PHYS 265 Lab Report 2

April 10, 2025 ■ Erika Falco

Introduction

When operating one of the deepest mine shafts on Earth, it is critical that we have a method of measuring the shaft's depth. This report investigates the feasibility of using the fall time of a test mass dropped into the shaft to determine its depth, outlining the physical complications of calculating said mass's trajectory due to gravitational, drag, and Coriolis forces. It explores the implications of shafts of increased depth, extrapolating to the case of a shaft that tunnels completely through the Earth, ultimately investigating the influence of planetary density on a mass's trajectory.

I. Fall time

To measure the depth of the mine shaft, it has been proposed that we drop a test mass into the shaft and measure the time it takes to reach the bottom of the shaft. The desired shaft depth is 4 km, necessitating the calculation of the fall time corresponding with that depth. Two key factors dictate the expected fall time of the test mass: gravitational acceleration and drag. Defining the distance below the Earth's surface as y, with y < 0 as the mass descends, the trajectory of our test mass can be modeled by Equation 1,

$$\frac{d^2y}{dt^2} = -g + \alpha\left|\frac{dy}{dt}\right|^{\gamma} \quad \textbf{(1)}$$

with gravitational acceleration g, drag coefficient α, and speed dependence γ. When solving this differential equation, it was assumed that the test mass started at rest at y = 0 km at t = 0 seconds. "Fall time" is the time at which y = -4 km, the estimated depth of the mine shaft.

Equ. 1 was initially solved numerically assuming constant gravitational acceleration g = 9.81 m/s^2 and negligible air resistance ($\alpha = 0$), which yielded a fall time of 28.6 seconds. This is consistent with the analytic solution of the kinematic equation $t = \sqrt{2\Delta y/g}$, which proposes t = 28.6 seconds.

However, gravitational acceleration is not constant for all depths and is better modeled as a function of r, distance from Earth's center, as $g(r) = g_o(r/R_{\oplus})$, assuming a spherical Earth of uniform density with radius $R_{\oplus}$ = 6378.1 km. Solving Eq. 1 with $g = g(r)$ yielded a fall time 0.0015 seconds greater than when using a constant gravitational acceleration, with the fall time still rounding to 28.6 seconds. This discrepancy is due to gravitational acceleration decreasing in magnitude as the object falls, corresponding with a less rapid increase in velocity and a subsequently longer fall time. It is largely negligible in this case, since 4 km is shallow relative to the Earth's radius.

To further improve our model, we must now consider the drag force experienced by the test mass due to air resistance. The drag coefficient α = 0.004 was calibrated under the assumption that the test mass would reach a terminal velocity of -50 m/s, and $\gamma = 2$ was used to calculate drag. Equ. 1 was solved with $g = g(r)$, $\alpha = 0.004$, and $\gamma = 2$, yielding a fall time of 84.3 seconds. Fall time increased by 194.8% when drag was taken into consideration because drag and gravitational forces oppose one another as the object accelerates, causing it to reach a constant terminal velocity of -50 m/s, as shown in Figure 1.

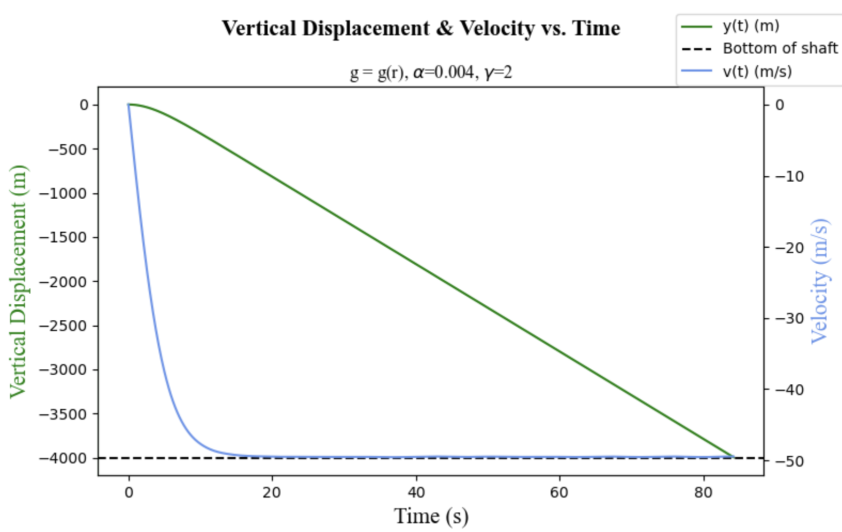

Figure 1: Test mass vertical displacement and velocity as functions of time, considering air resistance and position dependent gravity.

II. Feasibility of Depth Measurement Approach

Measuring the fall time of a test mass is not a feasible method for calculating the depth of this particular shaft due to width constraints. In addition to gravitational acceleration and air resistance, the trajectory of the test mass is influenced by the Coriolis force, resulting from the Earth's rotation. Defining x as the mass's transverse position relative to the center of the shaft (where x = 0 at the center and x > 0 indicates displacement East), the x and y components of the Coriolis force at the equator are modeled by $F_{cx} = +2mv_y\Omega$ and $F_{cy} = -2mv_x\Omega$ respectively, where m = 1 kg is the mass of the test mass and $\Omega = 7.272 * 10^{-5}$ rad/s is Earth's angular speed at the equator. Accounting for these additional forces yields an updated system of differential equations expressed by Equations 2 and 3:

$$\frac{d^2x}{dt^2} = 2\frac{dy}{dt}\Omega \quad (2) \qquad \frac{d^2y}{dt^2} = -g + \alpha|\frac{dy}{dt}|^{\gamma} - 2\frac{dx}{dt}\Omega \quad (3)$$

This system was solved with initial conditions x = 0 and y = 0, assuming that the object started from rest. In addition to fall time, the time elapsed before the mass hits the side of the 5 m wide shaft (x = 2.5 m) was calculated, referred to as "side impact time." Equations 2 and 3 were solved with $g = g(r)$ and neglecting drag ($\alpha = 0$), yielding a side impact time of 21.9 seconds and a fall time of 28.6 seconds. When x = -2.5 m, the object has only fallen 2,353.9 m, 59% of the total shaft depth. This indicates that the mass will hit the side of the shaft before it reaches the bottom, disrupting its trajectory and therefore making fall time an infeasible method for determining shaft depth.

Equations 6 and 7 were then numerically solved accounting for air resistance with $\alpha = 0.004$ and $\gamma = 2$, yielding a fall time of 84.3 seconds and a side impact time of 29.7 seconds. When x = -2.5 m, the object has only fallen 1,296.6 m, about 32% of the total shaft depth. The mass's trajectory is illustrated with and without drag in Figure 2. Accounting for drag exacerbates the discrepancy between side impact time and fall time, indicating that our shaft would need to be significantly wider for the mass to reach the bottom before it hits the side. In this model, the mass will have drifted -23.5 m in the transverse direction when it hits the bottom of the shaft, indicating that the shaft would need to be widened by 21 m on each side to make this method feasible. This method of calculating shaft depth is clearly impractical given the current dimensions.

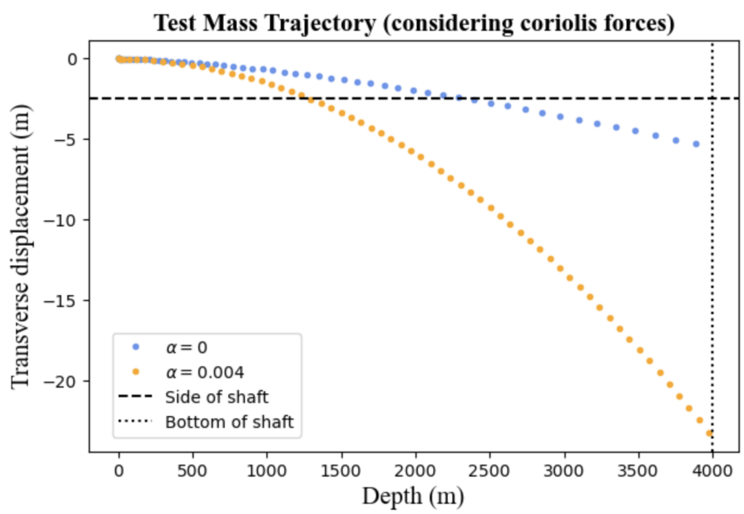

Figure 2: Test mass trajectory under the influence of Coriolis forces. Transverse displacement is relative to the center of the well with x>0 indicating displacement east of center.

III. Planetary Crossing Time

As our future work aims to dig deeper, it will become essential that we also consider the inhomogeneity of Earth's mass distribution, since density is a key factor in determining an object's fall time. A planet's density can be modeled as a function of r, the distance from its center, by Equation 4, with different models of density distribution corresponding with different exponents n with normalizing coefficients ρ_n described by Equation 5, where M and R are the planet's mass and radius, respectively.

$$\rho(r) = \rho_n(1 - (r^2/R^2))^n \quad (4) \qquad \rho_n = \frac{M}{4\pi}\left(\int_0^R r^2(1 - (r^2/R^2))^n\, dr\right)^{-1} \quad (5)$$

The gravitational acceleration $g'(r)$ experienced by an object dropped into a shaft tunneling through a planet depends on the mass beneath it, requiring a summation of density over volume: $g'(r) = (4\pi G/r^2)\int_0^r r^2\rho(r)\, dr$. Equation 1 was solved under the previously described initial conditions for a hypothetical shaft tunneling between the Earth's poles (from y = 0 to y = $2R_\oplus$), a case that eliminates Coriolis forces. Assuming g = g'(r) and neglecting air resistance ($\alpha = 0$), the crossing time required to traverse this shaft was calculated for several values of n, each describing a different density distribution.

For n = 0, a homogenous mass distribution, a crossing time of 2,373.1 seconds was calculated. For n = 9, the crossing time was 1,726.7 seconds. This 27.2% decrease in crossing time speaks to the importance of considering density distribution when anticipating an object's trajectory as it passes through the Earth. The mass's expected trajectory, under the stated assumptions, is depicted for n = 0 and n = 9 in Figure 3.

The crossing time through a given planet of uniform density is equivalent to half the period T of a mass orbiting the planet's radius, as given by $T = 2\pi\sqrt{R^3/GM}$. T = 5069.4 seconds was calculated for the Earth, approximately twice its crossing time for n = 0. Using $M = \rho(4\pi R^3/3)$ for a uniform sphere, it can be shown that $T = \sqrt{3\pi/G\rho}$ for a planet of density ρ. It follows that for two planets of uniform density, the ratio of their orbital times, and therefore of their crossing times, can be expressed by $t_1/t_2 = \sqrt{\rho_2/\rho_1}$. This relationship is demonstrated by comparing the crossing times of the Earth and Moon, assuming homogeneous mass distribution (n = 0). Equation 1 was solved for the Moon, neglecting air resistance and Coriolis forces, with g = g'(r), taking R = 1,738.1 km and M = $7.35 * 10^{22}$ kg. A crossing time of 9685.3 seconds was calculated for a shaft tunneling through the Moon (y = -1738.1 km). The ratio of the Earth to Moon crossing times is approximately 0.7798. Using Equation 4, assuming n = 0, the Earth's density was calculated to be 3,341.8 kg/m^3 and the Moon's density was calculated to be 5494.9 kg/m^3. The inverse square of their density ratio was calculated to be 0.7798, confirming the derived relationship between density and crossing time.

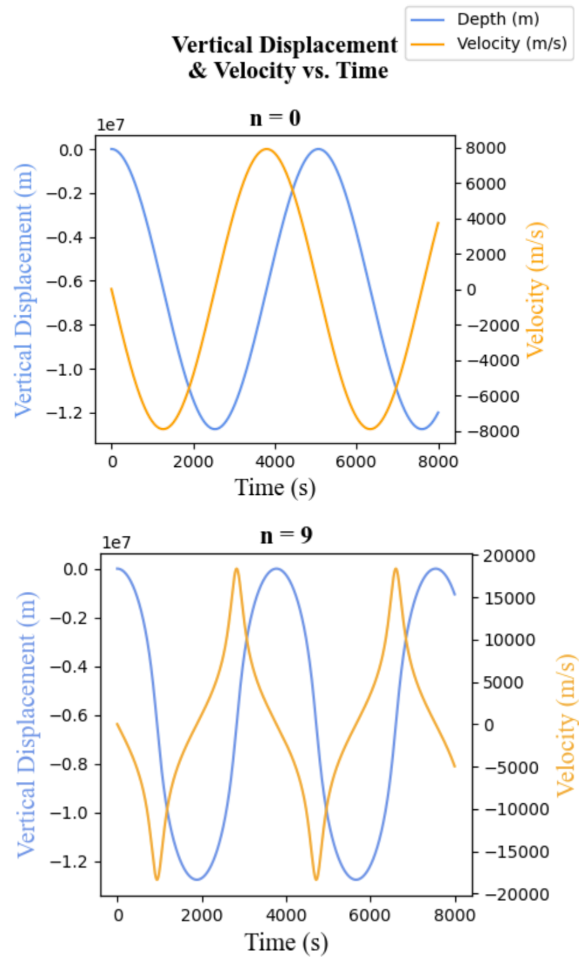

Figure 3: Vertical displacement and velocity of a mass dropped into a shaft tunneling between the poles of the earth for density distributions modeled by Equation 4, using gravitational acceleration g'(r) and ignoring effects of air resistance and Coriolis forces.

Discussion and Future Work

After analyzing the fall time of a test mass under the influence of gravitational, drag, and Coriolis forces, it has been determined that using fall time as a method of calculating shaft depth is not possible given the current shaft dimensions. Mathematical models have been proposed to model an object's motion as it traverses a deeper shaft located at the Earth's poles, providing useful insights for future projects aiming to dig deeper into the Earth. A number of approximations were made in the above calculations that limit the accuracy of these findings. The Earth was approximated as spherical, oversimplifying our models of density and mass distribution used in calculations of fall time. Additionally, α was approximated to be 0.004 based on a relatively arbitrary calibration based on the assumption that the mass's terminal velocity will be 50 m/s. Realistically, drag force depends on several factors, such as the test mass's geometry, that were not taken into account. The drag coefficient will also not be a constant due to variations in air pressure that were ignored in these calculations. Finally, the models discussed are designed specifically for mine shafts located at the Earth's equator and poles. Such precision in location is unrealistic, and thus the forces acting on the test mass will likely be more complex than modeled here. Future work will produce more robust mathematical models of the drag, gravitational, and Coriolis forces acting on the test mass by addressing these oversimplifications. Future work will also focus on developing alternative methods to measuring shaft depth, as the proposed method has been deemed unfeasible.