1.3 a) $\underline{\sigma = -1}$:

$$\begin{cases} \dot{x} = 2x + 4y \stackrel{!}{=} 0 \\ \dot{y} = -\frac{9}{4}x - 4y = 0 \end{cases} \Rightarrow \begin{cases} x = -2y, \quad -\frac{9}{4}(-2y) - 4y = \frac{9}{2}y - \frac{8}{2}y = \frac{1}{2}y = 0 \\ \Rightarrow y = 0 \Rightarrow x = 0 \Rightarrow \text{f.P. at } (x, y)^T = (0,0)^T \end{cases}$$

$\underline{\sigma = 0}$:

$$\begin{cases} \dot{x} = 3x + 4y = 0 \\ \dot{y} = -\frac{9}{4}x - 3y = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{-4}{3}y, \quad -\frac{9}{4}\left(\frac{-4}{3}y\right) - 3y = 3y - 3y = 0 \\ \Rightarrow \text{"f.P.s." for any } x \text{ and } y \text{ so that} \\ x = -\frac{4}{3}y \end{cases}$$

$\underline{\sigma = 1}$

$$\begin{cases} \dot{x} = 4x + 4y = 0 \\ \dot{y} = -\frac{9}{4}x - 2y = 0 \end{cases} \Rightarrow \begin{matrix} x = -y, \quad -\frac{9}{4}(-y) - 2y = \frac{9}{4}y - \frac{8y}{4} = \frac{1}{4}y = 0 \\ \Rightarrow y = 0 \Rightarrow x = 0 \Rightarrow \text{f.P. at } (x, y)^T = (0,0)^T \end{matrix}$$

1.3

$$\begin{cases} \dot{x} = (\sigma+3)x + 4y \\ \dot{y} = -\frac{9}{4}x + (\sigma-3)y \end{cases} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} (\sigma+3) & 4 \\ -\frac{9}{4} & (\sigma-3) \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

b) $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$, $\begin{cases} \Delta = \det M \\ \tau = \operatorname{Tr} M \end{cases}$

$$\Delta = (\sigma+3)(\sigma-3) - \left(4 \cdot \frac{-9}{4}\right) = \sigma^2 - 3\sigma + 3\sigma - 9 + 9 = \sigma^2$$

$$\tau = \sigma + 3 + \sigma - 3 = 2\sigma$$

$$\lambda_{1,2} = \frac{2\sigma \pm \sqrt{4\sigma^2 - 4\sigma^2}}{2} = \sigma \quad \underline{\text{Ans}}: \lambda_{1,2} = \begin{bmatrix} \sigma \\ \sigma \end{bmatrix}$$

c) $0 = M - I\lambda = \left(\begin{array}{cc|c} (\sigma+3-\sigma) & 4 & 0 \\ -\frac{9}{4} & (\sigma-3-\sigma) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 3 & 4 & 0 \\ -\frac{9}{4} & -3 & 0 \end{array}\right) =$

$R_1 \to R_1 \cdot \frac{1}{3}$

$$= \left(\begin{array}{cc|c} 1 & \frac{4}{3} & 0 \\ -\frac{9}{4} & -3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & \frac{4}{3} & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x + \frac{4}{3}y = 0$$

$R_2 \to R_2 - \left(\frac{-9}{4}\right)R_1$

$$\Rightarrow x = \frac{-4}{3}y, \; y = y = 1 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$$

Normalise to unit vector: use euclidean distance:

$$1 = \frac{\sqrt{(-4/3 - 0)^2 + (1-0)^2}}{\sqrt{(-4/3-0)^2 + (1-0)^2}} = \sqrt{\frac{16}{9} + 1}$$

$\underline{\text{Ans}}$: Eigenvector: $\sqrt{\frac{16}{9} + 1} \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$

d) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$M^{-1} = \frac{1}{(\sigma+3)(\sigma-3) - \left(\frac{-9}{4} \cdot 4\right)} \begin{pmatrix} (\sigma-3) & -4 \\ \frac{9}{4} & (\sigma+3) \end{pmatrix} = \frac{1}{\sigma^2 - 9 + 9} \begin{pmatrix} (\sigma-3) & -4 \\ \frac{9}{4} & (\sigma+3) \end{pmatrix}$$

$\underline{\text{Ans}}$: $M^{-1} = \frac{1}{\sigma^2} \begin{pmatrix} (\sigma-3) & -4 \\ \frac{9}{4} & (\sigma+3) \end{pmatrix}$

e) Ans: If $\sigma = 0$ then M^{-1} cannot exist; division by zero.

$$\begin{cases} \dot{x} = (\sigma - cd)x + d^2 y \\ \dot{y} = -c^2 x + (\sigma + cd)y \end{cases}$$

f) If $c = 3/2$ and $d = -2$ then we get same as before

$$\begin{cases} \dot{x} = (\sigma - \frac{3}{2} \cdot -2)x + (-2)^2 y = (\sigma + 3)x + 4y \\ \dot{y} = -(\frac{3}{2})^2 x + (\sigma + \frac{3}{2} \cdot -2)y = -\frac{9}{4}x + (\sigma - 3)y \end{cases}$$

Ans: $c = 3/2$, $d = -2$

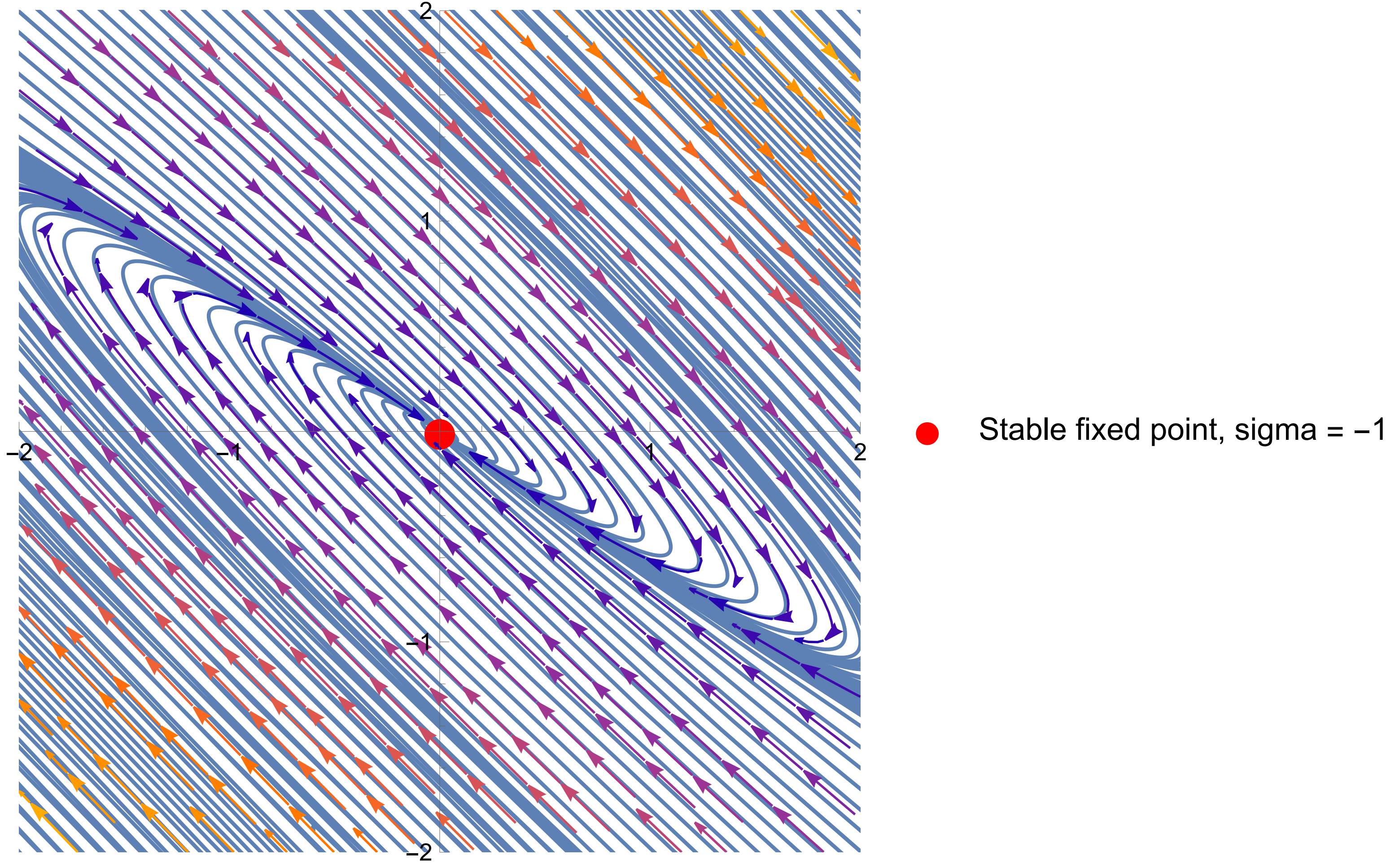
g) $\lambda_{1,2} = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$, $M = \begin{bmatrix} (\sigma - cd) & d^2 \\ -c^2 & (\sigma + cd) \end{bmatrix}$

$$\begin{cases} \tau = \operatorname{Tr} M = 2\sigma \\ \Delta = \det M = (\sigma - cd)(\sigma + cd) - (d^2 \cdot -c^2) = \sigma^2 - c^2 d^2 + c^2 d^2 = \sigma^2 \end{cases}$$

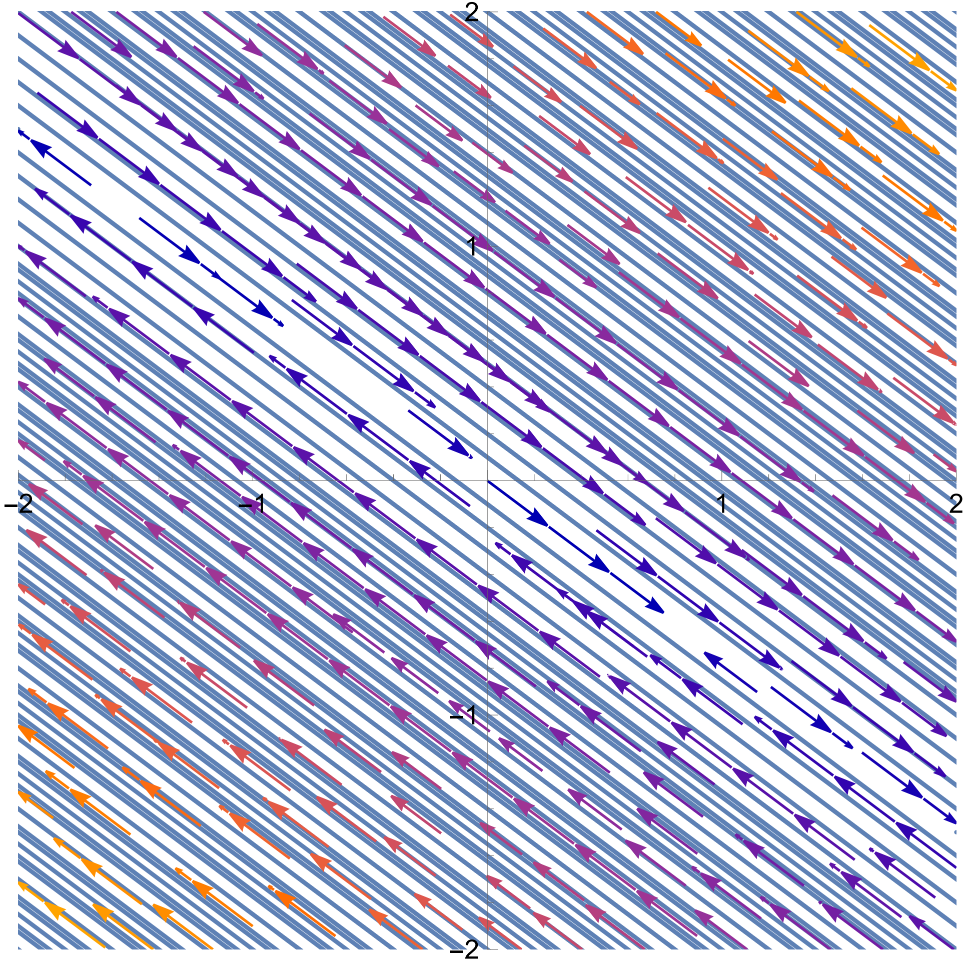
$\Rightarrow \lambda_{1,2} = \dfrac{2\sigma \pm \sqrt{4\sigma^2 - 4\sigma^2}}{2} = \sigma$

Ans: $\lambda_{1,2} = \begin{bmatrix} \sigma \\ \sigma \end{bmatrix}$

Out[•]=

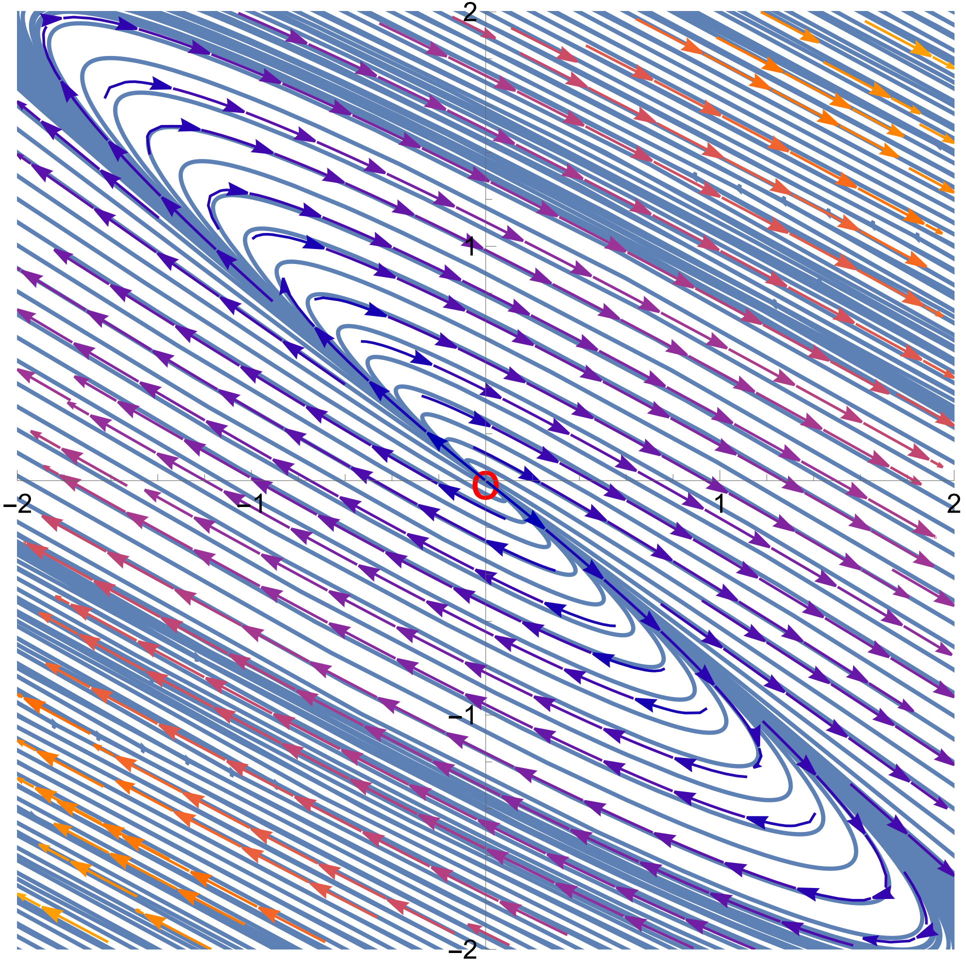


Out[ ]=



Line of fixed fixed point, sigma = 0

Out[●]=



In[135]:=

```
Clear[minx, miny, maxx, maxy]
minx = -2;
miny = -2;
maxx = 2;
maxy = 2;
```

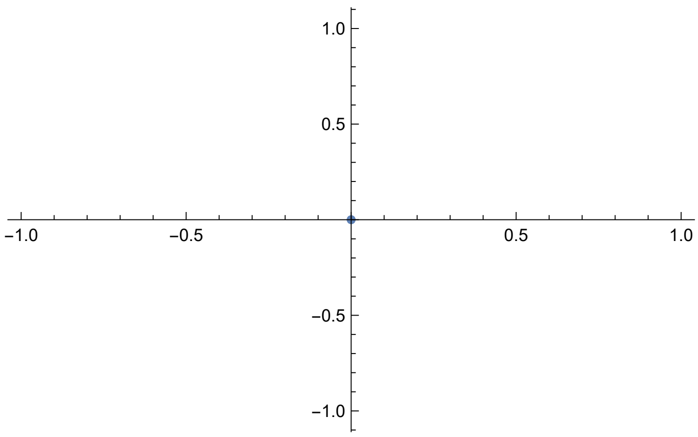
In[39]:=

```
Clear[sol, x, y, t]
sol[x0_, y0_] := NDSolve[
  {x'[t] == 3 * x[t] + 4 * y[t],
   y'[t] == -9/4 * x[t] - 3 * y[t],
   x[0] == x0, y[0] == y0},
  {x, y}, {t, -10, 10}]
```

In[130]:=

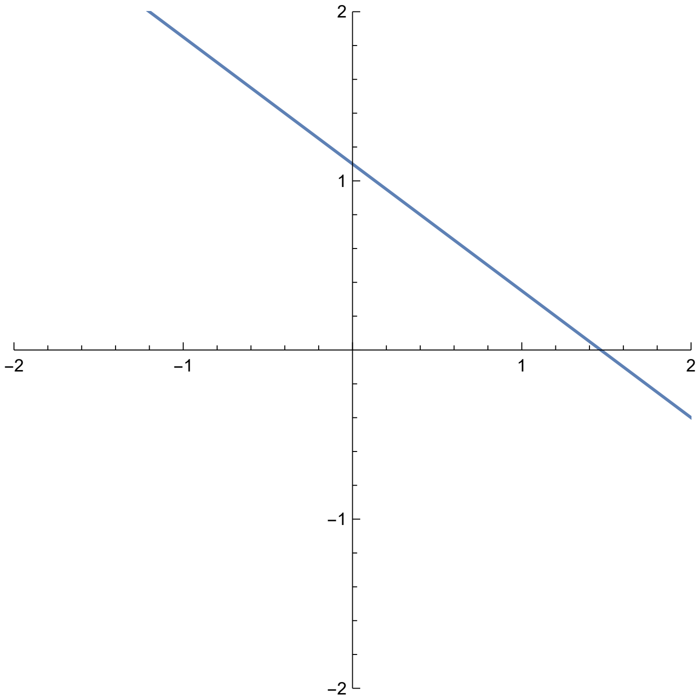
```
initialCond = Join[
   (*Table[{minx , y } , {y,miny , maxy,0.1}],
   Table[{maxx, y} , {y, miny , maxy,0.1}],
   Table[{x , miny},{x , minx , maxx,0.1}],
   Table[{x,maxy},{x,minx , maxx,0.1}]*)

   Table[{x , miny}, {x , minx , maxx, 0.1}],
   Table[{x, maxy}, {x, minx , maxx, 0.1}],
   Table[{minx , y } , {y, miny , maxy, 0.1}],
   Table[{maxx, y} , {y, miny , maxy, 0.1}]
  ];
```

Out[ ]=



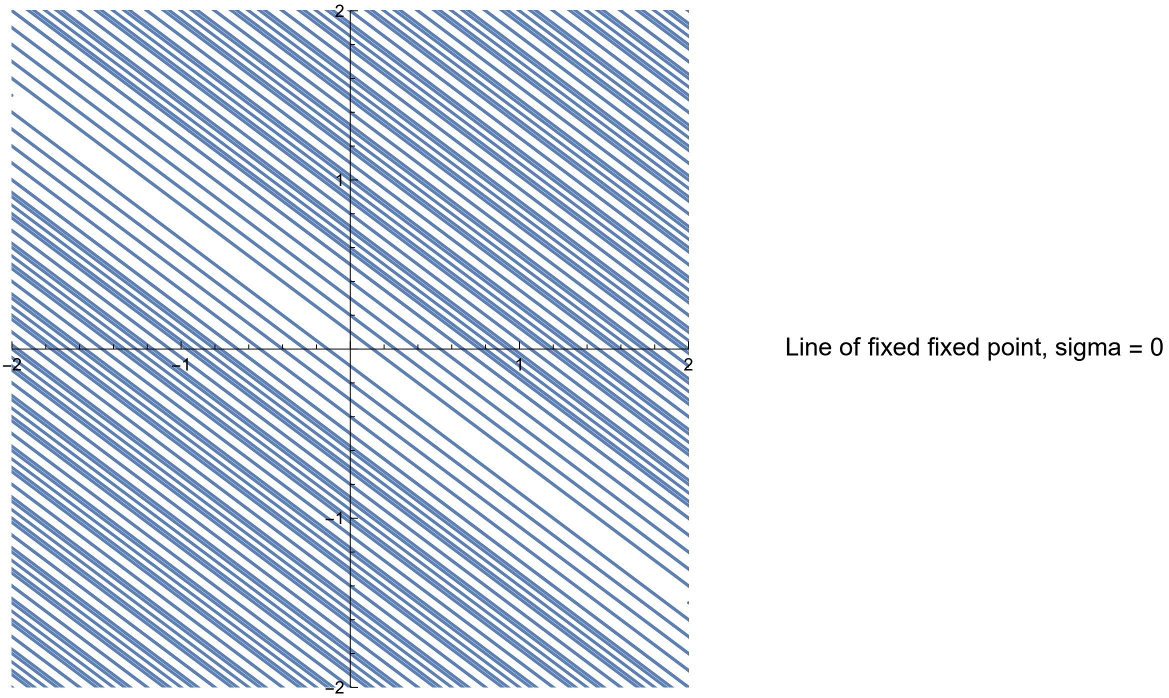
In[41]:=
```
ParametricPlot[
 Evaluate[{x[t], y[t]} /. sol[initialCond[[50, 1]], initialCond[[50, 2]]]],
 {t, -10, 10}, PlotRange → {{minx, maxx}, {miny, maxy}}]
```

Out[41]=



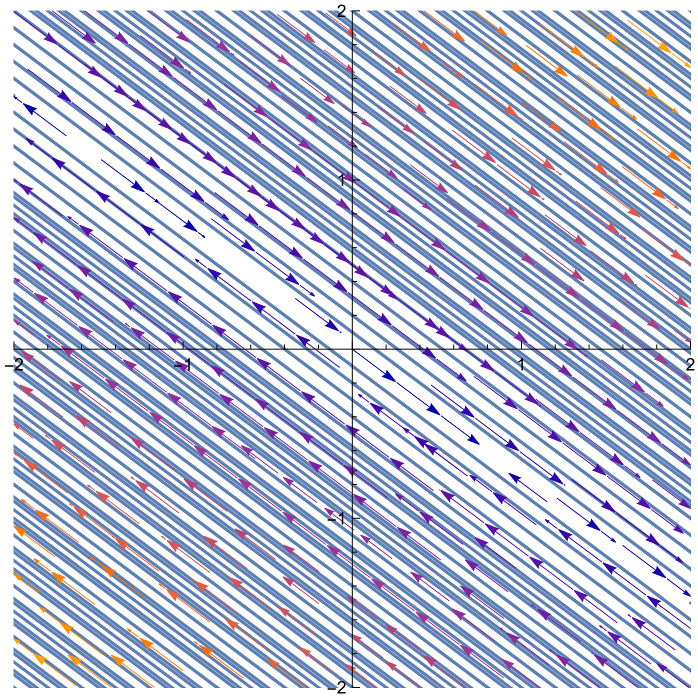
In[42]:=
```
p2 = Show[
  Table[
   ParametricPlot[
    Evaluate[{x[t], y[t]} /. sol[initialCond[[i, 1]], initialCond[[i, 2
        ]]]], {t, -10, 10}, PlotRange → {{minx, maxx}, {miny, maxy}}],
   {i, 1, Length[initialCond]}],
  ListPlot[{{0, 0}}, PlotStyle → {PointSize[0.03], Red},
   PlotMarkers → {"", Large}, PlotLegends → {"Line of fixed fixed point, sigma = 0"}]
 ]
```

Out[42]=



In[43]:=
```
Show[p2, StreamPlot[{3 * x + 4 * y, -9/4 * x - 3 * y}, {x, -2, 2}, {y, -2, 2}],
 PlotRange → {{minx, maxx}, {miny, maxy}}]
```

Out[43]=



Line of fixed fixed point, sigma = 0