Ch 1

Ex 1.1.1

We plan to use Proposition B.19. Let $g_r(\zeta) = f(\zeta, re^{i\theta})$ for $r \in [0,1)$. Then let $r_j \in [0,1)$ be some sequence which converges to 1. Each g_{r_j} is holomorphic on $\mathbb{D}$, as f is holomorphic on $\mathbb{D}^2$ and $r_j e^{i\theta} \in \mathbb{D}$. The g_{r_j} converge towards g_1, and we need to show that the g_{r_j} (or some subsequence) converge uniformly on compact subsets of $\mathbb{D}$, after which an application of Proposition B.19 tells us that g_1 is holomorphic.

We will use Theorem B.20 for this. As f is continuous on the compact set $\overline{\mathbb{D}^2}$, it is bounded, and attains some maximal modulus M. It follows that the g_{r_j} are uniformly bounded on compact subsets of $\mathbb{D}$, and Theorem B.20 gives us some some subsequence of g_{r_j} which converges uniformly on compact subsets of $\mathbb{D}$. By the previous paragraph, it follows that g_1 is holomorphic, and g_2 as well by analogous arguments.

Ex 1.1.2

As f is holomorphic in the first variable on Δ_1, and continuous in the first variable on $\overline{\Delta}_1$, it follows from the univariate Cauchy integral formula that

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta_1} \frac{f(\zeta_1, z_2, z_3, \ldots, z_n)}{z_1 - \zeta_1} d\zeta_1.$$

But then for a fix $\zeta_1 \in \partial\Delta_1$, we showed above that $f(\zeta_1, z_2, z_3, \ldots, z_n)$ is holomorphic in the second variable by Exercise 1.1.1. We can apply Cauchy's integral formula again, and get

$$f(z) = \frac{1}{(2\pi i)^2} \int_{\partial \Delta_1} \int_{\partial \Delta_2} \frac{f(\zeta_1, \zeta_2, z_3, \ldots, z_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2.$$

Continuing this way, we see that we eventually get

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2) \ldots (\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_n.$$

Ex 1.1.3

As Γ is compact and f is continuous on Γ, f attains its maximal modulus M for some $z_m \in \Gamma$. It now follows from the Cauchy integral formula that

$$\begin{aligned}
f(z) &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)\ldots(\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_n \\
&\leq \frac{1}{(2\pi i)^n} \int_\Gamma \frac{|f(\zeta_1, \zeta_2, \ldots, \zeta_n)|}{(\zeta_1 - z_1)(\zeta_2 - z_2)\ldots(\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_n \\
&\leq \frac{1}{(2\pi i)^n} \int_\Gamma \frac{M}{(\zeta_1 - z_1)(\zeta_2 - z_2)\ldots(\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_n \\
&= M \frac{1}{(2\pi i)^n} \int_\Gamma \frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)\ldots(\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_n \\
&= M,
\end{aligned}$$

where

$$1 = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)\ldots(\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_n$$

since the RHS is the expansion of the constant function $(z_1, \ldots, z_n) \mapsto 1$ via the Cauchy integral formula.

Ex 1.1.4

Let $f_p(\boldsymbol{z}) = \boldsymbol{z} + p$. Then f_p is holomorphic everywhere. By considering the f_p as a function on real space $\mathbb{R}^{2n} \to \mathbb{R}^2$, we see that the image of f_p is a $2n$ sphere centered about p, which has a strictly furtherest point $2p$ from the origin. Hence the function attains a strict maximum at p for values in the unit ball.

Ex 1.1.5

Fix y and let $f_a(x) = \frac{ax}{x^2+a^2}$. Then if $a \neq 0$, we have f_a differentiable everywhere with derivative

$$f_a'(x) = \frac{a(x^2 + a^2) - 2ax^2}{(x^2 + a^2)^2} = \frac{a^3 - ax^2}{(x^2 + a^2)^2}.$$

If $a = 0$, then $f_a = 0$, and is infinitely smooth everywhere on $\mathbb{R}$. The same story holds when we fix x and partial derivatives in y. Thus we've shown that partial derivatives of f exist everywhere. Moreover, f is locally bounded as $|xy| < |x^2 + y^2|$ for all $x, y \in \mathbb{R}^2$, hence $|f| \leq 1$.

Ex 1.1.6

First suppose that f is locally bounded and that $\zeta \mapsto f(\zeta a + b)$ is holomorphic for every $a, b \in \mathbb{C}^n$ on the set W such that $\zeta \in W$ whenever $\zeta a + b \in U$. Then picking $a = e_k^c = (c_1, c_2, \ldots, c_{k-1}, 1, c_{k+2}, \ldots, c_n)$ and $b = 0$ yields that f is holomorphic in the k-th coordinate when we fix the remaining coordinates to some $c \in \pi_{k^*}(U)$ (where π_{k^*} is the projection onto all but the k-th coordinate). Running over all possible choices of c shows that f is holomorphic in the k-th variable on all points of U, then running through all ks, shows that f is holomorphic in every variable on U. Combining this with the fact that f is locally bounded yields that f is holomorphic on U.

Now suppose that f is holomorphic. Then f is locally bounded by definition. Moreover, $f \circ (\zeta \mapsto a\zeta + b)$ is clearly continuous and differentiable as it is a composition of such functions, and the Wirtinger equation is given by

$$\begin{aligned}\frac{\partial}{\partial \overline{z}} f(a\zeta + b) &= \frac{\partial}{\partial x} f(a\zeta + b) + i \frac{\partial}{\partial x} f(a\zeta + b) \\ &= \sum_{k=1}^{n} f_{x_k}(a\zeta + b)a + i f_{y_k}(a\zeta + b)a \\ &= \sum_{k=1}^{n} a_k \frac{\partial}{\partial \overline{z}} f \bigg|_{a\zeta + b} \\ &= 0.\end{aligned}$$

Ex 1.2.5

First, let's prove the following lemma which allows us to restrct checking normal convergence to checking uniform convergence on polydiscs.

Lemma 0.1. Let D be a domain. Then a sequence of functions f_n is normally convergent on D if it converges uniformly on all closed polydiscs $\overline{\Delta} \subset D$.

Proof. Let $K \subset D$ be a compact subset. Then there is some neighbourhood $U \subset D$ containing D. Write D as a union of open polydiscs. Now pick a finite subset of the polydiscs which cover K. As K is closed, we may pick these polydiscs to be closed as well (either by adding the boundary, or by shrinking them). Denote this cover of K by closed polydiscs as $\overline{\Delta}_n, n \in [1..N]$

Now, let $\epsilon > 0$. As the f_n converge uniformly on each $\overline{\Delta}_n, n \in [1..N]$, to say g_n, we can pick N_n for each $n \in N$ such that $f_n - g_n < \epsilon$ for all $n > N_n$. Let $M = \max_{N_n}$. Then $f_n - g < \epsilon$ for all $n > M$ where $g = g_n$ on each $\overline{\Delta}_n$, and so f_n converges uniformly on K. □

We are now ready to prove the theorem.

Proof. Let $\overline{\Delta}$ be a polydisc in K. Then

$$\left| \frac{df}{dz_k}(z) \right| \leq \frac{1}{\rho_k} \|f\|_{\Gamma}$$

for all $z \in \overline{\Delta}$. Uniform convergence is equivalent to uniform cauchy convergense, and so if we let $F_n = f_{n+1} - f_n$, we see that uniform cauchy convergence of the f_n imply that $F_n \to 0$ uniformly. It follows that

$$\left| \frac{df_{n+1}}{dz_k}(z) - \frac{df_n}{dz_k}(z) \right| \leq \frac{1}{\rho_k} \|f_{n+1} - f_n\|_\Gamma \to 0$$

uniformly, hence the derivatives are uniformly cauchy convergent, and so uniformly convergent.

It now follows that the derivatives of the f_n converge to the derivative of f, as in Exercise 1.2.3, and so f is holomorphic just as in Exercise 1.2.3.

We may now repeat the procedure, taking derivatives of the sequence of derivatives, as many times as we need, to obtain the desired result. □

Ex 1.2.13

Proof. Pick two points, $a, b \in \mathbb{C}^n$ and let $c = f(a)$. Consider the function which fixes the last n coordinates of f to a, namely $z_1 \mapsto f(z_1, a_2, \ldots, a_n)$. This is a bounded holomorphic function $\mathbb{C} \to \mathbb{C}$, hence constant by Liouville's Theorem in one variable, and so $f(b_1, a_2, \ldots, a_n) = c$. Now consider the function, $z_2 \mapsto f(b_1, z_2, a_3, \ldots, a_n)$. Again, this is a constant function and so $f(b_1, b_2, a_3, \ldots, a_n) = c$. Continuing this way we see that $f(a) = f(b)$. We've shown that f is constant on two arbitrary points in $\mathbb{C}^n$, whence f is constant. □

Ex 1.2.15

Proof. First of, it's enough to prove uniform convergence on polydiscs, since any compact set can be covered by finitely many polydiscs.

$\mathbb{C}^n$ is separable as it's homeomorphic to a countable product of $\mathbb{R}$. Thus, by Arzela-Ascoli, it will suffice to show that the sequence is equicontinuous.

Let M be a bound on the f_j. Then for any $f = f_j$ and a, b in some polydisc

$\Delta \subset U$, we have

$$\begin{aligned}
|f(a) - f(b)| &= \left| \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - a} d\zeta - \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - b} d\zeta \right| \\
&= \left| \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)(\zeta - b) + f(\zeta)(\zeta - a)}{(\zeta - a)(\zeta - b)} d\zeta \right| \\
&= \left| \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)(b - a)}{(\zeta - a)(\zeta - b)} d\zeta \right| \\
&\leq \left| \frac{M(b-a)}{(2\pi i)^n} \int_\Gamma \frac{1}{(\zeta - a)(\zeta - b)} d\zeta \right| \\
&\leq \left| \frac{M(b-a)}{\rho_1 \rho_2 \dots \rho_n} \right|.
\end{aligned}$$

We can pick a neighbourhood U about a which is small enough such that for all $b \in U$, we have

$$\left| \frac{M(b-a)}{\rho_1 \rho_2 \dots \rho_n} \right| \leq \epsilon$$

for any $\epsilon > 0$, and so the sequence is equicontinuous. □

Ex 1.2.15

Suppose that U is disconnected, and that V is a connected component of U. Then we can pick two holomorphic non-zero functions, one which is non-zero on V, and zero on $U \setminus V$, and one which is zero on V and non-zero on $U \setminus V$, and their product is the zero function.

Now suppose that U is connected, and that $f, g \in \mathcal{O}(U)$ are such that $fg = 0$, or in other words, $V(f) \cup V(g) = U$. If $f \neq 0$, then the vanishing set of f, $V(f) = \{p \in U : f(p) = 0\}$ does not contain any open set by the uniqueness theorem. It follows that every single open subset of U intersects $V(g)$, since otherwise it would be contained in $V(f)$ per $V(f) \cup V(g) = U$. Thus $V(g)$ is dense in U. But $V(g)$ is also closed as g is continuous, hence $V(g) = U$ and $g = 0$.

Ex 1.4.3

Suppose that $f(\partial U) \subset \partial V$. Then let $\{p_k\}$ be a sequence of points in U converging to $p \in \partial U$. As f is continuous, the points $g(p_k) = f(p_k)$ converge to $f(p)$ which lies in ∂V by hypothesis. Hence g is proper by Lemma 1.4.7.

Suppose now that g is proper, and let $p \in \partial U$. Let $\{p_k\}$ be a sequence of points in U converging to p. As f is continuous, the points $g(p_k) = f(p_k)$ converge to $f(p)$, and as g is proper, $f(p) \in \partial V$. As p was an arbitrary point in ∂U, it follows that $f(\partial U) \subset \partial V$

Ex 1.4.7

Ex 1.4.8