If $\{|\varphi_n\rangle\}$ and $\{|\chi_n\rangle\}$ are two complete and orthonormal bases of the same Hilbert space, we can expand each ket of one basis as a linear combination of kets from the other basis:

$$|\varphi_n\rangle = \left(\sum_m |\chi_m\rangle\langle\chi_m|\right)|\varphi_n\rangle = \sum_m |\chi_m\rangle\langle\chi_m|\varphi_n\rangle \equiv \sum_m U_{mn}|\chi_m\rangle$$

$$|\chi_m\rangle = \left(\sum_n |\varphi_n\rangle\langle\varphi_n|\right)|\chi_m\rangle = \sum_n |\varphi_n\rangle\langle\varphi_n|\chi_m\rangle \equiv \sum_n V_{nm}|\varphi_n\rangle$$

where $U_{mn} \equiv \langle\chi_m|\varphi_n\rangle$ and $V_{nm} \equiv \langle\varphi_n|\chi_m\rangle = \langle\chi_m|\varphi_n\rangle^* = (U^*)_{mn} \rightarrow U = V^\dagger$

Since $\sum_l U_{ml}V_{ln} = \sum_l \langle\chi_m|\varphi_l\rangle\langle\varphi_l|\chi_n\rangle = \langle\chi_m|\left(\sum_l |\varphi_l\rangle\langle\varphi_l|\right)|\chi_n\rangle = \langle\chi_m|\chi_n\rangle = \delta_{mn}$
V is the inverse of U. $U^{-1} = V = U^\dagger$ so U is unitary.

Now let $|\psi'\rangle = U|\psi\rangle$ and $A|\psi\rangle = |\varphi\rangle$. We want to find the relation between A and the operator A' such that $A'|\psi'\rangle = |\varphi'\rangle$.

$|\varphi'\rangle = A'|\psi'\rangle = A'U|\psi\rangle$

$|\varphi'\rangle = U|\varphi\rangle = UA|\psi\rangle$

Thus

$$A'U = UA \rightarrow A' = UAU^\dagger$$