FYS4170 Problem set 2

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Problem 11: Classical Field Theory

We consider two massive non-interacting scalar fields $\phi_1, \phi_2 : x^\mu \to \mathbb{R}$ described by the Lagrangian (sum over μ and i implied)

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi_i\right)\left(\partial_\mu \phi_i\right) - \frac{1}{2}m^2\phi_i\phi_i - \lambda\phi_i^2\phi_i^2.$$

a)

Pre-factors:

The pre-factors of $1\,/\,2$ are purely for convenience when taking derivatives of the Lagrangian. λ is a measure of the self-interaction strength.

Signs:

The reference point of the energy is the vacuum energy, $E(\text{vacuum}) = 0$. The fact that the fields evolve over space and time implies that they contain positive energy relative to the vacuum. This is expressed by the term $\frac{1}{2}\left(\partial^\mu \phi_i\right)\left(\partial_\mu \phi_i\right)$ which thus has to be positive.

The following is pure speculation and most likely incorrect: The terms $-\frac{1}{2}m^2\phi_i\phi_i$ represents the energy associated with having massive fields in the universe at all. The fact that there is something rather than nothing implies that the state $\left|\text{something}\right\rangle$ has less energy than $\left|\text{nothing}\right\rangle$ (hopefully anyway, or else we might cease to exist at any moment). So this term has to be negative.

More speculation: The term $-\lambda\phi_i^2\phi_i^2$ represents the interaction of a field with itself. The sign signifies that the interaction is "attractive" in some sense relative to the non-interacting vacuum.

Units:

When using natural units $\hbar = c = 1$, energy has the same dimension as mass as we can see from the relativistic dispersion relation $E^2 = p^2 + m^2$. From $E = hf$ we can deduce that frequency also has dimension of mass in the natural units system, and thus time has the dimension of inverse mass. Since $c = 1$, space has dimension of inverse mass as well.

The Lagrangian (not the density) has dimension energy (and thus mass), and since $L = \int d^3x\mathcal{L}$ this means that the Lagrangian density has dimension

$$\text{mass} = \text{mass}^{-3}[\mathcal{L}] \to [\mathcal{L}] = \text{mass}^4.$$

From the term $-\frac{1}{2}m^2\phi_i\phi_i$ we can find the dimension of the fields:

$$\text{mass}^4 = \text{mass}^2[\phi]^2 \to [\phi] = \text{mass}$$

The factor λ is then dimensionless.

Now we consider the single-field Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_i)^2 - \frac{1}{2}m^2\phi_i^2 - \lambda\phi_i^4$$

where $(\partial_\mu\phi_i)^2$ is shorthand for $\partial^\mu\phi_i\partial_\mu\phi_i$ (sum over i not implied), and we assume that both ϕ_1, ϕ_2 satisfy the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right] = 0,$$

$$-m^2\phi_i - 4\lambda\phi_i^3 - \partial_\mu\partial^\mu\phi_i = 0.$$

The question is if a general linear combination $\phi = a\phi_1 + b\phi_2$ also satisfy the Euler-Lagrange equation

$$-m^2\phi - 4\lambda\phi^3 - \partial_\mu\partial^\mu\phi \stackrel{?}{=} 0.$$

The answer is clearly not, because this equation of motion is non-linear due to the term $-4\lambda\phi^3$ containing a cubed power of ϕ. So a general linear combination $\phi = a\phi_1 + b\phi_2$ is a solution of the equation of motion only when $\lambda = 0$, i.e no self-interaction.

b)

The action is given by

$$S[\phi_i] = \int d^4x \mathcal{L}(\phi_i, \partial_\mu\phi_i),$$

where $\phi_i = (\phi_1, \phi_2, \phi_3, \,\dots)$ is an arbitrary number of fields. An important assumption is that all fields are "dead" at spatial and temporal infinity (i.e the fields and their derivatives are zero at spacetime infinities). When changing the fields infinitesimally $\phi_i \to \phi_i + \delta\phi_i$, the infinitesimal change in the action is (sum over i implied)

$$\delta S = \int d^4x \delta\mathcal{L}$$

$$= \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta(\partial_\mu\phi_i) \right\}$$

$$= \int d^4x \left\{ \left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \right] \right)\delta\phi_i + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i \right) \right\}.$$

The integral of the last term

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i \right)$$

over all space is zero due to the divergence theorem

$$\int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right) = \oint d\mathcal{A} n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i,$$

where $\mathcal{A}$ is the spacetime surface of the integration boundary and n_μ is the normal four-vector. Since the variation in the field $\delta\phi_i$ is also dead at infinity we end up with

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right] \right) \delta\phi_i.$$

Setting $\delta S = 0$ leads to the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right] = 0.$$

c)

When $\lambda = 0$, the equation of motions for ϕ_1, ϕ_2 (as already shown) are

$$-m^2 \phi_i - \partial_\mu \partial^\mu \phi_i = 0$$

- more conventionally written as

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi_i = 0,$$

which is the Klein-Gordon equation.

d)

Now we define the complex scalar field

$$\Phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2).$$

Then we can express the original fields as

$$\phi_1 = \frac{1}{2}\left(\Phi + \Phi^*\right),\ \phi_2 = \frac{1}{2i}\left(\Phi - \Phi^*\right),$$

which can be put into the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m^2\phi_2^2$$

$$= \frac{1}{2}\left[(\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2\right] - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right).$$

We have that

$$(\partial_\mu \phi_1)^2 = \partial^\mu \phi_1 \partial_\mu \phi_1$$

$$= \frac{1}{4}\left(\partial^\mu \Phi + \partial^\mu \Phi^*\right)\left(\partial_\mu \Phi + \partial_\mu \Phi^*\right),$$

$$(\partial_\mu \phi_2)^2 = -\frac{1}{4}\left(\partial^\mu \Phi - \partial^\mu \Phi^*\right)\left(\partial_\mu \Phi - \partial_\mu \Phi^*\right).$$

Adding these results gives

$$(\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 = \frac{1}{4}\left(2\partial^\mu \Phi \partial_\mu \Phi^* + 2\partial^\mu \Phi^* \partial_\mu \Phi\right) = \frac{1}{2}\left(\partial^\mu \Phi \partial_\mu \Phi^* + \partial^\mu \Phi^* \partial_\mu \Phi\right)$$

$$= \frac{1}{2}\left(\partial^\mu \Phi \partial_\mu \Phi^* + \partial_\mu \Phi^* \partial^\mu \Phi\right) = \partial^\mu \Phi \partial_\mu \Phi^*.$$

Furthermore,

$$\phi_1^2 = \frac{1}{4}\left(\Phi + \Phi^*\right)^2 = \frac{1}{4}\left(\Phi^2 + {\Phi^*}^2 + 2\Phi\Phi^*\right),$$

$$\phi_2^2 = -\frac{1}{4}\left(\Phi - \Phi^*\right)^2 = -\frac{1}{4}\left(\Phi^2 + {\Phi^*}^2 - 2\Phi\Phi^*\right).$$

Adding these results gives

$$\phi_1^2 + \phi_2^2 = \frac{1}{4}\left(2\Phi\Phi^* + 2\Phi\Phi^*\right) = \Phi\Phi^*.$$

So the Lagrangian for a complex scalar field is

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right)$$

$$= \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi^* - \frac{1}{2}m^2\Phi\Phi^*.$$

Multiplying the Lagrangian by some number doesn't change the equations of motion, so we might as well write

$$\mathcal{L} = \partial^\mu\Phi\partial_\mu\Phi^* - m^2\Phi\Phi^*.$$

e)

The Euler-Lagrange equations are

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right]$$

$$= -m^2\Phi^* - \partial_\mu\partial^\mu\Phi^* = 0,$$

$$\frac{\partial\mathcal{L}}{\partial\Phi^*} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi^*\right)}\right]$$

$$= -m^2\Phi - \partial_\mu\partial^\mu\Phi = 0.$$

So Φ and Φ^* satisfy the Klein-Gordon equation, which comes as no shock since they are linear combinations of ϕ_1, ϕ_2 and the Klein-Gordon equation is linear.

f)

$$\mathcal{L} = \partial^\mu \Phi \partial_\mu \Phi^* - m^2 \Phi \Phi^*$$

$$= \dot{\Phi}\dot{\Phi}^* - \nabla\Phi \cdot \nabla\Phi^* - m^2 \Phi\Phi^*$$

The Hamiltonian (density) is given by

$$\mathcal{H} = \pi_\Phi \dot{\Phi} + \pi_{\Phi^*} \dot{\Phi}^* - \mathcal{L},$$

where

$$\pi_\Phi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \dot{\Phi}^*,$$

$$\pi_{\Phi^*} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^*} = \dot{\Phi}.$$

Putting it all together gives

$$\mathcal{H} = \pi_\Phi \pi_{\Phi^*} + \pi_{\Phi^*} \pi_\Phi - \mathcal{L}$$

$$= 2\pi_\Phi \pi_{\Phi^*} - \left(\pi_\Phi \pi_{\Phi^*} - \nabla\Phi \cdot \nabla\Phi^* - m^2 \Phi\Phi^*\right)$$

$$= \pi_\Phi \pi_{\Phi^*} + \nabla\Phi \cdot \nabla\Phi^* + m^2 \Phi\Phi^*.$$

It is clearly not Lorentz-invariant (no contractions between four-vectors or tensors or whatever in sight), as it shouldn't be, since the Hamiltonian density is the 00-component of the energy-momentum tensor.

g)

$$\mathcal{L} = \partial^\mu \Phi \partial_\mu \Phi^* - m^2 \Phi \Phi^*$$

Substituting $\Phi \to \Phi' = e^{i\alpha}\Phi$ we get

$$\mathcal{L} = \partial^\mu e^{i\alpha}\Phi \partial_\mu e^{-i\alpha}\Phi^* - m^2 e^{i\alpha}\Phi e^{-i\alpha}\Phi^*$$

$$= e^{i\alpha}e^{-i\alpha}\partial^\mu \Phi \partial_\mu \Phi^* - e^{i\alpha}e^{-i\alpha} m^2 \Phi \Phi^*$$

$$= \partial^\mu \Phi \partial_\mu \Phi^* - m^2 \Phi \Phi^*.$$

So the Lagrangian is invariant with respect to this transformation, $\partial \mathcal{L} / \partial \alpha = 0$.

The "current" which gives us the conserved quantity is given by

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi')} \frac{\partial \Phi'}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi'^*\right)} \frac{\partial \Phi'^*}{\partial \alpha},$$

where

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi')} = \partial^\mu \Phi'^*,$$

$$\frac{\partial \Phi'}{\partial \alpha} = i\Phi',$$

$$J^\mu = \partial^\mu \Phi'^* i \Phi' + \partial^\mu \Phi' (-i) \Phi^{*\prime}$$

$$= i\left(\Phi' \partial^\mu \Phi'^* - \Phi^{*\prime} \partial^\mu \Phi^{*\prime}\right).$$

Since α is arbitrary, we might as well set $\alpha = 0$ and write

$$J^\mu = i\left(\Phi \partial^\mu \Phi^* - \Phi^* \partial^\mu \Phi\right).$$

According to Noether's theorem, this current satisfies $\partial_\mu J^\mu = 0$. To find the conserved quantity we have

$$\partial_t J^0 - \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

$$\partial_t J^0 = \boldsymbol{\nabla} \cdot \mathbf{J}$$

$$\int d^3x \partial_t J^0 = \int d^3x \boldsymbol{\nabla} \cdot \mathbf{J}$$

$$\partial_t \int d^3x J^0 = \oint \mathbf{J} \cdot \mathbf{n} dA.$$

Defining $Q \equiv \int d^3x J^0$ we get

$$\frac{\partial Q}{\partial t} = \oint \mathbf{J} \cdot \mathbf{n} dA$$

which shows that the quantity Q is locally conserved within the closed surface A, unless some current $-\mathbf{J}$ (note the sign) flows out of it. In particular, if $\int d^3x$ is an integral over all space, then we can recall the assumption that fields die at infinity. Since $\mathbf{J}$ is a function of the fields, it dies at infinity as well. The integral on the right hand side gives zero, and we're left with

$$\frac{\partial Q}{\partial t} = 0,$$

i.e Q is a conserved quantity within all space. The direct expression for Q is of course

$$Q = \int d^3x J^0$$

$$= \int d^3x \; i\Big(\Phi \partial_t \Phi^* - \Phi^* \partial_t \Phi\Big).$$