Compulsory project 1 - Babha scattering

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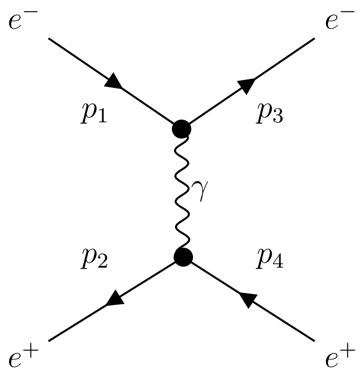
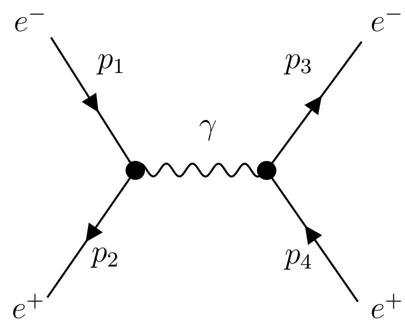
Problem 1

The two lowest-order Feynman diagrams for $e^-e^+ \to e^-e^+$ scattering are

$-i\mathcal{M}_1 =$



$-i\mathcal{M}_2 =$



According to the QED Feynman rules in Thomson p. 124, the matrix elements of diagrams 1 and 2 are, respectively

$$-i\mathcal{M}_1 = \left[\overline{u}(p_3)\left(ie\gamma^\mu\right)u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\left[\overline{v}(p_2)\left(-ie\gamma^\nu\right)v(p_4)\right],$$

$$-i\mathcal{M}_2 = \left[\overline{v}(p_2)\left(-ie\gamma^\mu\right)u(p_1)\right]\frac{-ig_{\mu\nu}}{q'^2}\left[\overline{u}(p_3)\left(-ie\gamma^\nu\right)v(p_4)\right].$$

Or, after simplifying,

$$\mathcal{M}_1 = \frac{e^2}{q_1^2}\left[\overline{u}(p_3)\gamma^\mu u(p_1)\right]\left[\overline{v}(p_2)\gamma_\mu v(p_4)\right],$$

$$\mathcal{M}_2 = -\frac{e^2}{q_2^2}\left[\overline{v}(p_2)\gamma^\mu u(p_1)\right]\left[\overline{u}(p_3)\gamma_\mu v(p_4)\right].$$

The four-momenta of the virtual photons are $q_1 \equiv p_1 - p_3$ and $q_2 \equiv p_1 + p_2$.

Since the differential cross section will later be plotted at a center-of-momentum energy of $\sqrt{s} =$ 14 GeV, the electron/positron mass $m_e \approx 0.511$ MeV, which is roughly $28,000$ times smaller, is utterly negligible in comparison. Therefore I will in the following neglect the mass of the electron/positron.

Problem 2

The spin-averaged matrix element is

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2$$

where $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ is the total matrix element to leading order. Hence, in terms of $\mathcal{M}_1$ and $\mathcal{M}_2$,

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{\text{spin}}|\mathcal{M}_1 + \mathcal{M}_2|^2$$

$$= \frac{1}{4}\sum_{\text{spin}}(\mathcal{M}_1 + \mathcal{M}_2)^\dagger(\mathcal{M}_1 + \mathcal{M}_2)$$

$$= \frac{1}{4}\left(\sum_{\text{spin}}|\mathcal{M}_1|^2 + \sum_{\text{spin}}|\mathcal{M}_2|^2 + \sum_{\text{spin}}\mathcal{M}_1^\dagger\mathcal{M}_2 + \sum_{\text{spin}}\mathcal{M}_1\mathcal{M}_2^\dagger\right).$$

I will now go through each of these terms in turn, expressing them in terms of traces of matrices.

1)

$\mathcal{M}_1$ and $\mathcal{M}_1^\dagger$ are given by

$$\mathcal{M}_1 = \frac{e^2}{q^2}\left[\overline{u}(p_3)\gamma^\mu u(p_1)\right]\left[\overline{v}(p_2)\gamma_\mu v(p_4)\right],$$

$$\mathcal{M}_1^\dagger = \frac{e^2}{q^2}\left[\overline{u}(p_1)\gamma^\mu u(p_3)\right]\left[\overline{v}(p_4)\gamma_\mu v(p_2)\right],$$

where I've used the identity $[\overline{\psi}\Gamma\phi]^\dagger = \overline{\phi}\Gamma\psi$ which is valid for all standard model vertices. Then the matrix element squared is

$$|\mathcal{M}_1|^2 = \mathcal{M}_1^\dagger\mathcal{M}_1$$

$$= \frac{e^2}{q^2}\left[\overline{u}(p_1)\gamma^\mu u(p_3)\right]\left[\overline{v}(p_4)\gamma_\mu v(p_2)\right] \times \frac{e^2}{q^2}\left[\overline{u}(p_3)\gamma^\nu u(p_1)\right]\left[\overline{v}(p_2)\gamma_\nu v(p_4)\right]$$

$$= \frac{e^4}{q^4}\left[\overline{u}(p_1)\gamma^\mu u(p_3)\overline{u}(p_3)\gamma^\nu u(p_1)\right]\left[\overline{v}(p_2)\gamma_\nu v(p_4)\overline{v}(p_4)\gamma_\mu v(p_2)\right].$$

Summing over all spin configurations, $\sum_{\text{spin}} = \sum_{s,s'}\sum_{r,r'}$ where s, s' labels the spins of the initial- and final-state electron and r, r' labels the spin of the initial-and final-state positron, we have

$$|\mathcal{M}_1|^2 = \frac{e^4}{q^4}\sum_{s,s'}\overline{u}^s(p_1)\gamma^\mu u^{s'}(p_3)\overline{u}^{s'}(p_3)\gamma^\nu u^s(p_1)$$
$$\times\sum_{r,r'}\overline{v}^r(p_2)\gamma_\nu v^{r'}(p_4)\overline{v}^{r'}(p_4)\gamma_\mu v^r(p_2).$$

The first factor is

$$\sum_{s,s\prime} \overline{u}^s(p_1)\gamma^\mu u^{s'}(p_3)\overline{u}^{s'}(p_3)\gamma^\nu u^s(p_1)$$

$$= \sum_{s,s\prime} \overline{u}_i^s(p_1)\gamma_{ij}^\mu u_j^{s'}(p_3)\overline{u}_m^{s'}(p_3)\gamma_{mn}^\nu u_n^s(p_1)$$

$$= \left[\sum_s u_n^s(p_1)\overline{u}_i^s(p_1)\right]\left[\sum_{s'} u_j^{s'}(p_3)\overline{u}_m^{s'}(p_3)\right]\gamma_{ij}^\mu\gamma_{mn}^\nu$$

$$= \left(\not{p}_1\right)_{ni}\left(\not{p}_3\right)_{jm}\gamma_{ij}^\mu\gamma_{mn}^\nu$$

$$= \left(\not{p}_1\right)_{ni}\gamma_{ij}^\mu\left(\not{p}_3\right)_{jm}\gamma_{mn}^\nu$$

$$= \mathrm{Tr}\left[\left(\not{p}_1\right)\gamma^\mu\left(\not{p}_3\right)\gamma^\nu\right].$$

The second one is

$$\sum_{r,r'} \overline{v}^r(p_2)\gamma_\nu v^{r'}(p_4)\overline{v}^{r'}(p_4)\gamma_\mu v^r(p_2)$$

$$= \sum_{r,r'} \overline{v}_i^r(p_2)(\gamma_\nu)_{ij}v_j^{r'}(p_4)\overline{v}_m^{r'}(p_4)(\gamma_\mu)_{mn}v_n^r(p_2)$$

$$= \left[\sum_r v_n^r(p_2)\overline{v}_i^r(p_2)\right]\left[\sum_{r'} v_j^{r'}(p_4)\overline{v}_m^{r'}(p_4)\right](\gamma_\nu)_{ij}(\gamma_\mu)_{mn}$$

$$= \left(\not{p}_2\right)_{ni}\left(\not{p}_4\right)_{jm}(\gamma_\nu)_{ij}(\gamma_\mu)_{mn}$$

$$= \left(\not{p}_2\right)_{ni}(\gamma_\nu)_{ij}\left(\not{p}_4\right)_{jm}(\gamma_\mu)_{mn}$$

$$= \mathrm{Tr}\left[\left(\not{p}_2\right)\gamma_\nu\left(\not{p}_4\right)\gamma_\mu\right].$$

where m is the electron/positron mass. Thus, we have

$$\sum_{\text{spin}} |\mathcal{M}_1|^2 = \frac{e^4}{q_1^4}\mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_2\gamma_\nu\not{p}_4\gamma_\mu\right].$$

2)

Proceeding as before, we have

$$\mathcal{M}_2 = -\frac{e^2}{q_2^2}\left[\overline{v}(p_2)\gamma^\mu u(p_1)\right]\left[\overline{u}(p_3)\gamma_\mu v(p_4)\right],$$

$$\mathcal{M}_2^\dagger = -\frac{e^2}{q_2^2}\left[\overline{u}(p_1)\gamma^\nu v(p_2)\right]\left[\overline{v}(p_4)\gamma_\nu u(p_3)\right],$$

$$|\mathcal{M}_2|^2 = \frac{e^4}{q_2^4}\left[\overline{v}(p_2)\gamma^\mu u(p_1)\overline{u}(p_1)\gamma^\nu v(p_2)\right]\left[\overline{u}(p_3)\gamma_\mu v(p_4)\overline{v}(p_4)\gamma_\nu u(p_3)\right],$$

$$\begin{aligned}\sum_{\text{spin}}|\mathcal{M}_2|^2 = \frac{e^4}{q_2^4}&\sum_{s,r}\overline{v}^r(p_2)\gamma^\mu u^s(p_1)\overline{u}^s(p_1)\gamma^\nu v^r(p_2)\\ &\times\sum_{s',r'}\overline{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\overline{v}^{r'}(p_4)\gamma_\nu u^{s'}(p_3).\end{aligned}$$

The first factor is

$$\sum_{s,r}\overline{v}^r(p_2)\gamma^\mu u^s(p_1)\overline{u}^s(p_1)\gamma^\nu v^r(p_2)$$

$$=\sum_{s,r}\overline{v}_i^r(p_2)\gamma^\mu_{ij}u_j^s(p_1)\overline{u}_m^s(p_1)\gamma^\nu_{mn}v_n^r(p_2)$$

$$=\left[\sum_r v_n^r(p_2)\overline{v}_i^r(p_2)\right]\left[\sum_s u_j^s(p_1)\overline{u}_m^s(p_1)\right]\gamma^\mu_{ij}\gamma^\nu_{mn}$$

$$=\left(\not{p}_2\right)_{ni}\left(\not{p}_1\right)_{jm}\gamma^\mu_{ij}\gamma^\nu_{mn}$$

$$=\left(\not{p}_2\right)_{ni}\gamma^\mu_{ij}\left(\not{p}_1\right)_{jm}\gamma^\nu_{mn}$$

$$=\mathrm{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right].$$

The second one is

$$\sum_{s',r'}\overline{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\overline{v}^{r'}(p_4)\gamma_\nu u^{s'}(p_3)$$

$$=\sum_{s',r'}\overline{u}_i^{s'}(p_3)(\gamma_\mu)_{ij}v_j^{r'}(p_4)\overline{v}_m^{r'}(p_4)(\gamma_\nu)_{mn}u_n^{s'}(p_3)$$

$$=\left[\sum_{s'}u_n^{s'}(p_3)\overline{u}_i^{s'}(p_3)\right]\left[\sum_{r'}v_j^{r'}(p_4)\overline{v}_m^{r'}(p_4)\right](\gamma_\mu)_{ij}(\gamma_\nu)_{mn}$$

$$=\left(\not{p}_3\right)_{ni}\left(\not{p}_4\right)_{jm}(\gamma_\mu)_{ij}(\gamma_\nu)_{mn}$$

$$=\left(\not{p}_3\right)_{ni}(\gamma_\mu)_{ij}\left(\not{p}_4\right)_{jm}(\gamma_\nu)_{mn}$$

$$=\mathrm{Tr}\left[\left(\not{p}_3\right)\gamma_\mu\left(\not{p}_4\right)\gamma_\nu\right].$$

Thus, we have

$$\sum_{\text{spin}} |\mathcal{M}_2|^2 = \frac{e^4}{q_2^4} \text{Tr}\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right] \text{Tr}\left[\not{p}_3 \gamma_\mu \not{p}_4 \gamma_\nu\right].$$

1) & 2)

What remains are the cross terms. Since $\mathcal{M}_1^\dagger \mathcal{M}_2$ and $\mathcal{M}_1 \mathcal{M}_2^\dagger$ are complex conjugates, we can calculate one and get the other for free. Again, proceeding as before,

$$\mathcal{M}_1^\dagger = \frac{e^2}{q_1^2} \left[\overline{u}(p_1)\gamma^\mu u(p_3)\right] \left[\overline{v}(p_4)\gamma_\mu v(p_2)\right],$$

$$\mathcal{M}_2 = -\frac{e^2}{q_2^2} \left[\overline{v}(p_2)\gamma^\nu u(p_1)\right] \left[\overline{u}(p_3)\gamma_\nu v(p_4)\right],$$

$$\mathcal{M}_1^\dagger \mathcal{M}_2 = -\frac{e^4}{q_1^2 q_2^2} \left[\overline{u}(p_1)\gamma^\mu u(p_3)\overline{u}(p_3)\gamma_\nu v(p_4)\overline{v}(p_4)\gamma_\mu v(p_2)\overline{v}(p_2)\gamma^\nu u(p_1)\right],$$

$$\sum_{\text{spin}} \mathcal{M}_1^\dagger \mathcal{M}_2 = -\frac{e^4}{q_1^2 q_2^2} \sum_{s,s',r,r'} \overline{u}^s(p_1)\gamma^\mu u^{s'}(p_3)\overline{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)\overline{v}^{r'}(p_4)\gamma_\mu v^r(p_2)\overline{v}^r(p_2)\gamma^\nu u^s(p_1).$$

Now, ignoring the spin-independent pre-factor,

$$\sum_{s,s',r,r'} \left[\overline{u}^s(p_1)\gamma^\mu u^{s'}(p_3)\overline{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)\overline{v}^{r'}(p_4)\gamma_\mu v^r(p_2)\overline{v}^r(p_2)\gamma^\nu u^s(p_1)\right]$$

$$= \sum_{s,s',r,r'} \left[\overline{u}_i^s(p_1)\gamma_{ij}^\mu u_j^{s'}(p_3)\overline{u}_k^{s'}(p_3)(\gamma_\nu)_{kl} v_l^{r'}(p_4)\overline{v}_m^{r'}(p_4)(\gamma_\mu)_{mn} v_n^r(p_2)\overline{v}_a^r(p_2)\gamma_{ab}^\nu u_b^s(p_1)\right]$$

$$= \sum_{s,s',r,r'} \left[u_b^s(p_1)\overline{u}_i^s(p_1)u_j^{s'}(p_3)\overline{u}_k^{s'}(p_3)v_l^{r'}(p_4)\overline{v}_m^{r'}(p_4)v_n^r(p_2)\overline{v}_a^r(p_2)\gamma_{ij}^\mu(\gamma_\nu)_{kl}(\gamma_\mu)_{mn}\gamma_{ab}^\nu\right]$$

$$\begin{aligned}
&= \sum_s u_b^s(p_1)\overline{u}_i^s(p_1) \\
&\times \sum_{s'} u_j^{s'}(p_3)\overline{u}_k^{s'}(p_3) \\
&\times \sum_{r'} v_l^{r'}(p_4)\overline{v}_m^{r'}(p_4) \\
&\times \sum_r v_n^r(p_2)\overline{v}_a^r(p_2) \\
&\times \gamma_{ij}^\mu(\gamma_\nu)_{kl}(\gamma_\mu)_{mn}\gamma_{ab}^\nu
\end{aligned}$$

$$= \left(\not{p}_1\right)_{bi} \left(\not{p}_3\right)_{jk} \left(\not{p}_4\right)_{lm} \left(\not{p}_2\right)_{na} \gamma_{ij}^\mu(\gamma_\nu)_{kl}(\gamma_\mu)_{mn}\gamma_{ab}^\nu$$

$$= \left(\not{p}_1\right)_{bi} \gamma_{ij}^\mu \left(\not{p}_3\right)_{jk} (\gamma_\nu)_{kl} \left(\not{p}_4\right)_{lm} (\gamma_\mu)_{mn} \left(\not{p}_2\right)_{na} \gamma_{ab}^\nu$$

$$= \mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma_\nu\not{p}_4\gamma_\mu\not{p}_2\gamma^\nu\right].$$

Thus, the cross terms can be expressed as

$$\sum_{\text{spin}}\mathcal{M}_1^\dagger\mathcal{M}_2 + \text{c.c} = -\frac{e^4}{q_1^2 q_2^2}\mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma_\nu\not{p}_4\gamma_\mu\not{p}_2\gamma^\nu\right] + \text{c.c.}$$

Now it remains to calculate the following quantities:

$$\begin{aligned}
T_1 =& \mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_2\gamma_\nu\not{p}_4\gamma_\mu\right] \\
T_2 =& \mathrm{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu\right] \\
T_{12} =& \mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma_\nu\not{p}_4\gamma_\mu\not{p}_2\gamma^\nu\right]
\end{aligned}$$

T_1 :

The first factor of T_1 is

$$\mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu\right]$$

$$= \mathrm{Tr}\left[p_{1\rho}\gamma^\rho\gamma^\mu p_{3\sigma}\gamma^\sigma\gamma^\nu\right]$$

$$= p_{1\rho}p_{3\sigma}\mathrm{Tr}\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right]$$

$$= 4p_{1\rho}p_{3\sigma}\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\right)$$

$$= 4\left(p_1^\mu p_3^\nu - g^{\mu\nu}p_1\cdot p_3 + p_1^\nu p_3^\mu\right).$$

The second factor of T_1 is obtained by replacing $p_1 \to p_2,\ p_3 \to p_4,\ \mu \leftrightarrow \nu$.

$$\mathrm{Tr}\left[\not{p}_2\gamma_\nu\not{p}_4\gamma_\mu\right] = 4(p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2\cdot p_4 + p_{2\mu}p_{4\nu}).$$

Multiplying the factors together we have

$$T_1 = 16\left(p_1^\mu p_3^\nu - g^{\mu\nu}p_1\cdot p_3 + p_1^\nu p_3^\mu\right)\left(p_{2\nu}p_{4\mu} - g_{\mu\nu}p_2\cdot p_4 + p_{2\mu}p_{4\nu}\right)$$

$$\begin{aligned}
= 16\times&(p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_2)(p_3\cdot p_4) \\
&- (p_1\cdot p_3)(p_2\cdot p_4) + 4(p_1\cdot p_3)(p_2\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) \\
&+ (p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)
\end{aligned}$$

$$= 16[2(p_1\cdot p_4)(p_2\cdot p_3) + 2(p_1\cdot p_2)(p_3\cdot p_4)]$$

$$= 32[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4)].$$

T_2:

The first factor of T_2 is immediately obtained from the first factor of T_1 by substituting $p_1 \to p_2,\ p_3 \to p_1$:

$$\mathrm{Tr}\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right] = 4\left(p_2^\mu p_1^\nu - g^{\mu\nu} p_2 \cdot p_1 + p_2^\nu p_1^\mu\right)$$

The second factor of T_2 is also obtained from the first factor of T_1 by substituting $p_1 \to p_3,\ p_3 \to p_4$:

$$\mathrm{Tr}\left[\not{p}_3 \gamma_\mu \not{p}_4 \gamma_\nu\right] = 4(p_{3\mu} p_{4\nu} - g_{\mu\nu} p_3 \cdot p_4 + p_{3\nu} p_{4\mu})$$

Multiplying them together we have

$$T_2 = 16\left(p_2^\mu p_1^\nu - g^{\mu\nu} p_2 \cdot p_1 + p_2^\nu p_1^\mu\right)\left(p_{3\mu} p_{4\nu} - g_{\mu\nu} p_3 \cdot p_4 + p_{3\nu} p_{4\mu}\right)$$

$$\begin{aligned} = 16\times & (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) \\ & - (p_1 \cdot p_2)(p_3 \cdot p_4) + 4(p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) \\ & + (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \end{aligned}$$

$$= 16[2(p_1 \cdot p_4)(p_2 \cdot p_3) + 2(p_1 \cdot p_3)(p_2 \cdot p_4)]$$

$$= 32[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4)].$$

T_{12}:

$$T_{12} = \mathrm{Tr}\left[\not{p}_1 \gamma^\mu \not{p}_3 \gamma_\nu \not{p}_4 \gamma_\mu \not{p}_2 \gamma^\nu\right]$$

$$= \mathrm{Tr}\left[p_{1\rho} \gamma^\rho \gamma^\mu p_{3\sigma} \gamma^\sigma \gamma_\nu p_{4\alpha} \gamma^\alpha \gamma_\mu p_{2\beta} \gamma^\beta \gamma^\nu\right]$$

$$= p_{1\rho} p_{3\sigma} p_{4\alpha} p_{2\beta} \mathrm{Tr}\left[\gamma^\rho \gamma^\mu \gamma^\sigma \gamma_\nu \gamma^\alpha \gamma_\mu \gamma^\beta \gamma^\nu\right].$$

To make progress we can use the contraction identity $\gamma_\nu \gamma^\alpha \gamma_\mu \gamma^\beta \gamma^\nu = -2\gamma^\beta \gamma_\mu \gamma^\alpha$.

$$T_{12} = -2 p_{1\rho} p_{3\sigma} p_{4\alpha} p_{2\beta} \mathrm{Tr}\left[\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\beta \gamma_\mu \gamma^\alpha\right].$$

Now using $\gamma^\mu \gamma^\sigma \gamma^\beta \gamma_\mu = 4g^{\sigma\beta}$ we get

$$T_{12} = -8 g^{\sigma\beta} p_{1\rho} p_{3\sigma} p_{4\alpha} p_{2\beta} \mathrm{Tr}\left[\gamma^\rho \gamma^\alpha\right].$$

Finally, since $\mathrm{Tr}\left[\gamma^\rho \gamma^\alpha\right] = 4g^{\rho\alpha}$ we arrive at

$$T_{12} = -32 g^{\sigma\beta} g^{\rho\alpha} p_{1\rho} p_{3\sigma} p_{4\alpha} p_{2\beta}$$

$$= -32(p_1 \cdot p_4)(p_2 \cdot p_3).$$

Since this quantity turned out to be real, we get both of the cross terms by multiplying by 2. Thus, the leading order contributions to the spin-averaged matrix element are

$$\sum_{\text{spin}} |\mathcal{M}_1|^2 = \frac{e^4}{q_1^4} \times 32[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4)],$$

$$\sum_{\text{spin}} |\mathcal{M}_2|^2 = \frac{e^4}{q_2^4} \times 32[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4)],$$

$$\sum_{\text{spin}} \mathcal{M}_1^\dagger \mathcal{M}_2 + \text{c.c} = 2 \times \left(-\frac{e^4}{q_1^2 q_2^2}\right) \times [-32(p_1 \cdot p_4)(p_2 \cdot p_3)].$$

Since $q_1^2 \approx -2(p_1 \cdot p_3)$ and $q_2^2 \approx 2(p_1 \cdot p_2)$ when neglecting the electron/positron mass we can write

$$\sum_{\text{spin}} |\mathcal{M}_1|^2 = 8e^4 \frac{(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4)}{(p_1 \cdot p_3)^2},$$

$$\sum_{\text{spin}} |\mathcal{M}_2|^2 = 8e^4 \frac{(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4)}{(p_1 \cdot p_2)^2},$$

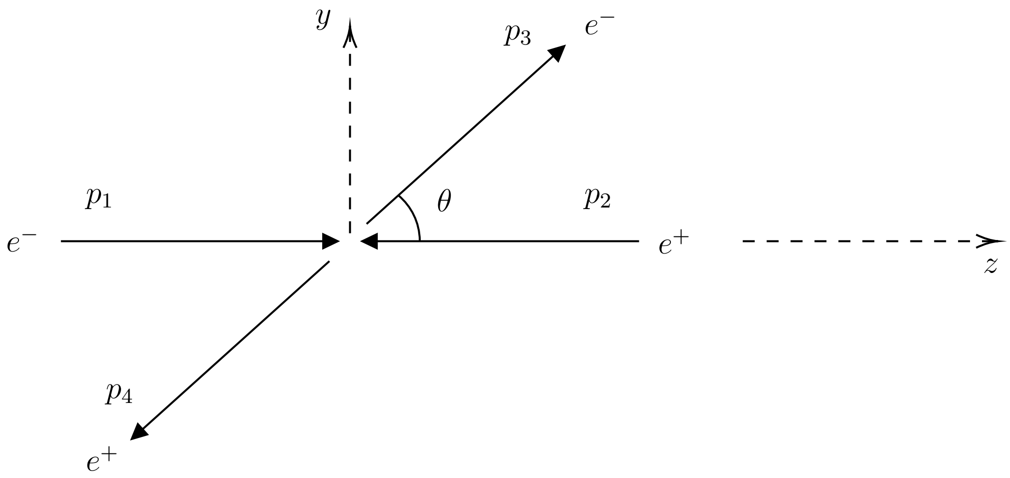
$$\sum_{\text{spin}} \mathcal{M}_1^\dagger \mathcal{M}_2 + \text{c.c} = -16e^4 \frac{(p_1 \cdot p_4)(p_2 \cdot p_3)}{(p_1 \cdot p_3)(p_1 \cdot p_2)}.$$

Putting it all together we have

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4}\left[\sum_{\text{spin}} |\mathcal{M}_1|^2 + \sum_{\text{spin}} |\mathcal{M}_2|^2 + \sum_{\text{spin}} \mathcal{M}_1^\dagger \mathcal{M}_2 + \sum_{\text{spin}} \mathcal{M}_1 \mathcal{M}_2^\dagger\right]$$

$$\begin{aligned}
=&2e^4 \frac{(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4)}{(p_1 \cdot p_3)^2} \\
+&2e^4 \frac{(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4)}{(p_1 \cdot p_2)^2} \\
-&4e^4 \frac{(p_1 \cdot p_4)(p_2 \cdot p_3)}{(p_1 \cdot p_3)(p_1 \cdot p_2)}.
\end{aligned}$$

Problem 3



In the center-of-momentum frame, the four-momenta are

$$
\begin{aligned}
p_1 &= (E, 0, 0, E) \\
p_2 &= (E, 0, 0, -E) \\
p_3 &= (E, 0, E\sin\theta, E\cos\theta) \\
p_4 &= (E, 0, -E\sin\theta, -E\cos\theta),
\end{aligned}
$$

where E is the initial-state energy of the electron and of the positron respectively, which is the same as the final-state energies of the electron and the positron due to energy conservation, and θ is the scattering angle of the final-state electron.

The dot products between the four-momenta are

$$p_1 \cdot p_2 = 2E^2,\ p_1 \cdot p_3 = E^2(1 - \cos\theta),\ p_1 \cdot p_4 = E^2(1 + \cos\theta),$$

$$p_2 \cdot p_3 = E^2(1 + \cos\theta),\ p_2 \cdot p_4 = E^2(1 - \cos\theta),$$

$$p_3 \cdot p_4 = 2E^2.$$

Then the spin-averaged matrix element in terms of E and $\cos\theta$ is

$$
\begin{aligned}
\left\langle |\mathcal{M}|^2 \right\rangle =& 2e^4 \frac{E^4(1+\cos\theta)^2 + 4E^4}{E^4(1-\cos\theta)^2} \\
&+ 2e^4 \frac{E^4(1+\cos\theta)^2 + E^4(1-\cos\theta)^2}{4E^4} \\
&- 4e^4 \frac{E^4(1+\cos\theta)^2}{2E^4(1-\cos\theta)}
\end{aligned}
$$

$$= e^4 \left[2\frac{(1+\cos\theta)^2 + 4}{(1-\cos\theta)^2} + \left(1 + \cos^2\theta\right) - 2\frac{(1+\cos\theta)^2}{1-\cos\theta} \right].$$

In terms of the fine-structure constant $\alpha = e^2/4\pi$,

$$\langle|\mathcal{M}|^2\rangle = 16\pi^2\alpha^2\left[2\frac{(1+\cos\theta)^2+4}{(1-\cos\theta)^2} + \left(1+\cos^2\theta\right) - 2\frac{(1+\cos\theta)^2}{1-\cos\theta}\right].$$

The differential cross section in the center-of-momentum frame is given by Eq. (3.50) in Thomson,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle|\mathcal{M}|^2\rangle.$$

Thus, for $e^-e^+ \to e^-e^+$ scattering to leading order and in the limit where the electron/positron mass can be neglected,

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^-e^+\to e^-e^+} = \frac{\alpha^2}{4s}\left[2\frac{(1+\cos\theta)^2+4}{(1-\cos\theta)^2} + \left(1+\cos^2\theta\right) - 2\frac{(1+\cos\theta)^2}{1-\cos\theta}\right].$$

Note that the first term in the square brackets stems from scattering process in diagram 1 alone, the second term stems from the annihilation process in diagram 2 alone, and the third term stems from the cross terms and represents the interference between diagrams 1 and 2.

The differential cross section in the center-of-momentum frame for the $e^-e^+ \to \mu^-\mu^+$ process to leading order and in the limit where the muon mass can be neglected is given by Eq. (6.23) in Thomson,

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^-e^+\to \mu^-\mu^+} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right).$$

Note that this is the same expression as the contribution from diagram 2 to $e^-e^+ \to e^-e^+$ scattering. The muon is identical to the electron in all but mass, which has been taken to be neglectable. Hence, diagram 2 and the single leading order Feynman diagram for the $e^-e^+ \to \mu^-\mu^+$ process are identical in the massless limit and yields identical contributions to the cross section.

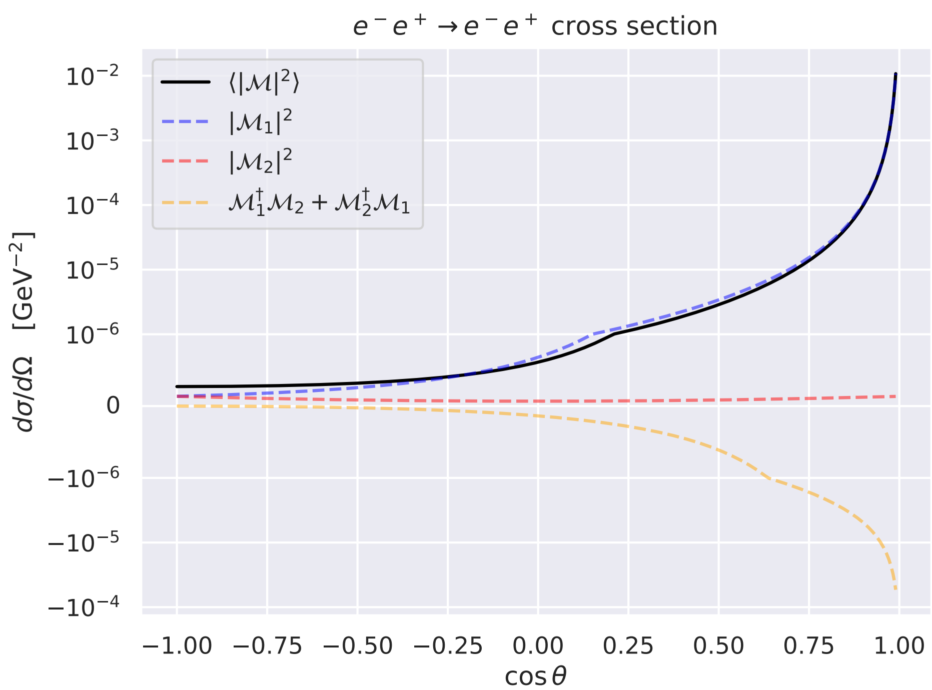


Figure 1

The differential cross section for $e^-e^+ \to e^-e^+$ scattering to leading order at $\sqrt{s} = 14$ GeV is shown in Figure 1 (black), along with the contributions from diagram 1 (blue), diagram 2 (red) and the interference contribution (yellow). The black line is the sum of the blue, red and yellow lines. Unlike the $e^-e^+ \to \mu^-\mu^+$ process, where the azimuthal distribution of produced muons is symmetric in the forward and backward hemispheres, there are orders of magnitudes more electrons scattered out in the forward hemisphere than in the backward hemisphere. The reason being that the contribution from diagram 1 dominates over the contribution from diagram 2 across all azimuthal angles θ. The differential cross section largely follows the contribution from diagram 1 alone, with some small deviations mostly stemming from the destructive interference contribution between diagrams 1 and 2. Although both of these contributions blow up as $\theta \to 0$, the interference contribution has a simple pole while the contribution from diagram 1 has a second order pole, and thus dominates for $\theta \approx 0$.

Problem 4

Assuming a detector acceptance of $-a < \cos\theta < a$, the total cross section is

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega}$$

$$= \int d\varphi \int_{-a}^{a} d\cos\theta \frac{d\sigma}{d\Omega}$$

$$= \frac{2\pi\alpha^2}{4s} \int_{-a}^{a} dx\, f(x),$$

where

$$f(x) \equiv 2\frac{(1+x)^2+4}{(1-x)^2} + \left(1+x^2\right) - 2\frac{(1+x)^2}{1-x}.$$

The result is (after using a program for analytical integration)

$$\int_{-a}^{a} dx\, f(x) = \frac{2a\left(a^4+26a^3-75\right)}{3\left(a^2-1\right)} + 16\ln\left(\frac{a-1}{-a-1}\right).$$

The total cross section for the $e^-e^+ \to e^-e^+$ in terms of the detector acceptance is then

$$\sigma_{e^-e^+\to e^-e^+} = \frac{\pi\alpha^2}{s}\left[\frac{a\left(a^4+26a^3-75\right)}{3\left(a^2-1\right)} + 8\ln\left(\frac{a-1}{-a-1}\right)\right].$$

Similarly, for the $e^-e^+ \to \mu^-\mu^+$ process,

$$\sigma_{e^-e^+\to\mu^-\mu^+} = \frac{\pi\alpha^2}{s}\frac{a\left(a^2+3\right)}{3}.$$

Both cross sections has the same x^{-2} dependence on the center-of-momentum energy for $\sqrt{s} \gg m_\mu$, just with different constants of proportionality.

Figure 2 shows the total cross sections for both processes with a detector acceptance of $a = 0.98$ for center-of-momentum energies of $\sqrt{s} \sim 10$ GeV.

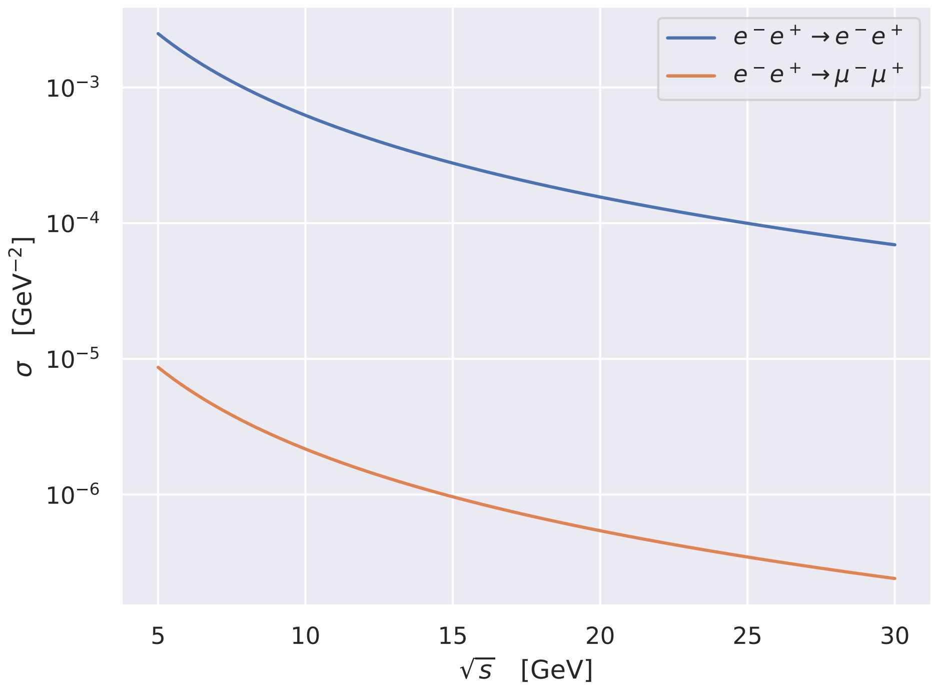


Figure 2

The total number of expected events is given by

$$N = \sigma\varepsilon \int dt \mathcal{L}(t)$$

where ε is the detector efficiency and $\int dt\mathcal{L}(t)$ is the integrated luminosity. At $\sqrt{s} = 14$ GeV, the total cross sections are

$$\sigma_{e^-e^+\to e^-e^+} = 1.2 \cdot 10^5 \text{ pb},$$

$$\sigma_{e^-e^+\to \mu^-\mu^+} = 430\text{pb}.$$

So for an efficiency of $\varepsilon = 50\%$ and an integrated luminosity of $\int dt\mathcal{L}(t) = 10$ pb^{-1}, the number of events for each process are

$$N_{e^-e^+\to e^-e^+} = 6.2 \cdot 10^5,$$

$$N_{e^-e^+\to \mu^-\mu^+} = 2.2 \cdot 10^2.$$

So there are roughly 3000 times more $e^-e^+ \to e^-e^+$ events than $e^-e^+ \to \mu^-\mu^+$ events.