FYS4170 Problem set '0'

Problem 1: Subtleties of Dirac Delta Functions

$$f_1(t) = g(t)\delta(t-a)$$

$$f_2(t) = g(a)\delta(t-a)$$

$$T_{f_i}[\phi] = \int_{-\infty}^{\infty} dt f_i(t)\phi(t)$$

$$\phi(t) \to 0 \text{ as } t \to \pm\infty$$

a)

The first functional is

$$T_{f_1}[\phi] = \int_{-\infty}^{\infty} dt f_1(t)\phi(t) = \int_{-\infty}^{\infty} g(t)\delta(t-a)\phi(t) = g(a)\phi(a)$$

by one of the defining properties of the delta function. The second functional is

$$T_{f_2}[\phi] = \int_{-\infty}^{\infty} dt f_2(t)\phi(t) = \int_{-\infty}^{\infty} dt g(a)\delta(t-a)\phi(t) = g(a)\int_{-\infty}^{\infty} dt\delta(t-a)\phi(t) = g(a)\phi(a).$$

Thus $f_1(t) = f_2(t)$ in the distributional sense.

b)

$$T_{f'(t)}[\phi] = \int_{-\infty}^{\infty} dt f'(t)\phi(t) = [f(t)\phi(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dt f(t)\phi'(t)$$

$f(t)\phi(t)$ vanishes at $\pm\infty$ since $f(t)$ is proportional to a delta function, so we're left with

$$T_{f'(t)}[\phi] = -\int_{-\infty}^{\infty} dt f(t)\phi'(t) = -T_{f(t)}\left[\frac{\partial\phi}{\partial t}\right].$$

c)

$$h_1(t) = g(t)\delta'(t-a)$$

$$h_2(t) = g(a)\delta'(t-a)$$

$$T_{h_1}[\phi] = \int_{-\infty}^{\infty} dt h_1(t)\phi(t) = \int_{-\infty}^{\infty} dt\delta'(t-a)g(t)\phi(t)$$

$$= [\delta(t-a)g(t)\phi(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dt\delta(t-a)\frac{\partial}{\partial t}[g(t)\phi(t)] = -\left(\frac{\partial}{\partial t}[g(t)\phi(t)]\right)_{t=a}$$

$$= -\phi(a)\left(\frac{\partial g(t)}{\partial t}\right)_{t=a} - g(a)\left(\frac{\partial\phi(t)}{\partial t}\right)_{t=a}$$

$$T_{h_2}[\phi] = \int_{-\infty}^{\infty} dt h_2(t)\phi(t) = \int_{-\infty}^{\infty} dt\delta'(t-a)g(a)\phi(t)$$

$$= [\delta(t-a)g(a)\phi(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dt\delta(t-a)\frac{\partial}{\partial t}[g(a)\phi(t)] = -\left(\frac{\partial}{\partial t}[g(a)\phi(t)]\right)_{t=a}$$

$$= -g(a)\left(\frac{\partial\phi(t)}{\partial t}\right)_{t=a} \neq T_{h_1}[\phi]$$

$h_1(t)$ and $h_2(t)$ are **not equal** in the distributional sense.

d)

$$f_1'(t) = g'(t)\delta(t-a) + g(t)\delta'(t-a)$$

$$f_2'(t) = g(a)\delta'(t-a)$$

$$T_{f_1'}[\phi] = \int_{-\infty}^{\infty} dt f_1'(t)\phi(t)$$

$$= \int_{-\infty}^{\infty} dt g'(t)\delta(t-a)\phi(t) + \int_{-\infty}^{\infty} dt g(t)\delta'(t-a)\phi(t) = \phi(a)\left(\frac{\partial g(t)}{\partial t}\right)_{t=a} + T_{h_1}[\phi]$$

$$= \phi(a)\left(\frac{\partial g(t)}{\partial t}\right)_{t=a} - \phi(a)\left(\frac{\partial g(t)}{\partial t}\right)_{t=a} - g(a)\left(\frac{\partial \phi(t)}{\partial t}\right)_{t=a} = -g(a)\left(\frac{\partial \phi(t)}{\partial t}\right)_{t=a}$$

$$T_{f_2'}[\phi] = \int_{-\infty}^{\infty} dt f_2'(t)\phi(t)$$

$$= \int_{-\infty}^{\infty} dt g(a)\delta'(t-a)\phi(t) = T_{h_2}[\phi]$$

$$= -g(a)\left(\frac{\partial \phi(t)}{\partial t}\right)_{t=a} = T_{f_1'}[\phi]$$

$f_1'(t)$ and $f_2'(t)$ **are equal** in the distributional sense.

e)

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

$$T_\theta[\phi] = \int_{-\infty}^{\infty} dt \theta(t)\phi(t)$$

Using $T_{\theta'}[\phi] = -T_\theta[\partial_t \phi]$ we have that

$$T_{\theta'}[\phi] = -\int_{-\infty}^{\infty} dt \theta(t) \frac{\partial \phi(t)}{\partial t} = -\int_0^{\infty} dt \frac{\partial \phi(t)}{\partial t} = -[\phi(t)]_0^{\infty} = \phi(0).$$

Meanwhile, defining $f(t) = \delta(t)$ we have that

$$T_f[\phi] = \int_{-\infty}^{\infty} dt \delta(t) \phi(t) = \phi(0).$$

So $\theta'(t) = \delta(t)$ in the distributional sense.

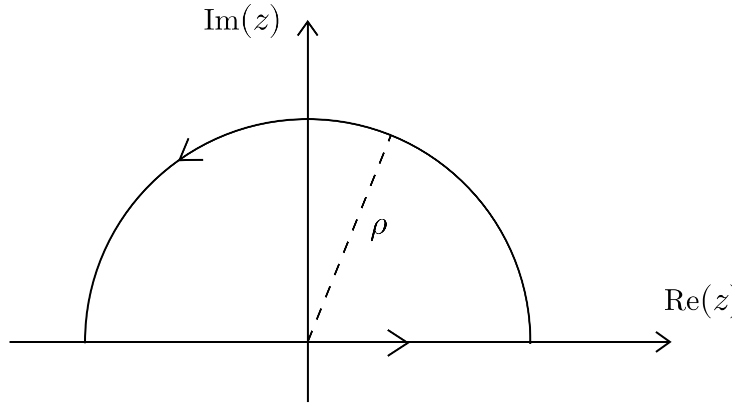
Problem 2: Contour Integration

$$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2} dx = \frac{\pi}{a} e^{-|k|a}, \ a > 0 \tag{1}$$

To verify Eq. (1) for positive k we can evaluate the integral

$$\tilde{I} = \oint_C \frac{e^{ikz}}{z^2 + a^2} dz$$

where C is the closed boundary of the semicircle in the figure below.



The integral $\widetilde{I}$ can be written as sum of contributions, from $-\rho$ to ρ along the real axis and from the semicircle,

$$\widetilde{I} = \int_{-\rho}^{\rho} \frac{e^{ikx}}{x^2+a^2} dx + \int_{\text{semicircle}} \frac{e^{ikz}}{z^2+a^2} dz.$$

Taking the limit $\rho \to \infty$ makes the contribution along the real axis approach I. Since

$$\left| \int_{\text{semicircle}} \frac{e^{ikz}}{z^2+a^2} dz \right| \leq \int_{\text{semicircle}} \left| \frac{e^{ikz}}{z^2+a^2} \right| |dz| = \int_{\text{semicircle}} \frac{\left| e^{ikz} \right|}{\left| z^2+a^2 \right|} |dz|,$$

where

$$\left| e^{ikz} \right| = \left| e^{ik(x+iy)} \right| = \left| e^{ikx} e^{-ky} \right| = e^{-ky} \left| e^{ikx} \right| = e^{-ky} \leq 1,$$

$$\left| z^2 + a^2 \right| = \left| \rho^2 e^{2i\theta} + a^2 \right| = \rho^2 \left| e^{2i\theta} + (a/\rho)^2 \right| \propto \rho^2,$$

$$|dz| = \left| i\rho e^{i\theta} d\theta \right| = \rho d\theta \propto \rho,$$

the absolute value of the integrand is less than or proportional to $\rho/\rho^2 = 1/\rho \to 0$, which makes the integral vanish in the $\rho \to \infty$ limit.

The residue theorem states that

$$\oint_C f(z)dz = 2\pi i \sum_i \mathrm{Res}(f, z_i)$$

where the sum goes over all singularities of $f(z)$ inside C and C is oriented anti-clockwise. Since

$$\frac{e^{ikz}}{z^2+a^2} = \frac{e^{ikz}}{(z+ia)(z-ia)} = \frac{e^{ikz} \,/\, (z+ia)}{z-ia},$$

the only singularity of the integrand inside C is a simple pole at $z = ia$ with residue

$$\frac{e^{ik\cdot ia}}{ia+ia} = \frac{e^{-ka}}{2ia},$$

and the integral is thus

$$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx = 2\pi i \cdot \frac{e^{-ka}}{2ia} = \frac{\pi}{a} e^{-ka}.$$

For $k < 0$ we can integrate along the semicircle in the lower complex plane in the clockwise direction. Only the contribution along the real axis survives by a similar argument and the only singularity of the integrand is a simple pole at $z = -ia$ with residue

$$\frac{e^{ik\cdot(-ia)}}{-ia-ia} = -\frac{e^{ika}}{2ia}.$$

For $k < 0$ we thus have

$$I = -2\pi i \cdot \left(-\frac{e^{ika}}{2ia}\right) = \frac{\pi}{a} e^{ka}.$$

Both results can be combined by writing $-k = -|k|$ when k is positive and $k = -|k|$ when k is negative.

$$I = \frac{\pi}{a} e^{-|k|a}$$

Problem 3: Green's Functions

A Green's function $G(x, y)$ of a linear differential operator $\mathcal{D}$ (only dependent on x) is one that satisfies

$$\mathcal{D}G(x, y) = \delta(x - y).$$

Green's functions can be used to solve differential equations on the form

$$\mathcal{D}u(x) = f(x)$$

where $f(x)$ is given and $u(x)$ is unknown. This is shown by the following:

$$\mathcal{D}u(x) = f(x) = f(x) \cdot 1 = f(x) \cdot \int dy \delta(x - y) = \int dy f(y) \delta(x - y)$$

$$= \int dy f(y) \mathcal{D}G(x, y) = \mathcal{D} \int dy f(y) G(x, y)$$

where the operator $\mathcal{D}$ can be taken outside the integral because it is linear and only dependent on x. We can thus identify

$$u(x) = \int dy f(y) G(x, y).$$

$u(x)$ is only uniquely determined when boundary conditions are provided, and since $f(y)$ is given, this puts some conditions on the Green's function.

In the problem we have the partial differential equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)u(\mathbf{x}, t) = f(\mathbf{x}, t)$$

where the Green's function $G(\mathbf{x}, t, \mathbf{x}', t')$ satisfies

$$G(\mathbf{x}, t=0, \mathbf{x}', t') = 0,\ \ \partial_t G(\mathbf{x}, t=0, \mathbf{x}', t') = 0,\ \ \lim_{|\mathbf{x}|\to\infty} G(\mathbf{x}, t, \mathbf{x}', t') = 0,$$

in order to match some unspecified conditions on $u(\mathbf{x}, t)$.

a)

$u(\mathbf{x}, t)$ is given by

$$u(\mathbf{x}, t) = \int d^3x' \int dt' G(\mathbf{x}, t, \mathbf{x}', t') f(\mathbf{x}', t'),$$

where

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)u(\mathbf{x}, t) = \int d^3x' \int dt' f(\mathbf{x}', t')\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)G(\mathbf{x}, t, \mathbf{x}', t') = f(\mathbf{x}, t).$$

We thus require

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)G(\mathbf{x}, t, \mathbf{x}', t') = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t').$$

Since the right side only depends on the relative differences $\mathbf{x}-\mathbf{x}'$ and $t-t'$ and the differential operator can be written as derivatives with respect to $\mathbf{x}-\mathbf{x}'$ and $t-t'$ (with $\mathbf{x}'$, t' held constant), the Green's function can only depend on the relative differences, i.e $G = G(\mathbf{x}-\mathbf{x}', t-t')$. For the moment, $\mathbf{x}'$ and t' are set to zero (and recovered at the end to not clutter the notation). We then have

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)G(\mathbf{x}, t) = \delta(\mathbf{x})\delta(t),$$

or in four-vector notation,

$$\partial_\mu \partial^\mu G(x^\nu) = \delta(x^\nu)$$