Supersymmetry

Lecture notes for FYS5190/FYS9190

Paul Batzing and Are Raklev

November 27, 2019

Contents

Chapter 1

Introduction

The goal of these lecture notes is to introduce the basics of low-energy models of supersymmetry (SUSY) using the Minimal Supersymmetric Standard Model (MSSM) as our main example. The notes are based on lectures given at the University of Oslo in 2011, 2013, 2015, 2017, and 2019, and lectures at the NORDITA Winter School on Theoretical Particle Physics in 2012. The notes were originally taken by Paul Batzing in 2011, but has since been embellished somewhat.

Oslo, November 2019
Are Raklev

Chapter 2

Groups and algebras

Rather than starting with the current problems of the Standard Model of particle physics, and how supersymmetry can solve these, we will focus on the algebraic origin of supersymmetry in the sense of an extension of the symmetries of Einstein's Special Relativity (SR). This was the original motivation for work on what we today call supersymmetry. We first need to introduce some basic concepts used in physics for exploring symmetries, mainly groups and Lie algebras.

2.1 Group definition

A group is an abstract mathematical structure that consists of a set of objects (elements), and a multiplication rule acting between pairs of these objects. As we will see, it is closely tied to the concept of symmetries in physics, and we shall almost exclusively discuss symmetries in terms of groups. We define a group as follows.

Definition: The set of elements $G = \{g_i\}$ and operation $\bullet$ (sometimes called multiplication) form a **group** if and only if for $\forall\, g_i \in G$:

i) $g_i \bullet g_j \in G$, (closure)

ii) $(g_i \bullet g_j) \bullet g_k = g_i \bullet (g_j \bullet g_k)$, (associativity)

iii) $\exists e \in G$ such that $g_i \bullet e = e \bullet g_i = g_i$, (identity element)

iv) $\exists g_i^{-1} \in G$ such that $g_i \bullet g_i^{-1} = g_i^{-1} \bullet g_i = e$. (inverse)

A simple example of a group is $G = \mathbb{Z}$ (the integers) with standard addition as the operation. Then $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to $\mathbb{Z}_n$, where the operation is addition modulo n. In this group, $g_i^{-1} = n - g_i$ and the unit element is again $e = 0$.[1] Here $\mathbb{Z}$ is an example of an **infinite** group, the set has an infinite number of members, while $\mathbb{Z}_n$ is **finite**, with **order** n, meaning n members. Both are **abelian** groups, meaning that the elements commute: $g_i \bullet g_j = g_j \bullet g_i$.

[1] Note that we here use e for the identity in an abstract group, while we later use I or 1 as the identity matrix in matrix representations of groups.

The simplest, non-trivial, of these $\mathbb{Z}_n$ groups is $\mathbb{Z}_2$ which only has the members $e = 0$ and 1. The operation is defined by $0 + 0 = 0$, $0 + 1 = 1$ and $1 + 1 = 0$. Now, compare this to the set $G = \{-1, 1\}$ with the ordinary multiplication operation. Here all the possible operations are $1 \cdot 1 = 1$, $1 \cdot (-1) = -1$ and $(-1) \cdot (-1) = 1$. This has exactly the same structure as $\mathbb{Z}_2$, only that the identity element is 1. We say that these two groups are **isomorphic**, and in fact consider them as the same group.[2] We will return to this point in more detail in Sec. 2.3.

A somewhat more sophisticated example of a group can be found in the Taylor expansion of a function f, where

$$\begin{aligned} f(x+a) &= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \ldots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{dx^n} f(x) \\ &= e^{a\frac{d}{dx}} f(x). \end{aligned}$$

The last equality uses the formal definition of the exponential series, but may drive some mathematicians crazy.[3] The resulting operator $T_a = e^{a\frac{d}{dx}}$ is called the **translation operator**, in this case in one dimension, since it shifts the coordinate x by an amount a. Together with the (natural) multiplication operation $T_a \bullet T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where $T_a^{-1} = T_{-a}$.[4] In n dimensions the group $T(n)$ has the elements $T_{\vec{a}} = e^{\vec{a}\cdot\vec{\nabla}}$. Whereas we say that the groups $\mathbb{Z}$ and $\mathbb{Z}_n$ are **discrete groups**, since we can count the number of elements, T_a is a **continuous group** since a can be any real number.

2.2 Matrix groups

We next define some groups that are very important in physics and to the discussion in these notes. They have in common that they are defined in terms of matrices.

Definition: The **general linear group** $GL(n)$ is defined by the set of invertible $n \times n$ matrices A under matrix multiplication. If we additionally require that $\det(A) = 1$ the matrices form the **special linear group** $SL(n)$.

We usually take these matrices to be defined over the field of complex numbers $\mathbb{C}$. If we want to specify the field we may use the notation $GL(n, \mathbb{R})$, signifying that the group is defined over the real numbers.

[2] This generalises to the set $G = \{e^{2\pi ik/n}; k = 1, \ldots, n-1\}$, the **n-th roots of unity**, together with the multiplication operation, which is isomorphic to $\mathbb{Z}_n$.

[3] We will not discuss this further, but there is a deep question here whether the operator formed by this exponentiation is well defined.

[4] We could instead have defined the operation between two group elements to be ordinary multiplication and used that to show the relationship $T_a \bullet T_b = T_{a+b}$. However, it is important to notice that showing this is not entirely trivial because ordinary arithmetic rules for exponentials fail for operators. In this case the proof is fairly simple, but this is in general not so.

Definition: The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U, *i.e.* matrices such that $U^\dagger U = 1$ or $U^{-1} = U^\dagger$. If we additionally require that $\det(U) = 1$ the matrices form the **special unitary group** $SU(n)$.

It is these groups that form the gauge symmetry groups of the Standard Model $SU(3)$, $SU(2)$ and $U(1)$. The group $U(1)$ makes perfect sense, this is simply the set of all complex numbers of unit length with ordinary multiplication, *i.e.* $U(1) = \{e^{i\alpha}; \alpha \in \mathbb{R}\}$, but notice that $SU(1)$ is trivial.

The unitary group has has the neat property that for $\forall \vec{x}, \vec{y} \in \mathbb{C}^n$ multiplication by a unitary matrix leaves scalar products unchanged:

$$
\begin{aligned}
\vec{x}' \cdot \vec{y}' &\equiv \vec{x}'^\dagger \vec{y}' = (U\vec{x})^\dagger U \vec{y} \\
&= \vec{x}^\dagger U^\dagger U \vec{y} = \vec{x}^\dagger \vec{y} = \vec{x} \cdot \vec{y}.
\end{aligned}
$$

Thus, its members do not change the length of the vectors they act on.

Definition: The **orthogonal group** $O(n)$ is the group of real $n \times n$ orthogonal matrices O, *i.e.* matrices where $O^T O = 1$. If we additionally require that $\det(O) = 1$ the matrices form the **special orthogonal group** $SO(n)$.

For $\vec{x} \in \mathbb{R}^n$ the orthogonal group has the same property as the unitary group of leaving the length invariant. The special orthogonal groups $SO(2)$ and $SO(3)$ are much used because their elements represent rotations around a point and axis, respectively, and $SO(n)$ represents the symmetries of a sphere in n dimensions.

2.3 Group properties

We now extend our vocabulary for groups by defining the **subgroup** of a group G.

Definition: A subset $H \subset G$ is a **subgroup** if and only if:[a]

i) $h_i \bullet h_j \in H$ for $\forall h_i, h_j \in H$, (closure)

ii) $h_i^{-1} \in H$ for $\forall h_i \in H$. (inverse)

[a] An alternative, equivalent, and more compact way of writing these two requirements is the single requirement $h_i \bullet h_j^{-1} \in H$ for $\forall h_i, h_j \in H$. This is often utilised in proofs.

There is a very important type of subgroup called the **normal** subgroup. The importance will become clear in a moment.

Definition: H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$. A subgroup H is a **normal** (invariant) subgroup, if and only if for $\forall g \in G$,[a]

$$ghg^{-1} \in H \text{ for } \forall h \in H.$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no abelian normal subgroup.

[a] Another, pretty but slightly abusive, way of defining a normal group is to say that $gHg^{-1} = H$.

Up to this point things hopefully seem pretty natural, if not exactly easy. We will now become slightly more cryptic by defining **cosets**.

Definition: A **left coset** of a subgroup $H \subset G$ with respect to $g \in G$ is the set $\{gh : h \in H\}$, and a **right coset** of the subgroup is the set $\{hg : h \in H\}$. These are sometimes written gH and Hg, respectively. For normal subgroups H it can be shown that the left and right cosets coincide and form the **coset group**[a] G/H. This has as its members the *sets* $\{gh : h \in H\}$ for $\forall g \in G$ and the binary operation $*$ with $gh * g'h' \in \{(g \bullet g')h : h \in H\}$.

[a] Sometimes called the **factor** or **quotient group**.

The abstract looking coset group has a heuristic understanding as a *division* of groups, where the structure of the normal subgroup is removed from the larger group, hench the symbol.

Now that we have introduced group division, we also need to introduce products of groups.

Definition: The **direct product** of groups G and H, $G \times H$, is defined as the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet h_j)$. $G \times H$ is then a group and G and H can be shown to be normal subgroups of $G \times H$.

Because it has at least one guest star appearance in this text we also need the definition of the **semi-direct product**.

Definition: The **semi-direct product** $G \rtimes H$, where G is a mapping $G : H \to H$, is defined by the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet g_i(h_j))$. Here H is not a normal subgroup of $G \rtimes H$, but G is.

The famous Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ is an example of a direct product. Direct products are “trivial” structures because there is no “interaction” between the subgroups, the elements of each group act only on elements of the same group. This is not true for semi-direct products.

Given both the definition of a coset group and the direct product of groups, we can see how to both make and remove products of groups. The normal subgroups can, again heuristically, be viewed as factors in groups. In extension of this the group G and the group $(G \times H)/H$ should be the same group. However, to decide such questions we also need a clear notion of when two groups are the same.

Definition: Two groups G and H are **homomorphic** if there exists a map between the elements of the groups $\rho : G \to H$, such that for $\forall g, g' \in G$, $\rho(g \bullet g') = \rho(g) \bullet \rho(g')$.

For homomorphic groups we say that the mapping conserves the structure of the group, or in other words, all the rules for the group operation/multiplication. This leads to our notion of group equality, namely **isomorphic** groups:

Definition: Two groups G and H are **isomomorphic**, written $G \cong H$, if they are homomorphic and the relevant mapping is one-to-one.

So, for isomorphic groups there are corresponding elements in each group.

2.4 Representations

The previous section was dominantly a mathematical approach to groups. Physicists are usually more interested in groups G where the elements of G *act* to transform some elements of a set $s \in S$, $g(s) = s' \in S$.[5] Here, the members of S can for example be the state of a system, say a wave-function in quantum mechanics or a field in quantum field theory. To be usefull in physics, we would like that the result of the group operation $g_i \bullet g_j$ acts as $(g_i \bullet g_j)(s) = g_i(g_j(s))$ and the identity acts as $e(s) = s$.

We begin with the abstract definition of a representation that ensures these properties.

Definition: A **representation** of a group G on a vector space V is a map $\rho : G \to GL(V, K)$, where $GL(V, K)$ is the **general linear group** on V, *i.e.* the invertible matrices of the field K of V,[a] such that for $\forall g_i, g_j \in G$,

$$\rho(g_i \bullet g_j) = \rho(g_i)\rho(g_j). \text{ (homomorphism)}$$

If this map is isomorphic, we say that the representation is **faithful**.

[a] Technically, we can only be sure that we can write $GL(V, K)$ as matrices as long as V is a finite dimensional vector space. However, we shall do our best not to ass around with infinite dimensional representations.

From a physicist point of view, the underlying point here is that (the members of) our groups will be used on quantum mechanical states, or fields in field theory, which can be just complex numbers (functions) or multi-component vectors of such. They are thus members of a vector space, and the definition of representations force the transformation properties of the group to be written in terms of matrices. Furthermore, that the mapping from the group, or, if you like, the concrete way of writing the abstract group elements, must be homomorphic (structure preserving), meaning that if we can write a group element as the product of two others, the matrix for that element must be the product of the two matrices for the individual group elements it can be written in terms of.

You may by now have realized that the matrix groups defined in Sec. 2.2 have the property that they are defined in terms of one of their representations. These are called the **fundamen-**

[5] As a result mathematics courses in group theory are not always so relevant to a physicist. This part of group theory is called representation theory.

tal or **defining representations**. However, we will also have use for other representations, *e.g.* the **adjoint representation** which we will introduce below.

Let us now take a few examples that connect to our definition. For $U(1)$ the group members can be written as the complex numbers on the unit circle $e^{i\alpha}$, which can be used as phase transformations on wavefunctions $\psi(x)$ that form a one dimensional vector space over the complex numbers. For $SU(2)$ we can show that the group elements can be written in the fundamental representation as $e^{i\alpha_i\sigma_i}$, with σ_i being the Pauli matrices, which in the Standard Model is applied to weak doublets of fields, *e.g.* the electron–neutrino doublet $\psi = (\nu_e, e)$ that form a two-dimensional vector space, as the $SU(2)_L$ gauge transformation.[6]

For later use we need to know when two representations are equivalent.

Definition: Two representations ρ and ρ' of G on V and V' are **equivalent** if and only if $\exists A : V \to V'$, that is one-to-one, such that for $\forall g \in G$, $A\rho(g)A^{-1} = \rho'(g)$.

The building blocks of representations are so-called **irreducible representations**, also called **irreps**. These are the essential ingredients in representation theory, and are defined as follows:

Definition: An **irreducible representation** ρ is a representation where there is *no* proper subspace $W \subset V$ that is closed under the group, *i.e.* there is no $W \subset V$ such that for $\forall w \in W$, $\forall g \in G$ we have $\rho(g)w \in W$.[a]

[a] In other words, we can not split the matrix representation of G in two parts that do not "mix".

Let us take an example to try to clear up what a reducible representation means. The representation $\rho(g)$ for $g \in G$ acts on a vector space V as matrices. If these matrices $\rho(g)$ can all be decomposed into $\rho_1(g)$, $\rho_2(g)$ and $\rho_{12}(g)$ such that for $\vec{v} = (\vec{v}_1, \vec{v}_2) \in V$

$$\rho(g)\vec{v} = \begin{bmatrix} \rho_1(g) & \rho_{12}(g) \\ 0 & \rho_2(g) \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix},$$

then ρ is **reducible**. The subspace of V spanned by $\vec{v}_1$ violates the irreducibility condition above.

If we also have $\rho_{12}(g) = 0$ we say that the representation is **completely reducible**. It can be shown that in most cases a reducible representation is also completely reducible. In fact, representations for which this is not true tend to be mathematical curiosities. As a result, there is a tendency in physics to use the term "reducible" where we should use the term "completely reducible". In the case of a completely reducible representation we can split the vector space V into two vector spaces $V = V_1 \oplus V_2$, where $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$, and define a representation of G on each of them using ρ_1 and ρ_2, which in turn could either be reduced more, or would be irreducible.

We end this section with an important theorem that helps us decide whether a representation is irreducible, and ultimately gives a property identifying the representation. As many

[6] We will return to why we want to write the group as this particular exponential in Sec. 2.6, however, a quick count of the number of free parameters in $SU(2)$ should convince us that we need three real numbers to parametrise the group elements, here the α_i: each matrix consists of four complex numbers, or eight real, the unitarity requirement removes four free parameters, while the requirement on the determinant one more. Since the three Pauli matrices are linearly independent $\alpha_i\sigma_i$ should then span $SU(2)$ and the exponentiation does not change that.

important theorems, it is called a lemma.

Theorem: (Schur's Lemma)
If we have an irreducible representation ρ of a group G, all matrices A that commute with $\rho(g)$ for $\forall g \in G$ are proportional to the identity, $A = \lambda I$.

Here the λ are constants that label the representation.

2.5 Lie groups

In physics we are particularly interested in a special type of group, the **Lie group**, a class of continuous groups that we can parametrise and which are the basic tool we use to describe continuous symmetries. In order to define Lie groups we will need to use the technical term (smooth) manifold, meaning a mathematical object (formally a topological space) that locally[7] can be parametrised as a function of $\mathbb{R}^n$ or $\mathbb{C}^n$. We will describe a Lie group G in terms of a parameterisation of the members $g(\vec{a}) \in G$, where $\vec{a} \in \mathbb{R}^n$ (or $\mathbb{C}^n$). Additionally, in order to describe continous symmetries these parameterisations need to be *smooth*, also in the technical sense of smooth, which means infinitely differentiable.

Definition: A **Lie group** G is a finite-dimensional **smooth manifold** where group multiplication and inversion are smooth functions, meaning that given elements $g(\vec{a}), g'(\vec{a}') \in G$, $g(\vec{a}') \bullet g'(\vec{a}') = g''(\vec{b})$ where $\vec{b}(\vec{a}, \vec{a}')$ is a smooth function of $\vec{a}$ and $\vec{a}'$, and $g^{-1}(\vec{a}) = g'(\vec{a}')$ where $\vec{a}'(\vec{a})$ is a smooth function of $\vec{a}$.

In terms of the situation we are often interested in, a Lie group G acting on a vector space V through a representation, it can be shown (for finite-dimensional representations) that we can write the map of the representation $G \times V \to V$ for $\vec{x} \in V$ as $x_i \to x_i' = f_i(\vec{x}, \vec{a})$ where f_i is analytic[8] in x_i and a_i. Additionally f_i has an inverse.

We immediately see that the translation group $T(1)$ with the parameterisation $g(a) = e^{a\frac{d}{dx}}$ is a Lie group since $g(a) \cdot g(a') = g(a + a')$ and $a + a'$ is an analytic function of a and a'. Here we can write the action of the group on the vector space $\mathbb{R}^1$ as $f(x, a) = x + a$. The $SU(n)$ groups are also Lie groups as they have a fundamental representation $e^{i\vec{\alpha}\vec{\lambda}}$ where λ is a set of $n \times n$-matrices, and $f_i(\vec{x}, \vec{\alpha}) = [e^{i\vec{\alpha}\vec{\lambda}}\vec{x}]_i$.

By the analyticity we can always construct the parametrization so that $g(0) = e$ or $x_i = f_i(x_i, 0)$. By an infinitesimal transformation da_i we then get the following Taylor expansion[9]

$$\begin{aligned} x_i' &= x_i + dx_i = f_i(x_i, da_i) \\ &= f_i(x_i, 0) + \frac{\partial f_i}{\partial a_j} da_j + \ldots \\ &= x_i + \frac{\partial f_i}{\partial a_j} da_j \end{aligned}$$

[7]This insistence on local means that the parameterisation is not necessarily the same for the whole group.

[8]Meaning infinitely differentiable and in possession of a convergent Taylor expansion. As a result analytic functions (on $\mathbb{R}$) are smooth, but the reverse does not hold.

[9]The fact that f_i is analytic means that this Taylor expansion must converge in some radius around $f_i(x_i, 0)$.

This is the transformation by the member of the group that in the parameterisation sits $d\vec{a}$ from the identity. If we now let F be a function from the vector space V to either the real $\mathbb{R}$ or complex numbers $\mathbb{C}$, then the group transformation defined by $d\vec{a}$ changes F by

$$\begin{aligned} dF &= \frac{\partial F}{\partial x_i} dx_i \\ &= \frac{\partial F}{\partial x_i}\frac{\partial f_i}{\partial a_j} da_j \\ &\equiv da_j X_j F \end{aligned}$$

where the operators defined by

$$X_j \equiv \frac{\partial f_i}{\partial a_j}\frac{\partial}{\partial x_i}$$

are called the n **generators** of the Lie group. It is these generators X that define the action of the Lie group in a given representation as the a's are mere parameters. We can say that the generators determine the local structure of the group.

As an example of the above we can now go in the opposite direction and look at the two-parameter transformation *defined* by

$$x' = f(x) = a_1 x + a_2,$$

which gives

$$X_1 = \frac{\partial f}{\partial a_1}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x},$$

which is the generator for **dilation** (scale change), and

$$X_2 = \frac{\partial}{\partial x},$$

which is the generator for $T(1)$. Note that $[X_1, X_2] = X_1 X_2 - X_2 X_1 = -X_2$.

The commutator of the generators of the Lie group satisfy $[X_i, X_j] = C_{ij}^k X_k$, where C_{ij}^k are the **structure constants** of the group. We can easily see that these are antisymmetric in i and j, $C_{ij}^k = -C_{ji}^k$. In what is called Lie's third theorem, Sophus Lie [1] showed that there is a **Jacobi identity** among the generators,

$$[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0. \tag{2.1}$$

This immediately leads to the following identity for the structure constants: $C_{ij}^k C_{kl}^m + C_{jl}^k C_{ki}^m + C_{li}^k C_{kj}^m = 0$.

We touched on the fundamental representation of a matrix based group earlier. These representations have the lowest possible dimension. Another important representation is the **adjoint**. This consists of the matrices:

$$(M_i)_j^k = -C_{ij}^k,$$

where C_{ij}^k are the structure constants. From the Jacobi identity we have $[M_i, M_j] = C_{ij}^k M_k$, meaning that the adjoint representation fulfills the same algebra as the fundamental (generators). Note that the dimension of the fundamental representation n for $SO(n)$ and $SU(n)$ is equal to the degrees of freedom, $\frac{1}{2}n(n-1)$ and $n^2 - 1$, respectively.

2.6 Lie algebras

We begin this section by defining algebras, which extend familiar vector spaces by adding a multiplication operation for the vectors.

Definition: An **algebra** A on a field (say $\mathbb{R}$ or $\mathbb{C}$) is a linear vector space with a binary operation $\circ : A \times A \to A$.

As a very simple example, the vector space $\mathbb{R}^3$ together with the standard cross-product constitutes an algebra.

Definition: A **Lie algebra** L is an algebra where the binary operator $[\,,\,]$, called the **Lie bracket**, has the properties that for $x, y, z \in L$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$):

i) (bilinearity)
$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y]$$

ii) (anti-commutation)
$$[x, y] = -[y, x]$$

iii) (Jacobi identity)
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Again $\mathbb{R}^3$ with $[\vec{x}, \vec{y}] = \vec{x} \times \vec{y}$ is a simple example of a Lie algebra.

We usually restrict ourselves to algebras of linear operators where the Lie bracket is the commutator $[x, y] = xy - yx$, where these properties follow automatically. The generators of an n-dimensional Lie group with the commutator as the binary operation then form a unique n-dimensional Lie algebra. However, the reverse is not true. There can be multiple Lie groups with the same algebra. The often quoted example is $SO(3)$ and $SU(2)$, which have the same algebra.

Now that we have discussed the algebra as the local structure of the group, we can finally look at how the group (and matrix representation) is reconstructed from the algebra. For this we use what is called the exponential map.

Definition: The **exponential map** from the Lie algebra L of the general linear group $GL(n)$ is defined by $\exp : L \to GL$, where

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}. \tag{2.2}$$

This is nothing than the formal definition of an exponential of a matrix. For any subgroup G of GL, the Lie algebra of G is mapped into G by the exponential map, meaning that any group that can be written in terms of matrices, can be reconstructed from the algebra in this manner. For groups that can not be written as matrices the exponential map must be generalized, however, this is somewhat beyond the scope of these notes.

2.7 Exercises

Exercise 2.1 Show that $T_a^{-1} = T_{-a}$ and that $T(1)$ is group.

Solution: Given the group multiplication definition the identity element must be $e = T_0 = 1$ since $T_a \bullet T_0 = T_{a+0} = T_a$. Let $T_a^{-1} = T_b$. Since $T_a \bullet T_b = T_{a+b} = 1$, we find $b = -a$. (This does not show that the inverse is unique, but this is not a requirement.) We have now demonstrated the required existence of an identity element (iii) and an inverse (iv) for $T(1)$ as required by the group definition (the order of operations can obviously be reversed). The closure of the multiplication operation (i) is true by its definition. The associativity (ii) can be demonstrated as follows $(T_a \bullet T_b) \bullet T_c = T_{a+b} \bullet T_c = T_{a+b+c} = T_a \bullet T_{b+c} = T_a \bullet (T_b \bullet T_c)$.

Exercise 2.2 Show that for a subset $H \subset G$, if $h_i \bullet h_j^{-1} \in H$ for $\forall h_i, h_j \in H$, then H is a subgroup of G.

Exercise 2.3 Show that if H is a subgroup of G, then $h_i \bullet h_j^{-1} \in H$ for $\forall h_i, h_j \in H$.

Solution: Since H is a subgoup of G then by point ii) in the definition since $h_j \in H$ we must also have $h_j^{-1} \in H$. With $h_i, h_j^{-1} \in H$ by point i) in definition $h_i \bullet h_j^{-1} \in H$.

Exercise 2.4 Show that $SU(n)$ is a proper subgroup of $U(n)$ and that $U(n)$ is not simple.

Solution: Let $U_i, U_j \in SU(n)$, then

$$\det(U_i U_j^{-1}) = \det(U_i)\det(U_j^{-1}) = 1.$$

This means that $U_i U_j^{-1} \in SU(n)$. In other words, $SU(n)$ is a proper subgroup of $U(n)$. Let $V \in U(n)$ and $U \in SU(n)$, then $VUV^{-1} \in SU(n)$ because:

$$\det(VUV^{-1}) = \det(V)\det(U)\det(V^{-1}) = \frac{\det(V)}{\det(V)}\det(U) = 1.$$

In other words, $SU(n)$ is also a normal subgroup of $U(n)$, thus $U(n)$ is not simple.

Exercise 2.5 If H is a normal subgroup of G, show that its left and right cosets are the same, and show that the set formed of the cosets is a group under an appropriate group operation.

Exercise 2.6 Show that the factors in a direct product of groups are normal groups to the product.

Exercise 2.7 Show that the group G and the group $(G \times H)/H$ are isomorphic.

Exercise 2.8 Show that $U(1) \cong \mathbb{R}/\mathbb{Z} \cong SO(2)$.

Exercise 2.9 Find the dimensions of the fundamental and adjoint representations of $SU(n)$.

Exercise 2.10 Find the fundamental representation for $SO(3)$ and the adjoint representation for $SU(2)$. What does this say about the groups and their algebras?

Exercise 2.11 Find the generators of $SU(2)$ and their commutation relationships. *Hint:* One answer uses the Pauli matrices, but try to derive this from an infinitesimal parametrization.

Exercise 2.12 What are the structure constants of SU(2)?

Exercise 2.13 Show that $\mathbb{R}^3$ with the binary operator $[\vec{x}, \vec{y}] = \vec{x} \times \vec{y}$ is a Lie algebra.

Exercise 2.14 Let A_i be the generators of the group G and B_j be the generators of group$H$. Explain in what sense the A_i and B_j are generators of the direct product group $G \times H$ and show that $[A_i, B_j] = 0$.

Chapter 3

The Poincaré algebra and its extensions

We now take a look at the groups behind Special Relativity (SR), the Lorentz and Poincaré groups. We will first see what sort of states transform properly under SR, which has surprising connections to already familiar physics. We will then look for ways to extend these external symmetries to internal symmetries, *i.e.* the symmetries of gauge groups.

3.1 The Lorentz Group

A point in the Minkowski space-time manifold $\mathbb{M}_4$ is given by $x^\mu = (t, x, y, z)$ and Einstein's requirement in Special Relativity was that the laws of physics should be invariant under rotations and/or boosts between different reference frames. These transformations are captured in the Lorentz group.

Definition: The **Lorentz group** L is the group of linear transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that $x^2 = x_\mu x^\mu = x'_\mu x'^\mu$ is invariant. The **proper orthochronous Lorentz group** $L_+^\uparrow$ is a subgroup of L where $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$. [a]

[a] This guarantees that time moves forward, and makes space and time reflections impossible, so that the group describes only proper boosts and rotations.

From the discussion in the previous section any $\Lambda \in L_+^\uparrow$ can be written as

$$\Lambda^\mu{}_\nu = \left[\exp\left(-\frac{i}{2}\omega^{\rho\sigma} M_{\rho\sigma}\right)\right]^\mu{}_\nu, \tag{3.1}$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ are the parameters of the transformation and $M_{\rho\sigma}$ are the generators of the group L. The elements of $M_{\rho\sigma}$ form the basis of the Lie algebra for L, and are given by:

$$M = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix},$$

where K_i and J_i are generators of boost and rotation respectively. These fulfil the following algebra:[1]

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k, && (3.2)\\
[K_j, J_i] &= i\epsilon_{ijk} K_k, && (3.3)\\
[K_i, K_j] &= -i\epsilon_{ijk} J_k. && (3.4)
\end{aligned}$$

In terms of M these commutation relations can be written:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \tag{3.5}$$

3.2 The Poincaré group

We extend L by translation to get the Poincaré group, where translation : $x^\mu \to x'^\mu = x^\mu + a^\mu$. This leaves lengths $(x-y)^2$ invariant in $\mathbb{M}_4$.

Definition: The **Poincaré group** P is the group of all transformations of the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$

We can also construct the **restricted Poincaré group** $P_+^\uparrow$, by restricting the matrices Λ in the same way as in $L_+^\uparrow$.

We see that the composition of two elements in the group is:

$$(\Lambda_1, a_1) \bullet (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1).$$

This tells us that the Poincaré group is **not** a direct product of the Lorentz group and the translation group, but a **semi-direct product** of L and the translation group $T(1,3)$, $P = L \rtimes T(1,3)$. The translation generators P_μ have a trivial commutation relationship:[2]

$$[P_\mu, P_\nu] = 0 \tag{3.6}$$

One can show that:[3]

$$[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \tag{3.7}$$

Equations (3.5)–(3.7) form the **Poincaré algebra**, a Lie algebra.

3.3 The Casimir operators of the Poincaré group

Definition: The **Casimir operators** of a Lie algebra are the operators that commute with all elements of the algebra [a]

[a] Technically we say they are members of the centre of the universal enveloping algebra of the Lie algebra. Whatever that means.

[1] Notice that (3.2) is the $SU(2)$ algebra.

[2] This means that the translation group in Minkowski space is abelian. This is obvious, since $x^\mu + y^\mu = y^\mu + x^\mu$. One can show that the differential representation is the expected $P_\mu = -i\partial_\mu$.

[3] For a rigorous derivation of this see Chapter 1.2 of [2]

A central theorem in representation theory for groups and algebras is **Schur's lemma**:

> **Theorem**: (Schur's Lemma)
> In any irreducible representation of a Lie group, the Casimir operators are proportional to the identity.

This has the wonderful consequence that the constants of proportionality can be used to classify the (irreducible) representations of the Lie algebra (and group). Let us take a concrete example to illustrate: $P^2 = P_\mu P^\mu$ is a Casimir operator of the Poincaré algebra because the following holds:

$$\left[P_\mu, P^2\right] = 0, \tag{3.8}$$
$$\left[M_{\mu\nu}, P^2\right] = 0. \tag{3.9}$$

This allows us to label the irreducible representation of the Poincaré group with a quantum number m^2, writing a corresponding state as $|m\rangle$, such that:[4]

$$P^2|m\rangle = m^2|m\rangle.$$

The number of Casimir operators is the **rank** of the algebra, *e.g.* rank $SU(n) = n-1$. It turns out that $P_+^\uparrow$ has rank 2, and thus two Casimir operators. To demonstrate this is rather involved, and we won't make an attempt here, but note that it can be shown that[5] $L_+^\uparrow \cong SU(2)\times SU(2)$ because of the structure of the boost and rotation generators, where $SU(2)$ can be shown to have rank 1. Furthermore, $L_+^\uparrow \cong SL(2,\mathbb{C})$. We will return to this relationship between $L_+^\uparrow$ and $SL(2,\mathbb{C})$ in Section 3.5, where we use it to reformulate the algebras we work with in supersymmetry.

So, what is the second Casimir of the Poincaré algebra?

> **Definition**: The **Pauli-Ljubanski polarisation vector** is given by:
>
> $$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{3.10}$$

Then $W^2 = W_\mu W^\mu$ is a Casimir operator of $P_+^\uparrow$, *i.e.*:

$$\left[M_{\mu\nu}, W^2\right] = 0, \tag{3.11}$$
$$\left[P_\mu, W^2\right] = 0. \tag{3.12}$$

Again, because W^2 is a Casimir operator, we can label all states in an irreducible representation (read particles) with quantum numbers m, s, such that:

$$W^2|m,s\rangle = -m^2 s(s+1)|m,s\rangle$$

[4]This quantum number looks astonishingly like mass and P^2 like the square of the 4-momentum operator. However, we note that in general m^2 is not restricted to be larger than zero.

[5]Here $\cong$ means homomorfic, that is structure preserving.

The m^2 appears because there are two P_μ operators in each term. However, what is the significance of the s, and why do we choose to write the quantum number in that (familiar?) way? One can easily show using ladder operators that $s = 0, \frac{1}{2}, 1, \ldots$, *i.e.* can only take integer and half integer values. In the rest frame (RF) of the particle we have:[6]

$$P_\mu = (m, \vec{0})$$

Using that $WP = 0$ this gives us $W_0 = 0$ in the RF, and furthermore:

$$W_i = \frac{1}{2}\epsilon_{i0jk} m M^{jk} = m S_i,$$

where $S_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$ is the **spin operator**. This gives $W^2 = -\vec{W}^2 = -m^2\vec{S}^2$, meaning that s is indeed the spin quantum number.[7]

The conclusion of this subsection is that anything transforming under the Poincaré group, meaning the objects considered by SR, can be classified by two quantum numbers: mass and spin.

3.4 The no-go theorem and graded Lie algebras

Since we now know the Poincaré group and its representations well, we can ask: Can the external space-time symmetries be extended, perhaps also to include the internal gauge symmetries? Unfortunately no. In 1967 Coleman and Mandula [3] showed that any extension of the Pointcaré group to include gauge symmetries is isomorphic to $G_{SM} \times P_+^\uparrow$, *i.e.* the generators B_i of standard model gauge groups all have

$$[P_\mu, B_i] = [M_{\mu\nu}, B_i] = 0.$$

Not to be defeated by a simple mathematical proof this was countered by Haag, Łopuszański and Sohnius (HLS) in 1975 in [4] where they introduced the concept of graded Lie algebras to get around the no-go theorem.

Definition: A ($\mathbb{Z}_2$) **graded Lie algebra** or **superalgebra** is a vector space L that is a direct sum of two vector spaces L_0 and L_1, $L = L_0 \oplus L_1$ with a binary operation $\bullet : L \times L \to L$ such that for $\forall x_i \in L_i$

i) $x_i \bullet x_j \in L_{i+j \bmod 2}$ (grading)[a]

ii) $x_i \bullet x_j = -(-1)^{ij} x_j \bullet x_i$ (supersymmetrization)

iii) $x_i \bullet (x_j \bullet x_k)(-1)^{ik} + x_j \bullet (x_k \bullet x_i)(-1)^{ji} + x_k \bullet (x_i \bullet x_j)(-1)^{kj} = 0$ (generalised Jacobi identity)

This definition can be generalised to $\mathbb{Z}_n$ by a direct sum over n vector spaces L_i, $L = \oplus_{i=0}^{n-1} L_i$, such that $x_i \bullet x_j \in L_{i+j \bmod n}$ with the same requirements for supersymmetrization and Jacobi identity as for the $\mathbb{Z}_2$ graded algebra.

[a]This means that $x_0 \bullet x_0 \in L_0$, $x_1 \bullet x_1 \in L_0$ and $x_0 \bullet x_1 \in L_1$.

[6]This does not loose generality since physics should be independent of frame.

[7]Observe that this discussion is problematic for massless particles. However, it is possible to find a similar relation for massless particles, when we chose a frame where the velocity of the particle is mono-directional.

We can start, as HLS, with a Lie algebra ($L_0 = P_+^\uparrow$) and add a new vector space L_1 spanned by four operators, the Majorana spinor charges Q_a. It can be shown that the superalgebra requirements are fulfilled by:

$$[Q_a, P_\mu] = 0 \tag{3.13}$$
$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu} Q)_a \tag{3.14}$$
$$\{Q_a, \bar{Q}_b\} = 2\not{P}_{ab} \tag{3.15}$$

where $\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and as usual $\not{P} = P_\mu \gamma^\mu$ and $\bar{Q}_a = (Q^\dagger \gamma_0)_a$.[8]

Unfortunately, the internal gauge groups are nowhere to be seen. They can appear if we extend the algebra with Q_a^α, where $\alpha = 1, \ldots, N$, which gives gives rise to so-called $N > 1$ supersymmetries. This introduces extra particles and does not seem to be realised in nature due to an extensive number of extra particles.[9] This extension, including $N > 1$, can be proven, under some reasonable assumptions, to be the **largest possible** extension of SR.

3.5 Weyl spinors

Previously we claimed that there is a homomorphism between the groups $L_+^\uparrow$ and $SL(2, \mathbb{C})$. This homomorphism, with $\Lambda^\mu{}_\nu \in L_+^\uparrow$ and $M \in SL(2, \mathbb{C})$, can be explicitly given by:[10]

$$\Lambda^\mu{}_\nu(M) = \frac{1}{2}\mathrm{Tr}[\bar{\sigma}^\mu M \sigma_\nu M^\dagger], \tag{3.16}$$
$$M(\Lambda^\mu{}_\nu) = \pm\frac{1}{\sqrt{\det(\Lambda^\mu{}_\nu \sigma_\mu \bar{\sigma}^\nu)}}\Lambda^\mu{}_\nu \sigma_\mu \bar{\sigma}^\nu, \tag{3.17}$$

where $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and $\sigma^\mu = (1, \vec{\sigma})$.

This two-to-one correspondence means that $L_+^\uparrow \cong SL(2, \mathbb{C})/\mathbb{Z}_2$. Thus we can look at the representations of $SL(2, \mathbb{C})$ instead of the Poincaré group, with its usual Dirac spinors, when we describe particles, but what are those representations? It turns out that there exist two inequivalent fundamental representations of $SL(2, \mathbb{C})$:

i) The self-representation $\rho(M) = M$ working on an element ψ of a representation space F:

$$\psi'_A = M_A{}^B \psi_B, \quad A, B = 1, 2.$$

ii) The complex conjugate self-representation $\rho(M) = M^*$ working on $\bar{\psi}$ in a space $\dot{F}$:[11]

$$\bar{\psi}'_{\dot{A}} = (M^*)_{\dot{A}}{}^{\dot{B}} \bar{\psi}_{\dot{B}}, \quad \dot{A}, \dot{B} = 1, 2.$$

---

[8] Alternatively, (3.15) can be written as $\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu$.

[9] Note that $N > 8$ would include particles with spin greater than 2.

[10] The sign in Eq. (3.17) is the reason that this is a homomorphism, instead of an isomorphism. Each element in $L_+^\uparrow$ can be assigned to two in $SL(2, \mathbb{C})$.

[11] The dot on the indices is just there to help us remember which sum is which and does not carry any additional importance.

Definition: ψ and $\bar{\psi}$ are called **left- and right-handed Weyl spinors**.

Indices can be lowered and raised with:

$$\epsilon_{AB} = \epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\epsilon^{AB} = \epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The relationship between ψ and $\bar{\psi}$ can be expressed with:[12]

$$\bar{\sigma^0}^{\dot{A}A}(\psi_A)^* = \bar{\psi}^{\dot{A}}$$

Note that from the above:

$$(\psi_A)^\dagger = \bar{\psi}_{\dot{A}}$$

$$(\bar{\psi}_{\dot{A}})^\dagger = \psi_A$$

We define contractions of Weyl spinors as follows:

Definition: $\psi\chi \equiv \psi^A\chi_A$ and $\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{A}}\bar{\chi}^{\dot{A}}$.

These quantities are invariant under $SL(2,\mathbb{C})$. With this in hand we see that

$$\psi^2 \equiv \psi\psi = \psi^A\psi_A = \epsilon^{AB}\psi_B\psi_A = \epsilon^{12}\psi_2\psi_1 + \epsilon^{21}\psi_1\psi_2 = \psi_2\psi_1 - \psi_1\psi_2.$$

This quantity is zero if the Weyl spinors commute. In order to avoid this we make the following assumption which is consistent with how we treat fermions (and Dirac spinors):

Postulate: All Weyl spinors anticommute:[a] $\{\psi_A, \psi_B\} = \{\bar{\psi}_{\dot{A}}, \bar{\psi}_{\dot{B}}\} = \{\psi_A, \bar{\psi}_{\dot{B}}\} = \{\bar{\psi}_{\dot{A}}, \psi_B\} = 0$.

[a]This means that Weyl spinors are so-called **Grassmann numbers**.

This means that

$$\psi^2 \equiv \psi\psi = \psi^A\psi_A = -2\psi_1\psi_2.$$

Weyl spinors can be related to Dirac spinors ψ_a as well:[13]

$$\psi_a = \begin{pmatrix} \psi_A \\ \bar{\chi}^{\dot{A}} \end{pmatrix}.$$

We see that in order to describe a Dirac spinor we need both handedness of Weyl spinor. For Majorana spinors we have:

$$\psi_a = \begin{pmatrix} \psi_A \\ \bar{\psi}^{\dot{A}} \end{pmatrix}.$$

[12]This is a bit daft, as $\bar{\sigma^0}^{\dot{A}A} = \delta_{\dot{A}A}$, and we will in the following omit the matrix and write $(\psi_A)^* = \bar{\psi}^{\dot{A}}$.

[13]Note that in general $(\psi_A)^* \neq \bar{\chi}^{\dot{A}}$.

We can now write the super-Poincaré algebra (superalgebra) in terms of Weyl spinors. With

$$Q_a = \begin{pmatrix} Q_A \\ \bar{Q}^{\dot{A}} \end{pmatrix}, \tag{3.18}$$

for the Majorana spinor charges, we have

$$\begin{aligned}
\{Q_A, Q_B\} &= \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 0, && (3.19)\\
\{Q_A, \bar{Q}_{\dot{B}}\} &= 2\sigma^{\mu}_{A\dot{B}} P_\mu, && (3.20)\\
[Q_A, P_\mu] &= [\bar{Q}_{\dot{A}}, P_\mu] = 0, && (3.21)\\
[Q_A, M^{\mu\nu}] &= \sigma^{\mu\nu}{}_A{}^B Q_B, && (3.22)
\end{aligned}$$

where now $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$.

3.6 The Casimir operators of the super-Poincaré algebra

It is easy to see that P^2 is still a Casimir operator of the superalgebra. From Eq. (3.21) P_μ commutes with the Qs, so in turn P^2 must commute.[14] However, W^2 is not a Casimir because of the following result:[15]

$$[W^2, Q_a] = W_\mu (\not{P} \gamma_\mu \gamma^5 Q)_a + \frac{3}{4} P^2 Q_a.$$

We want to find an extension of W that commutes with the Qs while retaining the commutators we alread have. The construction

$$C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu,$$

where

$$B_\mu \equiv W_\mu + \frac{1}{4} X_\mu,$$

and with

$$X_\mu \equiv \frac{1}{2} \bar{Q} \gamma_\mu \gamma^5 Q,$$

has the required relation:

$$[C_{\mu\nu}, Q_a] = 0.$$

We can show that C^2 indeed commutes with all the generators in the algebra:

$$\begin{aligned}
[C^2, Q_a] &= 0, && \text{(trivial)}\\
[C^2, P_\mu] &= 0, && \text{(excessive algebra)}\\
[C^2, M_{\mu\nu}] &= 0. && \text{(because } C^2 \text{ is a Lorentz scalar)}
\end{aligned}$$

Thus C^2 is a Casimir operator for the superalgebra.

[14] Although the fact that Eq. (3.21) holds crucially depends on Q_a being four-dimensional. P_μ and Q_a would not commute if there had been five Qs.

[15] Which, by the way, is really hard work!

3.7 Representations of the superalgebra

What sort of particles transform under the super-Poincaré group? Or, in other words, what are the irreducible representations of the group? Let us again assume without loss of generality that we are in the rest frame, *i.e.* $P_\mu = (m, \vec{0})$.[16] As was the case for the original Poincaré group, states are labeled by m, where m^2 is the eigenvalue of P^2. For C^2 we have to do a bit of calculation:

$$\begin{aligned} C^2 &= 2B_\mu P_\nu B^\mu P^\nu - 2B_\mu P_\nu B^\nu P^\mu \\ &\overset{RF}{=} 2m^2 B_\mu B^\mu - 2m^2 B_0^2 \\ &= 2m^2 B_k B^k, \end{aligned}$$

and from the definition of B_μ we get:

$$\begin{aligned} B_k &= W_k + \frac{1}{4} X_k \\ &= mS_k + \frac{1}{8}\bar{Q}\gamma_k\gamma^5 Q \equiv mJ_k. \end{aligned}$$

The operator we just defined, $J_k \equiv \frac{1}{m} B_k$, is an abstraction of the ordinary spin operator, and fulfills the angular momentum algebra (just like the spin operator):

$$[J_i, J_j] = i\epsilon_{ijk} J_k.$$

and has $[J_k, Q_a] = 0$.[17] This gives us

$$C^2 = 2m^4 J_k J^k,$$

such that:

$$C^2 |m, j, j_3\rangle = -m^4 j(j+1)|m, j, j_3\rangle,$$

where $j = 0, \frac{1}{2}, 1\ldots$ and $j_3 = -j, -j+1, \ldots, j-1, j$, because J_k fulfils the angular momentum algebra.[18] So, the irreducible representations of the superalgebra can be labeled by (m, j), and any given set of m and j will give us $2j+1$ states with different j_3.[19]

In the following we will construct all the states for a given representation labeled by the set (m, j). To do this it is very usefull to write the generators Q in terms of two-component Weyl spinors instead of four-component Dirac spinors, making explicit use of their Majorana nature, as we did in Section 3.5. We note that from the above discussion

$$[J_k, Q_A] = [J_k, \bar{Q}_{\dot{B}}] = 0.$$

We begin by claiming that for any state with a given value of j_3 there must then exist a state (possibly the same) $|\Omega\rangle$ that has the same value of j_3 and for which

$$Q_A|\Omega\rangle = 0. \tag{3.23}$$

[16] We can carry out a similar argument in a different frame for massless particles.

[17] Again the proof is algebraically extensive, and again I suggest the interested reader to pursue [2].

[18] The interested reader can check that the proof seen in any quantum mechanics course using ladder operators for spin holds also for J since it does not depend on any properties but the algebra.

[19] Make sure you remember that that j is not the spin, but a generalization of spin. J_3 is not a Casimir, so strictly speaking j_3 does not label the irrep, rather, for given values of m and j the irrep has $2j+1$ independent states.

This state is called the **Clifford vacuum**.[20]

To show this, start with $|\beta\rangle$, a state with j_3. Then the construction

$$|\Omega\rangle = Q_1 Q_2 |\beta\rangle,$$

has these properties. First we show that (3.23) holds:

$$Q_1 Q_1 Q_2 |\beta\rangle = -Q_1 Q_1 Q_2 |\beta\rangle = 0,$$

and

$$Q_2 Q_1 Q_2 |\beta\rangle = -Q_1 Q_2 Q_2 |\beta\rangle = Q_1 Q_2 Q_2 |\beta\rangle = -Q_2 Q_1 Q_2 |\beta\rangle = 0.$$

For this state we then have:

$$\begin{aligned} J_3|\Omega\rangle &= J_3 Q_1 Q_2 |\beta\rangle \\ &= Q_1 Q_2 J_3 |\beta\rangle = j_3 |\Omega\rangle, \end{aligned}$$

in other words, $|\Omega\rangle$ has the same value for j_3 as the $|\beta\rangle$ it was constructed from and the Clifford vacuum exists.

We can now use the explicit expression for J_k

$$J_k = S_k - \frac{1}{4m} \bar{Q}_{\dot{B}} \bar{\sigma}_k^{\dot{B}A} Q_A,$$

in order to find the spin for this state:

$$J_k|\Omega\rangle = S_k|\Omega\rangle = j_k|\Omega\rangle,$$

meaning that $s_3 = j_3$ and $s = j$ are the eigenvalues of S_3 and S^2 for the Clifford vacuum $|\Omega\rangle$.

We can construct three more states from the Clifford vacuum:[21]

$$\bar{Q}^{\dot{1}}|\Omega\rangle, \quad \bar{Q}^{\dot{2}}|\Omega\rangle, \quad \bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle.$$

This means that there are four possible states that can be constructed out of any state with the quantum numbers m, j, j_3. Taking a look at:

$$J_k \bar{Q}^{\dot{A}}|\Omega\rangle = \bar{Q}^{\dot{A}} J_k|\Omega\rangle = j_k \bar{Q}^{\dot{A}}|\Omega\rangle,$$

this means that all these states have the same j_3 (and j) quantum numbers.[22] From the superalgebra (3.22) we have:

$$[M^{ij}, \bar{Q}^{\dot{A}}] = -(\sigma^{ij})^{\dot{A}}{}_{\dot{B}} \bar{Q}^{\dot{B}},$$

so that:

$$S_3 \bar{Q}^{\dot{A}}|\Omega\rangle = \bar{Q}^{\dot{A}} S_3|\Omega\rangle - \frac{1}{2}(\bar{\sigma}_3 \sigma^0)^{\dot{A}}{}_{\dot{B}} \bar{Q}^{\dot{B}}|\Omega\rangle = \left(j_3 \mp \frac{1}{2}\right) \bar{Q}^{\dot{A}}|\Omega\rangle,$$

[20] It is called the Clifford vacuum because the operators satisfy a Clifford algebra $\{Q_A, \bar{Q}_{\dot{B}}\} = 2m\sigma^0_{A\dot{B}}$. Do not confuse this with a vacuum state, it is only a name.

[21] All other possible combinations of Qs and $|\Omega\rangle$ give either one of the other four states, or the zero state which is trivial and of no interest.

[22] The same can easily be shown for $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$.

where $-$ is for $\dot{A}=\dot{1}$ and $+$ is for $\dot{A}=\dot{2}$. We can similarly show that

$$S_3\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle = j_3\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle.$$

This means that each set of quantum numbers m, j, j_3 gives 2 states with $s_3=j_3$, and two with $s_3=j_3\pm\frac{1}{2}$, giving two bosonic and two fermionic states, with the same mass.

The above explains the much repeated statement that any supersymmetry theory has an equal number of bosons and fermions, which, incidentally, is not true.

Theorem: For any representation of the superalgebra where P_μ is a one-to-one operator there is an equal number of boson and fermion states.

To show this, divide the representation into two sets of states, one with bosons and one with fermions. Let $\{Q_A,\bar{Q}_{\dot{B}}\}$ act on the members of the set of bosons. $\bar{Q}_{\dot{B}}$ transforms bosons to fermions and Q_A does the reverse mapping. If P_μ is one-to-one, then so is $\{Q_A,\bar{Q}_{\dot{B}}\} = 2\sigma^\mu{}_{A\dot{B}}P_\mu$. Thus there must be an equal number in both sets.[23]

3.7.1 Examples of irreducible representations

Finally, let us briefly look at two examples of irreducible representations for a fixed non-zero m.

$j=0$

For $j=0$, we must have $j_3=0$ and as a result the Clifford vacuum $|\Omega\rangle$ has $s=0$ and is a bosonic state. There are two states $\bar{Q}^{\dot{A}}|\Omega\rangle$ with $s=\frac{1}{2}$ and $s_3=\mp\frac{1}{2}$ and one state $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$ with $s=0$ and $s_3=0$. In total there are two scalar states and two spin-$\frac{1}{2}$ fermion states. We will later represent this set of states by the so-called **scalar superfield**.

We should use be carefull about using the term particle about these states since what we have found are in fact Weyl spinor states. A real Dirac fermion can only be described by a $j=0$ representation together with a different complex conjugate representation, thus consisting of four states, or four degrees of freedom (d.o.f.). In field theory, when the fermion is on-shell, two of these states are eliminated by the Dirac equation, thus we get the expected two d.o.f. for a spin-$\frac{1}{2}$ fermion. The situation for the scalars is the same, from the total four scalar d.o.f., two are eliminated by the equations of motion, resulting in two scalar particles. The complex conjugate representation of the first representation together with the self-representation of the second then form the anti-particle of the fermion, and provide an additional two scalars. So the particle count from the two irreducible representations is a fermion–anti-fermion pair, and four scalars. Note that all of the resulting particles have the same mass m.

$j=\frac{1}{2}$

For $j=\frac{1}{2}$ we have two Clifford vacua $|\Omega\rangle$ with $j_3=\pm\frac{1}{2}$, and with $s=\frac{1}{2}$ and $s_3=\pm\frac{1}{2}$ (thus they are fermionic states). For the moment we label them as $|\Omega;\frac{1}{2}\rangle$ and $|\Omega;-\frac{1}{2}\rangle$. From each of these we can construct two further fermion states $\bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega;\pm\frac{1}{2}\rangle$ with $s_3=\mp\frac{1}{2}$. In addition to

[23] Observe that this tells us that there must be an equal number of states in both sets, not particles.

this we have the states $\bar{Q}^{\dot{1}}|\Omega; \frac{1}{2}\rangle$ and $\bar{Q}^{\dot{2}}|\Omega; -\frac{1}{2}\rangle$ with $s_3 = 0$, the state $\bar{Q}^{\dot{2}}|\Omega; \frac{1}{2}\rangle$ with $s_3 = 1$, and the state $\bar{Q}^{\dot{1}}|\Omega; -\frac{1}{2}\rangle$ has $s_3 = -1$. Together these states can form two fermions with $s = \frac{1}{2}$ and $s_3 = \pm\frac{1}{2}$, one massive vector particle with $s = 1$, and $s_3 = 1, 0, -1$, and one scalar with $s = 0$.[24] We will later refer to this set of states as the **vector superfield**.

3.8 Exercises

Exercise 3.1 Show that

$$\begin{aligned}\left[P_\mu, P^2\right] &= 0, \qquad (3.24)\\ \left[M_{\mu\nu}, P^2\right] &= 0. \qquad (3.25)\end{aligned}$$

Exercise 3.2 Using:[25]

$$W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M^{\rho\sigma}M_{\nu\sigma}P_\rho P^\nu,$$

show that

$$\begin{aligned}\left[M_{\mu\nu}, W^2\right] &= 0, \qquad (3.26)\\ \left[P_\mu, W^2\right] &= 0. \qquad (3.27)\end{aligned}$$

Exercise 3.3 Show that $L_+^\uparrow$ and $SL(2,\mathbb{C})$ are indeed **homomorphic**, *i.e.* that the mapping defined by (3.16) or (3.17) has the property that $\Lambda(M_1 M_2) = \Lambda(M_1)\Lambda(M_2)$ or $M(\Lambda_1\Lambda_2) = M(\Lambda_1)M(\Lambda_2)$.

Exercise 3.4 Show that the generalization of the spin operator, $J_k \equiv S_k + \frac{1}{8m}\bar{Q}\gamma_\mu\gamma^5 Q$, fulfils the algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$

Exercise 3.5 What are the states for $j = 1$?

[24] For massless particles, $m = 0$, we can form a vector particle with $s_3 = \pm 1$ and one extra scalar.

[25] This is non-trivial to demonstrate, see Chapter 1.2 of [2].

Chapter 4

Superspace

In this chapter we will introduce a very handy notation system for considering supersymmetry transformations effected by the Q elements of the superalgebra, or, more correctly, the elements of the super-Poincaré group and their representations. This notation is called superspace, and allows us to define so-called superfields as a replacement of ordinary field theory fields. This mirrors the Lorentz invariance built into relativistic field theory by using four-vectors. In order to do this we first need to know a little about the properties of Grassman numbers.

4.1 Superspace calculus

Grassman numbers θ are numbers that anti-commute with each other but not with ordinary numbers. We will here use four such numbers and in addition we want to place them in Weyl spinors, indexed by A and $\dot{A}$:[1]

$$\{\theta^A, \theta^B\} = \{\theta^A, \bar{\theta}^{\dot{B}}\} = \{\bar{\theta}^{\dot{A}}, \theta^B\} = \{\bar{\theta}^{\dot{A}}, \bar{\theta}^{\dot{B}}\} = 0.$$

From this we get the relationships:[2]

$$\begin{aligned} \theta_A^2 &= \theta_A\theta_A = -\theta_A\theta_A = 0, & (4.1)\\ \theta^2 &\equiv \theta\theta \equiv \theta^A\theta_A = -2\theta_1\theta_2, & (4.2)\\ \bar{\theta}^2 &\equiv \bar{\theta}\bar{\theta} \equiv \bar{\theta}_{\dot{A}}\bar{\theta}^{\dot{A}} = 2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}}. & (4.3)\end{aligned}$$

Notice that if we have a function f of a Grassman number, say θ_A, then the all-order expansion of that function in terms of θ_A, is

$$f(\theta_A) = a_0 + a_1\theta_A, \tag{4.4}$$

as there are simply no more terms because of (4.1).

We now need to define differentiation and integration on these numbers in order to create a calculus for them.

[1] We can already see how this can be handy: if we consistently use $\theta^A Q_A$ and $\bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}}$ instead of only Q_A and $\bar{Q}^{\dot{A}}$ in Eqs. (3.19)–(3.22) we can actually rewrite the superalgebra as an ordinary Lie algebra, but with Grassman elements, because of these commutation properties.

[2] There is no summation implied in the first line. These are of course the same relations we already used for the Weyl spinors.

Definition: We define differentiation by:[a]

$$\partial_A \theta^B \equiv \frac{\partial}{\partial \theta^A} \theta^B \equiv \delta_A{}^B,$$

with a product rule

$$\begin{aligned}
\partial_A(\theta^{B_1}\theta^{B_2}\theta^{B_3}\ldots\theta^{B_n}) \quad \equiv \quad & (\partial_A\theta^{B_1})\theta^{B_2}\theta^{B_3}\ldots\theta^{B_n} \\
& -\theta^{B_1}(\partial_A\theta^{B_2})\theta^{B_3}\ldots\theta^{B_n} \\
& + \ldots + (-1)^{n-1}\theta^{B_1}\theta^{B_2}\ldots(\partial_A\theta^{B_n}). \qquad (4.5)
\end{aligned}$$

[a]Note that this has no infinitesimal interpretation.

Definition: We define integration by $\int d\theta_A \equiv 0$ and $\int d\theta_A \theta_A \equiv 1$ and we demand linearety:

$$\int d\theta_A [af(\theta_A) + bg(\theta_A)] \equiv a \int d\theta_A f(\theta_A) + b \int d\theta_A g(\theta_A).$$

This has one surprising property. If we take the integral of (4.4) we get:

$$\int d\theta_A f(\theta_A) = a_1 = \partial^A f(\theta_A),$$

meaning that differentiation and integration has the same effect on Grassman numbers.

To integrate over multiple Grassman numbers we define volume elements as

Definition:

$$d^2\theta \equiv -\frac{1}{4} d\theta^A d\theta_A,$$

$$d^2\bar{\theta} \equiv -\frac{1}{4} d\bar{\theta}_{\dot{A}} d\bar{\theta}^{\dot{A}},$$

$$d^4\theta \equiv d^2\theta d^2\bar{\theta}.$$

This definition is made so that

$$\int d^2\theta \; \theta\theta = 1,$$

$$\int d^2\bar{\theta} \; \bar{\theta}\bar{\theta} = 1,$$

$$\int d^4\theta \; (\theta\theta)(\bar{\theta}\bar{\theta}) = 1.$$

Delta functions of Grassmann variables are given by:

$$\delta(\theta_A) = \theta_A,$$

$$\delta^2(\theta_A) = \theta\theta,$$

$$\delta^2(\bar{\theta}^{\dot{A}}) = \bar{\theta}\bar{\theta},$$

and these functions satisfy, just as the usual definition of delta functions:

$$\int d\theta_A f(\theta_A)\delta(\theta_A) = f(0).$$

4.2 Superspace definition

Superspace[3] is a coordinate system where supersymmetry transformations are manifest, in other words, the action of elements in the super-Poincaré group (SP) based on the superalgebra are treated like Lorentz-transformations are in Minkowski space.

> **Definition**: **Superspace** is an eight-dimension manifold that can be constructed from the **coset space** of the super-Poincaré group (SP) and the Lorentz group (L), SP/L, by giving coordinates $z^\pi = (x^\mu, \theta^A, \bar{\theta}^{\dot{A}})$, where x^μ are the ordinary Minkowski coordinates, and where θ_A and $\bar{\theta}^{\dot{A}}$ are four Grassman (anti-commuting) numbers, being the parameters of the Q-operators in the algebra.

To see this we begin by writing a general element of SP, $g \in SP$, as[4]

$$g = \exp[-ix^\mu P_\mu + i\theta^A Q_A + i\bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}} - \frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}],$$

where x^μ, θ^A, $\bar{\theta}_{\dot{A}}$ and $\omega_{\rho\nu}$ constitute the parametrisation of the group, and P_μ, Q_A, $\bar{Q}^{\dot{A}}$ and $M_{\rho\nu}$ are the generators. We can now parametrise SP/L simply by setting $\omega_{\mu\nu} = 0$.[5] The remaining parameters of SP/L then span superspace.

As we are physicists we also want to know the dimensions of our new parameters. To do this we first look at Eq. (3.20):

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2\sigma^\mu{}_{A\dot{B}}P_\mu$$

we know that P_μ has mass dimension $[P_\mu] = M$. This means that $[Q^2] = M$ and $[Q] = M^{\frac{1}{2}}$. In the exponential, all terms must have mass dimension zero to make sense. This means that $[\theta Q] = 0$, and therefore $[\theta] = M^{-\frac{1}{2}}$.

In order to show the effect of supersymmetry transformations, we begin by noting that any SP transformation can effectively be written in the following way:

$$L(a, \alpha) = \exp[-ia^\mu P_\mu + i\alpha^A Q_A + i\bar{\alpha}^{\dot{A}}\bar{Q}_{\dot{A}}],$$

[3] Introduced by Salam & Strathdee [5].

[4] We have already discussed this way of reconstructing the group by an **exponential map** of the Lie algebra to the Lie group in Section 2.6. Technically this only provides a **local cover** of the group around small values of the parameters, but we shall not go into more details here.

[5] SP/L is in reality not a coset group as defined previously, because L is not a normal subgroup of SP, but its parametrisation still forms a vector space (the coset space) which we call superspace.

because one can show that[6]

$$\exp\left[-\frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right]L(a,\alpha) = L(\Lambda a, S(\Lambda)\alpha)\exp\left[-\frac{i}{2}\omega_{\rho\nu}M^{\rho\nu}\right], \tag{4.6}$$

i.e. all that a Lorentz boost does is to transform spacetime coordinates by $\Lambda(M)$ and Weyl spinors by $S(\Lambda(M))$, which is a spinor representation of $\Lambda(M)$. Thus, we can pick frames, do our thing with the transformation, and boost back to any frame we wanted. In addition, since P_μ commutes with all the Qs, when we speak of the supersymmetry transformation we usually mean just the transformation

$$\delta_S = \alpha^A Q_A + \bar{\alpha}_{\dot{A}}\bar{Q}^{\dot{A}}. \tag{4.7}$$

We can now find the transformation of superspace coordinates under a supersymmetry transformation, just as we have all seen the transformation of Minkowski coordinates under Lorentz transformations. The effect of $g_0 = L(a,\alpha)$ on a superspace coordinate $z^\pi = (x^\mu, \theta^A, \bar{\theta}_{\dot{A}})$ is defined by the mapping $z^\pi \to z'^\pi$ given by $g_0 e^{iz^\pi K_\pi} = e^{iz'^\pi K_\pi}$ where $K_\pi = (P_\mu, Q_A, \bar{Q}^{\dot{A}})$. We have[7]

$$\begin{aligned} g_0 e^{iz^\pi K_\pi} &= \exp(-ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}})\exp(iz^\pi K_\pi) \\ &= \exp(-ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}} + iz^\pi K_\pi \\ &\quad -\frac{1}{2}[-ia^\nu P_\nu + i\alpha^B Q_B + i\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}}, iz^\pi K_\pi] + \ldots) \end{aligned}$$

Here we take a closer look at the commutator:[8]

$$\begin{aligned} [\,,\,] &= [\alpha^B Q_B, \bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}}] + [\bar{\alpha}_{\dot{B}}\bar{Q}^{\dot{B}}, \theta^A Q_A] \\ &= -\alpha^B\bar{\theta}_{\dot{A}}\epsilon^{\dot{A}\dot{C}}\{Q_B, \bar{Q}_{\dot{C}}\} - \bar{\alpha}_{\dot{B}}\theta^A\epsilon^{\dot{B}\dot{C}}\{\bar{Q}_{\dot{C}}, Q_A\} \\ &= -2\alpha^B\bar{\theta}_{\dot{A}}\epsilon^{\dot{A}\dot{C}}\sigma^\mu{}_{B\dot{C}}P_\mu - \bar{\alpha}_{\dot{B}}\theta^A\epsilon^{\dot{B}\dot{C}}\sigma^\mu{}_{A\dot{C}}P_\mu \\ &= (-2\alpha^B\bar{\theta}^{\dot{C}}\sigma^\mu{}_{B\dot{C}} - 2\bar{\alpha}^{\dot{C}}\theta^A\sigma^\mu{}_{A\dot{C}})P_\mu \end{aligned}$$

We can relabel $B = A$ and $\dot{C} = \dot{A}$ which leads to

$$-\frac{1}{2}[\,,\,] = (\alpha^A\sigma^\mu{}_{A\dot{A}}\bar{\theta}^{\dot{A}} - \theta^A\sigma^\mu{}_{A\dot{A}}\bar{\alpha}^{\dot{A}})P_\mu.$$

The commutator is proportional with P_μ, and will therefore commute with all operators, in particular the higher terms in the Campbell-Baker-Hausdorff expansion, meaning that the series reduces to

$$\begin{aligned} &g_0 e^{iZ^\pi K_\pi} \\ = \; &\exp[i(-x^\mu - a^\mu + i\alpha^A\sigma^\mu{}_{A\dot{A}}\bar{\theta}^{\dot{A}} - i\theta^A\sigma^\mu{}_{A\dot{A}}\bar{\alpha}^{\dot{A}})P_\mu + i(\theta^A + \alpha^A)Q_A + i(\bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}})\bar{Q}^{\dot{A}}]. \end{aligned}$$

[6]Fortunately we are not going to do this because it is messy, but it can be done using the algebra of the group and the series expansion of the exponential function. Note, however, that the proof rests on the Ps and Qs forming a closed set, which we saw in the algebra Eqs. (3.19)–(3.22).

[7]Here we use Campbell-Baker-Hausdorff expansion $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}-\frac{1}{2}[\hat{A},\hat{B}]+\cdots}$ where the next term contains commutators of the first commutator and the operators $\hat{A}$ and $\hat{B}$.

[8]Using that P_μ commutes with all elements in the algebra, as well as $[\theta^A Q_A, \xi^B Q_B] = \theta^A\xi^B\{Q_A, Q_B\} = 0$, and the same for $\bar{Q}^{\dot{B}}$.

So superspace coordinates transform under supersymmetry transformations as:

$$(x^\mu, \theta^A, \bar{\theta}_{\dot{A}}) \to f(a^\mu, \alpha^A, \bar{\alpha}_{\dot{A}}) = (x^\mu + a^\mu - i\alpha^A \sigma^\mu{}_{A\dot{A}} \bar{\theta}^{\dot{A}} + i\theta^A \sigma^\mu{}_{A\dot{A}} \bar{\alpha}^{\dot{A}}, \theta^A + \alpha^A, \bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}}). \quad (4.8)$$

As a by-product we can now write down a differential representation for the supersymmetry generators by applying the standard expression for the generators X_i of a Lie algebra, given the functions f_π for the transformation of the parameters:

$$X_j = \frac{\partial f_\pi}{\partial a_j} \frac{\partial}{\partial z_\pi}$$

which gives us:[9]

$$\begin{aligned} P_\mu &= i\partial_\mu & (4.9) \\ iQ_A &= -i(\sigma^\mu \bar{\theta})_A \partial_\mu + \partial_A & (4.10) \\ i\bar{Q}^{\dot{A}} &= -i(\bar{\sigma}^\mu \theta)^{\dot{A}} \partial_\mu + \partial^{\dot{A}} & (4.11) \end{aligned}$$

4.3 Covariant derivatives

Similar to the properties of covariant derivatives for gauge transformations in gauge theories, it would be nice to have a derivative that is invariant under supersymmetry transformations, *i.e.* commutes with supersymmetry operators. Obviously $P_\mu = i\partial_\mu$ does this, but more general covariant derivatives can be made.

Definition: The following covariant derivatives commute with supersymmetry transformations:

$$\begin{aligned} D_A &\equiv \partial_A + i(\sigma^\mu \bar{\theta})_A \partial_\mu, & (4.12) \\ \bar{D}_{\dot{A}} &\equiv -\partial_{\dot{A}} - i(\theta\sigma^\mu)_{\dot{A}} \partial_\mu. & (4.13) \end{aligned}$$

These can be shown to satisfy relations that are useful in calculations:

$$\begin{aligned} \{D_A, D_B\} &= \{\bar{D}_{\dot{A}}, \bar{D}_{\dot{B}}\} = 0 & (4.14) \\ \{D_A, \bar{D}_{\dot{B}}\} &= -2\sigma^\mu_{A\dot{B}} P_\mu & (4.15) \\ D^3 = \bar{D}^3 &= 0 & (4.16) \\ D^A \bar{D}^2 D_A &= \bar{D}_{\dot{A}} D^2 \bar{D}^{\dot{A}} & (4.17) \end{aligned}$$

4.4 Superfields

Using the superspace coordinates we can now define functions of these. Naturally we should call these superfields.

Definition: A **superfield** Φ is an operator valued function on superspace $\Phi(x, \theta, \bar{\theta})$.

[9] We define the generators X_i as $-iP_\mu$, iQ_A and iQ_B respectively.

We can expand any Φ in a power series in θ and $\bar{\theta}$. In general:[10]

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= f(x)+\theta^A\varphi_A(x)+\bar{\theta}_{\dot{A}}\bar{\chi}^{\dot{A}}(x)+\theta\theta m(x)+\bar{\theta}\bar{\theta}n(x)\\ &\quad +\theta\sigma^\mu\bar{\theta}V_\mu(x)+\theta\theta\bar{\theta}_{\dot{A}}\bar{\lambda}^{\dot{A}}(x)+\bar{\theta}\bar{\theta}\theta^A\psi_A(x)+\theta\theta\bar{\theta}\bar{\theta}d(x).\end{aligned} \tag{4.18}$$

The properties of the component fields of a superfield can be deduced from the requirement that Φ must be a Lorentz scalar or pseudoscalar. This is shown in Table 4.1

Component field	Type	d.o.f.
$f(x)$, $m(x)$, $n(x)$	Complex (pseudo) scalar	2
$\psi_A(x)$, $\varphi_A(x)$	Left-handed Weyl spinors	4
$\bar{\chi}^{\dot{A}}(x)$, $\bar{\lambda}^{\dot{A}}(x)$	Right-handed Weyl spinors	4
$V_\mu(x)$	Lorentz 4-vector	8
$d(x)$	Complex scalar	2

Table 4.1: Field content of a general superfield.

One can show (tedious) that under supersymmetry transformations these component fields transform linearly into each other, thus superfields are **representations** of the supersymmetry (super-Poincaré) algebra, albeit *highly reducible* representations![11] We can recover the known irreducible representations, see Section 3.7, by some rather *ad hoc* restrictions on the fields:[12]

$$\bar{D}_{\dot{A}}\Phi(x,\theta,\bar{\theta}) = 0 \quad \text{(left-handed scalar superfield)} \tag{4.19}$$

$$D_A\Phi^\dagger(x,\theta,\bar{\theta}) = 0 \quad \text{(right-handed scalar superfield)} \tag{4.20}$$

$$\Phi^\dagger(x,\theta,\bar{\theta}) = \Phi(x,\theta,\bar{\theta}) \quad \text{(vector superfield)} \tag{4.21}$$

Products of same-handed superfields are also superfields with the same handedness since

$$\bar{D}_{\dot{A}}(\Phi_i\Phi_j) = (\bar{D}_{\dot{A}}\Phi_i)\Phi_j + \Phi_i(\bar{D}_{\dot{A}}\Phi_j) = 0.$$

This is important when creating a **superpotential**, the supersymmetric precursor to a full Lagrangian.[13]

4.4.1 Scalar superfields

What is the connection of the scalar superfields to the $j=0$ irreducible representation? We use a cute[14] trick: Change to the variable $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$. Then:

$$D_A = \partial_A + 2i\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}}\frac{\partial}{\partial y^\mu}, \tag{4.22}$$

$$\bar{D}_{\dot{A}} = -\partial_{\dot{A}}. \tag{4.23}$$

[10] Note that any superfield commutes with any other superfield, because all Grassmann numbers appear in pairs. Equation (4.18) can be shown to be closed under supersymmetry transformations, meaning that a superfield transforms into another superfield under the transformations of the previous section.

[11] Indeed, they are linear representations since a sum of superfields is a superfield, and the differential supersymmetry operators act linearly.

[12] Note that it is $\Phi^\dagger$ which is the right handed superfield in Eq. (4.20), not Φ.

[13] Supersymmetry transformations can also be shown to transform left-handed superfields into left-handed superfields and right-handed superfields into right-handed superfields.

[14] Here cute is used in the widest sense.

This means that a field fulfilling $\bar{D}_{\dot{A}}\Phi = 0$ in the new set of coordinates must be independent of $\bar{\theta}$. Thus we can write:

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y),$$

and looking at the field content we get the result in Table 4.2.

Component field	Type	d.o.f.
$A(x)$, $F(x)$	Complex scalar	2
$\psi(x)$	Left-handed Weyl spinors	4

Table 4.2: Fields contained in a left-handed scalar superfield.

We can undo the coordinate change and get:[15]

$$\Phi(x,\theta,\bar{\theta}) = A(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu A(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x).$$

By doing the transformation $y^\mu \equiv x^\mu - i\theta\sigma^\mu\bar{\theta}$ we can show a similar field content for the right handed scalar superfield. The general form of a right handed scalar superfield is then:

$$\Phi^\dagger(x,\theta,\bar{\theta}) = A^*(x) - i(\theta\sigma^\mu\bar{\theta})\partial_\mu A^*(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A^*(x) + \sqrt{2}\bar{\theta}\bar{\Psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\Psi}(x) + \bar{\theta}\bar{\theta}F^*(x).$$

Compare the above to the $j = 0$ representation with two scalar states and two fermionic states (d.o.f.). After applying the equations of motions (e.o.m.) the (**auxillary**) field $F(x)$ can be eliminated as it does not have any derivatives. The e.o.m. also eliminates two of the fermion d.o.f. Thus we are left with the same states as in the $j = 0$ representation.

However, the scalar superfields will not correspond directly to particle states for the known SM particles since a Weyl spinor on its own cannot describe a Dirac fermion. When we construct particle representations we will take one left-handed scalar superfield and one different right-handed scalar superfield. These will form a fermion and two scalars (and their antiparticles) after application of the e.o.m. We see from (4.19) and (4.20) that if Φ is left handed, then $\Phi^\dagger$ is right handed and vice versa with the dagger signifying hermitian conjugation.

4.4.2 Vector superfields

We take the general superfield and compare Φ and $\Phi^\dagger$. We see that the following is the structure of a general vector superfield:

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= f(x) + \theta\varphi(x) + \bar{\theta}\bar{\varphi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}m^*(x) \\ &+\theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}d(x).\end{aligned}$$

and looking at the component fields we find the results in Table 4.3.

Component field	Type	d.o.f.
$f(x)$, $d(x)$	Real scalar field	1
$\varphi(x)$, $\lambda(x)$	Weyl spinors	4
$m(x)$	Complex scalar field	2
$V_\mu(x)$	Real Lorentz 4-vector	4

Table 4.3: Field content of a general vector superfield.

[15] Just by expanding the above in powers of θ and $\bar{\theta}$.

One example of a vector superfield is the product $V = \Phi^\dagger\Phi$ where we easily see that $V^\dagger = (\Phi^\dagger\Phi)^\dagger = \Phi^\dagger(\Phi^\dagger)^\dagger = \Phi^\dagger\Phi$. Note that sums and products of vector superfields are also vector superfields:

$$(V_i + V_j)^\dagger = V_i^\dagger + V_j^\dagger = V_i + V_j,$$

and

$$(V_iV_j)^\dagger = V_j^\dagger V_i^\dagger = V_iV_j.$$

You may now be a little suspicious that this vector superfield does not correspond to the promised degrees of freedom in the $j = \frac{1}{2}$ representation of the superalgebra. Gauge-freedom comes to the rescue.

4.5 Supergauge

We begin with the definition of an abelian (super) gauge transformation on a vector superfield[16]

> **Definition**: Given a vector superfield $V(x,\theta,\bar\theta)$, we define the **abelian supergauge transformation** as
>
> $$\begin{aligned} V(x,\theta,\bar\theta) \to V'(x,\theta,\bar\theta) &= V(x,\theta,\bar\theta) + \Phi(x,\theta,\bar\theta) + \Phi^\dagger(x,\theta,\bar\theta) \\ &\equiv V(x,\theta,\bar\theta) + i(\Lambda(x,\theta,\bar\theta) - \Lambda^\dagger(x,\theta,\bar\theta)) \end{aligned}$$
>
> where the parameter of the transformation Φ (or Λ) is a scalar superfield.

One can show that under supergauge transformations the vector superfield components transform as:

$$\begin{aligned} f(x) &\to f'(x) = f(x) + A(x) + A^*(x) && (4.24)\\ \varphi(x) &\to \varphi'(x) = \varphi(x) + \sqrt{2}\psi(x) && (4.25)\\ m(x) &\to m'(x) = m(x) + F(x) && (4.26)\\ V_\mu(x) &\to V'_\mu(x) = V_\mu(x) + i\partial_\mu(A(x) - A^*(x)) && (4.27)\\ \lambda(x) &\to \lambda'(x) = \lambda(x) && (4.28)\\ d(x) &\to d'(x) = d(x) && (4.29) \end{aligned}$$

Notice that from the above the standard field strength for a vector field, $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, is supergauge invariant. With the newfound freedom of gauge invariance we can choose component fields of Φ to eliminate some remaining reducibility.

> **Definition**: The **Wess-Zumiono (WZ) gauge** is a supergauge transformation of a vector superfield by a scalar superfield with
>
> $$\begin{aligned} \psi(x) &= -\frac{1}{\sqrt{2}}\varphi(x), && (4.30)\\ F(x) &= -m(x), && (4.31)\\ A(x) + A^*(x) &= -f(x). && (4.32) \end{aligned}$$

[16]And promise we will get back to the corresponding definition for a scalar superfield.

A vector superfield in the WZ gauge can be written:

$$V_{WZ}(x,\theta,\bar{\theta}) = (\theta\sigma^\mu\bar{\theta})[V_\mu(x) + i\partial_\mu(A(x) - A^*(x))] + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}d(x),$$

which, considered carefully, contains one real scalar field d.o.f., three gauge field d.o.f.[17] and four fermion d.o.f., corresponding to the representation $j = \frac{1}{2}$.[18]

Notice that the WZ gauge is particularly convenient for calculations because:

$$V_{WZ}^2 = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[V_\mu(x) + i\partial_\mu(A(x) - A^*(x))][V^\mu(x) + i\partial^\mu(A(x) - A^*(x))]$$

and

$$V_{WZ}^3 = 0,$$

so that

$$e^{V_{WZ}} = 1 + V_{WZ} + \frac{1}{2}V_{WZ}^2.$$

4.6 Exercises

Exercise 4.1 Check that Eqs. (4.9)–(4.11) fulfil the superalgebra in Eqs. (3.19)–(3.21).

Exercise 4.2 Show the vector superfield component field transformation properties, using the redefinitions:

$$\lambda(x) \to \lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\varphi}(x),$$

$$d(x) \to d(x) - \frac{1}{4}\Box f(x).$$

[17] Hang on, where did that last d.o.f. go from $V(x)$? We have a remaining gauge freedom in the choice of $A(x) - A^*(x)$, which is the ordinary gauge freedom of a $U(1)$ field theory. This can be used to eliminate one d.o.f. from the vector field.

[18] Note that supersymmetry transformations break this gauge.

Chapter 5

Construction of a low-energy supersymmetric Lagrangian

We would now like to construct a model that is invariant under supersymmetry transformation, much in the same way that the Standard Model Lagrangian is invariant under Poincaré transformations.

5.1 Supersymmetry invariant Lagrangians and actions

As should be well known the **action**

$$S \equiv \int_R d^4x \, \mathcal{L}, \tag{5.1}$$

is invariant under supersymmetry transformations if this transforms the Lagrangian by a total derivative term $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial^\mu f(x)$, where $f(x) \to 0$ on $S(R)$ (the surface of the integration region R). The question then becomes: how can we construct a Lagrangian from superfields with this property?

We can show that the highest order component fields in θ and $\bar{\theta}$ of a superfield always transform in this way, *e.g.* for the general superfield the highest order component field $d(x)$ transforms under the supersymmetry transformation

$$\delta_s = \alpha Q + \bar{\alpha}\bar{Q}, \tag{5.2}$$

where the constant α is the supersymmetry transformation parameter,[1] as

$$\delta_s d(x) = d'(x) - d(x) = \frac{i}{2}(\partial_\mu \psi(x)\sigma^\mu \bar{\alpha} - \partial_\mu \bar{\lambda}(x)\sigma^\mu \alpha), \tag{5.3}$$

For a scalar (chiral) superfield it is the F-field which has this property

$$\delta_s F(x) = -i\sqrt{2}\partial_\mu \psi(x)\sigma^\mu \bar{\alpha}. \tag{5.4}$$

These highest power component can be isolated by using the projection property of integration in Grassman calculus so that

$$S = \int_R d^4x \int d^4\theta \, \mathcal{L},$$

[1]Note that this is a **global** SUSY transformation. Replacing $\alpha \to \alpha(x)$ gives a **local** SUSY transformation, which, it turns out, leads to supergravity.

where $\mathcal{L}$ is a function of superfields, is guaranteed to be supersymmetry invariant. Note that this constitutes a redefinition of what we mean by $\mathcal{L}$, and one should be careful when counting the dimension of terms.[2]

We can now write down a generic form for the supersymmetry Lagrangian of scalar (chiral) superfields, where the indices indicate the highest power of θ in the term:

$$\mathcal{L} = \mathcal{L}_{\theta\theta\bar{\theta}\bar{\theta}} + \theta\theta\mathcal{L}_{\bar{\theta}\bar{\theta}} + \bar{\theta}\bar{\theta}\mathcal{L}_{\theta\theta}.$$

Here $\mathcal{L}_{\theta\theta}$ ($\mathcal{L}_{\bar{\theta}\bar{\theta}}$) is a function of left-handed (right-handed) scalar superfields where we project out the F-field — called the **superpotential** — while $\mathcal{L}_{\theta\theta\bar{\theta}\bar{\theta}}$ is a real valued function of the scalar superfields where we project out the d-field, called the **Kähler potential**.

The requirement of renormalizability puts further restrictions on the fields in $\mathcal{L}$. We can at most have three powers of scalar superfields, for details see *e.g.* Wess & Bagger [6]. Since the action must be real, the (almost) most general supersymmetry Lagrangian that can be written in terms of scalar superfields is:

$$\mathcal{L} = \Phi_i^\dagger\Phi_i + \bar{\theta}\bar{\theta}W[\Phi] + \theta\theta W[\Phi^\dagger].$$

Here the first term is called the **kinetic term**[3], and W is the symbol for the **superpotential** which is restricted to

$$W[\Phi] = g_i\Phi_i + m_{ij}\Phi_i\Phi_j + \lambda_{ijk}\Phi_i\Phi_j\Phi_k. \tag{5.5}$$

This means that to specify a supersymmetric Lagrangian we only need to specify the superpotential. Dimension counting for the couplings give $[g_i] = M^2$, $[m_{ij}] = M$ and $[\lambda_{ijk}] = 1$. Notice also that m_{ij} and λ_{ijk} are symmetric.

5.2 Abelian gauge theories

We would ultimately like to have a gauge theory like that of the SM, so we start with an abelian warm-up, by finally definig what we mean by an (abelian) supergauge transformation on a scalar superfield.

> **Definition**: The $U(1)$ **(super)gauge transformation** (local or global) on left handed scalar superfields is defined as:
>
> $$\Phi_i \to \Phi_i' = e^{-i\Lambda q_i}\Phi_i$$
>
> where q_i is the $U(1)$ charge of Φ_i and Λ, or $\Lambda(x)$, is the parameter of the gauge transformation.

For the definition to make sense Φ_i' must be a left-handed scalar superfield, thus

$$\bar{D}_{\dot{A}}\Phi_i' = 0,$$

[2]Looking at the mass dimensions we have, since $\int d\theta\,\theta = 1$ from superspace calculus (see Section 4.1), $[\theta] = M^{-1/2}$ which leads to $[\int d\theta] = M^{1/2}$. We then have $[\int d^4\theta] = M^2$. Since we must have $[\int d^4\theta\,\mathcal{L}] = M^4$ for the action to be dimensionless, we need $[\mathcal{L}] = M^2$.

[3]The constant in front can always be chosen to be one because we can rescale the whole Lagrangian. Notice that the kinetic terms are vector superfields.

and this requires:

$$\begin{aligned}\bar{D}_{\dot{A}}\Phi_i' &= \bar{D}_{\dot{A}}e^{-i\Lambda q_i}\Phi_i = e^{-i\Lambda q_i}\bar{D}_{\dot{A}}\Phi_i - iq_i(\bar{D}_{\dot{A}}\Lambda)e^{-i\Lambda q_i}\Phi_i \\ &= -iq_i(\bar{D}_{\dot{A}}\Lambda)\Phi_i' = 0.\end{aligned}$$

Thus we must have $\bar{D}_{\dot{A}}\Lambda = 0$, which by definition means that Λ itself is a left-handed superfield. This is of course completely equivalent for right-handed scalar fields.

We will of course now require not only a supersymmetry invariant Lagrangian, but also a gauge invariant Lagrangian. Let us first look at the transformation of the superpotential W under the gauge transformation:

$$W[\Phi] \to W[\Phi'] = g_i e^{-i\Lambda q_i}\Phi_i + m_{ij}e^{-i\Lambda(q_i+q_j)}\Phi_i\Phi_j + \lambda_{ijk}e^{-i\Lambda(q_i+q_j+q_k)}\Phi_i\Phi_j\Phi_k$$

For $W[\Phi] = W[\Phi']$ we must have:

$$g_i = 0 \text{ if } q_i \neq 0 \tag{5.6}$$

$$m_{ij} = 0 \text{ if } q_i + q_j \neq 0 \tag{5.7}$$

$$\lambda_{ijk} = 0 \text{ if } q_i + q_j + q_k \neq 0 \tag{5.8}$$

This puts great restrictions on the form of the superpotential and the charge assignments of the superfields (as in ordinary gauge theories). What then about the kinetic term?

$$\Phi_i^\dagger\Phi_i \to \Phi_i^\dagger e^{i\Lambda^\dagger q_i}e^{-i\Lambda q_i}\Phi_i = e^{i(\Lambda^\dagger-\Lambda)q_i}\Phi_i^\dagger\Phi_i.$$

As in ordinary gauge theories we can introduce a **gauge compensating vector (super)field** V with the appropriate gauge transformation to make the kinetic term invariant under supersymmetry transformations. We can write the kinetic term as $\Phi_i^\dagger e^{q_iV}\Phi_i$, which gives us:

$$\Phi_i^\dagger e^{q_iV}\Phi_i \to \Phi_i^\dagger e^{i\Lambda^\dagger q_i}e^{q_i(V+i\Lambda-i\Lambda^\dagger)}e^{-i\Lambda q_i}\Phi_i = \Phi_i^\dagger e^{q_iV}\Phi_i$$

This definition of gauge transformation can be shown to recover the SM **minimal coupling** for the component fields through the covariant derivative

$$D_\mu^i = \partial_\mu - \frac{i}{2}q_iV_\mu,$$

where V_μ is the vector component field of the vector superfield.

In case you were worried: we can use the WZ gauge to show that the new kinetic term $\Phi_i^\dagger e^{q_iV}\Phi_i$ has no term with dimension higher then four, and is thus renormalizable.

5.3 Non-Abelian gauge theories

How do we extend the above to deal with much more complicated non-abelian gauge theories? Let us take a group G with the Lie algebra of group generators t_a that fulfil

$$[t_a, t_b] = if_{ab}{}^c t_c, \tag{5.9}$$

where $f_{ab}{}^c$ are the structure constants. For an element g in the group G we want to write down a unitary[4] representation $U(g)$ that transforms a scalar superfield Ψ by $\Psi \to \Psi' = U(g)\Psi$.

[4]By unitary we mean, as usual, that $U^\dagger = U^{-1}$ so that $U^\dagger U = 1$.

With an exponential map we can write the representation as $U(g) = e^{i\lambda^a t_a}$, as you may perhaps have expected.[5,6] Thus, we simply copy the abelian structure (as in ordinary gauge theories), and transform superfields as

$$\Psi \to \Psi' = e^{-iq\Lambda^a t_a}\Psi,$$

where q is the charge of Ψ under G.[7] Again we can easily show that we must require that the Λ^a are left-handed scalar superfields for Ψ to transform to a left-handed scalar superfield.

For the superpotential to be invariant we must now have:

$$g_i = 0 \quad \text{if} \quad g_i U_{ir} \neq g_r \tag{5.10}$$
$$m_{ij} = 0 \quad \text{if} \quad m_{ij} U_{ir} U_{js} \neq m_{rs} \tag{5.11}$$
$$\lambda_{ijk} = 0 \quad \text{if} \quad \lambda_{ijk} U_{ir} U_{js} U_{kt} \neq \lambda_{rst} \tag{5.12}$$

where the indices on U are its matrix indices. We also want a similar construction for the kinetic terms as for abelian gauge theories, $\Psi^\dagger e^{qV^aT_a}\Psi$, to be invariant under non-abelian gauge transformations.[8] Now

$$\Psi^\dagger e^{qV^aT_a}\Psi \to \Psi'^\dagger e^{qV'^aT_a}\Psi' = \Psi^\dagger e^{iq\Lambda^{a\dagger}T_a} e^{qV'^aT_a} e^{-iq\Lambda^aT_a}\Psi,$$

so we have to require that the vector superfield V transforms as:[9]

$$e^{qV'^aT_a} = e^{-iq\Lambda^{a\dagger}T_a} e^{qV^aT_a} e^{iq\Lambda^aT_a}. \tag{5.13}$$

When we look at this as an infinitesimal transformation in Λ we can show that

$$V'^a = V^a + i(\Lambda^a - \Lambda^{a\dagger}) - \frac{1}{2} q f_{bc}{}^a V^b(\Lambda^{c\dagger} + \Lambda^c) + \mathcal{O}(\Lambda^2),$$

which reduces to the abelian definition for abelian groups. If we look at the component vector fields, V_μ^a, these transform just like in a standard non-abelian gauge theory:

$$V_\mu^a \to V_\mu'^a = V_\mu^a + i\partial_\mu(A^a - A^{a*}) - q f_{bc}{}^a V_\mu^b(A^c - A^{c*}),$$

in the adjoint representation of the gauge group.

The supergauge transformations of vector superfields can be written more efficiently in a representation independent way as

$$e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda},$$

and the inverse transformation is then given by

$$e^{-V'} = e^{-i\Lambda} e^{-V} e^{i\Lambda^\dagger},$$

where $\Lambda \equiv q\Lambda^a T_a$ and $V \equiv qV^aT_a$, such that $e^V e^{-V} = e^{V'}e^{-V'} = 1$.[10]

[5] Since we demanded a unitary representation the generators t_a must be hermitian.

[6] Of, course, you may ask, how do we even know that we can find a unitary representation for a particular Lie group? It turns out that this is alwyas true for a subset of Lie groups, called **compact Lie groups**. These are the Lie groups where the parameters vary over a closed interval.

[7] At this point can choose a representation different from the fundamental, reflected in a different choice for t_a. Since we are almost exclusively interested in groups defined by a matrix representation $U(g)$ will be a matrix with dimension fixed by the dimension chosen for the representation.

[8] We have chosen some specific representation T_a of the generators t_a of the Lie algebra (5.9).

[9] This is independent of our choice of representation for the gauge group for the supergauge transformation.

[10] Notice that despite the non-commutative nature of the matrices involved, the identity $e^Ae^{-A} = 1$ holds.

5.4 Supersymmetric field strength

There is one missing type of term for the supersymmetric Lagrangian, namely field strength terms, *e.g.* terms to describe the electromagetic field strength.

Definition: **Supersymmetric field strength** is defined by the spinor (matrix) scalar superfields given by

$$W_A \equiv -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_A e^V,$$

and

$$\bar{W}_{\dot{A}} \equiv -\frac{1}{4}DDe^{-V}\bar{D}_{\dot{A}}e^V,$$

where $V = V^a T_a$.

We can show that W_A is a left-handed superfield and that $\mathrm{Tr}[W^A W_A]$ (and $\mathrm{Tr}[\bar{W}_{\dot{A}}\bar{W}^{\dot{A}}]$) is supergauge invariant and potential terms in the supersymmetry Lagrangian. Firstly

$$\bar{D}_{\dot{A}}W_A = -\frac{1}{4}\bar{D}_{\dot{A}}\bar{D}\bar{D}e^{-V}D_A e^V = 0,$$

because from Eq. (4.16) $\bar{D}^3 = 0$. Under a supergaugetransformation we have:

$$\begin{aligned}
W_A \to W'_A \quad &= -\frac{1}{4}\bar{D}\bar{D}e^{-i\Lambda}e^{-V}e^{i\Lambda^\dagger}D_A e^{-i\Lambda^\dagger}e^V e^{i\Lambda} \\
(\bar{D}_{\dot{A}}\Lambda = 0) \quad &= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}e^{i\Lambda^\dagger}D_A e^{-i\Lambda^\dagger}e^V e^{i\Lambda} \\
(D_A\Lambda^\dagger = 0) \quad &= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}D_A e^V e^{i\Lambda} \\
&= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}[(D_A e^V)e^{i\Lambda} + e^V(D_A e^{i\Lambda})] \\
&= e^{-i\Lambda}W_A e^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}D_A e^{i\Lambda}. \qquad (5.14)
\end{aligned}$$

We are free to add zero to (5.14) in the form of $-\frac{1}{4}e^{-i\Lambda}\bar{D}D_A\bar{D}e^{i\Lambda} = 0$,[11] giving

$$\begin{aligned}
W'_A &= e^{-i\Lambda}W_A e^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\bar{D}\{\bar{D}, D_A\}e^{i\Lambda} \\
&= e^{-i\Lambda}W_A e^{i\Lambda} + \frac{1}{2}e^{-i\Lambda}\bar{D}_{\dot{A}}\sigma^\mu{}_{A\dot{B}}\epsilon^{\dot{A}\dot{B}}P_\mu e^{i\Lambda} \\
&= e^{-i\Lambda}W_A e^{i\Lambda},
\end{aligned}$$

where we have used Eq. (4.15) to replace the anti-commutator. This means that the trace is gauge invariant:

$$\begin{aligned}
\mathrm{Tr}[W'^A W'_A] &= \mathrm{Tr}[e^{-i\Lambda}W^A e^{i\Lambda}e^{-i\Lambda}W_A e^{i\Lambda}] \\
&= \mathrm{Tr}[e^{i\Lambda}e^{-i\Lambda}W^A W_A] = \mathrm{Tr}[W^A W_A].
\end{aligned}$$

[11] Which is zero because Λ is a left-handed scalar superfield, $\bar{D}_{\dot{A}}\Lambda = 0$.

If we expand W_A in the component fields we find, as we might have hoped, that it contains the ordinary field strength tensor:

$$F^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + q f_{bc}{}^a V^b_\mu V^c_\mu$$

and that the trace indeed contains terms with $F^a_{\mu\nu} F^{\mu\nu a}$.

5.5 The (almost) complete supersymmetric Lagrangian

We can now write down the Lagrangian for a supersymmetric theory with (possibly) non-abelian gauge groups:[12]

$$\mathcal{L} = \Phi^\dagger e^V \Phi + \delta^2(\bar\theta) W[\Phi] + \delta^2(\theta) W[\Phi^\dagger] + \frac{1}{2T(R)} \delta^2(\bar\theta) \mathrm{Tr}[W^A W_A], \tag{5.15}$$

where $T(R)$ is the Dynkin index that appears to correctly normalize the energy density for the chosen representation R of the gauge group. Note that since W_A is spanned by T_a for a given representation, we can write $W_A = W^a_A T_a$. Then

$$\mathrm{Tr}[W^A W_A] = W^{aA} W^b_A \mathrm{Tr}[T_a T_b] = W^{Aa} W^b_A \delta_{ab} T(R) = T(R) W^{aA} W^a_A. \tag{5.16}$$

5.6 Spontaneous supersymmetry breaking

As we have seen above, supersymmetry predicts scalar partner particles with the same mass as the known fermions (and new fermions for the known vectors). These, somewhat unfortunately, contradict experiment by not existing. In the SM we have a similar problem: the vector bosons should remain massless under the gauge symmetry of the model. Yet, they are observed to be very massive. This is solved with the introduction of the Higgs mechanism and **spontaneous symmetry breaking** in the **scalar potential**.[13] The idea is that while there is a symmetry of the Lagrangian (in the SM the gauge symmetry), this may not be a symmetry of the vacuum state, thereby allowing the properties of the vacuum to supply the masses. Would it not be great if we could have *spontaneous symmetry breaking* in order to break supersymmetry this way and boost the masses of supersymmetric particles beyond current limits?

From exercise 5.13 we can see that the Lagrangian of (5.15) written in terms of component field contains no kinetic (derivative) terms for the $F(x)$ scalar fields. These are then what we call **auxilary fields** and can be eliminated by the e.o.m. we get from solving the Euler-Lagrange equation for this field:[14]

$$\frac{\partial \mathcal{L}}{\partial F^*_i(x)} = F_i(x) + W^*_i = 0,$$

[12] Note that there is no hermitian conjugate of the trace term, and an odd normalisation. This is because the term can be proven to be real, although this is sometimes overlooked in the literature.

[13] The potential of the Lagrangian are those terms not containing derivatives of the fields (kinetic terms). The scalar potential are such terms that contain only scalar fields.

[14] We remind the reader that the Euler-Lagrange equation for a field ϕ is the result of minimizing the action and is given in terms of the Lagrangian as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{5.17}$$

where

$$W_i \equiv \frac{\partial W[A_1, ..., A_n]}{\partial A_i}. \tag{5.18}$$

This allows us to rewrite the action as (ignoring gauge interactions):

$$S = \int d^4x \{ i\partial_\mu \bar{\psi}_i \sigma^\mu \psi_i - A_i^* \Box A_i - \frac{1}{2} W_{ij} \psi_i \psi_j - \frac{1}{2} W_{ij}^* \bar{\psi}_i \bar{\psi}_j - |W_i|^2 \}$$

with[15]

$$W_{ij} \equiv \frac{\partial^2 W[A_1, ..., A_n]}{\partial A_i \partial A_j}. \tag{5.19}$$

Thus the scalar potential of the Lagrangian is

$$V(A_i, A_i^*) = \sum_{i=1}^{n} \left| \frac{\partial W[A_1, ..., A_n]}{\partial A_i} \right|^2. \tag{5.20}$$

In the SM figuring out a scalar potential that breaks $SU(2)_L \times U(1)_Y$ is a little messy. In supersymmetry the argument goes like this: First, notice that we can write the supersymmetric Hamiltonian as

$$H = \frac{1}{4}(Q_1 \bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}} Q_1 + Q_2 \bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}} Q_2).$$

To see this, consider

$$\begin{aligned} \{Q_A, \bar{Q}_{\dot{B}}\} \bar{\sigma}^{\nu \dot{B} A} &= 2\sigma^\mu{}_{A\dot{B}} \bar{\sigma}^{\nu \dot{B} A} P_\mu \\ &= 2\,\mathrm{Tr}[\sigma^\mu \bar{\sigma}^\nu] P_\mu \\ &= 4 g^{\mu\nu} P_\mu = 4P^\nu. \end{aligned}$$

Now,

$$\begin{aligned} H &= P^0 = \frac{1}{4}\{Q_A, \bar{Q}_{\dot{B}}\} \bar{\sigma}^{0\dot{B}A} \\ &= \frac{1}{4}(Q_1 \bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}} Q_1 + Q_2 \bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}} Q_2). \end{aligned}$$

As discussed in Section 3.5 we have $Q_A^\dagger = \bar{Q}_{\dot{A}}$. Thus the Hamiltonian is **semipositive definite**, *i.e.* $\langle \Psi | H | \Psi \rangle \geq 0$ for any state $|\Psi\rangle$.

Imagine now that there exists some lowest lying states (possibly degenerate), the ground state(s) $|0\rangle$, that have vanishing energy $\langle 0 | H | 0 \rangle = 0$. These are supersymmetric since, to fulfill the energy assumption, we must have

$$Q_A |0\rangle = \bar{Q}_{\dot{A}} |0\rangle = 0 \quad \text{for} \quad \forall A, \dot{A}, \tag{5.21}$$

and are thus invariant under the supersymmetry transformations given by (4.7)

$$\delta_S |0\rangle = (\alpha^A Q_A + \bar{\alpha}_{\dot{A}} \bar{Q}^{\dot{A}})|0\rangle = 0. \tag{5.22}$$

[15]This is called the **fermionic mass matrix**.

This means that at this supersymmetric minimum of the potential the scalar potential must contribute zero

$$V(A, A^*) = 0 \quad \text{and thus} \quad \frac{\partial W}{\partial A_i} = 0.$$

Conversely, if the scalar potential does contribute in the vacuum (ground state) $|0\rangle$, meaning

$$\frac{\partial W}{\partial A_i} \neq 0 \quad \text{and thus} \quad V(A, A^*) > 0,$$

in the minimum of the potential for some A_i, then *supersymmetry must be broken*! As in the SM, the Lagrangian is still (super)symmetric, but $|0\rangle$ is not because (5.21) can no longer hold for all the Qs.

The **O'Raifeartaigh model** (1975) [7] is an example of a model that spontaneously breaks supersymmetry with three scalar superfields X, Y, Z, and the superpotential

$$W = \lambda YZ + gX(Z^2 - m^2), \tag{5.23}$$

where λ, g and m are real non-zero parameters. The scalar potential is

$$\begin{aligned} V(A, A^*) &= \left|\frac{\partial W}{\partial A_X}\right|^2 + \left|\frac{\partial W}{\partial A_Y}\right|^2 + \left|\frac{\partial W}{\partial A_Z}\right|^2 \\ &= |g(A_Z^2 - m^2)|^2 + |\lambda A_Z|^2 + |\lambda A_Y + 2gA_XA_Z|^2, \end{aligned} \tag{5.24}$$

which can never be zero because setting $A_Z = 0$, which is needed for the second term, gives a non-zero contribution g^2m^4 from the first term. Since the expectation value at the minimum that breaks supersymmetry is $\langle 0|\frac{\partial W_i}{\partial A_i}|0\rangle$, and $F_i = \frac{\partial W_i}{\partial A_i}$, the condition for spontaneous $\cancel{\text{SUSY}}$ (supersymmetry breaking) with the O'Raifertaigh mechanism can be written

$$\langle F_i \rangle \equiv \langle 0|F_i(x)|0\rangle > 0, \tag{5.25}$$

hence it is given the name **F-term breaking**. In F-term breaking it is the vacuum expectation value (vev) of the auxilary field of a scalar superfield that supplies the breaking.

In a gauge theory, a similar mechanism is found by adding a term $\mathcal{L}_{FI} \sim 2kV$ where V is a vector superfield. The vev of the $d(x)$ auxiliary field will create a non-zero scalar potential.[16] This is called the **Fayet-Iliopolous model**, or **D-term breaking**.

5.7 Supertrace

Unfortunately, the above does not work in practice with all particles at a low energy scale. The problem is that at *tree level* the **supertrace**, STr, the weighted sum of eigenvalues of the mass matrix $\mathcal{M}$, can be shown to vanish, $\mathrm{STr}\,\mathcal{M}^2 = 0$.[17]

Definition: The **supertrace** is given by

$$\mathrm{STr}\,\mathcal{M}^2 \equiv \sum_s (-1)^{2s}(2s+1)\,\mathrm{Tr}\,M_s^2 \tag{5.26}$$

where $\mathcal{M}$ is the mass matrix of the Lagrangian, s is the spin of particles and M_s is the mass matrix of all spin–s particles.

[16] It is **always** the auxiliary fields fault!

[17] See Ferrara, Girardello and Palumbo (1979) [8].

For a theory with only scalar superfields, with two fermionic and two bosonic degrees of freedom each, and with, respectively, mass matrices $M_{1/2}$ and M_0 after spontaneous supersymmetry breaking, this means that $\mathrm{Tr}\,\{M_0^2 - 2M_{1/2}^2\} = 0$, *i.e.* the sum of scalar particle masses (squared) is equal to the fermion masses (squared).[18] The consequence is that not all the scalar partners can be heavier than our known fermions.[19]

5.8 Soft breaking

What we can do instead is to add explicit supersymmetry breaking terms to the Lagrangian parametrizing our ignorance of the true (spontaneous) supersymmetry breaking on some higher scale $\sqrt{\langle F \rangle}$ that we do not have access to where the supertrace relation is fullfilled,[20] for which there are many alternatives in the literature, *e.g.*:

- Planck-scale Mediated Symmetry Breaking (PMSB)
- Gauge Mediated Symmetry Breaking (GMSB)
- Anomaly Mediated Symmetry Breaking (AMSB)

However, we cannot simply add arbitrary terms to the Lagrangian. The terms we can add are so-called **soft terms** with couplings of mass dimension one or higher. The dis-allowed terms with smaller mass dimension are terms that can lead to divergences in loop contributions to scalar masses (such as the Higgs) that are quadratic or worse (because of the high dimensionality of the fields in the loops). We will return to this issue in a moment. The allowed terms are in superfield notation as follows:

$$\begin{aligned}
\mathcal{L}_{\text{soft}} \;=\; & -\frac{1}{4T(R)} M\theta\theta\bar{\theta}\bar{\theta}\mathrm{Tr}\{W^A W_A\} - \frac{1}{6} a_{ijk}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j\Phi_k \\
& -\frac{1}{2} b_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j - t_i\theta\theta\bar{\theta}\bar{\theta}\Phi_i + h.c. \\
& -m_{ij}^2\theta\theta\bar{\theta}\bar{\theta}\Phi_i^\dagger\Phi_j.
\end{aligned} \tag{5.27}$$

Note that these terms are *not* supersymmetric. From the $\theta\theta\bar{\theta}\bar{\theta}$-factors we see that only the lowest order component fields of the superfields contribute. There are also some terms that are called "maybe-soft" terms:

$$\mathcal{L}_{\text{maybe}} = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta} c_{ijk}\Phi_i^\dagger\Phi_j\Phi_k + \text{h.c.} \tag{5.28}$$

This last—oft ignored—type of term is soft as long as none of the scalar superfields is a singlet under all gauge symmetries. It is, however, quite difficult to get large values for c_{ijk} with spontaneous $\not{S}\not{U}\not{S}\not{Y}$. In the above terms we have not specified any gauge symmetry, which will, in the same way as it did for the superpotential, severely restrict the allowed terms. However, it turns out that soft-terms are responsible for most of the parameters in supersymmetric theories!

[18]Remember that there are two scalar particles for each fermion.

[19]Strong coupling, meaning tree level is a bad approximation, may help, but life is still difficult.

[20]Remember that $[\Phi] = M$ and $[\theta] = M^{-\frac{1}{2}}$ so that the component field must have $[F] = M^2$.

We can write the soft terms in terms of their component fields as[21]

$$\begin{aligned}\mathcal{L}_{\text{soft}} &= -\frac{1}{2}M\lambda^A\lambda_A - (\frac{1}{6}a_{ijk}A_iA_jA_k + \frac{1}{2}b_{ij}A_iA_j + t_iA_i + \frac{1}{2}c_{ijk}A_i^*A_jA_k + c.c.) \\ &\quad -m_{ij}^2A_i^*A_j\end{aligned}$$

Note that to be viable $\cancel{\text{SUSY}}$ should to predict (universal) structures for the many soft-term parameters involved. Non-diagonal parameters tend to lead to flavor changing neutral currents (FCNC) or CP-violation in violation of measurement and should be avoided.

5.9 The hierarchy problem

Take a scalar particle, say the Higgs h. If we calculate loop-corrections to its mass in self-energy diagrams like the ones shown in Fig. 5.1, where f is a fermion and s some other scalar, they diverge, meaning they are infinite. This then needs what is called regularization in field theory in order to yield a finite answer. There are different ways of achiving this. Since we know that the SM is an incomplete theory, at least when we go up to Planck scale energies where we need an unknown quantum theory of gravity, we can introduce a cut-off regularization limiting the integral in the loop-correction to energies below a scale Λ_{UV}. Then the loop-correction to the Higgs mass is, at leading order in Λ_{UV},

$$\Delta m_h^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{UV}^2 + \frac{\lambda_s}{16\pi^2}\Lambda_{UV}^2 + \ldots \tag{5.29}$$

where λ_f and λ_s are the couplings of f and s to the Higgs, respectively, and Λ_{UV} is the high energy cut-off scale, suggestively the Planck scale, $\Lambda_{UV} = M_P = 2.4 \times 10^{18}$ GeV. Now, in order to keep $m_h \sim 125\,\text{GeV}$ as measured there must then be a crazy cancellation of 10^{16} times larger terms. This is known as the **hierarchy problem**.[22]

Enter supersymmetr to the rescue: with unbroken supersymmetry we find that we automatically have $|\lambda_f|^2 = \lambda_s$ and exactly twice as many scalar as fermion degrees of freedom running around in loops. This provides a magic cancellation of the quadratic divergence in Eq. (5.29). To see that this relation between the couplings holds, remember that $W \sim \lambda_{ijk}\Phi_i\Phi_j\Phi_k$ gives Lagrangian terms of the form $\lambda_{ijk}\psi_i\psi_jA_k$, and from the scalar potential we have terms of the form

$$V(A, A^*) \sim \left|\frac{\partial W}{\partial A_i}\right|^2 = |\lambda_{ijk}|^2A_j^*A_k^*A_jA_k. \tag{5.30}$$

When the scalar field A_k is the Higgs field, the fermion is represented by $\psi_i = \psi_j$ and the second scalar by A_j, these two terms are responsible for the two types of vertices in Fig. 5.1 with $\lambda_f = \lambda_{ijk}$ and $\lambda_s = |\lambda_{ijk}|^2$. Note that the argument above applies to *any* scalar in the theory.

Now, we have unfortunately already broken supersymmetry, so what happens in $\cancel{\text{SUSY}}$? This is the reason for restricting ourselves to soft supersymemtry breaking terms in the

[21] We have omitted terms that have the form $-\frac{1}{2}m_{ij}\psi_i\psi_j$, because these can be absorbed by a redefinition of the superpotential.

[22] What about choosing dimensional regularization instead where there is no cut-off scale? That could in principle work, however, as soon as you introduce *any* new particle (significantly) heavier than the Higgs this results in a quadratic correction with the new particle mass, meaning that we cannot complete the SM at a higher scale without reintroducing the problem!

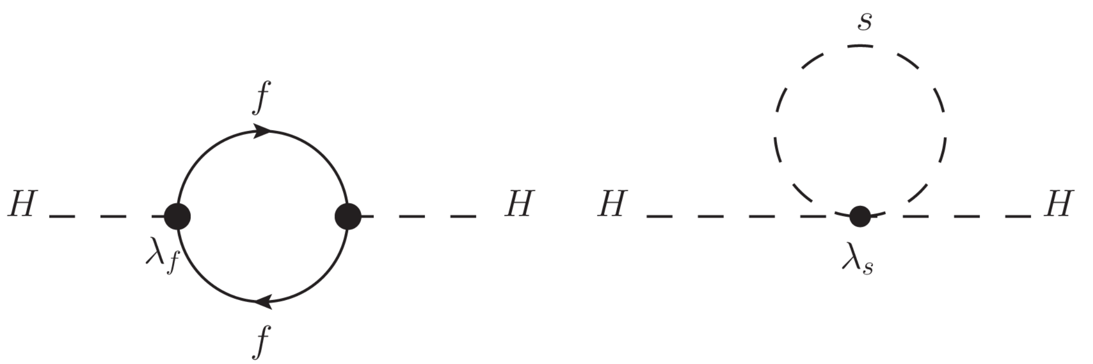

Figure 5.1: One loop contributions to the Higgs mass from a fermion (left) and scalar (right) loop.

previous section. This guarantees that we end up with contributions to the Higgs mass of at most

$$\Delta m_h^2 = -\frac{\lambda_s}{16\pi^2} m_s^2 \ln \frac{\Lambda_{UV}^2}{m_s^2} + \ldots, \tag{5.31}$$

at the leading order in Λ_{UV}, where m_s is the mass scale of the soft term. This is the most important argument in favour of supersymmetry existing at low energy scales where we can detect it, because m_s can not be too large if we want the above corrections to be small. This is called the **little hierarchy problem** and means that we want $m_s \sim \mathcal{O}(1\text{ TeV})$ in order to keep cancellations reasonable.

5.10 The non-renormalization theorem

With our generic supersymmetric Lagrangian in Eq. (5.15) we should really ask ourselves whether we can regularize the theory, *i.e.* is there a finite number of renormalisation constants/counter terms to make all measurable predictions finite? And if so, what are they?

You may not be so surprised that the answer is yes, and indeed we have already used one of the restrictions this gives on the possible terms in our superpotential construction. Furthermore, we can prove the following theorem with a funny name…

Theorem: **Non-renormalisation theorem** (Grisaru, Roach and Siegel, 1979 [9])
All higher order contributions to the effective supersymmetric action S_{eff} can be written:

$$S_{\text{eff}} = \sum_n \int d^4x_i ... d^4x_n d^4\theta \, F_1(x_1, \bar{\theta}, \theta) \times ... \times F_n(x_1, \bar{\theta}, \theta) \times G(x_1, ..., x_n), \tag{5.32}$$

where F_i are products of the external superfields and their covariant derivatives, and G is a supersymmetry invariant function.

So, why is the name funny? Well, mainly because it is not about not being able to renormalize the theory, but about about not *needing* to renormalize certain parts of it. The theorem has two important consequences:[23]

1. The couplings of the superpotential do not need separate normalization.
2. There is zero vacuum energy in global unbroken SUSY. In other words, $\Lambda = 0$ in general relativity.
3. Quantum corrections cannot (perturbatively) break supersymmetry.

Let us try to argue how these consequences come about. From the non-renormalization theorem we know that there are no counter terms needed for superpotential terms, because superpotential terms have lower θ integration than found in all the possible higher order contributions in the non-renormalisation theorem. This means that we can relate the bare fields Φ_0 and couplings g_0, m_0 and λ_0 to the renormalized fields Φ and couplings g, m and λ, by

$$\begin{aligned} g_0\Phi_0 &= g\Phi, && (5.33)\\ m_0\Phi_0\Phi_0 &= m\Phi\Phi, && (5.34)\\ \lambda_0\Phi_0\Phi_0\Phi_0 &= \lambda\Phi\Phi\Phi. && (5.35) \end{aligned}$$

If we let scalar superfields be renormalized by the **counterterm** Z, $\Phi_0 = Z^{1/2}\Phi$, vector superfields by Z_V, $V_0 = Z_V^{1/2}V$, coupling constant g by Z_g, $g_0 = Z_g g$, m by Z_m, $m_0 = Z_m m$, and λ by Z_λ, $\lambda_0 = Z_\lambda\lambda$, then

$$\begin{aligned} Z_g Z^{1/2} &= 1 && (5.36)\\ Z_m Z^{1/2} Z^{1/2} &= 1 && (5.37)\\ Z_\lambda Z^{1/2} Z^{1/2} Z^{1/2} &= 1 && (5.38) \end{aligned}$$

This set of equations can be solved for Z_g, Z_m and Z_λ in terms of $Z^{1/2}$ so no separate renormalization except for the superfields Φ and V is needed.

The second consequence comes about because vaccum diagrams have no external fields. This means that the integration $\int d^4\theta$ in S_{eff} gives zero for the contribution from these diagrams. The same argument leads to $V(A, A^*) = 0$ after quantum corrections.

In practice the regularisation of supersymmetric models is tricky. Using so-called DREG (dimensional regularisation) with modified minimal subtraction ($\overline{MS}$) fails because working in $d = 4 - \epsilon$ dimensions violates the supersymmetry in the Lagrangian. In practice DRED (dimensional reduction) with $\overline{DR}$ is used, where all the algebra is done in four dimensions, but integrals are done in $d = 4 - \epsilon$ dimensions. However, this leads to its own problems with potential ambiguities in higher loops.

5.11 Renormalisation group equations

Renormalisation, the removal of infinities from field theory predictions, introduces a fixed scale μ at which the parameters of the Lagrangian, the couplings, are defined. For example,

[23]The theorem is for unbroken supersymmetry.

the charge of the electron is not simply the bare charge e, but a charge at a given energy scale μ, $e(\mu)$, which is the scale at which the theory describes the electron, and which we can measure in an experiment at that scale. Scattering an electron at very high energy will require a different value of $e(\mu)$ than at a low energy. This is an experimentally well verified fact.[24]

However, since μ is not an observable *per se* but in principle a choice of how to write down the theory (at which energy to write down the Lagrangian), the action should be invariant under a change of μ, which is expressed as:

$$\mu \frac{d}{d\mu} S(Z\Phi, \lambda, \mu) = 0, \tag{5.39}$$

where λ are the couplings of the theory and Φ represents the (super)fields that have been renormalised.[25] This equation can be re-written in terms of partial derivatives

$$\left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \lambda}{\partial \mu} \frac{\partial}{\partial \lambda}\right) S(Z\Phi, \lambda, \mu) = 0, \tag{5.40}$$

which is the **renormalisation group equation** (RGE).

We can look at the behavior of a Lagrangian parameter λ as a function of the energy scale μ away from the value where it was defined, often denoted μ_0. This is controlled by the **β-function**:

$$\beta_\lambda \equiv \mu \frac{\partial \lambda}{\partial \mu}. \tag{5.41}$$

These β-functions can be found from the counterterm Z. As an example, take a **gauge coupling constant** g_0 defined (taken from measurement) at some scale μ_0. At a different scale μ, g_0 is given by (in $d = 4 - \epsilon$ dimensions):[26]

$$g_0 = Z g \mu^{-\epsilon/2}$$

Then, differentiating both sides with respect to μ,

$$\begin{aligned} 0 &= \frac{\partial Z}{\partial \mu} g \mu^{-\epsilon/2} + Z \frac{\partial g}{\partial \mu} \mu^{-\epsilon/2} - \frac{\epsilon}{2} Z g \mu^{-\epsilon/2-1} \\ \mu \frac{\partial g}{\partial \mu} &= \frac{\epsilon}{2} g - \frac{g\mu}{Z} \frac{\partial Z}{\partial \mu} \\ \mu \frac{\partial g}{\partial \mu} &= \frac{\epsilon}{2} g - g\mu \frac{\partial}{\partial \mu} \ln Z, \end{aligned}$$

and taking the limit $\epsilon \to 0$:

$$\beta_g = \mu \frac{\partial g}{\partial \mu} = -g\gamma_g,$$

where we have defined the anomalous dimension of g

$$\gamma_g = \mu \frac{\partial}{\partial \mu} \ln Z. \tag{5.42}$$

[24] It is also impossible to avoid if we accept that the electron is a point particle. Since the potential has the form $V(r) \propto e/r$ an infinte energy would appear unless we somehow were to modify the charge at high energies, or equivalently short distances.

[25] In the previous section we showed that we did not need to renormalise the coupling constants of the superpotential.

[26] The factor $\mu^{-\epsilon/2}$ is there to ensure that the scale of g is correct, see the exercise below.

It is often practical to rewrite $\beta_g = \frac{\partial g}{\partial t}$ with $t = \ln \mu$ so that $\mu \frac{\partial}{\partial \mu} = \frac{\partial}{\partial t}$.

Z can now be calculated to the required loop-order to find the β-function to that order and in turn the running of the coupling constant with μ. By evaluating one-loop super graphs we can find that for our particular example

$$\gamma_g \,|_{1-\text{loop}} = \frac{1}{16\pi^2}\, g^2 \left(\sum_R T(R) - 3C(A) \right), \tag{5.43}$$

where the sum is over all superfields that transform under a representation R of the gauge group and $C(A)$ is the Casimir invariant of the adjoint representation A of R. This expression is particularly important since it will later lead us to the concept of gauge coupling unification. Notice both that the running of the couplings with scale μ is very slow because the β-function is a logarithmic function of μ and that the anomolous dimension may be negative for some gauge groups.

5.12 Vacuum energy

We saw in the Section 5.10 that a globaly supersymmetric theory has $\Lambda = 0$. This is to be compared to the measured value of the dark energy density, which can be interpreted as vacuum energy and is $\Lambda_{DE} \sim 10^{-3}\,\text{eV}$, and the value in the SM which is $\Lambda \sim M_P \simeq 10^{18}\,\text{GeV}$.[27] Clearly models with supersymmetry are doing a bit better than the SM in predicting this. Now, what about ~~SUSY~~?

The scale of the contribution has to be the mass scale of the supersymmetric particles, so with $m_{SUSY} \geq 1\,\text{TeV}$ we have $m_{SUSY}/\Lambda_{DE} \geq 10^{15}$ which is twice as good as $M_P/\Lambda_{DE} = 10^{30}$ but still a bit off the measured value. This problem is the **hierachy problem for vacuum energy**.

However, in supergravity something interesting happens. Introducing a local supersymmetry the scalar potential is not simply given by the superpotential derivatives in (5.20), but instead is (ignoring the effects of gauge fields)

$$V(A, A^*) = e^{K/M_P} \left[K_{ij}(D_i W)(D_j W^*) - \frac{3}{M_P^2} |W|^2 \right], \tag{5.44}$$

where $K_{ij} = \partial_i \partial_j K(A, A^*)$ is the **Kähler metric** and the derivatives are with respect to the scalar fields in the Kähler potential K, and D_i the **Kähler derivative** $D_i = \partial_i + \frac{1}{M_P^2}(\partial_i K)$. In the $M_P \to \infty$ limit, the low energy limit, we see that we recover the flat space result of Eq. (5.20). What is important to notice is that there is now a second *negative* term in the potential that can in principle cancel the ~~SUSY~~ contribution, however, this will come at the price of fantastic fine-tuning unless some mechanism can be found where this is natural.

5.13 Excercises

Exercise 5.1 Write down the Lagrangian and find the action of the simplest possible supersymmetric field theory with a single scalar superfield, without gauge transformations, in

[27]The origin of this is just the same as the quadratic divergence for the Higgs mass. It is the same type of diagrams contributing, only without external legs.

terms of component fields, and show that it contains no kinetic terms for the $F_i(x)$ fields. Then show how they can be eliminated by the equations of motion. Challenge: Repeat for a gauge theory (here $d(x)$ can be eliminated). *Hint:* The action is

$$S = \int d^4x \left\{ -A^*(x)\Box A(x) + |F(x)|^2 + i(\partial_\mu \psi(x))\bar{\sigma}^\mu \psi(x) \right\}. \tag{5.45}$$

Solution: The Lagrangian for a single scalar superfield Φ is

$$\mathcal{L} = \Phi^\dagger \Phi + \delta^2(\bar{\theta})(g\Phi + m\Phi\Phi + \lambda\Phi\Phi\Phi) + \delta^2(\theta)(g\Phi^\dagger + m\Phi^\dagger\Phi^\dagger + \lambda\Phi^\dagger\Phi^\dagger\Phi^\dagger). \tag{5.46}$$

Using the following expressions for the scalar superfield in terms on component fields

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) &= A(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu A(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} \\
&\quad + \theta\theta F(x). \\
\Phi^\dagger(x,\theta,\bar{\theta}) &= A^*(x) - i(\theta\sigma^\mu\bar{\theta})\partial_\mu A^*(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) \\
&\quad + \bar{\theta}\bar{\theta}F^*(x).
\end{aligned}$$

we have, ignoring the superpotential terms,

$$\begin{aligned}
\int d^4\theta\, \Phi^\dagger\Phi &= -\frac{1}{4}A\Box A^* - \frac{1}{4}A^*\Box A + |F|^2 \\
&\quad + \int d^4\theta \left\{ (\theta\sigma^\mu\bar{\theta})\partial_\mu A^*(\theta\sigma^\nu\bar{\theta})\partial_\nu A + i\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}\theta\psi - i\bar{\theta}\bar{\psi}\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta} \right\},
\end{aligned}$$

where we have removed the factor $\theta\theta\bar{\theta}\bar{\theta}$ by integration. Using the identities

$$\begin{aligned}
(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) &= \frac{1}{2}g^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta}, && (5.47)\\
\theta\sigma^\mu\partial_\mu\bar{\psi}\theta\psi &= -\frac{1}{2}\psi\sigma^\mu\partial_\mu\bar{\psi}\theta\theta, && (5.48)\\
\partial_\mu\psi\sigma^\mu\bar{\theta}\bar{\theta}\bar{\psi} &= -\frac{1}{2}\partial_\mu\psi\sigma^\mu\bar{\psi}\bar{\theta}\bar{\theta}, && (5.49)
\end{aligned}$$

gives

$$\begin{aligned}
\int d^4\theta\, \Phi^\dagger\Phi &= -\frac{1}{4}A\Box A^* - \frac{1}{4}A^*\Box A + \frac{1}{2}\partial^\mu A\partial_\mu A^* + |F|^2 \\
&\quad - \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi}.
\end{aligned}$$

Now, since

$$\frac{1}{2}\partial^\mu A\partial_\mu A^* = \frac{1}{2}\partial^\mu(A\partial_\mu A^*) - \frac{1}{2}A\Box A^*, \tag{5.50}$$

and

$$\begin{aligned}
-\frac{1}{4}A\Box A^* &= -\frac{1}{4}\partial_\mu(A\partial^\mu A^*) + \frac{1}{4}\partial_\mu A\partial^\mu A^* \\
&= -\frac{1}{4}\partial_\mu(A\partial^\mu A^*) + \frac{1}{4}\partial^\mu((\partial_\mu A)A^*) - \frac{1}{4}A^*\Box A,
\end{aligned}$$

we can write

$$-\frac{1}{4}A\Box A^* - \frac{1}{4}A^*\Box A + \frac{1}{2}\partial^\mu A\partial_\mu A^* = -A^*\Box A + \text{total derivatives.} \tag{5.51}$$

Using $\phi\sigma^\mu\chi = -\bar{\chi}\bar{\sigma}^\mu\phi$, we can write

$$\begin{aligned} -\frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} &= \frac{i}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \\ &= i\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{i}{2}\partial_\mu(\bar{\psi}\bar{\sigma}^\mu\psi). \end{aligned} \tag{5.52}$$

Finally,

$$S = \int d^4x \int d^4\theta\, \mathcal{L} = \int d^4x \left\{-A^*(x)\Box A(x) + |F(x)|^2 + i(\partial_\mu\psi(x))\bar{\sigma}^\mu\psi(x)\right\}. \tag{5.53}$$

Exercise 5.2 For fun, and ten points, prove the scale factor in $g_0 = Zg\mu^{-\epsilon/2}$. *Hint:* what are the dimensions of stuff in the Lagrangian in $d = 4 - \epsilon$ dimensions?

Chapter 6

The Minimal Supersymmetric Standard Model (MSSM)

The Minimal Supersymmetric Standard Model (MSSM) is a minimal model in the sense that it has the smallest field (and gauge) content consistent with the known SM fields. We will now construct this model on the basis of the previous chapters, and look at some of its consequences.

6.1 MSSM field content

Previously we learnt that each (left-handed) scalar superfield S has a (left-handed) Weyl spinor ψ_A and a complex scalar $\tilde{s}$ since they are a $j = 0$ representation of the superalgebra.[1] Given an application of the equations of motion these have two fermionic and two bosonic degree of freedom remaining each (the auxiliary field has been eliminated and with it two fermionic d.o.f.).

In order to construct a Dirac fermion, which are plentiful in the SM, we need a right-handed Weyl spinor as well. We can aquire the needed right-handed Weyl spinor from the $\bar{T}^\dagger$ of a different scalar superfield $\bar{T}$ with the right-handed Weyl spinor $\bar{\varphi}_{\dot{A}}$.[2] With these four fermionic d.o.f. we can construct *two* Dirac fermions, a particle–anti-particle pair, and four scalars, two particle–anti-particle pairs.

We use these two superfield ingredients to construct all the known fermions:

- To get the SM leptons we introduce the superfields l_i and $\bar{E}_i$ for the charged leptons (i is the generation index) and ν_i for the neutrinos, where we form $SU(2)_L$ doublet vectors $L_i = (\nu_i, l_i)$. We do not introduce $\bar{N}_i$.[3] These would contain right-handed neutrino spinors needed for massive Dirac neutrinos, but are omitted as they do not couple to anything, being SM singlets.[4] This is a convention (MSSM is older than neutrino mass),

[1] With all posssible appologies, we have now changed notation for these fields to what is conventional in phenomenology (as opposed to pure theory) and we will try to use the tilde notation for the scalar component fields, while the superfields are denoted by latin letters.

[2] The bar here is used to (not) confuse us, it is part of the name of the superfields and does not denote any hermitian or complex conjugate.

[3] The anti-neutrino contained in the superfield $\nu_i^\dagger$ is right-handed consistent with experiment.

[4] They can't be colour-charged, they are right-handed singlets under $SU(2)_L$ thus they have zero weak isospin, but since they should also have zero electric charge the hypercharge must also be zero.

and including $\bar{N}_i$ fields has some interesting consequences.[5]

- For quarks the situation is similar. Up-type and down-type quarks get the superfields u_i, $\bar{U}_i$ and d_i, $\bar{D}_i$, forming the $SU(2)_L$ doublets $Q_i = (u_i, d_i)$.[6]

Additionally we need vector superfields, which after the e.o.m. contain a *massless* vector boson with two scalar d.o.f. and two Weyl-spinors, one of each handedness λ and $\bar{\lambda}$, with two fermionic degrees of freedom. Together these form a $j = \frac{1}{2}$ representation of the superalgebra. If the vector superfield is neutral, the fermions can form a Majorana fermion, if not they can be combined with the Weyl-spinors from other fields to form Dirac fermions.

Looking at the construction $V \equiv qt^aV^a$ in the supersymmetric Lagrangian we see that, as expected, we need one superfield V^a per generator t^a of the algebra, giving the normal $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ vector bosons. We call these superfields C^a, W^a and B^0.[7] In order to be really confusing, we use the following symbols for the fermions constructed from the respective Weyl-spinors: $\tilde{g}$, $\tilde{W}^0$ and $\tilde{B}^0$. The tilde here is supposed to tells us that hey are supersymmetric partners (often just called **sparticles**) of the known SM particles.

We also need Higgs superfields. Now life gets interesting. The usual Higgs $SU(2)_L$ doublet sclar field H in the SM cannot give mass to all fermions because it relies on the $H^C \equiv -i(H^\dagger\sigma_2)^T$ construction to give masses to up-type quarks (and possibly neutrinos). The superfield version of this cannot appear in the superpotential because it would mix left- and right-handed superfields. The minimal Higgs content we can get away with are two Higgs superfield $SU(2)_L$ doublets, which we will call H_u and H_d, indexing the quarks they give mass to.[8] These must have (more on that in a little bit) weak hypercharge $y = \pm 1$ for H_u and H_d respectively, so that we have the doublets:

$$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}, \quad H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}. \tag{6.1}$$

6.2 The kinetic terms

It is now straight forward to write down the kinetic terms of the MSSM Lagrangian giving matter-gauge interaction terms

$$\begin{aligned} \mathcal{L}_{kin} &= L_i^\dagger e^{\frac{1}{2}g\sigma W - \frac{1}{2}g'B} L_i + Q_i^\dagger e^{\frac{1}{2}g_s\lambda C + \frac{1}{2}g\sigma W + \frac{1}{3}\cdot\frac{1}{2}g'B} Q_i \\ &+ \bar{U}_i^\dagger e^{\frac{1}{2}g_s\lambda C - \frac{4}{3}\cdot\frac{1}{2}g'B} \bar{U}_i + \bar{D}_i^\dagger e^{\frac{1}{2}g_s\lambda C + \frac{2}{3}\cdot\frac{1}{2}g'B} \bar{D}_i \\ &+ \bar{E}_i^\dagger e^{2\frac{1}{2}g'B} \bar{E}_i + H_u^\dagger e^{\frac{1}{2}g\sigma W + \frac{1}{2}g'B} H_u + H_d^\dagger e^{\frac{1}{2}g\sigma W - \frac{1}{2}g'B} H_d, \end{aligned} \tag{6.2}$$

where g', g and g_s are the couplings of $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$. As a convention we assign the charge under $U(1)$, hypercharge, in units of $\frac{1}{2}g'$. All non-singlets of $SU(2)_L$ and $SU(3)_C$ have the same charge, the factor $\frac{1}{2}$ here is used to get by without accumulation of numerical factors since the algebras for the Pauli and Gell-Mann matrices are:

$$\left[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j\right] = i\epsilon_{ijk}\frac{1}{2}\sigma_k,$$

[5] Note that component fields in the same superfield must have the same charge under all the gauge groups, *i.e.* the scalar partner of the electron has electric charge $-e$, so it cannot be a neutrino.

[6] Here we should really also include a color index a such that u_i^a is a component in a $SU(3)_C$ vector. We omit these for simplicity.

[7] And there we have another W.

[8] In some further insanity some authors prefer H_1 and H_2 so that you have no idea which is which.

and

$$\left[\frac{1}{2}\lambda_i, \frac{1}{2}\lambda_j\right] = if_{ijk}\frac{1}{2}\lambda_k.$$

These conventions lead to the SM gauge transformations for fermion component fields and the familiar relations after electroweak symmetry breaking,[9] $Q = \frac{y}{2} + T_3$, where Q is the unit of electric charge, y is hypercharge and T_3 is weak charge, and $e = g\sin\theta_W = g'\cos\theta_W$.

We mentioned earlier that the two Higgs superfields have opposite hypercharge. This is needed for so-called **anomaly cancellation** in the MSSM. Gauge anomaly is the possibility that at loop level contributions to processes such as in Fig. 6.1 break gauge invariance and ruins the predictability of the theory. This miraculously does not happen in the SM becuase it has the field content it has, so that all gauge anomalies cancel (we don't know of a deeper reason). If we have *one* Higgs doublet this does not happer for the MSSM. With two Higgs doublets, with opposite hypercharge, it does.

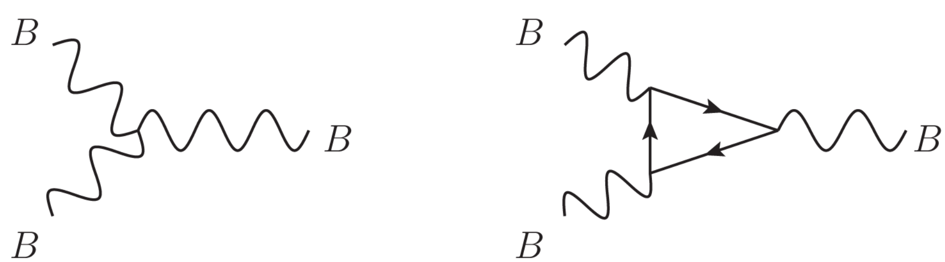

Figure 6.1: Possible three gauge boson B couplings a one-loop fermion contribution.

6.3 Gauge terms

The pure gauge terms with supersymmetric field strengths are also fairly easy to write down:

$$\mathcal{L}_V = \frac{1}{2}\mathrm{Tr}\{W^A W_A\}\bar{\theta}\bar{\theta} + \frac{1}{2}\mathrm{Tr}\{C^A C_A\}\bar{\theta}\bar{\theta} + \frac{1}{4}B^A B_A\bar{\theta}\bar{\theta} + \text{h.c.} \tag{6.3}$$

where we have used

$$T(R)_L = \mathrm{Tr}\left[\frac{1}{2}\sigma^1 \cdot \frac{1}{2}\sigma^1\right] = \frac{1}{2},$$

and

$$T(R)_C = \mathrm{Tr}\left[\frac{1}{2}\lambda^1 \cdot \frac{1}{2}\lambda^1\right] = \frac{1}{2},$$

in the normalization of the terms, and where the field strengths are given as:

$$W_A = -\frac{1}{4}\bar{D}\bar{D}e^{-W}D_A e^W , \qquad W = \frac{1}{2}g\sigma^a W^a, \tag{6.4}$$

$$C_A = -\frac{1}{4}\bar{D}\bar{D}e^{-C}D_A e^C , \qquad C = \frac{1}{2}g_s\lambda^a C^a, \tag{6.5}$$

$$B_A = -\frac{1}{4}\bar{D}\bar{D}D_A B , \qquad B = \frac{1}{2}g' B^0. \tag{6.6}$$

[9] Getting ahead of ourselves a little here.

6.4 The MSSM superpotential

With the same gauge structure as in the SM in place we are ready to write down all possible terms in the superpotential. First, we notice that there can be no **tadpole terms** (terms with only one superfield), since there are no superfields that are singlets (zero charge) under all SM gauge groups. The only alternative would be right-handed neutrino superfields $\bar{N}_i$.

We have seen that possible mass terms must fulfill $m_{ij}U_{ir}U_{js} = m_{rs}$ to preserve gauge invariance. For the abelian gauge group $U(1)_Y$ this reduces to $Y_i + Y_j = 0$, which is easier to check so this is where we start. In Table 6.1 we see that the only possible contributions are particle–anti-particle combinations such as $l_{iL}\bar{l}_{iR}$, but these come from superfields with different handedness and cannot be used together.

Superfield	L_i	$\bar{E}_i^\dagger$	Q_i	$\bar{U}_i^\dagger$	$\bar{D}_i^\dagger$
Particle	ν_{iL}, l_{iL}	l_{iR}	u_{iL},d_{iL}	u_{iR}	d_{iR}
Hypercharge	-1	-2	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$
Superfield	$L_i^\dagger$	$\bar{E}_i$	$Q_i^\dagger$	$\bar{U}_i$	$\bar{D}_i$
Anti-particle	$\bar{\nu}_{iR}$, $\bar{l}_{iR}$	$\bar{l}_{iL}$	$\bar{u}_{iR}$,$\bar{d}_{iR}$	$\bar{u}_{iL}$	$\bar{d}_{iL}$
Hypercharge	1	2	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$

Table 6.1: MSSM superfields with SM fermion content and their hypercharge.

The exception is for the two Higgs superfields that have opposite hypercharge. In order to also be invariant under $SU(2)_L$ we have to write this superpotential term as

$$\mathcal{L}_{\text{mass}} = \mu H_u^T i\sigma^2 H_d, \tag{6.7}$$

where μ is the Lagrangian mass parameter.[10] This is invariant under $SU(2)_L$ because, with the gauge transformations $H_d \to e^{ig\frac{1}{2}\sigma^k W^k} H_d$ and $H_u^T \to H_u^T e^{ig\frac{1}{2}\sigma^{kT} W^k}$, we get

$$\begin{aligned} H_u^T i\sigma_2 H_d &\to H_u^T e^{ig\frac{1}{2}\sigma^{kT} W^k} i\sigma_2 e^{ig\frac{1}{2}\sigma^k W^k} H_d \\ &= H_u^T i\sigma^2 e^{-i\frac{1}{2}g\sigma^k W^k} e^{i\frac{1}{2}g\sigma^k W^k} H_d = H_u^T i\sigma^2 H_d, \end{aligned}$$

since $\sigma^{kT}\sigma^2 = -\sigma^2\sigma^k$. Usually we ignore the $SU(2)_L$ specific structure and write terms like this as $\mu H_u H_d$, confusing the hell out of anyone that is not used to this convention since we really do mean Eq. (6.7). Notice that if we write (6.7) in terms of component fields we get

$$H_u^T i\sigma^2 H_d = H_u^+ H_d^- - H_u^0 H_d^0,$$

which we should have been able to guess because the Lagrangian must also conserve electric charge.

If you have paid very close attention to the argument above you may have noticed that there is one more possibility, namely

$$\mu_i' L_i H_u \equiv \mu_i' L_i^T i\sigma^2 H_u = \mu_i'(\nu_i H_u^0 - l_i H_u^+),$$

where μ' is some other mass parameter in the superpotential. This is clearly an allowable term (and we will return to it below), however, it also raises a very interesting question:

[10]Must not be confused with the RGE scale!

Could we have $L_i \equiv H_d$? Could the lepton superfields L_i play the rôle of Higgs superfields, thus reducing the field content needed to describe the SM particles in a supersymmetric theory? While not immediately forbidden, this suggestions unfortunately leads to problems with anomaly cancelation, processes with large lepton flavor violation (LFV) and much too massive neutrinos, and has been abandoned.

We have now found all possible mass terms in the superpotential. What about the Yukawa terms? The hypercharge requirement is $Y_i + Y_j + Y_k = 0$. From our table of hypercharges only the following terms are viable:

$$L_iL_j\bar{E}_k\ , \qquad L_iH_d\bar{E}_j\ , \qquad L_iQ_j\bar{D}_k\ , \qquad Q_iH_u\bar{U}_j\ , \qquad \bar{U}_i\bar{D}_j\bar{D}_k \quad \text{and} \quad Q_iH_d\bar{D}_i.$$

For all these terms we can simultaneously keep $SU(2)_L$ invariance with the $i\sigma^2$ construction implicitly inserted between any superfield doublets.

For $SU(3)_C$ to be conserved, we need to have colour singlets. Some of these terms are colour singlets by construction since they do not contain any coloured fields. The terms with two quark superfields contain left-handed Weyl spinors for quarks and anti-quarks, which are $SU(3)_C$ singlets if the superfields come in colour–anti-colour pairs. In representation language they are in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations of $SU(3)_C$. Written with all indices explicit we have *e.g.* $L_iQ_j\bar{D}_k = L_iQ_j^{\alpha}i\sigma^2\bar{D}_{k\alpha}$, where α is the colour index. The final term $\bar{U}_i\bar{D}_j\bar{D}_k$ is a colour singlet once we demand that it is totally anti-symmetric in the colour indices: $\bar{U}_i\bar{D}_j\bar{D}_k \equiv \epsilon^{\alpha\beta\gamma}\bar{U}_{i\alpha}\bar{D}_{j\beta}\bar{D}_{k\gamma}$.

Our complete superpotential is then:

$$\begin{aligned} W \;=\; & \mu H_uH_d + \mu_i' L_iH_u + y_{ij}^e L_iH_dE_j + y_{ij}^u Q_iH_u\bar{U}_j + y_{ij}^d Q_iH_d\bar{D}_j \\ & + \lambda_{ijk}L_iL_j\bar{E}_k + \lambda'_{ijk}L_iQ_j\bar{D}_k + \lambda''_{ijk}\bar{U}_i\bar{D}_j\bar{D}_k, \end{aligned} \tag{6.8}$$

where we have named and indexed the couplings in a natural way.[11]

6.5 R-parity

The superpotential terms LH_u, LLE and $LQ\bar{D}$ that we have written down all violate lepton number conservation, and $\bar{U}\bar{D}\bar{D}$ violates baryon number conservation. Allowing such terms leads to, among other phenomenological problems, processes like proton decay $p \to e^+\pi^0$ as shown in Fig. 6.2.

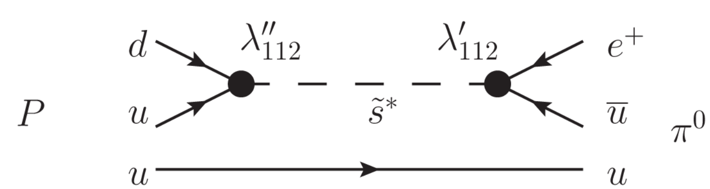

Figure 6.2: Feynman diagram for proton decay with RPV couplings.

[11] For some peculiar opinion of what is natural.

We can estimate the resulting proton life-time by noting that the scalar particle (a strange squark $\tilde{s}$) creates an effective Lagrangian term $\lambda\bar{u}\bar{\bar{d}}eu$ with coupling

$$\lambda = \frac{\lambda'_{112}\lambda''_{112}}{m_{\tilde{s}}^2}, \tag{6.9}$$

where the sparticle mass $m_{\tilde{s}}$ comes from the scalar propagator in the diagram. The resulting matrix element for the process must then be proportional to $|\lambda|^2$. Since the mass scale involved in the problem is the proton mass m_p the phase space integration part of a calculation of the proton decay width must be of the order of m_p^5. We then have

$$\Gamma_{p\to e^+\pi^0} \sim |\lambda|^2 m_p^5 = \frac{|\lambda'_{112}\lambda''_{112}|^2}{m_{\tilde{s}}^4} m_p^5. \tag{6.10}$$

The measured lower limit on the lifetime from watching a lot of protons not decay is $\tau_{p\to e^+\pi^0} > 1.6\cdot 10^{33}$ y or $\tau_{p\to e^+\pi^0} > \pi\cdot 10^7$ s/y$\times 1.6\cdot 10^{33}$ y $= 5.0\cdot 10^{40}$ s, which gives $\Gamma_{p\to e^+\pi^0} < 1.3\cdot 10^{-65}$ GeV, so that with we have the following very strict limit on the combination of two couplings

$$|\lambda'_{112}\lambda''_{112}| < 3.6\cdot 10^{-26}\left(\frac{m_{\tilde{s}}}{1\text{ TeV}}\right)^2. \tag{6.11}$$

To avoid all such couplings Fayet (1975) [10] introduced the conservation of R-partity.

Definition: **R-parity** is a multiplicatively conserved quantum number given by

$$R = (-1)^{2s+3B+L}$$

where s is a particle's spin, B its baryon number and L its lepton number.

For all SM particles $R = 1$, while the superpartners all have $R = -1$. One usually *defines* the MSSM as conserving R-parity. The consequence of this somewhat *ad hoc* definition is that in all interactions supersymmetric particles are only created or annihilated in pairs. This leads to the following very important phenomenological consequences:

1. The lightest supersymmetric particle (LSP) is absolutely stable.
2. Every other sparticle must decay down to the LSP (possibly in multiple steps).
3. Sparticles will always be produced in pairs in collider experiments.

For the MSSM this excludes the terms LH_u, $LL\bar{E}$, $LQ\bar{D}$ and $\bar{U}\bar{D}\bar{D}$ from the superpotential.

6.6 SUSY breaking terms

We can use our previous arguments on gauge invariance that we used when discussing the superpotential on the general soft-breaking terms in Eq. (5.28) to determine which terms are allowed. Terms

$$-\frac{1}{4T(R)}M\theta\theta\bar{\theta}\bar{\theta}\mathrm{Tr}\{W^A W_A\},$$

are allowed because they have the same gauge structure as the field strength terms. In component fields these are for the MSSM:

$$-\frac{1}{2}M_1\tilde{B}\tilde{B} - \frac{1}{2}M_2\tilde{W}^a\tilde{W}^a - \frac{1}{2}M_3\tilde{g}^a\tilde{g}^a + c.c$$

where the M_i are potentially complex-valued. This gives six new parameters. Terms

$$-\frac{1}{6}a_{ijk}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j\Phi_k,$$

are allowed when corresponding terms exist in the superpotential (are gauge invariant and not disallowed by R-parity). In component fields the allowed terms are

$$-a^e_{ij}\tilde{L}_iH_d\tilde{e}^*_{jR} - a^u_{ij}\tilde{Q}_iH_u\tilde{u}^*_{jR} - a^d_{ij}\tilde{Q}_iH_d\tilde{d}^*_{jR} + c.c.$$

where the H here refers to scalar parts of the Higgs superfields. The couplings a_{ij} are all potentially complex valued, so this gives us 54 new parameters. The terms

$$-\frac{1}{2}b_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi_i\Phi_j,$$

are only allowed for corresponding terms in the superpotential, *i.e.* $-bH_uH_d + c.c.$, where b is potentially complex valued, which gives us 2 new parameters.[12] Tadpole terms

$$-t_i\theta\theta\bar{\theta}\bar{\theta}\Phi_i,$$

are not allowed, as there are no tadpoles in the superpotential. Mass terms

$$-m^2_{ij}\theta\theta\bar{\theta}\bar{\theta}\Phi^\dagger_i\Phi_j,$$

are allowed because they have the same gauge structure as kinetic terms. In component fields they are:

$$\begin{aligned}&-(m^L_{ij})^2\tilde{L}^\dagger_i\tilde{L}_j - (m^e_{ij})^2\tilde{e}^*_{iR}\tilde{e}_{jR} - (m^Q_{ij})^2\tilde{Q}^\dagger_i\tilde{Q}_j - (m^u_{ij})^2\tilde{u}^*_{iR}\tilde{u}_{jR} - (m^d_{ij})^2\tilde{d}^*_{iR}\tilde{d}_{jR}\\&-m^2_{H_u}H^\dagger_uH_u - m^2_{H_d}H^\dagger_dH_d,\end{aligned} \tag{6.12}$$

where the m^2_{ij} are complex valued, however, also hermetic. This gives rise to 47 new parameters. Despite being allowed the MSSM ignores the "maybe-soft" terms in Eq. (5.28).

In total, after using our freedom to choose our basis wisely in order to remove what freedom we can, the MSSM has 105 new parameters compared to the SM, 104 of these are soft-breaking terms and μ is the only new parameter in the superpotential.

6.7 Radiative EWSB

In the SM the vector bosons are given mass spontaneous by electroweak symmetry breaking (EWSB), which is induced by the shape of the scalar potential for a scalar field Φ:

$$V(\Phi) = \mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2, \tag{6.13}$$

[12] The coupling b is sometimes written $B\mu$ where B is a unitless constant that indicates how different the coupling is from the corresponding coupling in the superpotential.

where the requirement for EWSB is that $\lambda > 0$ and $\mu^2 < 0$.[13] The first of these requirements ensures that the potential is **bounded from below**, *i.e.* that in the limit of large field values the potential does not turn to negative infinity. The second ensures that the minimum of the potential, the vacuum, is not given by zero field values, *i.e.* that the fields have **vacuum expectation values** (vevs).

In supersymmetry we have the scalar potential

$$V(A, A^*) = \sum_i \left|\frac{\partial W}{\partial A_i}\right|^2 + \frac{1}{2}\sum_a g^2(A^*T^aA)^2 > 0, \tag{6.14}$$

when we have extended Eq. (5.20) by including also gauge interactions and vector superfields.[14] For the scalar Higgs component fields (not superfields!) this gives the MSSM potential

$$\begin{aligned} V(H_u, H_d) \;=\;& |\mu|^2(|H_u^0|^2 + |H_u^+|^2 + |H_d^0|^2 + |H_d^-|^2) && \text{(from } F\text{-terms)} \\ &+\frac{1}{8}(g^2+g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2 && \text{(from } D\text{-terms)} \\ &+\frac{1}{2}g^2|H_u^+H_d^{0*} + H_u^0H_d^{-*}|^2 && \\ &+m_{H_u}^2(|H_u^0|^2 + |H_u^+|^2) + m_{H_d}^2(|H_d^0|^2 + |H_d^-|^2) && \text{(from soft breaking terms)} \\ &+[b(H_u^+H_d^- - H_u^0H_d^0) + c.c] && (6.15) \end{aligned}$$

This potential has 8 d.o.f. from 4 complex scalar fields H_u^+, H_u^0, H_d^0 and H_d^-.

We now want to do as in the SM and break $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$ in order to give masses to gauge bosons and SM fermions.[15] To do this we need to show that (6.15) has: i) a minimum for finite, *i.e.* non-zero, field values, ii) that this minimum has a remaining $U(1)_{\text{em}}$ symmetry and iii) that the potential is bunded from below, which are the essential properties of Eq. (6.13). We restrict our analysis to tree level, ignoring loop effects on the potential.

We start by using our $SU(2)_L$ gauge freedom to rotate away any field value for H_u^+ at the minimum of the potential, so without loss of generality we can use $H_u^+ = 0$ in what follows. At the minimum we must have $\partial V/\partial H_u^+ = 0$, and by explicit differentiation of the potential one can show that $H_u^+ = 0$ then leads to $H_d^- = 0$. This is good since it guarantees our item ii), that $U(1)_{\text{em}}$ is a symmetry for the minimum of the potential, since the charged fields then have no vev. We are then left with the potential

$$\begin{aligned} V(H_u^0, H_d^0) \;=\;& (|\mu|^2 + m_{H_u}^2)|H_u^0|^2 + (|\mu|^2 + m_{H_d}^2)|H_d^0|^2 \\ &+\frac{1}{8}(g^2+g'^2)(|H_u^0|^2 - |H_d^0|^2)^2 - (bH_u^0H_d^0 + c.c.) \end{aligned} \tag{6.16}$$

Since we can absorb a phase in H_u^0 or H_d^0 we can take b to be real and positive. This does not affect other terms because they are protected by absolute values. The minimum must also have $H_u^0H_d^0$ real and positive, to get a as large as possible negative contribution from the b term. Thus the vevs $v_u = \langle H_u^0\rangle$ and $v_d = \langle H_d^0\rangle$ must have opposite phases. By the remaining

[13] The **Mexican hat** or **wine bottle** potential, depending on preferences.

[14] The last term is due to the elimination of auxillary d-fields from vector superfields giving a contribution $d^ad^a = g^2(A^*T^aA)^2$ where T^a is the corresponding generator. The sum is taken over all the vector superfields with their respective couplings g.

[15] The soft-terms are unable to provide masses to these particles because they deal mostly with scalar fields.

$U(1)_Y$ symmetry, we can transform v_u and v_d so that they are real and have the same sign. For the potential to have a negative mass term, and thus fulfill point i) above, we must then have

$$b^2 > (|\mu|^2 + m_{H_u}^2)(|\mu|^2 + m_{H_d}^2). \tag{6.17}$$

Since the potential has $\cancel{\text{SUSY}}$ we must also check that it is actually bounded from below, our point iii), which was guaranteed for the SUSY vacuum. For large $|H_u^0|$ or $|H_d^0|$ the quartic gauge term blows up to save the potential, except for $|H_u^0| = |H_d^0|$, the so-called d-flat directions. This means that we must also require

$$2b < 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2. \tag{6.18}$$

Negative values of $m_{H_u}^2$ (or $m_{H_d}^2$) help satisfy (6.17) and (6.18), but they do not guarantee EWSB. If we assume that $m_{H_d} = m_{H_u}$ at some high scale (GUT) then (6.17) and (6.18) cannot be simultaneously be satisfied *at that scale*. However, to 1-loop the RGE running of these mass parameters is:

$$16\pi^2 \beta_{m_{H_u}^2} \equiv 16\pi^2 \frac{dm_{H_u}^2}{dt} = 6|y_t|^2(m_{H_u}^2 + m_{Q_3}^2 + m_{u_3}^2) + ...$$

$$16\pi^2 \beta_{m_{H_d}^2} \equiv 16\pi^2 \frac{dm_{H_d}^2}{dt} = 6|y_b|^2(m_{H_d}^2 + m_{Q_3}^2 + m_{d_3}^2) + ...$$

where y_t and y_b are the top and bottom quark Yukawa couplings, and $m_{Q_3} = m_{33}^Q$, $m_{u_3} = m_{33}^u$, $m_{d_3} = m_{33}^d$ in our previous notation. Because $y_t \gg y_b$, m_{H_u} runs down much faster than m_{H_d} as we go to the electroweak scale, and may become negative, see Fig. 6.3. It is this property that is termed **radiative EWSB** (REWSB). Thus, in the MSSM with soft terms there is an explanation why EWSB happens, it is not put in by hand in the potential as it is in the SM!

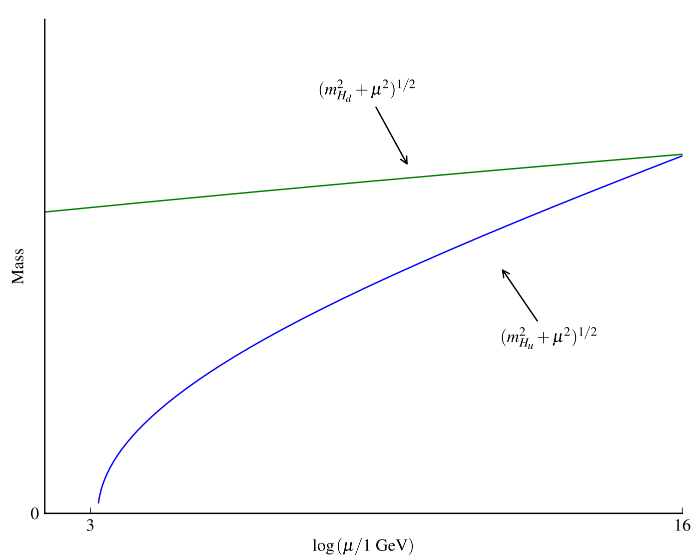

Figure 6.3: Sketch of the RGE running of the two soft Higgs mass parameters $m_{H_u}^2$ and $m_{H_d}^2$ as a function of the energy scale

To get the familiar vector boson masses, we need to satisfy the electroweak constraint:

$$v_u^2 + v_d^2 \equiv v^2 = \frac{2m_Z^2}{g^2 + g'^2} \approx (174 \text{ GeV})^2,$$

which comes from experiment. Thus we have one free parameter coming from the Higgs vevs. We can write this as

$$\tan\beta \equiv \frac{v_u}{v_d},$$

where by convention $0 < \beta < \pi/2$. Using the condition for the existence of an extremal point (minimum)

$$\partial V/\partial H_u^0 = \partial V/\partial H_d^0 = 0, \tag{6.19}$$

b and $|\mu|$ can be eliminated as free parameters from the model, however, not the sign of μ. Alternatively, we can choose to eliminate $m_{H_u}^2$ and $m_{H_d}^2$. You can look at this as giving away the freedom of these parameters to the vevs, and then fixing one vev by the electroweak constraint, and using $\tan\beta$ for the other.

Let us make a little remark here on the parameter μ. We have what is called the μ **problem**. The soft terms all get their scale from some common mechanism at some common high energy scale, it is assumed, however, μ is a mass term in the superpotential (the only one) and could *a priori* take *any* value, even M_P. Why is μ then of the order of the soft terms allowing us to achieve REWSB?[16]

6.8 Higgs boson properties

Of the 8 d.o.f. in the scalar potential for the Higgs component fields three are Goldstone bosons that get eaten by Z and $W^\pm$ to give masses. The remaining 5 d.o.f. form two neutral scalars h, H, two charged scalars $H^\pm$ and one neutral pseudo-scalar (CP-odd) A.[17] At tree level one can show that these have the masses:

$$\begin{aligned} m_A^2 &= \frac{2b}{\sin 2\beta} = 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2, && (6.20)\\ m_{h,H}^2 &= \frac{1}{2}\left(m_A^2 + m_Z^2 \mp \sqrt{(m_A^2 - m_Z^2)^2 + 4m_Z^2 m_A^2 \sin^2 2\beta}\right), && (6.21)\\ m_{H^\pm}^2 &= m_A^2 + m_W^2. && (6.22)\end{aligned}$$

As a consequence m_A and $\tan\beta$ can be used to parametrize the Higgs sector (at tree level), and H, $H^\pm$ and A are in principle unbounded in mass since they grow as $b/\sin 2\beta$. However, at tree level the lightest Higgs boson is restricted to

$$m_h < m_Z|\cos 2\beta|. \tag{6.23}$$

In contrast we have the Higgs boson discovery with a mass of $m_h = 125.7 \pm 0.3\,(\text{stat.}) \pm 0.3\,(\text{sys.})$ GeV from the LHC [11].

[16] This problem can be solved in extensions of the MSSM such as the Next-to-Minimal Supersymmetric Standard Model (NMSSM).

[17] In addition to the scalars, the Higgs supermultiplets contain four fermions, $\tilde{H}_u^0$, $\tilde{H}_d^0$, $\tilde{H}_u^+$ and $\tilde{H}_d^-$ (higgsinos). These will mix with the fermion partners of the gauge bosons (gauginos).

Fortunately there are large loop-corrections or the MSSM would have been excluded already.[18] Because of the size of the Yukawa couplings the largest corrections to the mass come from stop and top loops (see Fig. 5.1 for the relevant Feynman diagrams). In the limit $m_{\tilde{t}_R}, m_{\tilde{t}_L} \gg m_t$, and with stop mass eigenstates close to the chiral eigenstates (more on this later), we get the dominant loop correction:

$$\Delta m_h^2 = \frac{3}{4\pi^2} \cos^2\alpha \, y_t^2 m_t^2 \ln\left(\frac{m_{\tilde{t}_L} m_{\tilde{t}_R}}{m_t^2}\right), \tag{6.24}$$

where α is a mixing angle for h and H with respect to the superfield component fields H_u^0 and H_d^0, given by

$$\frac{\sin\alpha}{\sin\beta} = -\frac{m_H^2 + m_h^2}{m_H^2 - m_h^2}, \tag{6.25}$$

at tree level.

With this and other corrections the bound is weaker:

$$m_h \leq 135 \text{ GeV},$$

assuming a common sparticle mass scale of $m_{\text{SUSY}} \leq 1\,\text{TeV}$. Higher values for the sparticle masses give large fine-tuning and weaken the bound very little because of the logarithm in Eq. (6.24). The bound can be further weakened by adding extra field content to the MSSM, *e.g.* as in the NMSSM, but for $m_{\text{SUSY}} \approx 1\,\text{TeV}$ there is an upper pertubative limit of $m_h \approx 150\,\text{GeV}$.

It is very interesting to discuss what the Higgs discovery implies for low-energy supersymmetry. As can be seen from the above it requires rather large squark masses even in the favourable scenario with $\tan\beta > 10$. A naive estimate from Eq. (6.24) gives $m_{\tilde{t}} > 1$ TeV. However, this does not take into account negative contributions to the Higgs mass from heavy gauginos, and possible increases in the stop contribution due to tuning of the mixing of the chiral eigenstates in the mass eigenstates.

Since the lightest stop quark is expected to be the lightest squark in scenarios with common GUT scale soft masses—because of the large downward RGE running of m_{33}^Q due to the large top Yukawa coupling—the expected sparticle spectrum lies mostly above 1 TeV, with the possible exception of gauginos/higgsinos. This points to so-called **Split-SUSY** scenarios with heavy scalars and light gauginos, and a relatively large degree of fine-tuning. If one can live with this little hierarchy problem, it will explain why no signs of supersymemtry have been seen yet at the LHC. With squark masses above 1 TeV any hints of SUSY are not likely to come before the machine has been upgraded to 14 TeV in 2014.

If you are willing to accept fine-tuning of the stop mixing instead, or come up with a good reason for why the mixing should be just-so to give a maximal Higgs mass, you can keep fairly light stop quarks. With the addition of light higgsinos and a light gluino the model is then technically natural, these scenarios are called **Natural SUSY** and should be within the current or near future reach of the LHC.

In Split-SUSY scenarios with a neutralino dark matter candidate (see below) the lightest neutralino typically has a significant higgsino component. This means that its should be relatively accessible in direct detection experiments due to its large coupling to normal matter,

[18] It is worth pointing out that the MSSM, despite its many parameters, is a falsifiable theory in that had the Higgs boson mass been $\sim 15\,\text{GeV}$ higher, which is allowed in the SM, the MSSM would have been excluded.

and in the indirect search for neutrinos from captured dark matter annihilation in the Sun. Both types of experiments may very soon see first indications of a signal if this scenario is indeed realised in nature.

To do calculations with the Higgs bosons in the MSSM we need the Feynman rules that result from the relevant Lagrangian terms. Since these have been listed elsewhere we will not repeat them here, but recommend in particular the PhD-thesis of Peter Richardson [12], where they can be found in Appendix A.6, including all interactions with fermions and sfermions. These can also be found, together with all gauge and self-interactions, in the classic paper by Gunion and Haber [13]. Note that in this paper a complex Higgs singlet appears which can safely be ignored.

6.9 The gluino $\tilde{g}$

The **gluino** is a color octet Majorana fermion. As such it has nothing to mix with in the MSSM (even with RPV) and at tree level the mass is given by the soft term M_3. The one complication for the gluino is that it is strongly interacting so $M_3(\mu)$ runs quickly with energy. It is useful to instead talk about the scale-independent **pole-mass**, *i.e.* the pole of the renormalized propagator, $m_{\tilde{g}}$. Including one loop effects due to gluon exchange and squark loops, see Fig. 6.4, in the $\overline{DR}$ scheme we get:

$$m_{\tilde{g}} = M_3(\mu)\left[1 + \frac{\alpha_s}{4\pi}\left(15 + 6\ln\frac{\mu}{M_3} + \sum_{\text{all }\tilde{q}} A_{\tilde{q}}\right)\right],$$

where the squark loop contributions are

$$A_{\tilde{q}} = \int_0^1 dx\, x \ln\left(x\frac{m_{\tilde{q}}^2}{M_3^2} + (1-x)\frac{m_q^2}{M_3^2} - x(1-x) - i\epsilon\right).$$

Due to the 15-factor the correction can be significant (colour factor).

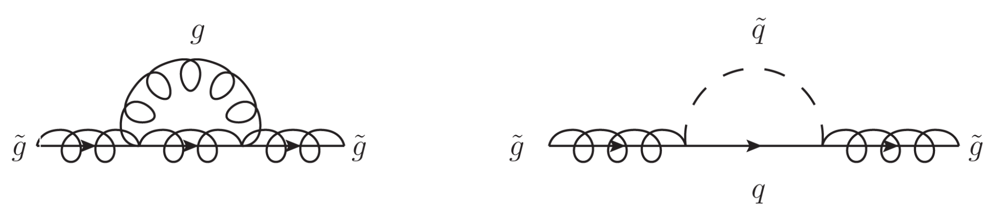

Figure 6.4: One loop contributions to the gluino mass.

Complete Feynman rules for gluinos can be found in Appendix C of the classic MSSM reference paper of Haber & Kane [14]. A more comprehensible alternative may be Appendix A.3 from the PhD-thesis of M. Bolz [15]. This also provides a description of how to handle clashing fermion lines that can appear with Majorana fermions.

6.10 Neutralinos & Charginos

We have a bunch of fermion fields that can mix because electroweak symmetry is broken and we do not have to care about $SU(2)_L \times U(1)_Y$ charges, only the $U(1)_{\rm em}$ charges matter. The candidates are:

$$\tilde{B}^0, \quad \tilde{W}^0, \quad \tilde{W}^\pm, \quad \tilde{H}_u^+, \quad \tilde{H}_u^0, \quad \tilde{H}_d^- \quad \text{and} \quad \tilde{H}_d^0.$$

The only requirement we have is that only fields with equal electromagnetic charge can mix. The neutral (Majorana) gauginos mix as

$$\tilde{\gamma} = N'_{11}\tilde{B}^0 + N'_{12}\tilde{W}^0 \quad \text{(photino)} \tag{6.26}$$
$$\tilde{Z} = N'_{21}\tilde{B}^0 + N'_{22}\tilde{W}^0 \quad \text{(zino)} \tag{6.27}$$

where the mixing is inherited from the gauge boson mixing. More generally, they also mix with the higgsinos to form four **neutralinos**:[19]

$$\tilde{\chi}_i^0 = N_{i1}\tilde{B}^0 + N_{i2}\tilde{W}^0 + N_{i3}\tilde{H}_d^0 + N_{i4}\tilde{H}_u^0, \tag{6.28}$$

where N_{ij} indicates size of the component of each of the fields in the gauge eigenstate basis

$$\tilde{\psi}^{0T} = \left(\tilde{B}^0,\, \tilde{W}^0,\, \tilde{H}_d^0,\, \tilde{H}_u^0\right). \tag{6.29}$$

In this basis the neutralino mass term can be written as

$$\mathcal{L}_{\chi-\rm mass} = -\frac{1}{2}\tilde{\psi}^{0T} M_{\tilde{\chi}} \tilde{\psi}^0 + \text{c.c.}$$

where the mass matrix is found from the Lagrangian to be

$$M_{\tilde{\chi}} = \begin{bmatrix} M_1 & 0 & -\frac{1}{\sqrt{2}}g'v_d & \frac{1}{\sqrt{2}}g'v_u \\ 0 & M_2 & \frac{1}{\sqrt{2}}gv_d & -\frac{1}{\sqrt{2}}gv_u \\ -\frac{1}{\sqrt{2}}g'v_d & \frac{1}{\sqrt{2}}gv_d & 0 & -\mu \\ \frac{1}{\sqrt{2}}g'v_u & -\frac{1}{\sqrt{2}}gv_u & -\mu & 0 \end{bmatrix}$$

In this matrix, the upper left diagonal part comes from the soft terms for the $\tilde{B}^0$ and the $\tilde{W}^0$, the lower right off diagonal matrix comes from the superpotential term $\mu H_u H_d$, while the remaining entries come from Higgs-higgsino-gaugino terms from the kinetic part of the Lagrangian, *e.g.* $H_u^\dagger e^{\frac{1}{2}g\sigma W + g'B} H_u$.

With the Z-mass condition on the vevs we can also write

$$\frac{1}{\sqrt{2}}g'v_d = \cos\beta \sin\theta_W m_Z, \tag{6.30}$$
$$\frac{1}{\sqrt{2}}g'v_u = \sin\beta \sin\theta_W m_Z, \tag{6.31}$$
$$\frac{1}{\sqrt{2}}gv_d = \cos\beta \cos\theta_W m_Z, \tag{6.32}$$
$$\frac{1}{\sqrt{2}}gv_u = \sin\beta \cos\theta_W m_Z. \tag{6.33}$$

[19] The neutral higgsinos are also Majorana fermions despite coming from scalar superfields. Unlike the (s)fermion superfields the Higgs superfields have no $\bar{H}$ chiral partners to supply the left-right Weyl spinor combinations required for Dirac fermions. Thus the neutralinos are Majorana fermions.

The mass matrix can now be diagonalized to find the $\tilde{\chi}_i^0$ masses.[20] If N is the diagonalization matrix, then $NM_{\tilde{\chi}}N^{-1} = D$, where $D = (m_{\tilde{\chi}_1^0}, m_{\tilde{\chi}_2^0}, m_{\tilde{\chi}_3^0}, m_{\tilde{\chi}_4^0})$ is the diagonal matrix containing the neutralino masses.

One particularly interesting solution to the diagonalization is in the limit where EWSB is a small effect, $m_Z \ll |\mu \pm M_1|$, $|\mu \pm M_2|$, and when $M_1 < M_2 \ll |\mu|$, $\mu \in \mathbb{R}$. Then $\tilde{\chi}_1^0 \approx \tilde{B}^0$, $\tilde{\chi}_2^0 \approx \tilde{W}^0$, $\tilde{\chi}_{3,4}^0 \approx \frac{1}{\sqrt{2}}(\tilde{H}_d^0 \pm \tilde{H}_d^0)$ and

$$\begin{aligned}
m_{\tilde{\chi}_1^0} &= M_1 + \frac{m_Z^2 \sin^2\theta_W \sin 2\beta}{\mu} + \ldots && (6.34)\\
m_{\tilde{\chi}_2^0} &= M_2 - \frac{m_W^2 \sin 2\beta}{\mu} + \ldots && (6.35)\\
m_{\tilde{\chi}_{3,4}^0} &= |\mu| + \frac{m_Z^2}{2\mu}(\operatorname{sgn}\mu \mp \sin 2\beta) + \ldots && (6.36)
\end{aligned}$$

Since the LSP is stable in R-parity conserving theories the lightest neutralino is an excellent candidate for dark matter. In particular since a 100 GeV neutralino has a natural relic density close to the measured dark matter density of the Universe. We will return to this issue later.

From the charged fermions we can make **charginos** $\tilde{\chi}_i^\pm$ that are Dirac fermions with mass terms

$$\mathcal{L}_{\chi^\pm-\mathrm{mass}} = -\frac{1}{2}\tilde{\psi}^{\pm T} M_{\chi^\pm}\tilde{\psi}^\pm + \mathrm{c.c.}$$

where $\tilde{\psi}^{\pm T} = (\tilde{W}^+, \tilde{H}_u^+, \tilde{W}^-, \tilde{H}_d^-)$ and

$$M_{\tilde{\chi}^\pm} = \begin{bmatrix} 0 & 0 & M_2 & gv_d \\ 0 & 0 & gv_u & \mu \\ M_2 & gv_u & 0 & 0 \\ gv_d & \mu & 0 & 0 \end{bmatrix}.$$

Here the M_2 terms come from the soft terms for the $W^\pm$, the μ terms come from the superpotential as above, while the remainder come from the kinetic terms. We have

$$\begin{aligned}
gv_d &= \sqrt{2}\cos\beta\, m_W, && (6.37)\\
gv_u &= \sqrt{2}\sin\beta\, m_W. && (6.38)
\end{aligned}$$

The eigenvalues of this matrix are doubly degenerated (to give the same masses to particles and their anti-particles), and are given as:

$$m_{\tilde{\chi}_{1,2}^\pm} = \frac{1}{2}\left(|M_2|^2 + |\mu|^2 + 2m_W^2 \mp \sqrt{(|M_2|^2 + |\mu|^2 + 2m_W^2)^2 - 4|\mu M_2 - m_W^2 \sin^2\beta|^2}\right).$$

In the limit of small EWSB discussed above we have $\tilde{\chi}_1^\pm \approx \tilde{W}^\pm$ and $\tilde{\chi}_2^\pm \approx \tilde{H}_u^+/\tilde{H}_d^-$ with

$$\begin{aligned}
m_{\tilde{\chi}_1^\pm} &= M_2 - \frac{m_W^2}{\mu}\sin 2\beta, && (6.39)\\
m_{\tilde{\chi}_2^\pm} &= |\mu| + \frac{m_W^2}{\mu}\operatorname{sgn}\mu. && (6.40)
\end{aligned}$$

[20] Note that we are perfectly happy with negative or even complex eigenvalues, as this is just a phase for the corresponding mass eigenstate in (6.28). Redefinition of fields can rotate away either the M_1 or M_2 phase, to make the parameter real and positive, but not both and not the μ-phase, which gives rise to problematic CP-violation. Therefore these are often just assumed to be real in order not to violate experimental bounds.

Note that in this limit $m_{\tilde{\chi}_2^0} \approx m_{\tilde{\chi}_1^+}$.

We should mention that some authors prefer other symbols for the neutralinos and charginos. Common examples are $\tilde{N}_i$ or $\tilde{Z}_i$ for neutralinos, and $\tilde{C}_i$ or $\tilde{W}_i$ (again!) for charginos.

Feynman rules for charginos & neutralinos can again be found in Haber & Kane [14].

6.11 Sleptons & Squarks

There are multiple contributions to sfermion masses from the MSSM Lagrangian. We make the following list:

i) Under the reasonable assumption that soft masses are (close to) diagonal[21] the sfermions get contributions $-m_F^2 \tilde{F}_i^\dagger \tilde{F}_i$ and $-m_f^2 \tilde{f}_{iR}^* \tilde{f}_{iR}$ from the soft terms.[22]

ii) There are so-called *hyperfine* terms that come from d-terms $\frac{1}{2}\sum g_a^2(A^* T^a A)^2$ in the scalar potential that give Lagrangian terms of the form (sfermion)^2(Higgs)^2 when one of the scalar fields A is a Higgs field. Under EWSB, when the Higgs field gets a vev these become mass terms. They contribute with a mass

$$\Delta_F = (T_{3F} g^2 - Y_F g'^2)(v_d^2 - v_u^2) = (T_{3F} - Q_F \sin^2\theta_W)\cos 2\beta\, m_Z^2,$$

where the weak isospin, T_3, hypercharge, Y, and electric charge, Q, are for the left-handed supermultiplet F to which the sfermion belongs. However, these contributions are usually quite small.

iii) There are also so-called F-term contributions that come from Yukawa terms in the superpotential of the form $y_f F H \bar{K}$. From the contribution $\sum |W_i|^2$ to the scalar potential these give Lagrangian terms $y_f^2 H^{0*} H^0 \tilde{f}_{iL}^* \tilde{f}_{iL}$ and $y_f^2 H^{0*} H^0 \tilde{f}_{iR}^* \tilde{f}_{iR}$. With EWSB we get the mass terms $m_f^2 \tilde{f}_{iL}^* \tilde{f}_{iL}$ and $m_f^2 \tilde{f}_{iR}^* \tilde{f}_{iR}$ since $m_f = v_{u/d}\, y_f$. These are only significant for large Yukawa coupling y_f.

iv) Furthermore, there are also F-terms that combine scalars from the $\mu H_u H_d$ term and Yukawa terms $y_f F H \bar{K}$ in the superpotential. These give Lagrangian terms $-\mu^* H^{0*} y_f \tilde{f}_L \tilde{f}_R^*$. With a Higgs vev this gives mass terms $-\mu^* v_{u/d}\, y_f \tilde{f}_R^* \tilde{f}_L + \text{c.c.}$

v) Finally, the soft Yukawa terms of the form $a_f \tilde{F} H \tilde{f}_R^*$ with a Higgs vev give mass terms $a_f v_{u/d} \tilde{f}_L \tilde{f}_R^* + \text{c.c.}$[23]

For the first two generations of sfermions, terms of type iii)–v) are small due to small Yukawa couplings. Then the sfermion masses are *e.g.*

$$m_{\tilde{u}_L}^2 = m_{Q_1}^2 + \Delta \tilde{u}_L, \tag{6.41}$$

$$m_{\tilde{d}_L}^2 = m_{Q_1}^2 + \Delta \tilde{d}_L, \tag{6.42}$$

$$m_{\tilde{u}_R}^2 = m_{u_1}^2 + \Delta \tilde{u}_R. \tag{6.43}$$

[21] This is of course to avoid flavor changing neutral currents (FCNCs).

[22] Here, and in the following, $\tilde{F}_i$ represents an $SU(2)_L$ doublet with generation index i, while $\tilde{f}_{iR}$ represents a singlet.

[23] We often assume that $a_f = A_0 y_f$ in order to further reduce the FCNC, meaning that there is a global constant A_0 with unit mass relating the Yukawa couplings and the trilinear A-term couplings.

Mass splitting between same generation slepton/squark is then given by

$$m^2_{\tilde{e}_L} - m^2_{\tilde{\nu}_L} = m^2_{\tilde{d}_L} - m^2_{\tilde{u}_L} = -\frac{1}{2}g^2(v_d^2 - v_u^2) = -\cos 2\beta\, m_W^2,$$

since they have the same hypercharge, see Table 6.1. For $\tan\beta > 1$ this gives $m^2_{\tilde{e}_L} > m^2_{\tilde{\nu}_L}$ and $m^2_{\tilde{d}_L} > m^2_{\tilde{u}_L}$.

The **third generation sfermions** $\tilde{t}$, $\tilde{b}$ and $\tilde{\tau}$ have a more complicated mass matrix structure, *e.g.* in the gauge eigenstate basis $(\tilde{t}_L, \tilde{t}_R)$ for stop quarks the mass term is

$$\mathcal{L}_{\text{stop}} = -\begin{pmatrix}\tilde{t}_L & \tilde{t}_R\end{pmatrix} m^2_{\tilde{t}} \begin{pmatrix}\tilde{t}_L \\ \tilde{t}_R\end{pmatrix},$$

where the mass matrix is given by

$$m^2_{\tilde{t}} = \begin{bmatrix} m^2_{Q_3} + m_t^2 + \Delta\tilde{u}_L & v(a_t^* \sin\beta - \mu y_t \cos\beta) \\ v(a_t \sin\beta - \mu^* y_t \cos\beta) & m^2_{u_3} + m_t^2 + \Delta\tilde{u}_R \end{bmatrix}, \tag{6.44}$$

where the diagonal elements come from i), ii) and iii), while the off-diagonal elements come from iv) and v). To find the particle masses, we must diagonalize this matrix, writing it in terms of the mass eigenstates $\tilde{t}_1$ and $\tilde{t}_2$, aquiring also a mixing matrix for the mass eigenstates in terms of the gauge eigenstates $\tilde{t}_L$ and $\tilde{t}_R$:

$$\begin{pmatrix}\tilde{t}_1 \\ \tilde{t}_2\end{pmatrix} = \begin{bmatrix} c_{\tilde{t}} & -s^*_{\tilde{t}} \\ s_{\tilde{t}} & c_{\tilde{t}} \end{bmatrix} \begin{pmatrix}\tilde{t}_L \\ \tilde{t}_R\end{pmatrix}, \tag{6.45}$$

where $m^2_{\tilde{t}_1} < m^2_{\tilde{t}_2}$ are the eigenvalues of (6.44) and $|c_{\tilde{t}}|^2 + |s_{\tilde{t}}|^2 = 1$. The matrices for $\tilde{b}$ and $\tilde{t}$ have the same structure.

6.12 Gauge coupling unification

We have already discussed the 1-loop β-functions of gauge couplings in a generic model, which were given in Eq. (5.43). With the MSSM field content and the gauge couplings:[24]

$$g_1 = \sqrt{\frac{5}{3}}g', \qquad g_2 = g, \qquad g_3 = g_s,$$

we arrive at

$$\beta_{g_i}|_{1-\text{loop}} = \frac{1}{16\pi^2}\, b_i g_i^3, \tag{6.46}$$

with

$$b_i^{MSSM} = \left(\frac{33}{5}, 1, -3\right).$$

The values of b_i are found from the Casimir invariant and the Dynkin index of the gauge group representations

$$C(A)_{SU(3)} = 3, \quad C(A)_{SU(2)} = 2, \quad C(A)_{U(1)} = 0,$$

[24]The normalisation choice for g_1 may seem a bit strange, however, this is the correct numerical factor when breaking *e.g.* SU(5) or SO(10) down to the SM group. This factor might be different with a different unified group.

using the definition $C(A)\delta_{ij} = (T^a T^b)_{ij}$, and

$$T(R)_{SU(3)} = \frac{1}{2}, \qquad T(R)_{SU(2)} = \frac{1}{2}, \qquad T(R)_{U(1)} = \frac{3}{5}y^2,$$

from the definition $T(R)\delta_{ab} = \mathrm{Tr}\{t_a t_b\}$, *e.g.* $b_3 = \frac{1}{2}\cdot 12 - 3\cdot 3 = -3$ because we have twelve quark/squark scalar superfields transforming under $SU(3)_C$.

At one-loop order we can do a neat rewrite using $\alpha_i \equiv \frac{g_i^2}{4\pi}$. Since

$$\frac{d}{dt}\alpha_i^{-1} = -2\frac{4\pi}{g_i^3}\frac{d}{dt}g_i,$$

we have:

$$\beta_{\alpha_i^{-1}} \equiv \frac{d}{dt}\alpha_i^{-1} = -\frac{8\pi}{g_i^3}\frac{1}{16\pi^2}g_i^3 b_i = -\frac{b_i}{2\pi}.$$

Thus α^{-1} runs linearly with t at one loop.

By running the α_i^{-1} from the EW scale measured values to high energies it is observed that in the MSSM the coupling constants intersect at a single point, which they do not naturally do in the SM. See Fig. 6.5, taken from Martin [16]. The assumption is then that a unified gauge group, *e.g.* $SU(5)$ or $SO(10)$, is broken at that scale, called the grand unifications scale or GUT-scale, down to the SM gauge group. This scale is $m_{\mathrm{GUT}} \approx 2\cdot 10^{16}$ GeV, about two orders of magnitude below the Planck scale.

Something funny happens to the gaugino mass parameters M_i if we look at their running. The one-loop β functions turn out to be

$$\beta_{M_i}|_{1-\mathrm{loop}} \equiv \frac{d}{dt}M_i = \frac{1}{8\pi^2}g_i^2 M_i b_i. \tag{6.47}$$

As a consequence all three ratios M_i/g_i^2 are scale independent at one loop. To see this let $R = M_i/g_i^2$, then

$$\beta_R \equiv \frac{dR}{dt} = \frac{\frac{d}{dt}M_i g_i^2 - M_i\frac{d}{dt}g_i^2}{g_i^4} = \frac{\frac{1}{8\pi^2}g_i^2 M_i b_i\cdot g_i^2 - M_i\cdot 2g_i\cdot\frac{1}{16\pi}g_i^3 b_i}{g_i^4} = 0. \tag{6.48}$$

If we now assume the coupling constants unify at the GUT scale to the coupling g_u, and that the gauginos have a common mass at the same scale $m_{1/2} = M_1(m_{\mathrm{GUT}}) = M_2(m_{\mathrm{GUT}}) = M_3(m_{\mathrm{GUT}})$, it follows that

$$\frac{M_1}{g_i^2} = \frac{M_2}{g_2^2} = \frac{M_3}{g_3^2} = \frac{m_{1/2}}{g_u^2}, \tag{6.49}$$

at all scales! (At one-loop.) This is a very powerful and predictive assumption. It leads to the following relation

$$M_3 = \frac{\alpha_s}{\alpha}\sin^2\theta_W M_2 = \frac{3}{5}\frac{\alpha_s}{\alpha}\cos^2\theta_W M_1, \tag{6.50}$$

which numerically predicts

$$M_3 : M_2 : M_1 = 6 : 2 : 1$$

at a scale of 1 TeV. Comparing to our previous discussion for neutralinos and charginos this predicts the masses $m_{\tilde g} \simeq 6m_{\tilde\chi_1^0}$, $m_{\tilde\chi_2^0} \simeq m_{\tilde\chi_1^\pm} \simeq 2m_{\tilde\chi_1^0}$. However, it is important to remember that this often used relationship is based on the *conjecture* of gauge coupling unification!

In Fig. 6.6, again taken from Martin [16], we show the running of the gaugino mass parameters M_i (solid black), the Higgs mass parameters $m^2_{H_{d/u}}$ (dot-dashed green), the third generation sfermion soft terms m_{d_3}, m_{Q_3}, m_{u_3}, m_{L_3} and m_{e_3} (dashed red and blue, listed from top to bottom) and the corresponding first and second generation terms (solid lines).

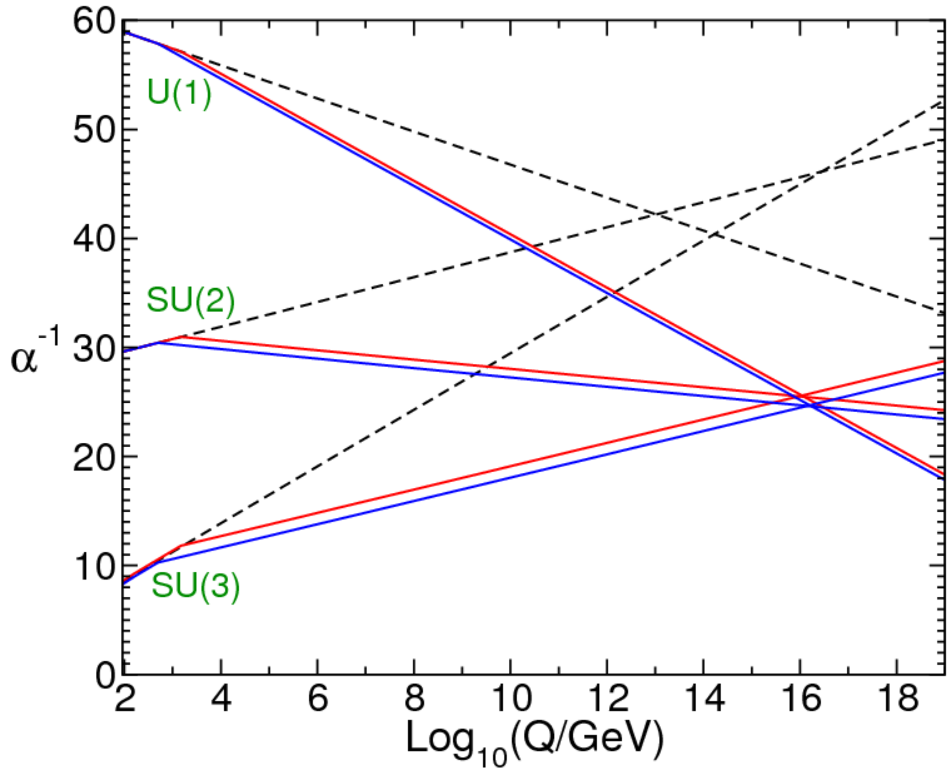

Figure 6.5: RGE evolution of the inverse gauge couplings $\alpha_i^{-1}(Q)$ in the SM (dashed lines) and the MSSM (solid lines). In the MSSM case, the sparticle mass thresholds are varied between 250 GeV and 1 TeV and $\alpha_3(m_Z)$ between 0.113 and 0.123 to create the bands shown by the red and blue lines. Two-loop effects are included.

6.13 Excercises

Exercise 6.1 Using the explicit form of the $SU(3)_C$ transformations with the Gell-Mann matrices, show that with our definition of the superpotential term $\bar{U}_i\bar{D}_j\bar{D}_k$ this is invariant under $SU(3)_C$.

Exercise 6.2 Show how you can eliminate the parameters $|\mu|$ and b by using the properties of the minimum of the potential in Eq. (6.16).

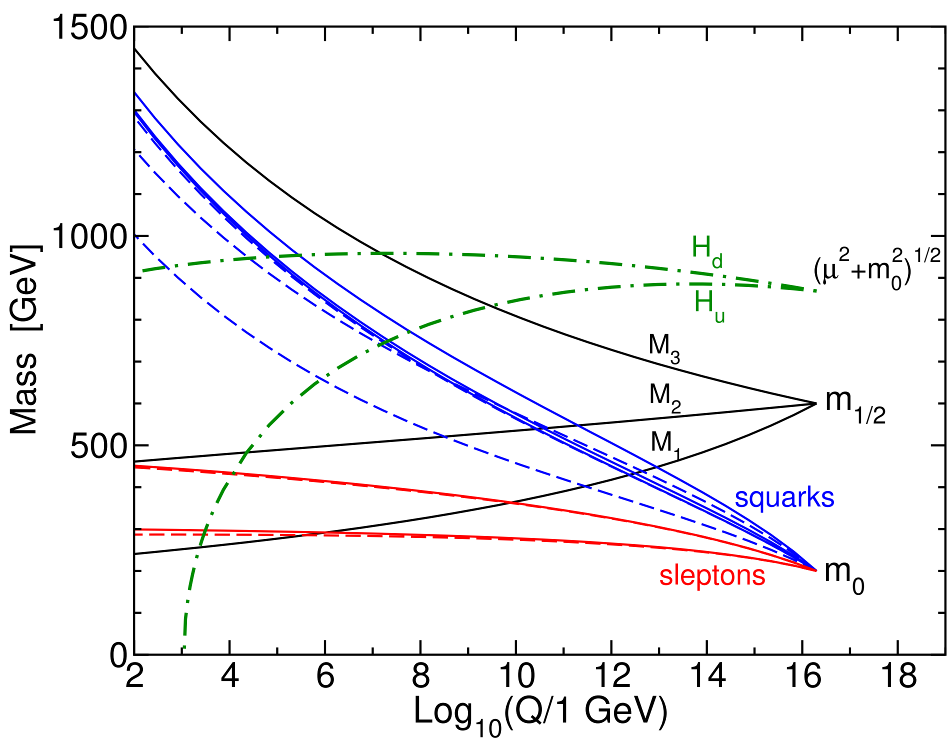

Figure 6.6: RGE evolution of scalar and gaugino mass parameters in the MSSM with typical minimal supergravity-inspired boundary conditions imposed at 2×10^{16} GeV. The parameter values used for this illustration were $m_0 = 200\,\text{GeV}$, $m_{1/2} = -A_0 = 600\,\text{GeV}$, $\tan\beta = 10$, and $\text{sgn}(\mu) = +$. The parameter $\mu^2 + m^2_{H_u}$ runs negative, provoking EWSB.

Chapter 7

Sparticle phenomenology

In this chapter we discuss the phenomenology of supersymmetric models and how to search for supersymmetry in experiments. We begin by returning to supersymmetry breaking in order to define some reasonable and (partially) motivated subsets of the 124 MSSM parameters which can be used to define more constrained models. We then discuss supersymmetry at hadron and lepton colliders, and finally look at precision measurements that are indirectly sensitive to the existence of sparticles.

7.1 Models for supersymmetry breaking

Let us take a little closer look at the models we use to motivate supersymmetry breaking, $\not{\text{SUSY}}$-models, and what their phenomenological consequences are. This is important to keep in mind as most searches for supersymmetry are interpreted under certain assumptions on the $\not{\text{SUSY}}$-mechanism.

Generically such models can be illustrated as shown in Fig. 7.1. There is one or more **hidden sector** (HS) scalar superfield X — by hidden we mean that it has no or very small direct couplings to the MSSM fields — that has an *effective (non-renormalizable) coupling* to the MSSM scalar fields Φ_i of the form

$$\mathcal{L}_{HS} = -\frac{1}{M}(\bar{\theta}\bar{\theta})X\Phi_i\Phi_j\Phi_k, \tag{7.1}$$

where M is some large scale, *e.g.* the Planck scale, that suppresses the interaction. Figure 7.2 shows an interaction that can lead to such terms, where M is the mass scale of some mediating particle Y. If the hidden sector is constructed so that X develops a vev for its auxillary F-component field, F_X,

$$\langle X\rangle = \theta\theta\langle F_X\rangle, \tag{7.2}$$

it breaks supersymmetry, see the discussion of Eq. (5.25). Then (7.1) will produce a soft-term of the form of the second term in Eq. (5.28),

$$\mathcal{L}_{\text{soft}} = -\frac{\langle F_X\rangle}{M}A_iA_jA_k, \tag{7.3}$$

with the soft mass

$$m_{\text{soft}} = \frac{\langle F_X\rangle}{M}.$$

This has reasonable limits in that $m_{\text{soft}} \to 0$ as $\langle F_X \rangle \to 0$, which is the limit of no ~~SUSY~~, and $m_{\text{soft}} \to 0$ as $M \to \infty$, where the scale of the HS interaction is decoupled (the mediating particle Y becomes too heavy to have any influence). We will now look at two possible ways to construct such a hidden sector called Planck-scale Mediated Supersymmetry Breaking (PMSB) and Gauge Mediated Supersymmetry Breaking (GMSB).

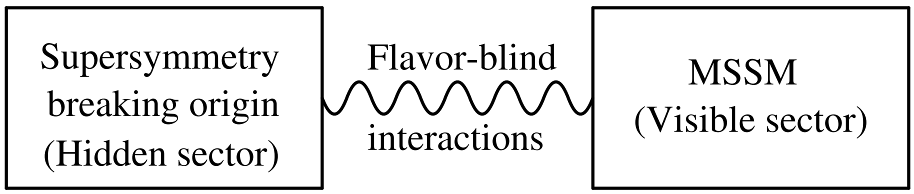

Figure 7.1: A generic illustration of how to generate soft breaking terms [16].

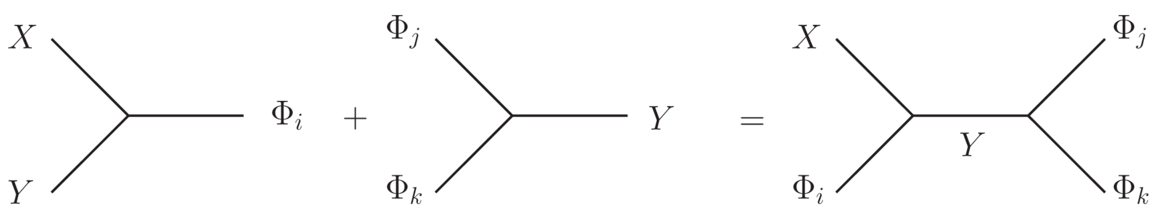

Figure 7.2: Interactions leading to effective 4-particle couplings in our example.

7.1.1 Planck-scale Mediated Supersymmetry Breaking (PMSB)

In Planck-scale mediated ~~SUSY~~ (PMSB) we blame some gravity mechanism for mediating the ~~SUSY~~ from the hidden sector to the MSSM so that the scale of the breaking is $M = M_P = 2.4 \cdot 10^{18}\,\text{GeV}$. Then we need to have[1] $\sqrt{\langle F \rangle} \sim 10^{10} - 10^{11}\,\text{GeV}$ in order to get $m_{\text{soft}} \simeq 100 - 1000\,\text{GeV}$, which is of the right magnitude not to re-introduce the hierarchy problem. The complete soft terms can then be shown to be

$$\begin{aligned}\mathcal{L}_{\text{soft}} &= -\frac{\langle F_X \rangle}{M_P}\left(\frac{1}{2} f_a \lambda^a \lambda^a + \frac{1}{6} y'_{ijk} A_i A_j A_k + \frac{1}{2}\mu'_{ij} A_i A_j + \frac{\langle F_X \rangle^*}{M_P^2} x_{ijk} A_i^* A_j A_k + \text{c.c.}\right) \\ &\quad -\frac{|\langle F_X \rangle|^2}{M_P^2} k_{ij} A_i A_j^*. \end{aligned} \tag{7.4}$$

Incidentally, we can now see why we assumed the maybe-soft breaking terms to be unimportant, as in this model they are suppressed by $\langle F_X \rangle^* / M_P^2$ compared to the other masses. If one assumes a minimal form for the parameters at the GUT scale, motivated by the wish for unification, *i.e.* $f = f_a$, $y'_{ijk} = \alpha y_{ijk}$, $\mu'_{ij} = \beta \mu, k_{ij} = k \delta_{ij}$ then all the soft terms are fixed by

[1]The use of $\sqrt{\langle F \rangle}$ is just a conventional shorthand notation for the magnitude of the vev of whichever F-term that breaks supersymmetry. This is called the supersymmetry breaking scale.

just four parameters

$$m_{1/2} = f\frac{\langle F_X\rangle}{M_P}, \qquad m_0^2 = k\frac{|\langle F_X\rangle|^2}{M_P^2}, \qquad A_0 = \alpha\frac{\langle F_X\rangle}{M_P}, \qquad B_0 = \beta\frac{\langle F_X\rangle}{M_P}.$$

The resulting phenomenology is called **minimal supergravity**, **mSUGRA/CMSSM**, minimal in the sense of the form of the parameters, and is the most studied, but perhaps not best motivated, version of the MSSM. Often B_0 and $|\mu|$ are exchanged for $\tan\beta$ at low scales using the EWSB condition in Eq. (6.19), so it is common to say that there are four and a half parameters in the model: $m_{1/2}$, m_0, A_0, $\tan\beta$ and $\operatorname{sgn}\mu$.

7.1.2 Gauge Mediated Supersymmetry Breaking (GMSB)

An alternative to PMSB is gauge-mediated ~~SUSY~~ where soft terms come from loop diagrams with **messenger** superfields that get their own mass by coupling to the HS ~~SUSY~~ vev, and that have SM gauge interactions. By dimensional analysis we must have

$$m_{\text{soft}} = \frac{\alpha_i}{4\pi}\frac{\langle F\rangle}{M_{\text{messenger}}}.$$

If now $\sqrt{\langle F\rangle}$ and $M_{\text{messenger}}$ are roughly comparable in size then $\sqrt{\langle F\rangle} \simeq 10\,\text{TeV}$ can give a viable sparticle spectrum. Notice that there is now a lot less RGE running for the parameters since the soft masses are given at a rather low scale.

One way of thinking about how these mass terms appear is that the messenger field(s) get masses from HS vevs and contribute to *e.g.* gaugino mass terms through diagrams such as the one in Fig. 7.3, where messenger scalars and fermions run in the loop. Note that scalars can only get mass contributions like this at two-loop order. To keep GUT unification messengers are often assumed to have small mass splittings and come in N_5 complete $\mathbf{5}+\overline{\mathbf{5}}$ representations of $SU(5)$.

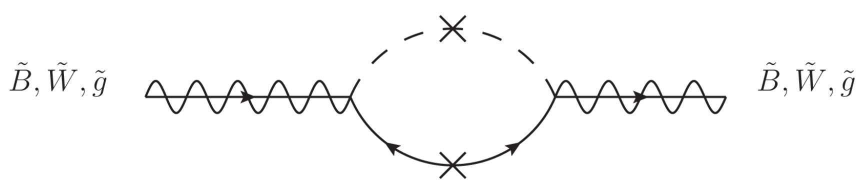

Figure 7.3: Diagram for GMSB. The messenger scalars and fermions run in the loop.

The minimal parametrization of GMSB models is in terms of $\Lambda = \frac{\langle F\rangle}{M_{\text{messenger}}}$, $M_{\text{messenger}}$, N_5 and $\tan\beta$ for the EWSB criterion (instead of μ). This gives the soft masses

$$M_i = \frac{\alpha_i}{4\pi}\Lambda N_5, \tag{7.5}$$

$$m_j^2 = 2\Lambda^2 N_5 \sum C(A)_i \left(\frac{\alpha_i}{4\pi}\right)^2. \tag{7.6}$$

While this looks independent of $M_{\text{messenger}}$, the messenger scale sets the starting point of the RGE running of the sparticle masses, and thus influences their magnitude. One should notice that this gives the same hierarchy of gaugino masses as in mSUGRA, $M_3 > M_2 > M_1$, since

(7.5) is ordered in terms of the strength of the gauge couplings α_i. The origin of the hierarchy is different since in mSUGRA it comes from the running of the parameters down from the GUT scale.

7.2 Supersymmetry at hadron colliders

Let us first point out some more or less obvious points.[2]

1) Hadron colliders collide quarks and gluons. This means that we get large cross sections only for QCD charged sparticles, *i.e.* squarks and gluinos, provided their masses are low enough.

2) As discussed earlier, with R-parity conservation (RPC) sparticles are produced in pairs and both decay to the LSP.

3) Illustrated in Fig. 7.4 these sparticles can decay to the LSP in many different, and potentially complicated, cascades. The possible decays for a particular MSSM model point called SPS1a is shown in Fig. 7.5. We should realize that many of these decays are hard to distinguish from ordinary SM (background) processes, or just undetectable.

4) Standard Model backgrounds have much, much bigger cross sections. Figure 7.6 shows the expected backgrounds and signals produced in different channels at the 14 TeV LHC for different particle masses.

5) R-parity conservation gives you **missing transverse energy** $\not{E}_T$ at hadron colliders due to the escaping LSPs, *i.e.* an imbalance in the directional sum of all energy deposits transverse to the beam direction. There is no logitudinal energy balance in a hadron colllider because the energies of the colliding partons are not known.

The consequences of the above is that we search for events with jet activity—squarks/gluinos decaying to the LSP—and missing energy from two LSPs. One simple way to do this is to define the **effective mass**

$$M_{\text{eff}} = \sum p_T^{\text{jet}} + \not{E}_T, \tag{7.7}$$

and search for deviations from SM expectations. Figure 7.7 shows a simulation of such a supersymmetry signal at the LHC for a benchmark MSSM model called LHC Point 2. However, there are models where this is ineffective. Imagine a scenario where only the lightest stop $\tilde{t}_1$ is copiously produced. If $m_{\tilde{t}_1} - m_{\tilde{\chi}_1^0} < m_W$ then $\tilde{t}_1 \to c\tilde{\chi}_1^0$ or $\tilde{t}_1 \to bl\nu\tilde{\chi}_1^0$ decays dominate, where all final state particles have low energy (p_T), so-called **soft particles**. This is very difficult to discover with standard techniques.

One alternative to jets and lots of missing energy is to look for leptons (and some small missing energy) from gaugino pair production and decays. Searching the lepton and missing energy channels is a very effective way to isolate any production of sparticles from SM backgrounds, but for setting bounds it is bad since the only model independent production is **Drell-Yan** production, *e.g.* $q\bar{q} \to (Z/\gamma)^* \to \tilde{\chi}_1^0\tilde{\chi}_2^0, \tilde{\chi}_1^+\tilde{\chi}_1^-, \tilde{l}_L^*\tilde{l}_L, \tilde{l}_R^*\tilde{l}_R$, and $q'\bar{q} \to W^* \to \tilde{\chi}_2^0\tilde{\chi}_1^\pm$, which all have low cross sections due to the smaller electroweak coupling and the smaller anti-quark content of the proton. The expected bounds from such searches for the mSUGRA model is compared to other searches in Fig. 7.8.

[2] You might find these very obvious, they are, however, quite important and some theory people seem oblivious to them.

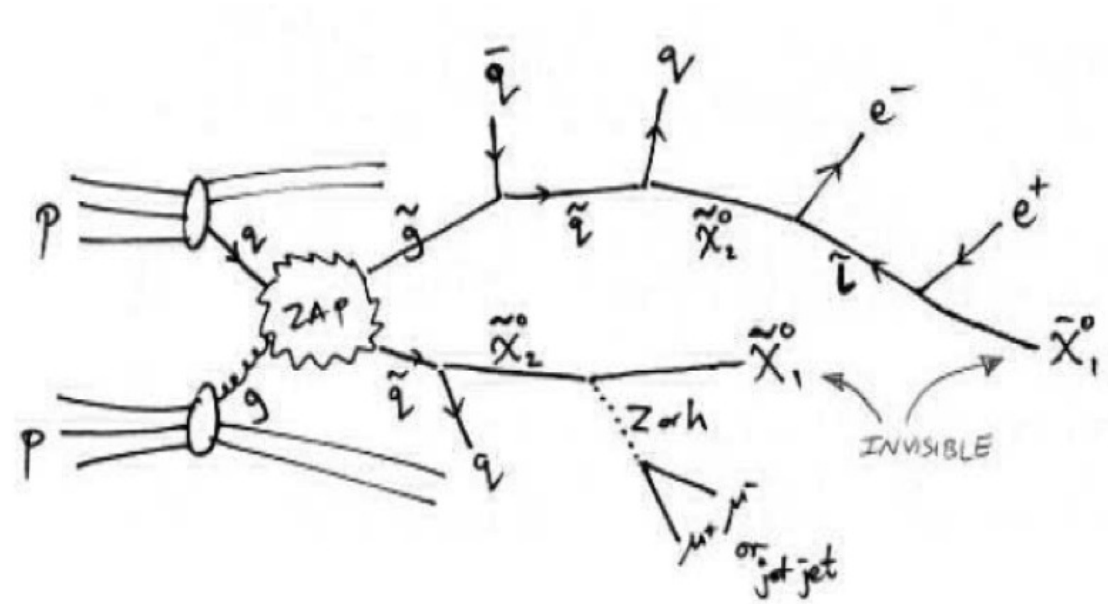

Figure 7.4: Diagram of a possible collision process in the LHC for an RPC model. Illustration by C. Lester [17].

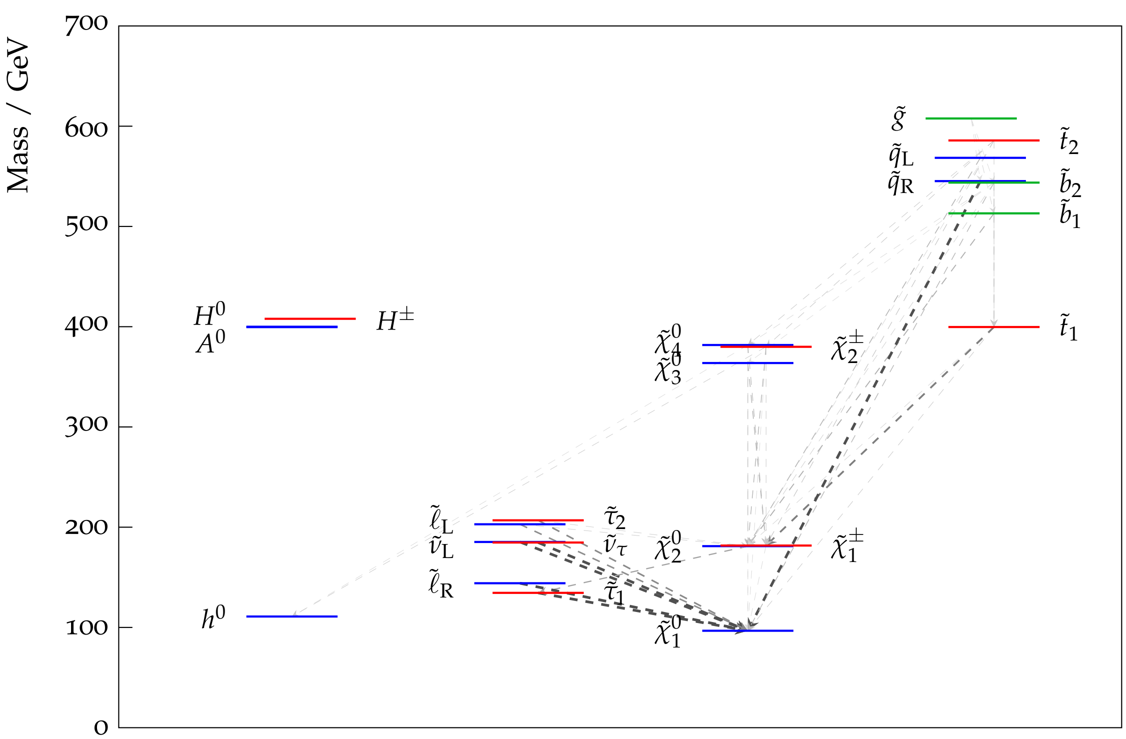

Figure 7.5: Possible sparticle cascades for the SPS1a model point. Only decays with branching rations above 5% are shown. The line width indicates relative branching ratios. The plot was generated using `PySLHA 3.0.1` [18].

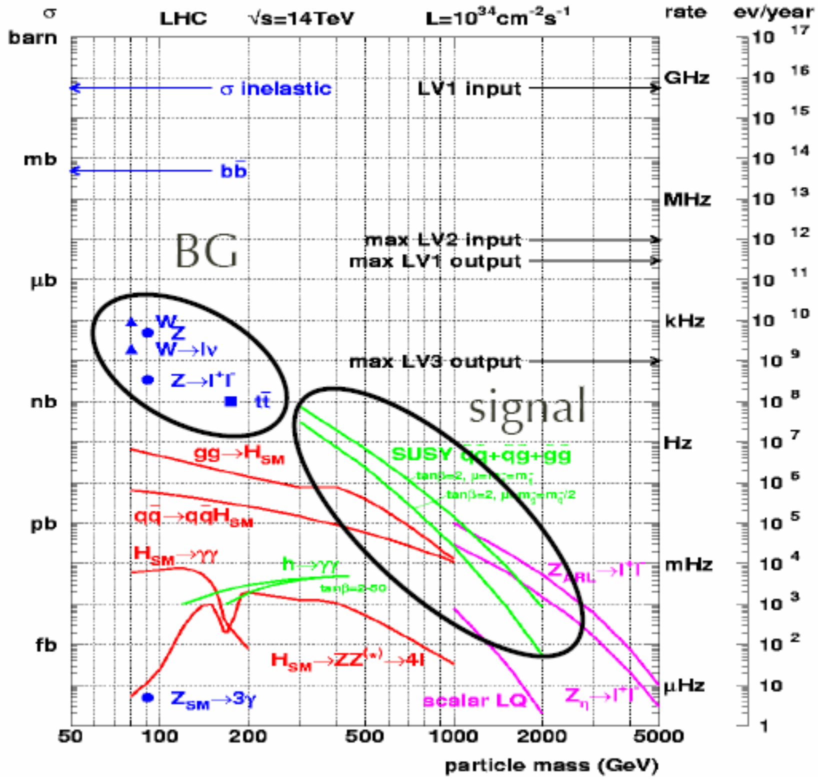

Figure 7.6: Plot of the expected signals for various processes at the 14 TeV LHC plotted against the mass of the particles. The current Run II of the LHC has collected around 4 fb^{-1} of data, so the fb scale indicates processes where $\mathcal{O}(1)$ events are expected.

You may ask, why not look for the production of $\tilde{\chi}_1^0\tilde{\chi}_1^0$? To first order the answer might be that with nothing else in the event, we cannot measure the missing energy as that requires an *imbalance* in momentum. However, given sufficient QCD radiation from the initial quark/gluon[3] a single jet recoiling against missing energy could potentially be measured, and this, so-called **mono-jet search**, is indeed a search channel for dark matter production at the LHC. However, for neutralino dark matter this does not work all that well for other reasons. The $Z\tilde{\chi}_i^0\tilde{\chi}_j^0$ vertex shown in Fig. 7.9 has the Feynman rule

$$\frac{ig}{2\cos\theta_W}\gamma^\mu\left[\left(N_{i3}N_{j3}^* - N_{i4}N_{j4}^*\right)P_L - \left(N_{i3}^*N_{j3} - N_{i4}^*N_{j4}\right)P_R\right], \tag{7.8}$$

which depends only on the higgsino components of the neutralinos, N_{i3} and N_{i4}. This can be understood from the fact that there are no ZZZ or $Z\gamma\gamma$ vertices in the SM that can be supersymmetrized, only a Zhh vertex. For the photon there is no tree level coupling to the neutralinos at all since there are no direct couplings between the higgs and the photon in the SM. Thus, only neutralinos with significant higgsino components can be produced this way. To top it off, a light higgsino with a mass dominated by the μ parameter would have very similar values of N_{i3} and N_{i4}, thus canceling the coupling.

Should some excess be discovered in any search, we need some smoking duck in order to confirm that this is indeed supersymmetry. We would like to identify and measure the

[3]For an e^+e^- collider this would be photon radiation from the initial electron/positron.

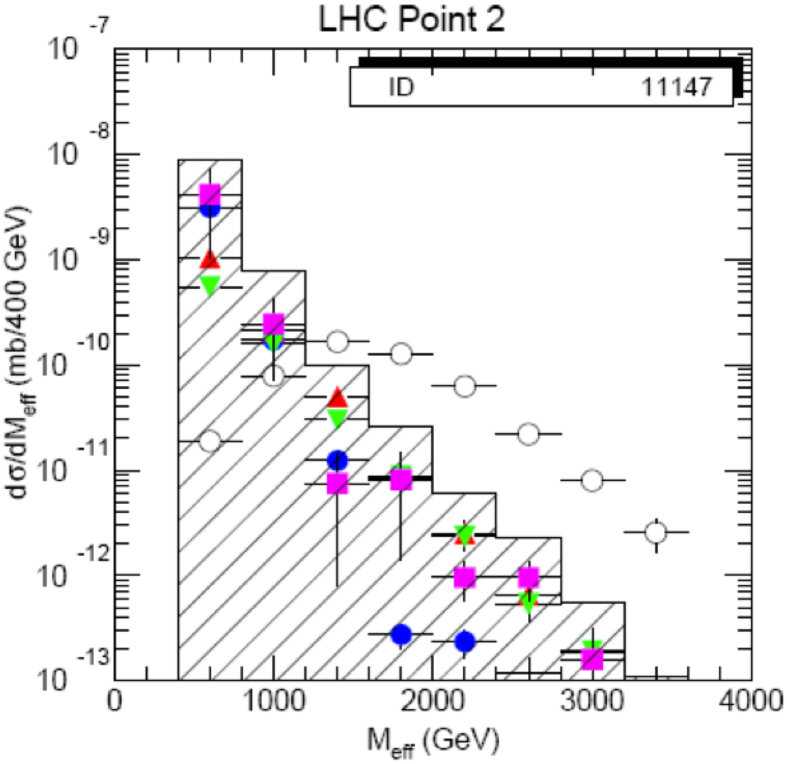

Figure 7.7: Plot of the differential cross section with respect to effective mass, plotted against the effective mass of the final state particles as given in (7.7). The colored data points represent different SM processes, and the histogram is the sum of all SM contributions, while the white circles represent a possible supersymmetry scenario. The position of the supersymmetry signal maximum is correlated to the masses of $\tilde{\chi}$ and $\tilde{q}$, but there is large variance.

masses of as many new particles as possible, and hopefully also their spin. To do this, a multitude of techniques have been invented, all facing the problem of how to deal with the loss of information from the LSP. Figure 7.11 shows an example of one such technique where sequential two-body decays of sparticles are used. For the generic decay chain shown in Fig. 7.10 with three sequential two-body decays we can measure the invariant mass between two detectable end-products, a and b, m_{ab}. Even if the particle A at the end of the chain is invisible one can show that the invariant mass distribution for m_{ab} has a triangular shape with a sharp endpoint at the maximum

$$(m_{ab}^{\max})^2 = \frac{\left(m_C^2 - m_B^2\right)\left(m_B^2 - m_A^2\right)}{m_B^2}, \tag{7.9}$$

where we have assumed that a and b are massless.[4] A measurement of this endpoint position gives us one realtionship between the three unknown (sparticle) masses. If we have a chain with three sequential two-body decays we can repeat this measurement with three more

[4]A more complicated expression covers the massive case.

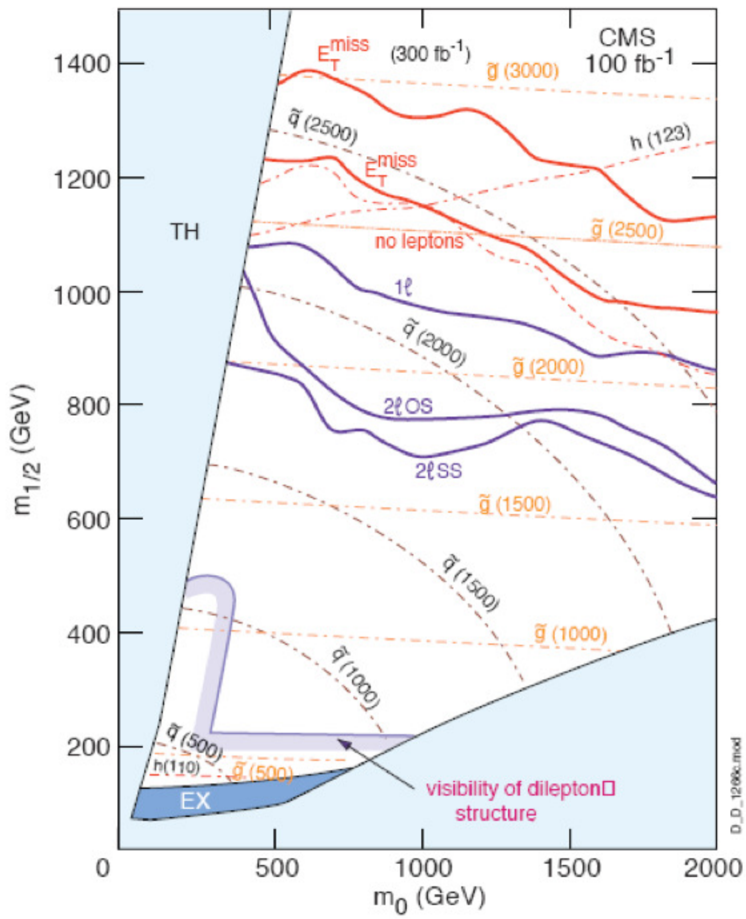

Figure 7.8: Plot of the projected discovery reach for different values of $m_{1/2}$ and m_0 in the mSUGRA model with 100 fb^{-1} or 300 fb^{-1} of data at the Compact Muon Spectrometer (CMS). The light blue area represents theoretical restrictions on the parameter space. The dark blue area is the parameter space that was probed by the Tevatron. The red lines represents a pure jets pluss $\not{E}_T$ search at 14 TeV. The blue lines represent searches using leptons. The dotted lines show the masses of different sparticles in this parameter space.

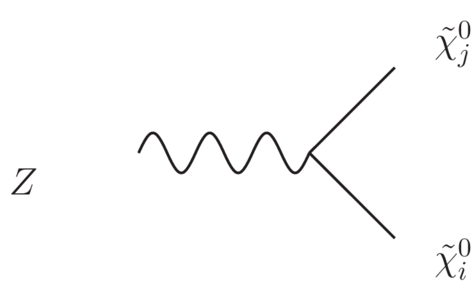

Figure 7.9: Coupling $Z\tilde{\chi}^0_i\tilde{\chi}^0_j$.

possible invariant mass combinations, arriving at four equations with four unknown, which can in principle at least be solved for the masses involved.

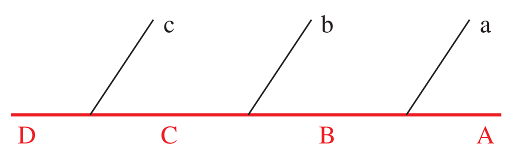

Figure 7.10: Generic cascade decay $D \to Cc \to Bb \to Aabc$ [19].

As alternatives to these standard searches we have searches for decaying LSPs when R-parity is violated, or the production of single sparticles.[5] There is the possibility of **massive**

[5]Single sparticle production requires rather large RPV couplings for the $LQ\bar{D}$ or $\bar{U}\bar{D}\bar{D}$ operators, of the order of $\lambda > 10^{-2}$.

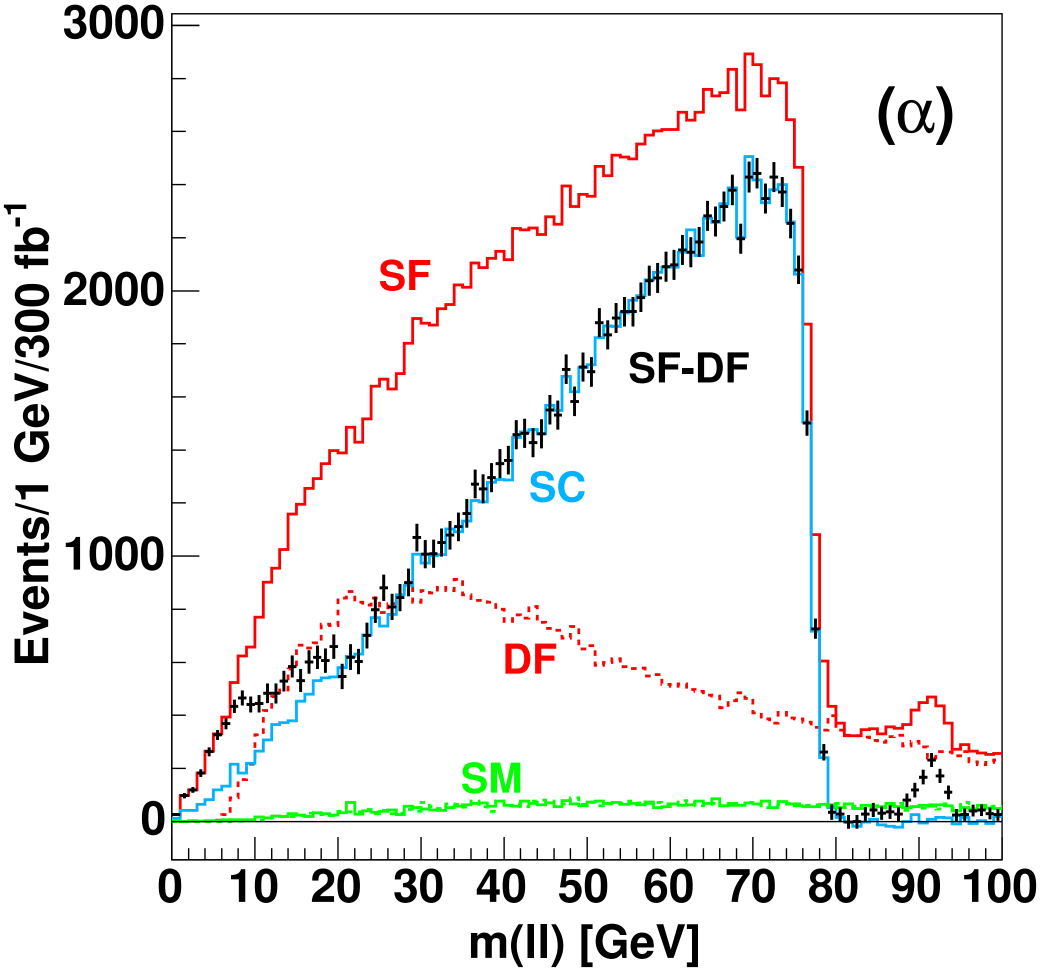

Figure 7.11: Invariant mass distribution of opposite sign same flavour (OSSF) dileptons for the mSUGRA benchmark model point SPS1a [20].

metastable charged particles (MMCPs), typically in scenarios with a gravitino LSP, where the next-to-lightest supersymmetric particle (NLSP) is charged and long-lived because the decay to the gravitino is via a very weak gravitational coupling. The latter also includes so-called **R-hadrons** if the NLSP has color charge, which means that it will hadronize after production and be a short-lived but very massive meson or baryon. We should also mention the searches for the extra Higgs states predicted in the MSSM.[6]

7.3 Current bounds on sparticle masses

With the LHC running and collecting data the details in this section are continuously becoming out-of-date. We will still try to make some general remarks on the current limits. Most of these limits are from Run I of the LHC at 8 TeV with analysis using up to 20 fb^{-1} of data. This is strongest current limits are on the squark and gluino masses simply because of the production cross section. Bounds on EW gauginos and sleptons exist, but these are

[6]But we really don't have time.

either model dependent (depend on squark/gluino mass assumptions and cascade decays), or weaker if the rely only on electroweak production. Direct bounds from the LHC experiments ATLAS and CMS now superseed bounds from other colliders (Tevatron and LEP) in almost all channels.

7.3.1 Squarks and gluinos

In Fig. 7.12 we show the most recent limits from ATLAS in the jets plus missing energy channel, using all currently available data at the highest energy of 8 TeV. The limit has been interpreted within the mSUGRA model, where the parameters $\tan\beta$ and A_0 have been chosen in order to give relatively large Higgs masses for small values of $m_{1/2}$ and m_0. The figure also shows the corresponing first and second generation squark masses, the gluino mass and the higgs mass for these parameter values. From ATLAS we then have the following approximate bounds in mSUGRA: $m_{\tilde{q}} > 1600\,\text{GeV}$ and $m_{\tilde{g}} > 1100\,\text{GeV}$.

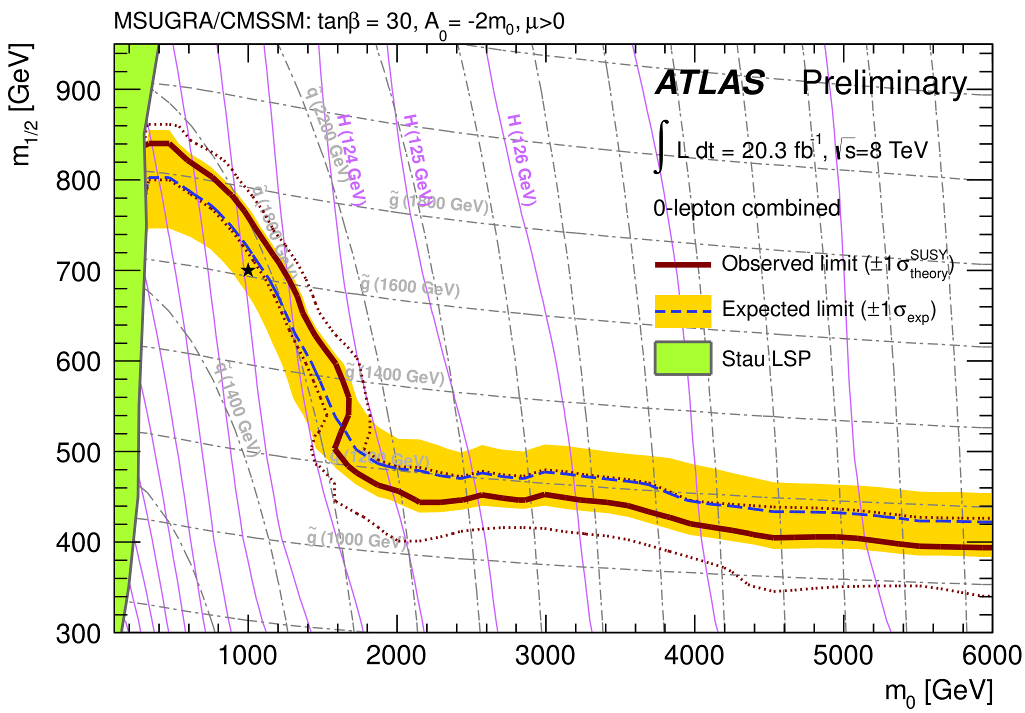

Figure 7.12: Plot of the excluded area in the $m_{1/2}$-m_0 plane of the mSUGRA parameter space for $\tan\beta = 30$, $A_0 = -2m_0$ and $\mu > 0$. The limit is the red line. The green area is theoretically forbidden because it has a charged LSP (the stau) [21].

Notice that in the figure the direct squark mass bound is almost equivalent to the mass required for a sufficiently heavy higgs, thus the direct search does not yet constrain the squarks masses significantly more than the indirect constraint from the higgs mass.

An important question is how these bounds change as we move away from the mSUGRA assumptions. By pushing the gluino up in mass using M_3 the production cross section falls significantly. Limits of at most $m_{\tilde{q}} > 850\,\text{GeV}$ assuming only squark production were quoted in the summer 2013 conferences, and the limit falls away entirely if $m_{\tilde{\chi}_1^0} > 300\,\text{GeV}$ becuase the decay products of the squark (quarks) have too little energy.[7] Should one squark generation or flavour be significantly lighter than the others this means a further reduction in the production cross section and thus an even weaker bound. It is also fairly clear that removing R-parity, meaning that the LSP decays, also weakens the above conclusions due to the possible absence of significant missing energy. Thus, despite popular optinion, the generic squark mass bounds outside of specific scnearios like mSUGRA, are currently still fairly weak, in particular compared to indirect bounds via the higgs.

The gluino mass bound is somewhat more robust. Pushing up squark masses and assuming only gluino production gives $m_{\tilde{g}} > 1200\,\mathrm{GeV}$ (when also including CMS results), however, the limit again disappears for $m_{\tilde{\chi}_1^0} > 480\,\mathrm{GeV}$.

7.3.2 Sbottom

The above bounds on the first and second generation squarks do not apply to the third generation as they are generically lighter and can have more complicated decay signatures. In Fig. 7.13 we see current best limits from ATLAS on the lightest sbottom taken from [22]. Note that this limit assumes 100% branching ratio for $\tilde{b}_1 \to b\tilde{\chi}_0^1$. If this branching ratio is reduced to 60% the excluded upper limit on the sbottom mass for $m_{\tilde{\chi}_1^0} < 150\,\mathrm{GeV}$ is reduced to 520 GeV. Similarly for $m_{\tilde{b}_1} = 250\,\mathrm{GeV}$, the upper limit on $m_{\tilde{\chi}_1^0}$ is reduced by 30 GeV.

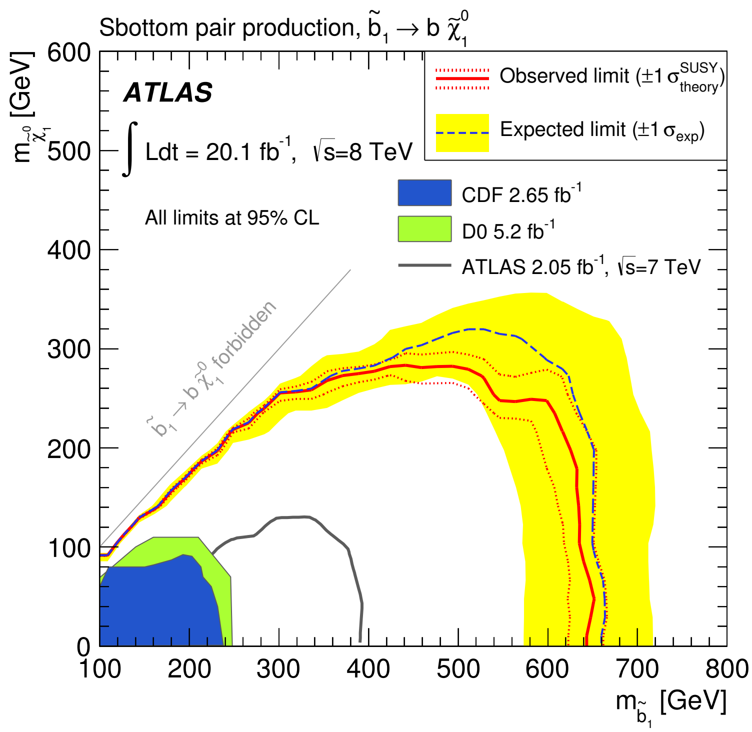

Figure 7.13: Plot of the excluded area in the $(m_{\tilde{b}_1},m_{\tilde{\chi}_1^0})$ plane. The limit from ATLAS is the red line, while the green and blue colored areas are excluded from different Tevatron experiments [22].

[7]The technical term for this is **soft** decay products.

7.3.3 Stop

For the stop there are many possible competing decay chanels, meaning that any limit set is very model dependent. The two main decay categories for the lightest stop are via the chargino, if available, $\tilde{t}_1 \to b\tilde{\chi}_1^{\pm}$, and directly to the neutralino $\tilde{t}_1 \to t\tilde{\chi}_1^0/Wb\tilde{\chi}_1^0/c\tilde{\chi}_1^0$, where the dominant decay mode depends on the stop–neutralino mass difference. A summary of (the many) current ATLAS limits for the stop is found in Fig. 7.14. It is important to notice the surviving possibility of quite light stops in conjunction with a light neutralino or chargino.

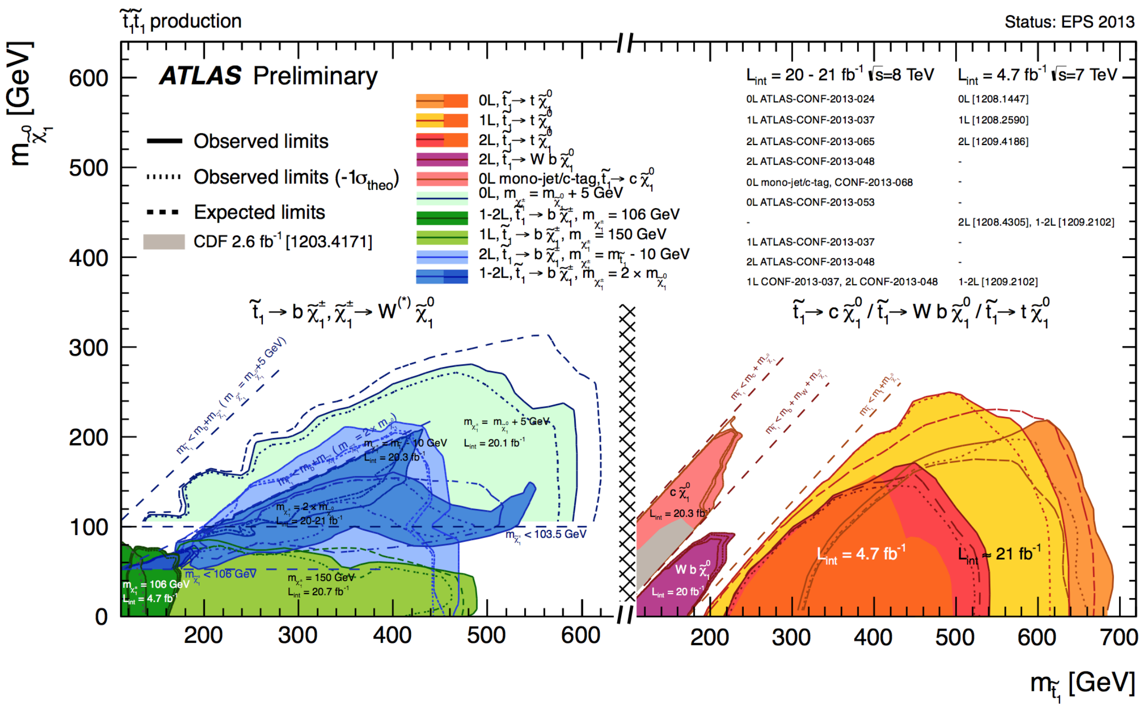

Figure 7.14: Plot of the excluded area in the ($m_{\tilde{t}_1}$,$m_{\tilde{\chi}_1^0}$) plane for the two main decay categories. References for the individual analysis given in figure.

7.3.4 Sleptons

As mentioned above the mass bounds on sleptons will be very dependent on the assumed production mechanism. The most model independent bounds come from assuming only electroweak pair production as in [23], which presents the results of a search for two opposite-sign same-flavour (OSSF) leptons with missing energy. The result for degenerate right- and left-handed smuons and selectrons, assuming 100% branching ratio in the neutralino, is shown in Fig. 7.15. Individual selectron and smuon limits are significantly weaker. Limits from this kind of search in complete models, such as mSUGRA, are typically much weaker than those that come from searches for jets and missing energy, *e.g.* see Fig. 7.8.

There are currently no constraining searches for direct pair production of staus.

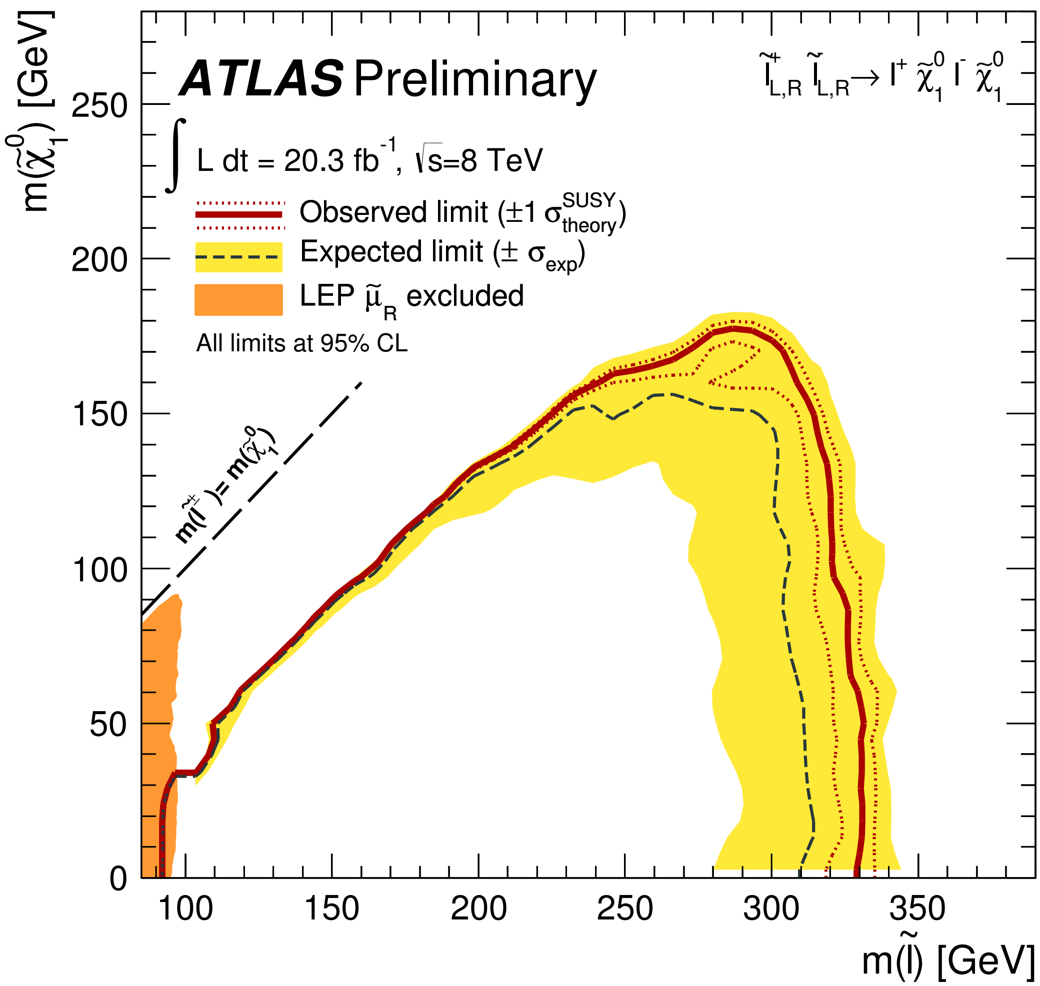

Figure 7.15: Plot of the excluded area in the $(m_{\tilde{l}}, m_{\tilde{\chi}^0_1})$ plane for mass degenerate right- and left-handed smuons and selectrons [23].

7.3.5 Charginos and neutralinos

As for the sleptons, bounds are dependent on the production process assumed. With chargino pair production, $\tilde{\chi}^+_1\tilde{\chi}^-_1$, the search for two OSSF leptons discussed in the previous subsection again applies because the chargino can decay via a slepton or sneutrino [23]. We show the results assuming $m_{\tilde{l}} = m_{\tilde{\nu}} = (m_{\tilde{\chi}^\pm_1} + m_{\tilde{\chi}^0_1})/2$, and again 100% branching ratio, in Fig. 7.16.

We can also search for $\tilde{\chi}^0_2\tilde{\chi}^\pm_1$ production with three leptons and missing energy. The results from [24], where 100% branching ratio into vector bosons is assumed, are shown in Fig. 7.17.

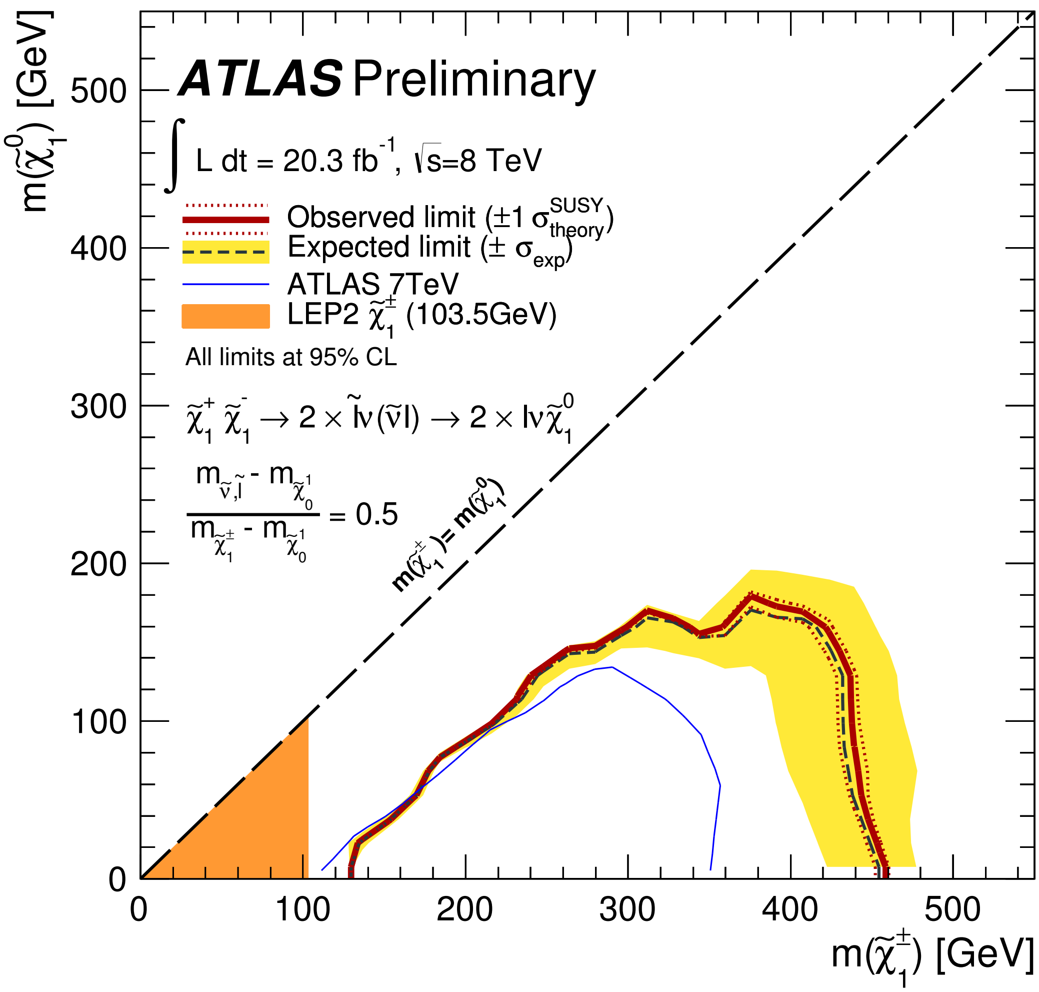

Figure 7.16: Plot of the excluded area in the $(m_{\tilde{\chi}_1^{\pm}}, m_{\tilde{\chi}_1^0})$ plane [23].

7.4 Supersymmetry at lepton colliders

Most lepton colliders are e^+e^--colliders, although plans are being made for a muon collider where there is less bremsstrahlung because of the higher muon mass, meaning that higher energies can be reached. The highest energy so-far at an e^+e^--collider was 209 GeV CoM-energy at LEP2 in 2000.

Most supersymmetry searches at lepton colliders rely on pair production from $e^+e^- \to \gamma^*/Z^*$ to set limits, and for R-parity conserving supersymmetry we (again) rely on misisng energy $\not{E}$ as an essential signature, however, since the longitudinal momentum is now exactly known full energy conservation can in principle be used. In practice this is challenging at high energies because of collinear Bremsstrahlung. This will be a particularly difficult for a future $0.5-3.0$ TeV CoM International Linear Collider (ILC) or the Compact LInear Collider

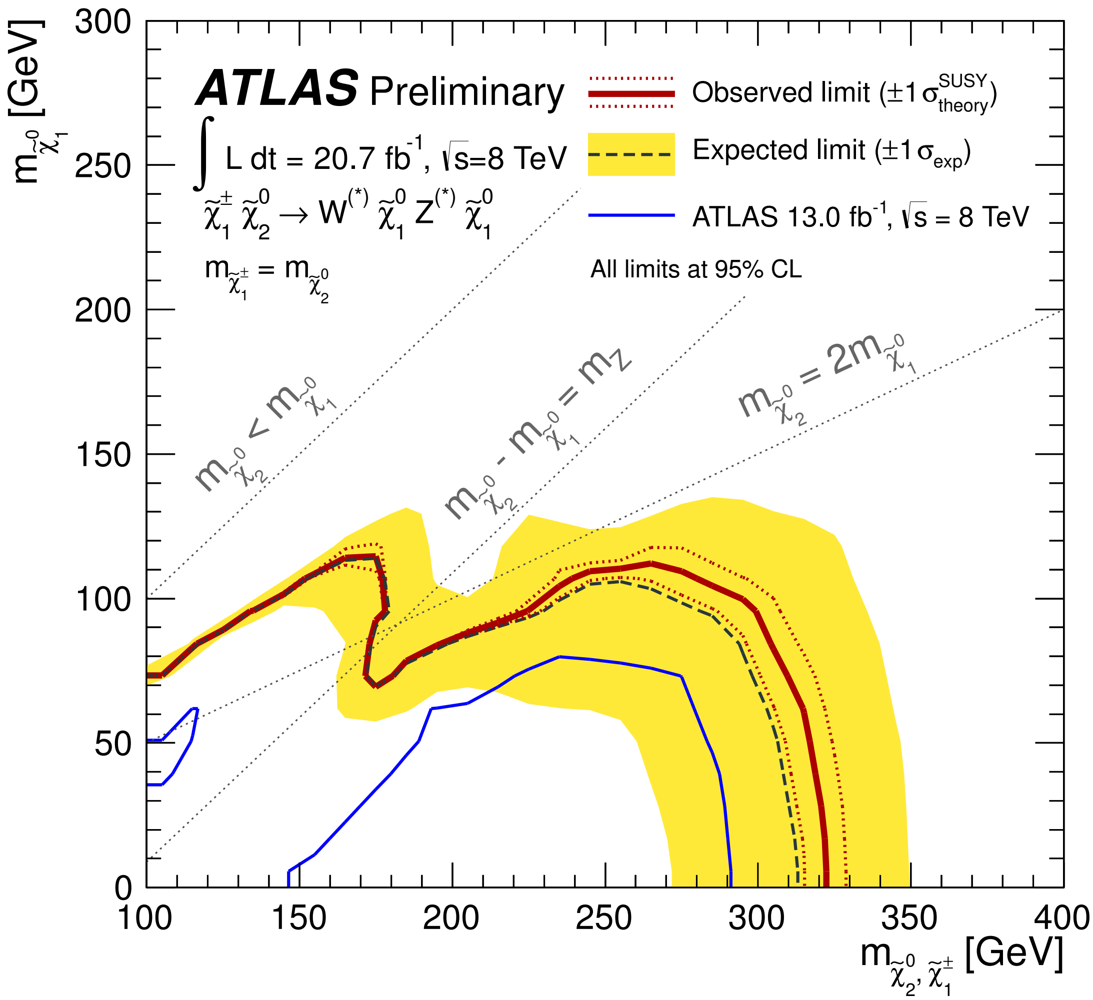

Figure 7.17: Plot of the excluded area in the $(m_{\tilde{\chi}_1^\pm}, m_{\tilde{\chi}_1^0})$ plane assuming $m_{\tilde{\chi}_1^\pm} = m_{\tilde{\chi}_2^0}$ [24].

(CLIC) project.[8]

We can estimate the amplitude of the sfermion pair production process shown in Fig. 7.18. We can write down the matrix element as:

$$\mathcal{M} = \bar{v} i e \gamma^\mu u \frac{-i g_{\mu\nu}}{k^2 + i\epsilon} [-ie \cdot e_f (p_1 - p_2)^\nu], \tag{7.10}$$

which gives a squared matrix element of, assuming that the CoM s is much greater than m_Z and taking into account both the photon and the Z:

$$|\mathcal{M}|^2 \simeq \frac{g^4 e_f^2}{8 \cos\theta_W} \frac{st + (m_{\tilde{f}}^2 - t)^2}{s^2} \times (1 + (4\sin^2\theta_W - 1)^2). \tag{7.11}$$

[8]For more information on these projects see the websites for the International Linear Collider `http://www.linearcollider.org/` and the Compact LInear Collider `http://clic-study.org/`

We take safely take $(1+(4\sin^2\theta_W-1)^2)\simeq 1$. The complete differential cross-section is then:

$$\frac{d\sigma}{dt}=\frac{1}{32\pi}\frac{1}{s^2}|\mathcal{M}|^2. \tag{7.12}$$

This cross section is small due to the coupling factor g^4 and sfermion mass suppression.

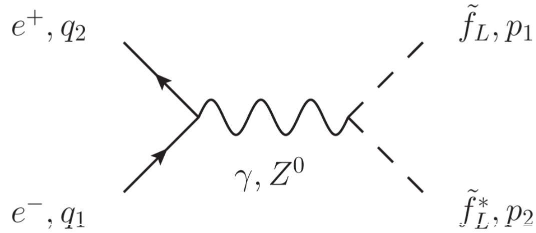

Figure 7.18: Feynman diagram for the pair production of left-handed sfermions in the s-channel at a linear collider.

For charginos and neutralinos, as in the case of hadron colliders, the production cross section depends on their wino, bino and higgsino components. The selectron and electron sneutrino have a special rôle for e^+e^- colliders due to t-channel diagrams. Figure 7.19 shows the t-channel diagrams that are important in pair production at a e^+e^- collider. We show an example of the slepton pair production cross section including the Z-resonance at low energies and the t-channel contributions from neutralinos in Fig. 7.20. Neutralino pair production with t-channel selectron exchange does not suffer from the same problems as neutralino pair production at a hadron collider in the s-channel. However, the process depends on the selectron mass as $m_{\tilde{e}}^{-4}$ for large mass values.

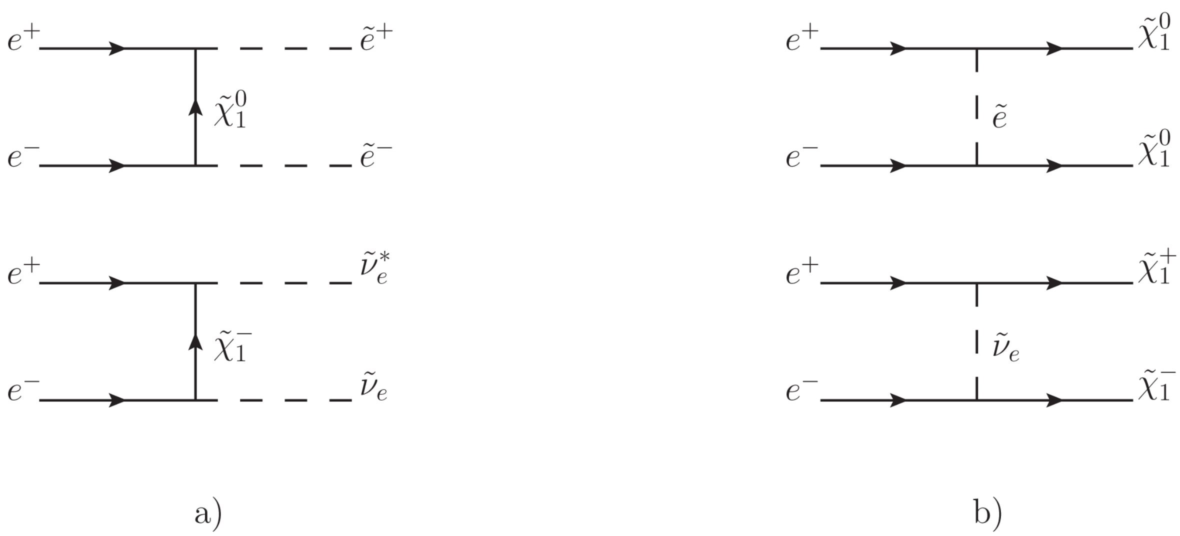

Figure 7.19: The t-channel diagrams for pair production of selectrons and electron sneutrinos a) and gauginos b).

Should a signal be found the parameters of the new particles can be *precisely measured* at a lepton collider. either through threshold scans of cross section where the cross section is measured as a function of $\sqrt{s}$. Or, through kinematical distributions, *e.g.* in $e^+e^- \to \tilde{l}^+\tilde{l}^- \to l^+l^-\tilde{\chi}_1^0\tilde{\chi}_1^0$ the energy distribution for the final state leptons is a uniform distribution between $E_{\min}$ and $E_{\max}$ where

$$E_{\max/\min} = \frac{\sqrt{s}}{4}\left(1 - \frac{m_{\tilde{\chi}_1^0}^2}{m_{\tilde{l}}^2}\right)\left(1 \pm \left(1 - \frac{4m_{\tilde{l}}^2}{s}\right)^{1/2}\right). \tag{7.13}$$

7.4.1 Current bounds at lepton colliders

The below bounds are all from the LEP (Large Electron Positron) collider, running from 1989 until 2000, which outdated all previous bounds with a top energy of $\sqrt{s} = 209$ GeV, recording an integrated luminosity of 233 pb^{-1} above 204 GeV. Results exist from all four LEP experiments ALEPH, DELPHI, L3 and OPAL.[9] The numbers are all taken from the PDG (Particle Data Group) review [25]. While these bounds often come from pair-production of the relevant sparticles, and thus are less modell dependent than the hadron collider bounds, there remains some model dependence in many results, which, unfortunately, is sometimes ignored in the litterature. Complicating matters is a reliance by the LEP experiments on theoretical assumptions such as GUT-scale coupling and gaugino mass unification.

- Selectron: $m_{\tilde{e}_L} > 107$ GeV and $m_{\tilde{e}_R} > 73$ GeV (ALEPH 2002) in searches for acoplanar di-electrons.[10] The limit is the result of a scan over MSSM parameter space assuming a common m_0 and $m_{1/2}$ at GUT scale. Interpreted in mSUGRA with $A_0 = 0$ the bounds are 152 GeV and 95 GeV, respectively. Due to strict limits on the measured Z-width, there is a model independent limit of $m_{\tilde{e}_{L/R}} > 40$ GeV.[11]

- Smuon: $m_{\tilde{\mu}_R} > 94$ GeV (DELPHI 2003). The limit is obtained as in the MSSM scenario for the selectron.

- Stau: $m_{\tilde{\tau}_1} > 81.9$ GeV (DELPHI 2003) assuming exclusive $\tilde{\tau}_1 \to \tau\tilde{\chi}_1^0$ and $m_{\tilde{\tau}_1} - m_{\tilde{\chi}_1^0} > 15$ GeV.

- Sneutrinos: From the Z-width we can obtain the model independent limit $m_{\tilde{\nu}} > 44.7$ GeV. From collider experiments we have $m_{\tilde{\nu}} > 94$ GeV (DELPHI 2003) in neutralino & slepton searches. This assumes $m_{\tilde{e}_R} - m_{\tilde{\chi}_1^0} > 10$ GeV.

- Neutralino: $m_{\tilde{\chi}_1^0} > 46$ GeV (DELPHI 2003). This limit is derived from the direct searches for $\tilde{\chi}_1^\pm\tilde{\chi}_2^0$ and $\tilde{\chi}_1^0\tilde{\chi}_2^0$. This assumes gauge coupling unification and a common gaugino mass $m_{1/2}$ at GUT scale. Even in the Z-decays, the contribution depends on the higgsino part in the lightest neutralino, so $m_{\tilde{\chi}_1^0} \simeq 0$ GeV is in principle allowed [26].

- From the Z-width we can extract a strict limit of $m_{\tilde{\chi}_1^\pm} \geq 45$ GeV. We also have $m_{\tilde{\chi}_1^\pm} \geq 94$ GeV (DELPHI 2003), assuming GUT scale universality of m_0 and $m_{1/2}$ and using multiple direct search channels from production of charginos, neutralinos and sleptons. It also assumes either no third generation mixing or $m_{\tilde{\chi}_1^\pm} - m_{\tilde{\chi}_1^0} > 6$ GeV.

[9] Most of which are silly acronyms of course.

[10] The observant reader will notice that two electrons are always in the same plane, however, when experimentalists say acoplanar, they mean not in one plane with the beam axis.

[11] Similar model independent limits around half the Z-mass exists for all sparticles that couple to the Z.

7.5 Precision observables

A different way to exclude supersymmetric models is their indirect effect on very accurately measured SM processes, so-called precision observables, through loop diagrams with sparticles. We will here discuss four of the most sensitive probes: electroweak precision observables, the value of the anomolous magnetic moment of the muon $(g-2)_\mu$, the flavour changing neutral current (FCNC) process $b \to s\gamma$ and the very rare (and FCNC) process $B_s \to \mu\mu$.

7.5.1 Electroweak precision observables

When we talk about electroweak precision observables, we study parameters such as M_W (or M_Z), Γ_W, Γ_Z, m_t and $\sin\theta_W$, as well as the Higgs mass m_h and the properties of the Higgs such as its couplings to all the other particles (gauge and Yukawa couplings) and its self-coupling.

Up to last year we studied all of these as *functions* of the unknown Higgs mass, looking for devations that could be a sign of supersymmetry. We show a fit to all available electroweak data and direct exclusion bounds in Fig. 7.21 by the `Gfitter` collaboration just before the LHC started taking data, a fit pretty much indicating that the most probable SM Higgs mass was 125 GeV.

Figure 7.22 shows a similar plot for mSUGRA. At that time the absolute minimum of the fit, even taking into account the different number of parameters, gave a better a better fit for mSUGRA, $\min \chi^2_{\text{mSUGRA}} < \min \chi^2_{\text{SM}}$, but this changed quickly when the Higgs was found because of the position of the two minima.

Now all the parameters of the SM—neutrinos excepted—have been determined to some precission. Thus the SM is a completely constrained system. If we now do a electroweak fit the situation looks like that in Fig. 7.23, where we show the global fit compared to the measured values of the W and top masses. Clearly what we are seeing here is (still?) consistent with the SM.

7.5.2 $(g-2)_\mu$

The anomalous magnetic moment of the muon, $(g-2)_\mu$, has been very precisely measured by the E821 experiment at BNL [28] to be:

$$g_\mu = 2.00116592089(63),$$

or, in terms of a_μ which is the devation from 2,

$$a_\mu^{(\text{exp})} = 11659208.9(6.3) \cdot 10^{-10},$$

where the parenthesis indicates the uncertainty on the last digits. Figure 7.24 a) shows the lowest order $\mu \to \mu\gamma$ diagram. Loop corrections to this diagram give a_μ. In the SM we find the prediction

$$a_\mu^{\text{SM}} = 11659183.0(5.1) \cdot 10^{-10},$$

giving a difference with respect to the experimental value of

$$\delta_{a_\mu} \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (25.9 \pm 8.1) \cdot 10^{-10},$$

a value which is 3.2σ away from zero. This is probably the clearest discrepancy that exists today between the SM and measurements.

However, we should be aware that one of the SM contributions, the so called **hadronic vacuum polarization** as shown in Figure 7.24 b), involves hadronic loops where one has to rely on experimental information on low energy $e^+e^- \to \gamma^* \to$ hadrons in order to estimate a contribution of $a_\mu^{\mathrm{HVP}} = 10.5(2.6) \cdot 10^{-10}$, which is of the same order of magnitude as the discrepancy, and may be prone to errors in the interpretation.

One-loop corrections to $(g-2)_\mu$ in the MSSM are shown in Figure 7.24 c) and d). These contribute opposite sign terms $a_\mu(\tilde{\chi}^0)$ and $a_\mu(\tilde{\chi}^-)$. A thorough analysis shows that we need $\mu > 0$ in order to give a positive contribution that will close the gap between the experimnetal value and the prediction. In order to get a sufficiently large contribution the loop masses must be less than $500 - 600\,\mathrm{GeV}$ for $\tan\beta = 40 - 50$ and $200 - 300\,\mathrm{GeV}$ for $\tan\beta \simeq 10$.

7.5.3 $b \to s\gamma$

The process $b \to s\gamma$ is a FCNC process which must proceed through loops. Figure 7.25 a) shows the SM process. This is suppressed by the smallness of the CKM entries, and the large masses m_W and m_t.

The process has been measured in decays of the type $B \to X_s\gamma$, *e.g.* in $B \to K\gamma$, and calculated at NNLO to be $\mathrm{Br}(B \to X_s\gamma)_{\mathrm{SM}} = (3.36 \pm 0.23) \cdot 10^{-4}$ for $E_\gamma \geq 1.6\,\mathrm{GeV}$ [29, 30].[12]

Supersymmetry may contribute, *e.g.* with diagrams such as Fig. 7.25 b) where the $m_{bs}^2 \tilde{b}^* \tilde{s}$ mass term that changes a $\tilde{b}_1$ to a $\tilde{s}$ is a soft breaking off-diagonal term, often denoted δ_{23}. The main MSSM contributions are expected to come from chargino–stop[13] and charged higgs–top loops, as shown in Figs. 7.25 c) and d), respectively. However, there is little room for effects from superymmetry since the current experimental world average is $\mathrm{Br}(B \to X_s\gamma) = (3.55 \pm 0.26) \cdot 10^{-4}$ (PDG 2010). This means that either the charged Higgs is heavy enough and the stop-scharm soft mass term small enough, or that there are cancellations between the contributions.

7.5.4 $B_s \to \mu^+\mu^-$

The process $B_s \to \mu^+\mu^-$ is another FCNC process as either the bottom or the strange quark must change flavour in order to couple to the muons. The SM process is shown in Fig. 7.26 a), involving an intermediary Z-boson. There is additional suppression from a CKM factor in one of the W-vertices, in order to change a third generation quark to a second generation quark, or *vice versa*. On top of this, it also suffers from what is called **helicity suppression** in the SM. The Z-boson is spin-1, while the starting point meson B_s is spin-0 (pseudoscalar), meaning that the spins of the quarks are opposite. At some point in the diagram the helicity (chirality) must "flip". This introduces an extra suppression proportional to $m_\mu^2/M_{B_s}^2$, making the expected rate extremely small and sensitive to supersymmetry contributions. We get a similarly supressed process for B_d with a $\bar{d}$-quark instead of the $\bar{s}$ in the initial state.

The predicted SM branching ratios for these processes are [32]:

$$\begin{aligned} \mathrm{Br}(B_s \to \mu^+\mu^-) &= (3.65 \pm 0.23) \cdot 10^{-9}, &(7.14)\\ \mathrm{Br}(B_d \to \mu^+\mu^-) &= (1.06 \pm 0.09) \cdot 10^{-10}. &(7.15)\end{aligned}$$

[12]For the process $b \to d\gamma$ the SM calculation yields $\mathrm{BR}(B \to X_d\gamma) = 1.73^{+0.12}_{-0.22} \cdot 10^{-5}$.

[13]We usually expect a higher generation off-diagonal terms to be larger due to RGE running controlled by Yukawa couplings.

First evidence for the B_s decay was shown by the LHCb collaboration in 2012. The final observation required combining Run I data from both LHCb and CMS, and was published in 2014 [33]. The current values are:

$$\begin{aligned}
\mathrm{Br}(B_s \to \mu^+\mu^-) &= 2.8^{+0.7}_{-0.6} \cdot 10^{-9}, && (7.16)\\
\mathrm{Br}(B_d \to \mu^+\mu^-) &= 3.9^{+1.6}_{-1.4} \cdot 10^{-10}, && (7.17)
\end{aligned}$$

where one should keep in mind that the B_d decay has only evidence at 3.2σ significance.

In the MSSM there are contributions from process such as shown in Fig. 7.26 b). These contributions are proportional to $\tan^6\beta$, which makes the decay process highly sensitive to scenarios with large $\tan\beta$. To see this dependence, notice that μ couples to the mediating heavy higgses H/A^0 through the Yukawa term $y^l_{22}L_2H_d\overline{E}_2$ in the superpotential, and the Yukawa constant in this term, $y^l_{22} = y_\mu$, is connected to the fermion mass through $m_\mu = y_\mu v\cos\beta$. Thus this vertex is proportional to $1/\cos\beta$ or $\tan\beta$, giving a factor $\tan^2\beta$ in the amplitude squared.[14]

Furthermore, a chargino(higgsino)–stop loop can couple the strange and bottom quarks to the higgs. These couplings are proportional to the bottom Yukawa coupling y_b, from the superpotential terms $y^d_{33}Q_3H_d\bar{D}_3$, which appears in the stop–chargino–bottom vertex, and the $y^u_{32}Q_3H_d\bar{D}_2$, which appears in the strange–chargino–stop vertex. Both these Yukawa couplings are proportional to y_b and thus to $1/\cos\beta$, giving a further factor of $\tan^4\beta$ in the amplitude squared. This $\tan\beta$ dependence makes $B_s \to \mu^+\mu^-$ an excellent channel for discovering supersymmetry, and puts very stringent bounds on the sparticle masses in large $\tan\beta$ scenarios.

7.6 Excercises

Exercise 7.1 From relativistic kinematics, show Eq. (7.9). *Hint:* the choice of rest frame is very important in order to simplify the calculation.

Exercise 7.2 Find the total cross section for the process $q\bar{q} \to \tilde{q}\tilde{q}^*$ via an s-channel gluon shown in Fig. 7.27.

Solution: We will in the following take all outgoing momenta to go out of the vertex. The indices $abcd$ are gluon indices $(1, ..., 8)$, rs are spin indices $(1, 2)$, and $ijmn$ are color indices $(1, 2, 3)$. The relevant Feynman rules are as follows:

- Incoming quark: $u^r(k)$.
- Incoming antiquark: $\bar{v}^s(p)$.
- Gluon propagator: $-i\frac{g_{\mu\nu}}{s}\delta^{ab}$.
- Vertex $q\bar{q}g$: $-it^a_{ij}\gamma^\mu g_s$.

[14]Remember that in the limit of large $\tan\beta$

$$\cos\beta = \pm\frac{1}{\sqrt{1+\tan^2\beta}} = \pm\frac{1}{\tan\beta\sqrt{1+\frac{1}{\tan^2\beta}}} \simeq \pm\frac{1}{\tan\beta}. \qquad (7.18)$$

- Vertex $\tilde{q}\tilde{q}^* g$: $-it^b_{mn}(k'-p')^\nu g_s$.

We will assume the SM particles to have negligable mass compared to the squarks.

The matrix element is then given as

$$\mathcal{M} = -\frac{g_s^2}{s} t^a_{ij} t^b_{mn} \delta^{ab} \bar{v}^s \gamma^\mu u^r (k'-p')_\mu.$$

In the squared amplitude we average over all incoming spin and color, and sum over the outgoing:

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{4}\cdot\frac{1}{9}\frac{g_s^4}{s^2}\sum_{ab}\sum_{cd}\sum_{rs}\sum_{ijmn} t^a_{ij}t^b_{mn}t^c_{ij}t^d_{mn}\delta^{ab}\delta^{cd}\bar{v}^s\gamma^\mu u^r \bar{u}^r\gamma^\nu v^s (k'-p')_\mu (k'-p')_\nu \\
&= \frac{1}{4}\cdot\frac{1}{9}\frac{g_s^4}{s^2}\underbrace{\sum_{ijmn}\left(\sum_a t^a_{ij}t^a_{mn}\sum_c t^c_{ji}t^c_{nm}\right)}_{\equiv C_f}\sum_{rs}\bar{v}^s\gamma^\mu u^r \bar{u}^r\gamma^\nu v^s (k'-p')_\mu (k'-p')_\nu \\
&= \frac{1}{4}\cdot\frac{1}{9}\frac{g_s^4}{s^2} C_f p_\alpha k_\beta TR(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu)(k'-p')_\mu (k'-p')_\nu \\
&= \frac{1}{9}\frac{g_s^4}{s^2} C_f p_\alpha k_\beta (p^\mu k^\nu - p^\beta k_\beta \eta^{\mu\nu} + p^\nu k^\mu)(k'-p')_\mu (k'-p')_\nu \\
&= \frac{1}{9}\frac{g_s^4}{s^2} C_f (2p\cdot(k'-p')k\cdot(k'-p') - p\cdot k(k'-p')\cdot(k'-p')),
\end{aligned}$$

where we have isolated the color factors into the coefficient C_f.

In the centre of mass frame, we have $p = (E, \vec{p})$, $k = (E, -\vec{p})$, $p' = (E, \vec{p'})$ and $k' = (E, -\vec{p'})$. From this we find

$$\begin{aligned}
2p\cdot(k'-p')k\cdot(k'-p') - p\cdot k(k'-p')\cdot(k'-p') &= 2\cdot 4|\vec{p}|^2|\vec{p'}|^2(1-\cos^2\theta) \\
&= 2s|\vec{p'}|^2(1-\cos^2\theta),
\end{aligned}$$

where $\theta$is the acute angle between the incoming quarks, and $s = (p+k)^2 = 4E^2 = 4|\vec{p}|^2$. This gives the squared averaged amplitude

$$|\bar{\mathcal{M}}|^2 = \frac{2}{9}\frac{g_s^4}{s} C_f \cdot |\vec{p'}|^2(1-\cos^2\theta),$$

and the differential cross section

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{|\vec{p'}|^2}{32\pi^2\sqrt{s}s}|\bar{\mathcal{M}}|^2 \\
&= \frac{1}{144}\frac{g_s^4}{\pi^2\sqrt{s}s^2} C_f |\vec{p'}|^3(1-\cos^2\theta).
\end{aligned}$$

Integrating over the solid angle then gives the total cross section

$$\begin{aligned}
\sigma &= \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega \\
&= 2\pi \frac{1}{144} \frac{g_s^4}{\pi^2\sqrt{s}s^2} C_f |\vec{p'}|^3 \int_{-1}^{1} (1-\cos^2\theta) d(\cos\theta) \\
&= \frac{4}{3} 2\pi \frac{1}{144} \frac{g_s^4}{\pi^2\sqrt{s}s^2} C_f |\vec{p'}|^3 \\
&= \frac{1}{54\pi} \frac{g_s^4}{\sqrt{s}s^2} C_f |\vec{p'}|^3.
\end{aligned}$$

We can rewrite $|\vec{p'}|^3$ by noticing that

$$|\vec{p'}| = \sqrt{E^2 - m^2} = \frac{1}{2}\sqrt{4E^2 - 4m^2} = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4m^2}{s}}.$$

The color factor is calculated below and found to be $C_f = 2$, so the total cross section is

$$\sigma = \frac{g_s^4}{216\pi s}\sqrt{\left(1 - \frac{4m^2}{s}\right)^3}.$$

Using $\alpha_s = \frac{g_s^2}{4\pi}$, and assuming that both the left-, and right-handed squarks have the same mass, we arrive at the final expression

$$\sigma = \frac{4}{27}\frac{\pi\alpha_s^2}{s}\sqrt{\left(1 - \frac{4m^2}{s}\right)^3}.$$

To calculate the color factor C_f we use that the sum over the generators t is given (see for example [34]) as:

$$\sum_a t_{ij}^a t_{mn}^a = \frac{1}{2}\left(\delta_{in}\delta_{jm} - \frac{1}{N_C}\delta_{ij}\delta_{mn}\right),$$

and using that $\delta_{ij} = \delta_{ji}$, we have

$$\begin{aligned}
C_f &= \frac{1}{4}\sum_{ijmn}\left(\delta_{in}\delta_{jm} - \frac{1}{N_C}\delta_{ij}\delta_{mn}\right)\left(\delta_{jm}\delta_{in} - \frac{1}{N_C}\delta_{ji}\delta_{nm}\right) \\
&= \frac{1}{4}\sum_{ijmn}\left(\delta_{in}\delta_{jm}\delta_{jm}\delta_{in} - \frac{2}{N_C}\delta_{ij}\delta_{mn}\delta_{jm}\delta_{in} + \frac{1}{N_C^2}\delta_{ij}\delta_{mn}\delta_{ji}\delta_{nm}\right) \\
&= \frac{1}{4}\sum_{ijmn}\left(\delta_{in}\delta_{in}\delta_{jm}\delta_{jm} - \frac{2}{N_C}\delta_{ij}\delta_{mn}\delta_{jm}\delta_{in} + \frac{1}{N_C^2}\delta_{ij}\delta_{ij}\delta_{mn}\delta_{mn}\right) \\
&= \frac{1}{4}\left(N_C^2 - \frac{2}{N_C}N_C + \frac{1}{N_C^2}N_C^2\right) \\
&= \frac{1}{4}(N_C^2 - 1) = 2.
\end{aligned}$$

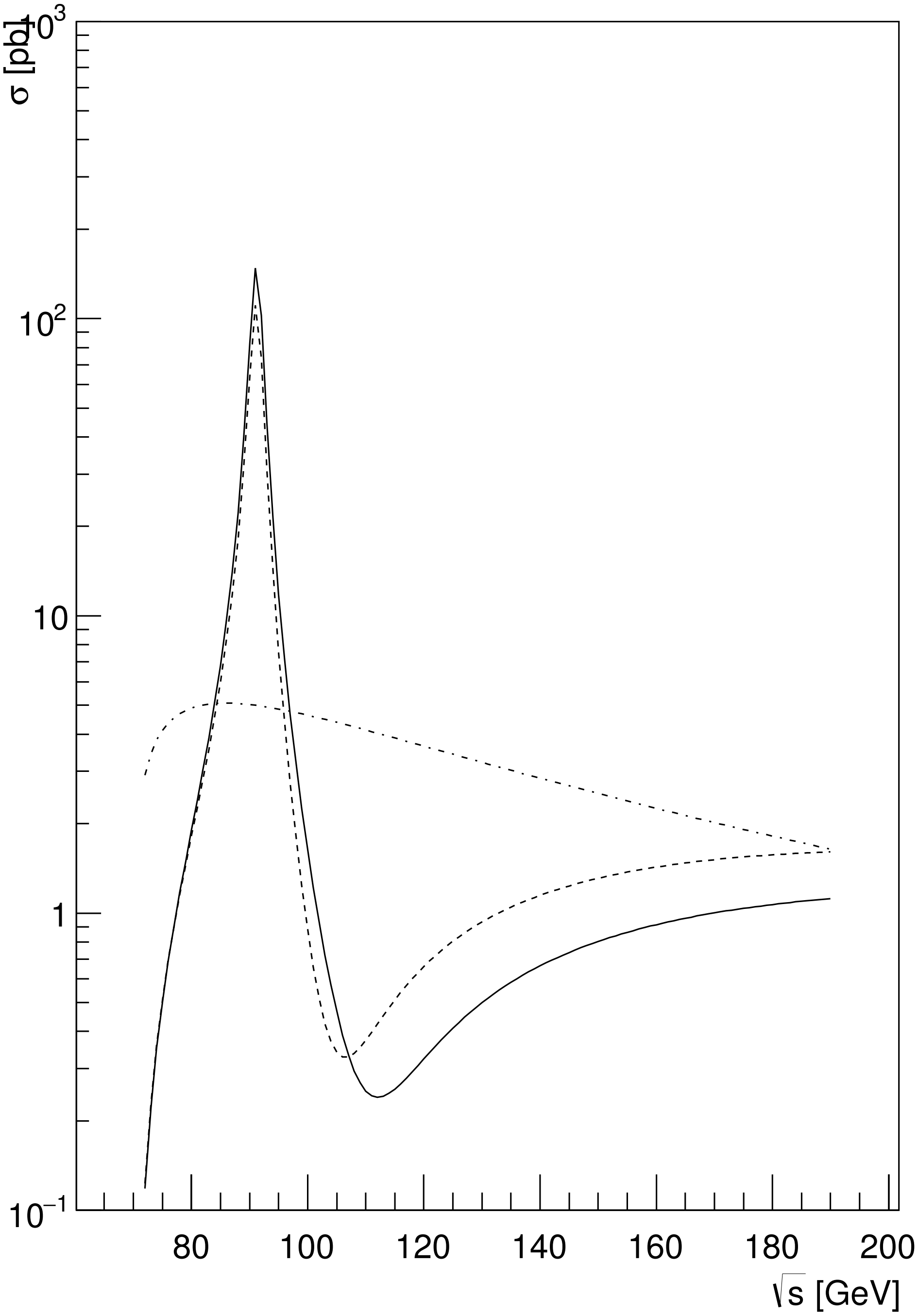

Figure 7.20: Cross sections for selectron pair production as a function of energy. The cross sections for $\tilde{e}_L^* \tilde{e}_L$ (solid line), $\tilde{e}_R^* \tilde{e}_R$ (dashed line), and $\tilde{e}_L^* \tilde{e}_R$ (dashed dotted line) are shown separately. The particular model point has a common slepton mass of $m_{\tilde{e}_{L/R}} = 35$ GeV.

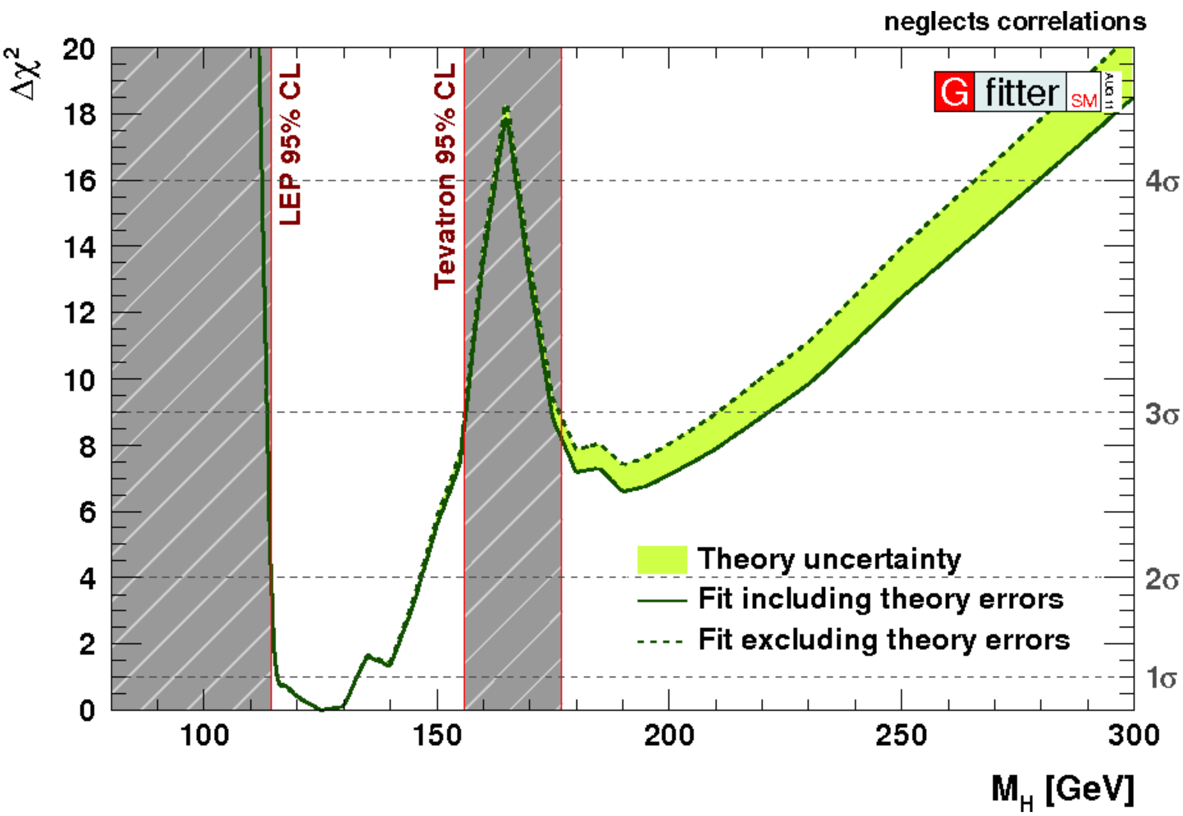

Figure 7.21: Plot of the total $\Delta\chi^2$ from all precision variable measurements and the direct exclusions bounds for the SM Higgs from LEP and the Tevatron, as a function of the Higgs mass.

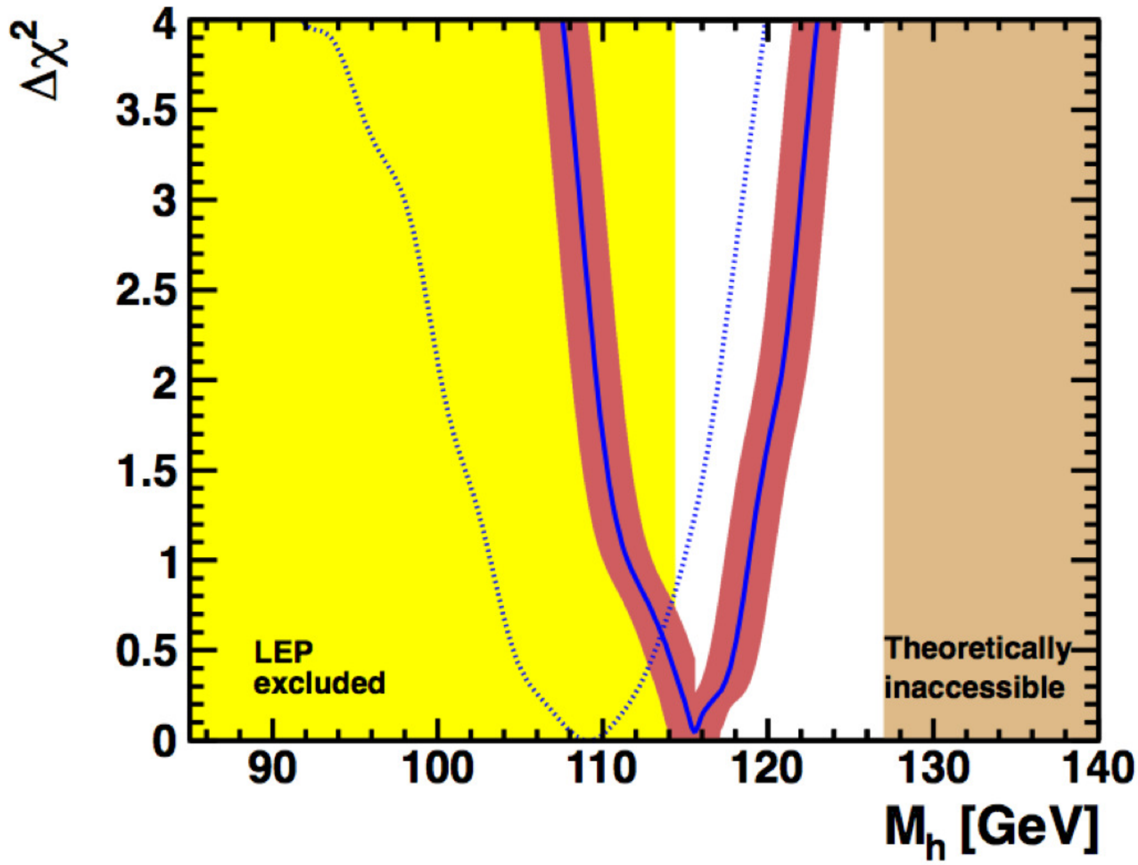

Figure 7.22: Plot of the $\Delta\chi^2$ from all precision variable measurements for mSUGRA as a function of the Higgs mass. The yellow area shows the experimentally excluded area, while the brown shows the theoretically inaccessible area.

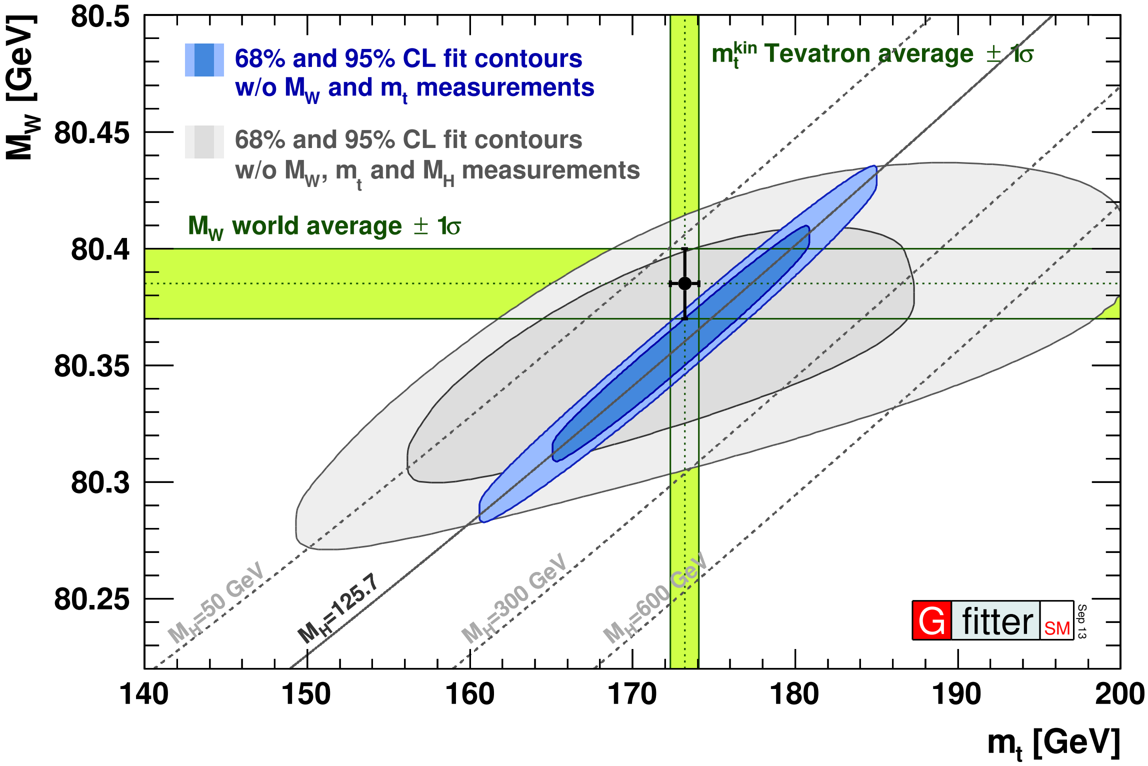

Figure 7.23: Electroweak fit excluding M_W and m_t (blue), compared to their measured values (green) [27].

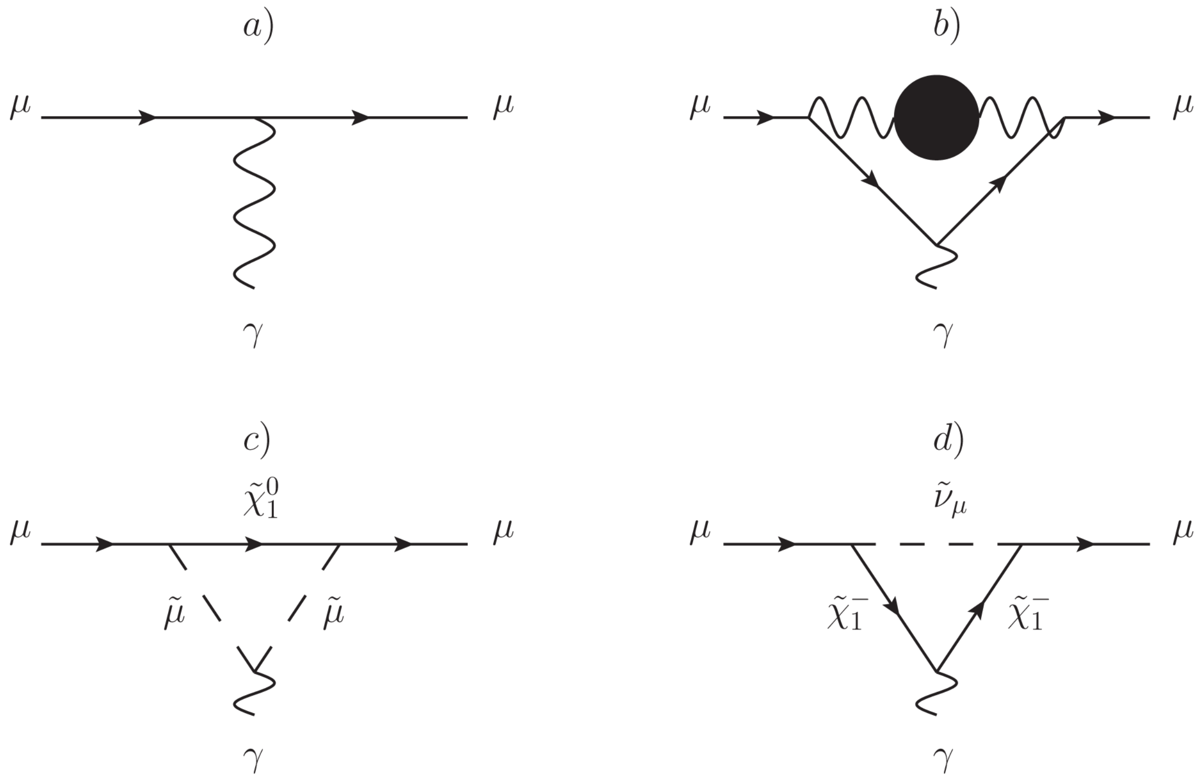

Figure 7.24: Diagrams for muon interaction with an electromagnetic field. Loop corrections to the tree level diagram a) give the value of a_μ. Diagram b) shows hadronic vacuum polarization where the blob contains QCD fields. Diagrams c) and d) show the lowest order MSSM contributions to a_μ.

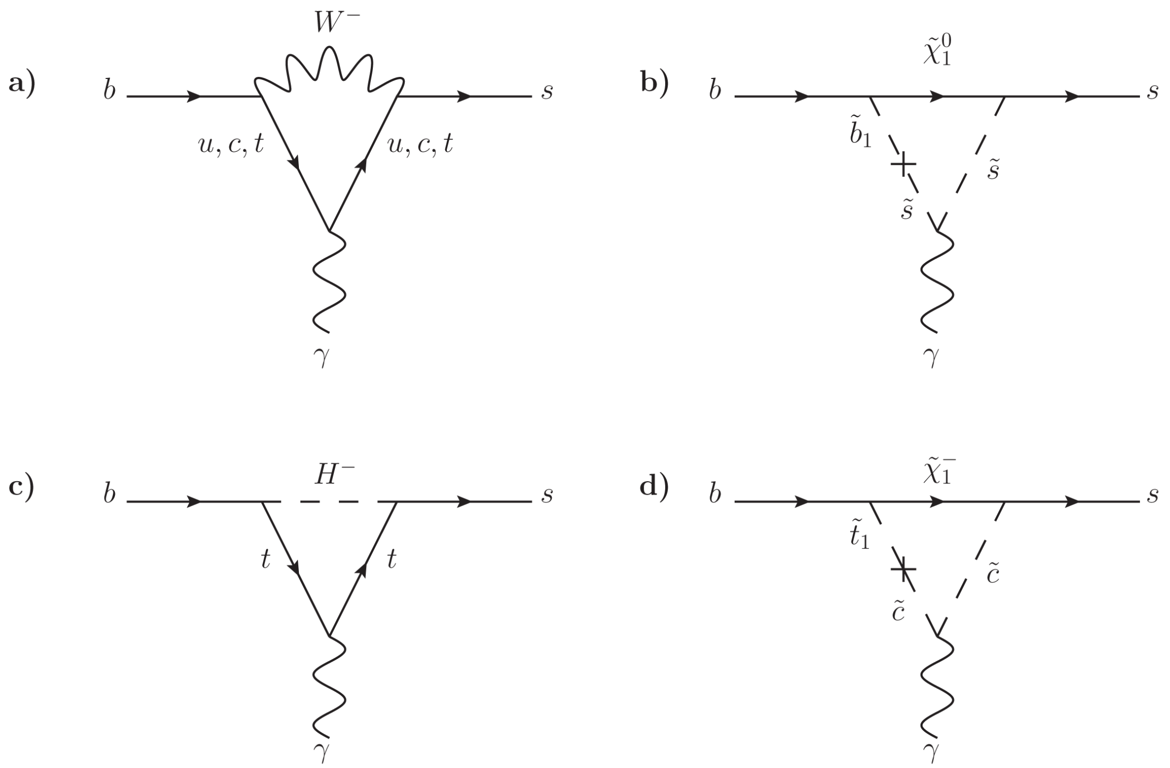

Figure 7.25: Diagrams for the process $b \to s\gamma$. a) shows the SM diagram while b), c) and d) show MSSM contributions.

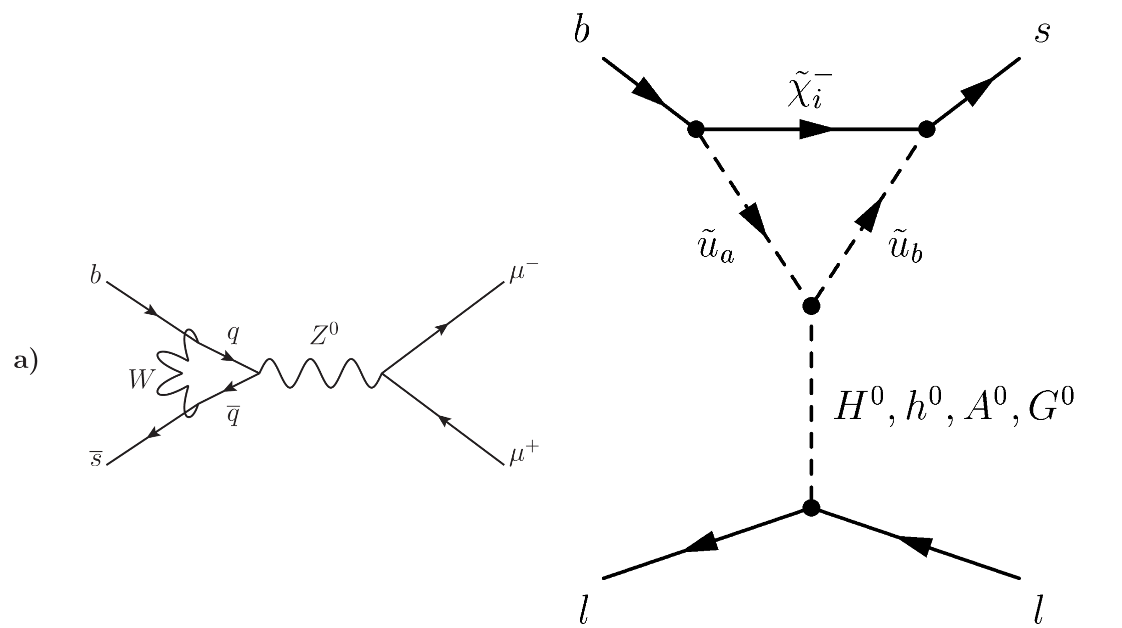

Figure 7.26: Diagrams for the process $B_s \to \mu^+\mu^-$. Diagram a) shows one of the leading SM contributions, while b) shows one contribution from the MSSM taken from [31].

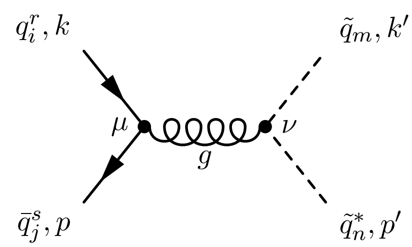

Figure 7.27: Strong SUSY production of two squarks through a gluon.

Chapter 8

Supersymmetric dark matter

8.1 Evidence for dark matter (DM)

The history of dark matter goes back quite a long way. Today we have evidence for the existence of dark matter through several effects where we observe its gravitational influence on ordinary matter. We list the evidence below:

1) Kinematics (Zwicky 1933 [35]): The motion of galaxies (velocity dispersion) cannot be explained by the visible matter. This was also observed on the scales of galaxies in their rotation curves (Rubin 1970 [36]).

2) Gravitational lensing (Tyson 1996 [37]). First observed in galactic clusters. Clusters show evidence of lensing not explained by luminous matter. Dark matter dynamics (non-interacting) are demonstrated by the Bullet cluster (Clowe 2006 [38]).

3) Large scale structures (clusters, superclusters, filaments and voids): The 2dFGRS (2-degree Field Galaxy Redshift survey Colles 2001 [39]) and SDSS (Sloan Digital Sky Survey Tegmark 2004 [40]) give a relative matter density of $\Omega_m \equiv \frac{\rho_m}{\rho_c} = 0.29$ where $\rho_c = 1.05 \cdot 10^{-5} h^2\,\text{GeV/cm}^3$ is the critical energy density for a flat universe.[1] They also imply that the majority of DM must be **cold** (non-relativistic), because warm DM would suppress clustering.

4) Big-Bang Nucleosynthesis (BBN): The formation of light elements in the period $t = 1 - 1000\,\text{s}$ after the Big Bang. Measurements of Early Universe abundance of light elements, mainly D and He, points to a baryonic matter density of $\Omega_b \approx 0.04$. This gives $\Omega_{\text{leftover}} \approx 0.25$.

5) Supernovae (Riess 1998 [41] and Perlmutter 1999 [42]): Measurements of type Ia supernovae (SNe Ia) were used as standard candles to show an accelerated expansion of the Universe. This fixes $\Omega_\Lambda - \Omega_m \simeq k$ where Ω_Λ is the energy density of dark energy/cosmological constant.

6) Cosmic Microwave Background (CMB) (Penzias & Wilson 1965 [43]): The temperature variation of the CMB over the sky of the order of $0.0002\,\text{K}$ is sensitive to all cosmological parameters, and gives $\Omega_\Lambda + \Omega_m \simeq k$, where k is some constant.

[1] h is defined through the Hubble constant H_0 as $H_0 = 100h\,\text{km/Mpc/s}$.

The evidence above can be used to constrain a minimal model of the Universe that can explain all the current measurements, the ΛCDM concordance model of cosmology, which has just a handful of ingredients such as baryonic and dark matter, radiation (photons) and dark energy. In Fig. 8.1 we show the effects of the SNe, CMB and large scale structure data (BAO) on this model.

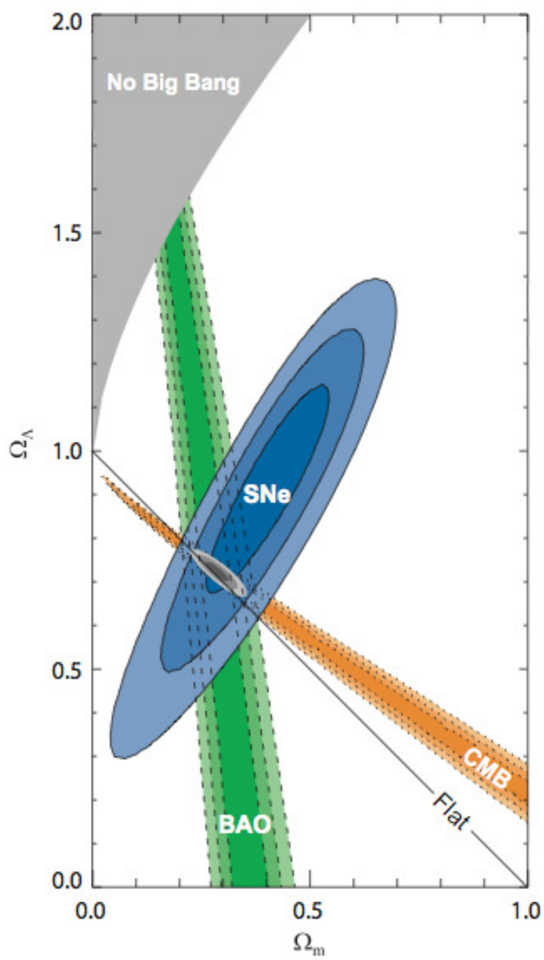

Figure 8.1: Limits from different experiments on the dark energy density, Ω_Λ, compared to the total mass density in the universe, Ω_m.

A maximum likelihood fit to a selected subset of the measurements gives the parameters for the model shown in Table 8.1.

Parameter	Ω_Λ	$\Omega_m h^2$	$\Omega_b h^2$	H_0 [km/Mpc/s]	t_0 [Gy]
Value	$0.685^{+0.018}_{-0.016}$	0.1426 ± 0.0025	0.02205 ± 0.00028	67.3 ± 1.2	13.817 ± 0.048

Table 8.1: Measured values for cosmological parameters [44].

8.2 WIMP magic

The very existence of a stable Weakly[2] Interacting Massive Particle (WIMP) χ automatically gives an additional component to the total energy density of the Universe. WIMPs are found in a number of theories, for example the lightest neutralino of the MSSM, the lightest Kaluza-Klein particle of a theory with extra dimensions or an inert Higgs boson.

This is due to the in equilibrium **thermal production** of the WIMP through the process $SM \times SM \to \chi\chi$, and the reverse annihilation process $\chi\chi \to SM \times SM$, in the early hot Universe ($T \gg m_\chi$). As the temperature decreases to $T < m_\chi$ and there is not enough energy in an average collision for the production of χ to occur, only the reverse process can take place, and the comoving density[3] falls with the temperature of the Universe.

The WIMPs then experience what is called a **chemical decoupling**, or loss of chemical equilibrium, due to the expansion of the Universe. This is when the WIMPs become so dilute, because of the expansion, that they in effect no longer interact inelastically, and this roughly happens when the expansion rate becomes larger than the rate of anihilation. The WIMPs then get a constant (comoving) density, we say that they experience a **freeze-out** at this temperature T_c. With weak-scale masses and couplings the freeze-out happens at $T_c \approx 0.05 m_\chi$, however, this is before or at the same time as **kinetic decoupling** where the WIMPs effectively lose elastic interactions, meaning that χ freezes-out with non-relativistic velocities and become so-called cold dark matter.

The exact time (temperature) of freeze-out is controlled by the annihilation cross section of χ, larger cross sections keep chemical equilibrium for longer, in turn resulting in lower dark matter relic abundance. This abundance, in number density, can be found from the Boltzmann equation

$$\frac{dn_\chi}{dt} = -3Hn_\chi - \langle\sigma v\rangle(n_\chi^2 - n_\chi^{eq2}), \tag{8.1}$$

where n_χ^{eq} and n_χ are the chemical equilibrium and actual comoving number densities, H is Hubble's constant for the expansion rate, and $\langle\sigma v\rangle$ the velocity averaged annihilation cross section for $\chi\chi \to SM \times SM$. In practice one must also often take into account co-annihilation with other particles with mass within $10\% - 20\%$ of the χ, and numerical codes such as `DarkSUSY` [45] or `MicrOMEGAs` [46, 47] are used.

For weak scale particles a rough approximation to the resulting dark matter density is

$$\Omega_\chi h^2 = 0.1 \times \frac{3 \times 10^{-26}\,\mathrm{cm^3 s^{-1}}}{\langle\sigma v\rangle},$$

and since the annihilation cross section can be shown to be

$$\langle\sigma v\rangle \approx \frac{\alpha_{weak}^2}{m_{weak}^2} \approx 10^{-25}\,\mathrm{cm^3 s^{-1}}, \tag{8.2}$$

the predicted DM density is

$$\Omega_\chi h^2 \approx 0.1 \times \left(\frac{g_{weak}}{g_\chi}\right)^4 \left(\frac{m_\chi}{m_{weak}}\right)^2.$$

[2]Weak as in electro-weak, meaning on the same scale as the weak force.

[3]Taking the expansion of the universe into account by looking at the number of particles in a volume expanding at the same rate.

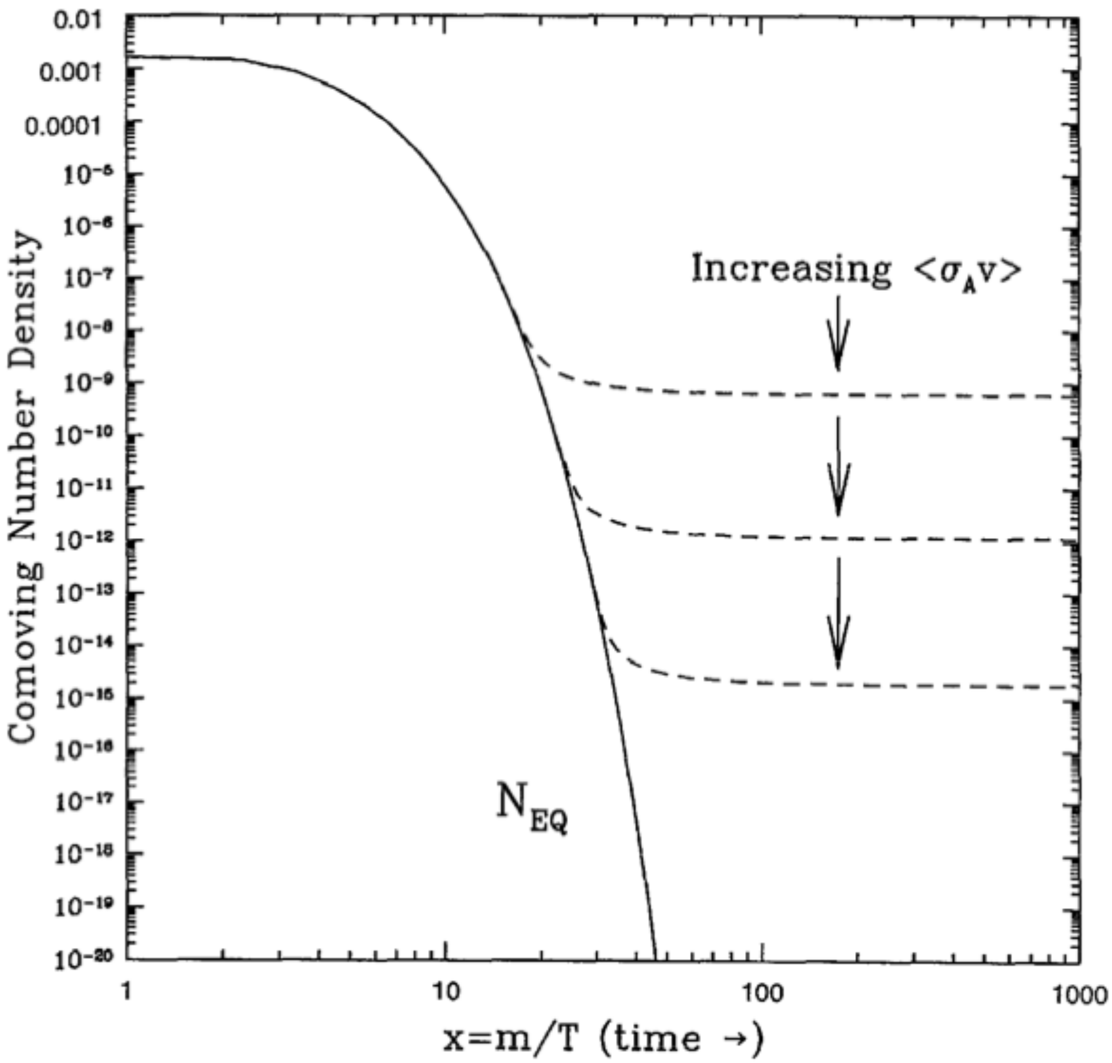

Figure 8.2: Ilustration of the freeze-out of the comoving number density of a WIMP, where the black line represents a model without chemical decoupling, and the dotted lines represent different freeze out temperatures for different velocity averaged annihilation cross sections.

When compared to the value in Table 8.1 this is called the **WIMP-miracle**.

For a more detailed discussion of the WIMP miracle, see the standard cosmology book by Kolb and Turner [48].

8.3 Dark matter candidates in supersymmetry

With R-parity conservation in place we have seen that any neutral LSP can be DM. Without R-parity only super-weakly coupling particles like gravitinos and axinos are candidates. Below we briefly discuss the various possibilities.

8.3.1 Neutralino

As soon as you have a stable neutralino LSP, you usually get into trouble trying to explain why there is so little dark matter. The neutralino in the standard mSUGRA bino like $\tilde{\chi}_1^0$ scenario gives a problematically high $\Omega_\chi h^2$ due to current lower bounds on the $\tilde{\chi}_1^0$ mass and the measured higgs mass. This is called the **bulk region** scenario which can be seen in Fig. 8.3 in the lower left corner. Alternatives to the bulk region scenario use co-annihilation or resonant annihilation to increase $\langle\sigma v\rangle$ and thus decrease the dark matter density.

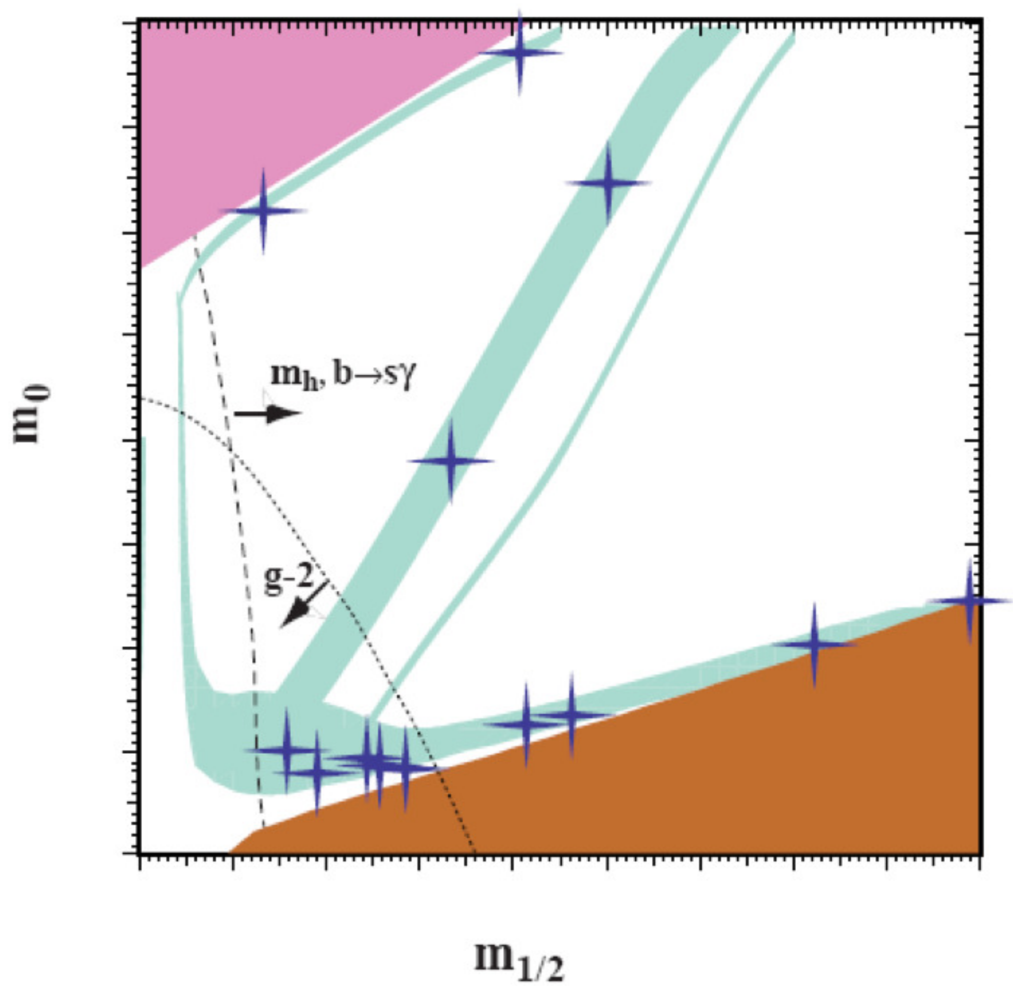

Figure 8.3: Generic illustration of the allowed neutralino DM regions (puke green) in the $(m_0, m_{1/2})$-plane for mSUGRA. Except for the low m_0 and $m_{1/2}$ regions the area outside of the allowed region gices too much drk matter. The dashed line shows the Higgs mass limit which pushes towards larger values of $m_{1/2}$, while the dotted line represents the limit from the anomalous magnetic moment of the muon.

The **stau-coannihilation region**, where $\tilde{\tau}_1\tilde{\chi}_1^0 \rightarrow SM \times SM$ is an efficient process, exists for small m_0 with $m_{\tilde{\tau}_1} - m_{\tilde{\chi}_1^0} \leq 10\,\text{GeV}$, which makes this scenario difficult to discover at collider experiments due to the production of soft (low-energy) taus. This is shown as the lower strip in Fig. 8.3 which follows the lower theoretical bound (brown) where the stau becomes the LSP.

The **stop-coannihilation region**, where $\tilde{t}_1\tilde{\chi}_1^0 \to SM \times SM$, exists for large values of $|A_0|$, small m_0 and $m_{1/2}$, and typically has $m_{\tilde{t}_1} - m_{\tilde{\chi}_1^0} \leq 25\,\text{GeV}$. Again, this is difficult to discover because of the soft decay products of the stop.

The **higgs funnel region** for $2m_{\tilde{\chi}_1^0} = m_{A,H}$ and large $\tan\beta$, where the neutralino has resonant annihilation through a heavy Higgs boson, is shown in Figure 8.3 as the diagonal structure roughly in the middle of the plot, rising as a funnel upwards.

The **focus point region** for large m_0 and low μ gives an higgsino-wino LSP with a more efficient higgsino annihilation channel for the LSP and thus a lower dark matter density within experimental bounds. This leads to so-called split-SUSY, as the sfermion masses need to be pressed up quite a bit. The focus point region can be seen in Fig. 8.3 following the upper theoretical bound where EWSB breaks down.

8.3.2 Sneutrinos

The left handed sneutrino $\tilde{\nu}_L$ is happily excluded as a DM particle due to the large cross section for $\tilde{\nu}_L q \to \tilde{\nu}_L q$ via Z-exchange.[4] The large cross means that it should already have been seen by direct detection experiments. It is also problematic to get $m_{\tilde{\nu}_L} < m_{\tilde{l}_L}$ due to hyperfine-splitting. However, $\tilde{\nu}_R$ couples very weakly and is still a viable candidate.

8.3.3 Gravitino

The gravitino is *not* a WIMP as it is never in chemical equilibrium. It can be created from NLSP decays giving, in RPC scenarios,

$$\Omega_{\tilde{G}} = \frac{m_{\tilde{G}}}{m_{NLSP}} \Omega_{NLSP},$$

however, these scenarios are problematic, because the NLSP is long-lived and creates potential trouble in BBN by injecting energy that changes the production of light elements. Alternatively, it can be created in non-thermal production as shown in Fig. 8.4 at reheating after inflation. The reverse process $\tilde{g}\tilde{G} \to gg$ is not efficient as the density of gravitinos and gluinos is never high enough given the small the cross section. This type of dark matter creation process is often called **freeze-in**. For the gravitino this gives a new magic formula:

$$\Omega_{\tilde{G}} h^2 \approx 0.5 \cdot \left(\frac{T_R}{10^{10}\,\text{GeV}}\right) \left(\frac{100\,\text{GeV}}{m_{\tilde{G}}}\right) \left(\frac{m_{\tilde{g}}}{1\,\text{TeV}}\right)^2, \tag{8.3}$$

where T_R is the reheating temperature. This is valid also for RPV scenarios. There the gravitino coupling $\propto \frac{1}{M_P}$ makes the gravitinos very long-lived, but not absolutely stable. One can also imagine an axino scenario, that would work just like the gravitino.

[4]For the neutralinos this problem only exists for a higgsino $\tilde{\chi}_1^0$ LSP, as the wino and bino do not couple to the Z

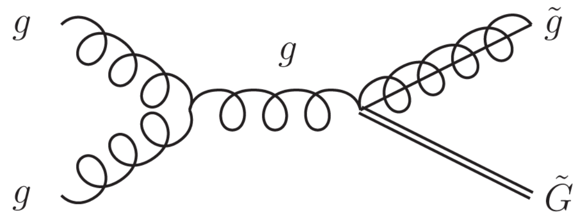

Figure 8.4: One possible diagram for the non-thermal production of gravitinos.

8.3.4 Others

One could even imagine color charged supersymmetric particles as DM, in particular the gluino, which, if stable, after hadronization form so-called **R-hadrons**. These have very strict limits from direct searches, but these limits are somewhat obfuscated by complications in R-hadron scattering.

8.4 Direct detection

In addition to the direct production of dark matter at colliders and the corresponding searches for missing energy, there are two other main ways to search for dark matter, direct and indirect detection. Here we briefly discuss direct detection.

Direct detection seeks to make weak DM interactions with SM matter visible by very low background searches in large volumes, using galactic halo DM interacting with ordinary matter. This is very dependent on the χ scattering cross section on nucleons (quarks), which can be calculated in a given model, but also the DM halo density distribution and velocity distribution, which have large uncertainties. This can be expressed in the differential scattering rate with respect to the recoil energy E_r of a scattered nucleon with mass M

$$\frac{dN}{dE_r} = \frac{\sigma\rho_{DM}}{2\mu^2 m_\chi}|F(q)|^2 \int_{v_{min}}^{v_{esc}} \frac{f(\vec{v})}{v} d^3v, \tag{8.4}$$

where σ is the DM scattering cross section off the nucleus in question, ρ_{DM} is the DM halo density at Earth, μ is the dark matter and nucleus reduced mass, $F(q)$ is a nuclear form factor dependent on the scattering momentum transfer $q = \sqrt{2ME_r}$, $f(\vec{v})$ is the velocity distribution in the halo, $v_{min} = \sqrt{ME_r/2\mu^2}$ is the minimal velocity that gives a recoil energy E_r and v_{esc} is the escape velocity from the halo.

There two main tactics followed in order to try to directly detect DM:

- Suppress (almost) all backgrounds, which is used in experiments such as XENON and CDMS.

- Look for an annular modulation, due to the Earth's movement in the galactic rest frame, in a small dark matter signal on top of a constant background, used in DAMA and CoGeNT.

Figure 8.5 shows results from the most important direct detection experiments. Observe that while DAMA and CoGeNT both have signals for detection, these are already excluded by XENON and not compatible with each other.

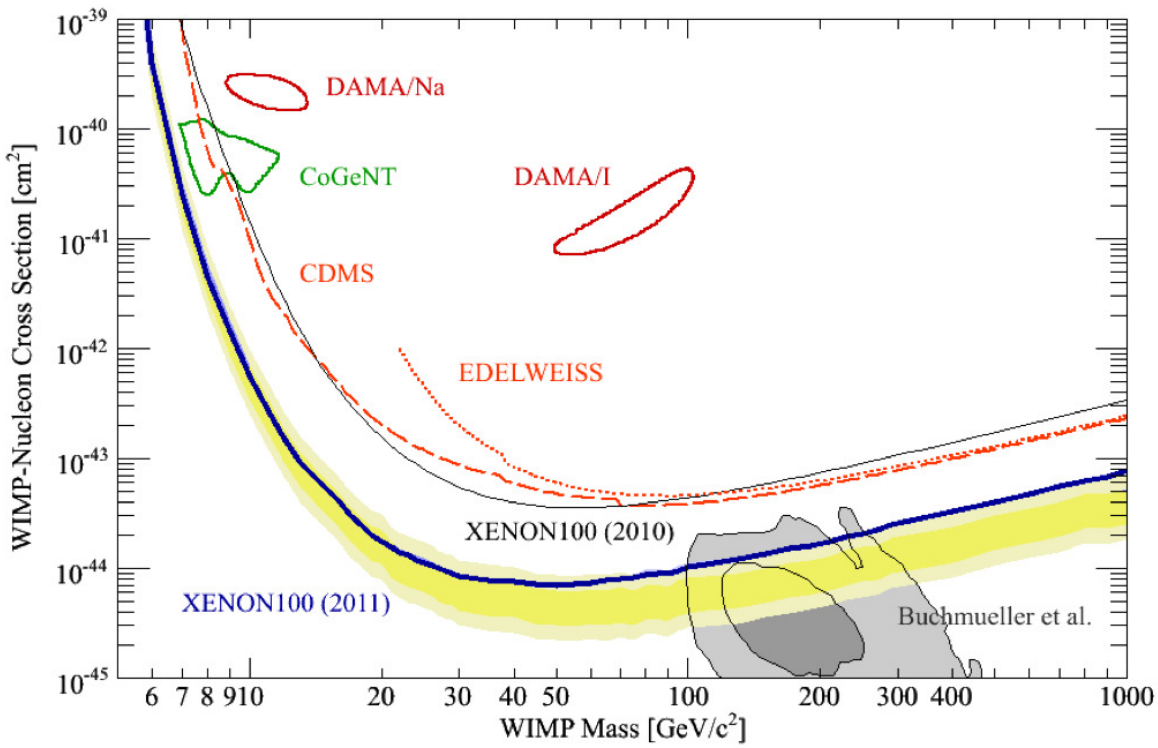

Figure 8.5: Plot of different exclusion and detection results for direct detection of DM in the WIMP mass versus WIMP–nucleon cross section plane. The grey area shows the expected mass and cross section in MSSM models, where we assume gauge unification at the GUT-scale.

8.5 Indirect detection

In indirect detection we look for annihilation or decay products from DM in multiple final (messenger) states in cosmic rays. Search channels must be stable SM particles, so that they can reach the Earth (or satelites in orbit). The messengers should also have as low backgrounds from ordinary astrophysical processes as possible, this makes searches with electrons and protons difficult. The remaining candidates are photons, neutrinos, positrons, antiprotons and antideuterons.

- **Photons:** These can either come from direct production processes such as $\chi\chi \to \gamma\gamma, Z\gamma$, which is easier to detect because the spectrum is a sharp line spectrum at exactly the mass of the DM, or γ from brehmsstrahlung or pion decays, which is a broad spectrum and hard to detect, but is expected to make up the majority of photons from dark matter. Photons from dark matter have the advantage that they point to the source so we can focus on areas with large ρ_{DM}, and thus reducing potential backgrounds relative tot he signal. We can also look for photons that are extragalactic in origin (but then we have to account for red-shifting the spectrum).

Dark matter annihilating in our own galaxy into photons should result in a flux at Earth given by

$$\frac{d\Phi}{dEd\Omega} = \frac{1}{8\pi m_\chi^2}\frac{dN_\gamma}{dE}\langle\sigma v\rangle\int_{\text{l.o.s.}}\rho_{DM}^2(l)dl, \tag{8.5}$$

where $N_\gamma(E)$ gives the number of photons with energy E in a single annihilation event. We see that the flux depends on the square of the DM density since annihilation requires two DM particles to be present. For decaying DM the corresponding expression is proportional to ρ_{DM}.

There have been som indications of an excess of photons above expected backgrounds from the galactic centre (a.k.a. the Hooperon [49]), however, no unambigeous DM signal has been confirmed. Current limits from the Fermi-LAT experiment seems to rule out most possible models a DM explanation for this excess, are sets a cross section limit close, and for some masses beyond, to the canonical limit

$$\langle\sigma v\rangle = 3\times 10^{-26}\,\text{cm}^3\text{s}^{-1},$$

see Fig. 8.6.

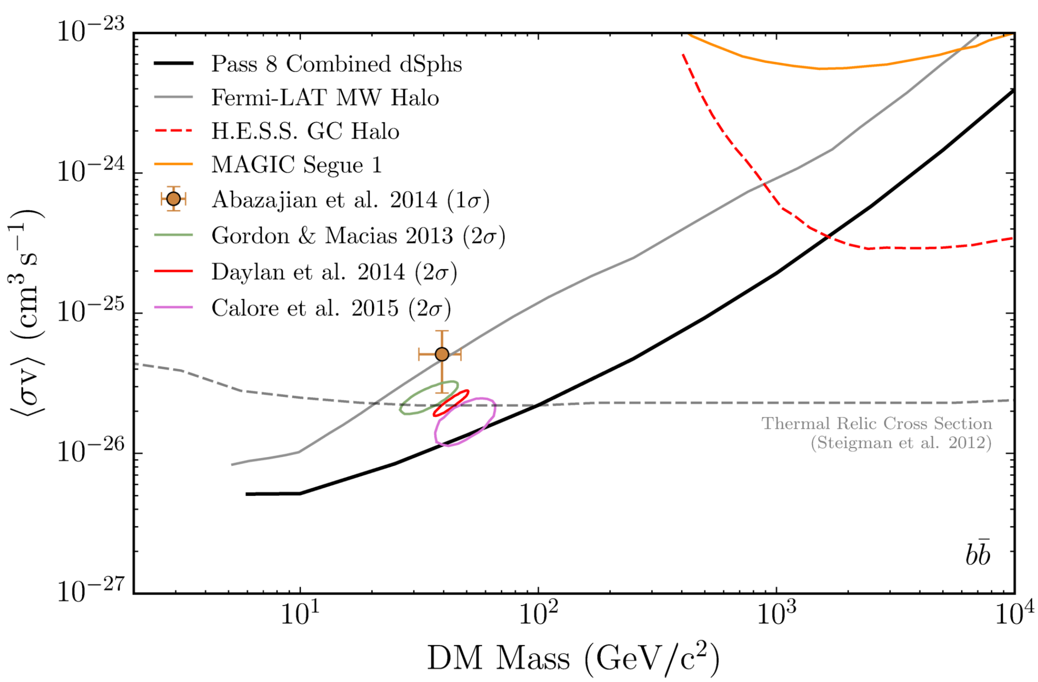

Figure 8.6: Results from Fermi-LAT indirect gamma-ray searches in the $\chi\chi \to b\bar{b}$ channel. Grey line shows limit from Milky Way halo search, black line from Milky Way dwarf spheroidal galaxy search with six years of data [50].

- **Neutrinos:** These also point to the source and can be extragalactic in origin just like the photons. The same flux calculation can be used, starting from the neutrino spectrum from dark matter annihilation. The astrophysical background is smaller, however, the neutrino signal is difficult to detect. The current leading experiment is IceCube at the South Pole. One interesting possibility is that DM matter scatters on ordinary matter sufficiently strongly that DM accumulates at the centre of the Sun (or possibly the Earth). When these DM particles annihilate the only decay products that can escape the Sun's interior are neutrinos.

- **Positrons:** Charged particles propagate in a complicated way through the galactic magnetic field, and they are therefore impossible to track back to the source. Sources outside of our own Galaxy cannot contribute significantly to the flux at Earth. This source has large astrophysical backgrounds, so experiments search for small excesses, mostly at high energies. Some potential excess has been seen by Fermi-LAT and PAMELA.

- **Antiprotons:** As the positrons these propagate in a complicated manner, but the backgrounds are under better control. PAMELA has set strict limits.

- **Antideuterons:** These have very low backgrounds because, however, the physics of the formation is quite complicated and hard to calculate. AMS-02 will provide new data soon.

8.6 Excercises

Exercise 8.1 Show that $\chi\chi \to Z \to f\bar{f}$ gives

$$\sigma v \approx \frac{g^4 E_\chi^2}{128\pi m_Z^2}, \tag{8.6}$$

which in the low-velocity limit can be shown to be

$$\langle \sigma v \rangle_0 \approx 10^{-25}\,\mathrm{cm}^3\mathrm{s}^{-1}.$$

Bibliography

[1] S. Lie, Math. Ann. **16**, 441 (1880).

[2] H. J. W. Müller-Kirsten and A. Wiedemann, Introduction to Supersymmetry, Second ed. (World Scientific Publishing Co. Pte. Ltd., 2010).

[3] S. R. Coleman and J. Mandula, Phys.Rev. **159**, 1251 (1967).

[4] R. Haag, J. T. Lopuszanski and M. Sohnius, Nucl.Phys. **B88**, 257 (1975).

[5] A. Salam and J. Strathdee, Phys.Rev. **D11**, 1521 (1975).

[6] J. Bagger and J. Wess, Phys.Lett. **B138**, 105 (1984).

[7] L. O'Raifeartaigh, Nucl.Phys. **B96**, 331 (1975).

[8] S. Ferrara, L. Girardello and F. Palumbo, Phys.Rev. **D20**, 403 (1979).

[9] M. T. Grisaru, W. Siegel and M. Rocek, Nucl.Phys. **B159**, 429 (1979).

[10] P. Fayet, Phys.Lett. **B58**, 67 (1975).

[11] CMS Collaboration, CERN Report No. CMS-PAS-HIG-13-005, 2013 (unpublished).

[12] P. Richardson, hep-ph/0101105, Ph.D. Thesis.

[13] J. Gunion and H. E. Haber, Nucl.Phys. **B272**, 1 (1986).

[14] H. E. Haber and G. L. Kane, Phys.Rept. **117**, 75 (1985).

[15] M. Bolz, (2000), DESY-THESIS-2000-013.

[16] S. P. Martin, hep-ph/9709356.

[17] C. G. Lester, A personal view of some of the recent (mostly BSM) results from ATLAS (and occasionally from CMS!), Conference talk, TOOLS 2012 Stockholm.

[18] A. Buckley, Eur. Phys. J. **C75**, 467 (2015), [1305.4194].

[19] D. J. Miller, P. Osland and A. R. Raklev, JHEP **03**, 034 (2006), [hep-ph/0510356].

[20] B. K. Gjelsten, D. J. Miller and P. Osland, JHEP **12**, 003 (2004), [hep-ph/0410303].

[21] ATLAS Collaboration, CERN Report No. ATLAS-CONF-2013-047, 2013 (unpublished).

[22] ATLAS, G. Aad et al., JHEP **1310**, 189 (2013), [1308.2631].

[23] ATLAS Collaboration, CERN Report No. ATLAS-CONF-2013-049, 2013 (unpublished).

[24] ATLAS Collaboration, CERN Report No. ATLAS-CONF-2013-035, 2013 (unpublished).

[25] Particle Data Group, K. A. Olive et al., Chin. Phys. **C38**, 090001 (2014).

[26] H. K. Dreiner et al., Eur. Phys. J. **C62**, 547 (2009), [0901.3485].

[27] M. Baak et al., Eur.Phys.J. **C72**, 2205 (2012), [1209.2716].

[28] Muon G-2 Collaboration, G. Bennett et al., Phys.Rev. **D73**, 072003 (2006), [hep-ex/0602035], Summary of E821 Collaboration measurements of the muon anomalous magnetic moment, each reported earlier in Letters or Brief Reports. Revised version submitted to Phys.Rev.D.

[29] M. Misiak et al., Phys. Rev. Lett. **114**, 221801 (2015), [1503.01789].

[30] M. Czakon et al., JHEP **04**, 168 (2015), [1503.01791].

[31] C. Bobeth, T. Ewerth, F. Kruger and J. Urban, Phys. Rev. **D64**, 074014 (2001), [hep-ph/0104284].

[32] C. Bobeth et al., Phys. Rev. Lett. **112**, 101801 (2014), [1311.0903].

[33] LHCb, CMS, V. Khachatryan et al., Nature **522**, 68 (2015), [1411.4413].

[34] A. Hiorth and J. O. Eeg, Eur. Phys. J.direct **C30**, 006 (2003), [hep-ph/0301118], [Eur. Phys. J.C32S1,69(2004)].

[35] F. Zwicky, Helv.Phys.Acta **6**, 110 (1933).

[36] V. C. Rubin and J. Ford, W.Kent, Astrophys.J. **159**, 379 (1970).

[37] P. Fischer, G. Bernstein, G. Rhee and J. Tyson, astro-ph/9608117, * Brief entry *.

[38] D. Clowe et al., Astrophys.J. **648**, L109 (2006), [astro-ph/0608407].

[39] E. Hawkins et al., Mon.Not.Roy.Astron.Soc. **346**, 78 (2003), [astro-ph/0212375].

[40] SDSS Collaboration, K. Abazajian et al., Astron.J. **129**, 1755 (2005), [astro-ph/0410239].

[41] Supernova Search Team, A. G. Riess et al., Astron.J. **116**, 1009 (1998), [astro-ph/9805201].

[42] S. Perlmutter, M. S. Turner and M. J. White, Phys.Rev.Lett. **83**, 670 (1999), [astro-ph/9901052].

[43] A. A. Penzias and R. W. Wilson, Astrophys.J. **142**, 419 (1965).

[44] Planck Collaboration, P. Ade et al., 1303.5076.

[45] P. Gondolo et al., JCAP **0407**, 008 (2004), [astro-ph/0406204].

[46] G. Belanger, F. Boudjema, A. Pukhov and A. Semenov, Comput. Phys. Commun. **149**, 103 (2002), [hep-ph/0112278].

[47] G. Belanger, F. Boudjema, A. Pukhov and A. Semenov, Comput. Phys. Commun. **174**, 577 (2006), [hep-ph/0405253].

[48] E. W. Kolb and M. S. Turner, The Early Universe (Addison-Wesley, 1990), Frontiers in Physics, 69.

[49] D. Hooper and L. Goodenough, Phys. Lett. **B697**, 412 (2011), [1010.2752].

[50] Fermi-LAT, M. Ackermann et al., 1503.02641.