Ch 1

Ex 1.20

$A(X)$ is a field if and only if $I(X)$ is maximal, which happens if and only if X is minimal, i.e a point.

Ex 1.21

Let $J = \langle f_1 = x_1^3 - x_2^6, f_2 = x_1x_2 - x_2^3 \rangle$. Then $f_2 = x_2(x_1 - x_2^2)$ is zero either when $x_2 = 0$ or when $x_1 = x_2^2$, meanwhile $f_1 = (x_1 - x_2^2)(x_1^2 + x_1x_2^2 + x_2^4)$ is zero when $x_1 = x_2^2$, but not when $x_2 = 0, x_1 \neq 0$. We see that $\sqrt{J} = I(V(J)) = \langle x_1 - x_2^2 \rangle$.

Ex 1.22

First, of the i-th coordinate axis can be written as the zero locust of all but the i-the hyperplanes of codimension 1. So, if X_i is the i-th coordinate axis, we have

$$X_i = \bigcap_{j \neq i} V(x_j) \qquad = V\left(\sum_{j \neq i} (x_j)\right) = V(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_n).$$

We are interested in the ideal of the union of all X_i in the case of $n = 3$

$$\begin{aligned} I(X) &= I\left(\bigcup X_i\right) \\ &= \bigcap I(a_i) \\ &= (x_1, x_2) \cap (x_2, x_3) \cap (x_1, x_3) \\ &= (x_1x_2, x_2x_3, x_1x_3). \end{aligned}$$

Another way to arrive at the same result, is to realise that the coordinate ring on a given axis is equal to the set of univariate polynomials in the given indeterminate, and the coordinate ring on all three axes is the vector space generated by all univariate monomials. This is exactly what $\mathbb{K}[\boldsymbol{x}]/(x_1x_2, x_2x_3, x_1x_3)$ is.

Now, since $x_1, x_2, x_3 \notin I(X)$, but $x_1x_2, x_2x_3, x_1x_3 \in I(X)$, we have that any generating set of $I(X)$ must have a linear span which includes each of the three monomials x_1x_2, x_2x_3, x_1x_3. But these three monomials are linearly independent, and thus we need atleast three generators.

Ex 1.23

(a)

Let J be an ideal in $A(Y)$. Then if $f \in J$, we have that $f(x) = 0$ implies that all $g \in \pi^{-1}(f) = f + \ker(\pi)$ satisfy $g(x) = 0$, since $x \in \ker(\pi)$.

For the other direction, if $f \in \pi^{-1}(J)$ and $f(x) = 0$, we have that $\pi(f)(x) = 0$.

So, the two sets on polynomials vanish on the same points, whence their varieties are equal by definition.

(b)

We have that $\pi^{-1}(I_Y(X))$ is the set of polynomials f, such that $\pi(f)$ vanishes on X. But since all the polynomials in $\ker(\pi)$ vanish on all of Y, and therefore X, we have that $\pi(f)$ vanishs on X if and only if f does, which means that $\pi^{-1}(I_Y(X)) = I(X)$.

(c)

We have

$$\begin{aligned} I_Y(V_Y(J)) &= I_Y(V(\pi^{-1}(J))) \\ &= \pi(I(V(\pi^{-1}(J)))) \\ &\subseteq \pi(\sqrt{\pi^{-1}(J)}) \\ &= \pi(\pi^{-1}(\sqrt{J})) \\ &= \sqrt{J}, \end{aligned}$$

where we used part (a), (b), the Nullstellensatz, and the fact that contraction commutes with taking the radical. We show the remaining inclusions as well.

Let $X \subseteq Y$ be a variety and let $x \in X$. Then every polynomial function in $I_Y(Y)$ vanishes on x, and $x \in V_Y(I_Y(X))$.

Let $J \trianglelefteq A(Y)$ and $f \in \sqrt{J}$. Then f^m vanishes on $V_Y(J)$ for some m, whence f does as well. It follows that $f \in I_Y(V_Y(J))$.

Finally, let $x \in V_Y(I_Y(X))$. Every variety is the zero locust of some ideal, so let $X = V_Y(J)$. Then $x \in V_Y(I_Y(V_Y(J))) = V_Y(\sqrt{J}) \subseteq V_Y(J) = X$.

Ex 2.17

We have $X = V\left(f_1 = x_1 - x_2x_3, f_2 = x_1x_3 - x_2^2\right)$. In this ring we have

$$\begin{aligned} x_1 &\equiv x_2x_3, \\ x_1x_3 &\equiv x_2^2, \end{aligned}$$

which in turn yields

$$\begin{aligned} x_1 &\equiv x_2 x_3, \\ x_2 x_3^2 &\equiv x_2^2, \end{aligned}$$

So we see that $A(X)$ is isomorphic to the algebra $A' = \mathbb{K}[x_2, x_3]/(x_2 x_3^2 - x_2^2)$. Let $(x_2(x_3^2 - x_2)) = J \trianglelefteq \mathbb{K}[x_2, x_3] = A(V(f_1))$. It's easy to see that J isn't prime. It is radical though as it is a principal ideal generated by a squarefree product of irreducibles. It follows that $J = (x_2) \cap (x_3^2 - x_2)$ since the intersection of two radical ideals equals the radical of their product, and we see that the minimal prime ideals belonging to J are (x_2) and $(x_3^2 - x_2)$. Since prime ideals in $A(V(f_1)) \cong \mathbb{K}[x_2, x_3]$ are in one to one correspondence with prime ideals in $\mathbb{K}[x_1, x_2, x_3]$ which contain (f_1), we have that the minimal ideals belonging to (f_1, f_2) are $(x_2, x_1 - x_2 x_3) = (x_1, x_2)$ and $(x_3^2 - x_2, x_1 - x_2 x_3)$, so $X = V(x_1, x_2) \cup V(x_3^2 - x_2, x_1 - x_2 x_3)$ is a decomposition of X into irreducible componenets.

This can be verified with the following Sage code (from python3)

```
import sage.all
from sage.rings.rational_field import QQ
from sage.rings.polynomial.polynomial_ring_constructor \
        import PolynomialRing

def ex2_17():
    R = PolynomialRing(QQ, ["x_1", "x_2", "x_3"])
    x1, x2, x3 = R.gens()

    f1 = x1 - x2 * x3
    f2 = x1 * x3 - x2 ** 2

    J = R * [f1, f2]

    for Q in J.primary_decomposition():
        print(Q.radical().gens())
```

Ex 2.18

$I(X)$ is the set of all polynomials vanishing on X, I.e it is maximal among sets of polynomials vanishing on X. Thus, $V(I(X))$ is minimal among the varieties containing X, which is the same as saying that $V(I(X))$ is the closure of X in the Zariski topology.

Ex 2.19

(a)

Let X be a non-empty topology such that it can't be written as a finite union of non-empty connected closed sets. Then in particular, X isn't connected, so we can write $X = X_1 \cup X_1'$ where $X_1 \cap X_1' = \emptyset$ and X_1, X_1' are non-empty closed sets. But then atleast one of these sets must be disconnected, say X_1, and we can write $X_1 = X_2 \cup X_2'$ like before. Continuing this way yields an infinite chain

$$X_1 \supsetneq X_2 \supsetneq X_3 \supsetneq \dots,$$

and X can't be Noetherian.

(b)

Let $X = \bigcup_{1 \leq i \leq r} X_i$. Then given any infinite strict decreasing chain of closed subsetes Y_i we get r infinite (possibly non-strict) decreasing chains from $Y_i \cap X_j$ for each $1 \leq j \leq r$. Taking the pairwise union of all r chains yields the original chain Y_i, and it follows that atleast one of the chains must contain infinitely many strict inclusions, and by removing duplicates from this chain, we get an infinite strictly decreasing chain of closed subsets. In other words, if X is non-Noetherian, then one of the X_i must be non-Noetherian.

Ex 2.20

(a)

$Y \cap A$ closed in the subspace topology by definition implies that there is some closed set Z in X such that $Z \cap A = Y$. Then $Y \subseteq Z$. Since $\overline{Y}$ is the intersection of all closeds set in X which contain Y, we have $\overline{Y} \subseteq Z$. It follows that $\overline{Y} \cap A \subseteq Z \cap A = Y$, but we also have $Y \subset A, Y \subset \overline{Y}$, so $\overline{Y} \cap A = Y$.

(b)

If $\overline{A} = U_1 \cup U_2$ with U_1, U_2 closed in the $\overline{A}$ subspace topology, then there are closed subsets X_i such that $U_i = X_i \cap \overline{A}$ and $A = (X_1 \cap A) \cup (X_2 \cap A)$. Moreover, if $X_i \cap A = A$, then we'd have $\overline{A} \subseteq X_i$, and $\overline{A} = U_i$, so the union $A = (X_1 \cap A) \cup (X_2 \cap A)$ is a non-trivial decomposition.

For the other direction, first note that $\overline{U_1} \cup \overline{U_2} \supseteq \overline{U_1 \cup U_2}$ since $\overline{U_1} \cup \overline{U_2}$ is a closed set which covers both U_1 and U_2.

If $A = U_1 \cup U_2$ with U_1, U_2 closed in the A subspace topology, then there are closed subsets X_i such that $U_i = X_i \cap A$. Then $X_i \cap \overline{A}$ must cover $\overline{U_i}$, and

$$\overline{A} \subseteq \overline{U_1} \cup \overline{U_2} \subseteq (X_1 \cap \overline{A}) \cup (X_2 \cap \overline{A}),$$

but it's clear that the inclusions must hold in the other direction as well since everything on the RHS is intersected with the LHS, and we see that $\overline{A}$ is reducible. Moreover, the two sets on the RHS are non-empty as the $U_i = X_i \cap A \subseteq X_i \cap \overline{A}$ are.

Ex 2.21

(a)

Assume that the cover U_i of X consists only of open connected sets and that they all pairwise intersect each other. Let X_1, X_2 be two closed proper subets of X such that $X_1 \cup X_2 = X$. Then X_1, X_2 are also both open as they complement each other. Now consider some U_1. If U_1 intersects both X_1 and X_2, we have that $U_1 \cap X_i$ is closed, since $U_1 \cap X_1 = (U_1^c \cup X_2)^c$, so $U_1 = (U_1 \cap X_1) \cup (U_1 \cap X_2)$ is a decomposition of U_1 into closed sets, whence $U_1 \cap X_1$ must intersect $U_2 \cap X_2$ by hypothesis of the U_i being connected. It follows that $X_1 \cap X_2 \neq \emptyset$ in this case.

Now consider the case where U_1 only intersects one of the X_i, say X_1. Let U_2 be a set which intersects X_2 (such exist since the U_i cover X). Then as U_1 and U_2 intersect each other, it follows that U_2 must intersect both X_1, X_2, and we can conclude like before that X_1 must intersect X_2 in this case as well.

(b)

Let X, U_i be as above but with the further stipulation that each U_i is irreducible. Assume towards a contradiction that X is reducible and let $X = X_1' \cup \ldots \cup X_r'$ where each X_i is a closed proper subset of X and no X_i is contained in the union of all $X_j, j \neq i$. Let $X_1 = X_1', X_2 = X_2 \cup \ldots X_r'$. Then $X = X_1 \cup X_2$ is a decomposition of X into two closed proper subsets.

Like above, if a U_i intersects both X_1, X_2 we get a decomposition of $U_i = (U_i \cap X_1) \cup (U_i \cap X_2)$ into non-empty closed proper subsets, which is imposible by our assumption that each U_i is irreducible. Thus all U_i must be contained in either X_1 or X_2 but not both. This is imposible since they all pairwise intersect each other (and cover both X_1, X_2).

Ex 2.22

(a)

If $f(X) = U_1 \cup U_2$ is a decomposition of $f(X)$ into non-intersecting closed proper subsets, then $X = f^{-1}(U_1) \cup f^{-1}(U_2)$ is the same for X. So $f(X)$ disconnected implies X disconnected.

(b)

Like above.

Ex 2.23

(a)

First note that $I(\overline{X}) = I(V(I(X)) = I(X)$, so we'll prove that $I(Y_1 \setminus Y_2) = I(Y_1) : I(Y_2)$.

Let $f \in I(Y_1 \setminus Y_2)$. Then f vanishes on Y_1, and we have that $fg \in I(Y_1)$ for all $g \in \mathbb{K}[\boldsymbol{x}]$, and in particular, for all $g \in I(Y_2)$, so $f \in I(Y_1) : I(Y_2)$.

Now let $f \in I(Y_1) : I(Y_2)$. Then for all $g \in I(Y_2)$, we have that $fg \in I(Y_1)$. Now, since Y_2 is a subvariety, it's the zero locust of $I(Y_2)$, and for all points $a \in Y_1 \setminus Y_2$, there is some polynomial g_a such that $g_a(a) \neq 0$. But $fg_a \in I(Y_1)$, so it must be that $f(a) = 0$, and we see that $f \in I(Y_1 \setminus Y_2)$.

(b)

Using the Nullstellensatz, part (a), and the fact that the J_i are radical, we get

$$\begin{aligned}
\overline{V(J_1) \setminus V(J_2)} &= V(I(\overline{V(J_1) \setminus V(J_2)})) \\
&= V(I(V(J_1)) : I(V(J_2))) \\
&= V(\sqrt{J_1} : \sqrt{J_2}) \\
&= V(J_1 : J_2).
\end{aligned}$$

Ex 2.24

Let $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ be irreducible affine varieties and assume towards a contradiction that $X \times Y = U_1 \cup U_2$ is a decomposition into closed proper subsets. Let π_X, π_Y be the projections onto X, Y repectively. First note that

$$\begin{aligned}
U_1 \cup U_2 &= X \times Y \\
&= \pi_X(X) \times \pi_Y(Y) \\
&= \pi_X(U_1 \cup U_2) \times \pi_Y(U_1 \cup U_2) \\
&= (\pi_X(U_1) \cup \pi_X(U_2)) \times (\pi_Y(U_1) \cup \pi_Y(U_2)) \\
&= (\pi_X(U_1) \times \pi_Y(U_1)) \cup (\pi_X(U_1) \times \pi_Y(U_2)) \\
&\cup (\pi_X(U_2) \times \pi_Y(U_1)) \cup (\pi_X(U_2) \times \pi_Y(U_2)),
\end{aligned}$$

from where it will follow that either $Y = \pi_Y(U_1) \cup \pi_Y(U_2)$, or $X = \pi_X(U_1) \cup \pi_X(U_2)$ are non-trivial decompositions into subsets. We will show that $\pi_X(U_i), \pi_Y(U_i)$ are closed, which will contradict X, Y being irreducible.

Let $f \in I(U_1) \subseteq \mathbb{K}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m]$ and fix the last m indeterminates of f to some point $y \in \pi_Y(U_1)$. Call the new polynomial $g_{f,y}$. Then for any $x \in \pi_X(U_1)$ we have $g_{f,y}(x) = f(x, y)$ and since $(x, y) \in U_1$ and $f \in I(U_1)$, we see that $g_{f,y}$ vanishes on $\pi_X(U_1)$. We claim that

$$\pi_X(U_1) = \bigcap_{f \in I(U_1), y \in \pi_Y(U_1)} V\left(g_{f,y}\right).$$

We've already shown one inclusion, for the other direction, suppose that $x \in X$ such that all $g_{f,y}$ vanish on x. Then $f(x,y) = 0$ for all $y \in Y, f \in I(U_1)$, which in turn means that $x \times Y \subset V(I(U_1)) = U_1$ so $x \in \pi_X(U_1)$. It follows that $\pi_X(U_1)$ is closed and we are done.

Ex 2.30

If U_i is some strict descending chain of closed irreducible sets in A, then by Ex 2.20 (a), we have that $\overline{U}_i$ is a strict descending chain of closed sets in X, which are irreducible by Ex 2.20 (b).

Ex 2.33

First, let's identify said matrices with $\mathbb{A}^6$ according to

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}.$$

The rank of a matrix is at most 1 if and only all of its minors of order ≥ 2 are zero. For our matrix above, this yields the following set of equations

$$\begin{aligned} a_1 a_5 &= a_2 a_4 \\ a_1 a_6 &= a_3 a_4 \\ a_2 a_6 &= a_3 a_5. \end{aligned}$$

If we let X denote our set in $\mathbb{A}^6$ identified with the given set of matrices, then it follows from the discussion above that

$$X = V(J = (x_1 x_5 - x_2 x_4, x_1 x_6 - x_3 x_4, x_2 x_6 - x_3 x_5)).$$

is a variety. To show that it's irreducible, note that we can surjectively parameterize X with t_1, t_2, t_3, u according to

$$\begin{gathered} a_1 = t_1, a_2 = t_2, a_3 = t_3, \\ a_4 = ut_1, a_5 = ut_2, a_6 = ut_3. \end{gathered}$$

Irreducibility follows from the following lemma.

Lemma 0.1. Let $X \subseteq \mathbb{A}^n$ be a variety that is parameterized by $a_i = f_i(y_1, y_2, \ldots, y_m)$. Then $I(X)$ is prime and X is irreducible.

Proof. $I(X)$ can be identified with the ideal of algebraic dependencies on the f_i. I.e the kernel of $\mathbb{K}[\boldsymbol{x}] \to \mathbb{K}[f_1, f_2, \ldots, f_n]$. This kernel is prime since the image of the map is a subring of $\mathbb{K}[y_1, y_2, \ldots, y_m]$ which is an integral domain. $\square$

Ex 2.34

(a)

Let $n \in \mathbb{N}$ be such that $\dim X \geq n$. Then there exist some chain

$$Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n$$

of closed irreducible subsets. Since Y_i is irreducible, any two non-empty open sets in Y_i must intersect. Let $V_i = Y_i \setminus Y_{i-1}$. Then each V_i is open and non-empty in Y_i, since Y_{i-1} is closed in Y_i.

Now, pick some U which intersects Y_0. Since $U \cap Y_1$ is open and non-empty in Y_1, it must intersect V_1, whence $U \cap Y_0 \subsetneq U \cap Y_1$. Moving on, $U \cap Y_2$ is open in Y_2, hence it must intersect V_2 and it follows that

$$U \cap Y_0 \subsetneq U \cap Y_1 \subsetneq U \cap Y_2.$$

This procedure can be repeated to show that

$$U \cap Y_0 \subsetneq U \cap Y_1 \subsetneq U \cap Y_2 \subsetneq \ldots \subsetneq U \cap Y_n \subset U$$

is an n-long chain of strict inclusions in Y. Moreover, each of the $U \cap Y_i$ is irreducible in U since a reduction of $Y_i \cap U = T_1 \cup T_2$ yields a reduction

$$Y_i = (Y_i \setminus (U \setminus T_1)) \cup (Y_i \setminus (U \setminus T_2)).$$

Hence $\dim X \leq \sup_{i \in I}(\dim U_i)$. The reversed inequality is immediate from Ex 2.30.

(b)

Let X be an irreducible affine variety and

$$Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset X$$

a chain of irreducible subvarieties. Then if U is non-empty and open in X, we can translate the chain in affine space $T_{\boldsymbol{c}}(a_1, a_2, \ldots, a_n) = (a_1 + c_1, \ldots, a_n + c_n)$ (this is a homeomorphism) such that $T_{\boldsymbol{c}}(Y_0)$ intersects U. We can then repeat the argument of (a) to show that

$$T_{\boldsymbol{c}}(Y_0) \cap U \subsetneq T_{\boldsymbol{c}}(Y_1) \cap U \subsetneq \ldots \subsetneq T_{\boldsymbol{c}}(Y_n) \cap U \subset U$$

is a chain of irreducible closed subvarieties in U.

This is not the case in arbitrary topological spaces. Consider for example the space $\mathbb{N}$ given the topology where the non-trivial closed subsets are of the form $[1..n]$ and $[1..\infty]$. Then $\{0\}$ is open in $\mathbb{N}$ but has dimension 0.

Ex 2.35

First suppose that we have some chain

$$\{a\} \subset Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset X$$

of irreducible closed subsets Y_i. Then $Y_n = \bigcup_{i=1}^{r} (Y_n \cap X_i)$, is a decomposition into closed sets, and since Y_n is irreducible, we need $Y_n \subseteq X_i$ for some i, whence $\max(\dim X_i : a \in X_i) \geq \operatorname{codim}\{a\}$.

For the other direction, let X_i be the irreducible variety containing a with maximal dimension. Then by Prop 2.28 (b), we have $\operatorname{codim}_{X_i}\{a\} = \dim X_i = \max(\dim X_j : a \in X_j)$.

Ex 2.36

(a)

X being Noetherian is equivalent to Let X be Noetherian and $U_i, i \in I$ be an open cover of X. Then X admits a decomposition into irreducible subsets $X = X_1 \cup X_2 \cup \ldots \cup X_r$. Let $U_i, i \in I_1$ be a subcover which covers X_1 where all $U_i, i \in I_1$ intersect X_1, and no U_i is contained in some other U_j. Then the $U_i, i \in I_1$ must pairwise intersect each other. Use the axiom of choice to pick a sequence of unique indices $i_1, i_2, \ldots$ in I_1, and construct the chain

$$U_{i_1} \supseteq U_{i_1} \cap U_{i_2} (\supseteq U_{i_1} \cap U_{i_2}) \cap U_{i_3} \supseteq \ldots,$$

Each intersection is non-empty, since it's an intersection non-empty as X_1 is irreducible. Moreover, since X is Noetherian, this must either be a finite chain so that I_1 is finite, or it must stabilize, so that

$$U_{i_n} \subset \bigcap_{j=1}^{n-1} U_{i_j}$$

for some n. This contradicts how I_1 was constructed.

(b)

Quick aside: I revisited an old solution to this problem after finding the reference to this problem in the start of Chapter 5. The old solution was wrong, so I've re-solved this problem using tools from Chapters 1 through 4. I also had trouble solving it the second time around as well, and after looking at this link, https://math.stackexchange.com/a/119349/887520 , I decided to try to do it using Noether Normalization. This was very difficult for me (took 2 days almost to show that the morphism f below is surjective). So, this is all just a heads up to say that this solution is a bit messy and uses a few theorems and lemmas from all over the place.

Let X be an irreducible subvariety in $\mathbb{C}[\boldsymbol{x}]$ of dimension $d \geq 1$. It follows that the Krull dimension of $A(X)$ is d. The Noether normalization lemma tells us that $A(X)$ is integral and finitely generated over some free polynomial algebra $\mathbb{C}[y_1, y_2, \ldots, y_s]$. We will show that $d = s$. First note that $d \geq s$, since we have a chain of length s of prime ideals $(0) \subsetneq (y_1) \subsetneq y_1, y_2) \subsetneq \ldots \subsetneq (y_1, y_2, \ldots, y_s)$ which can be extended to a chain of the same length in $A(X)$ by the Going Up Theorem. Now suppose that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_d$ is a chain of prime ideals in $A(X)$ of length d. Then any ideal $\mathfrak{p}_i \cap \mathbb{K}[y_1, y_2, \ldots, y_s]$ is prime, after which Corollary 5.9 in Atiyah-Macdonald tells us that the chain contracts to a chain of strict inclusions of prime ideals in $\mathbb{C}[y_1, y_2, \ldots, y_s]$. Exercise 11.7 in Atiyah-Macdonald tells us that this ring has dimension s, so $s \geq d$, and we have $s = d$.

Now consider the inclusion $f^* : \mathbb{C}[y_1, y_2, \ldots, y_d] \to A(X)$ where the y_i's are algebraically independent polynomials in $A(X)$. We can identify $\mathbb{C}[y_1, y_2, \ldots, y_d]$ with a free $\mathbb{C}$-algebra in d variables, and doing so induces a morphism of varieties. $f : X \to \mathbb{A}^d$. We claim that f is surjective. To show this, we will first construct f explicitly.

By the proof of Corollary 4.8, we have that $f = (\phi_1, \phi_2, \ldots, \phi_d)$ where $\phi_i = f^*(y_i)$, but f^* is just the inclusion, so $f = (y_1, y_2, \ldots, y_d)$. Now let $a \in \mathbb{A}^d$ and consider $J = (y_1 - a_1, y_2 - a_2, \ldots, y_d - a_d)$ as an ideal in $A(X)$. We want to show that there is some $b \in V(J)$, since doing so would imply that $y_i(b) - a_i = 0 \Leftrightarrow y_i(b) = a_i \Leftrightarrow f(b) = a$.

First of, note that $A(X)$ is a finitely generated $\mathbb{C}[y_1, y_2, \ldots, y_d]$-module by the Noether normalization lemma. We will consider $J' = (y_1 - a_1, y_2 - a_2, \ldots, y_d - a_d)$ as an ideal in $\mathbb{C}[y_1, y_2, \ldots, y_d]$. Note that $J \neq J'$, these two ideals have the same generators but different parent rings! Now let's show that $V(J) \neq \emptyset$.

Suppose towards a contradiction that $V(J) = \emptyset$. Then $\sqrt{J} = I(V(J)) = (1) \Rightarrow J = (1)$, so we can write

$$1 = \sum_{i=1}^{d} h_i(\boldsymbol{x}) y_i(\boldsymbol{x})$$

for some set of polynomials $h_i \in A(X)$. It follows that $J'A(X) = A(X)$. Now Nakayama's lemma grants us some $r \in \mathbb{C}[y_1, y_2, \ldots, y_d]$ such that $rA(X) = 0$ and $r - 1 \in J'$. It follows that $r(a) = 1$ so $r \neq 0$. But the y_i are algebraically independent, so $y_i r = 0$ is a contradiction. Hence $V(J) \neq \emptyset$ and we've shown that f is surjective.

f is a polynomial mapping, hence it's continuous in the classical topology. So we have a surjective continuous map $f : X \to \mathbb{A}^d = \mathbb{C}^d$. We have that $\mathbb{C}^d$ isn't compact (for $d \geq 1$), hence X isn't either (since X compact $\Rightarrow f(X) = \mathbb{C}^d$ compact, a contradiction, https://math.stackexchange.com/questions/26514/continuous-image-of-compact-sets-are-compact)

Ex 2.40

(a)

With $J = (x_1x_4 - x_2x_3)$, we recognize $V(J)$ as the set of 2 by 2 matrices with rank ≤ 1. This variety is parameterizeable with

$$a_1 = t_1, a_2 = t_2, a_3 = ct_1, a_4 = ct_2,$$

whence

$$R = \mathbb{K}[x_1, x_2, x_3, x_4]/J \cong \mathbb{K}[t_1, t_2, ct_1, ct_2] \subset \mathbb{K}[t_1, t_2, c]$$

so R is an integral domain. Moreover, R isn't a field, so J is a prime ideal which isn't maximal, whence it can at most have height $\dim(\mathbb{K}[x_1, x_2, x_3, x_4]) - 1 = 3$, and R can at most have dimension 3. But in $\mathbb{K}[t_1, t_2, ct_1, ct_2]$ we have the prime ideals

$$(0) \subsetneq (t_1, t_2) \subsetneq (t_1, t_2, ct_1, ct_2)$$

So R has dimension 3. We quickly verify that all the ideals are prime. (0) is prime since we're in a domain. (t_1, t_2, ct_1, ct_2) is prime since we get a field when we quotient by it. (t_1, t_2) is prime since any polynomial in $\mathbb{K}[t_1, t_2, ct_1, ct_2]$ has terms where the sum of the degrees of t_1, t_2 is greater than the degree of c, so $Q = \mathbb{K}[t_1, t_2, ct_1, ct_2]/(t_1, t_2)$ consists of all polynomials in $\mathbb{K}[t_1, t_2, ct_1, ct_2]$ with terms where the degree of c is equal to the sum of the degrees of t_1, t_2. Hence $Q \cong \mathbb{K}[ct_1, ct_2]$ is an integral domain. Note that (t_1) (or (t_2)) isn't a prime ideal, since it contains ct_1t_2 but neither ct_1 nor t_2 (or neither ct_2 nor t_1).

(b)

$x_1 | x_2x_3 = x_1x_4$, but $x_1 \nmid x_2$, $x_1 \nmid x_3$ (any element in J has deg ≥ 2, so x_1 won't divide any representatives of $x_2 + J$ or $x_3 + J$ since they all contain a term of x_2, or x_3 respectively).

(c)

Again, follows from $x_1 + J \nmid x_2 + J$, and similar.

(d)

Under the isomorphic map $x_1 \mapsto t_1, x_2 \mapsto t_2, x_3 \to ct_1, x_4 \mapsto ct_2$, we see that (x_1, x_2) is the contraction of the prime ideal (t_1, t_2), which has height 1, so (x_1, x_2) has height 1 also.

Ex 3.12

(a)

Suppose that $A(X)$ is a UFD, $Y \subset X$ is an irreducible subvariety, and that $\text{codim}_X Y \geq 2$. We will begin by showing that there are two elements $f_1, f_2 \in$

$I(Y)$ which are prime in $A(X)$.

Since Y is irreducible, $I(Y)$ is prime, and as $A(X)$ is a UFD, we must have some element $f_1 \in I(Y)$ which is prime in $A(X)$. Moreover, $I(Y)$ has height at least 2, so it's strictly bigger than (f_1) which has height 1 by Prop 2.28 (c), hence $I(Y)$ must contain some element m where $f_1 \not| m$. But again, $I(Y)$ is prime, so it contains some prime factor f_2 of m.

It follows that $(f_i) \subset I(Y) \Rightarrow V(f_i) \supset Y \Rightarrow D(f_i) \subset U$. Hence we can write $\phi = h_1/f_1^{k_1}$ on $D(f_1)$, and $\phi = h_2/f_2^{k_2}$ on $D(f_2)$ with $f_i \not| h_i$. On $D(f_1) \cap D(f_2)$ we have

$$h_1/f_1^{k_1} = h_2/f_2^{k_2} \Leftrightarrow h_1 f_2^{k_2} = h_2 f_1^{k_1},$$

but $V(h_1 f_2^{k_2} - h_2 f_1^{k_1})$ is closed, so $h_1 f_2^{k_2} = h_2 f_1^{k_1}$ must hold on $\overline{D(f_1) \cap D(f_2)}$. Our next claim is that $\overline{D(f_1) \cap D(f_2)}$ is all of $A(X)$. To see this, note that that $A(X)$ is a UFD, hence an integral domain, whence X is irreducible and any open subspace is dense. Thus $h_1 f_2^{k_2} = h_2 f_1^{k_1}$ on all of $A(X)$, and since $f_2^{k_1} | h_2 f_1^{k_1}$, but f_2 divides neither h_2 nor f_1, we have $k_1 = 0$, and $\phi = h_1 = h_2$ on $A(X)$.

For the other direction, we assume that $\operatorname{codim}_X(Y) = 1$, since if the codimension is 0 we'd have $Y = X$. Then Prop 2.38 $I(Y) = (f)$, with f a non-unit, so $Y = V(f) \Rightarrow U = D(f)$ and Corollary 3.10 tells us that $\mathcal{O}_X(U) = \mathcal{O}_X(D(f)) \cong A_f(X)$ which isn't isomorphic to $A(X)$ when f isn't a unit.

(b)

Consider Exercise 2.40 and Example 3.3. We know that (x_1, x_2) is a prime ideal of height 1 in $A(X)$, so $Y = V(x_1, x_2)$ has codimension 1 in X, but yet the example shows that $\mathcal{O}_X(X \setminus U)$ isn't $A(X)$.

Ex 3.20

We show that the corresponding localized coordinate rings are isomorphic, after which Lemma 3.19 gives the desired isomorphisms. Let $\mathbb{K}[\boldsymbol{x}]_{I(a)} \to A(X)_{I(a)}$ be given by

$$\phi\,(g/f) = \frac{g + I(X)}{f + I(X)}.$$

ϕ is well defined since $f \in \mathbb{K}[\boldsymbol{x}] \setminus I(a)$, and $a \in X \Rightarrow I(a) \supset I(X)$, so $f + I(X)$ is non-zero. Suppose that $\phi(g/f) = 0$. Then $(h + I(X))(g + I(X)) = hg + I(X) = 0$ for some $h + I(X) \in A(X) \setminus I(a)$, which in turn implies that $hg \in I(X)$. Hence $hg/1 \in I(X)\mathbb{K}[\boldsymbol{x}]_{I(a)}$, but $h(a) \neq 0$, so h is a unit in $\mathbb{K}[\boldsymbol{x}]_{I(a)}$, so $hh^{-1}g/1 = g/1 \in I(X)\mathbb{K}[\boldsymbol{x}]_{I(a)}$, whence $\ker(\phi) \subseteq I(X)\mathbb{K}[\boldsymbol{x}]_{I(a)}$.

Now suppose that $g \in I(X)$, then $\phi(g/1) = 0$, so all the generators of $I(X)\mathbb{K}[\boldsymbol{x}]_{I(a)}$ lie in $\ker(\phi)$ and we have equality of the two ideals. The desired result now follows from the first homomorphism theorem and Lemma 3.19.

Ex 3.21

(a)

Let $\epsilon \neq 0$ and define $f_\epsilon : \mathbb{R} \to \mathbb{R}$ as follows,

$$f_\epsilon(x) = \begin{cases} 0 \text{ for } x \in (a-\epsilon, a+\epsilon), \\ x-(a+\epsilon) \text{ for } x \in [a+\epsilon), \\ x-(a-\epsilon) \text{ for } x \in (a-\epsilon]. \end{cases}$$

Then f_ϵ is continuous, and agrees with the zero function on $(a-\epsilon, a+\epsilon)$, so they reside in the same germ. Since we can pick ϵ arbitrarily small, it follows that the functions in a given germ all agree on a only. Now let I denote the ideal of all germs which are 0 at a. We claim that $\mathcal{F}_a/I$ is isomorphic to $\mathbb{R}$. To see this, note that $f(a) = g(a)$ if and only if $f(a) - g(a) = 0$ and $f(a) - g(a) \in I$. Hence every equivalence class of $\mathcal{F}_a/I$ may be identified with the value which the stalks admit at a, and $\mathcal{F}_a/I \cong I$ is a field and I is maximal. Moreover, it's the only maximal ideal since it contains all non-units, indeed if $f(a) \neq 0$, then we can pick a small enough neighbourhood of a in which f is invertible.

(b)

The open subsets of $\mathbb{R}$ are all infinite, so if two polynomials agree on an open subset of $\mathbb{R}$ they must be equal. It follows that any stalk is isomorphic to the polynomial ring which is not local.

Ex 3.22

(a)

Let $a \in U$. Then since $(U, \phi) \sim (U, \psi)$, we have by definition some open $V_a \subset U$ containing a such that $\phi\big|_{V_a} = \psi\big|_{V_a}$. The $V_a : a \in U$ form an open cover of U, and $\phi = \psi$ on U by the gluing property.

(b)

The vanishing set of their difference is closed in X, and since their difference vanishes on some open subset $V_a \subset X$, it vanishes on $\overline{V_a} = X$.

(c)

Yes, consider for example the f_ϵ and the zero function from the solution to Ex 3.21 (a) with $U = \mathbb{R}$.

Ex 3.23

Let $\phi : A(X)_{I(Y)} \to \mathcal{O}_{X,Y}$ be given by

$$\phi(g/f) = \overline{(D(f), g/f)}.$$

ϕ is well defined, for if $g/f = g'/f'$, then we have $h \in A(X) \setminus I(Y)$ such that $h(gf' - g'f) = 0$. Hence g/f and g'/f' agree as regular functions on the open set $U = D(f) \cap D(f') \cap D(h) = D(ff'h)$. We have that U is contained in both $D(f), D(f')$. Moreover, U intersects Y as $f, f', h \notin I(Y)$ and $I(Y)$ is prime since Y is irreducible so $ff'h \notin I(Y)$, so there must be some point $y \in Y$ where all f, f', h are non-zero and $y \in U \cap Y$. We can conclude that $\overline{(D(f), g/f)} = \overline{(D(f'), g'/f')}$.

Now suppose that $g/f \in \ker(\phi)$. Then we have that $0/1$ and g/f agree as functions on some neighbourhood V which intersects Y. We can define g/f on all of $D(f)$. Vanishing sets of regular functions are closed, so g/f vanishes on the closure of V in $D(f)$, but Y is irreducible, so $\overline{D(f)} = Y$ and $D(f)$ is irreducible as well, whence g/f vanishes on all of $D(f)$. It follows that $f(1 \cdot g - 0 \cdot f) = 0$ as polynomial functions on X, hence $g/f = 0/1$ in $A(X)_{I(Y)}$.

Finally, to see that ϕ is surjective since, let $\overline{(U, \phi)} \in \mathcal{O}_{X,Y}$, $y \in U$ and V_y be an open subset of U containing y such that ϕ is given by g/f on V_y. Then $y \in D(f)$, so $D(f) \cap Y \neq \emptyset$, and $\phi(g/f) = \overline{(D(f), g/f)}$ is a stalk at Y which is equal to $\overline{(U, \phi)}$ since they agree on V_y (note that g/f is an element of the domain $A(X)_{I(Y)}$ since $f \notin I(Y)$ as $f(y) \neq 0$).

Ex 3.24

If $a \in V$, then $a \in U \cap V$, so any representative (V, ϕ) of a germ in the original stalk can be restricted down to U. The other direction is immediate since open subsets in U are open in X (by virtue of U being open). Passing up and down through the restriction doesn't change equivalence class since any regular function defined on V agrees with itself on $V \cap U$. So the restriction of stalks is bijective.

Since the original restriction maps in the sheaves are homomorphisms, it follows that if $(V, f), (V', f')$ are representatives of germs in $\mathcal{F}_a$, that their product

$$(V \cap V', f|_{V \cap V'} f'|_{V \cap V'})$$

restrict to the restrictions of their product,

$$(V \cap V' \cap U, (f|_{V \cap V'} f'|_{V \cap V'})\,|_U) = (V \cap V' \cap U, f|_{V \cap V' \cap U} f'|_{V \cap V' \cap U}).$$

The same is true for sums and any other algebraic operation, so we see that the restriction of the stalk is an isomorphism.

Ex 4.12

A general affine conic be given by

$$f(x,y) = a_5x^2 + a_4xy + a_3y^2 + a_2x + a_1y + a_0,$$

where the a_i are such that (f) is prime.

We will show that $A = \mathbb{K}[x,y]/(f)$ is isomorphic exactly one of either $B_1 = \mathbb{K}[x,y]/(g_1)$ or $B_2 = \mathbb{K}[x,y]/(g_2)$ where $g_1 = x^2 - y$ and $g_2 = xy - 1$.

First we show that $B_1 \not\cong B_2$. To see this, note that $B_1 \cong \mathbb{K}[x]$ whilst $B_2 \cong \mathbb{K}[y, 1/y]$. We see that any element in B_2 is a sum of units, whilst the only units in B_1 are scalars, so the two algebras can't be isomorphic.

Now consider f again. We will show that we can obtain either g_1 or g_2 from f by a linear change of variables.

case $a_5 = a_3 = 0$. Since f is quadric, we have $a_4 \neq 0$ in this case, so we can write

$$f(x,y) = a_4\left(xy + \frac{a_2}{a_4}x + \frac{a_1}{a_4}y\right) + a_0 = a_4\left(x + \frac{a_1}{a_4}\right)\left(y + \frac{a_2}{a_4}\right) - \frac{a_1a_2}{a_4} + a_0.$$

Since f is irreducible, we can assume that $\frac{a_1a_2}{a_4} - a_0 \neq 0$, and after relabeling we have

$$f = (ax+b)(y+c) - 1$$

where we were able to rescale f so that $\frac{a_1a_2}{a_4} - a_0 = 1$ since we only care about $V(f)$. We now have that $f(x,y) = g_2(ax+b, y+c)$, so $\phi : \mathbb{K}[\boldsymbol{x}] \to \mathbb{K}[\boldsymbol{x}]$ where $\phi(x,y) = (ax+b, y+c)$ takes $I(g_2)$ to $I(f)$, hence it induces a well defined homomorphism $\overline{\phi} : A(V(g_2)) \to A(V(f))$, where $\overline{\phi}(h(x,y) + I(g_2)) = h(ax+b, y+c) + I(f)$, which is invertible with $\overline{\phi}^{-1}(h(x,y) + I(f)) = h\left(\frac{x-b}{a}, y-c\right) + I(g_2)$. We've shown that $A(V(f)) \cong A(V(g_2))$ whence $V(f) \cong V(g_2)$ by Corollary 4.8.

case $a_5 \neq 0, a_4 \neq 0, a_3 = 0$. By permuting variables, this case covers when $a_3 \neq 0, a_4 \neq 0, a_5 = 0$ as well. We care only about $V(f)$ so we can assume that $a_5 = 1$ and

$$\begin{aligned} f(x,y) &= x^2 + a_4xy + a_2x + a_1y + a_0. \\ &= \left(x + a_4y + a_2 - \frac{a_1}{a_4}\right)\left(x + \frac{a_1}{a_4}\right) + a_0 - \frac{a_2a_1}{a_4} + \frac{a_1^2}{a_4^2}, \end{aligned}$$

and $a_0 - \frac{a_2a_1}{a_4} + \frac{a_1^2}{a_4^2} \neq 0$ since f is reducible so after relabeling and division we get

$$af(x,y) = (ax+by+c)(x+d) - 1,$$

which just like in the previous case gives us an isomorphism between the $A(V(g_2))$ and $A(V(af)) = A(V(f))$ given by

$$\phi(h(x,y) + (g_2)) = h(ax + by + c, x + d) + (f)$$

and

$$\phi^{-1}(h(x,y) + (f)) = h\left(y - d, \frac{x}{b} + \frac{by}{a} - \frac{db}{a} - \frac{c}{b}\right) + (g_2).$$

case $a_5 \neq 0, a_4 = a_3 = 0$. By permuting variables, this case covers when $a_3 \neq 0, a_4 = a_5 = 0$ as well. We care only about $V(f)$ so we can assume that $a_5 = 1$ and

$$f(x,y) = x^2 + a_2x + a_1y + a_0,$$

which gives us an isomorphism between the $A(V(g_1))$ and $A(V(f))$ given by

$$\phi(h(x,y) + (g_1)) = h(x, a_2x + a_1y + a_0) + (f)$$

and

$$\phi^{-1}(h(x,y) + (f)) = h\left(x, \frac{y}{a_1} - \frac{a_2}{a_1}x - \frac{a_0}{a_1}\right) + (g_1).$$

case $a_5 \neq 0, a_3 \neq 0$. We care only about $V(f)$ so we can assume that $a_5 = 1$ and

$$\begin{aligned} f(x,y) &= x^2 + a_4xy + a_3y^2 + a_2x + a_1y + a_0. \\ &= (x + Hy + P)(x + Ty + Q) + a_0 - QP \end{aligned}$$

where

$$H = \frac{a_4}{2} + \sqrt{\frac{a_4^2}{4} - a_3},\ T = \frac{a_4}{2} - \sqrt{\frac{a_4^2}{4} - a_3},$$

and

$$Q = \left(1 - \frac{H}{T}\right)^{-1}\left(a_2 - \frac{1}{T}a_1\right) P = \left(1 - \frac{T}{H}\right)^{-1}\left(a_2 - \frac{1}{H}a_1\right)$$

when $a_4^2 \neq 4a_3$. We get an isomorphism between $A(V(g_2))$ and $A(V(f))$ like in previous cases but we skip the final calculations. This leaves one more case.

case $a_5 \neq 0$**, and** $a_4^2 = 4a_3$**.** We can assume that $a_5 = 1$ and get

$$\begin{aligned} f(x,y) &= a_4x^2 + a_4^2xy + a_3y^2 + a_2x + a_1y + a_0 \\ &= (x + \sqrt{a_3}y)^2 + (a_4 - 2\sqrt{a_3})y + a_2x + a_1y + a_0 \\ &= (x + \sqrt{a_3}y)^2 + a_2x + a_1y + a_0, \end{aligned}$$

where we picked the square root of a_3 which yields cancelation as desired. We see that $A(V(f))$ is isomorphic to $A(V(g_1))$ in this case, and also that this case generalizes the $a_4 = a_3 = 0$ case.

Ex 4.13

Let **Aff** denote the category of affine varieties on, and **Coord** denote the category of coordinate rings.

Lemma 0.2. Let $F : \mathbf{Aff} \to \mathbf{Coord}$ denote the map which assigns each affine variety it's coordinate ring. Then F is an invertible contravariant functor, and $\mathbf{Aff^{op}} \cong \mathbf{Coord}$ as categories.

Proof. Remark 1.16 tells us that F is bijective on the objects, and Corollary 4.8 tells us that $f \mapsto f^*$ is bijective on the arrows. What remains to show is that $f \mapsto f^*$ preserves composition in a contravariant way. This is straightforward to verify. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be affine varieties and morphisms between them, and $h \in A(Z)$ be some regular function on Z. Then

$$(g \circ f)^*(h) = h \circ (g \circ f) = (h \circ g) \circ f = f^*(g^*(h)) = (f^* \circ g^*)(h)$$

□

The previous lemma tells us that epimorphisms (right cancelative maps), are mapped to monomorphisms (left cancelative) and vice versa. The problem though is that epimorphisms/monomorphisms need not be surjective/injective homomorphisms. The converse is true though, as we will soon see in the part (a).

(a)

Suppose that f is surjective. Then f is right cancelative in the category of **Set**, hence it's right cancelative and an epimorphism in the category of **Aff**. It follows that f^* is a monomorphism in the category **Coord**. Now let $g \in \ker(f^*)$, and consider the algebra homomorphism $\phi : \mathbb{K}[x] \to A(Y)$ given by $\phi(x) = g$. Then $f^* \circ \phi = f^* \circ 0$, which since f^* is a monomorphism yields that ϕ is the zero map, so $g = 0$ in $A(Y)$ and f^* is injective.

The converse is not true unfortunately, and this has to do with cases where the image of f is dense in Y, but not surjective. Consider for example $X = Y = \mathbb{A}^1$, and $f : X \to Y$ given by $f(x) = x^2$. Then f isn't surjective, but $\mathrm{im}(f)$ is dense in Y, so any $g_1, g_2 \in A(Y)$ that agree on $\mathrm{im}(f)$ agree on all of Y. Hence $f^*(g_1) = f^*(g_2) \Rightarrow g_1 = g_2$ and f^* is injective.

(b)

No, Example 4.9 exhibits a bijective morphism f of varieties $f : \mathbb{A}^1 \to V(x_1^2 - x_2^3)$, but the corresponding algebra homomorphism $f^* : A(V(x_1^2 - x_2^3)) \to \mathbb{K}[t]$ is not surjective.

(c)

Yes, isomorphisms $\mathbb{A}^1 \to \mathbb{A}^1$ correspond to polynomial maps which are invertible by polynomial maps, and these are always linear (in the univariate case).

(d)

No! Consider for example $f(x,y) = (x, x^2+y)$. This map is regular and bijective, and it's inverse is given by $f^{-1}(x,y) = (x, y-x^2)$, which is also regular and bijective.

Ex 4.19

(a) $\cong$ **(c)**. An isomorphism $f : \mathbb{A}^1 \setminus \{1\} \to V(x_2 - x_1^2, x_3 - x_1^3) \setminus \{0\}$ is given by

$$f(x) = \left((x-1), (x-1)^2, (x-1)^3\right),$$

where the inverse is

$$f^{-1}(x_1, x_2, x_3) = x_1 + 1.$$

(d) $\ncong$ **(a)**. We have that $V(x_1x_2)$ is reducible so $A(V(x_1x_2))$ is not an integral domain. Meanwhile, $\mathbb{A}^1 \setminus \{1\} \cong D(x-1)$ has coordinate ring $\mathbb{K}[x]_{x-1}$ which is an integral domain.

(a) $\ncong$ **(e)**. We just saw that the coordinate ring of $\mathbb{A}^1 \setminus \{1\}$ is local. This is not the case for $V(x_2^2 - x_1^3 - x_1^2)$ since every point on the (infinite) variety induces a maximal ideal.
(a) $\ncong$ **(f)**. We just saw that the coordinate ring of $\mathbb{A}^1 \setminus \{1\}$ is local. This is not the case for $V(x_2^2 - x_1^2 - 1)$ since every point on the (infinite) variety induces a maximal ideal.
(b) $\cong$ **(d)**. We have $V(x_1^2 + x_2^2) = V((x_1 - ix_2)(x_1 + ix_2))$, and if $a, b \in \mathbb{A}^2$ is such that $ab = 0$, then

$$ab = 0 \Leftrightarrow \left(\frac{a+b}{2} + i\frac{a-b}{2i}\right)\left(\frac{a+b}{2} - i\frac{a-b}{2i}\right) = 0,$$

so the isomorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ $f(x_1, x_2) = \left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2i}\right)$ sends $V(x_1x_2)$ to $V((x_1 + ix_2)(x_1 - ix_2))$.

(d) $\ncong$ **(e)**. We claim that $f = -x_2^2 + x_1^3 + x_1^2$ is irreducible. To see this, note that any factors would need to be of the form

$$(ax_1^2 + bx_1 + cx_2 + d)(Bx_1 + Cx_2 + D) =$$
$$aBx_1^3 + aCx_1^2x_2 + (aD + bB)x_1^2 + (bC + Bc)x_1x_2 + cCx_2^2 + (b+B)x_1 + (c+C)x_2 + dD$$

We get

$$aB = 1, cC = -1, aC = 0$$

which is impossible. It follows that $A(V(f))$ is an integral domain whilst $A(V(xy))$ isn't so the two varieties are not isomorphic.

(d) $\not\cong$ (f). We claim that $x_1^2 - x_2^2 - 1$ is irreducible. To see this, note that any factors would need to be of the form

$$(ax_1+bx_2+c)(Ax_1+Bx_2+C) = aAx_1^2+bBx_2^2+cC+(aB+Ab)x_1x_2+(aC+Ac)x_1+(bC+bC)x_2.$$

so we need $a, b, c, A, B, C \in \mathbb{C}$ such that

$$\begin{aligned} aA &= 1 \\ bB &= -1 \\ cC &= -1 \\ aB &= -Ab \\ aC &= -Ac \\ bC &= -Bc. \end{aligned}$$

If we fix $a = 1$ we get $A = 1$, then $B = -b$, $c = -C$, after which we get $b = 1, B = -1$ or $b = -1, B = 1$. Assume the first case, then $bC = -Bc$ turns into $C = c$, contradicting our previous equation $c = -C$ since $c \neq 0$ due to $cC = -1$. It follows that $A(V(x_1^2 - x_2^2 - 1))$ is an integral domain which $A(V(x_1x_2))$ is not.

(e) $\not\cong$ (f).
Let $X = V(x_1^2-x_2^2-1), Y = V(x_2^2-x_1^3-x_1^2)$. When graphing the projection of Y onto $\mathbb{R}$, we see that the resulting curve intersects itself at the origin. Meanwhile X is isomorphic to the unit circle by $(x_1, x_2) \mapsto (x_1, ix_2)$, which doesn't intersect itself. Thus we guess that investigating the behaviour of $A(X)$ near the origin might lead to a proof that the two varieties aren't isomorphic. We give it a try by looking at the completion of $A(X)$ by the ideal $I((0,0)) = (x, y)$. Proposition 10.13 in Atiyah MacDonald tells us that

$$\widehat{A(Y)} \cong \mathbb{K}[[x_1, x_2]] \otimes_{\mathbb{K}[x_1,x_2]} A(Y) \cong \mathbb{K}[[x_1, x_2]]/I(Y).$$

We now claim that f_Y factors in $\mathbb{K}[[x_1, x_2]]$, whence $\widehat{A(Y)}$ isn't a domain. We will use Hensel's Lemma to show this, more specifically, the variant given in Theorem 7.3 in Eisenbud. We use the lemma to find a square root of $x_1 + 1$ by searching for a solution of $g(z) = X^2 - (x_1 + 1) = 0$ in $\widehat{A(Y)}[z]$. We have that $g'(X)^2 = 4X^2$, and

$$g(1) = 1 - x_1 - 1 = x_1 \equiv 0 \bmod g'(1)(x_1, x_2) = (x_1, x_2),$$

whence applying the lemma gives us a root b of g in $\widehat{A(Y)}$, so that $x_1 + 1 = b^2$ and $f_Y = x_2^2 - x_1^2(x_1 + 1) = (x_2 - x_1b)(x_2 + x_1b)$.

Since X is isomorphic to the unit circle, we will just say that X is the unit circle from now on. We will now show that every completion of $A(X)$ is a domain, after which the non-isomorphism will follow. Consider the completion of $A(X)$ at the point p on X (I.e the maximal ideal $I(p)$). This is isomorphic to the completion of $A(I(f_X(x_1+p_1, x_2+p_2)))$ at the origin, where $f_X = x_1^2 + x_2^2 - 1$ (since we're just "moving" the circle such that p winds up at the origin). Denote the new polynomial by $f_{X,p}$. Then

$$\begin{aligned} f_{X,p} &= (x_1+p_1)^2 + (x_2+p_2)^2 - 1 \\ &= x_1^2 + 2x_1p_2 + p_1^2 + x_2^2 + 2x_2p_2 + p_2^2 - 1 \\ &= x_1^2 + 2x_1p_2 + x_2^2 + 2x_2p_2 \end{aligned}$$

since p is on the circle and $p_1^2 + p_2^2 = 1$. Furthermore,

$$f_{X,p} = x_1^2 + 2x_1p_2 + x_2^2 + 2x_2p_2 \qquad =$$

We are done if we can show that $f_{X,p}$ is irreducible in $\mathbb{K}[[x_1, x_2]]$ for all p. TODO Todo todo finnish

Conclusion: We have the following equivalence classes

$$\begin{gathered} (a) \cong (c), \\ (b) \cong (d), \\ (e), \\ (f), \end{gathered}$$

Ex 5.7

(a)

I was stuck here for a long time, so we'll solve this exercise in a perhaps overly detailed manner.

Let $X_1 = X_2 = \mathbb{A}^1$, and $U_i \subset X_i$ be $D(x)$, I.e $\mathbb{A}^1 \setminus 0$. Let $\mathbb{P}^1$ be X_1 glued with X_2 via the isomorphism $\phi : U_1 \to U_2$, where $\phi : x \mapsto 1/x$. Let i_1, i_2 be the injection maps from X_1, X_2 to $\mathbb{P}^1$.

Let Y be an prevariety and $f : Y \to \mathbb{P}^1$ be a map. Let $Y_1 = f^{-1}(i_1(X_1)), Y_2 = f^{-1}(i_2(X_2))$. We claim that f is a morphism precisely when both

$$\begin{aligned} f_1 : Y_1 \to \mathbb{A}^1, f_1 : y \mapsto i_1^{-1}(f(y)), \\ f_2 : Y_2 \to \mathbb{A}^1, f_2 : y \mapsto i_2^{-1}(f(y)). \end{aligned}$$

First of, we have that f is a morphism if and only if both

$$\begin{aligned} \hat{f}_1 : Y_1 \to \mathbb{P}^1, \hat{f}_1 : y \mapsto f(y), \\ \hat{f}_2 : Y_2 \to \mathbb{P}^1, \hat{f}_2 : y \mapsto f(y), \end{aligned}$$

are morphism by Remark 4.5 (b) and Lemma 4.6. Also, the injections i_1, i_2 are both isomorphisms onto their images (they are both their own inverse), hence each $\hat{f}_k$ is a morphism exactly when $f_k = i_k^{-1} \circ \hat{f}_k$ is a morphism.

Now, set $Y = \mathbb{A}^1 \setminus 0 = D(x)$. Then Y_1, Y_2 are both open in $\mathbb{A}^1$, and all such sets are distinguished. Thus we can suppose $Y_1 = D(h)$ and we can write

$$f_1 = \frac{g}{h^k}.$$

After canceling common factors and extracting all powers of x we are left with an expression of the form

$$f = x^m \frac{G}{H}$$

where $m \in \mathbb{Z}$ and $(G, H) = 1, x \not| G,\ x \not| H$. But i_1^{-1} is the identity, so $f = f_1$ on $D(h_1)$ and $Hf - x^m G = 0$ here. But $D(h_1)$ is dense in Y, so $Hf - x^m G = 0$ on all of $D(x)$ and

$$f = x^m \frac{G}{H}$$

Dividing by H might seem suspicious here, but this results in a well-defined function since $(G, H) = 1$. Indeed, we can send any root r of H to $r \mapsto i_2(0) = "\infty"$, after which chasing the definitions yields that $Y_1 = D(xH), Y_2 = D(xG)$ and $f_1 = f, f_2 = 1/f$ which are both well-defined morphisms on their corresponding domains.

If $m = 0$, then $f(0) \in \mathbb{A}^1 \setminus 0 = i_1(X_1) \cap i_2(X_2)$ and we can trivially extend the domain of f to all of $\mathbb{A}^1$. If $m > 0$, then $f(0) = 0$ and we can extend f_1 by sending $f_1 : 0 \mapsto 0$, whilst we can leave f_2 untouched as $0 \notin i_2(X_2)$ and $0 \notin Y_2$. If $m < 0$ we have $f(0) \notin i_1(X_1)$, so we only need to worry about f_2. Since $i_2^{-1} : x \mapsto 1/x$, we have that $f_2 = x^{-m} \frac{H}{G}$ and we can simply extend f_2 by sending $f_2 : 0 \mapsto 0$ again.

(b)

In part (a) we had that every regular function from $D(x) \to \mathbb{P}^1$ could be written as $x^m \frac{G}{H}$ coprime, and as G, H are univariate polynomials, they'll never be simultaneously zero. This is not the case for coprime polynomial in two variables.

Consider the regular function $f : \mathbb{A}^2 \setminus 0 = D(x) \cup D(y) \to \mathbb{P}^1$ given by

$$f(x, y) = \frac{y}{x}$$

Then f is regular because (using the notation of part (a) with $Y = D(x) \cup D(y)$), $Y_1 = D(x)$, $Y_2 = D(y)$ and $f_1 = \frac{y}{x}, f_2 = \frac{x}{y}$ are both quotient of polynomials with non-vanishing denominators on their domains.

Now suppose towards a contradiction that $\hat{f}$ is an extension of f to the affine plane and let $\hat{f}_k = i_k^{-1} \circ \hat{f}\Big|_{\hat{f}^{-1}(i_k(X_k))}$ as before. Then $(0,0)$ must lie in one or both of $\hat{f}^{-1}(i_1(X_1))$ or $\hat{f}^{-1}(i_2(X_2))$, and we assume the index 1 case. Now let U_0 be some open neighbourhood of $(0,0)$ such that $\hat{f}_1$ is a rational function on U_0. We have $\hat{f}_1 = y/x$ on $D(x) \cup D(y)$, hence on $U_0 \setminus (0,0)$, and there is no rational function on any open set U containing 0 which is y/x on $U \setminus (0,0)$, so we arrive in a contradiction.

(c)

Let $f : \mathbb{P}^1 \to \mathbb{A}^1$ and $f_1 = f \circ i_1, f_2 = f \circ i_2$. Then f_1, f_2 are both morphisms $\mathbb{A}^1 \to \mathbb{A}^1$ by the glueing construction of $\mathbb{P}^1$. On $\mathbb{A}^1 \setminus \{0\}$ we have that $f_1(x) = f_2(1/x)$, again by the construction of $\mathbb{P}^1$. Since f_1, f_2 are both in $A(\mathbb{A}^1) = \mathbb{K}[x]$, this is possible only if $f_1 = f_2 \in \mathbb{K}$.

Ex 5.8

We will solve these exercises using homogeneous coordinates. That is, we will write points $i_1(x) = [x : 1]$ and $i_2(x) = [1 : x]$ with the equivalence $[x_1 : x_2] = \lambda[x_1 : x_2]$ for all $\lambda \in \mathbb{K} \setminus 0$. Then i_1 is injective since $i_1(x_1) = i_1(x_2)$ implies $[x_1 : 1] = [x_2 : 1]$ which is the case only when $x_1 = x_2$. Moreover if $x \neq 0$,

$$i_1(x) = [x : 1] = 1/x[x : 1] = [1 : 1/x] = i_2(1/x)$$

and we see that the equivalence classes $[x_1 : x_2]$ correspond exactly to the points on $\mathbb{P}^1$.

(a)

We begin by retracing some of the steps of Exercise 5.7 (a) with this new notation to show the following lemma.

Lemma 0.3. Let $f : \mathbb{A}^1 \to \mathbb{P}^1$ be a morphism. Then f must be of the form $f(x) = [p(x) : q(x)]$, where $p, q \in \mathbb{K}[\boldsymbol{x}]$ are polynomials with no root in common.

Proof. Let $Y_1 = f^{-1}(i_1(\mathbb{A}^1)), Y_2 = f^{-1}(i_2(\mathbb{A}^1))$ and $f_1 = i_1^{-1} \circ f\big|_{Y_1}$, $f_2 = i_2^{-1} \circ f\big|_{Y_2}$. Then both f_k are morphisms $\mathbb{A}^1 \supseteq Y_k \to \mathbb{A}^1$, hence they can be written as

$$f_k = \frac{g_k}{h_k}$$

with $(g_k, h_k) = 1$. But then

$$f\big|_{i_1(Y_1)} = i_1 \circ f_1 = [g_1(x)/h_1(x) : 1] = [g_1(x) : h_1(x)],$$

and as $\mathbb{P}^1$ is irreducible and Y_1 is dense in $\mathbb{P}^1$, we have

$$f = [g_1(x) : h_1(x)]$$

on all of $\mathbb{P}^1$ and we are done. Note that it also follows that $g_1 = h_2, g_2 = h_1$. $\square$

Now let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an automorphism. From the lemma it follows that we can write

$$f\big|_{i_1(\mathbb{A}^1)}([x:1]) = [p_1(x) : q_1(x)],$$
$$f\big|_{i_2(\mathbb{A}^1)}([1:x]) = [p_2(x) : q_2(x)],$$

and as these expressions agree on the intersection $i_1(\mathbb{A}^1) \cap i_2(\mathbb{A}^2) = \{[x_1 : x_2] = [x_1/x_2 : 1] : x_1, x_2 \in \mathbb{K} \setminus 0\}$, we have

$$[p_1(x_1/x_2) : q_1(x_1/x_2)] = [p_2(x_1/x_2) : q_2(x_1/x_2)]$$

which in turn yields

$$p_1(x)q_2(x) = p_2(x)q_1(x)$$

whenever $x \notin V(q_1, q_2)$, but if $x \in V(q_1, q_2)$ then both sides are 0 so we have

$$p_1(x)q_2(x) = p_2(x)q_1(x)$$

everywhere and $(p_k, g_k) = 1$ yields $p_1 = cp_2, q_1 = cq_2$. If we now relabel $p = p_1, q = q_1$, we get

$$f([x_1 : x_2]) = [p(x_1/x_2) : q(x_1/x_2)]$$

when $x_2 \neq 0$ and

$$f([x_1 : x_2]) = [cp(x_2/x_1) : cq(x_2/x_1)] = [p(x_2/x_1) : q(x_2/x_1)]$$

when $x_1 \neq 0$. Moreover, since f is injective, we have that f is injective on $i_1(\mathbb{A}^1)$, hence p, q must have degree 1 and we are done.

(b)

Ex 5.9

For any ringed spaces X, Y, we always have that for any morphism $f : X \to Y$ the pull-back $f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is an algebra homomorphism. This is required by the definition of a morphism of a ringed space. Thus we will only focus on the other direction for both subproblems.

(a)

Let $f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ be an algebra homomorphism. Let $y_1, y_2, \ldots, y_n$ be the coordinate functions on Y, and $\phi_i = f^*(y_i)$. We claim that

$$f = (\phi_1, \phi_2, \ldots, \phi_n)$$

is a morphism $X \to Y$.

Let U_i be a cover of X by affine varieties. It follow from the definition of a sheaf that f^* followed by restriction to any U_i is an algebra homomorphism $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(U_i)$ and we write this as

$$f_i^* = ?\big|_{U_i} \circ f^*.$$

From the proof of Corollary 4.8, we know that f_i^* is the pullback of $f\big|_{U_i}$, hence $f\big|_{U_i}$ is a morphism $U_i \to X$ and f is a morphism by Lemma 4.6.

(b)

No, for example Exercise 5.7 (c) tells us that $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \cong \mathbb{K}$, so there is only one non-zero $\mathbb{K}$-algebra homomorphism $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \to \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) = \mathbb{K}[x]$ and that is the injection $\mathbb{K} \to \mathbb{K}[x], k \mapsto k$. Meanwhile, from the construction of $\mathbb{P}^1$ we already have two non-trivial homomorphism i_1, i_2 for $\mathbb{A}^1 \to \mathbb{P}^1$.

Ex 5.11

Let $Y \subseteq X$ be a closed subset and $U \subseteq X$ be an open affine subset. Let $i : U \to \mathbb{A}^n$ be an embedding into affine space. Then $i(U \cap Y)$ is closed in $\mathbb{A}^n$, hence a zero set of some polynomials $V(f_1, f_2, \ldots, f_m)$. In other words, $U \cap Y$ is isomorphic to $V(f_1, f_2, \ldots, f_m)$ via i as ringed spaces. Hence $U \cap Y$ is an affine open set of X.

It follows that if U_i is a finite cover of X by affine open sets, then $Y \cap U_i$ is a finite cover of Y by affine open sets, and Y is a prevariety.

Moreover, the structure sheaf on Y obtained from gluing the $U_i \cap Y$ together agrees with that of Construction 5.10 (b). Let $U \subset Y$ be an open set, $\mathcal{O}_y$ be the structure sheaf from Construction 5.10 (b) and $\mathcal{O}'_Y$ be the structure sheaf obtained from glueing $U_i \cap Y$.

We have $f \in \mathcal{O}'_Y(U)$ precisely when f is regular on all $U_i \cap Y$. Now let $a \in U$ and i be such that $a \in U_i$. Then let F be any element in the preimage of f under the quotient homomorphism $A(U_i) \to A(U_i \cap Y) = A(U_i) + I(Y)$. Then f and F agree on $U_i \cap Y$, so $f \in \mathcal{O}_Y(U)$ by Construction 5.10 (b).

Now let $f \in \mathcal{O}_Y(U)$. Then given a U_i and $a \in U_i \cap Y$, we have that there is some open $V_a \in X$ containing a such that there is some $g \in \mathcal{O}_X(V_a)$ and $f\big|_{V_a \cap U} = g\big|_{V_a \cap U}$. But then $f\big|_{V_a \cap U} \in \mathcal{O}_X(V_a \cap U)$, so given any $a \in U_i \cap U$, we can find V_a such that $f\big|_{V_a \cap U}$ is regular, hence f is regular on $U_i \cap U$ and finally on U by Lemma 4.6.

Ex 5.21

Lemma 0.4. Let V be a prevariety obtained by glueing the affine open sets U_i. Then Z is closed in V if and only if every $Z \cap U_i$ is closed.

Proof. We have that Z is closed if and only if $V \setminus Z$ is open, which happens if and only if every $U_i \cap (V \setminus Z) = U_i \setminus Z$ is open, and this is the case if and only if every $U_i \cap Z$ is closed. $\square$

Let i_1, i_2 be the injections into $\mathbb{P}^1$ and define $X_1 = i_1(\mathbb{A}^1), X_2 = i_2(\mathbb{A}^1)$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ can be obtained by glueing the patches $X_i \times X_j$, $i, j \in \{1, 2\}$. The diagonal $\Delta_{\mathbb{P}^1}$ intersects $X_1 \times X_1$ and $X_2 \times X_2$ as $\Delta_{\mathbb{A}^1}$ and is thus closed there. It intersects $X_1 \times X_2, X_2 \times X_2$ as $V(xy - 1)$ and is closed there as well. Hence $\Delta_{\mathbb{P}^1}$ is closed in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1$ is separated.

Ex 5.22

(a)

Let $\pi_{xx} : (X \times Y) \times (X \times Y) \to X \times X$ be the projection morphism onto the two X coordinates. Then the inverse image of the diagonal of X,

$$\pi_{xx}^{-1}(\Delta(X)) = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y) : x_1 = x_2\}$$

is closed in $(X \times Y) \times (X \times Y)$, and similarly, we have

$$\pi_{yy}^{-1}(\Delta(Y)) = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y) : y_1 = y_2\}$$

closed as well. Intersecting the two yields $\Delta_{X \times Y}$ and we are done.

(b)

Let U_i, V_i be finite open affine coverings of X, Y respectively. Then $U_i \times V_j$ is a finite open covering of $X \times Y$. Every U_i, V_j is irreducible, since if $A \cup B = U_i$ are two closed non-trivial sets in U_i, then $U_i \setminus A, U_i \setminus B$ are open in U_i and in X, whence $X \setminus (U_i \setminus A) = A \cup (X \setminus U_i)$ and $B \cup (X \setminus U_i)$ are two closed non-trivial sets. Their union is

$$(A \cup (X \setminus U_i)) \cup (B \cup (X \setminus U_i)) = (A \cup B) \cup (X \setminus U_i) = U_i \cup (X \setminus U_i) = X.$$

We now have that every $U_i \times V_j$ is irreducible by Exercise 2.24. If we can show that every $U_i \times V_j$ intersects every $U_r \times V_s$, then we can apply Exercise 2.21 which yields $X \times Y$ irreducible.

Every U_i, U_r and V_j, V_s intersect since they are open sets in the irreducible spaces X, Y, and it follows immediately that $U_i \times V_j$ and $U_r \times V_s$ intersect.

Ex 5.23

(a)

Let U, V be affine open sets in the variety X, and $\pi_U : U \times V \to X, \pi_V : U \times V \to X$ be the projections (followed by inclusions into X). Then it follows from Proposition 5.20 (b) that $\{(u, v) \in U \times V : \pi_U(u) = \pi_V(v)\} = \Delta_{U \cap V}$ is closed in $U \times V$.

Since both U, V are affine, we have that $U \times V$ is affine as well. Let $e : U \times V \xrightarrow{\sim} Z \subset \mathbb{A}^n$ be an embedding onto some Zariski closed set in affine space. Then e sends closed sets to closed sets. Hence $e(\Delta_{U \cap V})$ is closed in $\mathbb{A}^n$, so $\Delta_{U \cap V}$ is an affine variety.

Finally, $f : U \cap V \to \Delta_{U \cap V}, f : x \mapsto (x, x)$ is an isomorphism with inverse $f^{-1} : (x, x) \mapsto x$, and $U \cap V$ is affine as well.

(b)

Like above, we have that $X \cap Y \cong \Delta_{X \cap Y}$, so it's enough to consider the diagonal. Also like above, the diagonal $\Delta_{X \cap Y}$ is closed in $X \times Y$, and $X \times Y$ is affine so embeddable into some Zariski closed set in affine space, hence $\Delta_{X \cap Y}$ is as well. From now on we identify all of our varieties as embedded into affine space this way (that is we may assume that $\Delta_{X \cap Y}, X \times Y \subset \mathbb{A}^{2n}$ are Zariski closed).

Now, let U be some irreducible component of $\Delta_{X \cap Y}$. We have $X \times Y$ irreducible by Exercise 5.22, so we can apply Proposition 2.28 (b) to get

$$\begin{aligned} \dim \Delta_{X \cap Y} &= \dim X \times Y - \operatorname{codim}_{X \times Y}(U) \\ &= \dim X + \dim Y - \operatorname{codim}_{X \times Y}(U). \end{aligned}$$

Hence we are done if we can show that $\operatorname{codim}_{X \times Y}(U) \leq n$. This follows from Lemma XYZ which says that $\operatorname{codim}_{X \times Y}(U)$ is the same as the height of $I(U)$ in $A(X \times Y)$. Since U is an irreducible component of $\Delta_{X \times Y}$, we have that $I(U)$ is a minimal prime ideal of $I(\Delta_{X \times Y})$, hence $I(U)$ and $I(\Delta_{X \times Y})$ have the same height ?
but we claim that the height of $I(U)$
Now, $I(U)$ is a minimal ideal of $I(\Delta_{X,Y})$
, and since $U \subseteq \Delta_{X \cap Y}$, we have $I(U) \supseteq I(\Delta_{X \cap Y})$
$\cong A(X) \otimes A(Y)$.
TODO todo Todo : Finnish!

Ex 5.24

We will look locally enough where X is affine, and reduce this problem to the affine case which we already solved in Exercise 2.34 (b).

Let X be a variety, $U \subseteq X$ a dense open subset and

$$Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset X$$

a chain of irreducible closed subsets. Then let Z be a affine open set which intersects Y_0. Since U is dense, we have that U intersects Z as well. Let

$e : Z \to \mathbb{A}^n$ be an embedding into affine space such that $e(Z)$ is the zero locust of some set of polynomials. All our sets intersect Z, and we write $Z' = e(Z), Y'_i = e(Z \cap Y_i), U' = e(Z \cap U)$.

We can use the construction from Exercise 2.34 (a) to show that $Y'_{i-1} \subsetneq Y'_i$ are strict inclusions. In brief, we set $V_i = Y_{i-1} \setminus Y_i$, which is non-empty and open in Y_i and intersects $Z \cap Y_i$ since Y_i is irreducible, hence $\emptyset \neq e(Z \cap V_i) \subset Y'_i \setminus Y'_{i-1}$.

Now we claim that U' is dense in Z', which is equivalent to U' intersecting every open set of Z' (and this is the property we will need anyhow). Let V' be open in Z' and $V = e^{-1}(V)$. Then V is open in Z, and by the definition of the subspace topology, we have some open V'' in X such that $V'' \cap Z = V$. But Z is open, so V is open in X, and V intersects U since U is dense in X.

To recap, we now have a strict chain of closed irreducible subsets in affine space

$$Y'_0 \subsetneq Y'_1 \subsetneq \ldots \subsetneq Y'_n \subset Z'$$

and a dense open set $U' \subseteq Z'$, and we are free to use the translation of Exercise 2.34 (b) to conclude that $\dim U' = \dim U \geq n$.

The other inequality is Exercise 2.30.

Ex 6.14

Write $a = [a_0 : a_1 : \ldots : a_n]$ and $e_{i,j} = a_j x_i - a_i x_j$. We claim that $I_p(\{a\}) = (e_{i,j} : 0 \leq i < j \leq n)$. First note that $e_{i,j}(a) = a_j a_i - a_i a_j = 0$ for all i, j. Moreover, suppose that $b = [b_0 : b_1 : \ldots : b_n]$ is such that $e_{i,j}(b) = 0$ for all i, j. Then $a_i b_j = a_j b_i$ for all i, j, which is exactly the same thing as saying all of the minors of order 2 of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ b_0 & b_1 & \ldots & b_n \end{pmatrix}$$

vanish, which in turn happens if and only if the matrix has at most rank 1, I.e $a = b$ in $\mathbb{P}^n$. We've shown that $V((e_{i,j} : 0 \leq i < j \leq n)) = \{a\}$ and we are done.

Ex 6.29

In affine space, a line L_1 and a point a not on this line, span a plane P_1, and any line which intersect both L_1, a lies on this plane. If we then pick a line L_2 which doesn't intersect on this plane, we see that no line which intersects both L_1, a, also intersects L_2. Any lines L_2 which doesn't intersect P_1 is parallel to L_1, and since parallel lines in general don't intersect in $\mathbb{A}^3$, we see that it's impossible to find a third line L which intersects L_1, L_2, a when L_1, L_2 are parallel.

There is one more case where it's impossible to find such L. Let q be the point where L_2 meets P_1. If q, a spans a line which is parallel to L_1, then it's again impossible to find L which intersects L_1, L_2, a, since such a line would have to contain both q, a, whence it wont meet L_1.

This latter case correspond to when the plane P_2 containing L_2 and a is parallel to L_1.

Projective space is more flexible as we have points at infinity, and parallel lines intersect at infinity. Here if either P_1, L_2 or P_2, L_1 are parallel, then there is a line L through a in each corresponding plane P_i which intersects the other line L_j $i \neq j$ at infinity.

We solve the problem in $\mathbb{A}^4$. Here the lines L_1, L_2 corresponds to planes through the origin spanned by the vectors u_1, u_2 and v_1, v_2 respectively. a corresponds to a line through the origin spanned by a vector, which we will denote by a as well. The planes P_1, P_2 now correspond to three dimensional hypersurfaces spanned by a, u_1, u_2 and a, v_1, v_2 respectively. Their intersection is a plane which contains a, spanned by say a, w. Then the intersection plane meets the hypersurface corresponding to L_1 in at least one point, parameterised by say $sa + tw = s'a + t'u_1 + r'u_2$, and we have that $(s + s')a + tw = t'u_1 + r'u_2$, so the plane $\langle a, w\rangle$ meets the plane of $\langle u_1, u_2\rangle$. The same is of course true of the plane $\langle v_1, v_2\rangle$, hence the line L in $\mathbb{P}^3$ corresponding to $\langle a, w\rangle$ will meet all of a, L_1, L_2.

This line is unique, since the plane $\langle a, w\rangle$ is unique. Indeed, since $L_1 \neq L_2$, we have that the intersection $\langle a, u_1, u_2\rangle \cap \langle a, v_1, v_2\rangle$ is exactly a plane.

Ex 6.30

(a)

The only if part is immediate.

Let R be a homogeneous ring such that for any homogeneous $f, g \in R$, we have $fg = 0$ implies $f = 0$ or $g = 0$, and suppose that $p, q \in R$ are such that $pq = 0$. Write $r = \deg p, s = \deg q$, and let p_r, q_s be the leading homogeneous components. Since a graded ring is the direct sum of it's homogeneous components, every homogeneous component of pq must be zero, in particular the leading component $(pq)_{rs} = p_r q_s$. Our assumption on R then tells us that either $p_r = 0$ or $q_s = 0$, and since these are the leading components, $p = 0$ or $q = 0$, and R is an integral domain.

(b)

Suppose $f, g \in S(X)$ are such that $fg = 0$. Then $X = V(fg) = V(f) \cup V(g)$ and if X is irreducible, we must have either $V(f) = X$ or $V(g) = X$, whence $f = 0$ or $g = 0$ and $S(X)$ must be an integral domain.

Similarly, if X is reducible, let $X = U \cup V$ be a non-trivial decomposition and $f \in I_p(U), g \in I_p(V)$ be non-zero polynomials. Then fg vanishes on both U and V hence on X and $fg = 0$, so $S(X)$ isn't an integral domain.

Ex 6.31

(a)

By Lemma 6.18, any strictly increasing chain of varieties in $\mathbb{P}^n$ corresponds to a chain of cones in $\mathbb{A}^{n+1}$, to which we can append a point immediately after the empty variety a la

$$X_0 = \emptyset \subsetneq \{a\} \subsetneq C(X_1) \subsetneq C(X_2) \subsetneq \ldots \subsetneq C(X_m),$$

hence $\dim C(X) \geq 1 + \dim X$.

For the other direction, we take the opportunity to explore homogeneous coordinate ring and pass to the algebraic side. We provide a sequence of lemmas to this end.

Lemma 0.5. Let X be a projective variety. Then

$$S(X) = A(C(X)).$$

Proof. It follows from Remark 6.17 that $I_p(X) = I_a(C(X))$, after which the equality of the lemma follows from the definitions of homogeneous and affine coordinate rings. □

Lemma 0.6. Let X be a non-empty irreducible projective variety. Then $S(X)$ contains a homogeneous prime ideal J of height 1.

Proof. Let x_i be a non-zero coordinate on $C(X)$. Then (x_i) is a homogeneous ideal in $A(C(X))$. Moreover, (x_i) is prime, as it's the image of a prime ideal under the surjective ring homomorphism $\pi : \mathbb{K}[\boldsymbol{x}] \to A(C(X)), \pi : x \mapsto x + I_a(C(X))$. Finally, (x_i) is its own minimal ideal, and therefore has height at least one by Krull's Principal Ideal theorem, and height exactly one since X is irreducible, and $S(X) = A(C(X))$ is an integral domain so (0) is prime there. □

Corollary 0.7. If X is a projective variety, then $1 + \dim X \leq \dim C(X)$.

Proof. The respective coordinate rings are equal, and by the lemma, $A(C(X))$ contains a maximal chain of prime ideals which are all homogeneous, and since all of these but possibly the irrelevant ideals correspond to irreducible projective varieties, the corollary follows. □

(b)

We can reduce to the case where X, Y are pure dimensional by simply ignoring irreducible components of non-maximal dimension. It follows from part (a) that $\dim C(X) + \dim C(X) \geq n+2$, whence Exercise 5.23 (a) yields

$$\dim C(X) \cap C(Y) \geq \dim C(X) + \dim C(Y) - (n+1) \geq 1.$$

Now, if $x \in C(X) \cap C(Y)$ then $\lambda x \in C(X)$ and $\lambda x \in C(Y)$ for all λ, hence $x \in X \cap Y$ and $x \in C(X \cap Y)$. The other inclusion follows similarly, hence $C(X \cap Y) = C(X) \cap C(Y)$ has dimension at least 1, so it's at least a line in affine space, and least a point in projective space.

Ex 6.36

In the real plane, our surface is given by the graph $x_2 = \pm\sqrt{\frac{1}{x_1} + x_1^2}$ when $x_1 \in \mathbb{R}\backslash(-1, 0]$. For large x_1, we have two asymptotes where $x_2 \to x_1$ and $x_2 \to -x_1$. Hence we expect two points at infinity, namely $a = [0:1:1], b = [0:-1:1]$.

When x_1 goes to 0 from the positive side we see that $x_2 \to \pm\frac{1}{\sqrt{x_1}} \to \pm\infty$. So we might also expect the points at infinity $c = [0:0:1], d = [0:0:-1]$. However, they are not in the closure as we can see bellow. Perhaps because we don't have any points on the curve with $x_1 = 0$?

We see that a, b are the only points at infinity, since $\overline{X} = V_p(x_1^3 - x_1x_2^2 + x_0^3)$, which when intersected with $V_p(x_0)$ becomes $V_p(x_1^2 - x_2^2) = V_p((x_1 - x_2)(x_2 + x_1))$, and consists of a, b exactly.

Ex 7.3

(a)

Let $\mathbb{P}^1$ be the projective line as introduced in Example 5.5 (a), and $\mathbb{P}^{1'}$ be the projective line as given in Definition 6.1, using Notation 6.2, with structure sheaf as in Definition 7.2. We will show that the two are isomorphic.

As a set, we have that $\mathbb{P}^1 = X_0 \coprod X_1 / \sim$ where $X_0 = X_1 = \mathbb{A}^1$ and $x_0 \sim x_1$ for $x_0 \in X_0, x_1 \in X_1$ whenever $x_0, x_1 \neq 0$ and $x_0 = 1/x_1$.

We claim that the function $f : \mathbb{P}^1 \to \mathbb{P}^{1'}$ which sends $f(x_0) = [x_0 : 1], x_0 \in X_0$ and $f(x_1) = [1 : x_0], x_1 \in X_1$ is well-defined and bijective. To show that it's well-defined, let $x_0 \sim x_1$. Then $x_0, x_1 \neq 0$ and

$$f(x_1) = [1 : x_1] = 1/x_1[1 : x_1] = [1/x_1 : 1] = [x_0 : 1] = f(x_0).$$

Now suppose that $f(x_0) = f(x_1)$. This is impossible whenever either $x_0 = 0$ or $x_1 = 0$, so we have $x_0, x_1 \neq 0$. Then

$$[1 : x_1] = [x_0 : 1] = [1 : 1/x_0]$$

hence $x_1 = 1/x_0$ and $x_0 \sim x_1$. Finally, let $x = [x_0 : x_1] \in \mathbb{P}^{1'}$. If $x_0 = 0$, then $x = f(0_0)$, otherwise $x = f(x_1/x_0)$ with $x_1/x_0 \in X_1$. Hence f is a bijective function.

Now suppose that $Z \subseteq \mathbb{P}^1$ is a closed set. Then $Z_0 = Z \cap X_0, Z_1 = Z \cap X_1$ are both closed in $\mathbb{A}^1$. Let $J_0 = I(Z_0), J_1 = I(Z_1)$. Then let $g \in J_1$ and $x_1 \in Z_1$. We have that $g^h([1 : x_1]) = g(x_1) = 0$, hence g^h vanishes on $f(Z_1)$. Similarly, now let $g^h \in J_1^h$ and $[x_0 : x_1] \in V_p(J_0^h)$. First of, every term but the leading term of g^h has a factor x_0, hence $g^h([0 : 1]) \neq 0$ and $x_0 \neq 0$. Thus we can assume $x_0 = 1$. Now, $(g^h)^i = g$, so $g(x_1) = g^{hi}(x_1) = g^h([1 : x_1]) = 0$. It follows that $f(Z_k) = V_p(J_k^h)$ for $k = 0, 1$, and $f(Z) = V_p(J_0^h) \cup V_p(J_1)^h$ is closed.

Now let $Z \subseteq \mathbb{P}^{1'}$ be closed, and suppose $Z = V_p(J)$. Then let $g \in J$, $[1 : x_1] \in Z \cap U_0$. Before we move on, we need some new notation. Let $g^{i_0} = g(1, x_1)$ and $g^{i_1} = g(x_0, 1)$ be the dehomogenizations onto each coordinate. Then $0 = g([1 : x_1]) = g^{i_0}(x_1)$ so $f^{-1}(Z \cap U_0) \subseteq V_a(J^i)$. Now suppose that $x_1 \in V_a(J^{i_0})$. Then again, $g([1 : x_1]) = g^{i_0}(x_1) = 0$, so $f(x_1) \in Z \cap U_0$ and $f^{-1}(Z \cap U_0) = V_a(J^{i_0})$. We get

$$V_a(J^{i_0}) \cup V_a(J^{i_1}) = f^{-1}(Z \cap U_0) \cup f^{-1}(Z \cap U_1) = f^{-1}(Z).$$

Now, let i_0, i_1 be the injections $X_1, X_2 \to \mathbb{P}^1$. Then, $i_1^{-1}(V_a(J^{i_0}))$ is closed in X_1, as it i_1^{-1} leaves the x_1 coordinates unchanged and it's still the vanishing set of J^{i_0}. It follows from the lemma bellow that $i_1^{-1}(V_a(J^{i_1}))$ is closed as well.

Lemma 0.8. Let X_0, X_1, i_0, i_1 be as in the gluing construction of $\mathbb{P}^1$, and $Z \subseteq X_0$ be a closed set. Then $i_1^{-1}(i_0(X_0))$ is closed in X_1.

Proof. Let $f \in V(X_0)$. Then define $\hat{f}(x) = x^{\deg(f)} f(1/x) \in \mathbb{K}[x]$. We have that $V(\hat{f}) \setminus \{0\} = V(\hat{f}(x)) \cap i_1^{-1}(i_0(X_0)) = i_1^{-1}(i_0(V(f)))$, and $f(0) \neq 0$ since f has a constant term which is equal to the leading coefficient of f. It follows that $V(\hat{f}) = i_1^{-1}(i_0(V(f)))$, and by picking generators of X_0, hatting them and intersecting their vanishing sets, we see that $i_1^{-1}(i_0(X_0))$ is closed in X_1. $\square$

It follows that both $V_a(J^{i_k}), k = 1, 2$ are closed in both $X_k, k = 1, 2$, and $f^{-1}(Z)$ is closed in $\mathbb{P}^1$. Hence $f : \mathbb{P}^1 \to \mathbb{P}^{1'}$ is continuous.

It remains to show that f, f^{-1} are morphisms of ringed spaces. So, suppose that g is regular on $\mathbb{P}^{1'}$. Then g is locally the quotient of two homogeneous polynomials of the same degree, and

$$i_0^*(f^*(g)) = g(x_0, 1/x_0)$$

is also locally a quotient of polynomials, hence regular on X_0. Similarly, for X_1 as well. As $f^*(g)$ is regular on both X_0 and X_1, it's regular on $\mathbb{P}^1$.

Now suppose that g is regular on $\mathbb{P}^1$. This, by definition, is the case exactly when both $g_0 = i_0^*(g), g_1 = i_1^*(g)$ are regular. Let $f_0 = f \circ i_0, f_1 = f \circ i_1$. Then

$\mathrm{im}(f_0) = U_1$, (where $U_1 = \{[x_0 : 1] : x_0 \in \mathbb{A}^1\}$).

We have that

$$g \circ f^{-1}\big|_{U_1} = g \circ i_0 \circ i_0^{-1} \circ f^{-1}\big|_{U_1} = g_0 \circ f_0^{-1},$$

so by the gluing property of sheaves, we'll be done if we can show that $g_0 \circ f_0^{-1} : U_1 \to \mathbb{A}^1$ is regular. Locally, we can write $g_0 = p_0/q_0$ as a quotient of polynomials, which in turn is the restriction to U_1 of function which is locally a quotient of homogeneous polynomials $\mathbb{K}[\boldsymbol{x}][x_1, x_2]$ of the same degree via

$$g_0 \circ f_0^{-1}([x_0 : 1]) = g_0(x_0) = \frac{p_0(x_0)}{q_0(x_0)} = \left(\frac{p_0^h}{q_0^h} x_1^{\deg q_0 - \deg p_0}\right)([x_0 : 1]),$$

and we are done.

(b)

Let $\mathcal{O}_X$ be the structure sheaf defined as a closed subvariety via Construction 5.10 (b), and $\mathcal{O}'_X$ defined as in Definition 7.1. Let $U \subseteq X$ be an open set and suppose $f \in \mathcal{O}_X(U)$. Then given some $a \in U$, by Construction 5.10 (b), we have some V open in $\mathbb{P}^n$ containing a, and $g \in \mathcal{O}_{\mathbb{P}^n}(V)$ such that $f\big|_V = g\big|_U$, whence f can be written as a quotient of two homogeneous polynomials on $U \cap V \ni a$. Hence $f \in \mathcal{O}'_X(U)$.

Now suppose that $f \in \mathcal{O}'_X(U)$, and let $a \in U$. Then we have some open $U_a \subseteq U$ containing a such that f can be written as a quotient of two homogeneous polynomials of the same degree on U_a. Hence $f\big|_{U_a} \in \mathcal{O}_{\mathbb{P}^n}(U_a)$, and f satisfies Construction 5.10 b with $V = U_a, \Psi = f\big|_{U_a}$ for all $a \in U$.

Fleshing out the proof of Lemma 7.4

First note that f is continuous, since if $V(g_1, g_2, \ldots, g_m)$ is a closed set in $\mathbb{P}^m$, then

$$f^{-1}(V(g_1, g_2, \ldots, g_m)) = V(g_1 \circ f, g_2 \circ f, \ldots, g_m \circ f)$$

is closed in X, since compositions of quotients of homogeneous polynomials of the same degree are again quotients of homogeneous polynomials of the same degree. It follows that the U_i are open in $\mathbb{P}^n$.

The next problem we deal with is how to apply Proposition 4.7 in the end of the proof. For this we need U_i affine, and I can't see why this is known. Instead, define $V'_i \subset \mathbb{P}^n$ to be an affine cover of $\mathbb{P}^n$, just as we defined V_i to be an affine cover of $\mathbb{P}^m$. Then $U_i \cap V'_j$ is closed in every V'_j, hence affine, and we can apply Proposition 4.7 here and lift back up to U_i with Lemma 4.6 whence $f\big|_{U_i}$ is a morphism.