Lemma 0.1. Let M be a regular surface, $\gamma : I \to M$ be a differentiable curve on M, and $p \in \gamma(I)$. Furthermore, let $X : U \to X(U) \subset M$ be a local parameterization at $p \in X(U)$. Then there exist some interval $J \subseteq I$, and differentiable curve $\alpha : J \to U$ on U such that $\gamma\big|_J = X \circ \alpha$ and $p \in \alpha(J)$.

Proof. We assume without loss of generality that $\gamma(0) = X(0) = p$. Denote the components of X by $X(u, v) = (f(u, v), g(u, v), h(u, v))$. As X is regular, we have $X_u(0) \times X_v(0) \neq 0$, hence the differential

$$dX(0) = \begin{bmatrix} f_u(0) & f_v(0) \\ g_u(0) & g_v(0) \\ h_u(0) & h_v(0) \end{bmatrix}$$

has rank 2. It follows that there is some 2×2 minor of $dX(0)$ which has non-zero determinant. Suppose without loss of generality that one such minor is given by

$$\begin{bmatrix} f_u(0) & f_v(0) \\ g_u(0) & g_v(0) \end{bmatrix},$$

and let $F(u, v) = (f(u, v), g(u, v))$. Then let $p' = F(0, 0)$. As $dF(0)$ has full rank, F has a C^1 inverse about some neighbourhood $U_{p'}$ of p'. Now let $\pi_{x,y} : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the first two coordinates. Then let J be a small enough interval such that $\pi(\gamma(J)) \subset U_{p'}$. Then $0 \in J$ since $\pi(\gamma(0)) = \pi(p) = p'$. Now let $\alpha : J \to U$ be given by

$$\alpha = F^{-1} \circ \pi \circ \gamma\big|_J.$$

Then α is C^1 as it's a composition of C^1 functions. Moreover, as $F^{-1} \circ \pi$ coincides with X^{-1} on $\pi^{-1}(U_q)$ it follows that

$$\begin{aligned} X \circ \alpha &= X \circ F\big|_{U_q}^{-1} \circ \pi \circ \gamma\big|_J \\ &= \gamma\big|_J. \end{aligned}$$

$\square$

Corollary 0.2. Let M_1, M_2 be regular surfaces, and $\phi : M_1 \to M_2$ be a differentiable map. Let $\gamma : I \to M_1$ be a differentiable curve on M_1. Then $\phi \circ \gamma$ is a differentiable curve on M_2.

Proof. Let $p \in \gamma(I)$ and $q = \phi(p)$. Let $X : U \to X(U) \subset M_1, Y : V \to Y(V) \subset M_2$ be two local parameterizations such that $\phi(X(U)) \subset Y(V)$ and $p \in X(U)$. Let α be the factorization of $\gamma\big|_J$ through X. Then

$$\phi \circ \gamma\big|_J = Y \circ Y^{-1} \circ \phi \circ X \circ \alpha$$

and as $Y, Y^{-1} \circ \phi \circ X, \alpha$ are all C^1, so is their composition $\phi \circ \gamma\big|_J$. We've verified that $\phi \circ \gamma$ is C^1 at the arbitrary point $\gamma^{-1}(p)$, whence it's C^1 at all of its domain. $\square$