Exercise 2

1. b) This kind of opperation is called filtering. It computes the belief state given all evidence to date.

c) This is a prediction. It computes the future state given all evidence to date. When t increases the state converges to [0.6, 0.4].

d) This is called smoothing. It copmutes the posterior distribution over a past state, given all evidence up to the present.

e) This is called "Most likely explanation". It computes the most likely sequence of states to have generated the observations.

2b) $P(A_t \mid e_{1:t})$, for t 1, 2, 3, 4

$$P(A_1) = \sum_{a_0} P(A_1 \mid a_0) \cdot P(a_0)$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.7 + \langle 0.3, 0.7 \rangle \cdot 0.3$$
$$= \langle 0.65, 0.35 \rangle$$

$$P(A_1 \mid e_{T_1}, e_{F_1}) = \alpha \cdot P(e_{T_1}, e_{F_1} \mid A_1) \cdot P(A_1)$$

$$= \alpha \cdot \langle 0.65, 0.35 \rangle \cdot \langle 0.7 \cdot 0.3, 0.2 \cdot 0.1 \rangle$$

$$= \alpha \cdot \langle 0.1365, 0.007 \rangle \approx \underline{\underline{\langle 0.95, 0.05 \rangle}}$$

$$P(A_2) = \sum_{a_1} P(A_2 \mid a_1) \cdot P(a_1)$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.95 + \langle 0.3, 0.7 \rangle \cdot 0.05$$
$$= \langle 0.775, 0.225 \rangle$$

$$P(A_2 \mid e_{T2}, e_{F2}) = \alpha P(e_{T2}, e_{F2} \mid A_2) P(A_2)$$

$$= \alpha \cdot \langle 0.775, 0.225 \rangle \cdot \langle 0.3 \cdot 0.3, 0.8 \cdot 0.1 \rangle$$

$$= \alpha \cdot \langle 0.06975, 0.018 \rangle \approx \underline{\langle 0.79, 0.21 \rangle}$$

$$P(A_3) = \sum_{a_2} P(A_3 \mid a_2) \cdot P(a_2)$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.79 + \langle 0.3, 0.7 \rangle \cdot 0.21$$
$$= \langle 0.695, 0.305 \rangle$$

$$= P(A_3 \mid e_{T3}, e_{F3}) = \alpha P(e_{T3}, e_{F3} \mid A_3) P(A_3)$$

$$= \alpha \cdot (0.695, 0.305) \cdot \langle 0.3 \cdot 0.7, 0.8 \cdot 0.9 \rangle$$

$$\alpha \cdot \langle 0.14595, 0.2196 \rangle \approx \underline{\langle 0.4, 0.6 \rangle}$$

$$P(A_4) = \sum_{a_3} P(A_4 \mid a_3) \cdot P(a_3)$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.4 + \langle 0.3, 0.7 \rangle \cdot 0.6$$
$$= \langle 0.5, 0.5 \rangle$$

$$P(A_4 \mid e_{T4}, e_{F4}) = \alpha P(e_{T4}, e_{F4} \mid A_4) P(A_4)$$

$$= \alpha \cdot \langle 0.5, 0.5 \rangle \cdot \langle 0.7 \cdot 0.7,\ 0.2 \cdot 0.9 \rangle$$

$$= \alpha \cdot \langle 0.245, 0.09 \rangle \approx \underline{\underline{\langle 0.73, 0.27 \rangle}}$$

2 c) $P(X_t \mid e_{1:4})$, for $t = 5, 6, 7, 8$

$$P(A_5 \mid \overbrace{e_{T1:4}, e_{F1:4}}^{e_{TF}}) = \sum P(A_5, a_4) \cdot P(a_4 \mid e_{TF})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.73 + \langle 0.3, 0.7 \rangle \cdot 0.27$$
$$= \underline{\underline{\langle 0.665, 0.335 \rangle}}$$

$$P(A_6 \mid e_{TF}) = \sum P(A_6, a_5) P(a_5 \mid e_{TF})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.665 + \langle 0.3, 0.7 \rangle \cdot 0.335$$
$$\approx \underline{\underline{\langle 0.633, 0.367 \rangle}}$$

$$P(A_7 \mid e_{TF}) = \sum P(A_7, a_6) P(a_6 \mid e_{TF})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.633 + \langle 0.3, 0.7 \rangle \cdot 0.367$$
$$= \underline{\langle 0.617, 0.383 \rangle}$$

$$P(A_8 \mid e_{TF}) = \sum P(A_8, a_7) P(a_7 \mid e_{TF})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.617 + \langle 0.3, 0.7 \rangle \cdot 0.383$$
$$= \underline{\underline{\langle 0.608, 0.392 \rangle}}$$

d) $\lim\limits_{t \to \infty} P(A_t \mid e_{1:4})$:

$$\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$$0.8p + 0.3(1-p) = p$$

$$\Rightarrow p = 0.6$$

$$\Rightarrow \lim_{t \to \infty} P(A_t \mid e_{1:4}) = \begin{pmatrix} 0.6 \\ 1-0.6 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}}}$$

e) $P(X_t \mid e_{1:4})$, for $t = 0, 1, 2, 3$

Lets start by defining observation matrices for e:

$$O_1 = \begin{pmatrix} 0.7 \cdot 0.3, & 0 \\ 0, & 0.2 \cdot 0.1 \end{pmatrix} = \begin{pmatrix} 0.21, & 0 \\ 0, & 0.02 \end{pmatrix}$$

$$O_2 = \begin{pmatrix} 0.3 \cdot 0.3, & 0 \\ 0, & 0.8 \cdot 0.1 \end{pmatrix} = \begin{pmatrix} 0.09, & 0 \\ 0, & 0.08 \end{pmatrix}$$

$$O_3 = \begin{pmatrix} 0.3 \cdot 0.7, & 0 \\ 0, & 0.8 \cdot 0.9 \end{pmatrix} = \begin{pmatrix} 0.21, & 0 \\ 0, & 0.72 \end{pmatrix}$$

$$O_4 = \begin{pmatrix} 0.7 \cdot 0.7, & 0 \\ 0, & 0.2 \cdot 0.9 \end{pmatrix} = \begin{pmatrix} 0.49, & 0 \\ 0, & 0.18 \end{pmatrix}$$

Now lets calculate the backwards probabilities:

$$b_4 = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix}$$

$$b_3 = \alpha \overbrace{\begin{pmatrix} 0.8 , & 0.2 \\ 0.3 , & 0.7 \end{pmatrix}}^{\text{transition}} \overbrace{\begin{pmatrix} 0.49 , & 0 \\ 0 , & 0.18 \end{pmatrix}}^{O_4} \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{b_4}$$

$$= \alpha \begin{pmatrix} 0.428 \\ 0.273 \end{pmatrix} \approx \begin{pmatrix} 0.611 \\ 0.389 \end{pmatrix}$$

$$b_2 = \alpha \begin{pmatrix} 0.8 , & 0.2 \\ 0.3 , & 0.7 \end{pmatrix} \begin{pmatrix} 0.21 , & 0 \\ 0 , & 0.72 \end{pmatrix} \begin{pmatrix} 0.611 \\ 0.389 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 0.159 \\ 0.235 \end{pmatrix} = \begin{pmatrix} 0.404 \\ 0.596 \end{pmatrix}$$

$$b_1 = \alpha \begin{pmatrix} 0.8, 0.2 \\ 0.3, 0.7 \end{pmatrix} \begin{pmatrix} 0.09, 0 \\ 0, 0.08 \end{pmatrix} \begin{pmatrix} 0.404 \\ 0.596 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 0.0386 \\ 0.0443 \end{pmatrix} \approx \begin{pmatrix} 0.466 \\ 0.534 \end{pmatrix}$$

$$b_0 = \alpha \begin{pmatrix} 0.8, 0.2 \\ 0.3, 0.7 \end{pmatrix} \begin{pmatrix} 0.21, 0 \\ 0, 0.02 \end{pmatrix} \begin{pmatrix} 0.466 \\ 0.534 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 0.0804 \\ 0.0368 \end{pmatrix} \approx \begin{pmatrix} 0.686 \\ 0.314 \end{pmatrix}$$

Now we can finally find the smoothed values by multiplying the forward values with their backwards values.

$$\gamma_0 = \alpha \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} \circ \begin{pmatrix} 0.686 \\ 0.314 \end{pmatrix} = \alpha \begin{pmatrix} 0.4802 \\ 0.0942 \end{pmatrix}$$

$$\approx \underline{\underline{\begin{pmatrix} 0.836 \\ 0.164 \end{pmatrix}}}$$

$$\gamma_1 = \alpha \begin{pmatrix} 0.95 \\ 0.05 \end{pmatrix} \circ \begin{pmatrix} 0.466 \\ 0.534 \end{pmatrix} = \alpha \begin{pmatrix} 0.4427 \\ 0.0267 \end{pmatrix}$$

$$\approx \underline{\underline{\begin{pmatrix} 0.943 \\ 0.057 \end{pmatrix}}}$$

$$r_2 = \alpha \begin{pmatrix} 0.79 \\ 0.21 \end{pmatrix} \cdot \begin{pmatrix} 0.404 \\ 0.596 \end{pmatrix} = \alpha \begin{pmatrix} 0.319 \\ 0.125 \end{pmatrix}$$

$$\approx \underline{\underline{\begin{pmatrix} 0.718 \\ 0.282 \end{pmatrix}}}$$

$$r_3 = \alpha \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.611 \\ 0.389 \end{pmatrix} = \alpha \begin{pmatrix} 0.244 \\ 0.233 \end{pmatrix}$$

$$\approx \underline{\underline{\begin{pmatrix} 0.511 \\ 0.488 \end{pmatrix}}}$$