Ch 1

Ex 1.8

Suppose towards a contradiction that $F = y^2 + x - x^3$ was reducible via $F = GH$ with $\deg(G), \deg(H) > 0$. Then $\deg(F) = \deg(G) + \deg(H)$ and as neither of these are constant a unit, we can assume $\deg(G) = 2, \deg(H) = 1$. Thus we can write

$$G = x^2 + a_1xy + a_2y^2 + a_3x + a_4y + a_5,\ H = x + b_1y + b_2,$$

and by multiplying the two together and comparing to the coefficients of F we get

$$\begin{aligned} b_1 + a_1 &= 0 \\ a_1b_1 + a_2 &= 0, \\ a_2b_1 &= 0, \end{aligned}$$

so either a_2 or b_1 is zero. If it's a_2 we need $a_1b_1 = 0$ by the second equation, whence $b_1 = a_1 = 0$ by the first equation. If it's b_1, we get $a_1 = 0$ by the first equation and $a_2 = 0$ by the second equation. Hence $a_1 = a_2 = b_1 = 0$ and

$$G = x^2 + a_3x + a_4y + a_5,\ H = x + b_2.$$

But then GH can't possibly be F, since GH doesn't contain the term y^2, a contradiction.

Ex 1.16

(a)

We have that $L = y - tx - t$ and F intersect at the points where $0 = (tx+t)^2 + x^2 - 1$ which after moving things around gives

$$x^2 + \frac{2t^2}{t^2+1}x + \frac{t^2-1}{t^2+1} = 0$$

This quadratic has the solutions

$$x_1 = \frac{-t^2}{t^2+1} + \sqrt{\frac{t^4}{(t^2+1)^2} + \frac{1-t^2}{t^2+1}} = \frac{-t^2}{t^2+1} + \sqrt{\frac{1}{(t^2+1)^2}} = \frac{1-t^2}{t^2+1}$$

and

$$x_2 = -\frac{1+t^2}{t^2+1} = -1$$

Solving for $y = tx + t$ gives

$$y_1 = \frac{t-t^3}{t^2+1} + t = \frac{t - t^3 + t^3 + t}{t^2+1} = \frac{2t}{t^2+1},$$

and $y_2 = 0$. As t goes from $-\infty$ to ∞, it sweeps the circle, and we see that all points on the circle lie in the set

$$V(F) = \{0,1\} \cup \left\{ \left(\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1}, \right) : t \in K, 1+t^2 \neq 0 \right\}$$

(b)

(a,b,c) is a Pythagorean triple exactly when $(a/c, b/c) \in F$ where $K = \mathbb{Q}$. We can write $t = u/v$ with $u, v \in \mathbb{Z}$ whence

$$\begin{aligned} V(F) &= \{0,1\} \cup \left\{ \left(\frac{1-(u/v)^2}{(u/v)^2+1}, \frac{2(u/v)}{(u/v)^2+1}, \right) : u, v \in \mathbb{Z} \right\} \\ &= \{0,1\} \cup \left\{ \left(\frac{u^2-u^2}{u^2+v^2}, \frac{2uv}{u^2+v^2}, \right) : u, v \in \mathbb{Z} \right\} \end{aligned}$$

and the statement follows.

Ex 2.6

(a)

By Prop 1.12 (a), we have some $\hat{p}(x) \in (F,G)$, which since both F, G vanish at the origin, we can write $\hat{p}(x) = x^{n_x} p(x)$ for some $n_x \geq 1$ and $p(0) \neq 0$. Then in $\mathcal{O}_0$,

$$x^{n_x} = \frac{x^{n_x} p(x)}{p(x)} = \frac{\hat{p}(x)}{p(x)} \in (F,G)\mathcal{O}_0,$$

and it follows that $x^{n_x} = 0 \in \mathcal{O}_0/(F,G)$. Picking $n = \operatorname{lcm}(n_x, n_y)$ yields $x^n = y^n = 0$.

(b)

Let

$$\frac{1}{\hat{g}} \in \mathcal{O}_0(F,G),$$

and write $g = 1 - \frac{\hat{g}}{\hat{g}(0)}$. Note that $\hat{g}(0) \neq 0$ by the definition of our local ring. Then g doesn't have a constant term, and therefore, $g^{2n} = 0$, since all terms in g^{2n} has degree at least $2n$, and must contain either x^n or y^n which are equal to 0 by part (a) Let $k \in \mathbb{N}$ be the smallest natural number such that $g^{k+1} = 0$. Then

$$\frac{1}{1-\hat{g}(0)g} \sum_{i=0}^{k} (\hat{g}(0)g)^k = 1,$$

and

$$\left(\frac{1}{\hat{g}}\right)^{-1} = \left(\frac{1}{1-\hat{g}(0)g}\right) = \sum_{i=0}^{k} (\hat{g}(0)g)^k$$

is a polynomial representative.

(c)

By (a) and (b), every element in $\mathcal{O}_{(0,0)}/(F,G)$ is a linear combination of terms $x^i y^j$ with $i, j \leq n$. This is a finite set and it follows that $\mu_0(F,G) \in \mathbb{N}$.

Ex 2.7

(a)

Suppose towards a contradiction that the powers F^i are linearly dependent (over $\mathbb{K}$) in $\mathcal{O}_0/(G)$. Then let $\pi : \mathcal{O}_0/(G) \to \mathcal{O}_0/(F,G)$ be the canonical projection. Then $\ker(\pi) = (F)\mathcal{O}_0/(G)$ is generated by the powers F^i as a $\mathbb{K}$-vector space, hence finite dimensional by hypothesis. The Nullity-Rank Theorem now yields

$$\dim(\mathcal{O}_0/(G)) = \dim((F)\mathcal{O}_0/(G)) + \dim(\mathcal{O}_0/(F,G)),$$

but the two terms on the RHS are finite, whilst $\mathcal{O}_0/(G)$ is infinite dimensional by the following lemma.

Lemma 0.1. Let F be a curve. Then $\dim(\mathcal{O}_0/(F)) = \infty$

Proof. We have either $(F) \cap \mathbb{K}[x] = \emptyset$ or $(F) \cap \mathbb{K}[y] = \emptyset$ since $\mathbb{K} \in \mathbb{K}[x] \cap \mathbb{K}[y]$ and $\mathbb{K} \cap (F) = \emptyset$. Assume $(F) \cap \mathbb{K}[y] = \emptyset$. Then the powers of y are linearly independent in $\mathbb{K}[x,y]/(G)$. Indeed, a linear combination of powers in y over $\mathbb{K}$ is the same thing as a polynomial $p(y) \in \mathbb{K}[y]$, and no such polynomial lies in (G). Moreover, if $a(x,y) \in \mathbb{K}[x,y]$ is such that $a(x,y)p(y) \in (G)$, then $G|ap$, but G and p are coprime, so $G|a$ and $a(x,y) = 0$ in $\mathbb{K}[x,y]/(G)$. Hence p gets sent to a non-zero element when localizing at 0 by bullet point 2) in the text after Prop 3.1 in Atiyah-Macdonald, and the powers y remain linearly independent in $\mathcal{O}_0/(G)$.

$\square$

(b)

Let H be the common component. Then $(F,G) \subseteq (H)$, so

$$\dim(\mathcal{O}_0/(H)) \leq \dim(\mathcal{O}_0/(F,G)),$$

and we showed in part (a) that $\mathcal{O}_0/(H)$ is infinite-dimensional for any curve H.