N coupled harmonic oscillators

$$\mathrm{F}_{i,i+1} = -k(\mathrm{r}_i - \mathrm{r}_{i+1}) = -m\omega^2(\mathrm{r}_i - \mathrm{r}_{i+1})$$

$$V_{i,i+1} = -\frac{1}{2}m\omega^2(\mathrm{r}_i - \mathrm{r}_{i+1})^2$$

The Lagrangian is

$$L = K - V$$

$$= \sum_{i=1}^{N}\frac{1}{2}m\dot{\mathrm{r}}_i^2 + \sum_{i=1}^{N-1}\frac{1}{2}m\omega^2(\mathrm{r}_i - \mathrm{r}_{i+1})^2$$

$$= \sum_{i=1}^{N}\frac{1}{2}m\left(\dot{x}_i^2 + \dot{y}_i^2\right) + \sum_{i=1}^{N-1}\frac{1}{2}m\omega^2\left[(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2\right]$$

$$= \sum_{i=1}^{N}\frac{1}{2}m\dot{x}_i^2 + \sum_{i=1}^{N}\frac{1}{2}m\dot{y}_i^2 + \sum_{i=1}^{N-1}\frac{1}{2}m\omega^2(x_i - x_{i+1})^2 + \sum_{i=1}^{N-1}\frac{1}{2}m\omega^2(y_i - y_{i+1})^2$$

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

There are two equations of motion per particle.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = \frac{d}{dt}(m\dot{x}_j) + \frac{1}{2}m\omega^2\cdot 2(x_j - x_{j+1}) - \frac{1}{2}m\omega^2\cdot 2(x_{j-1} - x_j) = m\ddot{x}_j + m\omega^2(x_j - x_j$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_j}\right) - \frac{\partial L}{\partial y_j} = m\ddot{y}_j + m\omega^2(y_j - y_{j+1}) - m\omega^2(y_{j-1} - y_j) = 0$$

Which alternatively can be written as a system of four coupled first order differential equations.

$$\begin{aligned}
\dot{v}_{x,j} &= -\omega^2(x_j - x_{j+1}) + \omega^2(x_{j-1} - x_j) \\
\dot{v}_{y,j} &= -\omega^2(y_j - y_{j+1}) + \omega^2(y_{j-1} - y_j) \\
\dot{x}_j &= v_{x,j} \\
\dot{y}_j &= v_{y,j}
\end{aligned}$$

Simplified:

$$\begin{aligned}
\dot{v}_{x,j} &= \omega^2(x_{j-1} - 2x_j + x_{j+1}) \\
\dot{v}_{y,j} &= \omega^2(y_{j-1} - 2y_j + y_{j+1}) \\
\dot{x}_j &= v_{x,j} \\
\dot{y}_j &= v_{y,j}
\end{aligned}$$

Boundary conditions:

$$\dot{v}_{x,1} = \dot{v}_{x,N} = 0$$

Initial conditions:

$$x_i(0) = il,\ i = 0,\ 1,\ \ \dots,\ N-1$$

$$y_i(0) = 0\ \forall\ i$$

$$v_{x,j}(0) = 0\ \forall\ i$$

$$v_{y,j}(0) = \text{To be determined.}$$

$$\begin{bmatrix} \heartsuit & 0 \\ 0 & \heartsuit \end{bmatrix}$$