Show that $\dfrac{d}{dt}\mathbf{S}_H = \omega_0 \mathbf{n} \times \mathbf{S}_H$

Attempt:

The Hamiltonian is $H_0 = -\dfrac{e}{m}\mathbf{B}_0 \cdot \mathbf{S}$

Since $H_{0,H} = \mathcal{U}^\dagger H_0 \mathcal{U} = H_0 \mathcal{U}^\dagger \mathcal{U} = H_0$ and
$-\mathcal{U}^\dagger \dfrac{e}{m}\mathbf{B}_0 \cdot \mathbf{S}\mathcal{U} = -\dfrac{e}{m}\mathbf{B}_0 \cdot \mathcal{U}^\dagger \mathbf{S}\mathcal{U} = -\dfrac{e}{m}\mathbf{B}_0 \cdot \mathbf{S}_H$ we have

$$H_0 = -\frac{e}{m}\mathbf{B}_0 \cdot \mathbf{S}_H$$

so

$$\frac{d}{dt}\mathbf{S}_H = \frac{i}{\hbar}[H_0, \mathbf{S}_H]$$

$$= -\frac{i}{\hbar}\frac{eB_0}{m}[\mathbf{n}\cdot\mathbf{S}_H, \mathbf{S}_H] = \frac{i}{\hbar}\omega_0[\mathbf{n}\cdot\mathbf{S}_H, \mathbf{S}_H]$$

$$= \frac{i}{\hbar}\omega_0\{(\mathbf{n}\cdot\mathbf{S}_H)\mathbf{S}_H - \mathbf{S}_H(\mathbf{n}\cdot\mathbf{S}_H)\}$$

Problems with this expression:

- Cant get rid of the factor $i\,/\,\hbar$
- The content inside the curly brackets looks like a triple vector cross product i.e $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, but we want just $\mathbf{n}\times\mathbf{S}_H$.