Surface Area Optimisation Proof

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Understanding the Relationship between Radius-to-Height Ratio $\frac{r}{h}$ and Surface Area of a Cylinder

We are examining how the surface area (SA) of a cylinder changes with the radius-to-height ratio ($\frac{r}{h}$) for a fixed volume.

Step 1: Set Up the Equations

Volume of a Cylinder:

$$V = \pi r^2 h$$

For a fixed volume V, we can express h in terms of r and the ratio $\frac{r}{h}$:

Height in terms of radius:

$$h = \frac{V}{\pi r^2}$$

Surface Area of a Cylinder:

$$SA = 2\pi r^2 + 2\pi rh$$

Step 2: Express SA in Terms of the Ratio $\frac{r}{h}$

Let's introduce the ratio $k = \frac{h}{r}$, so $h = kr$.

Substituting $h = kr$ into the volume equation:

$$V = \pi r^2(kr) = \pi r^3 k$$

Solve for r in terms of k and V:

$$r = \left(\frac{V}{\pi k}\right)^{\frac{1}{3}}$$

Now, find h in terms of k:

$$h = kr = k\left(\frac{V}{\pi k}\right)^{\frac{1}{3}} = \left(\frac{V}{\pi k^2}\right)^{\frac{1}{3}}$$

Step 3: Substitute r and h in Terms of k into the SA Expression

Surface Area formula:

$$SA = 2\pi r^2 + 2\pi rh$$

First Term:

$$2\pi r^2 = 2\pi\left(\frac{V}{\pi k}\right)^{\frac{2}{3}}$$

Second Term:

$$2\pi rh = 2\pi\left(\frac{V}{\pi k}\right)^{\frac{1}{3}}\left(\frac{V}{\pi k^2}\right)^{\frac{1}{3}} = 2\pi\left(\frac{V^2}{\pi^2 k^3}\right)^{\frac{1}{3}}$$

Final Expression for SA

So, the total surface area SA in terms of k is:

$$SA = 2\pi \left(\frac{V}{\pi k} \right)^{\frac{2}{3}} + 2\pi \left(\frac{V^2}{\pi^2 k^3} \right)^{\frac{1}{3}}$$

Interpretation

This expression shows the surface area SA as a function of the ratio $k = \frac{h}{r}$ for a fixed volume V. To find the optimal ratio that minimizes the surface area, one can differentiate this expression with respect to k and solve for the value of k. It turns out the optimal ratio is $k = 2$, meaning the height is twice the radius ($\frac{h}{r} = 2$), for minimal surface area.