0 \\ \frac{40\pi r}{11} & 0 \end{pmatrix} = \left\{ r = \sqrt{\mu} = \frac{1}{\sqrt{10}} \right\} = \begin{pmatrix} \frac{-4\pi}{11} & 0 \\ \frac{40\pi}{11\sqrt{10}} & 0 \end{pmatrix}$$

$$\mathbb{M}_{cart} = G^{-1} \mathbb{M}_{polar} G$$

$$G = \begin{pmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial (x_1^2 + x_2^2)^{1/2}}{\partial x_1} & \frac{\partial (x_1^2 + x_2^2)^{1/2}}{\partial x_2} \\ \frac{\partial \tan^{-1}\left(\frac{x_2}{x_1}\right)}{\partial x_1} & \frac{\partial \tan^{-1}\left(\frac{x_2}{x_1}\right)}{\partial x_2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{x_1}{(x_1^2 + x_2^2)^{1/2}} & \frac{x_2}{(x_1^2 + x_2^2)^{1/2}} \\ -\frac{x_2}{x_1^2 + x_2^2} & \frac{x_1}{x^2 + y^2} \end{pmatrix} = \begin{Bmatrix} x_1 = \sqrt{\mu} = \frac{1}{\sqrt{10}} \\ x_2 = 0 \end{Bmatrix} = \begin{pmatrix} \frac{\frac{1}{\sqrt{10}}}{\frac{1}{\sqrt{10}}} & 0 \\ 0 & \frac{\frac{1}{\sqrt{10}}}{\frac{1}{10}} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{10} \end{pmatrix}$$

$$G^{-1} = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial \theta} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \left\{ \begin{array}{l} r = \sqrt{\mu} = \frac{1}{\sqrt{10}} \\ \theta = 0 \end{array} \right\} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$M_{cart} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{pmatrix} \underbrace{\exp\begin{pmatrix} -\frac{4\pi}{11} & 0 \\ \frac{40\pi}{11\sqrt{10}} & 0 \end{pmatrix}}_{\text{use mathematica}} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{10} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} e^{-\frac{4\pi}{11}} & 0 \\ \sqrt{10}\left(1-e^{-\frac{4\pi}{11}}\right) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{10} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-\frac{4\pi}{11}} & 0 \\ 1-e^{-\frac{4\pi}{11}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{10} \end{pmatrix} = \begin{pmatrix} e^{-\frac{4\pi}{11}} & 0 \\ 1-e^{-\frac{4\pi}{11}} & 1 \end{pmatrix}$$

<u>Ans:</u>

$$M = \begin{pmatrix} e^{-\frac{4\pi}{11}} & 0 \\ 1-e^{-\frac{4\pi}{11}} & 1 \end{pmatrix}$$

3.2 h)

$$\mathbb{M} = \begin{pmatrix} e^{\frac{-4\pi}{11}} & 0 \\ 1-e^{\frac{-4\pi}{11}} & 1 \end{pmatrix} \quad \begin{cases} \tau = 1 + e^{\frac{-4\pi}{11}} \\ \Delta = e^{\frac{-4\pi}{11}} \end{cases}$$

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \{\text{use mathematica}\} \Rightarrow$$

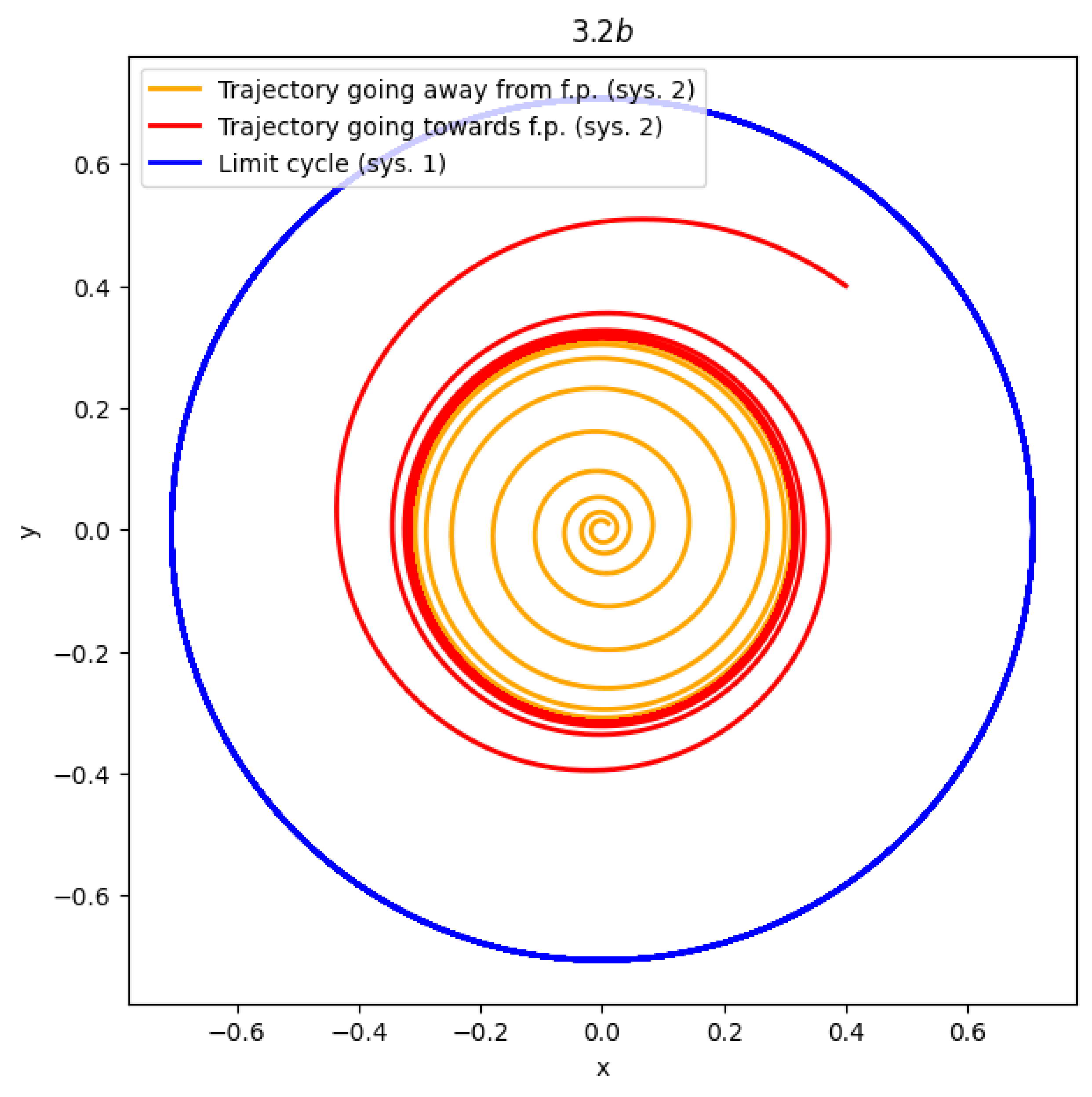
$$\Rightarrow \lambda_1 = e^{\frac{-4\pi}{11}}, \quad \lambda_2 = 1, \quad T = \frac{20\pi}{11}$$

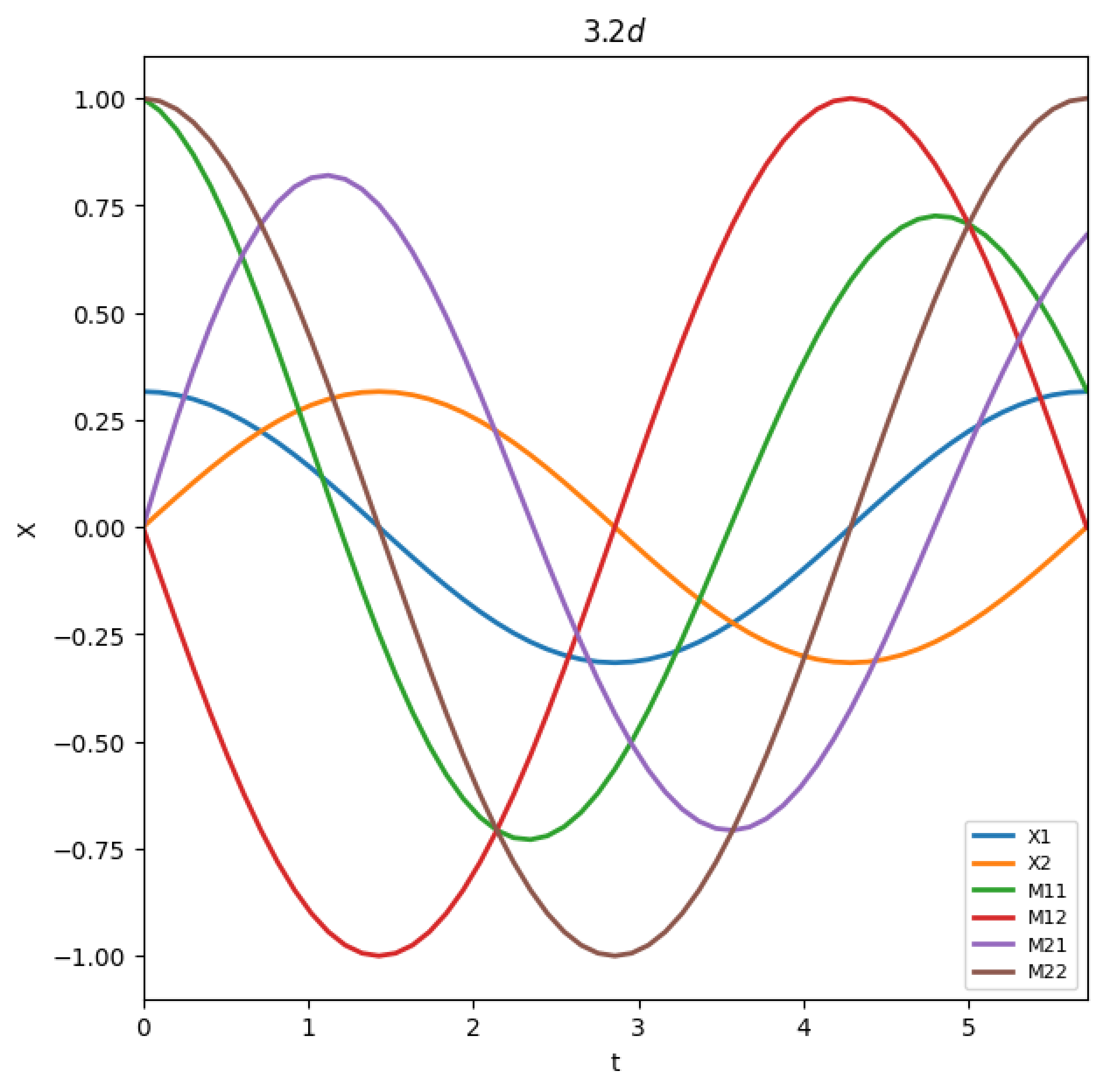
$$\sigma_i = \frac{1}{T} \ln(\text{eig}\,\mathbb{M}).$$

$$\sigma_1 = \frac{11}{20\pi} \ln\left(e^{\frac{-4\pi}{11}}\right) = \frac{11}{20\pi} \cdot \frac{-4\pi}{11} = -\frac{1}{5}$$

$$\sigma_2 = \frac{11}{20\pi} \ln(1) = 0$$

<u>Ans:</u> $\sigma = \left[-\frac{1}{5}, 0\right]$


3.2b
Trajectory going away from f.p. (sys. 2)
Trajectory going towards f.p. (sys. 2)
Limit cycle (sys. 1)
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
y
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
x


3.2d
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
X
0
1
2
3
4
5
t
X1
X2
M11
M12
M21
M22

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
import sys
import os

xmin = -1
ymin = -1
xmax = 1
ymax = 1

# Numerical integration
T = 100
t = np.linspace(0,T,T*100)

x1 = np.zeros(T)
y1 = x1.copy()
x1[0] = 0.01
y1[0] = 0.01

x2 = x1.copy()
y2 = x1.copy()
x2[0] = 0.4
y2[0] = 0.4

x3 = x1.copy()
y3 = x1.copy()

mu = 0.5
omega = 0.5
nu = 0.5
x3[0] = mu
y3[0] = mu

def dynamical_system1(xy, t):
    x = xy[0]
    y = xy[1]
    dxdt_integration = (1/10)*x-y**3-x*y**2-x**2*y-y-x**3
    dydt_integration = x+(1/10)*y+x*y**2+x**3-y**3-x**2*y
    return [dxdt_integration, dydt_integration]

def dynamical_system2(rtheta, t):
    r = rtheta[0]
    drdt_integration = mu*r-r**3
    dthetadt_integration = omega+nu*r**2
    return [drdt_integration, dthetadt_integration]

x0y01 = [x1[0],y1[0]]
xy1 = odeint(dynamical_system1, x0y01, t)
x1 = xy1[:,0]
y1 = xy1[:,1]

x0y02 = [x2[0],y2[0]]
xy2 = odeint(dynamical_system1, x0y02, t)
x2 = xy2[:,0]
y2 = xy2[:,1]

r0theta0 = [mu**0.5,0]
rtheta = odeint(dynamical_system2, r0theta0, t)
r = rtheta[:,0]
theta = rtheta[:,1]
x3 = r*np.cos(theta)
y3 = r*np.sin(theta)

fig, ax = plt.subplots(figsize=(7,7))
ax.plot(x1, y1,'-', color='orange', linewidth=2, label='Trajectory going away from f.p. (sys. 2)')
ax.plot(x2, y2,'-', color='red', linewidth=2, label='Trajectory going towards f.p. (sys. 2)')
ax.plot(x3, y3,'-', color='blue', linewidth=2, label='Limit cycle (sys. 1)')
ax.set_title('$3.2b$')
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_box_aspect(1)

plt.legend(loc="upper left")
plt.savefig('Dynamical systems/DS HW3/3.2/3.2b.png', bbox_inches='tight')
```

```
plt.show()
```

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
import sys
import os

# Numerical integration
omega = 1
nu = 1
mu = 1/10
T = 2*np.pi/(omega+nu*mu)
timesteps = int(T*10)
t_array = np.linspace(0,T,timesteps)
dt = T/len(t_array)
xmin = 0
xmax = T

# J11 = 1/10-X20**2-2*X10[0]*X20[0]-3*X10[0]
# J12 = -3*X20**2-2*X10[0]*X20[0]-X10[0]**2-1
# J21 = 1+X20**2+3*X10[0]**2-2*X10[0]*X20[0]
# J22 = 1/10+2*X10[0]*X20[0]-3*X20[0]**2-X10[0]**2

# J = np.array([[J11, J12],\
#               [J21, J22]])

def dynamical_system(IC, t):
    X1 = IC[0]
    X2 = IC[1]
    M11 = IC[2]
    M12 = IC[3]
    M21 = IC[4]
    M22 = IC[5]

    dX1 = (1/10)*X1-X2**3-X1*X2**2-X1**2*X2-X2-X1**3
    dX2 = X1+(1/10)*X2+X1*X2**2+X1**3-X2**3-X1**2*X2

    J11 = 1/10-X2**2-2*X1*X2-3*X1**2
    J12 = -3*X2**2-2*X1*X2-X1**2-1
    J21 = 1+X2**2+3*X1**2-2*X1*X2
    J22 = 1/10+2*X1*X2-3*X2**2-X1**2

    dM11 = J11*M11+J12*M21
    dM12 = J11*M12+J12*M22
    dM21 = J21*M11+J22*M21
    dM22 = J21*M12+J22*M22
    return [dX1, dX2, dM11, dM12, dM21, dM22]

X1 = mu**0.5
X2 = 0
M11 = 1
M12 = 0
M21 = 0
M22 = 1
IC = [X1,X2,M11,M12,M21,M22]
eqs = odeint(dynamical_system, IC, t_array)

X1 = eqs[:,0]
X2 = eqs[:,1]

M11 = eqs[:,2]
M12 = eqs[:,3]
M21 = eqs[:,4]
M22 = eqs[:,5]

M11_last = M11[-1]
M12_last = M12[-1]
M21_last = M21[-1]
M22_last = M22[-1]


print('M11: ', np.round(M11_last,4))
print('M12: ', np.round(M12_last,4))
print('M12: ', M12_last)
print('M21: ', np.round(M21_last,4))
print('M22: ', np.round(M22_last,4))

M = np.array([[M11_last,M12_last],[M21_last,M22_last]])
eig_values, eig_vectors = np.linalg.eig(M)
```

```python
print(eig_values)
stab_exps = np.log(eig_values)/T
print('Stability exponents: ', np.round(stab_exps,4))


# for t in (t_array-1):

#     t = int(t)
# J11 = 1/10-X2[t]**2-2*X1[t]*X2[t]-3*X1[-1]
# J12 = -3*X2[t]**2-2*X1[t]*X2[t]-X1[t]**2-1
# J21 = 1+X2[t]**2+3*X1[t]**2-2*X1[-1]*X2[-1]
# J22 = 1/10+2*X1[t]*X2[t]-3*X2[t]**2-X1[-1]**2

#     J = np.array([[J11, J12],\
#                   [J21, J22]])

#     M = M + np.matmul(J,M)*dt

#     M11[t+1] = M[0,0]
#     M12[t+1] = M[0,1]
#     M21[t+1] = M[1,0]
#     M22[t+1] = M[1,1]

fig, ax = plt.subplots(figsize=(7,7))
ax.plot(t_array, X1, '-', linewidth=2, label='X1')
ax.plot(t_array, X2, '-', linewidth=2, label='X2')
ax.plot(t_array, M11, '-', linewidth=2, label='M11')
ax.plot(t_array, M12, '-', linewidth=2, label='M12')
ax.plot(t_array, M21, '-', linewidth=2, label='M21')
ax.plot(t_array, M22, '-', linewidth=2, label='M22')

ax.set_title('$3.2d$')
ax.set_xlabel('t')
ax.set_ylabel('X')
ax.set_xlim(xmin,xmax)
ax.set_box_aspect(1)

plt.legend(loc="lower right", prop={'size': 8})
plt.savefig('Dynamical systems/DS HW3/3.2/3.2d.png', bbox_inches='tight')
# plt.show()
```

```
ClearAll["Global`*,"];
A = {{1, 0}, {0, 1/√10}};
B = MatrixExp[{{(-4*π)/11, 0}, {(40*π)/(11*√10), 0}}] // FullSimplify
AA = {{1, 0}, {0, 10/√10}};
```

Out[ ]=

$$\left\{\left\{e^{-4\pi/11}, 0\right\}, \left\{\sqrt{10}\left(1 - e^{-4\pi/11}\right), 1\right\}\right\}$$

In[ ]:=

```
M = {{E^(-4*π/11), 0}, {1 - E^(-4*π/11), 1}};
T = 20*π/11;
eig = Eigenvalues[M]
Log[eig]
```

Out[ ]=

$$\left\{1, e^{-4\pi/11}\right\}$$

Out[ ]=

$$\left\{0, -\frac{4\pi}{11}\right\}$$