FYS3120 Classical mechanics and electrodynamics
Midterm exam 2020

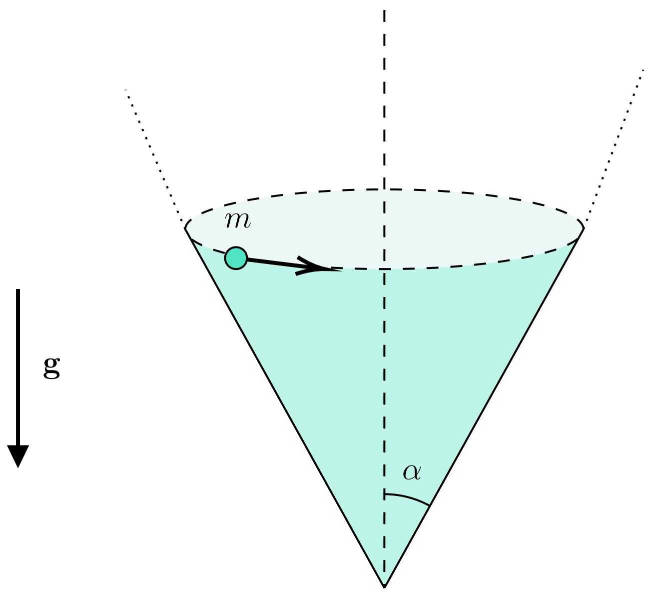
15112

September 8, 2021

Question 1

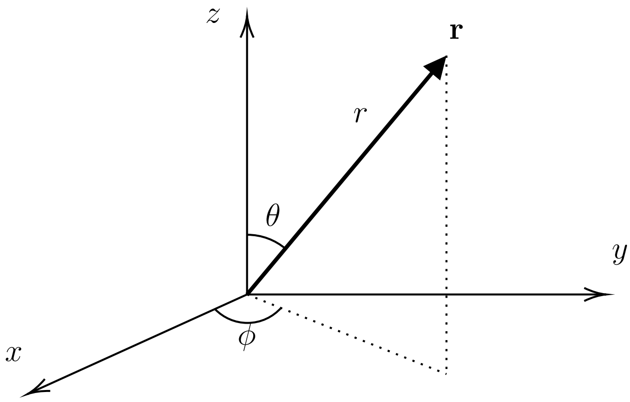
a)

The set-up of this problem looks something like this:



The above figure shows a point mass m constrained to move on the surface of a cone of half-angle α. In this problem I'm assuming that the cone has infinite height.

For $N = 1$ unconstrained particles moving in 3 dimensions, the number of degrees of freedom is $d = 3N = 3$. We could then have chosen the Cartesian coordinates x, y, z as the generalized coordinates. An equally valid choice of generalized coordinates is the spherical coordinates r, ϕ, θ, see the figure below.



The spherical coordinates are related to the Cartesian coordinates by the equations

$$x = r\cos\phi\sin\theta,\ \ y = r\sin\phi\sin\theta,\ \ z = r\cos\theta$$

as one might determine from the above figure (or see Rottmann p. 69). We might realize that by fixing the coordinate $\theta = \alpha$, the particle is constrained to move along the surface of a cone of half-angle α with its pointy end placed at the origin.

$\theta = \alpha$ is a holonomic constraint, which reduces the number of degrees of freedom by one. So in this problem the number of degrees of freedom is $d = 2$, and an appropriate choice of generalized coordinates is (r, ϕ). The relations between r and ϕ to the Cartesian coordinates are

$$x = r\cos\phi\sin\alpha,\ \ y = r\sin\phi\sin\alpha,\ \ z = r\cos\alpha$$

b)

The Lagrangian L is by definition the difference between the total kinetic energy T and the total potential energy V of the system.

$$L = T - V$$

The line element ds in spherical coordinates is given by

$$ds^2 = dr^2 + r^2\sin^2\theta d\phi^2 + r^2 d\theta^2$$

See Rottmann p. 69. Setting $\theta = \alpha = \text{const.}$ gives

$$ds^2 = dr^2 + r^2\sin^2\alpha d\phi^2$$

Dividing both sides by dt^2 gives the square of the velocity of the point mass.

$$\dot{\mathbf{r}}^2 = v^2 = \left(\frac{ds}{dt}\right)^2 = \dot{r}^2 + r^2\sin^2\alpha\dot{\phi}^2$$

So the total kinetic energy of this system is

$$T = \frac{1}{2}m\dot{\mathbf{r}}^2$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2 \sin^2 \alpha \dot{\phi}^2$$

Since the point mass is acted on by gravity it has potential energy (assuming the gravitational field is approximately constant in the vicinity of the point mass)

$$V = mgz$$

$$= mgr \cos \alpha$$

relative to the origin. So the Lagrangian of this system is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2 \sin^2 \alpha \dot{\phi}^2 - mgr \cos \alpha \tag{1}$$

c)

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

for each generalized coordinate q_i, $i = 1, \ldots, d$. We see that if the Lagrangian is independent of q_i, the quantity

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

called the conjugate momentum to q_i, is a constant of motion.

We see in Eq. (1) that the Lagrangian is independent of the generalized coordinate ϕ, so

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \alpha \dot{\phi} \equiv \ell = \text{const.} \tag{2}$$

My suspicion is that this has something to do with the z-component of the angular momentum of the point mass. Since gravity acts in the negative z-direction, the torque does not have a z-component and so the z-component of the angular momentum does not change. The angular momentum is

$$\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$$

where the velocity $\dot{\mathbf{r}}$ is given by

$$\dot{\mathbf{r}} = \frac{d}{dt}(r\cos\phi\sin\alpha\ \hat{\mathbf{i}} + r\sin\phi\sin\alpha\ \hat{\mathbf{j}} + r\cos\alpha\ \hat{\mathbf{k}})$$

$$= (\dot{r}\cos\phi - r\dot{\phi}\sin\phi)\sin\alpha\ \hat{\mathbf{i}} + (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)\sin\alpha\ \hat{\mathbf{j}} + \dot{r}\cos\alpha\ \hat{\mathbf{k}}$$

The z-component of the angular momentum is (which one can find by writing out the cross-product as a determinant)

$$\mathbf{L}\cdot\hat{\mathbf{k}} = m(xv_y - yv_x)$$

$$= m\left[r\cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)\sin^2\alpha - r\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)\sin^2\alpha\right]$$

$$= m\left(r^2\cos^2\phi\dot{\phi}\sin^2\alpha + r^2\sin^2\phi\dot{\phi}\sin^2\alpha\right)$$

$$= mr^2\sin^2\alpha\dot{\phi} = \ell$$

which confirms my suspicion.

We see in Eq. (1) that the Lagrangian has no explicit time dependence. This means that the quantity

$$H \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

called the Hamiltonian of the system, is a constant of motion, since the Hamiltonian satisfies

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

See Eq. (2.96) on p. 48 of the course book. Since $L = T - V$ and since the only constraint $\theta = \alpha$ is time independent, the Hamiltonian is the sum of the kinetic and potential energy.

$$H = T + V$$

See p. 49 in the course book. So the mechanical energy of the point mass is conserved.

d)

Two degrees of freedom will in general give two coupled equations of motion, one for each generalized coordinate. We've already found one of them in Eq. (2). The partial derivatives of the Lagrangian in Eq. (1) with respect to r and $\dot{r}$ are

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{\partial L}{\partial r} = mr\sin^2\alpha\dot{\phi}^2 - mg\cos\alpha$$

So the other equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \rightarrow m\ddot{r} - mr\sin^2\alpha\dot{\phi}^2 + mg\cos\alpha = 0$$

$$\ddot{r} - r\sin^2\alpha\dot{\phi}^2 + g\cos\alpha = 0 \tag{3}$$

We can combine Eq. (3) with Eq. (2) to eliminate $\dot{\phi}$. This gives the single equation of motion

$$\ddot{r} - \frac{\ell^2}{m^2\sin^2\alpha}\frac{1}{r^3} + g\cos\alpha = 0 \tag{4}$$

e)

The conjugate momentum to r is

$$p \equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

Hamilton's equations are then given by

$$\dot{r} = \frac{p}{m} \tag{5}$$

$$\dot{p} = \frac{\ell^2}{m\sin^2\alpha}\frac{1}{r^3} - mg\cos\alpha \tag{6}$$

where $\dot{p} = m\ddot{r}$ and Eq. (4) has been used. We can use Eq. (5) and (6) to visualize the phase space of the system as a stream plot, see Figure 1.

The parameters $m = g = 1$, $\ell = 10$ and $\alpha = \pi/4$ have been used to generate Figure 1. The units have been omitted since the stream plot is just to give a general sense of what the phase space looks like. We see that there is a stable equilibrium at $(r \approx 7, p = 0)$. The point mass is not at rest when it is in this stable equilibrium. However its r-coordinate is fixed and the conjugate momentum p to r is zero. The point mass is still moving on the inside of the cone with fixed angular velocity $\dot{\phi}$ and at a fixed distance r away from the origin (equivalently at a fixed height, since $z = r\cos\alpha$). This is a consequence of the conjugate momentum $\ell = mr^2\sin^2\alpha\dot{\phi}$ to ϕ being a conserved quantity. The point mass thus follows a completely circular path in a horizontal plane in physical space.

We can find an expression for the r-position of the stable equilibrium by setting $\dot{p} = 0$ in Eq. (6). This gives

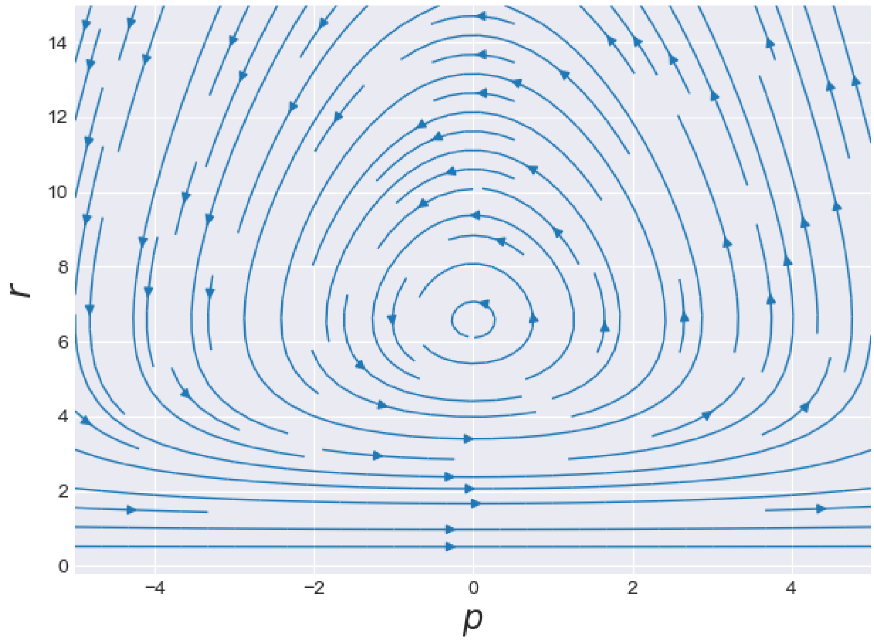


Figure 1

$$\frac{\ell^2}{m\sin^2\alpha}\frac{1}{r_{\text{eq}}^3} = mg\cos\alpha$$

$$r_{\text{eq}} = \left(\frac{\ell^2}{m^2 g\sin^2\alpha\cos\alpha}\right)^{1/3}$$

Putting our choice of numerical values for the parameters into this formula gives $r_{\text{eq}} = 6.56$. Just for the record, it's easy to see that this equilibrium point indeed is stable by examining Eq. (6). We see that $\dot{p}$ is negative whenever r is larger than r_{eq} and positive whenever r is less than r_{eq}.

f)

Figure 1 shows that the point mass will follow a closed loop in phase space when not in the stable equilibrium. It *must* follow a closed loop in phase space since the total energy H of the point mass is conserved. Every closed loop contains the equilibrium point. Physically this means that the point mass oscillates up and down the surface of the cone, around the height of stable equilibrium. The "amplitude" (max and min value of r on a particular loop in phase space) of these oscillations is fixed. In terms of ϕ and rotation around the z-axis the angular velocity $\dot{\phi}$ has its minimum value when the point mass is at the highest point in its trajectory on the surface of the cone, and its maximum value when the point mass is at the lowest point. This is because ℓ is a conserved quantity.

An explicit expression for the Hamiltonian of the point mass is

$$H = T + V$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\alpha\dot{\phi}^2 + mgr\cos\alpha$$

The natural variables of the Hamiltonian are the generalized coordinates and their conjugate momenta, so expressed in terms of its natural variables the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{\ell^2}{2m\sin^2\alpha}\frac{1}{r^2} + mgr\cos\alpha$$

This Hamiltonian is equivalent with a situation where a point mass is constrained to move in one dimension with r as the choice of generalized coordinate, subject to a potential

$$V_{\text{eff}}(r) \equiv \frac{\ell^2}{2m\sin^2\alpha}\frac{1}{r^2} + mgr\cos\alpha$$

In the context of a point mass constrained to move along the surface of a cone we refer to this as an effective potential. We can make a second order Taylor expansion of the effective potential in r around $r = r_{\text{eq}}$.

$$V_{\text{eff}}(r) \approx V_{\text{eff}}(r_{\text{eq}}) + \frac{dV_{\text{eff}}(r_{\text{eq}})}{dr}(r - r_{\text{eq}}) + \frac{1}{2}\frac{d^2V_{\text{eff}}(r_{\text{eq}})}{dr^2}(r - r_{\text{eq}})^2$$

We can add a constant $-V_{\text{eff}}(r_{\text{eq}})$ to the potential without changing the "force" that it represents. Also, since r_{eq} is an equilibrium point, the first derivative of the effective potential at r_{eq} must vanish.

The first and second derivatives of the effective potential are

$$\frac{dV_{\text{eff}}}{dr} = -\frac{\ell^2}{m\sin^2\alpha}\frac{1}{r^3} + mg\cos\alpha$$

$$\frac{d^2V_{\text{eff}}}{dr^2} = \frac{3\ell^2}{m\sin^2\alpha}\frac{1}{r^4}$$

Now substituting $1/r^4 \to 1/r_{\text{eq}}^4$.

$$\frac{d^2V_{\text{eff}}(r_{\text{eq}})}{dr^2} = \frac{3\ell^2}{m\sin^2\alpha}\left(\frac{m^2g\sin^2\alpha\cos\alpha}{\ell^2}\right)^{4/3}$$

$$= \left(\frac{3^3\ell^6}{m^3\sin^6\alpha}\frac{m^8g^4\sin^8\alpha\cos^4\alpha}{\ell^8}\right)^{1/3} = \left(\frac{27m^5g^4\sin^2\alpha\cos^4\alpha}{\ell^2}\right)^{1/3}$$

This means that near the equilibrium point the Hamiltonian describes a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}k\eta^2 \tag{7}$$

with

$$k \equiv \left(\frac{27m^5g^4\sin^2\alpha\cos^4\alpha}{\ell^2}\right)^{1/3}$$

as the "spring constant" and $\eta \equiv r - r_{\text{eq}}$. Since H is a constant of motion, Eq. (7) is the equation of an ellipse in Cartesian coordinates $(p,\ \eta)$. Near the equilibrium point the point mass will thus follow a trajectory in the phase space $(p,\ r)$ approximately shaped like an ellipse centered at $(p = 0, r = r_{\text{eq}})$. Physically the point mass will oscillate approximately harmonically up and down the surface of the cone around the equilibrium point. The angular frequency of the oscillations is $\omega = \sqrt{k/m}$ and the period is $\tau = 2\pi/\omega$.

The equation of motion of the harmonic oscillator is easily found by Newton's second law

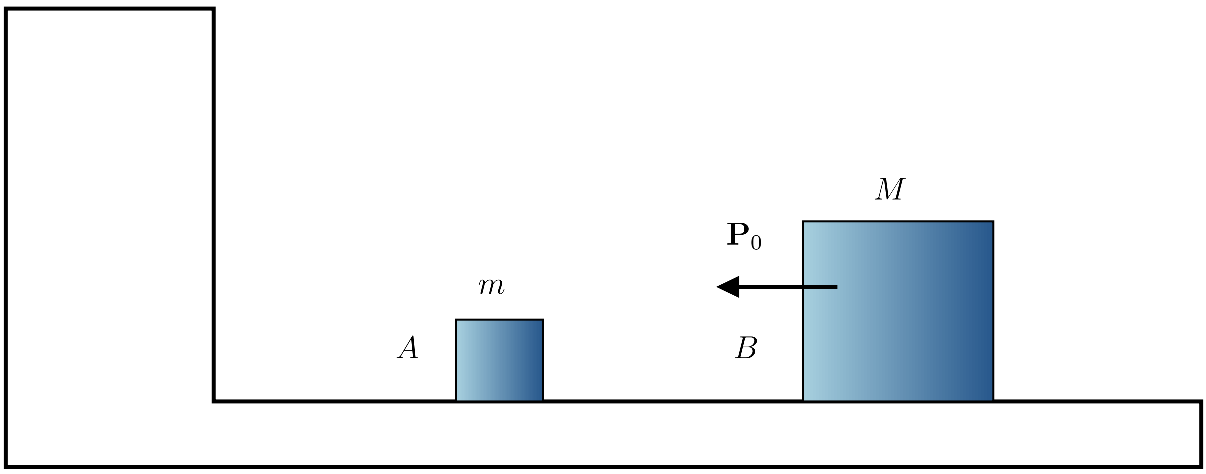
$$m\ddot{\eta} = -k\eta$$

which we can rewrite to the more standard form

$$\ddot{\eta} + \frac{k}{m}\eta = 0$$

Question 2

a)



The above figure shows the initial set up. The block A of mass m lies at rest between a block B of mass $M > m$ and a wall. Block B has initial momentum $\mathbf{P}_0$ in the direction of block A.

Since all block collisions and collisions with the wall are elastic, the total mechanical energy of the blocks are conserved in general. When block A is not colliding with the wall there are no external forces on the system consisting of blocks A and B, so during this time the total momentum of the system is conserved. The total momentum is not conserved during a collision with block A and the wall, but the momentum of A is simply reversed.

The first collision is between blocks A and B. Let p and P be their momenta after the collision. Conservation of momentum gives

$$P_0 = p + P \tag{8}$$

and conservation of energy gives

$$\frac{P_0^2}{2M} = \frac{p^2}{2m} + \frac{P^2}{2M}$$

However, when their masses are equal this equation reduces to

$$P_0^2 = p^2 + P^2 \tag{9}$$

By solving Eq. (8) for p and Eq. (9) for p^2 and combining the expressions we get the single equation

$$(P_0 - P)^2 = P_0^2 - P^2$$

$$(P_0 - P)^2 = (P_0 - P)(P_0 + P)$$

$$P_0 - P = P_0 + P$$

$$P = 0$$

By conservation of momentum the momentum of block A after the collision must be $p = P$. So block A simply absorbs the momentum of block B, leaving block B at rest and sending block A towards the wall.

The second collision between block A and the wall simply reverses the momentum of A, sending it towards block B which lies at rest. The third collision between blocks A and B is identical to the first collision. It will leave block A at rest and send B out into the void never to return.

So when the masses of blocks A and B are equal, there are 3 total collisions.

b)

Since the total mechanical energy of the system is conserved in general, each state of the system (i.e a point in the available phase space) will be projected (somewhere) onto the curve in (P, p) space given by the equation

$$\frac{P^2}{2M} + \frac{p^2}{2m} = E \tag{10}$$

where E is the total mechanical energy of the system. This is the equation for an ellipse in Cartesian coordinates (P, p) centered at the origin and with semi-major axis $\sqrt{2ME}$ and semi-minor axis $\sqrt{2mE}$.

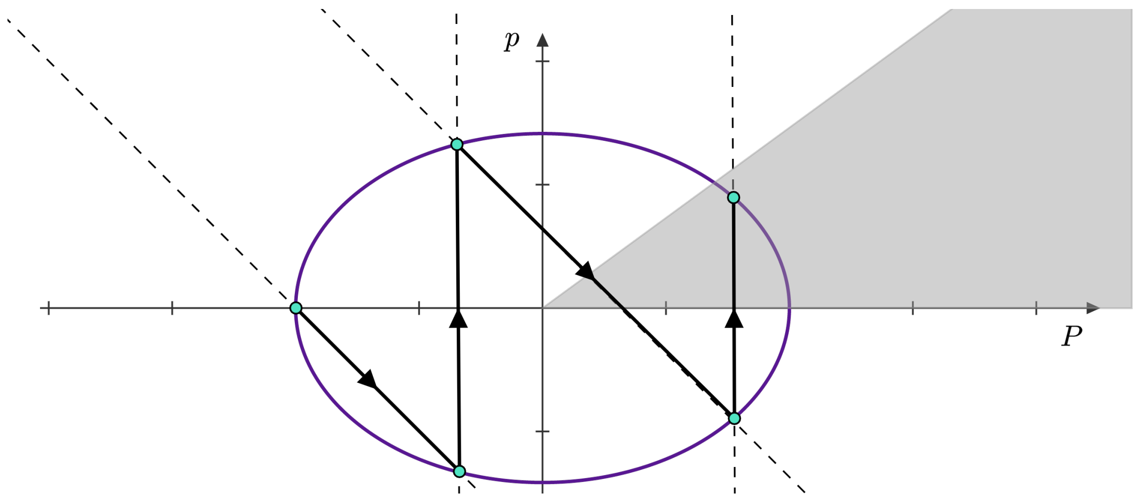
During a collision between the blocks the total momentum is conserved. This is described by the equation

$$p_0 + P_0 = p + P$$

With P and p interpreted as Cartesian coordinates this equation describes a line. Moving over P to the left side and differentiating both sides with respect to P gives the slope of the line

$$\frac{dp}{dP} = -1 \tag{11}$$

This means that during a collision, the system must transition from a point in (P, p) space to another point which is connected to the initial point by a line of slope -1 which also lies on the ellipse described by Eq. (10).



The projection of the available phase space onto the space (P, p) are the points represented by light blue dots in the figure above. Collisions between blocks A and B are represented by dark slanted lines, while collisions between block A and the wall are represented by dark vertical lines. So there are four total collisions represented in the figure above. The arrows indicate the sequence of the points.

The system initially starts in the leftmost point, where B has some momentum and A has no momentum. The system then transitions, through a collision between blocks A and B, to a point connected to the initial point by a line of slope -1 which lies on the ellipse. Then there is a transition through a collision between block A and the wall which reverses the momentum of A. The next point must therefore lie on the opposite side of the P-axis, mirroring the previous point. And so on. The system stops transitioning when both blocks have non-negative momentum with block B having a velocity larger than or equal to the velocity of block A. This requirement is given by

$$0 \leq \frac{p}{m} \leq \frac{P}{M} \tag{12}$$

This corresponds to the gray shaded area in the figure above.

c)

Defining $\tilde{P} \equiv P/\sqrt{2M}$, $\tilde{p} \equiv p/\sqrt{2m}$ we can write Eq. (10) as

$$\frac{\left(\tilde{P}\sqrt{2M}\right)^2}{2M} + \frac{\left(\tilde{p}\sqrt{2m}\right)^2}{2m} = E$$

$$\tilde{P}^2 + \tilde{p}^2 = E$$

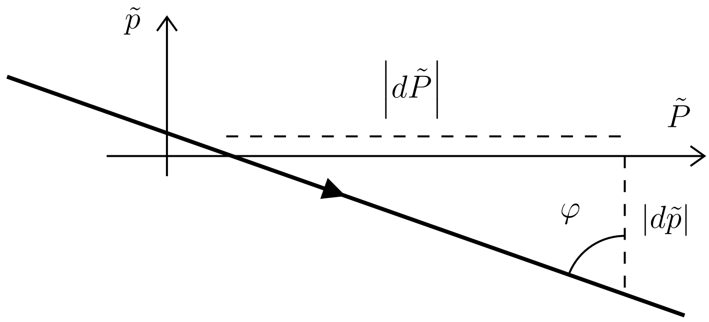
With $\tilde{P}$ and $\tilde{p}$ interpreted as Cartesian coordinates this equation describes a circle centered at the origin with radius $\sqrt{E}$.

Eq. (11) expressed in terms of $\tilde{P}$ and $\tilde{p}$ is

$$\frac{\sqrt{2m}d\tilde{p}}{\sqrt{2M}d\tilde{P}} = -1$$

$$\frac{d\tilde{p}}{d\tilde{P}} = -\sqrt{\frac{M}{m}}$$

The angle φ in the figure below is then given by $\tan\varphi = \left|d\tilde{P}/d\tilde{p}\right|$.

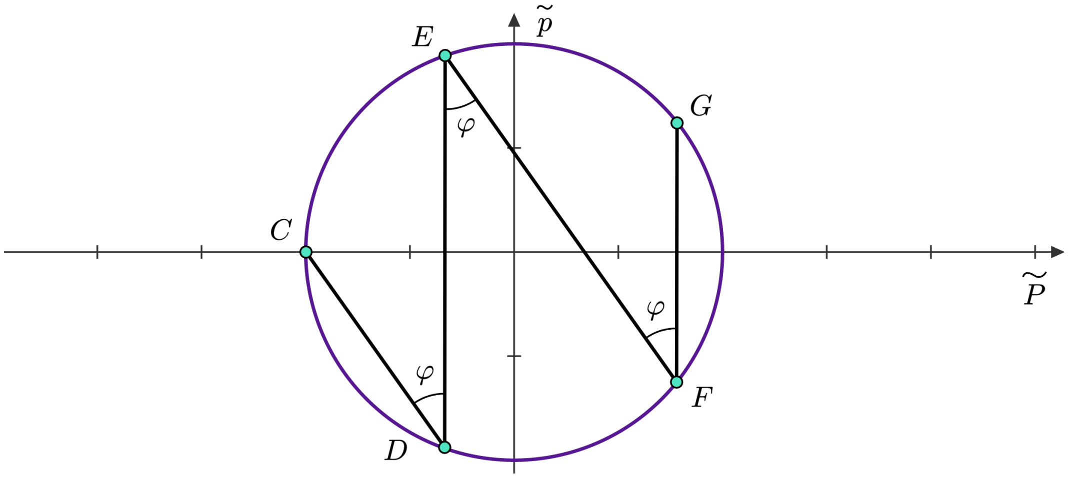


φ is then the angle between the lines DC and DE in the figure below. That the angle between the lines FE and FG is also φ follows from the fact that the lines DC and FE are parallel, and so are the lines DE and FG.

That the angle between the lines ED and EF is φ as well can be understood by making an argument in terms of similar triangles.

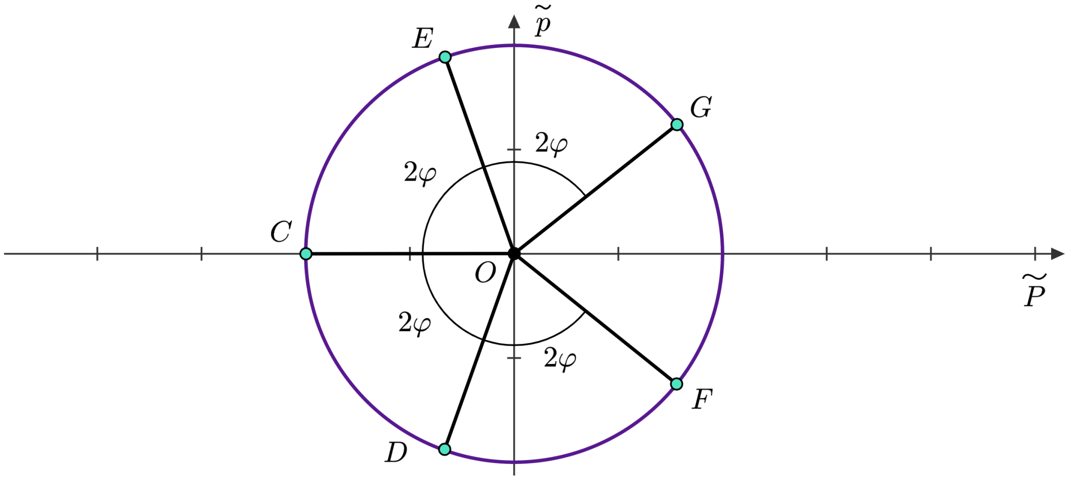
This means that the angle φ between two lines connecting three successive points on the circle in $\left(\tilde{P},\ \tilde{p}\right)$ is given by

$$\varphi = \arctan\left(\sqrt{\frac{m}{M}}\right)$$



Since both the number of collisions and φ are determined by the ratio of the masses m, M this formula holds in general.

d)



According to the inscribed angle theorem the angle between the lines OC and OE is 2φ. So is the angle between the lines OE and OG, and the angle between the lines OD and OF. Since the points D and E mirror each other about the $\tilde{P}$-axis, the angle between the lines OC and OD is 2φ as well.

We see in the figure above that the circle in $\left(\tilde{P},\ \tilde{p}\right)$ has been divided into five parts when there are four total collisions. Four of these parts have equal size. However, the angle between the lines OF and OG is not in general equal to 2φ, or anything at all really. The last transition of the system is between the point F and point G, where G can be arbitrarily close to the $\tilde{P}$-axis in accordance with the criterion in Eq. (12). So the angle between OF and OG can be arbitrarily small.

So to find the total number of collisions N we count the angles between all the points on the circle, except for the angle between the last two points. Since this angle is arbitrarily small, N is the largest integer that satisfies

$$N \cdot 2\varphi < 2\pi$$

$$N < \frac{\pi}{\arctan\left(\sqrt{\frac{m}{M}}\right)} \tag{13}$$

When $M \gg m$ we can make a first order Taylor expansion $\arctan\left(\sqrt{m/M}\right) \approx \sqrt{m/M}$ to write Eq. (13) as

$$N < \pi\sqrt{\frac{M}{m}} \tag{14}$$

If $M = 100^n m$ then $\sqrt{M/m} = \sqrt{100^n} = 10^n$. By taking the floor function of the right side of Eq. (14) we get an explicit expression for the number of collisions.

$$N = \lfloor 10^n \pi \rfloor$$

So when $M = 100^n m$ the number of collisions will be the concatenation of the first $n + 1$ digits of π.

The following table gives the number of collisions for a few selected ratios between the masses.

M/m	N
100^0	3
100^1	31
100^2	314
100^4	31415
100^{10}	31415926535