Assignment 3

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Problem 6

a)

A greedy algorithm that runs in $O(nlogn)$ is to start sorting the items by price in descending order. Then divide the set into triples starting from the beginning of the set. If the number of items is not divisible by 3, the last subset will contain either 1 or 2 items.

Pseudo code for this algorithm

```
Initialize a set A with all items
Sort A by price in descending order
Initialize an empty set F to contain all triples

While there are three or more items in A

    Make a triple out the first three items in the set A
    Add the triple to F
    Delete the first three items in A

    If there are two or less items left in A
        Make an incomplete triple out the remaining items in A
        Add the incomplete triple to F
```

The sorting runs in $O(nlogn)$, and the loop runs in $O(n)$ so the resulting running time for this algorithm is $(nlogn + n) \in O(nlogn)$.

b)

We prove that this algorithm is the most optimal one by induction. Let A be the set of all items and be sorted by price in descending order, F be a compatible set of items we get for free that is returned by the greedy algorithm also sorted by price in descending order, and O be an optimal compatible set of items we get for free sorted by price in descending order as well. Additionally, let i be items of F, j be items of O, and p be the price of items. By the rules, the two first items in A must be bought. It becomes clear that the most expensive item that we can get for free is the third item in A which the greedy algorithm guarantees. Comparing the price of the third item in A, i.e., item i_1 in F, with the most expensive item in O, i.e., item j_1, we must have that $p(i_1) \geq p(j_1)$ which becomes the base case for the induction proof. The induction hypothesis lets us assume that $p(i_r) \geq p(j_r)$, and we want to prove $p(i_{r+1}) \geq p(j_{r+1})$ for all indices $1 \leq r \leq k-1$ where $k = |F|$. Performing the induction step we have that

$$p(i_r) \geq p(j_r)$$

Items in F and O are sorted by price in descending order respectively. Thus,

$$p(i_r) \geq p(i_{r+1})$$
$$p(j_r) \geq p(j_{r+1})$$

This implies

$$p(i_r) \geq p(j_{r+1})$$

The greedy algorithm guarantees optimality at every step, and if the optimal solution at some point performs differently than the greedy algorithm it is not certain that it will offer the same guarantee at each step. We must therefore have that

$$p(i_r) \geq p(i_{r+1}) \geq p(j_{r+1})$$

Which fulfills the induction step.