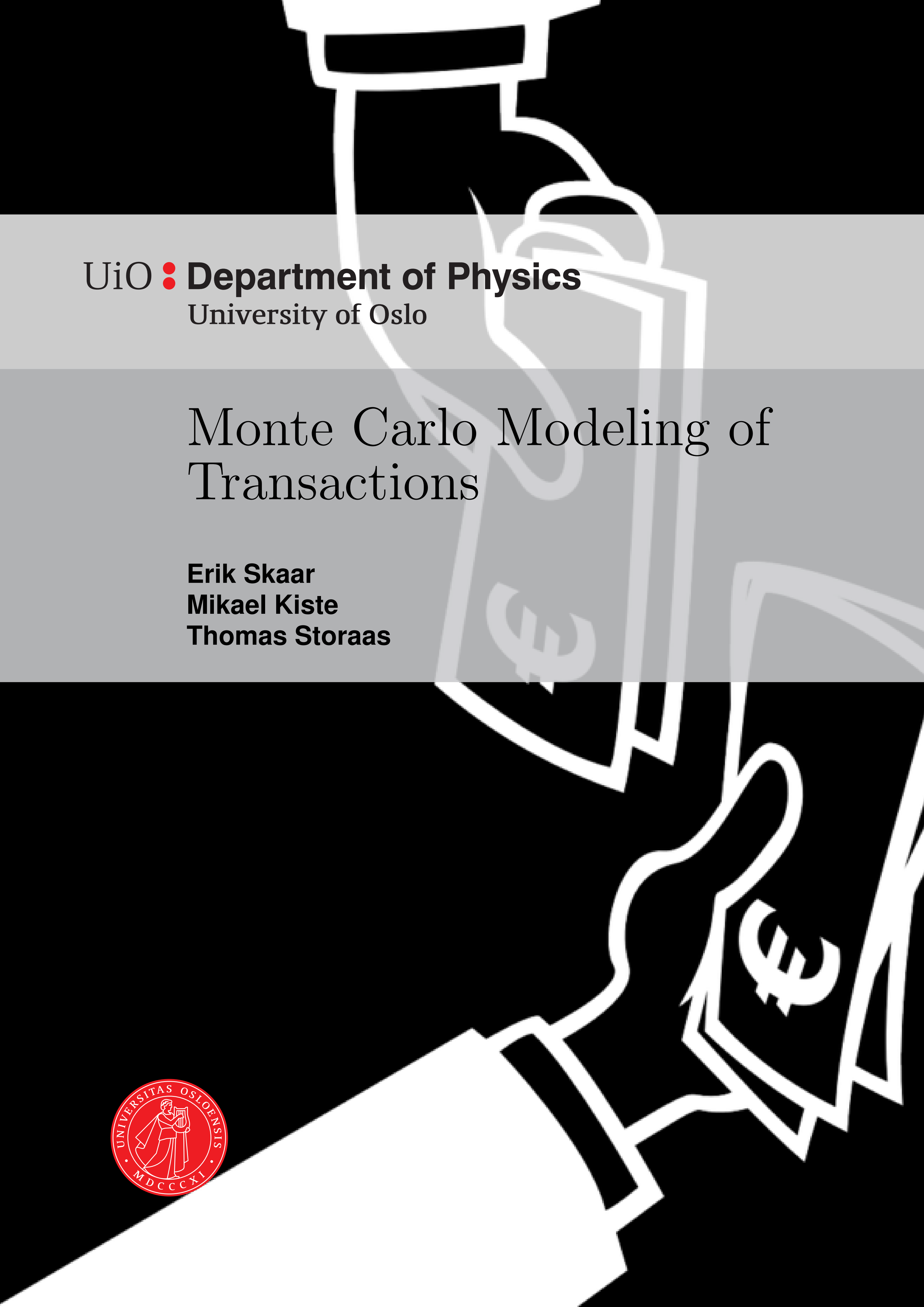

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Monte Carlo Modeling of Transactions

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Contents

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Abstract

I this project a model of a financial marked is build up from the basics, and follows two papers written on financial models. Three additional different levels of complexity is added to the system to expand the model. First the system only consists of agents that has to trade with each other. The resulting plots from this model is then compared to Pareto's law to verify the models basics. Then a savings term is introduced and several different savings factors are compared to make a comment on how saving influences the money distribution. Then a probability is added to the transaction stage. Here people with a similar amount of money are more willing to do a transaction. A scaling power factor is introduced and varied to influence to what extend the similarity in wealth influences the probability to trade. Here the results are commented and compared to Pareto's law. Finally a trust term is included in the probability term. This term increases the probability of making a transaction if the agents has previously done transactions. This term also has a scaling power factor which is varied to compare the influence of this term on the final wealth distribution. Lastly a modification is done to one of the scaling factors compared to the one in the papers followed in this project.

1 Introduction

Predictions of how the financial marked works has, in the modern age, become a large field of study[6]. With the knowledge of how stock exchanges work can a mathematical model be build up to simulate these exchanges in the hope that one can predict prices and trends in the future. These predictions are in turn used as a basis for investments and such. The accuracy and speed of these models are therefore of the utmost importance. When these models predicts the wrong outcome things can go very wrong as it did in 2009[2].

In this project a simulation is done on a simple closed economy model. The economy model consists of a group of financial agents with the ability to transfer money between themselves, using the Monte Carlo method. The simulation will be compared to the well established Pareto's law [4], and the analysis follows two papers. Patriarca with collaborates made one of the articles.[5] The second one was written by Goswami and Sen. [1]

In the first simulation all financial agents are made equal and there is a 100% probability of a transaction taking place when a trade is suggested. This it the simplest case and the model is then compared with a logarithmic plot to verify the trend. Then complexity is added by introducing saving. The model is further refined by adding a willingness to trade with agents with a comparative wealth to themselves. Lastly a favorable bias towards agents in which a trade has happened before is added. The whole job will be parallelized to speed up the simulations.

2 Physical theory

2.1 Basics

From empirical studies Pareto has been shown that the higher end of a distribution of money follows the distribution trend[4]:

$$\omega_m \propto m{-}1 - \alpha \tag{1}$$

Here $\alpha \in [1, 2]$. This trend will be analytically studied with a system of micro dynamical relations between a group of financial agents, and the resulting money distribution between them.

The system consists of N financial agents. They transfer money between them in pairs of (i, j) and they all start with a fixed amount of money m_0. The amount of money is arbitrary. So in this project the amount of money is given in amount of "wealth". Here one wealth is m_0 and the symbol used is \$. A result from this is that $\langle m \rangle = 1\$$

At a time step a pair of agents, (i, j), are chosen at random and let them transact money from j to i. The pair is chosen at random so for a specific pair m_i, m_j both (m_i, m_j) and (m_j, m_i) are possible. Money is conserved during the transaction which means that the money Δm transfered from j to i has to be equal. This means that the total expression for the transaction can be writen as

$$m_i + m_j = m'_i + m'_j \tag{2}$$

This transfer is done by a Randomized Number Generator and in this model none of the agents are left with a debt. To remove the debt from the picture, only a percentage of the money is transfered in stead of a fixed amount.

$$m'_j = \epsilon(m_i + m_j) \qquad \epsilon \in [0,1] \tag{3}$$

And similar for m'_i.

$$m'_j = (1-\epsilon)(m_i + m_j) \qquad \epsilon \in [0,1]$$

Due to the limitations of ϵ no agent gets debt, but it is likley that one agent is left with zero $m \geq 0$. This system now consists of a conserved amount of money with a transfer criteria. It can be shown that this system, given enough time, will relax towards a Gibbs distribution.[3]

$$\omega_m = \beta e^{-\beta m} \tag{4}$$

With

$$\beta = \frac{1}{\langle m \rangle} = \left[\sum_i \frac{m_i}{N}\right]^{-1} \tag{5}$$

This model uses $N = 500$. The 500 agents has to do enough transaction to achieve the above outlined distribution. The article suggests that 10^7 transactions is enough. [5] To find the final equilibrium distribution ω_m 10^4 runs were done and ω_m was taken as the average of all the recorded ω_m. The distribution of agents is exponential, so plotting the logarithm of the trend should yield a straight line.

$$\ln(\omega_m) = \ln\left(\beta e^{-\beta m}\right)$$
$$\ln(\omega_m) = \ln(\beta) - \beta m$$

2.2 Savings

To make the model more realistic complexity is added though a savings term. Now every agent will save a fraction of their wealth δ before the transaction takes place. The addition of this term will change the distribution of agents so it no longer makes a gibbs distribution.

$$m'_i = \lambda m_i + \epsilon(1-\lambda)(m_i + m_j) \tag{6}$$

and similar for m'_j.

$$m'_j = \lambda m_j + (1-\epsilon)(1-\lambda)(m_i + m_j) \tag{7}$$

The two terms in the expressions can be simplified by gathering both the savings term and the trading term in a variable $\delta m = (1-\lambda)(\epsilon m_j - (1-\epsilon)m_i)$

$$m'_i = \lambda m_i + \delta m$$

$$m'_j = \lambda m_j + \delta m$$

2.3 Willingness to trade

The model has up until now only varied how much is saved and not touched on if they want to trade. From observations it is know that the willingness to trade impacts if a trade takes place or not. Further refinement of the model includes two terms to include in the probability of trading. For this the model follows the work of.[1] The model now takes into account that there is a probability

$$p_{ij} \propto |m_i - m_j|^{-\alpha} \tag{8}$$

for an interaction to take place between two agents with respectively m_i and m_j wealth. The variable α is larger than zero. If α is zero one get the same model as in section 2.2. Several runs will be done with varying α to analyze the effect of the probability term on the expression.

2.3.1 Normalization

The equation 8 is not normalized. If one chooses a value less than one the expression will blow up and the probability becomes larger than one.

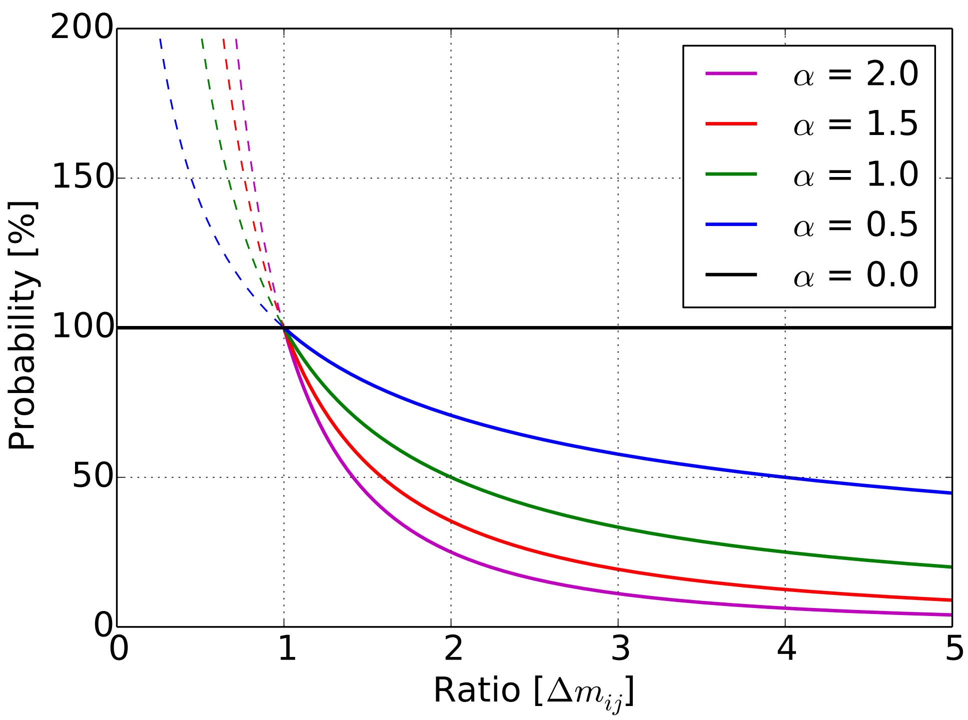


Figure 1: Figure shows the probability as a function of the difference in wealth between the two agents. The continuous line is the probability chosen in this project while the dashed line shows the trends without the criteria below one.

To prevent unrealistic probabilities above one a normalization criteria is introduced. The suggested normalization criteria on piazza is to normalize it with the expectation value of the agents $\langle m \rangle$.[1] From 2.1 the expectation value $\langle m \rangle = 1\$$. So dividing by this will not change this expression. The other change is to add a factor of two to the probability, $2|m_i - m_j|^{\alpha}/\langle m \rangle$ because the normalization gives an overall decrease in all the probabilities. This is not included because of the

[1] Piazza: Morten's suggestion

way the expectation value is defined here gives no change in the probability. The last piece in the normalization is to remove the probabilities higher than one, although it is not important for this part it will be important in the next section. All probabilities higher than one is synthetically sat to one. The physical representation is that if two agents has a difference in wealth that is less than one, a trade is guarantied.

$$p_{ij} = \begin{cases} |m_i - m_j|^{-\alpha} & |m_i - m_j| > 1 \\ 1 & |m_i - m_j| < 1 \end{cases} \tag{9}$$

2.4 Familiarity and trust

From observations it is known that it is more likely that two agents that know and trust each other trades. In this model this is introduced to the system by adding to the probability a increased probability between agents that has traded earlier.

$$p_{ij} \propto |m_i - m_j|^{-\alpha}(c_{ij} + 1)^{\gamma} \tag{10}$$

c_{ij} represents the number of previous transactions between the two agents. If $c_{ij} = 0$ then the $+1$ factor makes sure that the trust term does not influence the probability. Several runs will be done with varying γ to analyze the effect of the modified probability term on the expression.

2.4.1 Normalization

As for the term in2.3.1, the expression above has to be normalized. The normalization criteria suggested in piazza is to divide by the highest amount of trade two agents has traded.[2]

$$p_{ij} = |m_i - m_j|^{-\alpha}\left(\frac{c_{ij} + 1}{c_{\max} + 1}\right)^{\gamma} \tag{11}$$

By doing this one would make sure that no value supersedes one.

An alternative normalization criteria is to rather than taking the total of all the transactions one could normalize by the average of the two agents maximum value.

$$p_{ij} = |m_i - m_j|^{-\alpha}\left(\frac{c_{ij} + 1}{\langle c_{ij,\max}\rangle + 1}\right)^{\gamma} \tag{12}$$

The physical representation is that when an agent makes a trade it is with respect to what he has previously done, not what what the maximum of all the agents. This will increase the overall probability of a transaction taking place.

$$p_{ij} = \begin{cases} |m_i - m_j|^{-\alpha}\left(\frac{c_{ij}+1}{\langle c_{ij,\max}\rangle+1}\right)^{\gamma} & |m_i - m_j| > 1 \\ \left(\frac{c_{ij}+1}{\langle c_{ij,\max}\rangle+1}\right)^{\gamma} & |m_i - m_j| < 1 \end{cases} \tag{13}$$

[2]Piazza: Morten's suggestion

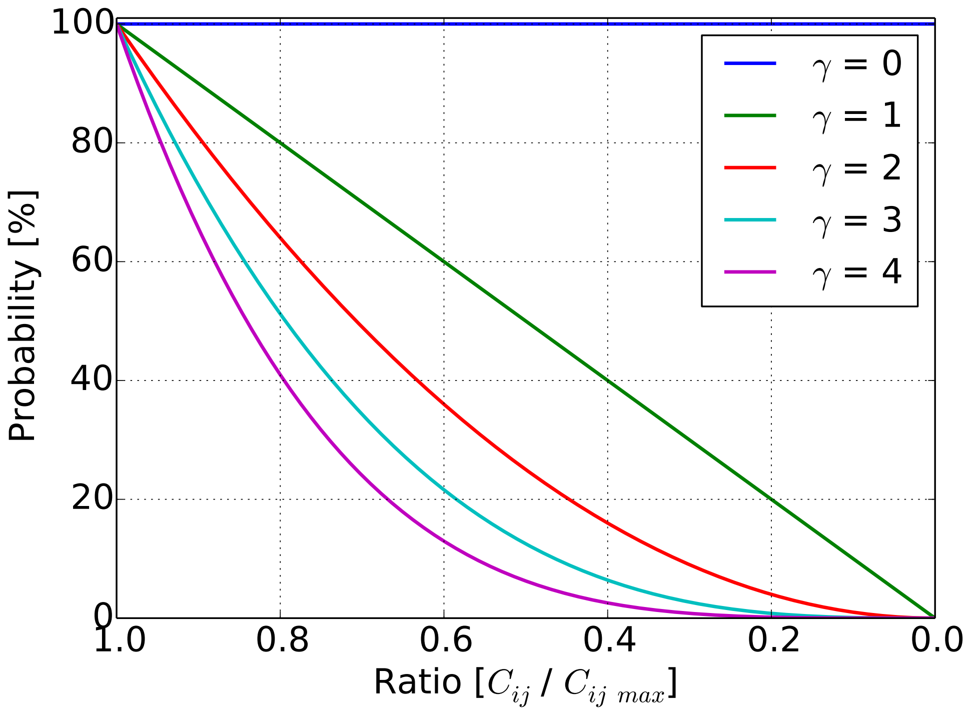


Figure 2: Figure shows the probability as a function of their trade ratio.

3 Computational theory

This project consists of four different cases with a different degree of complexity. For the basis case and the savings case it is enough to use the Monte carlo method. The Monte Carlo method is a method that utilizes the fact that a large number of experiments converges towards the expectation value. When the probability of doing a transaction is added the system utilizes the Metropolis Algorithm is implemented. The system has to do a random choice of acceptance or denial of the trade. This random choice is done with the Random Number Generator. A random number i created between $(0-1)$ If the number is less than the probability term, accept the new state. If the number is greater than the energy change then discard the change. The procedure is outlined in four steps.

- Chose two agents at random.
- Find the probability of acceptance for this new case
- A RNG is used to chose a number between $(0-1)$. Now if the probability term is less then the RNG number, reject the transaction and discard the trade. If not, accept the trade.
- Choose a new random number for how much is traded between the two(ϵ).
- Update the system with the transfered money.

4 Implementation

The algorithm was implemented as discussed in section 3 in the program called main.cpp. main.cpp has some functions for different versions of the stock market. The difference between the versions are how likely it is that two random agents will trade. For more insight see section 2.[3].

Many different simulations is needed for this report. This is why a parallelized version has been made. MPI is used since it is easy to implement. Most of the code in the parallelized version is the same as for the non-parallelized. The parallelized version should then be X times faster then the normal version with X cores. Each thread and runs the simulations for some given values for α, λ and γ. The values is determined by the rank of the thread and will therefor be unique to other thread's values.[4]

Table 1: The agents were given 10^7 opportunities to do a transaction and the system was simulated 160 times. The test ran on a macbook pro 15. It has a quad core CPU. Expected difference is 4. The difference was higher, then expected. This is due to the Hyper-Threading technology in this CPU.

Version	Normal	MPI	Expected difference	Actual difference
A	35.147 s	7.361 s	4.000	4.774
D	98.112 s	17.202 s	4.000	5.764
E	226.522 s	45.608 s	4.000	4.966

Table 2: For scaling we expect a time increased proportional to the size increase. The table shows how we expect the time to develop and how it actually it develops. There is a minor difference from expected and calculated and that comes from the fact that the program does more then just the algorithm and the fact that the algorithm has not been perfectly implemented.

Version	# Simulations	Expected time	Actual time	T_i / T_{10}
A	10	x	2.252 s	1.000
A	160	16x	35.147 s	15.60
D	10	x	6.229 s	1.000
D	160	16x	98.112 s	15.75
E	10	x	12.876 s	1.000
E	160	16x	226.522 s	17.59

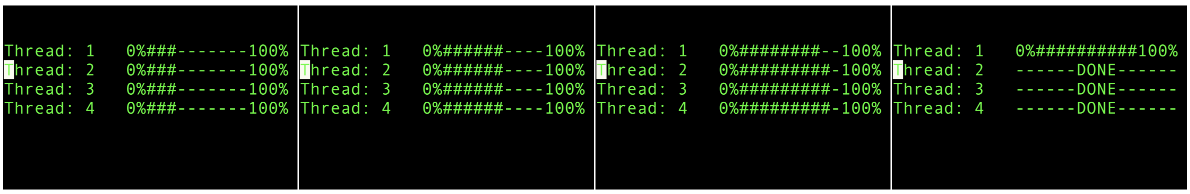


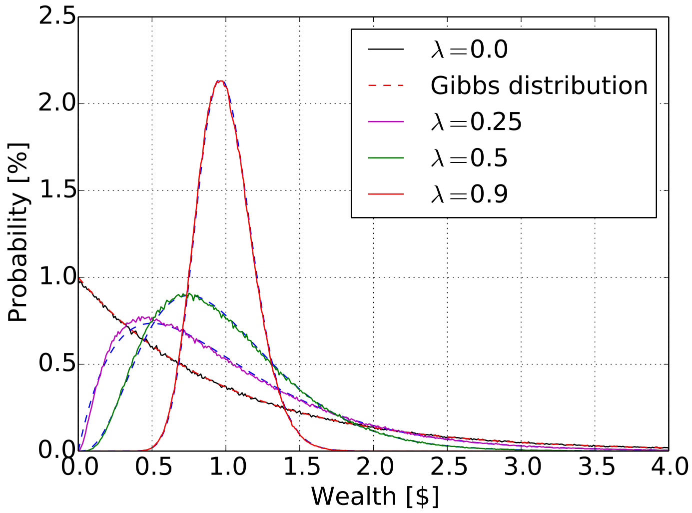
Figure 3: A progress bar was made for the program. It shows the progress for each individual thread. The progress bar works for 1 to 4 threads. More then 4 threads might look funny. By the way, we are A students in the newly announced PRO101 subject.

[5]

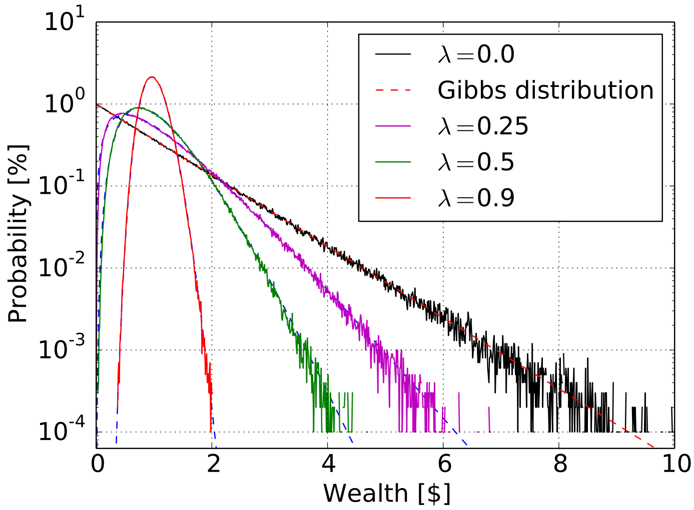
[3] All of the programs discussed in this section can be found at github
[4] Intel Hyper-Threading Technology
[5] PRO101: Procrastination for students

5 Result & Discussion

In this section we present and comment on our results from simulations of the three different models. Wealth distribution diagrams show the distribution of agents with respect to how much money they have. In most wealth distribution diagrams we also show the correlation of the data to various analytically known distributions, some of which are familiar and well studied, like Gibbs (Boltzmann) and Pareto. All simulations have a uniform distribution of wealth among the agents, where $m_0 = 1$, as the initial state. For the results to be representative it is important to have enough transactions to allow for the system to settle at some equilibrium. Such a macrostate is reached when the systems overall distribution does not change even if we perform many more transactions. The results are all controlled for this empirically by studying the evolution of the distribution through a range of montecarlo cycles (representing the time dependence of the system).



(a)



(b)

Figure 4: a) shows how the agents are distributed with respect to their wealth. b) log plot of the same distributions

Figure 4a shows how the distribution changes depending on the savings parameter λ (which is equal for all agents). When the agents always allow for their entire wealth to be part of a transaction we get a gibbs distribution. We see that a few agents are very wealthy compared to the majority who are rather poor. When the agents are allowed to save a certain amount of their wealth (i.e. exclude a fraction of their wealth from taking part of a transaction) there are both major and minor changes to the wealth distribution. The most striking feature being the reduction in number of very poor agents, even at low savings. The chance of loosing even more money if you are alrady poor is diminished if you save some of your money before agreeing to a transaction. At the other end of the scale we see that only a large λ will influence the distribution of very wealthy agents. As the saving parameter is increased further a peak is introduced to the distribution. As $\lambda \to 1$ the position of this peak approaches one (the average wealth). The deviation in the distribution from this peak is also narrowed. For the extreme case where $\lambda = 1$ no money is allowed to be part of transactions so the system remains in the initial state where every agent is equally wealthy. A fitting of these curves as given in the paper by Patriarca et.al. [5] was found to be reproducable in our results. This function is given by the equation

$$P_n(x) = a_n x^{n-1} \exp\{-nx\} \tag{14}$$

Where

$$n(\lambda) = 1 + \frac{3\lambda}{1-\lambda}, \qquad a_n = \frac{n^n}{\Gamma(n)}$$

this can be found as the dashed blue line in plot 4a and correlates strongly with the associated results from our computational simulation (λ being the same).

In the logarithmic plot for the same distributions, shown in figure 4b, a more detailed impression of the outliers in the distribution is obtained. As shown in section 2.1 we see that the gibbs distribution is linear; and our results for the no savings case follows this trend. The $\lambda \neq 0$ distributions however are not linear, again displaying a major distinction between them. At the very high ends however they all show a linear trend (admittedly with differing slopes that still allow for wildly varying probabilities of achieving these extreme values).

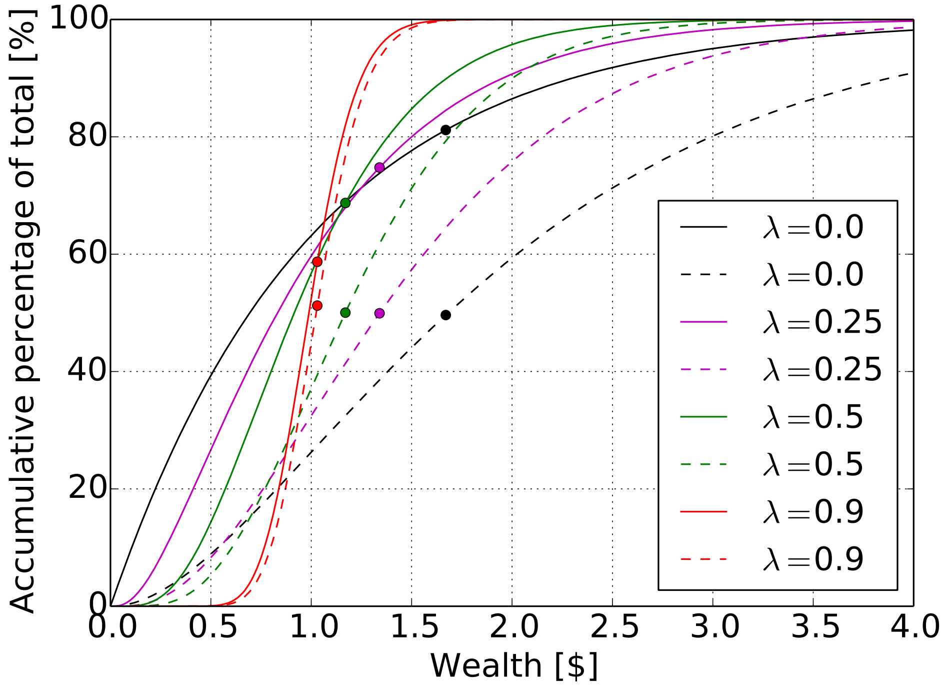


Figure 5: Accumulative percentage of total for both wealth and population. Dashed lines is for wealth, solid lines is population and dots corresponds to 50 % of total wealth.

Figure 5 gives a more thorough picture of the wealth distribution compared to the population distribution. The y-axis gives the percentage of the total of either the wealth or the number of agents up until that point (from left to right). The plots therefore converge towards 100% when nearing the end. Having both population and wealth in the same plot allows for a comparison of how these changes with respect to each other as you climb through the economic classes. The gap between each pair represents the disjunction between population and wealth distribution, meaning that the population up until that point does not own a fair share of the wealth. The closer the lines are to one another the more even the distribution. We see again see that this is the case for a higher value of λ, and vice versa. The slope (derivative) of the curves indicate how quickly new members are added to the total. In figure 7 is a similar plot but with respect to α instead of λ.

For the following wealth distributions we have $\alpha \neq 0$ so that there is a larger probability for people with similar amounts of money to perform transactions.

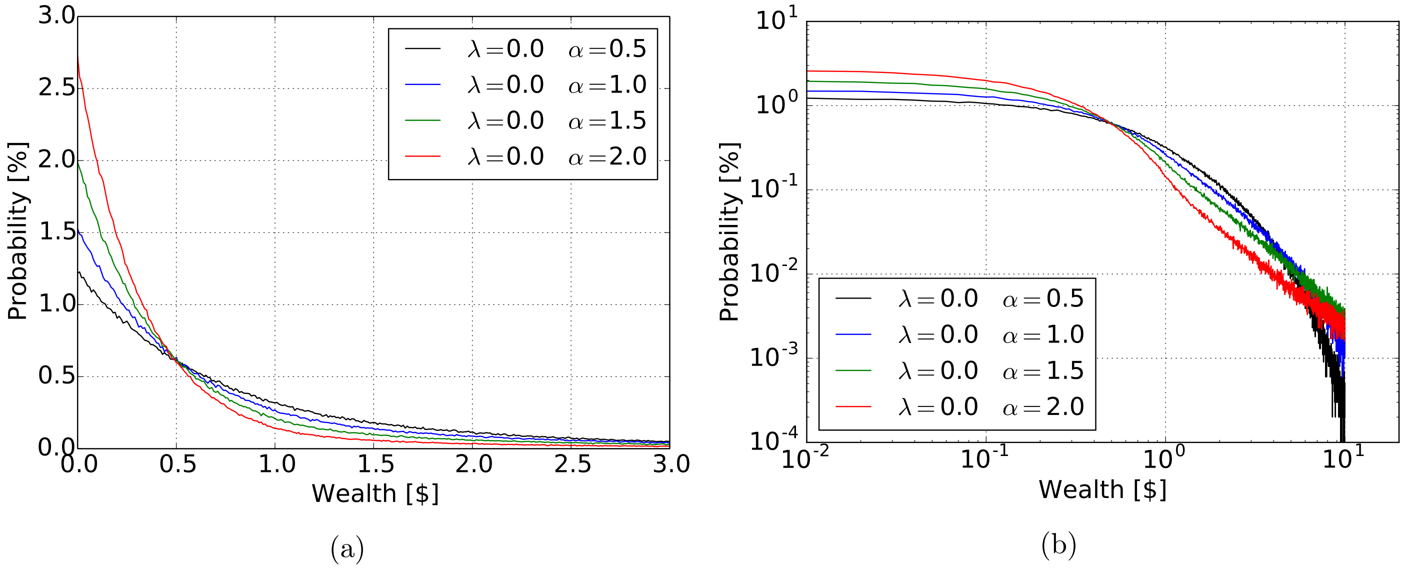


Figure 6: plots a) and b) shows the same four distributions where there is no saving and four different values for alpha. The latter is logarithmic while the former is not

The trends of the graphs in figure 6 clearly indicates that the new model leads to a greater difference in the low end vs the high end of the distribution. The number of poor people increases with α while the number of rich people decreases. Again, it is important to note that the plots represent histograms of agents and their wealth. Since there now are more poor people the sum of their wealth represents a larger amount of the total wealth than if there was a smaller population of poor people. This means, as is confirmed by summation of the data, that the money is still conserved even though there are more poor people and fewer rich people. The plots seems to intersect at around 0.5, setting the boundary at which the wealth is separated. By studying the logarithmic plot we see that distribution lines actually intersect a second time. This means that the extremely rich actually get even richer if the population has a strong bias towards transacting with agents in their financial class. [6]

The linear nature of the tails of the distributions in the logarithmic plot, particularly noticeable for higher values of α, is a strong indicator of a power law and its slope hints at the value of the exponent in this power law. Thus an increase in alpha means that the tail end of the distribution follows a power law with a smaller (less negative i.e. smaller absolute value) exponent.

[6]Due to computational constraints we exclude the distribution of agents with $m_j \geq 10$ which for this particular plot, unfortunately, would be of some interest. In any case it is still possible to get a decent idea of the distribution trends through the plot

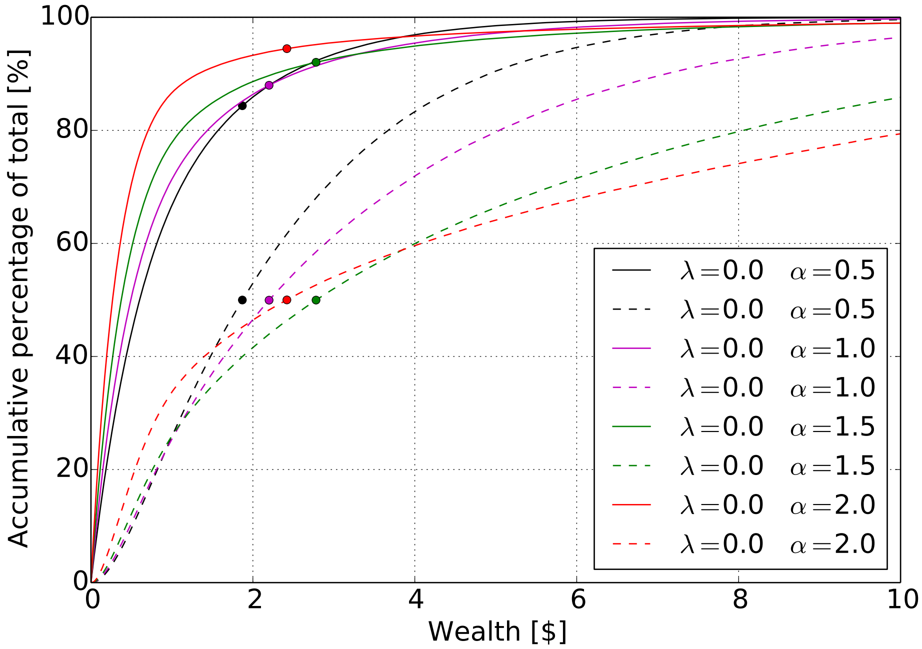
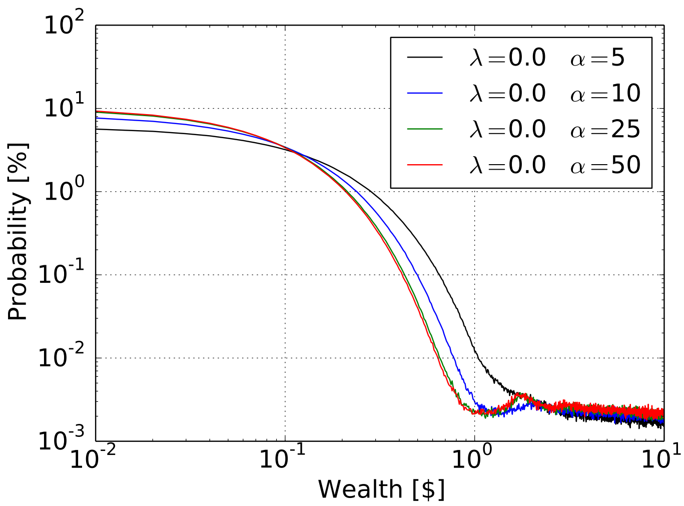
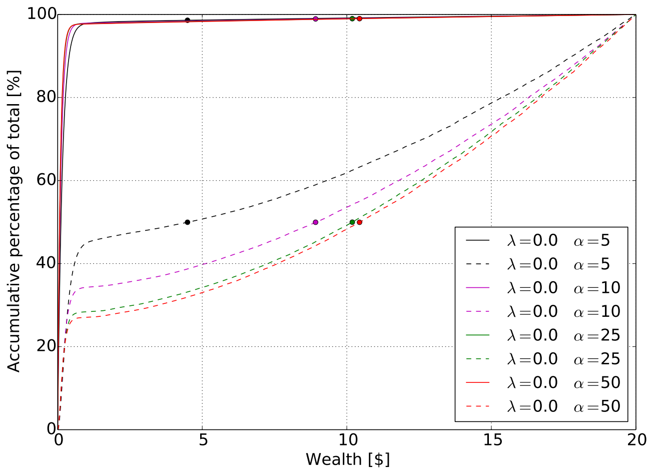


Figure 7: Accumulative percentage of total for both wealth and population. Dashed lines is for wealth, solid lines is population and dots corresponds to 50 % of total wealth.



(a) logarithmic plot wealth distribution



(b) Accumulative percentage of total for both wealth and population. Dashed line is for wealth while the others are for population

Figure 8: Plots showing trends when $\alpha >> 1$

In figure 8a we have the distributions resulting from simulations with even higher values of α. The number of people with almost no money seems to approach 100%. In general the curves seemingly congregate at some function limit.

In the plots below we combine the previous model with the new one, giving plots for different values of α but also $\lambda \neq 0$.

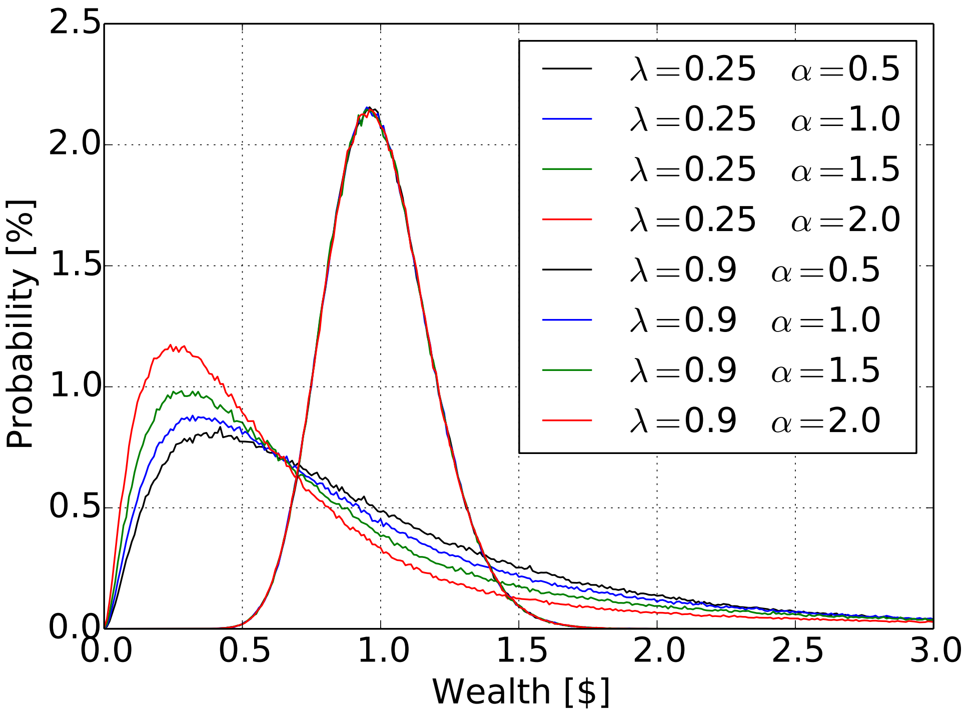


Figure 9

It seems as though the effect of agents saving money before committing to transactions has a greater impact on the distribution and dominates over the effect that preferences for same-wealth transactions has. If agents only trade using a tenth of their worth it is practically irrelevant whether they care about trading with other agents that has a similar amount of money.

In the third model we introduce a second factor in the expression that gives the probability of having two agents follow through on an interaction. There are a total of three degrees of freedom for our system. The three parameters λ, α and γ represent the weighting that's given to each of the corresponding models.

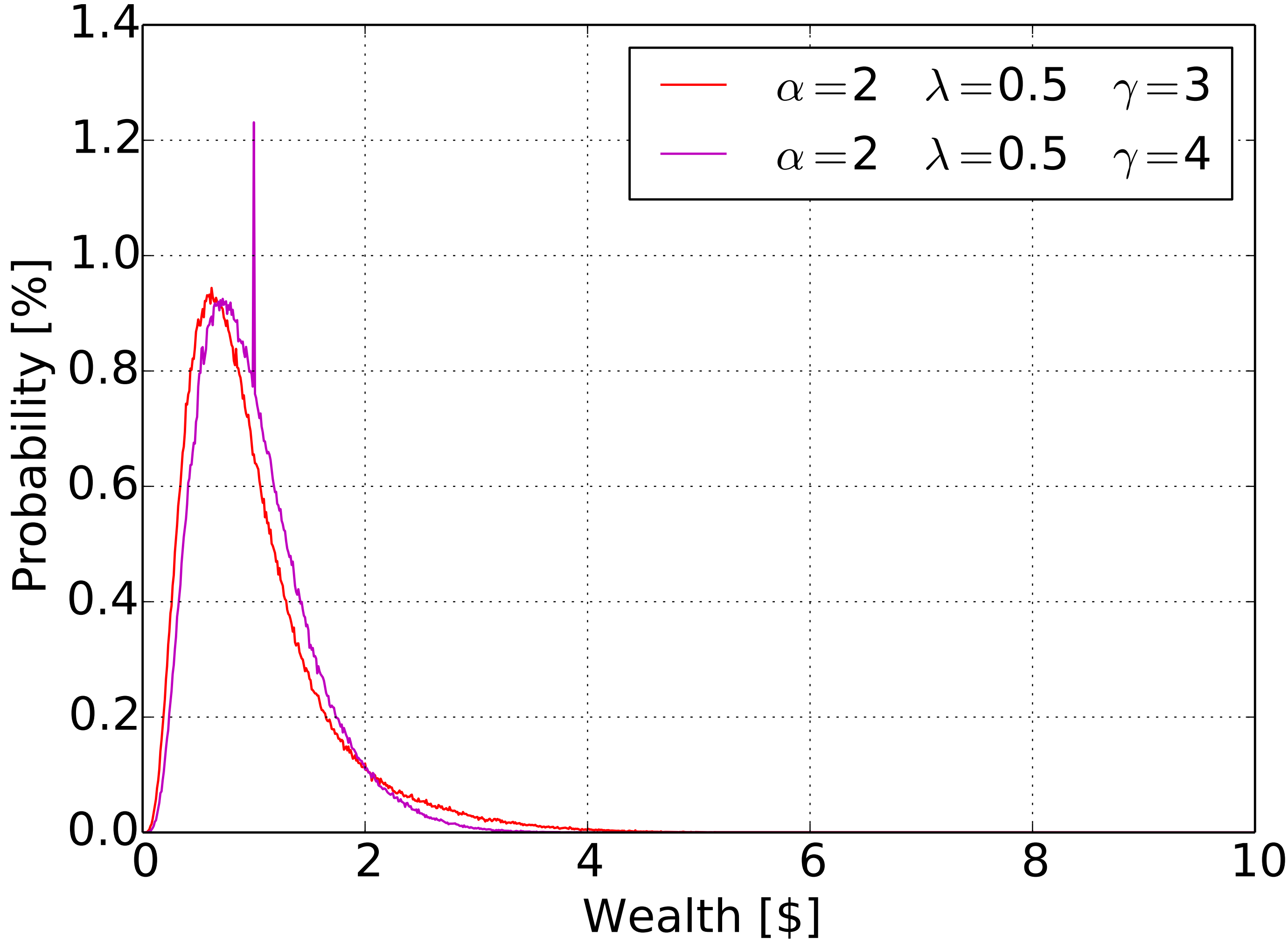
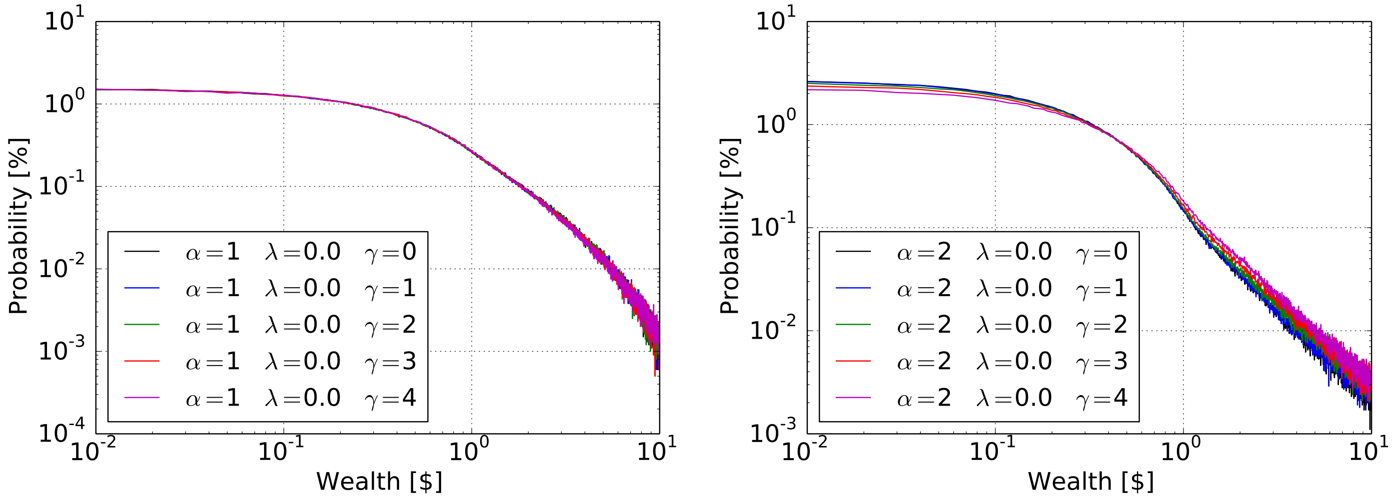


Figure 10: wealth distribution when using the number of transactions between the two agents with the highest number of transactions for the normalization

The specific implementation of the required normalization can be done in many different ways, as discussed in the normalization sections. When two agents interact and we calculate the probability for a transaction to take place we can divide the number of times they have previously traded with the number of times the most well known agents have traded (i.e. the couple who has the highest number of transactions among all agents). Figure 10 illustrates how this method leaves some agents in their initial state, never trading at all. This is because the agents who are not quick enough to trade in the beginning end up with a lower chance of finding new partners to transact with; this allows other agents to trade even more which further exacerbates the situation of the lonely agents ending in a vicious cycle of ever decreasing chances of participating in the socio-economic platform of interconnected agent-relations.

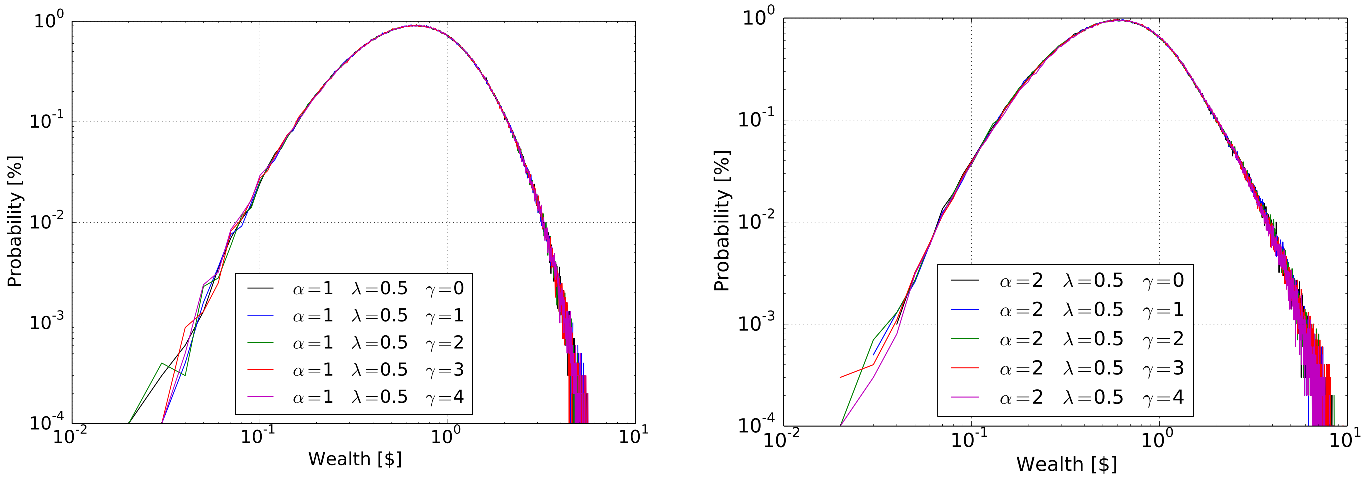
We therefore choose to normalize such that each agent only consider their own transactions when determining the probability of a new transaction. All of the results are from simulations using this method.



(a) (b)

Figure 11: logarithmic plots with no saving and four different values for γ. α is doubled for b) compared to a)

The shape is recognizable from figure 6 with the two values of α. As α is increased we see that the effect of changing γ is stronger, and thus has a stronger influence on the distribution.



(a) (b)

Figure 12: plots a) and b) shows the same four distributions where there is no saving and four different values for alpha.

When the agents are allowed to save money we again see that this effect dominates. From a technical point of view this makes sense as the savings is performed before calculating the probabilities in the simulations. λ therefore becomes a limiting factor for the other two variables.

6 Conclusion

Simulations of a system consisting of transacting agents was performed and analysed. Three different models allowed for different behaviour of the agents which greatly influenced the final wealth distribution of the population. In a first model we allow the agents to save a percentage of their money, represented by the variable λ, before comitting to a transaction. A greater value of λ resulted in a vastly more fair distribution in the sense that the spread in the distribution was low and most agents had roughly the same amount of money. In the two following models we introduce the two probability factors α and γ that influence the chance of two agents performing a transaction. In pracsis this was implemented by first picking a specific pair and then calculating the likelyhood of an interaction between these two. The pair would then skip the interaction if some randomly generated number between 0 and 1 was higher than this probability and vice versa. α represented the degree to which agents took into account the difference in their net worth. When α increased there was, on average, a larger discrepancy between the rich versus the poor. The number of agents whose net worth was lower than roughly half of the average was increased while the number of agents above this decreased. The γ variable represented the tendency for agents to transact with partners they had already transacted with before and this tendency increased with the variable. In the third model γ was implemented alongside α contributing to a secondary factor for the probability calculation. A larger α accentuated the effects of the γ variable and had otherwise only a small effect to the end wealth distribution. Double logarithmic plots displayed a linearity of the tail ends of the distribution, a strong indication of power laws. Qualitatively the exponent on the power law was negative and its absolute value decreased when α was increased. When we varied γ we again witnessed a linearity on the double logarithmic plots and the same trend was found. An increase in γ meant a decrease in the absolute value of the power law exponent.

For future work one should try to make the trade ratio time dependent. The denominator in the trade ratio becomes larger over time, but a more realistic model would consider a more dynamical view of trade relations where there are diminishing returns and/or a time dependancy on the relations. The effect of taxing or universal basic income would also be interesting to see, but this was too much work for this project.

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