

# Logic in Computer Science

## Föreläsning 2

Erik Sjöström

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### 1 Propositional logic

- $\wedge$  conjugation "and"
- $\vee$  disjunction "or"
- $\rightarrow$  implication "if ... then"
- $\perp$  absurdity
- $\neg$  negation "not"

Formulas are built from atomic formulas ( $p, q, \dots$ ) by using the connectives,  $\vee, \wedge, \dots$

**Exempel 1.1.**

$$(p \rightarrow q) \vee (q \wedge \pi)$$

The language of logic is built up from syntax, semantics, and proof rules.

#### 1.1 Proof rules

**Natural deduction** (*Gentzen 1934*)

There are two kinds of rules:

- Introduction rules
- Elimination rules

**Conjugation:**

$\wedge$ -introduction " $\wedge$ -i":

$$\frac{\phi \quad \psi}{\phi \wedge \psi}$$

$\wedge$ -elimination " $\wedge$ -e":

$$\frac{\phi \wedge \psi}{\phi} \quad \frac{\phi \wedge \psi}{\psi}$$

**Exempel 1.2.***Show:  $\phi \wedge \psi \vdash \psi \wedge \phi$* 

1.  $\phi \wedge \psi$     *Premise*
2.  $\psi$              $\wedge\text{-}e, 1$
3.  $\phi$              $\wedge\text{-}e, 1$
4.  $\psi \wedge \phi$      $\wedge\text{-}i, 2,3$

**Implication** $\rightarrow$ -introduction

1.  $\phi$
2.  $\vdots$
3.  $\psi$
4.  $\phi \rightarrow \psi$

 $\rightarrow$ -elimination

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi}$$

**Exempel 1.3.***Show:  $\phi \rightarrow (\psi_1 \wedge \psi_2) \vdash \phi \rightarrow \psi_1$* 

1.  $\phi \rightarrow (\psi_1 \wedge \psi_2)$             *Premise*
2.  $\phi$                                     *Assumption*
3.  $\psi_1 \wedge \psi_2$                      $\rightarrow\text{-}e, 1,2$
4.  $\psi_1$                                  $\wedge\text{-}e, 3$
5.  $\phi \rightarrow \psi_1$                      $\rightarrow\text{-}i, 2\text{-}4$

**Absurdity** $\perp$ -elimination

$$\frac{\perp}{\phi}$$

Definition of negation  $\neg\phi$ 

$$\neg\phi \stackrel{def}{=} \phi \rightarrow \perp$$

**Exempel 1.4.**

Show:  $\phi \vdash \neg\neg\phi$  i.e.  $\phi \vdash (\phi \rightarrow \perp) \rightarrow \perp$

- |    |  |                       |
|----|--|-----------------------|
| 1. | $Q$  | Premise               |
| 2. | $\phi \rightarrow \perp$                     | Assumption            |
| 3. | $\perp$                                      | $\rightarrow$ -e, 1,2 |
| 4. | $(\phi \rightarrow \perp) \rightarrow \perp$ | $\rightarrow$ -e, 2,3 |

**Exempel 1.5.**

Show:  $\phi \wedge \neg\phi \vdash \psi$

- |    |                          |                       |
|----|--------------------------|-----------------------|
| 1. | $\phi \wedge \neg\phi$   | Premise               |
| 2. | $\phi$                   | $\wedge$ -e, 1        |
| 3. | $\phi \rightarrow \perp$ | $\wedge$ -e, 1        |
| 4. | $\perp$                  | $\rightarrow$ -e, 2,3 |
| 5. | $\psi$                   | $\perp$ -e, 4         |

**Disjunction**  $\vee$ -introduction

$$\frac{\phi}{\phi \wedge \psi} \quad \frac{\psi}{\phi \wedge \psi}$$

$\vee$ -elimination

- |    |                  |
|----|------------------|
| 1. | $\phi \vee \psi$ |
| 2. | $\phi$           |
| 3. | $\vdots$         |
| 4. | $\chi$           |
| 5. | $\psi$           |
| 6. | $\vdots$         |
| 7. | $\chi$           |
| 8. | $\chi$           |

**Exempel 1.6.**

Show:  $\phi \vee \psi \vdash \psi \vee \phi$

- |    |                  |                        |
|----|------------------|------------------------|
| 1. | $\phi \vee \psi$ | <i>Premise</i>         |
| 2. | $\phi$           | <i>Assumption</i>      |
| 3. | $\psi \vee \phi$ | $\vee$ -i, 2           |
| 4. | $\psi$           | <i>Assumption</i>      |
| 5. | $\psi \vee \phi$ | $\vee$ -i, 4           |
| 6. | $\psi \vee \phi$ | $\vee$ -e, 1, 2-3, 4-5 |

### Proof by contradiction

- |    |            |
|----|------------|
| 1. | $\neg\phi$ |
| 2. | $\vdots$   |
| 3. | $\perp$    |
| 4. | $\phi$     |

### **Exempel 1.7.**

Show:  $\neg\neg\phi \vdash \phi$

- |    |                |                       |
|----|----------------|-----------------------|
| 1. | $\neg\neg\phi$ | <i>Premise</i>        |
| 2. | $\neg\phi$     | <i>Assumption</i>     |
| 3. | $\perp$        | $\rightarrow$ -e, 2,1 |
| 4. | $\phi$         | <i>PBC</i> , 2-3      |