1.1c)

$\dot{x} = x(r-x)+h$, F.P. at $\dot{x}=0$

$\Rightarrow xr - x^2 + h = 0 \Rightarrow x^2 - xr - h = 0$

$\Rightarrow x = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} + h} = \frac{r}{2} \pm \sqrt{\frac{r^2+4h}{4}}$

$\Rightarrow x = \frac{r \pm \sqrt{r^2+4h}}{2}$ (3D-curve for 1.1b)

x complex if $(r^2+4h) < 0$

x real, positive if $0 < (r^2+4h) < r^2$

x real, positive and negative if $r^2 < (r^2+4h)$

This follows that

0 equilibria if $(r^2+4h) < 0$

2 equilibria if $0 < (r^2+4h) < r^2$

2 equilibria if $r^2 < (r^2+4h)$

We can see that we have a bifurcation point at $r^2+4h=0$

$\Rightarrow h_c(r) = \frac{-r^2}{4}$ (Bifurcation diagram for 1.1a).

Ans: $\begin{bmatrix} h_c(r) \\ r \end{bmatrix} = \begin{bmatrix} \frac{-r^2}{4} \\ r \end{bmatrix}$

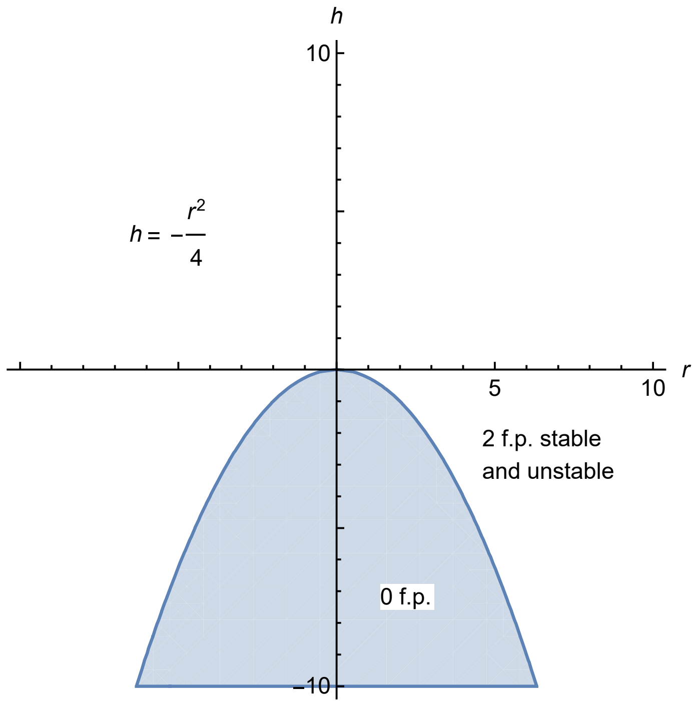
1.1d) Direction of bifurcation along $r \rightarrow$ unit vecter of the differential:

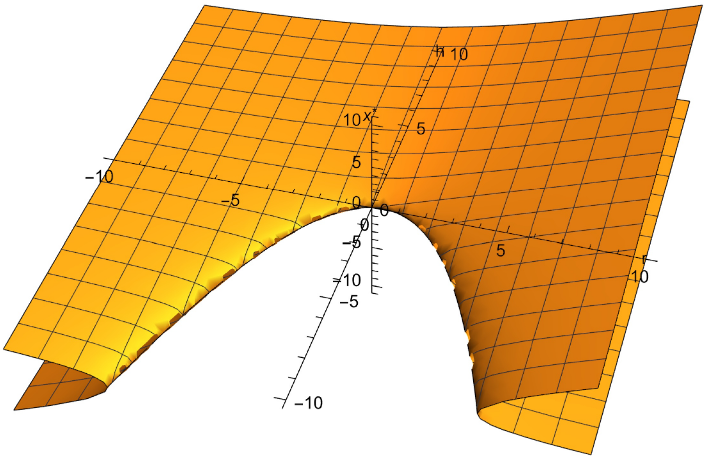
$h_c'(r) = -\frac{2r}{4} = \frac{-r}{2} \Rightarrow \begin{bmatrix} \frac{-r}{2} \\ 1 \end{bmatrix}$ ~~Normalise to unit vector:~~

Normalise to unit vector; use euclidean distance!

$1 = \frac{\sqrt{(\frac{-r}{2}-0)^2+(1-0)^2}}{\sqrt{(\frac{-r}{2}-0)^2+(1-0)^2}} = \frac{\sqrt{\frac{r^2}{2}+1}}{\sqrt{\frac{r^2}{2}+1}}$

Ans: $\begin{bmatrix} h_c(r) \\ r \end{bmatrix} = \frac{1}{\sqrt{\frac{r^2}{2}+1}} \begin{bmatrix} -\frac{r}{2} \\ 1 \end{bmatrix}$


h
10
$h = -\frac{r^2}{4}$
r
5
10
2 f.p. stable
and unstable
0 f.p.
−10

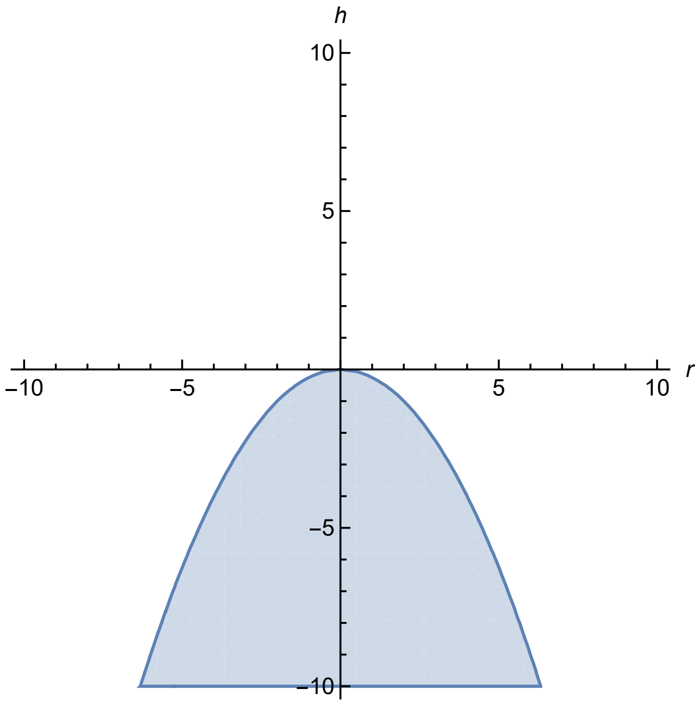

-10
-5
0
5
10
10
5
0
-5
-10
10
5
0
-5
-10
x

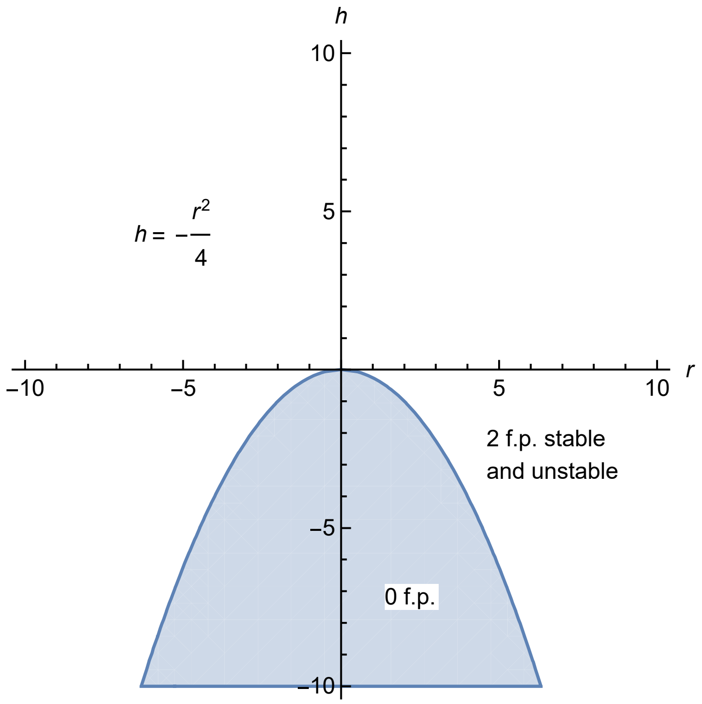
1.1 Imperfect transcritical bifurcation

a)

In[ ]:=
```
Clear["Global`*"]
RegionPlot[h < -r^2/4, {r, -10, 10}, {h, -10, 10},
 AxesLabel → Automatic,
 Axes → True,
 Frame → None,
 FrameLabel → {r, h},
 RotateLabel → False,
 LabelStyle → (FontSize → 12)]
```

Out[ ]=




h
10
5
$h = -\frac{r^2}{4}$
r
–10
–5
5
10
2 f.p. stable
and unstable
–5
0 f.p.
–10

b)

In[ ]:=
```
Clear["Global`*"]
x1 = (r +  r^2 + 4*h)/2;
x2 = (r -  r^2 + 4*h)/2;
Show[
 Plot3D[x1, {r, -10, 10}, {h, -10, 10},
  AxesOrigin → {0, 0, 0},
  PlotRange → {-11, 11}],
 Plot3D[x2, {r, -10, 10}, {h, -10, 10},
  AxesOrigin → {0, 0, 0},
  PlotRange → {-11, 11}],
 Graphics3D[{Text["r", {10, 0, 0}],
   Text["h", {0, 10, 0}],
   Text["x*", {0, 0, 11}]}],
 Boxed → False]
```

Out[ ]=