§9 Non-uniform variates

9.1 Inversion of the cumulative probability function

Inverse transform method / Inverse distribution function method

Recall: let X r.v.,

$F_X(x) = P(X \leq x)$ its distribution function

$F_X^{-1}(u) = \inf\{x : F_X(x) \geq u\}$ for $u \in [0,1]$ the generalized inverse of F_X.

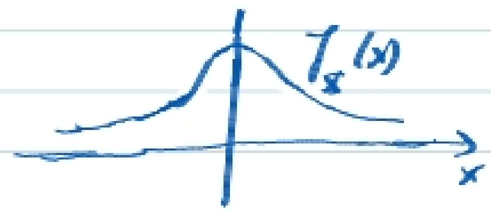
Then

(a) $u \leq F_X(x) \Leftrightarrow F_X^{-1}(u) \leq x$

(b) $U \sim R(0,1) \Rightarrow F_X^{-1}(U) =^d X$

(c) $x \mapsto F_X(x)$ cont $\Rightarrow F_X(X) \sim R(0,1)$

Ex (Cauchy distribution)



$f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2}$

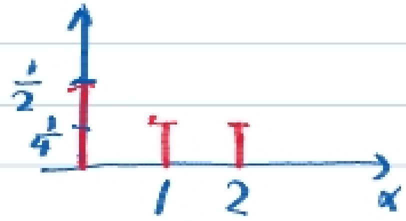
$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \frac{1}{\pi} \tan^{-1}(x) + 1/2$

$F_X^{-1}(u) = \tan \pi(u - 1/2)$, $u \in [0,1]$

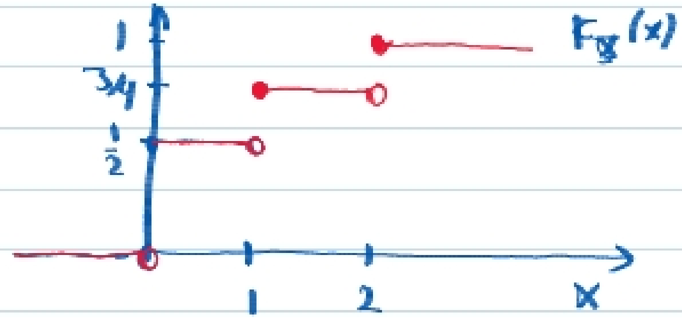
$U \sim R(0,1) \Rightarrow \tan \pi (U - 1/2) \sim$ Cauchy distributed

Ex (Discrete distribution)

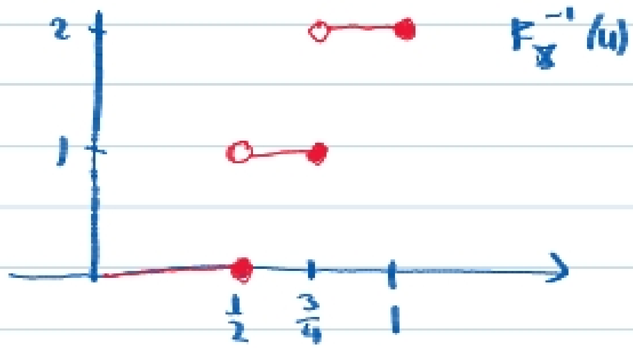
$$X = \begin{cases} 0, & \text{prob } 1/2 \\ 1, & \text{prob } 1/4 \\ 2, & \text{prob } 1/4 \end{cases}$$



$$F_X(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ 1/2, & 0 \le x < 1 \\ 3/4, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$



$$F_X^{-1}(u) = \inf\{x : F(x) \ge u\} = \begin{cases} 0, & 0 < u \le 1/2 \\ 1, & 1/2 < u \le 3/4 \\ 2, & 3/4 < u \le 1 \end{cases}$$



$$U \sim R(0,1) \Rightarrow \begin{cases} 0, & 0 < U \le 1/2 \\ 1, & 1/2 < U \le 3/4 \\ 2, & 3/4 < U \le 1 \end{cases} \stackrel{d}{=} X$$