Ch 1

Lemma 0.1. The series $\sum_{k=0}^{\infty} z^k$ converges normally absolutely on $\mathbb{D}$.

Proof. Let K be some compact subset of $\mathbb{D}$. Then K is contained in some disc of radius $\rho < 1$ centered at the origin such that K doesn't meet the boundary of that disc. Now,

$$\begin{aligned}\left(\sum_{k=0}^{\infty} |z|^k\right) - \left(\sum_{k=0}^{N} |z|^k\right) &= \sum_{k=N+1}^{\infty} |z|^k \\ &= |z|^{N+1} \sum_{k=0}^{\infty} |z|^k \\ &= \frac{|z|^{N+1}}{1-|z|}\end{aligned}$$

and

$$\frac{|z|^{N+1}}{1-|z|} < \frac{\rho^{N+1}}{1-\rho}$$

for all $z \in K$ since $|z| < \rho$ here. It follows that we can pick N large enough so that

$$\left(\sum_{k=0}^{\infty} |z|^k\right) - \left(\sum_{k=0}^{N} |z|^k\right)$$

is smaller than any $\epsilon > 0$ for all $z \in K$ whence the sum is normally absolutely convergent. $\square$

We generlize to the multivariate setting by the following lemma.

Lemma 0.2. Let $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ be two series of complex valued functions which converge absolutely normally on some domain $D \subset \mathbb{C}^n$. Then their product $(f_k g_k)_{k\in\mathbb{N}}$ converges absolutely on D as well.

Proof. Let $\epsilon > 0$. Pick N_f, N_g such that $|f| - |f_k| < \epsilon/2 \max_{z\in D}(f)$ for all $k > N_f$, and similarly for g. Then

$$\begin{aligned}|fg| - |f_k g_k| &= |fg| - |fg_k| + |fg_k| - |f_k g_k| \\ &= |f|(|g| - |g_k|) + (|f| - |f_k|)|g_k| \\ &\leq |f|\epsilon/2 \max_{z\in D}(f) + |g_k|\epsilon/2 \max_{z\in D}(g) \\ &\leq \epsilon\end{aligned}$$

for all $k > \max(N_f, N_g)$. $\square$

Lemma 0.3. The series $\sum_{\alpha\in\mathbb{N}_0^n} z^k$ converges normally absolutely on $\mathbb{D}^n$.

Proof. Let K be a compact subset of $\mathbb{D}^n$. Then the series convereges normally absolutely in each variable on K. $\square$

Now let's show that multivariate power series admit a polyradius of convergence.

Lemma 0.4. Let $\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha$ be a power series which converges at some $z_0 \in \mathbb{C}^n$, and let $z' \in \mathbb{C}^n$ be such that $|z'| < |z_0|$. Then the series converges absolutely at z'.

Proof. As the sum converges at z_0, the terms $c_\alpha z_0^\alpha$ tend to 0, and hence are bounded by some $M \in \mathbb{R}_+$. We now have that

$$\begin{aligned} |c_\alpha z'^\alpha| &= |c_\alpha z_0^\alpha| \left|\frac{z'^\alpha}{z_0^\alpha}\right| \\ &< M \left|\frac{z'^\alpha}{z_0^\alpha}\right| \\ &= M \left|\frac{z'}{z_0}\right|^\alpha \end{aligned}$$

where $\frac{z'}{z_0} = \left(\frac{z'_1}{(z_0)_1}, \dots, \frac{z'_n}{(z_0)_n}\right)$ and the sum

$$M \sum_\alpha \left|\frac{z'}{z_0}\right|$$

is absolutely convergent as it is a geometric sum and $\frac{z'}{z_0} \in \mathbb{D}^n$. It follows by Weierstrass M-test that $\sum_{\alpha\in\mathbb{N}_0^n} |c_\alpha z'^\alpha|$ is convergent. $\square$