Innlevering 3

Oppgave 1:

$\lambda_A = \frac{3}{73} = 0.0411$

$\lambda_B = \frac{1}{73} = 0.0137$

$X_A \to$ antall feil av typen A i løpet av 365 dager

$X_B \to$ antall feil av typen B i løpet av 365 dager

a) $E[X_A] = 365 \cdot 0.0411 \approx 15$

$P(X_A \leq 15) = 0.5681$ (Fra tabell)

$\lambda_B \cdot 365 = 5$

$x = 10$

$\mu = 5$

$P(X_A \leq 15, X_B \leq 10) = P(X_A \leq 15) \cdot P(X_B \leq 10) = 0.5681 \cdot 0.9863$
$= 0.5603$

A og B er uavhengig

$P(X_A \leq 15 \mid X_B \leq 10) = P(X_A \leq 15) = 0.5681$

b) $Z = X_A + X_B$

$M_A(t) = e^{\lambda_A(e^t - 1)}$

$M_B(t) = e^{\lambda_B(e^t - 1)}$

Momentgenererende funksjonen for summen av uavhengige stokastiske variabler er et produkt av deres momentgenererende funksjoner

$$M_Z(t) = M_A(t) \cdot M_B(t)$$
$$= e^{\lambda_A(e^t - 1)} \cdot e^{\lambda_B(e^t - 1)}$$
$$= e^{\lambda_A(e^t - 1) + \lambda_B(e^t - 1)}$$
$$= e^{(\lambda_A + \lambda_B)(e^t - 1)}$$
$$= e^{\lambda_Z(e^t - 1)}$$

der $\lambda_Z = \lambda_A + \lambda_B$ //

c) $f_{X_A|Z}(x_A|z) = P(X_A = x_a \mid Z = x)$

Siden vi ønsker et gitt antall feil av typen A gitt en mengde feil z følger denne en betinget binomisk fordeling.

Oppgave 2:

X er fødselsvekt [g] i Norge

$\mu = E[X] = 3500$
$\sigma = SD[X] = 570$

a) $P(X \geq 3000) = 1 - P(X \leq 3000)$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{3000-3500}{570}\right)$$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \leq -0.88\right)$$

$$= 1 - 0.1894 = 0.8106$$

$$P(3200 \leq X < 4000) = P(X < 4000) - P(X \leq 3200)$$

$$= P\left(\frac{X-\mu}{\sigma} < 0.88\right) - P\left(\frac{X-\mu}{\sigma} \leq -0.53\right)$$

$$= 0.8106 - 0.2981 = 0.5125$$

$$P(X \geq 3500 \mid X \geq 3000) = \frac{P(X \geq 3500 \cap X \geq 3000)}{P(X \geq 3000)} = \frac{1 - P(X \leq 3500)}{P(X \geq 3000)}$$

$$= \frac{0.5}{0.8106} = 0.6168$$

$$P(X \leq 4000 \mid X \geq 3200) = \frac{P(X \leq 4000 \cap X \geq 3200)}{P(X \geq 3200)}$$

$$= \frac{P(X \leq 4000) - P(X \leq 3200)}{1 - P(X \leq 3200)}$$

$$= \frac{P\left(\frac{X-\mu}{\sigma} \leq 0.88\right) - P\left(\frac{X-\mu}{\sigma} \leq 0.53\right)}{1 - P\left(\frac{X-\mu}{\sigma} \leq 0.53\right)}$$

$$= \frac{0.8106 - 0.2981}{1 - 0.2981} = 0.7302$$

b)

$$P(X < c) = 0.01$$

$n = 100$

Y e antall undervektige ved fødsel

$$P(Z > z_\alpha) = \alpha$$
$$P(Z > 2.326) = 0.01$$
$$\Downarrow$$
$$2.326 \cdot 570 = 1325.82$$
$$c = E[X] - 1325.8 = 3500 - 1325.8 = 2174.2$$

For at Y skal være binomisk fordelt må barnas vekter være uavhengige av hverandre.

$$P(Y \geq 1) = 1 - P(Y < 1)$$
$$= 1 - (1 - P(X < c))^N$$
$$= 1 - (1 - 0.01)^{100} = 1 - 0.99^{100}$$
$$= 1 - 0.366$$
$$= 0.634$$

$$P(Y \geq 2 \mid Y \geq 1) = \frac{P(Y \geq 2 \mid Y \geq 1)}{P(Y \geq 1)} = \frac{P(Y \geq 2)}{P(Y \geq 1)} = \frac{1 - P(Y=0) - P(Y=1)}{P(Y \geq 1)}$$
$$= \frac{1 - 0.344 - 0.3697}{0.634} = 0.264$$

$$P(Y=0) = (1 - 0.01)^{100} = 0.344$$
$$P(Y=1) = 100 \cdot 0.01 (1 - 0.1)^{99} = 0.99^{99} = 0.3697$$

Oppgave 3

$X_1 = Z_1$ og for $k = 2, 3, \dots$ $X_k = Z_k - Z_{k-1}$

a) $P(X_1 \geq 2) = 1 - F(2) = 1 - (1 - e^{-\lambda 2}) = e^{-2\lambda} = e^{-2 \cdot 0.5} = e^{-1} = 0.3679$

$P(X_1 + X_2 \geq 4)$

X_1: tid til 1. hendelse

X_2: tid mellom X_1 og X_2

$X_1 + X_2$: tid til 2. hendelse

$N(t)$: telle antall hendelser i $[0, t]$

$$N(t) \sim \text{Poi}(\lambda t)$$
$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$P(X_1 + X_2 \geq 4)$
$$= P(N(4) = 0 \cup N(4) = 1)$$
$$= P(N(4) = 0) + P(N(4) = 1)$$
$$= \frac{(4\lambda)^0}{0!} e^{-4\lambda} + \frac{(4\lambda)^1}{1!} e^{-4\lambda} = e^{-4\lambda}(1 + 4\lambda) = e^{-4 \cdot 0.5}(1 + 4 \cdot 0.5)$$
$$= e^{-2}(1 + 2) = 0.4060$$

$P(X_1 + X_2 \geq 4 \mid X_1 \geq 2)$

$= 1 - P(X_1 + X_2 \leq 4 \mid X_1 \geq 2)$

b) $\alpha = 1$, $\beta = \frac{1}{\lambda}$ Vi setter dette inn i sannsynlighet tetthets funksjonen (pdf) gamma fordelingen

$$f_Y(y) = \frac{1}{\Gamma(1)(\frac{1}{\lambda})^1} y^{1-1} e^{-(\frac{1}{\lambda})}$$

$\Gamma(1) = 1! = 1$ og $y^{1-1} = 1$

$$f_Y(y) = \frac{1}{\frac{1}{\lambda}} e^{-\frac{1}{\lambda}} = \boxed{\lambda e^{-\lambda y}}$$

Dette er identisk med eksponential funksjonen med parameter λ //

c) Vi antar Z_{n-1} er gammafordelt med parametre $\alpha = n-1$ og $b = \frac{1}{\lambda}$

$Z_n = Z_{n-1} + X_n$, Z_{n-1} og X_n er uavhengige

pdf for Z_n kan uttrykkes slik

$$f_{Z_n}(z) = \int_0^z f_{Z_{n-1}}(y) f_{X_n}(z-y)\,dy$$

Vi setter inn for $f_{Z_{n-1}}(y)$ som er gammafordeling med $\alpha = n-1$ og $\beta = \frac{1}{\lambda}$
og for $f_{X_n}(z-y)$ som er eksponential fordelt

$$f_{Z_n}(z) = \int_0^z \underbrace{\frac{1}{\Gamma(n-1)(\frac{1}{\lambda})^{n-1}} y^{n-2} e^{-\lambda y}}_{f_{Z_{n-1}}} \underbrace{\lambda e^{-\lambda(z-y)}}_{f_{X_n}}\,dy$$

$$= \frac{1}{\Gamma(n-1)(\frac{1}{\lambda})^{n-1}} \lambda e^{-\lambda z} \int_0^z y^{n-2}\,dy \qquad \int_0^z y^{n-2}\,dy = \frac{1}{n-1} y^{n-1} \Big|_0^z = \frac{z^{n-1}}{n-1} - \frac{0^{n-1}}{n-1} = \frac{z^{n-1}}{n-1}$$

$$= \frac{\lambda^n}{\Gamma(n-1)} e^{-\lambda z} \frac{z^{n-1}}{(n-1)} \qquad (n-1)\Gamma(n-1) = \Gamma(n)$$

$$\boxed{f_{Z_n}(z) = \frac{\lambda^n}{\Gamma(n)} e^{-\lambda z} z^{n-1}}$$

gammafordelingen med $\alpha = n$ og $\beta = \frac{1}{\lambda}$ //

Oppgave 4

$X \sim N(\mu, \sigma^2)$

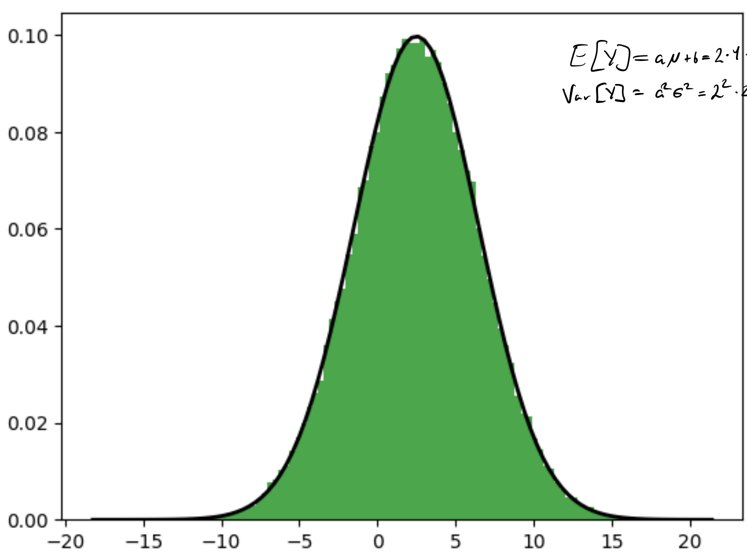
$Y = aX + b$

a) $E[Y] = aE[X] + b = a\mu + b$

$Var[Y] = a^2 Var[X] = a^2\sigma^2$

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

def sim_Y(n, mu, sigma, a, b):
    X = np.random.normal(mu, sigma, n)
    Y = a * X + b
    return Y

N = int(1e5)
mu = 1
sigma = 2
a = 2
b = 0.5

# Simulerer Y
Y = sim_Y(N, mu, sigma, a, b)

# Beregn forventningsverdi og varians for Y
E_Y = a * mu + b
Var_Y = (a * sigma)**2

# Histogram
plt.hist(Y, bins=100, density=True, alpha=0.7, color='g')

# Analytisk fordeling
xmin, xmax = plt.xlim()
x = np.linspace(xmin, xmax, 100)
p = norm.pdf(x, E_Y, np.sqrt(Var_Y))
plt.plot(x, p, 'k', linewidth=2)
```

✓ 2.3s



$E[Y] = a\mu + b = 2\cdot 1 + 0.5 = 2.5$

$Var[Y] = a^2\sigma^2 = 2^2 \cdot 2^2 = 16$

b)

$$f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\cdot\sigma^2}\right\}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi \, Var[y]}} \exp\left\{-\frac{(y-E[Y])^2}{2\cdot Var[Y]}\right\}$$

$$= \frac{1}{\sqrt{2\pi a^2 \sigma^2}} \exp\left\{-\frac{(y-(a\mu+b))^2}{2a^2\sigma^2}\right\} //$$

Oppgave 5:

$$M_X(t) = E[e^{tX}]$$

a)

$$M_{X_i}(t) = P(X_i = 1)e^t + P(X_i = -1)e^{-t} = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$$

$$M'_{X_i}(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$$

$$M'_{X_i}(0) = \frac{1}{2}e^0 - \frac{1}{2}e^0 = \frac{1}{2} - \frac{1}{2} = 0 \Rightarrow E[X_i] = 0$$

$$M''_{X_i}(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$$

$$M''_{X_i}(0) = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow E[X_i^2] = 1$$

$$Var[X] = E[X^2] - E[X]^2 = 1$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i \Rightarrow E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^n X_i\right] = \frac{1}{n}\sum_{i=1}^n E[X_i] = \frac{1}{n}\cdot n\cdot 0 = 0 //$$

$$Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^n X_i\right] = \frac{1}{n^2}\sum_{i=1}^n Var[X_i] = \frac{1}{n^2}\cdot n\cdot 1 = \frac{1}{n}$$

b)

$$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E\left[e^{t\left(\frac{1}{n}\sum_{i=1}^n X_i\right)}\right] = E\left[\prod_{i=1}^n e^{t/n\, X_i}\right] = \prod_{i=1}^n E[e^{t/n\, X_i}]$$

Vi vet at $E[e^{t/n\, X_i}] = \frac{1}{2}e^{t/n\, X_i} + \frac{1}{2}e^{-t/n\, X_i}$

$$M_{\bar{X}}(t) = \left(M_{X_i}\left(\frac{t}{n}\right)\right)^n = \left(\frac{1}{2}e^{t/n} + \frac{1}{2}e^{-t/n}\right)^n$$

$$U = \frac{\bar{X} - E[\bar{X}]}{\sqrt{Var[X]}} \xrightarrow[\text{fra a)}]{\text{med tall}} U = \frac{\bar{X} - 0}{\sqrt{\frac{1}{n}}} = \sqrt{n}\,\bar{X}$$

$$M_U(t) = E[e^{tU}] = E[e^{t\sqrt{n}\bar{X}}] = M_{\bar{X}}(t\sqrt{n})$$

$$M_U(t) = \left(\frac{1}{2}e^{\frac{t\sqrt{n}}{n}} + \frac{1}{2}e^{-\frac{t\sqrt{n}}{n}}\right)^n = \left(\frac{1}{2}e^{\frac{t}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{t}{\sqrt{n}}}\right)^n$$

$$\ln M_U(t) = \ln\left(\frac{1}{2}e^{\frac{t}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{t}{\sqrt{n}}}\right)^n = n\ln\left(\frac{1}{2}e^{\frac{t}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{t}{\sqrt{n}}}\right) //$$

c)

```python
# Definerer funksjonen for M_U(t) basert på den utledede formelen
def M_U(t, n):
    return np.exp(n * np.log((np.exp(t/np.sqrt(n)) + np.exp(-t/np.sqrt(n))) / 2))

# Definerer funksjonen for M_Z(t) basert på den momentgenererende funksjonen for en standard normalfordeling
def M_Z(t):
    return np.exp(t**2 / 2)

# Definerer t-verdier i intervallet [0, 1]
t_values = np.linspace(0, 1, 100)

# Lager plott for ulike n-verdier
n_values = [1, 2, 5]
plt.figure(figsize=(12, 8))
for n in n_values:
    plt.plot(t_values, M_U(t_values, n), label=f'M_U(t) for n={n}')

# Plotter M_Z(t)
plt.plot(t_values, M_Z(t_values), label='M_Z(t)', linestyle='--')
plt.title('Sammenligning av M_U(t) for ulike n og M_Z(t)')
plt.xlabel('t')
plt.ylabel('Momentgenererende funksjon')
```

✓ 0.1s