Problem set 1

Problem 1

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)^2 - \frac{1}{2}m_1^2\varphi_1^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - \frac{1}{2}m_2^2\varphi_2^2 - \lambda\varphi_1^2\varphi_2^2$$

$$S = \int d^4x \left\{ \frac{1}{2}(\partial_\mu \varphi_1)^2 - \frac{1}{2}m_1^2\varphi_1^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - \frac{1}{2}m_2^2\varphi_2^2 - \lambda\varphi_1^2\varphi_2^2 \right\}$$

$$\delta S = \int d^4x \left\{ \frac{1}{2}\delta\Big[(\partial_\mu \varphi_1)^2\Big] - \frac{1}{2}m_1^2\delta\Big[\varphi_1^2\Big] + \frac{1}{2}\delta\Big[(\partial_\mu \varphi_2)^2\Big] - \frac{1}{2}m_1^2\delta\Big[\varphi_2^2\Big] - \lambda\delta\Big(\varphi_1^2\varphi_2^2\Big) \right\}$$

$$= \int d^4x \left\{ \frac{1}{2}\frac{\partial\Big[(\partial_\mu\varphi_1)^2\Big]}{\partial(\partial_\nu\varphi_1)}\delta(\partial_\nu\varphi_1) - \frac{1}{2}m_1^2\frac{\partial\Big(\varphi_1^2\Big)}{\partial\varphi_1}\delta\varphi_1 + \frac{1}{2}\frac{\partial\Big[(\partial_\mu\varphi_2)^2\Big]}{\partial(\partial_\nu\varphi_2)}\delta(\partial_\nu\varphi_2) - \frac{1}{2}m_2^2\frac{\partial\Big(\varphi_2^2\Big)}{\partial\varphi_2}\delta\varphi_2 - \lambda\frac{\partial\Big(\varphi_1^2\varphi_2^2\Big)}{\partial\varphi_1}\delta\varphi_1 - \lambda\frac{\partial\Big(\varphi_1^2\varphi_2^2\Big)}{\partial\varphi_2}\delta\varphi_2 \right\}$$

The easy derivatives are $\frac{\partial\left(\varphi_i^2\right)}{\partial\varphi_i} = 2\varphi_i$ and $\frac{\partial\left(\varphi_i^2\varphi_j^2\right)}{\partial\varphi_i} = 2\varphi_i\varphi_j^2$. Since $(\partial_\mu\varphi_i)^2 = \partial_\mu\varphi_i\partial^\mu\varphi_i = g^{\mu\sigma}\partial_\mu\varphi_i\partial_\sigma\varphi_i$ we have

$$\frac{\partial\Big[(\partial_\mu\varphi_i)^2\Big]}{\partial(\partial_\nu\varphi_i)} = \frac{\partial}{\partial(\partial_\nu\varphi_i)}g^{\mu\sigma}\partial_\mu\varphi_i\partial_\sigma\varphi_i$$

$$= g^{\mu\sigma}\left[\partial_\sigma\varphi_i\frac{\partial}{\partial(\partial_\nu\varphi_i)}\partial_\mu\varphi_i + \partial_\mu\varphi_i\frac{\partial}{\partial(\partial_\nu\varphi_i)}\partial_\sigma\varphi_i\right] = g^{\mu\sigma}\Big(\partial_\sigma\varphi_i\delta_\mu^\nu + \partial_\mu\varphi_i\delta_\sigma^\nu\Big)$$

$$= g^{\mu\sigma}\partial_\sigma\varphi_i\delta_\mu^\nu + g^{\mu\sigma}\partial_\mu\varphi_i\delta_\sigma^\nu = \partial^\mu\varphi_i\delta_\mu^\nu + \partial^\sigma\varphi_i\delta_\sigma^\nu$$

$$= \partial^\nu\varphi_i + \partial^\nu\varphi_i = 2\partial^\nu\varphi_i$$

Thus

$$\delta S = \int d^4x \Big\{ \partial^\nu\varphi_1\delta(\partial_\nu\varphi_1) - m_1^2\varphi_1\delta\varphi_1 + \partial^\nu\varphi_2\delta(\partial_\nu\varphi_2) - m_2^2\varphi_2\delta\varphi_2 - 2\lambda\varphi_1\varphi_2^2\delta\varphi_1 - 2\lambda\varphi_1^2\varphi_2\delta\varphi_2 \Big\}$$

$$= \int d^4x \Big\{ \partial^\nu\varphi_1\delta(\partial_\nu\varphi_1) + \partial^\nu\varphi_2\delta(\partial_\nu\varphi_2) - \Big(m_1^2\varphi_1 + 2\lambda\varphi_1\varphi_2^2\Big)\delta\varphi_1 - \Big(m_2^2\varphi_2 + 2\lambda\varphi_1^2\varphi_2\Big)\delta\varphi_2 \Big\}$$

Using the fact that $\delta(\partial_\nu\varphi_i) = \partial_\nu(\delta\varphi_i)$ and the product rule backwards $\partial^\nu\varphi_i\partial_\nu(\delta\varphi_i) = \partial_\nu\Big(\partial^\nu\varphi_i\delta\varphi_i\Big) - \partial_\nu\Big(\partial^\nu\varphi_i\Big)\delta\varphi_i$ we get

$$\delta S = \int d^4x \Big\{ \partial_\nu\Big(\partial^\nu\varphi_1\delta\varphi_1\Big) - \partial_\nu\Big(\partial^\nu\varphi_1\Big)\delta\varphi_1 + \partial_\nu\Big(\partial^\nu\varphi_2\delta\varphi_2\Big) - \partial_\nu\Big(\partial^\nu\varphi_2\Big)\delta\varphi_2 - \Big(m_1^2\varphi_1 + 2\lambda\varphi_1\varphi_2^2\Big)\delta\varphi_1 - \Big(m_2^2\varphi_2 + 2\lambda\varphi_1^2\varphi_2\Big)\delta\varphi_2 \Big\}$$

$$= \int d^4x \Big\{ \partial_\nu\Big(\partial^\nu\varphi_1\delta\varphi_1\Big) + \partial_\nu\Big(\partial^\nu\varphi_2\delta\varphi_2\Big) - \Big[m_1^2\varphi_1 + 2\lambda\varphi_1\varphi_2^2 + \partial_\nu\Big(\partial^\nu\varphi_1\Big)\Big]\delta\varphi_1 - \Big[m_2^2\varphi_2 + 2\lambda\varphi_1^2\varphi_2 + \partial_\nu\Big(\partial^\nu\varphi_2\Big)\Big]\delta\varphi_2 \Big\}$$

Now let's examine the terms $\int d^4x \partial_\nu \left(\partial^\nu \varphi_i \delta\varphi_i\right)$. Since $d^4x = dx_0 \ldots dx_3$ we can write this term as

$$\int dx_{123} \int dx_0 \partial_0 \left(\partial^0 \varphi_i \delta\varphi_i\right) + \int dx_{023} \int dx_1 \partial_1 \left(\partial^1 \varphi_i \delta\varphi_i\right) + \int dx_{013} \int dx_2 \partial_2 \left(\partial^2 \varphi_i \delta\varphi_i\right) + \int dx_{012} \int dx_3 \partial_3 \left(\partial^3 \varphi_i \delta\varphi_i\right)$$

$$= \int dx_{123} \left[\partial^0 \varphi_i \delta\varphi_i\right]_{\Sigma_0} + \int dx_{023} \left[\partial^1 \varphi_i \delta\varphi_i\right]_{\Sigma_1} + \int dx_{013} \left[\partial^2 \varphi_i \delta\varphi_i\right]_{\Sigma_2} + \int dx_{012} \left[\partial^3 \varphi_i \delta\varphi_i\right]_{\Sigma_3}$$

where the indices Σ_j signifies that the expression is evaluated at some boundary. Since $\delta\varphi_i$ is assumed to be zero at the boundary all of these terms vanish. Thus

$$\delta S = -\int d^4x \left\{ \left[m_1^2 \varphi_1 + 2\lambda \varphi_1 \varphi_2^2 + \partial_\nu \left(\partial^\nu \varphi_1\right)\right] \delta\varphi_1 + \left[m_2^2 \varphi_2 + 2\lambda \varphi_1^2 \varphi_2 + \partial_\nu \left(\partial^\nu \varphi_2\right)\right] \delta\varphi_2 \right\}$$

If we set $\delta S = 0$ we can use the fact that $\delta\varphi_1$ and $\delta\varphi_2$ are not in general linearly dependent to claim that the "coefficients" in front of them are zero respectively.

$$m_1^2 \varphi_1 + 2\lambda \varphi_1 \varphi_2^2 + \partial_\nu \left(\partial^\nu \varphi_1\right) = 0$$

$$m_2^2 \varphi_2 + 2\lambda \varphi_1^2 \varphi_2 + \partial_\nu \left(\partial^\nu \varphi_2\right) = 0$$

We can rewrite these equations similar to the way the Klein-Gordon equation is conventionally written.

$$\left(\partial_\nu \partial^\nu + m_1^2 + 2\lambda \varphi_2^2\right) \varphi_1 = 0$$

$$\left(\partial_\nu \partial^\nu + m_2^2 + 2\lambda \varphi_1^2\right) \varphi_2 = 0$$

The Klein-Gordon equations for two non-interacting (i.e independent) fields I expect to be

$$\left(\partial_\nu \partial^\nu + m_1^2\right) \varphi_1 = 0$$

$$\left(\partial_\nu \partial^\nu + m_2^2\right) \varphi_2 = 0$$

We see that they only differ by two extra terms $2\lambda \varphi_1 \varphi_2^2,\ 2\lambda \varphi_2 \varphi_1^2$ which accounts for the interaction between the fields, while the other terms account for how φ_1 and φ_2 evolve by themselves.