Ch 2.10

Ex 3

(a)

Let

$$M = \begin{bmatrix} z & -y & x \\ y & -x & 1 \end{bmatrix}.$$

Then the determinants of the 2×2 minors of M are given by $y^2 - z, xy - z, x^2 - y$.

(b)

We have $[x, -y, z] \in \operatorname{im}(M)$, and indeed, $(y, -x, 1)$ is another syzygy on $(x^2 - y, xy - z, y^2 - xz)$. This is a general phenomena, as seen in the theorem below.

(c)

Theorem 0.1. Let A be an m by $m+1$ matrix over $k[x_1, \dots, x_n]$. Then let f_i be the determinant of the m by m minor obtained from deleting the i-th column from A, and suppose that A is such that some $f_i \neq 0$. Then $\operatorname{im}(A) = S(F)$ where $S(F)$ is the module of syzygies on $F = (f_1, \dots, f_{m+1})$.

Proof. Let $G = (g_1, \dots, g_{m+1}) \in (k[x_1, \dots, x_n])^{m+1}$. Then let

$$A' = \begin{bmatrix} G \\ M \end{bmatrix}.$$

The determinant of A' is then given as the sum over i of g_i times the determinant of the minor of A obtained by deleting the i-th column. I.e $\det(A') = \sum_{i=1}^{m+1} g_i f_i$ and we see that G is a syzygy on F if and only if $\det(A') = 0$. This happens if and only if either A has full rank and G is in the image of A, or if A doesn't have full rank, in which case all m by m minors of A have vanishing determinants, and $f_i = 0$ for all i. $\square$

Ch 3.1

Ex 2

(a)

Using the lexicographical order $x > y$ a Groebner basis for I is given by

$$y^3 - y, xy - y^2, x^2 + 2y^2 - 3.$$

It follows from the elimination theorem that $I_y \cap k[y] = (y^3 - y)$.

Using the lexicographical order $y > x$ a Groebner basis for I is given by

$$x^4 - 4x^2 + 3,\ 2y + x^3 - 3x$$

It follows that $I_x \cap k[x] = (x^4 - 4x^2 + 3) = (x^2 - 3)(x^2 - 1)$.

(b)

Using the ideal eliminating y, we see that

$$V(I_x) = \{\sqrt{3}, -\sqrt{3}, 1, -1\},$$

and plugging these values into $2y + x^3 - 3x = 2y + x(x^2 - 3)$ yields

$$V(I) = \{(\sqrt{3}, 0), (\sqrt{-3}, 0), (1, 1), (-1, -1)\}.$$

(c)

$$V(I) \cap \mathbb{Q}^2 = \{(1, -1), (-1, 2)\}.$$

(d)

$\mathbb{Q}(\sqrt{3})$.

Ex 3

Using the lexicographical order $y > x$ a Groebner basis for I is given by

$$3x^4 - 8x^2 + 4,\ 4y + 3x^3 - 6x$$

It follows from the Elimination Theorem that $I_x \cap k[x] = (3x^2 - 8x^2 + 4) = (x^2 - 2)(3x^2 - 2)$, whence

$$V(I_x) = \left\{\sqrt{2}, -\sqrt{2}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right\}.$$

Plugging these values into $4y + 3x(x^2 - 2)$, we see that

$$V(I) = \left\{(\sqrt{2}, 0), (-\sqrt{2}, 0), \left(\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right), \left(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right), \right\}.$$

None of the solutions are rational.

Ex 4

Using the lexicographical order $x > y > z$ a Groebner basis for I is given by

$$\begin{aligned}
&2z^4 - 3z^2 + 1, \\
&y^2 - z^2 - 1, \\
&x + 2z^3 - 3z
\end{aligned}$$

It follows from the Elimination Theorem that $I_2 \cap k[z] = (2z^4 - 3z^2 + 1) = (2z^2 - 1)(z^2 - 1)$, whence

$$V(I_2) = \left\{ 1, -1, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}} \right\}.$$

Plugging these values into the remaining generator of the elimination ideal $I_1 = I \cap k[y, z]$ yields

$$\begin{aligned}
V(I_1) = \Bigg\{ &\left(\sqrt{2}, 1\right), \left(-\sqrt{2}, 1\right), \left(\sqrt{2}, -1\right), \left(-\sqrt{2}, -1\right), \\
&\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}}\right), \left(\sqrt{\frac{3}{2}}, -\sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{1}{2}}\right), \Bigg\}.
\end{aligned}$$

Finally, plugging these values into $x + z(2z^2 - 3)$ yields,

$$\begin{aligned}
V(I_1) = \Bigg\{ &\left(1, \sqrt{2}, 1\right), \left(1, -\sqrt{2}, 1\right), \left(-1, \sqrt{2}, -1\right), \left(-1, -\sqrt{2}, -1\right), \\
&\left(\sqrt{2}, \sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}}\right), \left(-\sqrt{2}, \sqrt{\frac{3}{2}}, -\sqrt{\frac{1}{2}}\right), \left(\sqrt{2}, -\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}}\right), \left(-\sqrt{2}, -\sqrt{\frac{3}{2}}, -\sqrt{\frac{1}{2}}\right) \Bigg\}.
\end{aligned}$$

Ex 6

(a)

This is well-ordering as every monomial is ordered as greater than 1. To show that it is monomial ordering, let $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$ be such that $\alpha > \beta$. Then if

$$\alpha_1 + \ldots + \alpha_l > \beta_1 + \ldots + \beta_l$$

we have

$$\gamma_1 + \ldots + \gamma_l + \alpha_1 + \ldots + \alpha_l > \gamma_1 + \ldots + \gamma_l + \beta_1 + \ldots + \beta_l$$

and so $\gamma + \alpha > \gamma + \beta$. If instead

$$\alpha_1 + \ldots + \alpha_l = \beta_1 + \ldots + \beta_l,$$

and

$$\alpha >_{\text{grlex}} \beta,$$

we get

$$\gamma_1 + \ldots + \gamma_l + \alpha_1 + \ldots + \alpha_l = \gamma_1 + \ldots + \gamma_l + \beta_1 + \ldots + \beta_l$$

and

$$\alpha + \gamma >_{\text{grlex}} \beta + \gamma$$

since grlex is a monomial order.

Ex 8

Using $g_1 = z^2y^4 + z^4y^2 - z^2y^2 + 1$, we get an equation of y in terms of z by

$$y^2 = \frac{-z^2+1}{2} \pm \sqrt{\left(\frac{z^2-1}{2}\right)^2 - \frac{1}{z^2}}$$

Ch 3.2

Ex 3

(a)

In lex term order with $x > y$, a Groebner basis for I is given by x^2, y^2, and in particular $I_1 = (y^2)$.

(b)

We have $V(c_1, c_2, c_3) = V(y, y^3, y^2) = V(y)$, and in particular, $V(y) \cap V(I_1) = V(I_1)$, hence we don't have strict inequality $W \subsetneq V(I_1)$ which is promised by part (ii) of the closure theorem.

(c), (d) (e) - ish

As $V(c_i) \cap V(I_1) = V(I_1)$, we have $V(c_i) \supseteq V(I_1)$, but $V(I_1) \supseteq V(I)$ so

$$V(I) = V(c_i) \cap V(I) = V(c_i, I).$$

Thus whenever we don't have a strict equality and $V(c_i) \cap V(I_1) = V(I_1)$, we can let $\widetilde{I} = I \cup (c_i)$ and have $V(I) = V(\widetilde{I})$.

We can then cancel the leading x_1 terms from the generators of I, and repeat the procedure. As the degrees of the generators decrease at each iteration, this process terminates after finitely many steps, and in the end we are left with some $\widetilde{I}, \widetilde{c}_i$ such that $V(\widetilde{c}_i) \not\supseteq V(\widetilde{I}_1)$, whence we have a variety $W = V(\widetilde{c}_i) \cap V(\widetilde{I}_1)$ strictly smaller than $V(\widetilde{I}_1)$ such that $\pi_1(V(\widetilde{I})) \supseteq V(\widetilde{I}_1) \setminus W$.

Ex 6

By the closure theorem, $V(I_1)$ is the Zariski closure of $\pi_1(V)$. If $I_1 \neq (0)$, then $V(I_1) \neq \mathbb{C}$ by the Nullstellensatz, and so $V(I_1)$ is finite. Thus $\pi_1(V)$ is finite as well, hence closed, and $\pi_1(V) = V(I_1)$.

Ch 3.3

Ex 2

We know from equation (4) that $F(\mathbb{C}^m) = \pi_l(V(I))$ with $I = (x_1 - f_1, x_2 - f_2, \dots, x_n - f_n)$. The closure theorem tells us that $V(I_l)$ is the smallest variety containing $\pi_l(V(I))$ and that there exist some variety $W \subsetneq V(I_l)$ such that $V(I_l) \setminus W \subseteq \pi_l(V)$.

Ex 3

We will use the fact that any variety in $\mathbb{R}$ is either finite or all of $\mathbb{R}$, which follows from the fact that polynomials in $\mathbb{R}$ can be factored into linear and quadratic factors, where the quadratic factors have no solutions in $\mathbb{R}$.

Now, consider the parameterisation $f(t) = t^2$. The image of f is all non-negative real numbers $\mathbb{R}_{\geq 0}$. As this is an infinite set, the smallest variety V containing $\mathbb{R}_{\geq 0}$ is all of $\mathbb{R}$. But the complement $V \setminus R_{\geq 0} = \mathbb{R}_{<0}$ is also infinite, thus any subset $W \subset V$ such that $V \setminus W \subset R_{\geq 0}$ would have to be infinite, and if W is a variety, we then have $W = \mathbb{R}$, whence there is no strict inclusion W in V as $W = V = \mathbb{R}$.

Ex 6

(a)

Let $J = (s_0 - uv, s_1 - u^2, s_2 - v^2)$. A Groebner basis of J using Lex order $u > v > s_0 > s_1 > s_2$ is given by

$$\begin{aligned}
&s_0^2 - s_1 s_2,\\
&v^2 - s_2,\\
&us_2 - vs_0,\\
&us_0 - vs_1,\\
&uv - s_0,\\
&u^2 - s_1.
\end{aligned}$$

Hence we see that the variety $V(J_2) = V(s_0^2 - s_1 s_2)$ is the smallest variety containing the paramterized surface S.

(b)

Using the extension theorem, we see that every point of $V(J_2)$ extends to $V(J_1)$, as there is a polynomial $v^2 - s_2$ in the basis above in $J \cap \mathbb{C}[v, s_0, s_1, s_2]$ which has constant coefficient. We again see that all points in $V(J_1)$ extend to $V(J)$ since $u^2 - s_1$ has a constant coefficient.

As all points of $V(J_2)$ extend, it follows every point in $V(J_2)$ has non-empty preimage under π_2 and

$$S = \pi_2(V(J)) = V(J_2) = V.$$

Ex 7

(a)

Let $J = (s_0 - uv, s_1 - uv^2, s_2 - u^2)$. A Groebner basis of J using Lex order $u > v > s_0 > s_1 > s_2$ is given by

$$\begin{aligned}
&s_0^4 - s_1^2 s_2,\\
&v s_1 s_2 - s_0^3,\\
&v s_0 - s_1,\\
&v^2 s_2 - s_0^2,\\
&u s_1 - s_0^2,\\
&u s_0 - v s_2,\\
&uv - s_0,\\
&u^2 - s_2.
\end{aligned}$$

Hence we see that the variety $V(J_2) = V(s_0^4 - s_1^2 s_2)$ is the smallest variety containing the paramterized surface S.

(b)

We have that

$$\begin{aligned}
J_1 = \Big(&s_0^4 - s_1^2 s_2,\\
&v s_1 s_2 - s_0^3,\\
&v s_0 - s_1,\\
&v^2 s_2 - s_0^2\Big).
\end{aligned}$$

We see by the extension theorem, that every point in $V(J_2)$ not on $V(s_1 s_2, s_0, s_2) = V(s_2, s_0)$ extends to $V(J_1)$. Moreover, it's easy to see that the only point on $V(s_2, s_0) \cap V(J_2)$ which does extend is $(0, 0, 0)$, since any point $(0, a, 0)$ with $a \neq 0$ doesn't lie in $V(v s_0 - s_1)$.

As our Groebner basis for J contains a polynomial $u^2 - s_2$, it follows that every point on $V(J_1)$ extends to $V(J)$. Thus

$$S = V(s_0^4 - s_1^2 s_2) \setminus \{(0, a, 0) : a \in \mathbb{C}\}.$$

Ex 8

(a)

Let J be the elimination ideal of the parametric surface. A Groebner basis of J using Lex order $u > v > s_0 > s_1 > s_2$ is given by

$19683s_0^6 - 59049s_0^4s_1^2 + 10935s_0^4s_2^3 + 118098s_0^4s_2^2 - 59049s_0^4s_2 + 59049s_0^2s_1^4 + 56862s_0^2s_1^2s_2^3 + 118098s_0^2s_1^2s_2 + 129$

$8748vs_1^3s_2^2 + 648vs_1s_2^5 + 5832vs_1s_2^4 + 17496vs_1s_2^3 + 17496vs_1s_2^2 - 729s_0^4s_2 - 2187s_0^4 + 5832s_0^2s_1^2s_2 + 4374s_0^2s_1^2 -$

$27vs_0^2s_2 + 81vs_0^2 + 135vs_1^2s_2 - 81vs_1^2 + 8vs_2^4 + 96vs_2^3 + 216vs_2^2 + 81s_0^2s_1 - 81s_1^3 - 12s_1s_2^3 - 324s_1s_2,$

$4374vs_0^2s_1 + 8748vs_1^3s_2 - 4374vs_1^3 + 648vs_1s_2^4 + 5184vs_1s_2^3 + 17496vs_1s_2^2 - 729s_0^4 + 5832s_0^2s_1^2 - 189s_0^2s_2^3 - 2430$

$2187vs_0^4 + 69984vs_0^2 + 8748vs_1^4s_2 - 2187vs_1^4 + 648vs_1^2s_2^4 + 3240vs_1^2s_2^3 - 11664vs_1^2s_2^2 + 139968vs_1^2s_2 - 69984vs_1^2 -$

$18v^2s_2^2 + 54v^2s_2 - 54vs_1s_2 - 27s_0^2 + 27s_1^2 + s_2^3 - 18s_2^2 + 81s_2,$

$54v^2s_1s_2 + 27vs_0^2 - 27vs_1^2 + 8vs_2^3 + 72vs_2^2 - 9s_1s_2^2 - 27s_1s_2,$

$243v^2s_0^2 - 243v^2s_1^2 - 1296v^2s_2 - 108vs_1s_2^2 + 324vs_1s_2 + 108s_0^2s_2 + 648s_0^2 + 135s_1^2s_2 - 648s_1^2 - 4s_2^4 + 48s_2^3 + 108$

$2v^3 + vs_2 + 3v - s_1,$

$27us_1^2 - 8us_2^3 + 72us_2^2 - 36v^2s_0s_2 - 27vs_0s_1 - 24s_0s_2^2,$

$9us_0 + 6v^2s_2 - 9vs_1 + s_2^2 - 9s_2,$

$4uvs_2 - 3us_1 + 3vs_0,$

$9uvs_1 - 2us_2^2 + 18us_2 - 9v^2s_0 - 6s_0s_2,$

$6uv^2 - us_2 + 9u - 3s_0,$

$3u^2 - 3v^2 - s_2,$

Only the first polynomial lies in $\mathbb{C}[s_0, s_1, s_2]$, and so

$$\begin{aligned} V(J_2) = V(&19683s_0^6 - 59049s_0^4s_1^2 + 10935s_0^4s_2^3 + \\ &118098s_0^4s_2^2 - 59049s_0^4s_2 + 59049s_0^2s_1^4 + \\ &56862s_0^2s_1^2s_2^3 + 118098s_0^2s_1^2s_2 + 1 \\ &296s_0^2s_2^6 + 34992s_0^2s_2^5 + 174960s_0^2s_2^4 - \\ &314928s_0^2s_2^3 - 19683s_1^6 + 10935s_1^4s_2^3 - \\ &118098s_1^4s_2^2 - 59049s_1^4s_2 - 1296s_1^2s_2^6 + \\ &34992s_1^2s_2^5 - 174960s_1^2s_2^4 - 314928s_1^2s_2^3 - \\ &64s_2^9 + 10368s_2^7 - 419904s_2^5). \end{aligned}$$

Hence we see that the variety $V(J_2) = V(s_0^2 - s_1s_2)$ is the smallest variety containing the paramterized surface S.

(b)

The leading coefficients of the generators of $J_1 \setminus J_2$ interpreted as elements of $\mathbb{C}[s_0, s_1, s_2][v]$ are given by

$$
\begin{aligned}
&648s_1s_2^5 + 8748s_1^3s_2^2 + 5832s_1s_2^4 + 17496s_1s_2^3 + 17496s_1s_2^2,\\
&8s_2^4 + 27s_0^2s_2 + 135s_1^2s_2 + 96s_2^3 + 81s_0^2 - 81s_1^2 + 216s_2^2,\\
&648s_1s_2^4 + 8748s_1^3s_2 + 5184s_1s_2^3 + 4374s_0^2s_1 - 4374s_1^3 + 17496s_1s_2^2,\\
&648s_1^2s_2^4 - 192s_2^6 + 8748s_1^4s_2 + 3240s_1^2s_2^3 - 3456s_2^5 + 2187s_0^4 - 2187s_1^4 - 11664s_1^2s_2^2 - 15552s_2^4 + 139968s_1^2s_2 + 20\\
&18s_2^2 + 54s_2,\\
&54s_1s_2,\\
&243s_0^2 - 243s_1^2 - 1296s_2,\\
&2,
\end{aligned}
$$

and in particular, we see that one of them is 2. Hence all points of $V(J_2)$ extend to $V(J_1)$ by the extension theorem.

Similarly, the leading coefficients of the generators of $J \setminus J_1$ interpreted as elements of $\mathbb{C}[s_0, s_1, s_2, v][u]$ are given by

$$
\begin{aligned}
&-8s_2^3 + 27s_1^2 + 72s_2^2,\\
&3,\\
&-2s_2^2 + 9s_1v + 18s_2,\\
&4s_2v - 3s_1,\\
&6v^2 - s_2 + 9,
\end{aligned}
$$

which contains 3, and so all points of $V(J_2)$ extends to $V(J)$, whence we see that $V(J_2) = \pi_2(V(J)) = S$.

Ex 10

(a)

Let the curve be given by $S = F(t) = (f_1(t), \ldots, f_n(t))$, and let $J = (x_1 - f_1(t), \ldots, x_n - f_n(t))$. Using some elimination order where $t > x_i$ for all i, we see that J contains polynomials with leading terms that are pure powers of t (unless each f_i is constant and S is a point, whence the exercise solves trivially). It follows that a Groebner basis G for J must contain some polynomial $g \in G$ with a leading term that is a pure power of t. Thus the Extension Theorem, and more specifically Corollary 4 of §3.2 tells us that all $V(J_1) = \pi_1(V(J)) = S$.

(b)

The rational parameterization of the circle given in Chapter 1.§3 is given by

$$\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2},$$

and it never meets the point $(-1, 0)$.

The problem is that the ideal $J = (g_1(t)x_1 - f_1(t), \ldots, x_n - g_n(t)f_n(t), 1 - g(t)y)$ may not contain any polynomials that have a pure t-power as leading term. For example, using the parameterization above we get

$$J = ((1+t^2)x_1 - 1 + t^2, (1+t^2)x_2 - 2t, 1 - (1+t^2)y)$$

and if we compute it's Groebner basis we will find that no polynomial has a pure t-power leading term.

(c)

The image of the parameterization $t \mapsto t^2$ is all of the positive real numbers, the closure of which is all of $\mathbb{R}$.