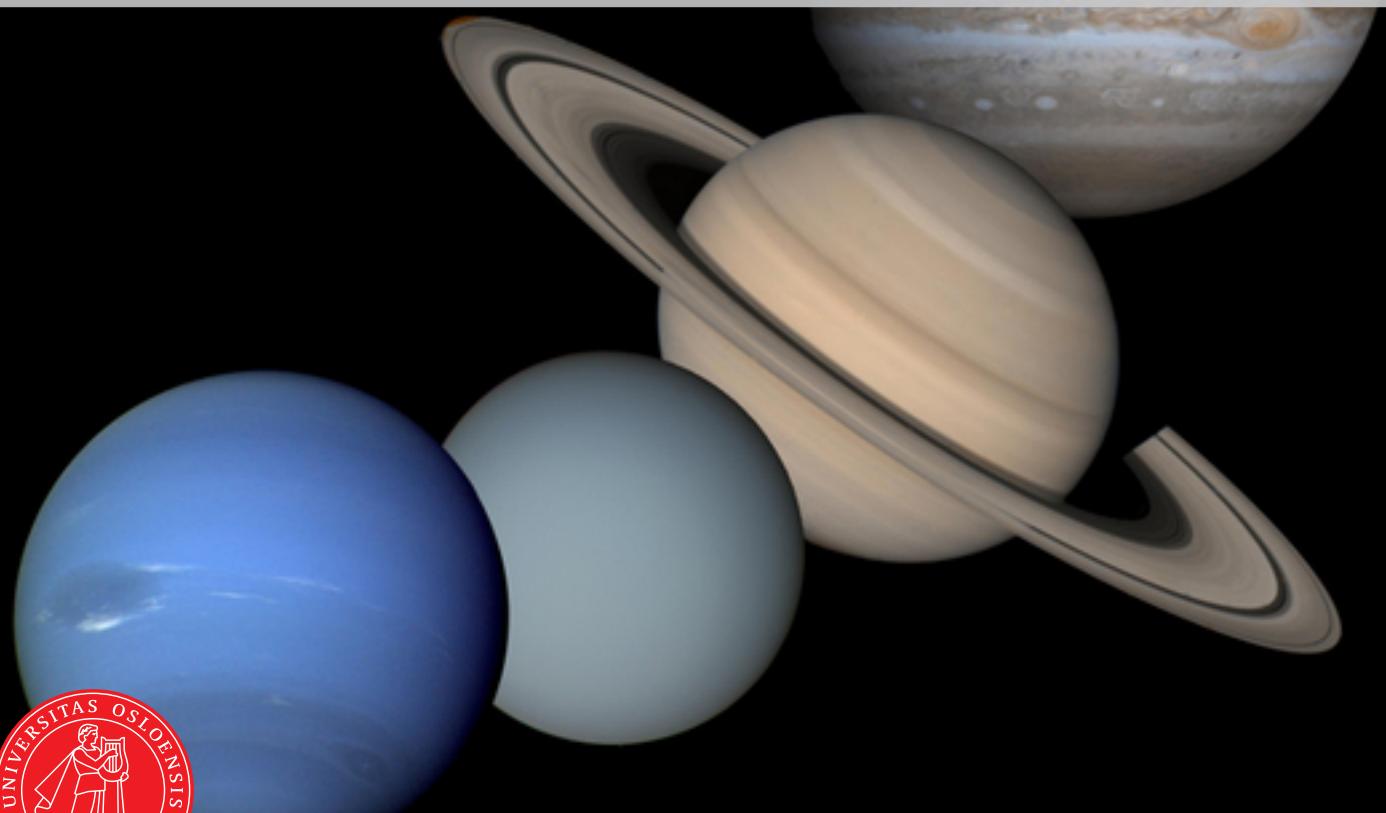


UiO : **Department of Physics**  
University of Oslo

# Solarsystem FYS3150/FYS4150

Erik Skaar  
Mikael Kiste  
Thomas Storaas



## Innhold

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Theory</b>	<b>3</b>
2.1	Gravitation . . . . .	3
2.2	Units . . . . .	3
2.2.1	Escape velocity . . . . .	4
2.3	perihelion . . . . .	5
2.4	Numerical methods . . . . .	5
2.4.1	Forward Euler . . . . .	5
2.4.2	Velocity-Verlet . . . . .	6
<b>3</b>	<b>Method</b>	<b>7</b>
3.1	Implementation of code . . . . .	7
<b>4</b>	<b>Result &amp; Discussion</b>	<b>8</b>
4.1	Earth-Sun system . . . . .	8
4.1.1	Stability . . . . .	8
4.1.2	Conserved quantities . . . . .	9
4.1.3	Escape velocity . . . . .	10
4.2	Three body system . . . . .	12
4.2.1	Fixed mass for jupiter . . . . .	12
4.2.2	Varying mass for jupiter . . . . .	12
4.3	Solar system . . . . .	13
4.3.1	Three planets and all moving . . . . .	13
4.3.2	Solar system all moving . . . . .	14
4.4	The perihelion precession of Mercury . . . . .	14
<b>5</b>	<b>Conclusion</b>	<b>15</b>
<b>6</b>	<b>Appendix</b>	<b>16</b>

## Abstract

## 1 Introduction

For thousand of years humankind have looked up to the beyond and wondered. Specifically our race has wondered about the motion of our solar system. Finally after Newton the mystery was solved. Newton developed the gravitational law, which made it possible to predict the motion of the planets. After a couple of centuries Einstein came with the theory of general relativity and made a small refinement to the law of motion that Newton proposed.

These laws are not enough to solve the motion of the planets. The laws makes differential equation which are not trivial or even impossible to solve. This is where our computational physics course comes in. With the tools developed in this course we can make a prediction to the motion of the planets in our solarsystem.

In this project we will make an object oriented code to solve the solar system with Forward Eulers method and Velocity Verlet. Both method will be derived, discussed, implemented and benchmarked for the Earth-Sun system.

Finally, a relativistic correction was made.....

REF  
KUR-  
SETS  
HJEM-  
MESIDE  
  
WRITE  
ABOUT  
THEIR  
PERFOR-  
MANCE

NEED TO  
IMPLE-  
MENT  
THIS  
AND  
GET  
FACTS

## 2 Theory

### 2.1 Gravitation

We will simulate the solar system with only the gravitational force affecting the planets. Newton's gravitational law is stated in equation (1). Where G is the gravitational constant ( $6.67 \cdot 10^{-11} \text{Nm}^2/\text{s}^2$ ), r is the distance between the planets, m is the mass of the object and M is the mass of the other object.

$$\vec{F}_G = \frac{GmM}{|\vec{r}|^3} \vec{r} \quad (1)$$

Thankfully for us the force is an attractive force. It is worth noting that if the sun is at origo distance is simply the norm position vector of the planet. From now on we will denoted the different masses with  $M_{\text{planetname}}$  except for the sun that has the special symbol  $M_{\odot}$ .

If you have n object attracting each other the total gravitational force,  $F_k$ , for each object is:

$$F_k = \sum_{i=1}^N \vec{F}_i = \frac{Gm_k m_i}{|\vec{r}_k - \vec{r}_i|^3} (\vec{r}_k - \vec{r}_i) (1 - \delta_{k,i}) \quad (2)$$

Where  $\delta_{k,i}$  is the function:

$$\delta_{k,i} = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

We can use Newton's second law to determine the acceleration of the planet. Newton's second law state  $F = m\vec{a}$ . Which translate to:

$$a_k = \sum_{i=1}^N \frac{Gm_i}{|\vec{r}_k - \vec{r}_i|^3} (\vec{r}_k - \vec{r}_i) (1 - \delta_{k,i}) \quad (3)$$

### 2.2 Units

As a famous person once said, If you use seconds and meters you will not finish within the deadline: I think our professor, Morten Hjorth-Jensen, is on to something. Seconds and meters are impractical in this project. Therefore we use more suitable units. The distance between the earth and sun, 149 597 870 691 meter, is defined as one astronomical unit (au). For time one year seems reasonable. These are the units that will be used in this project.

We can now express the gravitational constant with these units. To do this we use the formula for the acceleration for a circular orbit is  $\frac{v^2}{r}$ . Combine this with the gravitational law, we get:

$$\frac{v^2}{r} = \frac{GM_{\odot}}{|\vec{r}|^3} \vec{r} \implies G = \frac{v^2 r}{M_{\odot}} = \frac{\text{au}(2\pi \cdot \text{au/year})^2}{M_{\odot}} = \frac{4\pi^2}{M_{\odot}} \text{au}^3/\text{year}^2$$

Applying this to equation (3), we get:

$$a_k = \sum_{i=1}^N 4\pi \frac{m_i}{M_{\odot}} \frac{(\vec{r}_k - \vec{r}_i)}{|\vec{r}_k - \vec{r}_i|^3} (1 - \delta_{k,i}) \text{au}^3/\text{year}^2 \quad (4)$$

### 2.2.1 Escape velocity

Later in the project we are asked to find the escape velocity by trial and error. This question is basicly, when does the earth have enough kinetic energy to escape the potential. The potential is a integral from 0 to  $\infty$   $F_G$  over  $dr$ . For a given planet with only the sun in the solar system the integral is from  $r$  to  $\infty$ . This is shown in equation (6).

$$E_k = \int_r^{\infty} \frac{GmM}{|\vec{r}|^2} d\vec{r} \quad (5)$$

(6)

Let's solve it:

$$E_k = \int_r^\infty -\frac{GmM_\odot}{|\vec{r}|^2} d\vec{r}$$

$$\frac{1}{2}mv^2 = GmM_\odot \int_r^\infty -\frac{1}{|\vec{r}|^2} d\vec{r}$$

$G = 1$ , see section 2.2 for explanation.

$$\frac{1}{2}v^2 = M_\odot \int_r^\infty -\frac{1}{|\vec{r}|^2} d\vec{r}$$

$$\frac{1}{2}v^2 = M_\odot \left[ \frac{1}{|\vec{r}|} \right]_r^\infty$$

$$v^2 = 2M_\odot \frac{1}{|\vec{r}|}$$

$r = 1$ , see section 2.2 for explanation.

$$v = \sqrt{8\pi^2}$$

$$v \approx 8.8857 \text{ au/year}$$

## 2.3 prehelion

## 2.4 Numerical methods

Equation (4) initial conditions for velocity and position determines the orbit of the planets. These equations are for x,y,z direction and when written out you get a coupled set of differential equations. This set is near impossible to solve analytically. It might be impossible. We will use numerical methods to solve this set. More specifically we will use the Forward Euler method and the Velocity-Verlet method. A given  $t_i$  is equal to  $t_0 + ih$ , where  $h = (t_0 + t_n)/n$ .

since this is not implemented in the code yet nothing is written

### 2.4.1 Forward Euler

Both method use a Taylor polynomial approximation to solve the set of differential equation. The Forward Euler use the first order Taylor polynomial. With  $r'(t) = v(t)$  and  $v'(t) = a(t)$  the Forward Euler method result in:

$$\vec{r}_i(t+h) \approx \vec{r}_i(t) + h\vec{v}_i(t) \quad (7)$$

$$\vec{v}_i(t+h) \approx \vec{v}_i(t) + h\vec{a}_i(t) \quad (8)$$

Discretized version. i is the object number and j is the time step:

$$\begin{aligned}\vec{r}_{i,j+1} &\approx \vec{r}_{i,j} + h\vec{v}_{i,j} \\ \vec{v}_{i,j+1} &\approx \vec{v}_{i,j} + h\vec{a}_{i,j}\end{aligned}$$

The error for a first order taylor polynomial goes as  $\mathcal{O}(h^2)$ . This is the error for each step. The error is accumulated for each step and will thereby be proportional to  $h$ .

REF TO  
LECTU-  
RE NO-  
TES

#### 2.4.2 Velocity-Verlet

You guessed it, this is the second order Taylor polynomial. . As I said the Velocity-Verlet method is based on a Taylor polynomial approximation. And this is the second order.

$$\vec{r}_i(t+h) \approx \vec{r}_i(t) + h\vec{v}_i(t) + \frac{1}{2}h^2\vec{a}(t) \quad (9)$$

$$\vec{v}_i(t+h) \approx \vec{v}_i(t) + h\vec{a}(t) + \frac{1}{2}h^2\vec{a}'(t) \quad (10)$$

Since we have no formula for the derivative of  $a$ , we use the approximation:

$$\vec{a}'(t) \approx \frac{\vec{a}(t+h) - \vec{a}(t)}{h}$$

We simply update the position first. We keep the old acceleration and calculate the acceleration at the new position. Using this equation (10) we get:

$$\begin{aligned}\vec{v}_i(t+h) &\approx \vec{v}_i(t) + h\vec{a}(t) + \frac{1}{2}h(\vec{a}(t+h) - \vec{a}(t)) \\ \vec{v}_i(t+h) &\approx \vec{v}_i(t) + \frac{1}{2}h(\vec{a}(t+h) + \vec{a}(t))\end{aligned}$$

Discretized version. i is the object number and j is the time step:

$$\begin{aligned}\vec{r}_{i,j+1} &\approx \vec{r}_{i,j} + h\vec{v}_{i,j} + \frac{1}{2}h^2\vec{a}_{i,j} \\ \vec{v}_{i,j+1} &\approx \vec{v}_{i,j} + \frac{1}{2}h(\vec{a}_{i,j+1} + \vec{a}_{i,j})\end{aligned}$$

The error for a first order taylor polynomial goes as  $\mathcal{O}(h^2)$ . This is the error for each step. The error is accumulated for each step and will thereby be proportional to  $h^2$ . This is one order of magnitude better, then the Forward Euler method.

REF TO  
LECTU-  
RE NO-  
TES

## 3 Method

### 3.1 Implementation of code

The implementation is written in c++ and is object oriented. We found it natural to divide the project into two classes and one main program.

We made one class for the planet ([planet.cpp](#) and [planet.h](#)). This class has the variables for the position, velocity and mass for the specific planet. As well as a filename to write the output to. The class has methods for updating position, velocity and acceleration as well as methods for writing to a file.

The second class is for the solarsystem ([solarsystem.cpp](#) and [solarsystem.h](#)). The solar system contains a list with all the planets and has methods to make the planets move and react to each other. You can look at the solar system as a class with for-loops for the planets.

The [main.cpp](#) program is where all the inputs to the solar system is given. Here all the initial conditions for the planets are stated and organized for proper input to the solarsystem class.

All these programs are combined with [makefile](#). When this is done the [solsys.exe](#) will be made. Takes in three arguments. The first argument is the number of planets that you would like to simulate. This is a preset list of the bodies in the solarsystem. Where the Sun is the first element, Earth is the second and Jupiter is the third. After that they ascend based on radius. The second argument is the end time measured in years from now. The start time is set to 19th of October 2017. This is because this is the time that the initial values were obtained from NASA. Finally the third argument is the number of steps, n.

Finally, in our github repository there are directories where earlier versions can be found. For instance the [3a](#) directory is the first version with no object oriented code.

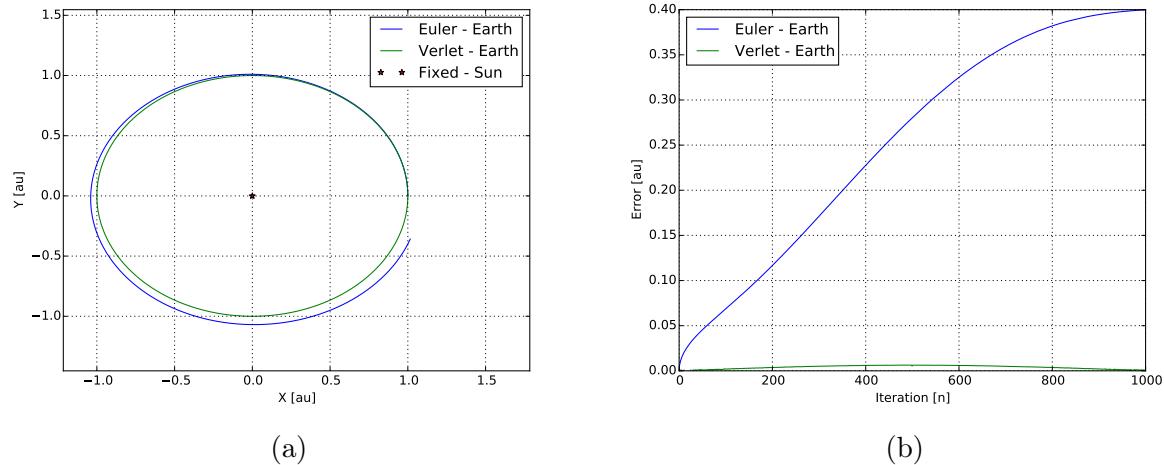
REF  
NASA

SJEKK  
AT  
DETTE  
STEM-  
MER  
NÅR VI  
LEVE-  
RER

## 4 Result & Discussion

### 4.1 Earth-Sun system

#### 4.1.1 Stability



Figur 1: a) show the orbit of earth around the sound. The intial velocity is set to  $2\pi$  in y direction and the start position to 1 au in x direction. b) shows how the error behaves. The intial values should give a perfect circular motion. So the error is calculated by  $r_i - r_0$ . It is clear that Verlet-Velocity method is superior. This simulation was with 1000 points with the end time of 1 year. Both simulations was produced by [plot\\_earth\\_sun.py](#)

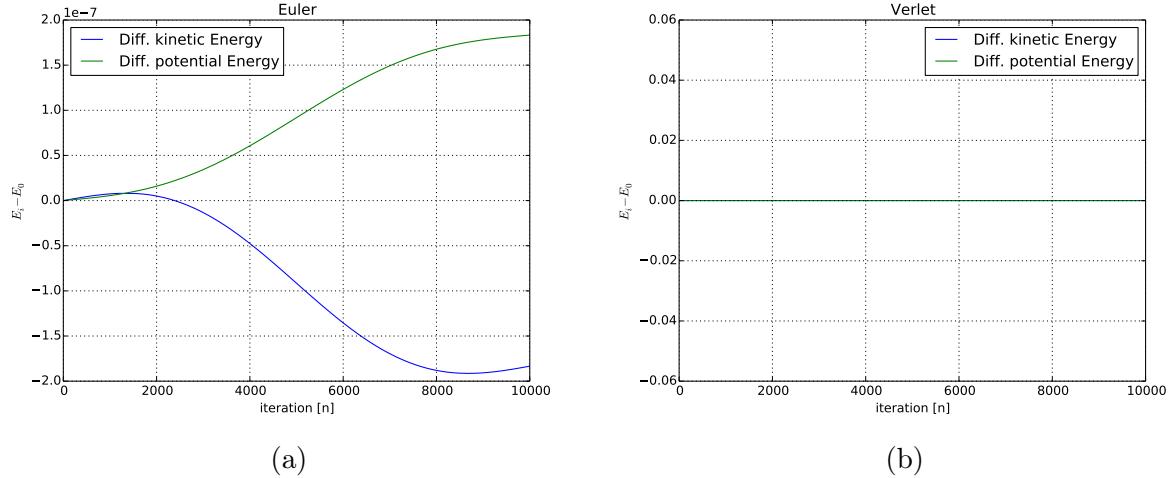
REF  
FLOPS  
SECTION

Tabell 1: Time table for the different algorithms. The algorithms use nearly the same time. This is not a shocker since the FLOPs are similar. Time grows very linear as expected from FLOPs. Disclaimer: this is only the result from one test, but several was done. Both algorithms were very close and it seem to be random which is fastest.

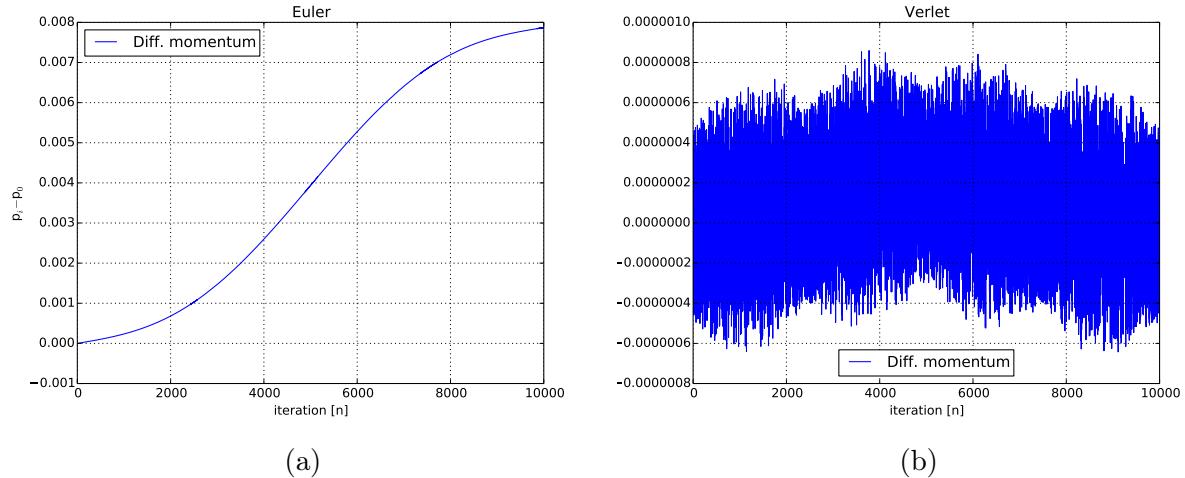
n	Forward-Euler	Verlet-Velocity	fastest	$\frac{\text{slowest}}{\text{fastest}}$
10	0.000136	0.000148	Euler	1.08823529412
100	0.000208	0.000179	Verlet	1.16201117318
1000	0.000392	0.000389	Verlet	1.00771208226
10000	0.002427	0.002426	Verlet	1.00041220115
100000	0.022931	0.022293	Verlet	1.02861884897
1000000	0.167022	0.175944	Euler	1.05341811258
10000000	1.58721	1.52666	Verlet	1.03966174525
100000000	15.1786	15.1176	Verlet	1.00403503202

### 4.1.2 Conserved quantities

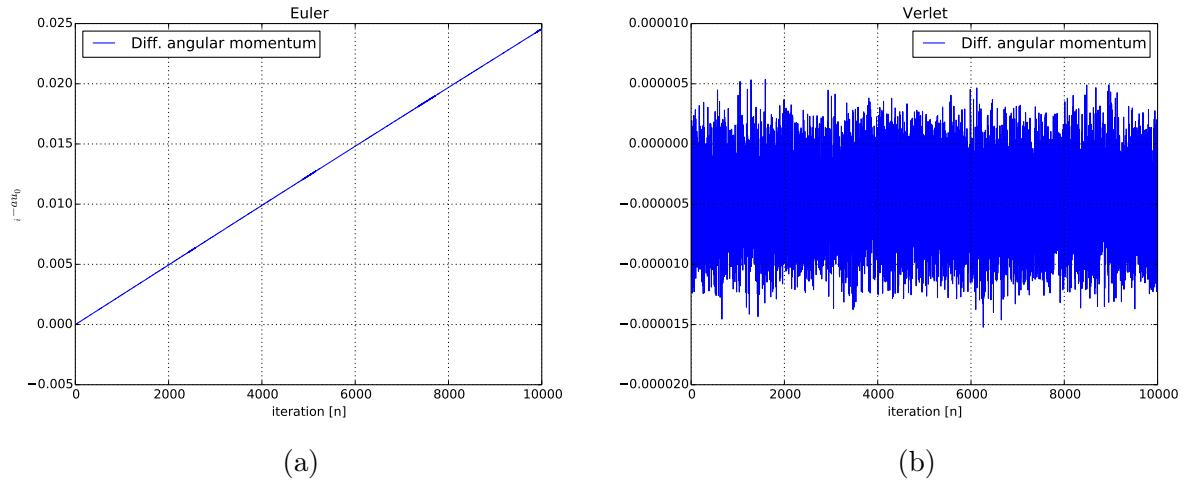
All the figures in this section was made from the data and python script in the directory [conserved-values](#).



Figur 2: Both are figures are graphs of the kinetic energy and potential energy and how it differ from they intial value. a) is the Forward Euler method and b) is the Verlet-Velocity method. As expected the energies are not conserved in the Forward Euler method, but is conserved in the Verlet-Velocity.



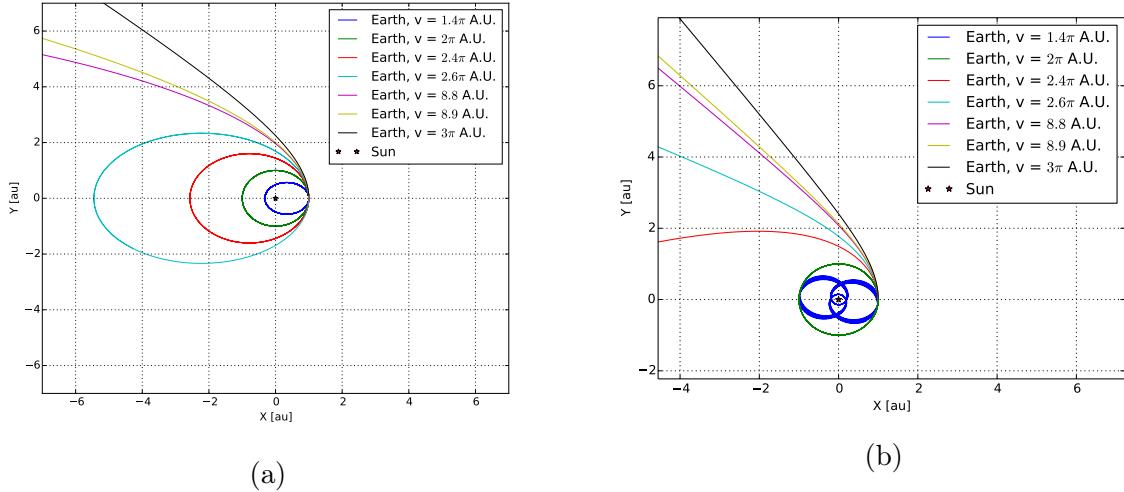
Figur 3: Both are figures are graphs of the momentum and how it differ from they intial value. a) is the Forward Euler method and b) is the Verlet-Velocity method. It should come as no suprise that momentum is not conserved for the Forward Euler method as the kinetic energy was not conserved, as the mass is a constant. Once again the Verlet-velocity method conserve the quantity.



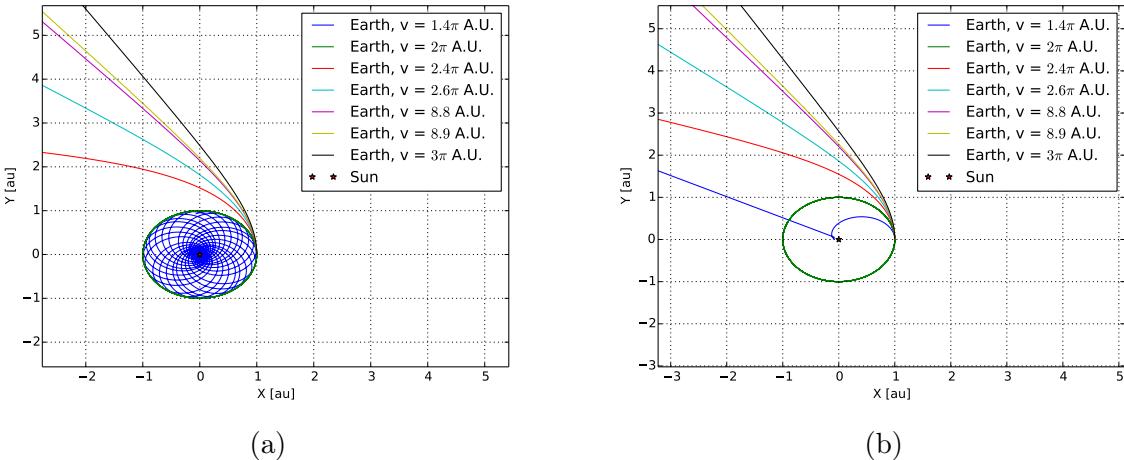
Figur 4: Both are figures are graphs of the angular momentum and how it differ from they intial value. a) is the Forward Euler method and b) is the Verlet-Velocity method. Forward Euler is once again not capable of conserving the value, but luckily for us the Verlet-Velocity method is.

#### 4.1.3 Escape velocity

The assignment was to find the escape velocity for the earth by trial and error. Fortunate for us that we know some math and can calculate it. See section 2.2.1 for this. But we started guessing randomly (winking Face emoji). Figure (5) a) shows these guesses. Where we can see that the velocities around 8.8 au/year shots out and never returns. The algorithms only runs for 15 years and will thereby not see the 8.8 au/year return to orbit even tho it should. The plots were made by the data and python scripts in the directory [escape-velocity](#).



Figur 5: a) Show how the orbits of earths with different initial velocity are. b) Shows the same as a) but this time the dependency of  $r$  in the denominator in equation (1) is set to 3.5.



Figur 6: a) Shows the same as figure (5) but this time the dependency of  $r$  in the denominator in equation (1) is set to 3.75. b) Shows the same as a) but this time the dependency of  $r$  in the denominator in equation (1) is set to 4.

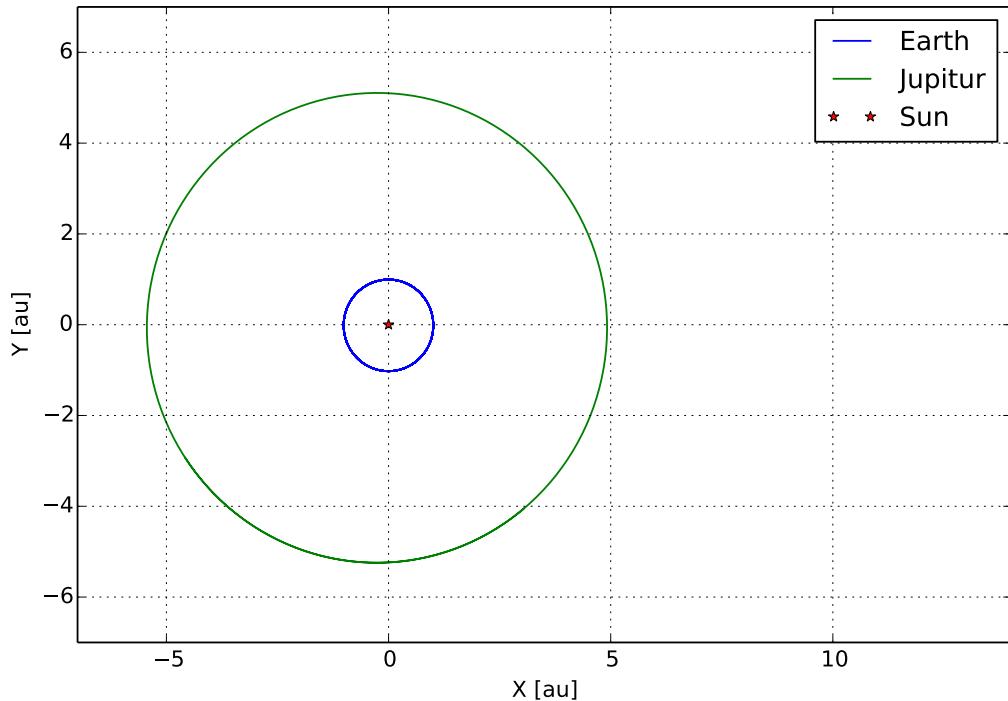
Personally I feel extremly lucky for living in a universe with a  $r$  dependency of 2, but then again i probably would not exist if the dependency was different. All the other dependencies are very unstable for even the slightest change in velocity from a perfect circle.

## 4.2 Three body system

All the figures in this section has been made from the directory `mass_jupiter`. In this directory there are different directories for the r dependency and a python script to generate graphs. The assignment said to try with mass multiplied with 10 and 1000. It was fun, so we did 10,100,1000 and 1100. Hope you enjoy the results.

### 4.2.1 Fixed mass for jupiter

For a simulation with jupiter original mass 100000 points over 15 years is sufficient to calculate the orbits of the planets. The figure (11) is quit smooth and you should not expect to get any major change in the result even with many more points per year.

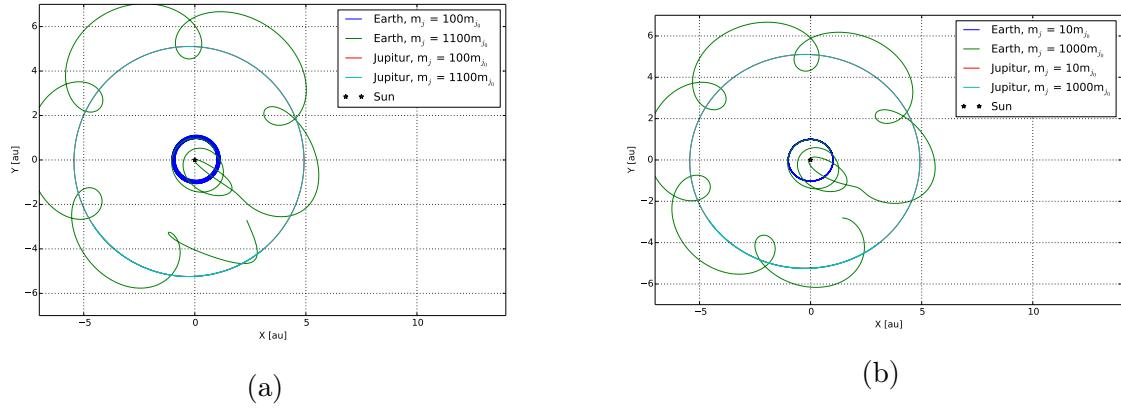


Figur 7: Plot of the three body system with Earth, Jupiter and the sun. The graph is discussed in the paragraph above.

### 4.2.2 Varying mass for jupiter

For varying mass it is mostly the same as for the original mass. For the mass multiplied with 10 and 100 the orbit is normal with some slight changes and you should not expect any major differences with higher amount of points per year. For the biggest masses the results vary much more on the number of steps per year. I found a couple of hundred thousand steps in total gave a good approximation. A bit higher step count makes earth

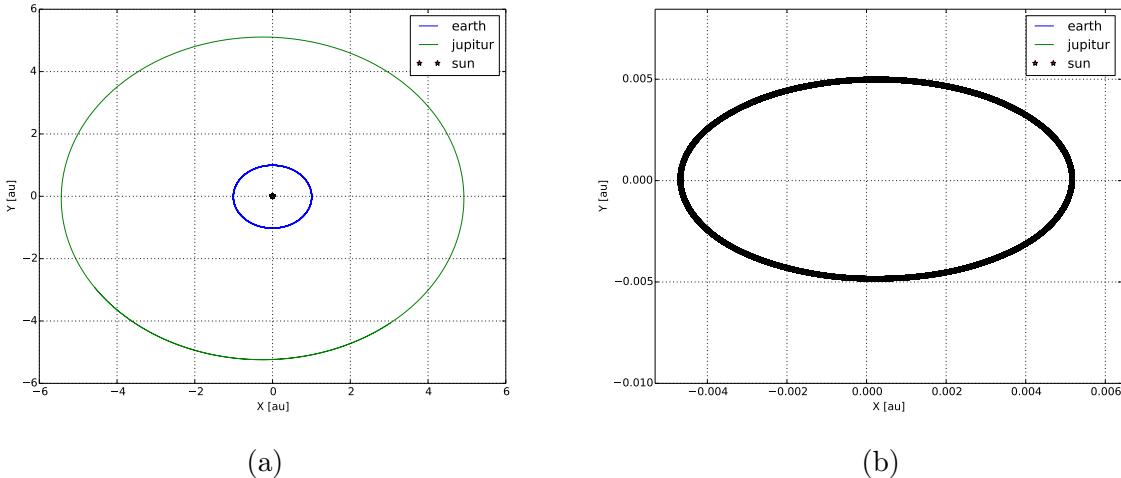
disappear and a way bigger step count makes the earth come back to orbit like shown below. This is basicly the best of both worlds. The graphs is few point, so easy for python to plot, and has a pretty good approximation.



Figur 8: Both plot has the sun as a fixed point. What the graphs represent is discussed above and legend should be pretty self explanatory.

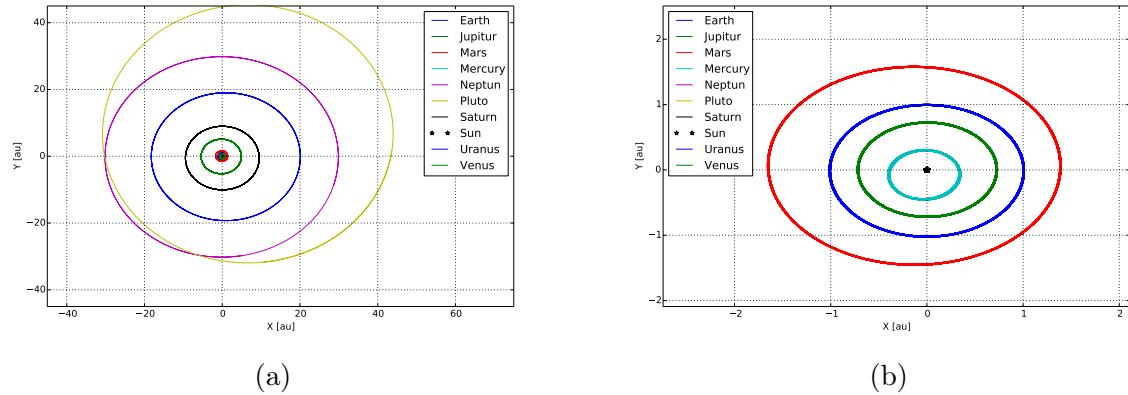
### 4.3 Solar system

#### 4.3.1 Three planets and all moving

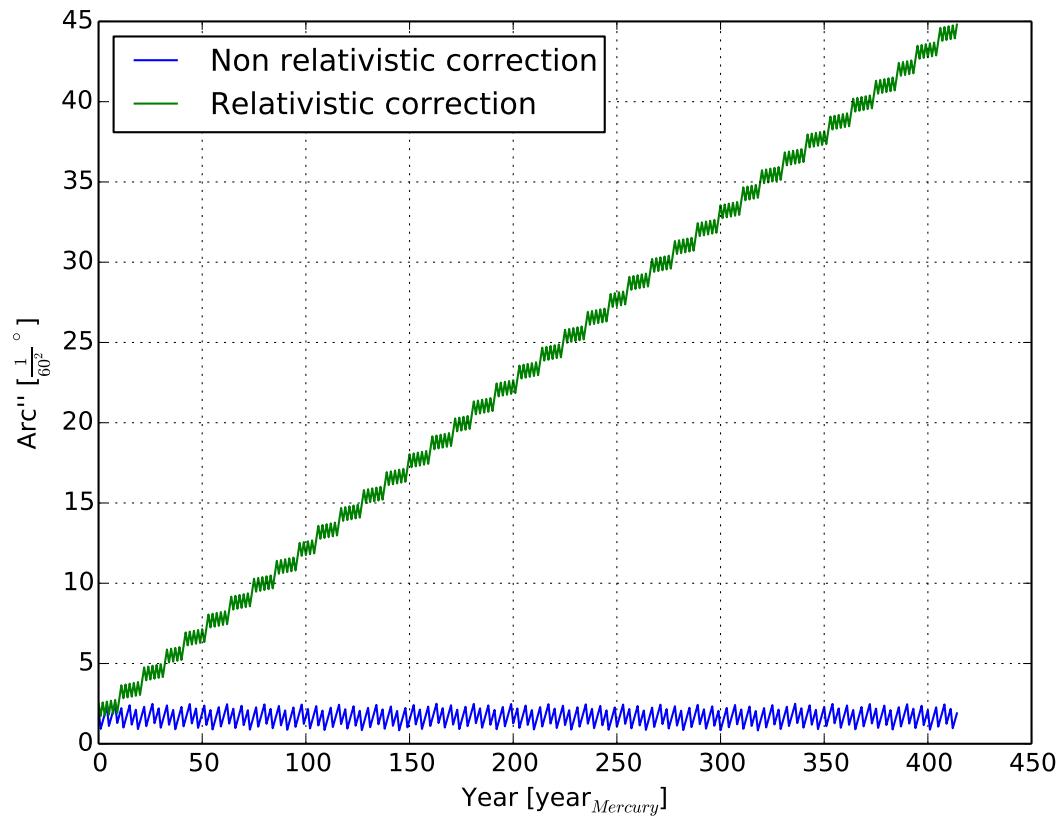


Figur 9:  $n = 10^5$  15 years

### 4.3.2 Solar system all moving

Figur 10:  $n = 10^6$  300 years

## 4.4 The perihelion precession of Mercury



Figur 11: EINSTEIN :D

## 5 Conclusion

write  
about this  
fig

## 6 Appendix