DFT och FFT

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Fourier Transform

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ut} dt$$

DFT

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} \quad F = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_N \end{bmatrix}$$

$$\begin{aligned} F_k &= \sum_{n=0}^{N-1} f_n e^{(-i2\pi nk)/N} \\ &= A_k + B_k i \end{aligned}$$

DFT

$N = 4$

$$
\begin{aligned}
F_k &= \sum_{n=0}^{3} f_n e^{(-i2\pi nk)/4} \\
&= f_0 w_4^{0k} + f_1 w_4^{1k} + f_2 w_4^{2k} + f_3 w_4^{3k}
\end{aligned}
$$

$$w_N = e^{-i2\pi/N}$$

DFT

$N = 4$

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$$w_N = e^{-i2\pi/N}$$

$$A = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^4 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

$$F = Af$$

DFT

$$F_k = \sum_{n=0}^{N-1} f_n e^{(-i2\pi nk)/N} = \sum_{n=0}^{N-1} f_n w^k$$

$$A = \begin{bmatrix} w_N^0 & w_N^0 & w_N^0 & \cdots & w_N^0 \\ w_N^0 & w_N^1 & w_N^2 & \cdots & w_N^{N-1} \\ w_N^0 & w_N^2 & w_N^4 & \cdots & w_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_N^0 & w_N^{N-1} & w_N^{2(N-1)} & \cdots & w_N^{(N-1)^2} \end{bmatrix}$$

$$F = Af$$

$$F_2 = f_0 e^{-i2\pi(2)0} + f_1 e^{-i2\pi(2)1} + \cdots + f_{N-1} e^{-i2\pi(2)(N-1)}$$

$N-1$ komplexa additioner och N komplexa multiplicationer för F_k Totalt N^2 komplexa multiplicationer. $O(N^2)$

FFT

$$F_k = \sum_{n=0}^{N-1} f_n \cdot e^{-2\pi ikn/N}$$

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$$F_k = \sum_{n=0}^{N-1} f_n \cdot w_N^{nk}$$

En N-punkt F_k DFT är N-periodisk

$$F_k = \sum_{n=0}^{N-1} f_n \cdot w_N^{nk}$$

$$F_{k+N} = \sum_{n=0}^{N-1} f_n \cdot w_N^{n(k+N)} = \sum_{n=0}^{N-1} f_n \cdot w_N^{nk} w_N^{nN} = \sum_{n=0}^{N-1} f_n \cdot w_N^{nk} e^{(\frac{-i2\pi}{N})nN}$$

$$= \sum_{n=0}^{N-1} f_n \cdot w_N^{nk} e^{(-i2\pi)n} = \sum_{n=0}^{N-1} f_n \cdot w_N^{nk} = F_k$$

Två $N/2$-punkt sekvenser med jämna och udda index

$$g_k = \{f_0, f_2, f_4, \ldots, f_{N-2}\}, \quad h_k = \{f_1, f_3, f_5, \ldots, f_{N-1}\}$$

$$\begin{aligned}
F_k &= \sum_{n=0}^{N/2-1} f_{2n} w_N^{2nk} + \sum_{n=0}^{N/2-1} f_{2n+1} w_N^{(2n+1)k} \\
&= \sum_{n=0}^{N/2-1} f_{2n} w_N^{2nk} + \sum_{n=0}^{N/2-1} f_{2n+1} w_N^{2nk} \cdot w_N^k \\
&= \sum_{n=0}^{N/2-1} f_{2n} w_N^{2nk} + w_N^k \sum_{n=0}^{N/2-1} f_{2n+1} w_N^{2nk} \\
&= \sum_{n=0}^{N/2-1} f_{2n} w_{N/2}^{nk} + w_N^k \sum_{n=0}^{N/2-1} f_{2n+1} w_{N/2}^{nk} \\
&= G_k + W_N^k H_k
\end{aligned}$$

FFT

$$F_k = G_k + W_N^k H_k$$

G_k och H_k är $N/2$-punkt DFTs och därmed $N/2$ periodiska så $G_{k+N/2} = G_k$ och $H_{k+N/2} = H_k$.
Dessutom har vi

$$w_N^{k+N/2} = w_N^k w_N^{N/2} = e^{-i2\pi} w_N^k = -w_N^k$$

Så att

$$F_k = G_k + w_N^k H_k$$
$$F_{k+N/2} = G_k - w_N^k H_k$$

FFT

Varje steg kräver

- $N/2$ komplexa multiplikationer
- N komplexa additioner

Komplexitet: $O(N \log_2 N)$

FFT: Implementation

```
function fft(f)
    N = length(f)
    if N == 1 return f end
    N_half = Int(N/2)
    w = exp(-2*im/N)
    G = fft(f[1:2:end])
    H = fft(f[2:2:end])
    F = zeros(Complex{Float64}, N)
    wk = 1;
    for k in 1:N_half
        wHk = wk*H[k]
        F[k] = G[k] + wHk
        F[k + N_half] = G[k] - wHk
        wk = wk*w
    end
    return F
end
```

Invers DFT

$$f_k = F_k^{-1} = \frac{1}{N}\sum_{n=0}^{N-1} F_k e^{i2\pi nk}$$

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$$\frac{1}{N}(\overline{F(\bar{F})})_k = \frac{1}{N}\sum_{n=0}^{N-1} F_k e^{\frac{i2\pi}{N}nk} = (F^{-1}(F))_k$$

Invers DFT: Implementation

```
function ifft(a)
    a = conj.(a)
    a = fft(a)
    a = conj.(a)
    a = a./length(a)
end
```