Ch 1

Ex 1.2.B

Let $a, b \in \text{Mor}(A, A)$ be invetible morphisms. Then (ab) is invertible with inverse $(ab)^{-1} = b^{-1}a^{-1}$, a^{-1} is invertible with inverse a, and the identity morphism is it's own inverse. Hence the set of invertible morphisms $A \to A$ form a group.

In Example 1.2.2, $\text{Aut}(A, A)$ is the set of bijections on A, hence the symmetric group S_A.

In Example 1.2.3, $\text{Aut}(V, V)$ is the set of invertible linear maps $V \to V$.

Let $A, B \in \mathcal{C}$ be two objects in some category, isomorphic via $f : A \to B$. Then if $a, b \in \text{Aut}(A)$, we have

$$\begin{aligned} e_B &= ff^{-1} = fe_Af^{-1} \\ fabf^{-1} &= faf^{-1}fbf^{-1} \\ (faf^{-1})^{-1} &= (f^{-1})^{-1}a^{-1}f^{-1} = fa^{-1}f^{-1}, \end{aligned}$$

hence $a \mapsto faf^{-1}$ is a group homomorphism $\text{Aut}(A) \to \text{Aut}(B)$. It's an isomorphism since its inverse is given by $a' \mapsto f^{-1}a'f$.

Ex 1.2.C

First of, let $\phi : A \to S^{-1}A$ be the canonical homomorphism. Then $a \in \ker(\phi)$ exactly when there is some $s \in S$ such that $s(a \cdot 1 - 0 \cdot 1) = 0$, and such a exists if and only if some $s \in S$ is a zero divisor.

Ch 2

Ex 2.2.J

$\mathcal{F}_p$ is the set of pairs (m, U) with U open in X containing p and $m \in \mathcal{F}(U)$ modulo the relation that if (m', U') is another such pair, and $V \in U \cap U', p \in V$ such that $m\big|_U = m'\big|_{U'}$, then $(m, U) \sim (m', U')$.

Now, let $(r, U) \in \mathcal{O}_p$. Then define $(r, U)(m, U) = (rm, U)$. This action is well defined, as if $(r', U') \sim (r, U)$ and $r = r'$ in $\mathcal{O}(V)$, then as $\mathcal{F}(V)$ is an $\mathcal{O}(V)$-module, $rm = r'm$ in $\mathcal{F}(V)$, and $(rm, V) \sim (r'm, V)$.

Ex 2.3.A

Let $\phi_p : (m, U) \to (\phi(m), U)$. Then ϕ_p is well defined, as if $(m, U) \sim (m', U')$ with $m = m'$ in $\mathcal{F}(V)$, then $\phi(m) = \phi(m')$ in $\mathcal{G}(V)$ since the morphisms in a morphism of sheaves commute with restrictions.

Ex 2.3.C

Given $V \subset U$, any morphism of sheaves $\mathcal{F}\big|_U \to \mathcal{G}\big|_U$ contains the data of a morphism of sheaves $\mathcal{F}\big|_V \to \mathcal{G}\big|_V$, so restrictions in $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ may be defined in this natural way, and it's clear that they compose as they should.

We now prove the identity axiom. Suppose that U is open in X and that $U_i, i \in I$ is an open cover of U such that $f, g \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})(U)$ agree on all U_i. Then let $V \subset U$ be open and $x \in \mathcal{F}(U)$. As f, g agree on all U_i, they agree on all $V \cap U_i$, hence $f\big|_{V\cap U_i}(x) = g\big|_{V\cap U_i}(x)$ in $\mathcal{G}(V \cap U_i)$ for all $i \in I$. Then commutativity of the morphisms tells us that $f(x)\big|_{V\cap U_i} = g(x)\big|_{V\cap U_i}$ on these sets, whence the identity axiom on $\mathcal{G}$ tells us that $f(x) = g(x)$. Hence $f = g$ on any open set $V \subset U$, and we have verified identity on $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

We now prove the gluability axiom. Suppose that U is open in X and that $U_i, i \in I$ is an open cover of U. Furthermore, suppose that $f_i \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})(U_i), i \in I$ is a set of sheaf morphisms that agree on the intersections $U_i \cap U_j$. Then, given some V open in U and $h \in \mathcal{F}(V)$, we have that the f_i all agree on the intersections $V \cap U_i$, hence $f_i(h\big|_{V\cap U_i})$ agrees on all intersections, and by the gluability axiom there exists a $h' \in \mathcal{G}(V)$ which restricts down to $f_i(h)$ in all $V \cap U_i$, and furthermore, identity tells us that this h' is unique. We can now define $f \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})(V)$ as $f : h \to h'$, and this f will restrict to each f_i on each U_i.

Ex 2.3.I

We show identity and glueing, beginning with identity.

Let U be open in $\ker(\phi)$ and $U_i, i \in I$ be an open cover of U. Let $f, g \in \ker(\phi)(U)$ be such that $f\big|_{U_i} = g\big|_{U_i}$ in every $\ker(\phi)(U_i), i \in I$. Then $f = g$ by the identity property in $\mathcal{F}$ and injectivity of $\ker(\phi)$ into $\mathcal{F}$.

We now show the glueing property. Let U be open in $\ker(\phi)$ and $U_i, i \in I$ be an open cover of U. Furthermore, suppose $f_i \in \ker(\phi)(U_i), i \in I$ is a set of elements which agree when restricted to intersections $U_i \cap U_j$. We can glue these f_i in $\mathcal{F}$ to find a $f \in \mathcal{F}(U)$ which restricts to f_i on each U_i. It remains to show that $f \in \ker(\phi)(U)$. To see this, note that $\phi(f)$ restricts to 0 on each U_i, since f restricts to f_i which lies in the kernel of ϕ. It follows that $\phi(f) = 0$ by the identity property of $\mathcal{G}$, and we are done.

Ex 2.4.A

Suppose that $f \in \mathcal{O}(U)$ is 0 at all stalks $p \in U$. Then for every $p \in U$, there exist some open set V_p containing p such that $f = 0$ in $\mathcal{O}(V_p)$. Then $V_p \cap U$ form an open cover of U, and $f = 0$ by the identity property.

Ex 2.4.B

Suppose that $(s_p)_{p\in U}$ is a set of compatible germs. Then we can find an open cover $U_i, i \in I$ of U and sections $f_i \in \mathcal{O}(U_i)$ such that the germ of any f_i at any point $q \in U_i$ is s_q. The f_i must then agree on intersections $U_i \cap U_j$ by Exercise 2.4.A, for they induce the same germs $s_q, q \in U_i \cap U_j$ here. It follows that we can glue the f_i to an $f \in \mathcal{O}(U)$ which restricts to the f_i, and this f has germ s_q at any point $q \in U$, and we are done.

Ex 2.4.C

If we show that the given diagram (2.4.3.1) commutes then we are done, as it would follow that if

$$\phi_1((s_p)_{p\in U}) = \phi_2((s_p)_{p\in U}),$$

then $\phi_1(s) = \phi_2(s)$ since sections are determined by germs (Exercise 2.4.A).

To see that the diagram commutes we just write out all the definitions. Let V be open in U and $s \in \mathcal{O}(V)$. Then

$$\phi(s_p) = \overline{(\phi(s), V)}_p = \phi(s)_p.$$

Ex 2.4.D

We begin with injectivity. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on the space X. Then if U is open in X and $s \in U$, we have $\phi(u) = 0$ in U if and only if the germ of $\phi(U)$ is 0 on every point of U. Hence $u \in \ker(\phi)$ if and only if $u \in \ker(\phi_p)$ for all $p \in U$. Hence a morphism of sheaves is injective if and only if it is injective on stalks.

We move on and treat surjectivity. Let $\phi : \mathcal{F} \to \mathcal{G}$ be an injective morphism of sheaves. Then if ϕ is surjective, and $\overline{(s, U)}$ is some germ at the stalk $\mathcal{G}(U)_p$, there exist some $s' \in \mathcal{F}(U)$ in the preimage of ϕ, hence $\phi_p(\overline{(s', U)}) = \overline{(\phi(s'), U)} = \overline{(s, U)}$ and the morphism is surjective at stalks as well.

For the final case, suppose that ϕ is injective on the level of sheaves, and is surjective at every stalk. Then by the previous part, we know that every ϕ_p is injective as well, hence an isomorphism. We will show that a compatible set of germs in $\mathcal{G}$ pull back to a compatible set of germs in $\mathcal{F}$ under ϕ, and doing this will require diligent bookkeeping.

Let U be open in X, and $(a_p)_{p\in U}$ be a compatible set of germs in $\mathcal{G}$. As ϕ_p is an isomorphism for every $p \in U$, we may write $\phi_p((b_p)_{p\in U}) = (a_p)_{p\in U}$ where $b_p \in \mathcal{F}_p$. Picking representatives, let $b_p = \overline{(c_p, U_p)}$ where $c_p \in \mathcal{F}(U_p)$. Then as Diagram 2.4.3.1 commutes (we showed this in Exercise 2.4.C), we have

$$\overline{(\phi(c_p), U_p)} = a_p.$$

Now, as $(a_p)_{p\in U}$ is a compatible set of germs, there exist an open cover $V_i, i \in I$ and $g_i \in \mathcal{G}(V_i)$ such that the germ $\overline{g_i}_p$ coincides with $a_p = \overline{(\phi(c_p), U_p)}$ for all $p \in V_i$. I.e, there exist some open $W_p^i \subseteq V_i \cap U_p$ where $\phi(c_p) = g_i$. Since there exist such a open set for every $p \in U$, just picking one for every $p \in U$ is enough to form an open cover of U, and forgetting the i-superscript, the W_p form an open cover of U, and for every $q \in W_p$ we have that

$$\overline{\phi(c_p)}_q = a_q = \phi_q(b_q),$$

whence commutativity of 2.4.3.1 yields

$$\phi_q(\overline{(c_p, W_p)}_q) = \overline{\phi(c_p)}_q = \phi_q(b_q),$$

and injectivity of ϕ_q that

$$\overline{(c_p, W_p)}_q = b_q.$$

To summarize, we have an open cover $W_p, p \in U$ of U, and sections $c_p \in \mathcal{F}(W_p)$ such that the germs of the c_p coincide with b_q for all $q \in W_p$. I.e, $(b_p)_{p\in U}$ is a compatible set of germs!

It now follows from Diagram 2.4.3.1 and Exercise 2.4.B that ϕ is surjective.

Ch 3

Ex 3.2.A

(a)

Let $R = \mathbb{K}[\epsilon]/(\epsilon^2)$. The set of prime ideals of R is the set of prime ideals of $\mathbb{K}[\epsilon]$ containing (ϵ^2). Hence $\mathrm{Spec}(R) = \{(\epsilon)\}$.

(b)

Let $R = \mathbb{K}[x]_{(x)}$. The set of prime ideals of R is the set of prime ideals of $\mathbb{K}[x]$ contained in (x). Hence $\mathrm{Spec}(R) = \{(x), (0)\}$.

Ex 3.2C

Let $R = \mathbb{Q}[x]$. Then R is a PID, hence prime ideals $\mathfrak{p} \in \mathrm{Spec}(R)$ correspond to irreducible polynomials in $\mathbb{Q}[x]$. I don't know any nice way to describe all irreducible polynomials over $\mathbb{Q}[x]$, but we certainly still do have all prime $(x - q) \in \mathrm{Spec}(R)$ for $q \in \mathbb{Q}$ which trace out a line.

Ex 3.2D

Suppose towards a contradiction that the set $Q \subset \mathbb{K}[x]$ of irreducible polynomials is finite, and let $p = 1 + \prod_{q\in Q} q$. Then $p \equiv 1 \mod q$ for all $q \in Q$, hence $p \notin$

(q). Since p isn't a unit (indeed it's a polynomial of degree $\deg(p) = \sum \deg(q)$), it's contained in some maximal ideal $p \in (q')$. But then q' is irreducible, and $q' \notin Q$, a contradiction!

Ex 3.2E

Since $\dim(\mathbb{C}[x, y]) = 2$, any non-maximal non-zero prime ideal has to have codimension 1. We claim that every prime $\mathfrak{p}$ of codimension 1 in a UFD is a principal ideal.

Let R be a Noetherian UFD (don't think we need Noetherian, but it simplifies things). Any prime ideal of R can be generated by irreducible elements, since if $\mathfrak{p} = (f_1, f_2, \ldots, f_k)$, then some irreducible factor of every f_i lies in $\mathfrak{p}$ by primality, and these factors generate $\mathfrak{p}$.

Now suppose that $\mathfrak{p} = (f_1, f_2, \ldots, f_k)$ is a prime ideal in R with $k > 1$ and the f_i form a minimal set of irreducible generators. Then each (f_i) is a prime ideal of codimension 1 strictly contained in $\mathfrak{p}$, and we see that $\mathfrak{p}$ has codimension ≥ 1.

I.e, if $\mathfrak{p}$ is some prime ideal of codimension 1, then it must have a minimal set of irreducible generators that only has one element.

It follows that every prime ideal of codimension 1 in $\mathbb{C}[x, y]$ is generated by an irreducible polynomial.

Ex 3.2.F

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{K}[\boldsymbol{x}]$. Then $\mathbb{K} \subset \mathbb{K}[\boldsymbol{x}]/\mathfrak{m}$ is a field extension which is finitely generated as an algebra (by $x_1, \ldots, x_n$), hence $\mathbb{K}[\boldsymbol{x}]/\mathfrak{m}$ has finite dimension as a $\mathbb{K}$-space by the Nullstellensatz. Since $\mathbb{K}$ is algebraically closed, and finite dimensional extensions are algebraic, we must have $\mathbb{K} \cong \mathbb{K}[\boldsymbol{x}]/\mathfrak{m}$ as fields. As $\mathbb{K} \subseteq \mathbb{K}[\boldsymbol{x}]/\mathfrak{m}$, we have an isomorphism $\mathbb{K} \to \mathbb{K}[\boldsymbol{x}]/\mathfrak{m}$ which is the identity on $\mathbb{K}$, as this is surjective, it follows that $x_i = a_i$ in $\mathbb{K}[\boldsymbol{x}]/\mathfrak{m}$ for all x_i, hence $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$.

Ex 3.2.G

Since ideals of A are subspaces, A must satisfy both chain conditions. Hence it's Artinian and Noetherian. The result now follows since Artinian domains are fields.

Ex 3.2H

First of, the ideal $I = (x^2 - 2, y^2 - 2)$ is not maximal, or even prime. To see this, note that $(x + y)(x - y) = x^2 - y^2 = -2 + 2 = 0 \in \mathbb{Q}[x, y]/I$

$(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$ corresponds to the maximal ideal $(x-y, y^2-2)$ with residue field $\mathbb{Q}[x,y]/(x-y, y^2-2) \cong \mathbb{Q}[\sqrt{2}]$.

$(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$ corresponds to the maximal ideal $(x+y, y^2-2)$ with residue field $\mathbb{Q}[x,y]/(x+y, y^2-2) \cong \mathbb{Q}[\sqrt{2}]$.

The two residue fields are isomorphic, but not equal, and they differ in whether $x = y$ or $x = -y$.

Ex 3.2.L

We begin by showing that localizations and quotients commute. This follows from the fact that localization is an exact functor, so both the following sequences are exact,

$$0 \longrightarrow S^{-1}I \longrightarrow S^{-1}R \longrightarrow S^{-1}(R/I) \longrightarrow 0$$

$$0 \longrightarrow S^{-1}I \longrightarrow S^{-1}R \longrightarrow (S^{-1}R)/(S^{-1}I) \longrightarrow 0,$$

hence the two rightmost modules/rings are isomorphic.

Now, in $\mathbb{C}[x,y]_x$ we have that $(xy)_x = (y)_x$. Hence $\mathbb{C}[x,y]_x/(xy)_x \cong \mathbb{C}[x]_x/(0)_x \cong C[x]_x$, where the last isomorphism follows since $C[x]_x$ is a domain. The sought after isomorphism now follows by applying that quotients and localizations commute.

Example 3.2.11

So we have

$$\phi : \mathrm{Spec}(\mathbb{C}[a,b]/(b-a^2)) \to \mathrm{Spec}(\mathbb{C}[x,y,z]/(y-x^2, z-y^2))$$

which sends a maximal ideal

$$\phi : (a-k, b-k^2) \mapsto (x-k, y-k^2, z-k^4).$$

Now consider the map

$$\psi : \mathbb{C}[x,y,z]/(y-x^2, z-y^2) \to \mathbb{C}[a,b]/(b-a^2)$$

which sends

$$\psi : x \mapsto a,\ \psi : y \mapsto b,\ \psi : z \mapsto b^2.$$

Then (as ψ is an algebra homomorphism and fixes $\mathbb{K}$)

$$\psi^{-1}(a-k) = x-k,\ \psi^{-1}(b-k^2) = y-k^2,\ \psi^{-1}(b^2-k^4) = z-k^4,$$

and we see that ψ^{-1} coincides with ϕ on maximal ideals.

Ex 3.2.O

Let $\phi : \mathbb{C}[y] \to \mathbb{C}[x]$ be the $\mathbb{C}$-algebra morphism given by $\phi(y) = x^2$. Then $\phi^{-1}(x^2 - a) = y - a$, and as $x^2 - a = (x - \sqrt{a})(x + \sqrt{a})$, we have $[(y - a)] = \phi^{-1}([x - \sqrt{a}])$ and $[(y - a)] = \phi^{-1}([x + \sqrt{a}])$. Moreover, if $\mathfrak{p}$ is a prime ideal in $\mathbb{C}[x]$ such that $[(y - a)] = \phi^{-1}(\mathfrak{p})$, then $\mathfrak{p}$ must contain $x^2 - a$, and as $\mathbb{C}[x]$ is a PID, we see that we've already found all such $\mathfrak{p}$.

Ex 3.2.P

(a)

We prove the statement for general rings A, B and a morphism $\phi : B \to A$. Note that if $\phi(J) \subseteq I$, then ϕ induces a map $B/I \to A/J$, and we get a map $\operatorname{Spec}(A/J) \to \operatorname{Spec}(B/I)$.

(b)

Suppose that $\phi : \mathbb{K}[y_1, y_2, \ldots, y_m] \to \mathbb{K}[\boldsymbol{x}]$ be given by $\phi : y_i \mapsto f_i$. Then let $a = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$. Then $\phi(y_i - f_i(a_1, a_2, \ldots, a_n)) = f_i(x_1, x_2, \ldots, x_n) - f_i(a_1, a_2, \ldots, a_n) \in a$ (the containment in a follows after evaluating at $(a_1, a_2, \ldots, a_n)$). Hence

$$(y_1 - f_1(a_1, a_2, \ldots, a_n), y_2 - f_2(a_1, a_2, \ldots, a_n), \ldots, y_m - f_m(a_1, a_2, \ldots, a_n)) \subseteq \phi^{-1}(a),$$

and equality follows from $(y_1 - f_1(\boldsymbol{a}), y_2 - f_2(\boldsymbol{a}), \ldots, y_m - f_m(\boldsymbol{a}))$ being a maximal ideal.

Ex 3.2.Q

The fiber $\pi^{-1}([p])$ consists of the set of all ideals in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ containing p. Any such ideal can be generated by p along with polynomials where all coefficients are $< p$, since we can subtract any terms of the form pkx^d. This gives us a map $\pi^{-1}([p]) \to \operatorname{Spec}(\mathbb{F}_p[x_1, x_2, \ldots, x_n])$.

Moreover, this map is invertible as follows. Given some ideal $I \in \operatorname{Spec}(\mathbb{F}_p[x_1, x_2, \ldots, x_n])$, and generators $f_1, f_2, \ldots, f_m$ for I - pick representatives $\overline{f}_i$ for each f_i in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$, and consider the ideal $J = (p, \overline{f}_1, \ldots, \overline{f}_n)$.

It's easy to see that these two maps are mutually inverse each other.

Ex 3.4.F

In $R \to R/I$, we have that the preimage of $\sqrt{0}$ is $\sqrt{I}$, and as $\sqrt{0}$ is the intersection of all prime ideals in R/I, which in turn correspond to all prime ideals in R containing I, the statement follows.

Ex 3.4.G

In a PID, all non-zero primes are maximal and are in one to one correspondence with irreducible elements. Hence, $\mathbb{A}^1_k$ corresponds to all irreducible polynomials in $\mathbb{K}[x]$. Given some set of functions $S \subset \mathbb{K}[x]$, $V(S)$ consists of all the common prime factors of elements in S. I.e $V(S)$ is a finite set of points. So, the situation is much like that of $\mathbb{A}^1_{\mathbb{C}}$, where the only closed subsets are finite sets of (non-zero) points, and no closed set other than the whole space contains (0). Hence every open set other than $\emptyset$ contains (0) as claimed in the exercise.

Ex 3.4.H

Let $V(I)$ be a closed set in B. Our objective is to show that $(\phi^*)^{-1}V(I)$ is a closed set in A. We have

$$\begin{aligned}(\phi^*)^{-1}(V(I)) &= \{\mathfrak{p} \in \mathrm{Spec}(A) : \phi^*(\mathfrak{p}) \in V(I)\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(A) : \phi^{-1}(\mathfrak{p}) \in V(I)\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subseteq \phi^{-1}(\mathfrak{p})\}.\end{aligned}$$

Now, if I is contained in $\phi^{-1}(\mathfrak{p})$, then certainly the ideal generated by $\phi(I)$ in A will be contained in $\mathfrak{p}$ and vice versa, so we have

$$\{\mathfrak{p} \in \mathrm{Spec}(A) : I \subseteq \phi^{-1}(\mathfrak{p})\} = \{\mathfrak{p} \in \mathrm{Spec}(A) : \phi(I)A \subseteq \mathfrak{p}\} = V(\phi(I)A)$$

which is a closed set in A.

Ex 3.4.I

(a)

$\mathrm{Spec}(B/I)$ corresponds to all prime ideals of B containing I, hence is given by $V(I)$. Meanwhile, $\mathrm{Spec}(B_f)$ contains all prime ideals of $\mathrm{Spec}(B)$ which don't meet f, I.e $\mathrm{Spec}(B)\backslash V(f)$. Finally, $\mathrm{Spec}(\mathbb{Q}) = \{(0)\}$ which is neither closed nor open as a set in $\mathrm{Spec}(\mathbb{Z})$, it's closure is all of $\mathrm{Spec}(\mathbb{Z})$, whilst it's complement isn't closed since the closed subsets of $\mathrm{Spec}(\mathbb{Z})$ are either the whole space or finite.

(b)

Let $V(J) \subseteq \mathrm{Spec}(B/I)$ be a closed subset, and suppose that $\mathfrak{p} \in V(J) = V(J \cup \{0\})$. Then $\phi^{-1}(J) \subseteq \phi^{-1}(\mathfrak{p})$, so $\phi^{-1}(\mathfrak{p}) \in V(\phi^{-1}(J \cup \{0\})) = V(\phi^{-1}J) \cap V(I)$. Now let $\mathfrak{q} \in V(\phi^{-1}(J) \cap V(I))$. Then $\phi(\mathfrak{q})$ is a prime ideal in B/I which contains $\phi\phi^{-1}(J)$, and $\phi\phi^{-1}(J) = J$ since ϕ is surjective. Hence $\phi(\mathfrak{q}) \in V(J)$.

Let $V(J) \subseteq \mathrm{Spec}(B_f)$ be a closed subset, and suppose that $\mathfrak{p} \in V(J)$. Then $\phi^{-1}(\mathfrak{p})$ is a prime ideal in B which contains $\phi^{-1}(J)$ and doesn't meet f, I.e it lies in $V(\phi^{-1}(J)) \setminus V(f)$. Similarly, let $\mathfrak{q} \in V(\phi^{-1}(J)) \setminus V(f)$. Then $\phi(\mathfrak{q})$ is a prime ideal in B_f which contains $\phi\phi^{-1}(J)$. But every ideal of B_f is an extended ideal (Atiyah Macdonald - Prop 3.11.i), so the ideal generated by $\phi\phi^{-1}(J)$ is J (AMD - Prop 1.17.iii), hence $J \subseteq \phi(\mathfrak{q})$ and $\phi(\mathfrak{q}) \in V(J)$.

Ex 3.4.J

f vanishes on $V(I)$ if and only if it vanishes on all prime ideals containing I if and only if it's contained in all prime ideals containing I if and only if it's contained in $\sqrt{I}$ by Exercise 3.4.F.

Ex 3.5.A

Let $U \in \mathrm{Spec}(A)$ be an open set, and suppose that $U = \mathrm{Spec}(A) \setminus V(J)$ (Any closed set can be written as $V(J)$ by Ex 3.4.C with J an ideal). Then let $f_i, i \in I$ be some generating set of J. Then

$$U = \mathrm{Spec}(A) \backslash V\left(\bigcup_{i\in I} f_i\right) = \mathrm{Spec}(A) \backslash \bigcap_{i\in I} V(f_i) = \bigcup_{i\in I} (\mathrm{Spec}(A) \setminus V(f_i)) = \bigcup_{i\in I} D_{f_i}.$$

Ex 3.5.B

From the previus exercise, we have that

$$\bigcup_{i\in I} D_{f_i} = \mathrm{Spec}(A) \setminus V\left(\bigcup_{i\in I} f_i\right).$$

Now, $A = (f_i, i \in I)$, if and only if $V(f_i, i \in I) = \emptyset$ (using Zorn's Lemma) if and only if $\bigcup_{i\in I} D_{f_i} = \mathrm{Spec}(A)$.

Ex 3.5.C

We have that $(f_i, i \in I) = A$ if and only if there are finitely many $a_i \in A$, $i \in J$ such that $\sum_{i\in J} a_i f_i = 1$, and these $f_i, i \in J$ therefore generate A as an ideal, so $\bigcup_{i\in J} D_{f_i} = A$ by Ex 3.5.B.

Ex 3.5.D

If $\mathfrak{p} \in D(f) \cap D(g)$ then $\mathfrak{p}$ contains neither f, g and primality, not fg hence $\mathfrak{p} \in D(fg)$. The other direction is trivial.

Ex 3.5.E

$D(f) \subset D(g)$ if and only if all prime ideals not meeting f also don't meet g if and only if all prime ideals containing g contain f as well if and only if $f \in \sqrt{(g)}$ (all prime ideals containing g contain (g), and the intersection of all prime ideals containing (g) is $\sqrt{(g)}$).

Now, $f \in \sqrt{g}$ if and only if $f^n \in (g)$ if and only if $gh = f^n$ if and only if $\frac{gh}{f^n} = 1$ in A_f if and only if g is invertible in A_f.

Ex 3.6.A

Let $\mathfrak{p}$ be a prime ideal in $A = A_1 \times A_2$. We will show that $\mathfrak{p}$ is of the form $\mathfrak{p}_1 \times A_2$ or $A_1 \times \mathfrak{p}_2$ for some arbitrary prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ in A_1, A_2 respectively. Suppose that $(a, b) \notin \mathfrak{p}$. Then $(a, 0)$ and $(0, b)$ can't both simultaneously lie in $\mathfrak{p}$, and we assume with out loss of generality that $(a, 0) \notin \mathfrak{p}$. Now suppose that $(a', b') \in \mathfrak{p}$. Then $(a, 0)(a', b') = (aa', 0) \in \mathfrak{p}$. Moreover, for any $b'' \in A_2$, we have $(a, 0)(a', b'') = (aa', 0) \in \mathfrak{p}$, and as $\mathfrak{p}$ is prime, we must have $(a', b'') \in \mathfrak{p}$. We've shown that if $(a', b') \in \mathfrak{p}$, then $(a', b'') \in \mathfrak{p}$ for any $b'' \in A_2$, hence $\mathfrak{p}$ is of the form $\mathfrak{p}_1 \times A_2$ for some set $\mathfrak{p}_1$. Moreover, $\mathfrak{p}_1$ must be a prime ideal in A_1, since $\mathfrak{p}_1 \times \{0\} \subset \mathfrak{p}_1 \times A_2$, which allows us to utilize primality of $\mathfrak{p}$ in A to obtain primality of $\mathfrak{p}_1$ in A_1.

This argument may be replicated to the case where A is the product of $A_i, i \in [1..n]$ to show that any prime ideal in A is of the form $A_1 \times A_2 \times \ldots \times \mathfrak{p}_j \times \ldots \times A_{n-1} \times A_n$.

This gives us a bijection between $\text{Spec}(A)$ and $\coprod_{i=1}^n \text{Spec}(A_i)$, which clearly respects inclusions, hence a homeomorphism.

Ex 3.6.B

(a)

Let $U \subset X$ be a non-empty open set in the irreducible topological space X. Then let $V = X \setminus U$. We have that $V \cup U = X$, hence $V \cup \overline{U} = X$. As U is non-empty, $V \neq X$, whence $\overline{U} = X$ by irreducibility of X.

(b)

Suppose that $Z = (A \cap Z) \cup (B \cap Z)$ with A, B closed in X (this is exactly what a reduction of Z into two closed subsets is in the subspace topology). Then $A \cup B$ is closed in X, and contains Z. Hence $\overline{Z} \subseteq A \cup B$ since $\overline{Z}$ is the smallest closed set containing Z. It follows that $\overline{Z} = (A \cap \overline{Z}) \cup (B \cap \overline{Z})$.

Now, Z is irreducible if and only if whenever $Z = (A \cap Z) \cup (B \cap Z)$ we have either $Z = (A \cap Z)$ or $Z = (B \cap Z)$, which is equivalent to $Z \subseteq A$ or $Z \subseteq B$. But again, as $\overline{Z}$ is contained in every closed subset containing Z, this is equivalent to $\overline{Z} \subseteq A$ or $\overline{Z} \subseteq B$, I.e irreducibility of $\overline{Z}$.

Ex 3.6.C

If A is an integral domain, then every prime ideal contains the minimal prime ideal (0). Since all closed subsets of spectra are of the form $V(S)$, it follows that any closed subset $V \subset \operatorname{Spec}(A)$ which contains $[(0)] \in V$, must contain every other prime ideal as well. Now, $V \cup U = \operatorname{Spec}(A)$ with V, U closed implies either $V = \operatorname{Spec}(A)$ or $U = \operatorname{Spec}(A)$ since one of these sets must contain $[(0)]$.

Ex 3.6.D

Proving the contrapositive is trivial. If $X = U \cup V$ where U, V are non-trivial, open and disjoint, then U, V are closed and non-trivial as well, hence X is reducible.

Ex 3.6.E

The union of the x and y-axes in $\mathbb{A}^2_{\mathbb{C}}$ seems like a good candidate. The corresponding ring is given by $A = \mathbb{C}[x, y]/(xy)$. This is reducible as it isn't an integral domain. To show that it's connected, we will show that there are no idempotents except 1. Suppose $f \in A$ is such that $f^2 = f$. Then if f has any pure x^m or y^m term, consider the term with highest degree, f^2 would have a x^{2m} or y^{2m} term since there couldn't be cancelation due the maximality of degree. It follows that $f^2 \neq f$. Hence f is constant, and $f = 1$ since $\mathbb{K}$ doesn't have idempotents $\neq 1$.

Ex 3.6.G

Let $\mathcal{U}$ be a cover of $\operatorname{Spec}(A)$. We can assume that every element of $\mathcal{U}$ is a distinguished open set, since these form a basis for the topology. Exercise 3.5.C tells us that each such cover by distinguished open sets adimts a finite subcover, hence $\operatorname{Spec}(A)$ is compact.

Ex 3.6.H

(a)

Let $X = \bigcap_{i=1}^{r} X_i$ with each X_i quasicompact, and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover for X. For each $i \in [1..r]$, let $\mathcal{U}_i$ be the restriction of $\mathcal{U}$ to X_i. Then each $\mathcal{U}_i$ admits a finite subcover with indexing set $J_i \subset I$, and it follows that X is covered by the finite subcover $\{U_i\}_{i \in J}$ with $J = \bigcap_{i=1}^{r} J_i$.

(b)

Let X be quaiscomapct and $V \subset X$ a closed non-trivial subset. Let $\mathcal{U} = \{U_i \cap V\}_{i \in I}$ be a open cover of V. Then $\mathcal{U} \cup \{X \setminus V\}$ is an open cover for X, hence admits a finite subcover, which is a finite subcover for V as well. So V is quasicompact.

Ex 3.6.I

Non-maximal prime ideals are not closed points, since if $\mathfrak{p} \in V(S)$, then $S \subseteq \mathfrak{p}$, and $S \subseteq \mathfrak{m}$ for any maximal ideal containing $\mathfrak{p}$, hence $\mathfrak{m} \in V(S)$. Meanwhile maximal ideals are closed, as $V(\mathfrak{m}) = [(\mathfrak{m})]$.

In particular, if $\mathfrak{p}$ lies in some closed set, then all prime ideals containing $\mathfrak{p}$ must lie in that set as well.

Ex 3.6.L

See remark from Ex 3.6.I above.

Ex 3.6.N

We prove the contrapositive. If $q \in X$, is such that some neighbourhood $U \ni q$ doesn't contain p, then $X \backslash U$ is a closed set which contains p, hence $\overline{\{p\}} \subseteq X \backslash U$, and in particular, $q \notin \overline{\{p\}}$.

In particular, this shows that $\overline{\{p\}}$ is dense, but this is fairly clear anyhow.

If $r \notin K$, then $X \setminus K$ is an open set containing r but not p.

Ex 3.6.P

First of all, the closure of any point must be irreducible by minimality of the closure. In §3.4.5 it is shown that the closed subsets of $\mathrm{Spec}(\mathbb{C}[x, y])$ are either the entire space, or finite unions of closure of points. So, given a chain

$$Z_0 \supset Z_1 \supset \ldots,$$

where $Z_0 \neq Z_1$, we know that Z_1 is a finite union of closures of points, hence admits only a finite amount of proper closed subsets. It follows that the chain must stabilize.

Ex 3.6.Q

Let C be a connected component. Then C is a union of irreducible components in the subspace topology. Moreover, the irreducible components of C are closed and irreducible in X since C is closed by Remark 3.6.13. They are also maximal since C is open. Indeed, if X_C is an irreducible component of C, and $V \supset X_C$ is closed, then $V \cap (X \setminus C)$ is closed, so V is not irreducible unless $V \subseteq C$.

Ex 3.6.R

PID's are UFD's, and so the ideals properly containing $(x) = (x_1 x_2 \ldots x_m)$ with each x_i irreducible are all the combinations of x_i. I.e there are finitely many such ideals, hence each chain must stabilize.

Ex 3.6.T

We prove the contrapositive. Any closed set in Spec(A) is of the form $V(I)$ with I a radical ideal. If

$$V(I_1) \supsetneq V(I_2) \supsetneq \dots$$

is a strictly decreasing sequence with each I_i radical, then

$$I_1 \subsetneq I_2 \subsetneq \dots$$

is a strictly increasing sequence of ideals by reverse inclusion.

The ring $\mathbb{K}[x_1, x_2, \dots]$ has a non-Noetherian spectrum since it has infinite strictly ascending chains of prime ideals, which turn into infinite strictly decreasing chains of closed irreducible sets under Spec.

Ex 3.6.U

First of all, if X is Noetherian, and U is an arbitrary subset of X, then U is Noetherian as well. To see this, let

$$U \cap U_1 \supsetneq U \cap U_2 \supsetneq \dots$$

be an infinite strictly decreasing chain of closed sets in U. Then

$$U_1 \supsetneq U_1 \cap U_2 \supsetneq U_1 \cap U_2 \cap U_3 \dots$$

is an infinite strictly decreasing chain of closed sets in X.

Now, suppose that X is Noetherian and U is an arbitrary subset of X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of U. Then for each $U_i \in \mathcal{U}$, let $V_i = U \setminus U_i$. Then every V_i is closed in U, and their intersection is empty. Now, first pick some $W_0 = V_0$. Then pick a V_1 which doesn't contain W_0. Such V_1 exists since the V_i have empty intersection. Now let $W_1 = W_0 \cap V_1$. If $W_1 \neq \emptyset$, continue and pick W_2 similarly. As X is Noetherian, so is U, hence the chain of W_i's must stabilize, but by construction this happens only when some $W_n = \emptyset$, giving us a finite subset of V_i's with empty intersection, whence the corresponding subset of U_i's is a finite subcover of $\mathcal{U}$.

Ex 3.6.X

Induction and Ex 3.6.W.

Ex 3.6.Y

Finitely generated modules are quotients of free modules, which are Noetherian by Ex 3.6.X, and their quotients are again Noetherian by Ex 3.6.W.

Ex 3.7.A

Let $f \in I(S)$. Then f vanishes on the x axis, hence is divisible by y and vanishes on $(0,1)$, hence lies in $(x, y-1)$. It follows that $f \in (y) \cap (x, y-1) = (xy, y^2 - y)$, and it's easy to see that $(xy, y^2 - y) \subseteq I(S)$, so there must be an equality.

Ex 3.7.B

The axes have ideals $(y,z), (x,z), (x,y)$ respectively, and their intersection is (yx, yz, xy). This is a radical ideal as the quotients has no nilpotents since elements of $\mathbb{C}[x,y,z]/(yx, yz, xy)$ have leading terms which are pure powers of either x, y, z, and are not nilpotent.

Ex 3.7.E

$f \in I(V(J))$ if and only if f vanishes on $V(J)$ if and only if f lies in all prime ideals containing J if and only if f lies in $\sqrt{J}$.

Using the old definitions from Gathmann, where $V(J)$ is the set of zeros in $\mathbb{K}$ of the ideal $J \in \mathbb{K}[\boldsymbol{x}]$, this statement required $\mathbb{K}$ algebraically closed. But now we don't. Let's see why.

Let J be an ideal in $\mathbb{R}[x]$, and $f \in I(V(J))$. Then f vanishes on $V(J)$, and w

Ex 3.7.D

Over $\mathbb{C}$, $V(J)$ corresponds to the intersection of the unit circle with the line $y = 1$. I.e geometrically, we should have that $V(J)$ corresponds to the point $(0,1)$, which is given by the maximal ideal $[(x, y-1)]$. It is indeed the case that $J \subset (x, y-1)$ since $x^2 + y^2 - 1 = x^2 + (y-1)(y+1)$. To see that no other ideals are contained in $V(J)$, just note that $\mathbb{C}[x,y]/J \cong \mathbb{C}[x]/(x^2)$ and the only prime ideal in $\mathbb{C}[x]/(x^2)$ is (x). It follows that $V(J) = [(x, y-1)]$, and that $I(V(J)) = (x, y-1)$, hence an element as requested may be given as

$$x \in I(V(J)) \setminus J.$$

Note that $x \in \sqrt{J}$ as expected (or proved really), since $x^2 = x^2 + y^2 - 1 - (y-1)(y+1) \in J$.

Ex 3.7.F

Now let $V(S) \subset \operatorname{Spec}(A')$ be an closed set. We can suppose that $V(S) = V(J)$ with J an ideal. Now, $V(J) = \operatorname{Spec}(A'/J)$, so we might as just well show that for some given ring A, $\operatorname{Spec}(A)$ is irreducible if and only if A is an integral domain. One direction was Exercise 3.6.C, so we focus on the other direction.

Let A be a ring such that $\operatorname{Spec}(A)$ is irreducible. Then any two non-trivial open subsets of $\operatorname{Spec}(A)$ intersect. I.e given non-zero $f_1, f_2 \in A$, there is always some $[\mathfrak{p}] \in D(f_1) \cap D(f_2)$. So $f_1 \notin \mathfrak{p}$ and $f_2 \notin \mathfrak{p}$, hence $f_1 f_2 \notin \mathfrak{p}$ by primality, and $f_1 f_2 \neq 0$. Since this is true of any two non-zero elements in A, A must be a domain.

Ex 3.7.G

By 3.7.F, irreducible subsets of $\operatorname{Spec}(A)$ are in inclusion-reversing bijection with prime ideals $\mathfrak{p} \subset A$. It follows by maximality of irreducible components, that they are in paired with minimal prime ideals of A.

Ex 3.7.H

Any prime ideal containing f must contain an irreducible factor of f by primality. Hence the minimal primes of (f) are exactly those that are generated by its irreducible factors, hence these make up the irreducible components of $V(f)$ as well.

Ex 3.7.I

We have $V(I) = \operatorname{Spec}(A/I)$ and it follows from Ex 3.7.H that the irreducible components of $\operatorname{Spec}(\mathbb{C}[x,y]/(xy))$ are given by $\{[(x)], [(y)]\}$.

Ex 4.1.A

Let $S = \{g \in A : D(f) \subset D(g)\}$. Then by Exercise 3.5.F and Definition 4.1.1, we have $\mathcal{O}(D(f)) = S^{-1}A$. Now let $\phi : A \to A_f$ and $\psi : S^{-1}A \to A$ be the two natural localization maps. Any element of S is a unit in A_f by Exercise 3.5.F, hence there is a unique morphism $\rho : S^{-1}A \to A_f$ such that $\phi = \rho \circ \psi$. Similarly, as $D(f) = D(f^k)$, we have $f^k \in S$, and there is a unique morphism $\theta : A_f \to S^{-1}A$ such that $\psi = \theta \circ \phi$. Hence $\phi = \rho \circ \theta \circ \phi$, and $\rho \circ \theta$ is the identity on $\operatorname{im}(\phi)$. But any homomorphism from $S^{-1}A$ is determined by the values it takes on $\operatorname{im}(\phi)$, hence $\rho \circ \theta = \mathrm{id}$ and $S^{-1}A \cong A_f$.

Ex 4.1.B

We have that $\operatorname{Spec}(A_f)$ consists of all the prime ideals of $\operatorname{Spec}(A)$ which don't contain f. I.e $\operatorname{Spec}(A_f) = D(f)$. Suppose that we have an open cover $D(f) = \bigcup_{i \in I} D(f_i)$ of distinguished open sets $D(f_i)$ in $\operatorname{Spec}(A)$. Then $D(f_i) \subseteq D(f)$, which implies that any prime ideal in $D(f_i)$ doesn't contain f, hence is a contracted ideal from A_f. Thus, the $D(f_i)$ are distinguished open sets in $\operatorname{Spec}(A_f)$ as well, since a prime ideal $\mathfrak{p}$ in A_f contains f_i if and only if its contraction in A contains f_i. It follows that the $D(f_i)$ cover $\operatorname{Spec}(A_f)$ as distinguished open sets in the topology of $\operatorname{Spec}(A_f)$, and the whole proof of 4.1.3 may be transposed to $\operatorname{Spec}(A_f)$.

Ex 4.1.C

Let $D(f_i), i \in I$ be an open cover for $D(f)$, and that $a_i/f_i^{k_i} \in D(f_i)$ are elements that agree on overlaps $D(f_i f_j)$. Then the $a_i/f_i^{k_i}$ certainly agree on the overlaps $D(f_i f_j)$ after localizing to A_f, and we may use the proof of base gluability for $\operatorname{Spec}(A)$ to find $r \in A_f = \mathcal{O}(D(f))$ such that $r = a_i/f_i^{k_i}$ in every $D(f_i)$.

Ex 4.1.D

We first show that $\widetilde{M}(D_f) = M_f$. To see this, note that any element in $S_f = \{1, f, f^2, \ldots\}$ doesn't vanish outside $V(f)$, hence act as isomorphisms on $\widetilde{M}(D_f)$, so we have a unique morphism $\phi : \widetilde{M}(D_f) \to M_f$ commuting with the canonical maps from M by the universal property. Now let S be the set of elements of A which doesn't vanish outside $V(f)$. We've shown before that this set is the set of elements that are invertible in A_f, and as M_f is an A_f-module, they act as isomorphisms on the elements of M_f, hence we get a unique morphism $\psi : M_f \to \widetilde{M}(D_f)$ commuting with the canonical maps from M by the universal property. These two maps compose to an automorphism of M_f which fixes the image of M under the canonical map $M \to M_f$, and only one such map can exist by the universal property, hence it's the identity map and the two modules are isomorphic.

It's now easy to see that $\widetilde{M}$ is a presheaf, that every $\widetilde{M}(D(f)) = M_f$ is an $\mathcal{O}(D(f)) = A_f$-module, and that restrictions commute with ring actions. We take this for granted, and dedicate our efforts to showing that the base identity and base gluability axioms hold.

We begin with identity. Let $D(f_i), i \in I$ be an open cover of $D(f)$. We can assume that $I = [1..n]$ by quasicompactness of $\operatorname{Spec}(A_f)$. Then suppose that $s \in \widetilde{M}(D(f))$ is such that $s\big|_{D(f_i)} = 0$. Then there is $k_i \in \mathbb{N}$ such that $f_i^{k_i} s = 0$ for each $i \in [1..n]$, and writing $g_i = f_i^{k_i}$, we have as before that the $D(g_i) = D(f_i)$ cover $\operatorname{Spec}(A)$, so $(g_1, \ldots, g_n) = 1$ in $A_f = \mathcal{O}(D_f)$, and we conclude identity by the same partition of unity argument as before,

$$s = \left(\sum_{i=1}^{n} r_i g_i \right) s = \sum_{i=1}^{n} r_i g_i s = 0,$$

where each $r_i \in A_f$.

For gluability, let $D(f_i), i \in I$ be an open cover of $D(f)$, and $a_i/f_i^{k_i} \in \widetilde{M}(D(f_i))$ be elements that agree on all intersections $D(f_i f_j)$. First, pick some finite subcover $[1..n] \subseteq I$. Then let $g_i = f_i^{k_i}$. The fact that a_i/g_i restricts to the same element as a_j/g_j on $D(g_i g_j)$ means that $(g_i g_j)^{m_{ij}}(a_i g_j - a_j g_i) = 0$, and after picking $m = \max(m_{ij})$ (using the fact that we picked a finite subcover), we have $(g_i g_j)^m (a_i g_j - a_j g_i) = 0$. Now let $b_i = a_i g^m$ and $h_i = g_i^{m+1}$. Then our previous equation becomes $b_i h_j = b_j h_i$. Note that this is an equality of module elements in $\widetilde{M}(D(f))$. Now let

$$1 = \sum_{i=1}^{n} r_i h_i$$

with $r_i \in A_f$ be a partition of unity (which exists since $D(h_i) = D(g_i) = D(f_i)$) in $A_f = \mathcal{O}(D(f))$, and set

$$s = \sum_{i=1}^{n} r_i b_i.$$

Then

$$sh_j = \sum_{i=1}^{n} r_i b_i h_j = \sum_{i=1}^{n} r_i b_j h_i = b_j,$$

and

$$s\big|_{D(f_j)} = h_j/b_j.$$

The same argument as in the proof of Theorem 4.1.2 can be used to widen gluability to the whole index set I.

Ex 4.1.E

Let $f \in A \setminus \mathfrak{p}$ and $m/f \in M_{\mathfrak{p}}$. Then we can send this to an equivalence class $(m/f, D(f)) \in \widetilde{M}_{[\mathfrak{p}]}$ since $[\mathfrak{p}] \in D(f)$ and $m/f \in \widetilde{M}(D_f) = M_f$. This map is well defined, since if $m/f = m'/f'$ in $M_{\mathfrak{p}}$, then $g(mf' - m'f) = 0$ for some $g \in A \setminus \mathfrak{p}$, and $m/f = m'/f'$ in $D(ff'g)$ so

$$(m/f, D(f)) = (m/f, D(ff'g)) = (m'/f', D(ff'g)) = (m'/f', D(f'))$$

in $\widetilde{M}_{[\mathfrak{p}]}$.

Similarly, let (m, U) be a representative of an element in $\widetilde{M}_{[\mathfrak{p}]}$. We can always restrict U down to a distinguished open set $D(f)$ which contains $[\mathfrak{p}]$, so we might as well suppose that we picked the representative $(m', D(f))$, and any element of $\widetilde{M}(D(f)) = M_f$ is of the form $m' = m/f$. So any element of $\widetilde{M}_{[\mathfrak{p}]}$ may be represented by a pair $(m/f, D(f))$ with some $f \in A \setminus \mathfrak{p}$. Hence such a representative can be sent to $m/f \in M_{\mathfrak{p}}$, and we will now show that this is a well-defined map $\widetilde{M}_{[\mathfrak{p}]} \to M_{\mathfrak{p}}$. Suppose that $(m/f, D(f)) \sim (m'/f', D(f'))$ in $\widetilde{M}_{[\mathfrak{p}]}$. Then there is some $U \in D(f) \cap D(f')$ such that $m/f = m'/f'$ on $\widetilde{M}_{[\mathfrak{p}]}(U)$, and we can further restrict to some $D(g) \subseteq U$ to see that $m/f = m'/f'$ on M_g. But them $m/f = m'/f'$ on $M_{\mathfrak{p}}$ as well since this is just a further restriction ($g \in A \setminus \mathfrak{p}$ since $[\mathfrak{p}] \in D(g)$), so the map is well-defined.

These two maps are clearly inverse each other, and both homomorphisms. Hence the two modules are isomorphic.

Ex 4.1.F

(a)

Denote the given map by ϕ. Let $m \in \ker(\phi)$. Then there is $s \in A \setminus \mathfrak{p}$ such that $sm = 0$ for every prime ideal $\mathfrak{p}$ of A. Hence there is $s \in A \setminus \bigcap \mathfrak{p}$ such that $sm = 0$ where the union runs over all prime ideals of A. But every non-unit of A lies in some maximal ideal. Hence s is a unit and $m = 0$.

(b)

By forming the Spec(A)-module $\widetilde{M}$, which has $[\mathfrak{p}]$ as points and $M_{\mathfrak{p}}$ as stalks by the previous exercise, an application of Exercise 2.4.A immediately shows that the given map is injective.

Ex 4.1.G

Any $\mathcal{O}$-module morphism $\widetilde{M} \to \widetilde{N}$ defines a $\mathcal{O}(\text{Spec}(A)) = A$-module morphism $M \to N$ via $M = \widetilde{M}(\text{Spec}(A) \to \widetilde{N}(\text{Spec}(A)) = N$.

For the other direction, let $\phi : M \to N$ be an A-module morphism. Then for open distinguished sets X_f in X, define $\phi_f : \widetilde{M}(U) \to \widetilde{N}(U)$ by

$$\phi_f : m/f^k \to \phi(m)/f^k.$$

This map clearly commutes with restrictions, and so we've constructed a morphism of sheaves by Exercise 2.5.C.

These constructions are also clearly inverses of each other, and we are done.

Ex 4.3.A

We begin with some lemmas that will make solving this exercise easier.

Lemma 0.1. Let A, A' be rings, and

$$(\pi, \phi) : (\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)}) \to (\text{Spec}(A'), \mathcal{O}_{\text{Spec}(A')})$$

be a isomorphism of locally ringed spaces. Then the value of $f \in \mathcal{O}_{\text{Spec}(A)}(\text{Spec}(A)) = A$ at a point $[\mathfrak{p}] \in \text{Spec}(A)$ is zero if and only if the value of $\phi(f) \in A'$ is zero at the point $\pi([\mathfrak{p}]) \in \text{Spec}(A')$.

Proof. We have the following chain of equivalences.

The value of f is zero at $[\mathfrak{p}]$. (The value of f as in the image of f in $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$)

$$\Leftrightarrow$$

f lies in the maximal ideal of $\mathcal{O}_{[\mathfrak{p}]} = A_{\mathfrak{p}}$

$$\Leftrightarrow$$

f is not invertible in $\mathcal{O}_{[\mathfrak{p}]}$ (since the stalk is local)

$$\Leftrightarrow$$

there exist no neighbourhood $V_{[\mathfrak{p}]}$ of $[\mathfrak{p}]$, and $g_{[\mathfrak{p}]} \in \mathcal{O}(V_{[\mathfrak{p}]})$ such that $fg_p[\mathfrak{p}] = 1$

$$\Leftrightarrow$$

there exist no neighbourhood $\pi(V_{[\mathfrak{p}]})$ of $\pi([\mathfrak{p}])$ and $\phi(g_{[\mathfrak{p}]}) \in \pi(V_{[\mathfrak{p}]})$ such that $\phi(f)\phi(g_{[\mathfrak{p}]}) = 1$.

It's clear that the chain can now be traversed back up using $\phi(f)$ and $\pi([\mathfrak{p}])$, from which see that vanishing in the two ringed spaces is equivalent. □

Lemma 0.2. Let A, A' be rings, and

$$(\pi, \phi) : (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \to (\mathrm{Spec}(A'), \mathcal{O}_{\mathrm{Spec}(A')})$$

be an isomorphism of ringed spaces. Then the isomorphism of global sections $\phi : \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \to \mathcal{O}_{\mathrm{Spec}(A')}(\mathrm{Spec}(A'))$ coincides with the map π on the spectra in the following sense. For any prime ideal $\mathfrak{p}$ of A, we have

$$[\phi(\mathfrak{p})] = \pi([\mathfrak{p}]).$$

Proof. The point $[\mathfrak{p}] \in \mathrm{Spec}(A)$ is characterized by the fact that $f([\mathfrak{p}]) = 0$ if and only if $f \in \mathfrak{p}$. By the previous lemma, we see that $\phi(f)(\pi([\mathfrak{p}])) = 0$ if and only if $f \in \mathfrak{p}$ which happens if and only if $\phi(f) \in \phi(\mathfrak{p})$. Hence $[\phi(\mathfrak{p})] = \pi([\mathfrak{p}])$. □

The previous lemma shows that given a isomorphism of affine schemes, the map of topologies is completely determined by the map of global sections. The next lemma will show that the map of rings over any distinguished open subset also is determined by the map of global sections. From these two facts it follows that isomorphisms of ringed spaces is equivalent to isomorphisms of rings, and more specifically the isomorphism of global sections.

Lemma 0.3. Let A, A' be rings, and

$$(\pi, \phi) : (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \to (\mathrm{Spec}(A'), \mathcal{O}_{\mathrm{Spec}(A')})$$

be an isomorphism of ringed spaces. Moreover, let $f \in A$ and $a/f^m \in \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$. Denote the map between the rings of $D(f)$ by $\phi_{D(f)} : \mathcal{O}_{\mathrm{Spec}(A)}(D(f)) \to \mathcal{O}_{\mathrm{Spec}(A')}(\pi(D(f)))$. Then

$$\phi_{D(f)}(a/f^m) = \phi(a)/\phi(f)^m.$$

Proof. We have $\phi(a) = \phi_{D(f)}(a)$ since the maps of ϕ have to commute with restrictions. Similarly $\phi(f) = \phi_{D(f)}(f)$, after which it follows that $1/\phi(f) = \phi_{D(f)}(1/f)$ since both of these elements are inverses to $\phi(f)$. □

Ex 4.3.B

We know from Exercise 3.4.I.(b) that the subspace topology $D(f) \subset \operatorname{Spec}(A)$ agrees with the topology of $\operatorname{Spec}(A_f)$. Hence it remains only to check that the structure sheaves agree. Let $D(g/f^m) = D(g)$ be a distinguished open set in $\operatorname{Spec}(A_f)$. Then $\mathcal{O}(\operatorname{Spec}(A_f)))(D(g)) = (A_f)_g = A_{fg}$. Meanwhile, $D(g) \subset \operatorname{Spec}(A_f)$ corresponds to $D(fg) \subset \operatorname{Spec}(A)$, and $\mathcal{O}_{\operatorname{Spec}(A)}(D(fg)) = A_{fg}$. Hence the two ringed spaces are isomorphic.

Ex 4.3.C

It's a ringed space by the definition of restriction of a sheaf. Moreover, given a point $[\mathfrak{p}] \in U$, we know that there exists an open neighbourhood $V \subset X$ of $[\mathfrak{p}]$ such that $\mathcal{O}\big|_V$ is affine. As distinguished open sets form a base, there must be some distinguished set $D(f)$ in $V \cap U$ which contains $\mathfrak{p}$. Then $\mathcal{O}\big|_{D(f)}$ is affine, as if $\mathcal{O}\big|_V = (\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$, then $\mathcal{O}\big|_{D(f)} = (\operatorname{Spec}(A_f), \mathcal{O}_{\operatorname{Spec}(A_f)})$ by Exercise 4.3.b.

Ex 4.3.D

Any points $[\mathfrak{p}] \in X$ lies in some affine open set, which in turn lies in some distinguished open set $D(f)$ inside that open set. If some $[\mathfrak{p}]$ simultaneously lies in two distinguished open sets $D(f), D(f')$, which in turn may lie in two different affine open sets, $D(f) \subset \operatorname{Spec}(A_f), D(f') \subset \operatorname{Spec}(A'_f)$, then in particular, $D(f) \cap D(f')$ is open in $\operatorname{Spec}(A_f)$, and we can find some $D(g) \subset D(f) \cap D(f')$ containing $[\mathfrak{p}]$ which is distinguished open in $\operatorname{Spec}(A_f)$. Hence the distinguished open sets of the affine cover form an basis for X.

Ex 4.3.F

(a)

Let $X, \mathcal{O}$ be a locally ringed space. Let $f \in \mathcal{O}(X)$, and U be the subset of X where the germ of f is invertible. I.e $p \in U$ if and only if there exist some open neighbourhood V_p of p such that $fg = 1$ in $\mathcal{O}(V_p)$. But then f is invertible on all of V_p, so we can write U as a union of open sets $U = \bigcap_{p \in U} V_p$, hence U is open. It follows that the space $W \subset X$ where the germ of f isn't invertible is closed. Now, a local ring is a disjoint union of its maximal ideal and set of units. Hence if the germ of f isn't invertible at some point $p \in W$, we have that f must lie in the maximal ideal of $\mathcal{O}_p$, whence f is zero in $\kappa(p)$.

(b)

Suppose that the germ of f doesn't vanish anywhere. It follows from part (a) that every germ of f is invertible. In other words, we have for each $p \in X$, some open neighbourhood V_p about p, and $g_p \in \mathcal{O}(V_p)$ such that $fg = 1$ in $\mathcal{O}(V_p)$. But then if q, p are two points such that V_p and V_q intersect, we must have $g_p = g_q$ on $V_p \cap V_q$ since inverses are uniquely determined. It follows by the gluing axiom that there is some $g \in \mathcal{O}(X)$ such that $gf = 1$ on $\mathcal{O}(X)$, and f is invertible.

Ex 4.4.A

We first specify how to construct the scheme $(X, \mathcal{O}_X)$ given $X_i, i \in I$, X_{ij} and $f_{ij} : X_{ij} \to X_{ji}$ as in the exercise description.

We construct the topological space X as the disjoint union of the X_i modulo the relation that $a \sim b$ whenever $a \in X_{ij}, b \in X_{ji}$ and $f_{ij}(a) = b$. Using this construction, we can assume that the X_i are embedded into X, and form an open cover of X with intersections $X_i \cap X_j = X_{ij} = X_{ji}$.

We construct the structure sheaf $\mathcal{O}_X$ as follows. Let U be open in X. Then we define $\mathcal{O}_X(U)$ as the subring of the product $\prod_{i\in I} \mathcal{O}_{X_i}(U \cap X_i)$ where for any $a \in \prod_{i\in I} \mathcal{O}_{X_i}(U \cap X_i)$ we have $a_i\big|_{X_{ij}\cap U} = a_j\big|_{X_{ji}\cap U}$ where a_i is the i-th entry of a. Then $\mathcal{O}_X$ is a presheaf with restriction maps defined component wise. To verify that it's a sheaf, we prove that the identity and gluing axioms hold.

Beginning with the identity axiom, let U be open in X, and $U_j, j \in J$ be a open cover of X. Then we may form another open cover according to $V_{ij} X_i \cap U_j, (i,j) \in I \times J$. Now suppose that $f, g \in \mathcal{O}_X(U)$ are such that $f\big|_{U_j} = g\big|_{U_j}$ for all $j \in J$. Then $f\big|_{V_{ij}} = g\big|_{V_{ij}}$ for all $i, j \in I \times J$, hence by identity in each X_i we have $f\big|_{X_i\cap U} = g\big|_{X_i\cap U}$ for all $i \in I$. It follows that $f = g$ since they have the same entry in every component $i \in I$.

For gluing, let U, U_j, V_{ij} be as in the previous paragraph. Then let $f_j \in \mathcal{O}_X(U_j)$ be a family of elements which agree on intersections $U_j \cap U_{j'}$. Then they agree on intersections $V_{ij} \cap V_{ij'}$, hence can be glued to elements f_i in each $\mathcal{O}_X(X_i)$. Then the element $f \in \prod_{i\in I} \mathcal{O}_{X_i}(X_i \cap U)$ which has f_i as it's i-th entry is such that $(f)_i = (f)_j$ on each intersection, hence it lies in $\mathcal{O}_X(U)$ by our construction.

Finally, note that the cocycle condition ensures that not only does each $(X_i, \mathcal{O}_{X_i})$ inject into $(X, \mathcal{O}_X)$, but it does so in a way which is compatible with the injection of $(X_j, \mathcal{O}_{X_j})$ on the intersection $(X_{ij}, \mathcal{O}_{X_{ij}})$ (don't have energy to finish this).

Ex 4.4.B

Let $(W, \mathcal{O}_W)$ denote the scheme of the affine line with doubled origin. First of, the topology W is given as elements from $\operatorname{Spec}(\mathbb{K}[t]) \coprod \operatorname{Spec}(\mathbb{K}[u])$ modulo the relation that $(t - a) \sim (u - a)$ for all non-zero a, and $(0) \sim (0)$. Hence, as a set we may view W as the affine line with an extra origin $\operatorname{Spec}(\mathbb{K}[t]) \cup (0_u)$. In $\operatorname{Spec}(\mathbb{K}[t])$ we have that all cofinite sets are open, and the same in $\operatorname{Spec}(\mathbb{K}[t])$, hence a set in W is open if and only if it's cofinite, since that's exactly the sets which correspond to cofinite sets in the original topologies.

Now let's calculate the ring of global sections. It's given as the subring of $\mathcal{O}_W \subset \mathbb{K}[u] \times \mathbb{K}[t]$ such that for any $(a_1, a_2) \in \mathcal{O}_W$, we have that

$$\phi \left(a_1\big|_{D(t)} \right) = a_2\big|_{D(u)}$$

where $\phi : t \mapsto u$. I.e if $a_1 = f(u)$, then $a_2 = f(t)$. Hence $\mathcal{O}_W \cong \mathbb{K}[t]$.

Suppose now towards a contradiction that $(W, \mathcal{O}_W)$ is affine, and equal to the scheme $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ for some ring A. Then as the ring of global sections of an affine scheme is equal to the ring, this would imply that $A = \mathbb{K}[x]$. But in $(\mathbb{A}^1_{\mathbb{K}}, \mathcal{O}_{\mathrm{Spec}(\mathbb{K}[x])})$, we have that $V(x-a) = [(x-a)]$, for all points, whilst in $(W, \mathcal{O}_W)$ we have $V(u) = v(t) = [(u), (t)]$. As any isomorphism between the two ringed spaces is determined by an isomorphism between the global sections, and any such isomorphism is determined by the image of $u = t$, such a map must send $u = t$ to a generator $x - a$ of $\mathbb{K}[x]$. But this is impossible to do in a way which respects vanishing sets, contradicting the first lemma in Exercise 4.3.A

Ex 4.4.C

We can pick U, V as the two copies of the affine plane. Their intersection is the punctured affine plane, which is not an affine scheme by Example XX.

Ex 4.4.D

First we verify that the gluing map $\phi_{ij} : U_i \to U_j$ is an isomorphism. To see this, note that

$$\phi_{ji} \circ \phi_{ij}(x_{k/i}) = \phi_{ji}(x_{k/j}/x_{i/j}) = \phi_{ji}(x_{k/j})/\phi_{ji}(x_{i/j}) = \frac{x_{k/i}/x_{j/i}}{x_{i/i}/x_{j/i}} = x_{k/i},$$

and if A is an integral domain, then any morphism from $K(A)$ is determined by how it maps A.

Now, to see that these isomorphisms agree on triple overlaps. Note that in U_i, the triple overlap is given as $D(x_{j/i}x_{k/i})$, and here the ring is given by

$$\mathbb{K}[x_{0/i}, \ldots, x_{n/i}, 1/x_{j/i}, 1/x_{k/i}]/(x_{i/i} - 1).$$

Similarly, on U_j and U_k the triple overlap rings are given by

$$\mathbb{K}[x_{0/j}, \ldots, x_{n/j}, 1/x_{i/j}, 1/x_{k/j}]/(x_{j/j} - 1),$$

and

$$\mathbb{K}[x_{0/k}, \ldots, x_{n/k}, 1/x_{j/k}, 1/x_{i/k}]/(x_{k/k} - 1)$$

respectively. Then

$$\phi_{jk}\circ\phi_{ij}(x_{l/i}) = \phi_{jk}(x_{l/j}/x_{i/j}) = \phi_{jk}(x_{l/j})/\phi_{jk}(x_{i/j}) = \frac{x_{l/k}/x_{j/k}}{x_{i/k}/x_{j/k}} = x_{l/k}/x_{i/k} = \phi_{ik}(x_{l/i}),$$

hence the gluing morphisms agree on triple overlaps.

Ex 4.4.E

Let $U = U_0 \cup U_1$. Then $\mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(U)$ is given by pairs from $(a_1, a_2) \in \mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(U_0) \times \mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(U_0)$ that agree on the overlap $D(x_{i/j})$ and $D(x_{j/i})$. But then a_1 is a polynomial in $x_{1/0}, x_{2/0}, \ldots, x_{n/0}$, and a_2 is a polynomial in $x_{0/1}, x_{2/1}, \ldots, x_{n/1}$, and these variables are identified according to $x_{i/0} = x_{i/1}/x_{0/1}$ on the overlap. No non-constant polynomial in $x_{1/1}/x_{0/1}, x_{2/1}/x_{0/1}, \ldots, x_{n/1}/x_{0/1}$ can be written as a polynomial in $x_{0/1}, x_{2/1}, \ldots, x_{n/1}$, hence $\mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(U) = \mathbb{K}$.

Ex 4.4.F

The closed points of each $\mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(U_i)$ are given by maximal ideals, which are of the form

$$\mathfrak{m}_{i,a} = (x_{0/i} - a_{0/i}, x_{1/i} - a_{1/i}, \ldots, x_{n/i} - a_{n/i})$$

where $a_{i/i}$ is fixed to 1. This closed point may be written as $[a_{0/i}, a_{1/i}, \ldots, a_{n/i}]$ where $a_{i/i} = 1$ again. As $a_{i/i} = 1$, we can think of this point as the equivalence class $[a_0, a_1, \ldots, a_n]$ where $a_i \neq 0$ and $[a_0, a_1, \ldots, a_n] \sim \lambda[a_0, a_1, \ldots, a_n]$ for all non-zero λ. A similar process can be repeated for any patch U_j, where the resulting point $[b_0, b_1, \ldots, b_n]$ has the coordinate b_j non-zero. Thus points of $\mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}$ can be thought of as equivalence classes $n+1$ tuples $[a_0, a_1, \ldots, a_n]$ where not all coordinates are zero, and two tuples are equivalent if they are scalar multiples of each other.

We will now verify, that on the overlap U_{ij}, forming point tuples from the two descriptions $\mathfrak{m}_{i,a}$ and $\phi_{ij}(\mathfrak{m}_{i,a})$ of the same ideal, result in equivalent point tuples.

On the overlap U_{ji}, the ideal/point $\mathfrak{m}_{i,a}$ may be identified as

$$\begin{gathered}(x_{0/j}/x_{i/j} - a_{0/i},\ x_{1/j}/x_{i/j} - a_{1/i},\ \ldots,\ x_{n/j}/x_{i/j} - a_{n/i}) = \\ (x_{0/j} - x_{i/j}a_{0/i},\ x_{1/j} - x_{i/j}a_{1/i},\ \ldots,\ x_{n/j} - x_{i/j}a_{n/i}),\end{gathered}$$

but the j-th entry in the generator list then reads $x_{j/j} - x_{i/j}a_{j/i}$, and $x_{j/j} = 1$ so we may replace $x_{i/j}$ with $1/a_{j/i}$ and get

$$\begin{gathered}(x_{0/j}/x_{i/j} - a_{0/i},\ x_{1/j}/x_{i/j} - a_{1/i},\ \ldots,\ x_{n/j}/x_{i/j} - a_{n/i}) = \\ (x_{0/j} - a_{0/i}a_{j/i},\ x_{1/j} - a_{1/i}/a_{j/i},\ \ldots,\ x_{n/j} - a_{n/i}/a_{j/i}),\end{gathered}$$

from which we can construct the point as

$$[a_{0/i}/a_{j/i}, a_{1/i}/a_{j/i}, \ldots, a_{n/i}/a_{n/i}]$$

which indeed is equivalent to the original point (as $x_{j/i} = x_{j/j}/x_{i/j} \neq 0$ on the intersection).

Ex 4.5.A

When $x_0, x_1 \neq 0$, we have that $x_{0/1} = x_{0/0}/x_{1/0} = 1/x_{1/0}$, and $x_{2/1} = x_{2/0}/x_{1/0}$.

Ex 4.5.B

On each of the affine patches U_0, U_1, U_2, we define schemes (by defining their global sections) as

$$\begin{aligned}
\mathcal{O}_{U_0} &= \mathbb{K}[x_{1/0}, x_{2/0}]/(1 + x_{1/0}^2 - x_{2/0}^2) \\
\mathcal{O}_{U_1} &= \mathbb{K}[x_{0/1}, x_{2/1}]/(x_{0/1}^2 + 1 - x_{2/1}^2) \\
\mathcal{O}_{U_2} &= \mathbb{K}[x_{0/2}, x_{1/2}]/(x_{0/2}^2 + x_{1/2}^2 - 1)
\end{aligned}$$

On the patch $U_{0,1}$ we have the isomorphism

$$\phi_{0,1} : \mathcal{O}_{U_0}(U_{0,1}) = \mathbb{K}[x_{1/0}, x_{2/0}, 1/x_{1/0}]/(1+x_{1/0}^2-x_{2/0}^2) \to \mathcal{O}_{U_1}(U_{1,0}) = \mathbb{K}[x_{0/1}, x_{2/1}, 1/x_{0/1}]/(x_{0/1}^2+1-x_{2/1}^2)$$

given by

$$\phi_{0,1} : x_{1/0} \mapsto 1/x_{0/1}, x_{2/0} \mapsto x_{2/1}/x_{0/1}.$$
$$\phi_{1,2} x_{0/1} \mapsto x_{0/2}/x_{1/2}, x_{2/1} \mapsto 1/x_{1/2}.$$

This map is indeed well-defined, as if $f = 1 + x_{1/0}^2 - x_{2/0}^2$, then

$$\phi_{0,1}(f) = 1 + (1/x_{0/1})^2 - (x_{2/1}/x_{0/1})^2 = (1/x_{0/1})^2(x_{0/1}^2 + 1 - x_{2/1}^2).$$

It's inverse, and all other intersection isomorphisms, are defined in a similar fashion.

Finally, we conclude by verifying the cocycle condition in one of the six directions

$$\begin{aligned}
\phi_{1,2} \circ \phi_{0,1}(x_{1/0}) &= \phi_{1,2}(1/x_{0/1}) = \frac{1}{x_{0/2}/x_{1/2}} = x_{1/2}/x_{0/2} = \phi_{0,2}(x_{1/0}) \\
\phi_{1,2} \circ \phi_{0,1}(x_{2/0}) &= \phi_{1,2}(x_{2/1}/x_{0/1}) = \frac{1/x_{1/2}}{x_{0/2}/x_{1/2}} = 1/x_{0/2} = \phi_{0,2}(x_{2/0}).
\end{aligned}$$

Ex 4.5.C

(a)

The ones where each term has the same degree, I.e polynomials of the form

$$-x_0^{k+5} + x_1^2 x_0^{k+3} + x_0^k x_2^5.$$

The "best" might be the one with $k = 0$ since this has lowest degree?

(b)

All points $(0, a, 0)$ satisfy the "best" homogeneous polynomial, and all points $(0, a, b)$ satisfy the remaining homogeneous polynomials. This justifies the word "best" from before, the "best" polynomial retains more information in this sense.

Ex 4.5.D

The parabola dehomogenises to $x_0x_2 - x_1^2$. This meets the line $x_0 = 0$ at infinity at all points $(0, a, 0)$.

On U_1 the equation becomes $x_0x_2 - 1$, and on U_2 it's $x_0 - x_1^2$

Ex 4.5.E

Given a set of homogeneous polynomials $f_i, i \in [1..m]$ of degrees $\deg(f_i) = d_i$, define $V_p(f_i, i \in [1..m])$ as the scheme glued together by the following gluing data. Subschemes $U_i, i \in [0..n]$ defined by global sections as

$$\mathcal{O}_{U_i} = \frac{A[x_{0/i}, \ldots, x_{n/i}]}{(x_{i/i} - 1, f_1(x_{0,i}, \ldots x_{n/i}), f_2(x_{0,i}, \ldots x_{n/i}), \ldots, f_m(x_{0,i}, \ldots x_{n/i}))}.$$

Let $U_{i,j} = \mathcal{O}_{U_i}(D(x_{j/i}))$ and morphisms $\phi_{i,j} : U_{i,j} \to U_{j,i}$ be given by $\phi_{i,j} : x_{k/i} \mapsto x_{k/j}/x_{i/j}$. These are well-defined maps as

$$\phi_{i,j}(x_{i/i} - 1) = x_{i/j}/x_{i/j} - 1 = 0,$$

and

$$\phi_{i,j}(f_k(x_{0,i}, \ldots x_{n/i})) = f_k(x_{0,j}/x_{i/j}, \ldots x_{n/j}/x_{i/j}) = f_k(x_{0,j}, \ldots x_{n/j})/x_{i/j}^{d_k}$$

and $x_{i/j}$ is a unit on the intersection. Moreover, they are isomorphisms as

$$\phi_{j,i} \circ \phi_{i,j}(x_{k/i}) = \phi_{j,i}(x_{k/j}/x_{i/j}) = \frac{x_{k/i}/x_{j/i}}{x_{i/i}/x_{j/i}} = x_{k/i}.$$

Finally, the cocycle condition is fulfilled as

$$\phi_{j,k} \circ \phi_{i,j}(x_{l/i}) = \phi_{j,k}(x_{l/j}/x_{i/j}) = \frac{x_{l/k}/x_{j/k}}{x_{i/k}/x_{j/k}} = x_{l/k}/x_{i/k} = \phi_{i,k}(x_{l/i}).$$

Ex 4.5.F

(a)

First, suppose that I is a homogeneous ideal of $S = \bigoplus_{i \in \mathbb{Z}} S_i$ which is generated by forms $g_i, i \in J$ of degrees $d_i \in \mathbb{Z}$. Then let $a = (a_i)_{i \in \mathbb{Z}} \in I$. It follows that a may be written as a S-linear combination of the generators

$$a = \sum_{i \in J} b_i g_i$$

with only finitely many b_i non-zero. Furthermore, for each b_i, let $(b_i)_d$ denote the degree d part of b_i. Then, as S is a direct sum of the S_i, we may decompose a according to

$$a_d = \sum_{i \in J} (b_i)^{d-d_i} g_i,$$

and as each $g_i \in I$, this sum, and therefore a_d, lies in I as well.

For the other direction, if I is generated by some elements $g_i, i \in I$, then I is generated by their homogeneous components as well.

(b)

Let I, I' be homogeneous ideals generated by forms $g_i, i \in J$ and $g'_i, i \in J'$. Then $I + I'$ is generated by the union of the two generating sets, hence is homogeneous, and II' is generated by the product set, hence also homogeneous. If $a \in I \cap I'$ then all the homogeneous components of a are in both I and I', hence in $I \cap I'$ and $I \cap I'$ is homogeneous.

Finally, if $a \in \sqrt{I}$, then $a^n \in I$ for some $n \in \mathbb{N}$. Let $i \in \mathbb{Z}$ be the maximal index such that a_i is non-zero. Then $(a^n)_{ni} = a_i^n$, and as I contains all its homogeneous components, it contains a_i^n. It follows that $a_i \in \sqrt{I}$, hence $a - a_i \in \sqrt{I}$ and has a lower highest degree than a. Repeating this procedure inductively with $a - a_i$ shows that every homogeneous component of a lies in $\sqrt{I}$.

(c)

Suppose that I is an ideal with the described property, and let $ab \in I$. Let a_n, b_m be the non-zero homogeneous components of greatest degree in a, b. Then $(ab)_{n+m} = a_n b_m$ lies in I as well, and a_n or b_m lies in I by hypothesis. Suppose $a_n \in I$, then $a_n b \in I$ and $ab - a_n b = (a - a_n)b \in I$. Repeat this process with $a' = a - a_n$ and $b' = b$. At each step, the degree of the product $a'b'$ lowers, until one of the factors is 0, showing that a or b lies in I and I is prime.

Ex 4.5.G

(a)

Suppose that S is finitely generated as an A-algebra by the generators $g_i, i \in [1..m]$. We assume that all algebras are unital, hence we may assume that none of the generators are in S_0. Any element in S can be written as a polynomial in the g_i. Now suppose that $a \in S^+$ and that $a = f(g_1, g_2, \ldots, g_m)$. Then as a doesn't have a component in S_0, f has no constant. Hence a can be written as a S-linear combination in the g_i, and the g_i generate S^+.

The other direction is similar.

(b)

Suppose that S is Noetherian. Then S^+ is finitely generated and S is finitely generated as an A-algebra by part (a). To see that A must be Noetherian as well, let I be an ideal in A. Then then the ideal IS in S is finitely generated by say $g_1, \ldots, g_m$. So, for any $a \in A = S_0$, we may write a as an S-linear combination of the g_i according to

$$a = \sum_{i=1}^{m} b_i g_i.$$

But as $\deg(a) = 0$, we have that all terms of positive degree in this sum must cancel and we can write a as the sum of degree 0 components

$$a = \sum_{i=1}^{m} (b_i)_0 (g_i)_0.$$

It follows that $(g_1)_0, \ldots, (g_m)_0$ is a generating set for I, hence S_0 is Noetherian.

Now suppose that $A = S_0$ is Noetherian and S is a finitely A-algebra. Then S is Noetherian by the Hilbert Basis Theorem, and the fact that quotients of Noetherian rings are Noetherian.

Ex 4.5.H

(a)

We use the notation suggested in the exercise. Let $\iota : A_0 \to A$ be the canonical injection. Suppose that $P \subset A$ is a homogeneous prime ideal. Then $P_0 = \iota^{-1}(P)$ is a prime ideal of A_0.

Now suppose that P_0 is a prime ideal of A_0, and define

$$P = \bigoplus_{i \in \mathbb{Z}} Q_i$$

where $a_i \in Q_i$ whenever $a_i^{\deg(f)}/f^i \in Q_0$. We will now show that P is a homogeneous ideal.

Let $a_i \in A_i$. Then $a_i \in Q_i$ if and only if $a_i^{\deg(f)/f^i \in Q_0}$ if and only if $(a_i^{\deg(f)}/f^i)^2 = (a_i^2)^{\deg(f)}/f^{2i} \in Q_0$ (as Q_0 is prime, hence radical) if and only if $a_i^2 \in Q_{2i}$. Now suppose that $a_i, b_i \in Q_i$. Then $a_i^{\deg(f)}/f^i \in Q_0$ and $b_i^{\deg(f)}/f^i \in Q_0$ hence $(a_i b_i)^{\deg(f)}/f^{2i} \in Q_0$ and $a_i b_i \in Q_{2i}$. Moreover, $a_i^2, b_i^2 \in Q_{2i}$, hence $(a_i + b_i)^2 \in Q_{2i}$ so $a_i + b_i \in Q_i$. Finally, suppose $a_i \in Q_i$ and $b_j \in A_j$. Then $a_i^{\deg(f)}/f^i \in Q_0$ and so

$$\left(b_j^{\deg(f)}/f^j\right)\left(a_i^{\deg(f)}/f^i\right) = (b_j a_i)^{\deg(f)}/f^{j+i} \in Q_0$$

and $b_j a_i \in Q_{i+j}$. Thus for any $b \in A$, we have $ba_i \in P$. We've shown that P is an ideal in S (radical ideal even), and it's homogeneous by construction. It remains to show that P is primality.

By Exercise 4.5.F.(c), it will suffice to show primality on graded components. Suppose that $a_i b_j \in Q_{i+j}$. Then

$$(b_j a_i)^{\deg(f)}/f^{j+i} = \left(b_j^{\deg(f)}/f^j\right)\left(a_i^{\deg(f)}/f^i\right) \in Q_0,$$

and by primality of Q_0, we may assume that

$$a_i^{\deg(f)}/f^i \in Q_0$$

whence $a_i \in Q_i$.

As $P_0 = Q_0$, these two constructions are inverse each other, and we've found the desired bijection.

(b)

The homogeneous prime ideals in the localization S_f are precisely those in S which do not meet f. As $f \in S^+$, these prime ideals do not contain the irrelevant ideal, and are elements of $\mathrm{Proj}(S)$. Hence the prime ideals of $(S_f)_0$ are in bijective correspondence with the subset of prime ideals in $\mathrm{Proj}(S)$ which do not contain f (I.e "$D(f)$" in some sense).

Ex 4.5.I

Solved in Exercise 4.5.H, part (b) in particular.

Ex 4.5.J

First of all, if $S = S_0$, then every ideal of S contains the irrelevant ideal and we're in a totally degenerate scenario so we ignore this case.

Let $U = \mathrm{Proj}(S) \setminus V(I)$ be a non-trivial open set in $\mathrm{Proj}(S)$ and $[\mathfrak{p}] \in U$. Then we claim that there is some homogeneous element of positive degree $f \in I$ such that $f \notin \mathfrak{p}$ Suppose towards a contradiction that all elements in $a_0 \in I \setminus \mathfrak{p}$ have degree $\deg(a_0) = 0$. Then for any $b \in S_+$, we have $a_0 b \in I$ so $a_0 b \in \mathfrak{p}$ as $\deg(a_0 b) \neq 0$, and as $a_0 \notin \mathfrak{p}$, we have $b \in \mathfrak{p}$, which leads to the contradiction $S_+ \subset I \Rightarrow V(I) = \emptyset$ (as elements of $\mathrm{Proj}(S)$ are not allowed to contain the irrelevant ideal). It follows that $I \setminus \mathfrak{p}$ contains an element of positive degree, and as I is homogeneous it contains it's components, at least one of which have positive degree.

So, for each $[\mathfrak{p}] \in U$, pick $f_{[\mathfrak{p}]} \in I$ of positive degree such that $f_{[\mathfrak{p}]} \notin \mathfrak{p}$. Then $D_+(f_{[\mathfrak{p}]})$ is an open set containing $[\mathfrak{p}]$ which lies in $D_+(f_{[\mathfrak{p}]}) \subset U$. It follows that U can be written as a union of distinguished open sets according to

$$U = \bigcup_{[\mathfrak{p}] \in U} D_+(f_{[\mathfrak{p}]}).$$

Ex 4.5.K

(a)

A prime ideal contains an ideal I if and only if it contains radical of that ideal $\sqrt{I}$. Hence $V(I) = V(\sqrt{I}) \subset V(f)$ if and only if $\{f\} \subseteq \sqrt{I}$.

(b)

Let

$$I(Z) = \{a \in S : a \in \mathfrak{p}, [\mathfrak{p}] \in Z\} = \bigcap_{[\mathfrak{p} \in Z]} \mathfrak{p}.$$

Then $I(Z)$ is homogeneous since the intersection of (an arbitrary amount of) homogeneous ideals is again a homogeneous ideal. It's clear that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$ by the same set theoretic rules that governed the affine case.

(c)

First of all, $V(I(Z))$ is a closed set, and it's clear that it contains Z. Hence $\overline{Z} \subseteq V(I(Z))$.

For the other inclusion, let $V(J)$ be a closed set containing Z, and $[\mathfrak{p}] \in V(I(Z))$. Then $I(Z)$ is contained in $\mathfrak{p}$, hence

$$\bigcap_{[\mathfrak{q}] \in Z} \mathfrak{q} \subseteq \mathfrak{p}.$$

As $Z \subseteq V(J)$, any prime ideal in Z contains J, hence

$$J \subseteq \bigcap_{[\mathfrak{q}] \in Z} \mathfrak{q},$$

and combining these inclusions we see that $J \subseteq \mathfrak{p}$ whence $[\mathfrak{p}] \in V(J)$ and $V(I(Z)) \subseteq V(J)$. We've shown that $V(I(Z))$ is contained in any closed set containing Z, hence $V(I(Z)) \subseteq \overline{Z}$ and we are done.

Ex 4.5.L

(a) $\Rightarrow$ (b). Suppose $V(I) = \emptyset$, and that $I = (f_i, i \in I) \subseteq S_+$. Then no ideal in $\mathrm{Proj}(S)$ contains all f_i. In other words, for every $[\mathfrak{p}] \in \mathrm{Proj}(S)$, there is some

f_i such that $[\mathfrak{p}] \in D_+(f_i)$ (note that $D_+(f_i)$ is well-defined as $f_i \in I \subseteq S_+$ has positive degree). It follows that

$$\mathrm{Proj}(S) = \bigcup_{i \in I} D_+(f_i).$$

(b) $\Rightarrow$ **(c)**. From

$$\mathrm{Proj}(S) = \bigcup_{i \in I} D_+(f_i).$$

it follows that no homogeneous prime ideal not containing S_+ contains all $f_i, i \in I$, hence I. Since $\sqrt{I}$ is the intersection of all minimal prime ideals of I, our desired result would follow if we could show that all minimal primes of I are homogeneous, and hence would have to contain S_+. This is equivalent to showing that all minimal primes of S/I are homogeneous, which follows from the following result.

Lemma 0.4. Let S be a graded ring and $\mathfrak{p}$ a prime ideal in S. Let $\mathfrak{p}^h \subseteq \mathfrak{p}$ be the ideal generated by all homogeneous elements of $\mathfrak{p}$. Then $\mathfrak{p}^h$ is prime.

Proof. $\mathfrak{p}^h$ is a homogeneous ideal by construction, and primality can be checked at homogeneous components. Let $a_i b_j \in \mathfrak{p}^h$. Then $a_i b_j \in \mathfrak{p}$, and we can assume that $a_i \in \mathfrak{p}$. As a_i is homogeneous, $a_i \in \mathfrak{p}^h$ as well. $\square$

(c) $\Rightarrow$ **(a)**. Since all prime ideals containing I contains it's radical, it follows from $\sqrt{I} \supset S_+$ that all prime ideals containing I contain S_+, hence are disqualified from $\mathrm{Proj}(S)$, and $V(I) = \emptyset$.

Ex 4.5.M

We've already shown that $\iota : \mathrm{Spec}((S_f)_0) \to \mathrm{Proj}(S)$ is an inclusion, and it only remains to show that it is a continuous map. Let $D_+(g)$ be an open set in $\mathrm{Proj}(S)$. Then $\iota^{-1}(D_+(g))$ is given by

$$\iota^{-1}(D_+(g)) = \{(\mathfrak{p}_f)_0 : \mathfrak{p} \in \mathrm{Proj}(S), g \notin \mathfrak{p}\}.$$

Now, $g \notin \mathfrak{p}$ if and only if $g^{\deg(f)}/f^{\deg(g)} \notin (\mathfrak{p}_f)_0$ by the construction from Exercise 4.5.H, hence

$$\iota^{-1}(D_+(g)) = D_+(g^{\deg(f)}/f^{\deg(g)})$$

is a distinguished open set in $\mathrm{Spec}((S_f)_0)$ and we are done.

Ex 4.5.N

We will show that the rings of global sections are isomorphic, after which it will follow from Exercise 4.3.A that the affine schemes are isomorphic. We will use the following lemma.

Lemma 0.5. Let S be a graded ring, and $f \in S_{d_f}, g \in S_{d_g}$. Then

$$((S_f)_0)_{g_{d_f}/f^{d_g}} \cong (S_{fg})_0$$

Proof. An arbitrary element in $(S_f)_0$ may be written as a/f^k, where $a \in S_{kd_f}$, and it follows that an arbitrary element in $((S_f)_0)_{g_{d_f}/f^{d_g}}$ can be written as $\frac{a/f^k}{g^{rd_f}/f^{rd_g}}$ for a, k like above and $r \in \mathbb{Z}$. This can then be mapped to $(S_{fg})_0$ according to

$$\frac{a/f^k}{g^{rd_f}/f^{rd_g}} \mapsto \frac{af^{rd_g+rd_f}g^k}{(gf)^{k+rd_f}}$$

An arbitrary element in $(S_{fg})_0$ can be written as $a/(fg)^k$ where $a \in S_{k(d_f+d_g)}$. We can map this according to

$$a/(fg)^k \mapsto \frac{ag^{kd_f-k}/f^{k+kd_g}}{g^{kd_f}/f^{kd_g}}.$$

The maps are well-defined and homomorphisms as they are just constructed by doing algebraic manipulations (we don't show this). The maps are inverse each other as

$$\begin{aligned}
\frac{ag^{kd_f-k}/f^{k+kd_g}}{g^{kd_f}/f^{kd_g}} &\mapsto \frac{ag^{kd_f-k}f^{kd_g+d_f}g^{k+kd_g}}{(gf)^{k+kd_g+kd_f}} \\
&= \frac{a(gf)^{kd_g+d_f}}{(gf)^{k+kd_g+kd_f}} \\
&= \frac{a}{(gf)^k}.
\end{aligned}$$

□

Now, open distinguished sets induce affine subschemes, and $\mathcal{O}_{\mathrm{Spec}((S_f)_0)}(D(g^{d_f}/f^{d_g}))$ is the ring of global sections of the affine subscheme $\mathcal{O}_{\mathrm{Spec}((S_f)_0)}\big|_{D(g^{d_f}/f^{d_g})}$, so by using the lemma, we get the following isomorphism of rings of global sections of affine schemes

$$\begin{aligned}
\mathcal{O}_{\mathrm{Spec}((S_f)_0)}(D(g^{d_f}/f^{d_g})) &= ((S_f)_0)_{g^{d_f}/f^{d_g}} \\
&\cong (S_{fg})_0 \\
&= \mathcal{O}_{\mathrm{Spec}((S_{fg})_0)}(\mathrm{Spec}(S_{fg})_0),
\end{aligned}$$

and we are done.

Ex 4.5.O

We begin by describing the triple overlaps in each patch. The subset $D_+(hg)$ of $\mathrm{Spec}(S_f)_0$ corresponds to the subset $D_+(hgf)$ of $\mathrm{Spec}(S)$, which again corresponds to the subset $D_+(fh)$ of $\mathrm{Spec}(S_f)_0$ and $D_+(fg)$ of $\mathrm{Spec}(S_h)_0$. The ring of sections on these patches are all given by

$$
\begin{aligned}
\mathcal{O}_{\mathrm{Spec}((S_f)_0)}(D_+(gh)) &= ((S_f)_0)_{(gh)^{d_f}/f^{d_g+d_h}} \cong (S_{fgh})_0 \\
\mathcal{O}_{\mathrm{Spec}((S_g)_0)}(D_+(fh)) &= ((S_g)_0)_{(fh)^{d_g}/g^{d_f+d_h}} \cong (S_{fgh})_0 \\
\mathcal{O}_{\mathrm{Spec}((S_h)_0)}(D_+(fg)) &= ((S_h)_0)_{(fg)^{d_h}/h^{d_f+d_g}} \cong (S_{fgh})_0.
\end{aligned}
$$

Now, in Exercise 4.5.N we constructed isomorphisms

$$\psi_{f,fg} : ((S_f)_0)_{g^{d_f}/f^{d_g}} \to (S_{fg})_0$$

for all $f, g \in S$, and the overlap isomorphisms are given by $\phi_{f,g} = \psi_{g,gf}^{-1} \circ \psi_{f,fg}$. This composition is given as follows (by Exercise 4.5.N)

$$\phi_{f,g} : \frac{a/f^k}{g^{rd_f}/f^{rd_g}} \mapsto \frac{af^{rd_g+rd_f}g^k}{(gf)^{k+rd_f}} \frac{af^{rd_g+rd_f}g^k f}{} \frac{af^{rd_g+rd_f}g^k}{(gf)^{k+rd_f}}$$

$$\phi_{g,h}\big|_{D_+(fh)} \circ \phi_{f,g}\big|_{D_+(gh)} = \psi_{h,gh}^{-1}\Big|_{D_+(f)} \circ \psi_{g,gh}\big|_{D_+(fh)} \circ \psi_{g,gf}^{-1}\Big|_{D_+(h)} \circ \psi_{f,fg}\big|_{D_+(gh)}$$

This leads to the following diagram

$$
\begin{tikzcd}
\mathcal{O}_{\mathrm{Spec}(\mathrm{Spec}(S_f)_0)}(D_+(g)) \arrow[r, hook, two heads, "\psi_{f,fg}"] \arrow[d, "\mathrm{res}|_{D_+(gh)}"] & \mathcal{O}_{\mathrm{Spec}(\mathrm{Spec}(S_{fg})_0)} \arrow[d, "\mathrm{res}|_{D_+(h)}"] & \mathcal{O}_{\mathrm{Spec}(\mathrm{Spec}(S_g)_0)}(D_+(f)) \arrow[l, hook, two heads, "\psi_{g,fg}"'] \arrow[d, "\mathrm{res}|_{D_+(fh)}"] \\
\mathcal{O}_{\mathrm{Spec}(\mathrm{Spec}(S_f)_0)}(D_+(gh)) \arrow[r, hook, two heads, "\psi_{f,fg}|_{D_+(gh)}"] & \mathcal{O}_{\mathrm{Spec}(\mathrm{Spec}(S_{fg})_0)}(D_+(h)) & \mathcal{O}_{\mathrm{Spec}(\mathrm{Spec}(S_g)_0)}(D_+(fh)) \arrow[l, hook, two heads, "\psi_{g,fg}|_{D_+(fh)}"']
\end{tikzcd}
$$

TODO: Finnish!

Ex 4.5.P

Ex 4.5.Q

First of all, the $D_+(x_i)$ cover $\mathbb{P}_A^n$ by Exercise 4.5.L as $(x_0, \ldots, x_n)$ generates the irrelevant ideal. The subschemes from §4.5.9 are precisely the $(S_{x_i})_0$, indeed the rings

$$A[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$$

and

$$(A[x_0, \ldots, x_n]_{x_i})_0$$

are isomorphic via

$$x_{j/i} \mapsto x_j/x_i.$$

Ex 4.5.R

Suppose the point $a = [1, a_1, \ldots, a_n] \in \mathbb{P}^n_k$ is given. In Exercise 4.4.F, we showed how this point corresponds to the ideal $(x_{0/i} - 1, x_{1/i} - a_1, \ldots, x_{n/i} - a_n)$ of $A[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$, and using the isomorphism of Exercise 4.5.Q, this corresponds to the maximal ideal $(x_0/x_i - 1, x_1/x_i - a_1, \ldots, x_n/x_i - a_n)$ of $(A[x_0, \ldots, x_n]_{x_i})_0$, which in turn via the isomorphism of Exercise 4.5.H corresponds to the homogeneous prime ideal $(x_0 - x_i, x_1 - x_i a_1, \ldots, x_n - x_i a_n)$ (the easiest way the last step is to verify this is to check that this ideal is sent to $(x_0/x_i - 1, x_1/x_i - a_1, \ldots, x_n/x_i - a_n)$ under the isomorphisms from Exercise 4.5.H).

Ex 4.5.S

We define the closed subscheme $V(f)$ as $\mathrm{Proj}(S/(f))$. To justify this, we show that the closed set $V(f) \subset \mathrm{Proj}(S)$ is homeomorphic to $\mathrm{Proj}(S/(f))$.

Prime ideals in $S/(f)$ correspond to prime ideals in S which contain f.

We will now show that ideals in $S/(f)$ are homogeneous if and only if their inverse images in S are homogeneous as well. Clearly $I + (f)$ is homogeneous when I is homogeneous, since any homogeneous generating set of I is a homogeneous generating set of $I + (f)$. Now suppose that $I + (f)$ is homogeneous, and that $\{a_i + (f)\}_{i \in I}$ is a homogeneous generating set for $I + (f)$. Then $\{f\} \cup \{a_i\}_{i \in I}$ is a homogeneous generating set for I.

Finally, we show that ideals $I + (f)$ contain $(S/(f))_+$ if and only if I contains S_+. Suppose that I contains S_+. Then I contains a generating set of S_+ whence $I + (f)$ contains a generating set of $S_+ + (f) = (S/(f))_+$. Now suppose that $I + (f)$ contains $S_+ + (f)$. Then as I contains (f), I also contains S_+.

Ex 5.1.A

Suppose that $\mathbb{P}^n_k = U \cup V$ with U, V closed. Then $D_+(x_0) = (U \cap D_+(x_0)) \cup (U \cap D_+(x_0))$, and as $D_+(x_0) \cong \mathbb{A}^n_k$ is irreducible, we have that some U or V contains $D_+(x_0)$, say U. Now we may take $V' = V \cap V(x_0) = V \setminus D_+(x_0)$ which is still a closed set, and we still have $\mathbb{P}^n_k = U \cup V'$. The same argument may be repeated to see that $D_+(x_1)$ is contained in U or V', but this time, since V' doesn't intersect $D_+(x_0)$ whilst $D_+(x_0)$, it must be that $D_+(x_1) \subset U$. Continuing this case shows that every open set of the affine cover $D_+(x_i), i \in [0..n]$ lies in U, hence $\mathbb{P}^n_k$ is irreducible.

This same argument may be repeated for any scheme which has an irreducible cover $U_i, i \in I$ (technically only when I is finite maybe) with non-trivial pairwise intersections.

Ex 5.1.B

The closure of a point is irreducible in any topological space. Indeed, if $\overline{x} = U \cup V$, with U, V closed, then either $x \in U$ or $x \in V$. If $x \in U$, then $\overline{x} \subseteq U$ since the closure of a set is the smallest closed set containing that set. It remains to show that all irreducible sets of a scheme arise this way.

Let $(X, \mathcal{O})$ be a scheme and $U \subset X$ an irreducible closed subset. Then if $\operatorname{Spec}(A_i), i \in I$ is an open affine cover, each $\operatorname{Spec}(A_i) \cap U$ is irreducible as well. Indeed, any two open subsets of U intersect, so any two open subsets of $\operatorname{Spec}(A_i) \cap U$ intersect as well. It follows that there exists a point $[\mathfrak{p}_i]$ such that $\operatorname{Spec}(A_i) \cap U = \overline{[\mathfrak{p}_i]}_{\operatorname{Spec}(A_i)}$ (here the subscript denotes the topology we are taking the closure in) for every $i \in I$ such that $\operatorname{Spec}(A_i)$ intersects U. Now, as U is a closed set containing $[\mathfrak{p}]$, we have $\overline{[\mathfrak{p}_i]}_X \subseteq U$ for every $\mathfrak{p}_i$ like above. But then since $\overline{[\mathfrak{p}_i]}_X \cap \operatorname{Spec}(A_i)$ is a closed set in $\operatorname{Spec}(A_i)$ containing $[\mathfrak{p}_i]$, we have

$$\operatorname{Spec}(A_i) \cap U = \overline{[\mathfrak{p}_i]}_{\operatorname{Spec}(A_i)} \subseteq \overline{[\mathfrak{p}_i]}_X \cap \operatorname{Spec}(A_i) \subseteq \overline{[\mathfrak{p}_i]}_X.$$

Hence $U = \overline{[\mathfrak{p}_i]}_X \cup (U \setminus \operatorname{Spec}(A_i))$ is a decomposition into U of closed sets, and as $\operatorname{Spec}(A_i)$ intersects U by assumption, it follows that $U = \overline{[\mathfrak{p}_i]}$.

Ex 5.1.C

Let $X = U \cup V$ with U, V open subsets that are Noetherian topological spaces. Then any descending chain of closed sets in X is a chain of open sets in U and V, hence must stabilize, and as the union of the chains in U, V is the original chain in X, it to stabilizes. This argument may be applied to any space that has a finite cover of Noetherian spaces.

Ex 5.1.D

Let $X, \mathcal{O}_X$ be a scheme and $U_i, i \in I$ be an open cover of X. If X admits a finite affine cover $\operatorname{Spec}(A_i), i \in [1..m]$, then each $\operatorname{Spec}(A_i)$ quasicompact and can be covered by finitely many U_i, and the union of all such U_i for all $\operatorname{Spec}(A_i)$ is a finite subcover of $U_i, i \in I$ of X.

Now suppose that X is quasicompact, and let $\operatorname{Spec}(A_i), i \in I$ be an affine open cover of X. It follows immediately from the definition of quasicompactness that some finite subcover of the $\operatorname{Spec}(A_i)$ cover X.

Ex 5.1.E

First of, we note that generalizations and specializations of points induce a transitive relation. Indeed, let x, y, z be points in a topological space such that $x \in \overline{y}$ and $y \in \overline{z}$, then any closed set containing y contains $\overline{y}$ hence x, and in particular, $x \in \overline{z}$.

Now, let $X, \mathcal{O}_X$ be a quasicompact scheme. Then it has a finite open affine cover $\mathrm{Spec}(A_i), i \in [1.m]$. Let $[\mathfrak{p}_0] \in X$ be an arbitrary point. Then let $i_0 \in [1..m]$ be some index such that $\mathrm{Spec}(A_{i_0})$ contains $[\mathfrak{p}_0]$. We can find a maximal ideal $\mathfrak{m}_0$ of A_{i_0} which contains $\mathfrak{p}_0$. Now consider the closure of $[\mathfrak{m}_0]$ in X. If

$$\overline{[\mathfrak{m}_0]} \neq [\mathfrak{m}_0],$$

let $[\mathfrak{p}_1] \in \overline{[\mathfrak{m}_0]} \setminus [\mathfrak{m}_o]$, and $[\mathfrak{m}_1]$ be a maximal ideal containing $\mathfrak{p}_1$ in some affine open set A_{i_1} which contains $[\mathfrak{p}_1]$. Since generalizations are transitive, we have $[\mathfrak{m}_1] \in \overline{[\mathfrak{m}_0]}$, and as $\mathfrak{m}_0$ is maximal in A_{i_0}, we can't have $[\mathfrak{m}_1] \in \mathrm{Spec}(A_{i_0})$ so $i_0 \neq i_1$. Continuing this way yields a sequence of points $[\mathfrak{m}_j]$ which are maximal ideals in rings A_{i_j}, but not contained in any $\mathrm{Spec}(A_{i_{j'}})$ for $j' \leq j$. Since there is a finite amount of A_{i_j}, this sequence must terminate, and the last element of the sequence yields a closed point in X.

The last two statements follow trivially from the first.

Ex 5.1.F

Let $X, \mathcal{O}_X$ be a quasiseparated scheme, and $U, V \subset X$ be two affine open subsets. Then U, V are both quasicompact by Exercise 5.1.D, hence their intersection is quasicompact, and again by Exercise 5.1.D, can be written as a finite union of affine subschemes.

Now, let $X, \mathcal{O}_X$ be a scheme that exhibits the described property, and $U, V \subset X$ be two quasicompact subschemes. Then U, V can be written as finite unions of affine open subschemes,

$$U = \bigcup_{i=0}^{n} \mathrm{Spec}(A_i),\ V = \bigcup_{i=0}^{m} \mathrm{Spec}(B_i),$$

and their intersection is given by

$$U \cap V = \left(\bigcup_{i=0}^{n} \mathrm{Spec}(A_i) \right) \cap \left(\bigcup_{i=0}^{m} \mathrm{Spec}(B_i) \right) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \mathrm{Spec}(A_i) \cap \mathrm{Spec}(B_j)$$

and as each $\mathrm{Spec}(A_i) \cap \mathrm{Spec}(B_j)$ is a finite union of open subschemes (which are quasicompact in particular), it follows that $U \cap V$ is as well.

Ex 5.1.G

We begin with generalizing Exercise 3.5.C

Lemma 0.6. Let $\mathrm{Spec}(B) \subset \mathrm{Spec}(A)$ be an affine subscheme, and $f_i \in A$ for $i \in I$ be a set of elements such that

$$\mathrm{Spec}(B) = \bigcup_{i \in I} D(f_i).$$

Then there is a finite subset $J \subset I$ such that

$$\mathrm{Spec}(B) = \bigcup_{i \in J} D(f_i).$$

Proof. Let $f \in A$ be such that $D(f) \subseteq \mathrm{Spec}(B)$. Then it's immediate from the definition of stalks that $f([\mathfrak{p}]) = f_B([\mathfrak{p}])$ for all $[\mathfrak{p}] \in \mathrm{Spec}(B)$. Hence $D(f) = D(f_B)$, since $f([\mathfrak{p}]) = 0$ if and only if $f \in \mathfrak{p}$ by 4.3.7.1.

Hence, after restricting we have a set of elements $(f_i)_B \in B, i \in I$ such that $D((f_i)_B), i \in I$ covers $\mathrm{Spec}(B)$, and Exercise 3.5.C tells us that there is a finite subset $J \subset I$ such that $D(f_i) = D((f_i)_B) \in B, i \in J$ covers $\mathrm{Spec}(B)$. □

Now let $\mathrm{Spec}(B), \mathrm{Spec}(C) \subset \mathrm{Spec}(A)$ be two affine open subschemes, and let $D(f_i), i \in [1..n]$ and $D(g_i), i \in [1..m]$ be finite open covers of the two respective schemes by distinguished open sets in $\mathrm{Spec}(A)$. Then

$$\mathrm{Spec}(B) \cap \mathrm{Spec}(C) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} D(f_i) \cap D(g_j) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} D(f_i g_j)$$

is a decomposition of $\mathrm{Spec}(B) \cap \mathrm{Spec}(C)$ into a union of finitely many affine subschemes. It now follows from Exercise 5.1.F that $\mathrm{Spec}(A)$ is quasiseparated.

Ex 5.1.H

The condition that there exist some open cover where pairwise intersections are finitely affine covered is weaker then the condition that the intersection of any two affine subsets is finitely affine covered. Hence Exercise 5.1.F implies one of the directions.

For the other direction, suppose that X is a scheme and $X = \bigcap_{i=1}^{n} \mathrm{Spec}(A_i)$ a finite affine open cover of X such that the intersection of any two $\mathrm{Spec}(A_i), \mathrm{Spec}(A_j)$ can be covered by a finite union of open subschemes

$$\mathrm{Spec}(A_i) \cap \mathrm{Spec}(A_j) = \bigcup_{r=1}^{n_{i,j}} \mathrm{Spec}(B_r^{i,j}).$$

Let $\mathrm{Spec}(C) \subset X$ be an open affine subscheme. Our first claim is that $\mathrm{Spec}(C) \cap \mathrm{Spec}(A_i)$ can be written as a finite union of open affine subschemes. For each $\mathrm{Spec}(A_i)$, let $D(f_j^i), j \in J^i$ be an open cover of $\mathrm{Spec}(C) \cap \mathrm{Spec}(A_i)$ by distinguished open sets for $f_j^i \in C$. Then as the $D(f_j^i), j \in J^i$ for all $i \in [1..n]$ cover $\mathrm{Spec}(C)$, a finite subset of them do, hence we can relabel the indices so that $D(f_j^i), j \in [1..n_i], i \in [1..n]$ is a finite open affine cover of $\mathrm{Spec}(C)$. Now it may be the case that $D(f_j^i), j \in [1..n_i]$ is not a cover of $\mathrm{Spec}(C) \cap \mathrm{Spec}(A_i)$ since $\mathrm{Spec}(C) \cap \mathrm{Spec}(A_i) \cap \mathrm{Spec}(A_j)$ may have been covered by sets from the cover of $\mathrm{Spec}(C) \cap \mathrm{Spec}(A_j)$. We will remedy this in the next paragraph.

To reiterate, we have

$$\operatorname{Spec}(C) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} D(f_j^i)$$

with $f_j^i \in C$. Intersecting with $\operatorname{Spec}(A_k)$ gives

$$\begin{aligned}
\operatorname{Spec}(C) \cap \operatorname{Spec}(A_k) &= \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \operatorname{Spec}(A_k) \cap D(f_j^i) \\
&= \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k) \cap D(f_j^i) \\
&= \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \bigcup_{r=1}^{n_{i,j}} \operatorname{Spec}(W_r^{i,k}) \cap D(f_j^i),
\end{aligned}$$

and as both $\operatorname{Spec}(W_r^{i,k}), D(f_j^i)$ are affine open subschemes of $\operatorname{Spec}(A_i)$, their intersection can be written as a finite union of affine open subschemes. It follows that $\operatorname{Spec}(C) \cap \operatorname{Spec}(A_k)$ can be written as a finite union of open affine subschemes.

Now, let $\operatorname{Spec}(C), \operatorname{Spec}(C') \subset X$ be affine open subschemes and

$$\operatorname{Spec}(C) \cap \operatorname{Spec}(A_i) = \bigcup_{j=1}^{n_i} \operatorname{Spec}(D_j),\ \operatorname{Spec}(C') \cap \operatorname{Spec}(A_i) = \bigcup_{j=1}^{n_i'} \operatorname{Spec}(D_j')$$

be finite covers by affine open subschemes. Then

$$\begin{aligned}
\operatorname{Spec}(C) \cap \operatorname{Spec}(C') &= \bigcup_{i=1}^{n} (\operatorname{Spec}(C) \cap \operatorname{Spec}(A_i)) \cap (\operatorname{Spec}(C') \cap \operatorname{Spec}(A_i)) \\
&= \bigcup_{i=1}^{n} \left(\bigcup_{j=1}^{n_i} \operatorname{Spec}(D_j) \right) \cap \left(\bigcup_{j=1}^{n_i'} \operatorname{Spec}(D_j') \right) \\
&= \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \bigcup_{j'=1}^{n_i'} \operatorname{Spec}(D_j) \cap \operatorname{Spec}(D_{j'}'),
\end{aligned}$$

and as both $\operatorname{Spec}(D_j), \operatorname{Spec}(D_{j'}')$ lie in the quasiseparated scheme $\operatorname{Spec}(A_i)$, and are affine open subschemes, their intersection can be written as a finite union of affine open subschemes by Exercise 5.1.F. It follows that $\operatorname{Spec}(C) \cap \operatorname{Spec}(C')$ can be written as a finite union of affine open subschemes as well.

Ex 5.1.I

Let $f_1, \ldots, f_m$ be a generating set for the graded ring S. We can furthermore suppose that the f_i are homogeneous, since otherwise we can just pick all the homogeneous componenets of each f_i. After relabeling, let $f_1, \ldots, f_k$ be the subset of homogenous generators of positive degree. Then $S_+ \subseteq (f_1, \ldots, f_k)$, and the affine open sets $D_+(f_i), i \in [1..k]$ cover $\operatorname{Proj}(S)$ by Exercise 4.5.L, and any intersection $D_+(f_i) \cap D_+(f_j) = D_+(f_i f_j)$ is also affine open. Hence $\operatorname{Proj}(S)$ qualify for the conditions of Exercise 5.1.H and is therefore qsqc.

Ex 5.1.K

Maybe later...

Ex 5.2.A

We begin by showing an affine scheme is reduced if and only if the ring of global sections is reduced. For the first direction, assume that A isn't reduced and let x be a non-zero nilpotent in A. Then let $I = (0 : x)$ be the ideal of zero-divisors of x. Then there exists some maximal ideal $\mathfrak{m}$ containing I. It follows that x is non-zero in $A_\mathfrak{m}$ since the kernel of a localization by a multiplicative set S is all the elements which have a zero divisor in S. As x is still a nilpotent in $A_\mathfrak{m}$, it follows that $\operatorname{Spec}(A)$ is not reduced since $A_\mathfrak{m}$ has non-zero nilpotents. For the other direction, suppose that $\operatorname{Spec}(A)$ isn't reduced. Then there is some prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$ and non-zero nilpotent $a/b \in A_\mathfrak{p}$. I.e, we have $sa^n = 0$ for some $s \in A \setminus \mathfrak{p}$ and $n \in \mathbb{N}$. But then $(sa)^n = 0$ and A isn't reduced.

Let $\operatorname{Spec}(A_i), i \in I$ be an affine cover of X. Then each $\mathcal{O}_X(\operatorname{Spec}(A_i))$ is reduced, and as $f - g \in \mathfrak{p}$ for all $[\mathfrak{p}] \in \operatorname{Spec}(A_i)$, it follows that $f = g$ in every $\operatorname{Spec}(A_i)$. It then follows from the identity axiom that $f = g$ in X.

Ex 5.2.B

See Exercise 5.2.A.

Ex 5.2.C

First of all, since being reduced is a stalk local property, it follows that any scheme that can be covered by reduced schemes is itself reduced.

$\mathbb{A}^n_k$ is reduced as $k[x_1, \ldots, x_n]$ is a UFD, and $\mathbb{P}^n_k$ is reduced as it's covered by $n+1$ reduced schemes $\mathbb{A}^n_k$.

Ex 5.2.D

We have that an element $f(x,y)/x^n \in (k[x,y]/(y^2, yx))_x$ is zero if and only if $x^m f(x,y) \in (y^2, yx)$ for some $m \in \mathbb{N}$, which happens if and only if $y | f(x,y)$. Hence $(k[x,y]/(y^2, yx))_x \cong k[x,y]/(y) \cong k[x]$ and $(k[x,y]/(y^2, yx))_x$ is reduced. Since this is the ring of sections on $D(x)$, it follows that the subscheme $D(x)$

is reduced (as it's affine), and the only possibly non-reduced stalks are those which correspond to points which lie in $V(x)$. But the only point in $V(x)$ is $[(x,y)]$ (it corresponds to the only prime ideal of $k[x,y]$ containing (y^2, yx) and (x)), and since we know that $k[x,y]/(y^2, yx)$ isn't reduced, it follows that the stalk at $[(x,y)]$ must not be reduced.

Ex 5.2.E

We begin by proving a useful lemma that cements the idea in the paragraph after Exercise 5.1.I in the text.

Lemma 0.7. Let X be a scheme and $[\mathfrak{p}] \in X$ be a point, and suppose that there exist some point $[\mathfrak{m}] \in \overline{[\mathfrak{p}]}$. Then any open neighbourhood $U_{[\mathfrak{m}]}$ of $[\mathfrak{m}]$ conatins $\mathfrak{p}$.

Proof. Suppose towards a contradiction that we had some $U_{[\mathfrak{m}]}$ not containing $[\mathfrak{p}]$. Then $X \setminus U_{[\mathfrak{m}]}$ is a closed set containing $[\mathfrak{p}]$, whence it also contains $\overline{[\mathfrak{p}]}$. Hence $U_{[\mathfrak{m}]}$ doesn't intersect $\overline{[\mathfrak{p}]}$, which contradicts $[\mathfrak{m}] \in \overline{[\mathfrak{p}]}$. □

It follows that when X is quasicompact, then checking that a local property (a property P such that when P holds for $[\mathfrak{p}]$, then P holds for a neighbourhood of $[\mathfrak{p}]$) holds on closed points suffices to check it on all $[\mathfrak{p}] \in X$ since every $[\mathfrak{p}]$ in a quasicompact scheme has closed points in its closure.

We don't do this though, as Remark 5.2.2 shows that reducedness isn't an open condition. We follow the hint instead.

Suppose that $[\mathfrak{p}]$ is a non-reduced point, and let $[\mathfrak{m}]$ be a closed point in the closure of $[\mathfrak{p}]$. It follows from the lemma that any affine open subscheme containing $[\mathfrak{m}]$ also contains $[\mathfrak{p}]$, hence we can suppose that they both lie in some $\mathrm{Spec}(A)$ where $[\mathfrak{m}]$ is a maximal ideal.

Since the stalk at $[\mathfrak{p}]$ is non-reduced, we have $A_{[\mathfrak{p}]}$ non-reduced. Hence there is some non-zero $(a/b) \in A_{[\mathfrak{p}]}$ such that $sa^n = 0$ for some $s \in A \setminus \mathfrak{p}$ whilst there is no $s' \in A \setminus \mathfrak{p}$ such that $s'a = 0$. It follows that sa is a non-zero nilpotent of A, which doesn't lie in the kernel of $A \to A_{[\mathfrak{p}]}$, and as this is a further localization of $A \to A_{[\mathfrak{m}]}$, sa is a non-zero in $A_{[\mathfrak{m}]}$ as well. It's clearly nilpotent here also, and we see that $A_{[\mathfrak{m}]}$ is non-reduced.

Ex 5.2.F

Let $\mathrm{Spec}(A_i), i \in [1..m]$ be a finite open affine cover of X, and suppose that $f \in \mathcal{O}_X$ is such that $f([\mathfrak{p}]) = 0$ for all $[\mathfrak{p}] \in X$. Then f is nilpotent in every $\mathrm{Spec}(A_i)$ with $f^{n_i} = 0$ in $\mathcal{O}_{\mathrm{Spec}(A_i)}$. Letting $n = \max_{i \in [1..m]}(n_i)$, we have that f^n is zero in $\mathcal{O}_X$ by the identity axiom.

For the other direction, let

$$X = \bigcup_{i \in \mathbb{N}} \operatorname{Spec}(A_i)$$

with $A = k[x_i]/x_i^i$. Then by the gluing axiom of sheaves, we have an element $\overline{x} \in \mathcal{O}_X$ which restricts to x_i on each patch $\operatorname{Spec}(A_i)$. This element is nilpotent on each patch and hence is contained in every prime ideal of every affine patch, but $\overline{x}^n \neq 0$ for any $n \in \mathbb{N}$ since $x_N^n \neq 0$ when $N > n$.

Ex 5.2.G

We begin by noting that an affine scheme $\operatorname{Spec}(A)$ is integral if and only if A is a domain. Indeed, if $\operatorname{Spec}(A)$ is integral, then the ring of sections of any open set is integral, and particularly the ring of global sections, A.

Now suppose that A is an integral domain and let U be open in $\operatorname{Spec}(A)$. Then $\operatorname{Spec}(A)$ is irreducible by Exercise 3.6.C. Suppose that $x, y \in \mathcal{O}_{\operatorname{Spec}(A)}(U)$ are such that $xy = 0$. Then let $f \in A$ be such that $D(f) \subseteq U$. We have that $\mathcal{O}_{\operatorname{Spec}(A)}(D(f)) = A_f$ is an integral domain as it's a localization of the integral domain A, and as $xy = 0$ on $D(f)$ it follows that either x or y is 0 here. Say it's x. Then the locus where x vanishes on U, $V(x)$ is closed in U. But U is irreducible as $\operatorname{Spec}(A)$ is, since any intersection of open sets in U is an intersection of open sets in $\operatorname{Spec}(A)$. Since $V(x) \neq \emptyset$, we have $V(x) = U$, and $x = 0$, so U is an integral domain, and $\operatorname{Spec}(A)$ is an integral scheme.

Now suppose that X is an integral scheme. Then in particular, every affine open set of X is integral, and these schemes are clearly reduced since they stem from integral domains. Now suppose towards a contradiction that X is reducible. Then there exist open subsets $U, V \subset X$ which are disjoint, so we can glue sections from these sets without any restriction and

$$\mathcal{O}_X(U) \times \mathcal{O}_X(V) \subset \mathcal{O}_X(U \cup V).$$

As the product isn't an integral domain, this is a contradiction to X being and integral scheme.

Now suppose that X is reduced and irreducible, and that $U \subset X$ is an open set where $x, y \in \mathcal{O}_X(U)$ are such that $xy = 0$. Let $\operatorname{Spec}(A_i), i \in I$ be an open affine cover of U. Then for a given $i \in I$, we have $xy = 0$ in A_i whence we can assume that $x = 0$ here. Now, U is irreducible as X is, hence the vanishing set of x, is all of U, and $\mathcal{O}_X(U)$ is an integral domain.