FYS4170 Problem set 1

Problem 5: Klein-Gordon Equation

a)

The Klein-Gordon equation is

$$\left(\partial^\mu \partial_\mu + m^2\right)\phi(x) = 0.$$

With a little bit of hindsight, we can use the plane wave ansatz
$\phi(x) = \exp\{-i(Et - \mathbf{p}\cdot\mathbf{x})\} = \exp\left\{-ik^\mu x_\mu\right\}$ where

$$k^\mu \equiv (E, \mathbf{p}),\ \ x^\mu \equiv (t, \mathbf{x}).$$

The derivatives of the plane wave are

$$\partial_\mu \phi(x) = \partial_\mu \exp\left\{-ik_\nu x^\nu\right\} = -ik_\mu \exp\left\{-ik_\nu x^\nu\right\} = -ik_\mu \phi(x),$$

$$\partial^\mu \partial_\mu \phi(x) = -ik_\mu \partial^\mu \exp\left\{-ik^\nu x_\nu\right\} = -k_\mu k^\mu \exp\left\{-ik^\nu x_\nu\right\} = -k_\mu k^\mu \phi(x),$$

where

$$-k^\mu k_\mu = -E^2 + \mathbf{p}^2.$$

Inserting this into the Klein-Gordon equation we get

$$\left(-E^2 + \mathbf{p}^2 + m^2\right)\phi(x) = 0.$$

So the plane wave is a solution of the Klein-Gordon equation if E and $\mathbf{p}$ are related by

$$E^2 = \mathbf{p}^2 + m^2$$

which is the relativistic energy-momentum relation. However, because the energy is squared, negative energy plane waves also satisfy the Klein-Gordon equation. Negative energy states is not necessarily a problem; energy is a relative quantity and the sign can be anything depending on the choice of reference. For instance, all energy states of the electron in the hydrogen atom are negative with the conventional reference point ($E = 0$ being a free electron). The problem with the Klein-Gordon equation is that the energy states are unbounded from below; there is no ground state. For any energy state that the electron is in, there will always be another lower energy state. Since energy is conserved, one can always gain energy from the electron by kicking it further down the energy ladder. If the Klein-Gordon equation described electrons, we'd have perpetual motion machines already by the 1930s when it was derived.

b)

See note on the Klein-Gordon equation.

Problem 7: Lorentz Group

a)

A Lorentz transformation is defined as a transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$

which leaves the "length" $x^2 \equiv x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu$ invariant. To this end we require that

$$x'^\mu x'_\mu = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma \stackrel{!}{=} g_{\rho\sigma} x^\rho x^\sigma$$

which means that $\Lambda^\mu{}_\nu$ must satisfy

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}. \tag{1}$$

If we rearrange this equation slightly,

$$\Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma},$$

we notice that $g_{\mu\nu}\Lambda^{\nu}{}_{\sigma}$ are the elements of the matrix product $g\Lambda$, and thus $\Lambda^{\mu}{}_{\rho}g_{\mu\nu}\Lambda^{\nu}{}_{\sigma}$ are the elements of the matrix product $\Lambda^T g\Lambda$. So in matrix form, the requirement in Eq. (1) is

$$\Lambda^T g\Lambda = g. \tag{2}$$

b)

The set of all Lorentz transformations is denoted

$$\mathcal{L} = \left\{\Lambda \in \mathrm{GL}(4, \mathbb{R}\} \mid \Lambda^T g\Lambda = g\right\},$$

i.e the set of all 4×4 invertible real matrices with the property $\Lambda^T g\Lambda = g$.

Closed under multiplication:

Let Λ and Γ be two Lorentz transformations. First define

$$\overline{x}^{\mu} \equiv \Gamma^{\mu}{}_{\nu}x'^{\nu},$$

$$x'^{\nu} \equiv \Lambda^{\nu}{}_{\rho}x^{\rho}.$$

Then

$$\overline{x}^{\mu} = \Gamma^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\rho}x^{\rho}.$$

The argument that $\Gamma\Lambda$ is also a Lorentz transformation is as simple as this: Since Λ is a Lorentz transformation, x'^{μ} has the same length as x^{μ}. Since Γ is a Lorentz transformation, $\overline{x}^{\mu}$ has the same length as x'^{μ}, which means that it has the same length as x^{μ}. Thus $\Gamma\Lambda$ conserves the length of a four-vector. The product of two 4×4 real invertible matrices is another 4×4 real invertible matrix, so $\Gamma\Lambda$ is a Lorentz transformation.

Unity element:

Setting $\Lambda = I$ in Eq. (2) trivially leads to

$$I^T g I = I g I = g.$$

So I is a Lorentz transformation, and $I\Lambda = \Lambda I = \Lambda$ for all Λ.

Associativity:

The multiplication operation of the Lorentz group is the ordinary matrix multiplication operation, which is well known to be an associative operation.

Inverse:

First note that

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \Lambda_{\nu\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}.$$

Multiplying both sides by $g^{\gamma\rho}$ leads to

$$g^{\gamma\rho}\Lambda_{\nu\rho}\Lambda^{\nu}{}_{\sigma} = g^{\gamma\rho}g_{\rho\sigma},$$

$$\Lambda_{\nu}{}^{\gamma}\Lambda^{\nu}{}_{\sigma} = g^{\gamma\rho}g_{\rho\sigma}.$$

The right hand side is simply the matrix product of the metric tensor with itself, which is the identity matrix. Thus,

$$\Lambda_{\nu}{}^{\gamma}\Lambda^{\nu}{}_{\sigma} = \delta^{\gamma}{}_{\sigma}.$$

By definition of the inverse

$$\left(\Lambda^{-1}\right)^{\gamma}{}_{\nu}\Lambda^{\nu}{}_{\sigma} = \delta^{\gamma}{}_{\sigma},$$

we can identify

$$\left(\Lambda^{-1}\right)^{\gamma}{}_{\nu} = \Lambda_{\nu}{}^{\gamma}$$

$$= g^{\gamma\rho} \Lambda_{\nu\rho} = g^{\gamma\rho} g_{\mu\nu} \Lambda^{\mu}{}_{\rho}.$$

Insert proof that $\left(\Lambda^{-1}\right)^{\gamma}{}_{\nu}$ preserves the length of four-vectors here, which makes it a Lorentz transformation

c)

Taking the determinant of both sides of Eq. (2), we have

$$\det\left(\Lambda^T g \Lambda\right) = \det(g),$$

$$\det\left(\Lambda^T\right) \det(g) \det(\Lambda) = \det(g).$$

Since $\det(g) \neq 0$ and $\det\left(A^T\right) = \det(A)$, we have

$$\det(\Lambda)^2 = 1,$$

which means that all Lorentz transformations have a determinant of ± 1. To show $\left(\Lambda^0{}_0\right)^2 \geq 1$ we can again start from

$$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}.$$

Then set $\rho = \sigma = 0$. In the $(-1, +1, +1, +1)$ metric we get

$$g_{\mu\nu} \Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{0} = -1,$$

$$\Lambda^{\mu}{}_{0} \Lambda_{\mu 0} = -1,$$

$$-\Lambda^{0}{}_{0} \Lambda^{0}{}_{0} + \Lambda^{i}{}_{0} \Lambda^{i}{}_{0} = -1,$$

$$\left(\Lambda^{0}{}_{0}\right)^2 - \Lambda^{i}{}_{0} \Lambda^{i}{}_{0} = 1.$$

The term $\Lambda^{i}{}_{0} \Lambda^{i}{}_{0}$ is a sum over products of identical elements. Each product must be non-negative, since the identical elements have the same sign. Thus the sum is also non-negative. If subtracting something non-negative from $\left(\Lambda^0{}_0\right)^2$ gives you 1, then

$$\left(\Lambda^0{}_0\right)^2 \geq 1.$$

Problem 8: Lorentz Algebra

A Lorentz transformation Λ can be written on the form

$$\Lambda^\rho{}_\sigma = \left(\exp\left\{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right\}\right)^\rho{}_\sigma,$$

where

$$J^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)$$

are the generators of the Lorentz group.

a)

The commutator between two generators is

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = \left[i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right), i\left(x^\rho\partial^\sigma - x^\sigma\partial^\rho\right)\right]$$

$$= i^2\left[x^\mu\partial^\nu - x^\nu\partial^\mu,\ x^\rho\partial^\sigma - x^\sigma\partial^\rho\right].$$

$$= i^2\left\{\left[x^\mu\partial^\nu, x^\rho\partial^\sigma\right] - \left[x^\mu\partial^\nu, x^\sigma\partial^\rho\right] - \left[x^\nu\partial^\mu, x^\rho\partial^\sigma\right] + \left[x^\nu\partial^\mu, x^\sigma\partial^\rho\right]\right\}.$$

Each term is on the same form with just a different set of indices. So we only need the commutator

$$\left[x^\mu \partial^\nu, x^\rho \partial^\sigma\right] = x^\mu \partial^\nu \left(x^\rho \partial^\sigma\right) - x^\rho \partial^\sigma \left(x^\mu \partial^\nu\right).$$

Since

$$\partial^\mu x^\nu = \frac{\partial x^\nu}{\partial x_\mu} = g^{\nu\rho} \frac{\partial x_\rho}{\partial x_\mu} = g^{\nu\rho} \delta^\mu_\rho = g^{\mu\nu},$$

we have by the use of the product rule that

$$\left[x^\mu \partial^\nu, x^\rho \partial^\sigma\right] = g^{\nu\rho} x^\mu \partial^\sigma + x^\mu x^\rho \partial^\nu \partial^\sigma - g^{\sigma\mu} x^\rho \partial^\nu - x^\rho x^\mu \partial^\sigma \partial^\nu$$

$$= g^{\nu\rho} x^\mu \partial^\sigma - g^{\sigma\mu} x^\rho \partial^\nu + x^\mu x^\rho \left[\partial^\nu, \partial^\sigma\right],$$

and since derivatives commute, we can simplify to

$$\left[x^\mu \partial^\nu, x^\rho \partial^\sigma\right] = g^{\nu\rho} x^\mu \partial^\sigma - g^{\sigma\mu} x^\rho \partial^\nu.$$

So for completion:

$$\left[x^\mu \partial^\nu, x^\rho \partial^\sigma\right] = g^{\nu\rho} x^\mu \partial^\sigma - g^{\sigma\mu} x^\rho \partial^\nu$$

$$\left[x^\mu \partial^\nu, x^\sigma \partial^\rho\right] = g^{\nu\sigma} x^\mu \partial^\rho - g^{\rho\mu} x^\sigma \partial^\nu$$

$$\left[x^\nu \partial^\mu, x^\rho \partial^\sigma\right] = g^{\mu\rho} x^\nu \partial^\sigma - g^{\sigma\nu} x^\rho \partial^\mu$$

$$\left[x^\nu \partial^\mu, x^\sigma \partial^\rho\right] = g^{\mu\sigma} x^\nu \partial^\rho - g^{\rho\nu} x^\sigma \partial^\mu$$

Collecting the terms into the original commutator:

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i^2 \left\{\left[x^\mu \partial^\nu, x^\rho \partial^\sigma\right] - \left[x^\mu \partial^\nu, x^\sigma \partial^\rho\right] - \left[x^\nu \partial^\mu, x^\rho \partial^\sigma\right] + \left[x^\nu \partial^\mu, x^\sigma \partial^\rho\right]\right\}$$

$$= i^2\Big\{g^{\nu\rho}x^\mu\partial^\sigma - g^{\sigma\mu}x^\rho\partial^\nu - \big(g^{\nu\sigma}x^\mu\partial^\rho - g^{\rho\mu}x^\sigma\partial^\nu\big) - \big(g^{\mu\rho}x^\nu\partial^\sigma - g^{\sigma\nu}x^\rho\partial^\mu\big) + g^{\mu\sigma}x^\nu\partial^\rho - g^{\rho\nu}x^\sigma\partial^\mu\Big\}$$

$$= i^2\Big\{g^{\nu\rho}x^\mu\partial^\sigma - g^{\sigma\mu}x^\rho\partial^\nu - g^{\nu\sigma}x^\mu\partial^\rho + g^{\rho\mu}x^\sigma\partial^\nu - g^{\mu\rho}x^\nu\partial^\sigma + g^{\sigma\nu}x^\rho\partial^\mu + g^{\mu\sigma}x^\nu\partial^\rho - g^{\rho\nu}x^\sigma\partial^\mu\Big\}$$

$$= i^2\Big\{g^{\nu\rho}\big(x^\mu\partial^\sigma - x^\sigma\partial^\mu\big) - g^{\mu\rho}\big(x^\nu\partial^\sigma - x^\sigma\partial^\nu\big) - g^{\nu\sigma}\big(x^\mu\partial^\rho - x^\rho\partial^\mu\big) + g^{\mu\sigma}\big(x^\nu\partial^\rho - x^\rho\partial^\nu\big)\Big\}.$$

Using the definition $J^{\mu\nu} = i\big(x^\mu\partial^\nu - x^\nu\partial^\mu\big)$ we thus have

$$\big[J^{\mu\nu}, J^{\rho\sigma}\big] = i\big(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\big).$$