Ex 1.20

$A(X)$ is a field if and only if $I(X)$ is maximal, which happens if and only if X is minimal, i.e a point.

Ex 1.21

Let $J = \langle f_1 = x_1^3 - x_2^6, f_2 = x_1x_2 - x_2^3 \rangle$. Then $f_2 = x_2(x_1 - x_2^2)$ is zero either when $x_2 = 0$ or when $x_1 = x_2^2$, meanwhile $f_1 = (x_1 - x_2^2)(x_1^2 + x_1x_2^2 + x_2^4)$ is zero when $x_1 = x_2^2$, but not when $x_2 = 0, x_1 \neq 0$. We see that $\sqrt{J} = I(V(J)) = \langle x_1 - x_2^2 \rangle$.

Ex 1.22

First, of the i-th coordinate axis can be written as the zero locust of all but the i-the hyperplanes of codimension 1. So, if X_i is the i-th coordinate axis, we have

$$X_i = \bigcap_{j \neq i} V(x_j) \qquad = V\left(\sum_{j \neq i} (x_j)\right) = V(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_n).$$

We are interested in the ideal of the union of all X_i in the case of $n = 3$

$$\begin{aligned} I(X) &= I\left(\bigcup X_i\right) \\ &= \bigcap I(a_i) \\ &= (x_1, x_2) \cap (x_2, x_3) \cap (x_1, x_3) \\ &= (x_1x_2, x_2x_3, x_1x_3). \end{aligned}$$

Another way to arrive at the same result, is to realise that the coordinate ring on a given axis is equal to the set of univariate polynomials in the given indeterminate, and the coordinate ring on all three axes is the vector space generated by all univariate monomials. This is exactly what $\mathbb{K}[\boldsymbol{x}]/(x_1x_2, x_2x_3, x_1x_3)$ is.

Now, since $x_1, x_2, x_3 \notin I(X)$, but $x_1x_2, x_2x_3, x_1x_3 \in I(X)$, we have that any generating set of $I(X)$ must have a linear span which includes each of the three monomials x_1x_2, x_2x_3, x_1x_3. But these three monomials are linearly independent, and thus we need atleast three generators.

Ex 1.23

(a)

Let J be an ideal in $A(Y)$. Then if $f \in J$, we have that $f(x) = 0$ implies that all $g \in \pi^{-1}(f) = f + \ker(\pi)$ satisfy $g(x) = 0$, since $x \in \ker(\pi)$.

For the other direction, if $f \in \pi^{-1}(J)$ and $f(x) = 0$, we have that $\pi(f)(x) = 0$.

So, the two sets on polynomials vanish on the same points, whence their varieties are equal by definition.

(b)

We have that $\pi^{-1}(I_Y(X))$ is the set of polynomials f, such that $\pi(f)$ vanishes on X. But since all the polynomials in $\ker(\pi)$ vanish on all of Y, and therefore X, we have that $\pi(f)$ vanishs on X if and only if f does, which means that $\pi^{-1}(I_Y(X)) = I(X)$.

(c)

We have

$$\begin{aligned} I_Y(V_Y(J)) &= I_Y(V(\pi^{-1}(J))) \\ &= \pi(I(V(\pi^{-1}(J)))) \\ &\subseteq \pi(\sqrt{\pi^{-1}(J)}) \\ &= \pi(\pi^{-1}(\sqrt{J})) \\ &= \sqrt{J}, \end{aligned}$$

where we used part (a), (b), the Nullstellensatz, and the fact that contraction commutes with taking the radical. We show the remaining inclusions as well.

Let $X \subseteq Y$ be a variety and let $x \in X$. Then every polynomial function in $I_Y(Y)$ vanishes on x, and $x \in V_Y(I_Y(X))$.

Let $J \trianglelefteq A(Y)$ and $f \in \sqrt{J}$. Then f^m vanishes on $V_Y(J)$ for some m, whence f does as well. It follows that $f \in I_Y(V_Y(J))$.

Finally, let $x \in V_Y(I_Y(X))$. Every variety is the zero locust of some ideal, so let $X = V_Y(J)$. Then $x \in V_Y(I_Y(V_Y(J))) = V_Y(\sqrt{J}) \subseteq V_Y(J) = X$.

Ex 2.17

We have $X = V\left(f_1 = x_1 - x_2x_3, f_2 = x_1x_3 - x_2^2\right)$. In this ring we have

$$\begin{aligned} x_1 &\equiv x_2x_3, \\ x_1x_3 &\equiv x_2^2, \end{aligned}$$

which in turn yields

$$\begin{aligned} x_1 &\equiv x_2x_3, \\ x_2x_3^2 &\equiv x_2^2, \end{aligned}$$

So we see that $A(X)$ is isomorphic to the algebra $A' = \mathbb{K}[x_2, x_3]/(x_2x_3^2 - x_2^2)$. Let $(x_2(x_3^2 - x_2)) = J \trianglelefteq \mathbb{K}[x_2, x_3] = A(V(f_1))$. It's easy to see that J isn't prime. It is radical though as it is a principal ideal generated by a squarefree product of irreducibles. It follows that $J = (x_2) \cap (x_3^2 - x_2)$ since the intersection of two radical ideals equals the radical of their product, and we see that the minimal prime ideals belonging to J are (x_2) and $(x_3^2 - x_2)$. Since prime ideals in $A(V(f_1)) \cong \mathbb{K}[x_2, x_3]$ are in one to one correspondence with prime ideals in $\mathbb{K}[x_1, x_2, x_3]$ which contain (f_1), we have that the minimal ideals belonging to (f_1, f_2) are $(x_2, x_1 - x_2x_3) = (x_1, x_2)$ and $(x_3^2 - x_2, x_1 - x_2x_3)$, so $X = V(x_1, x_2) \cup V(x_3^2 - x_2, x_1 - x_2x_3)$ is a decomposition of X into irreducible componenets.

This can be verified with the following Sage code (from python3)

```
import sage.all
from sage.rings.rational_field import QQ
from sage.rings.polynomial.polynomial_ring_constructor \
        import PolynomialRing

def ex2_17():
    R = PolynomialRing(QQ, ["x_1", "x_2", "x_3"])
    x1, x2, x3 = R.gens()

    f1 = x1 - x2 * x3
    f2 = x1 * x3 - x2 ** 2

    J = R * [f1, f2]

    for Q in J.primary_decomposition():
        print(Q.radical().gens())
```

Ex 2.18

$I(X)$ is the set of all polynomials vanishing on X, I.e it is maximal among sets of polynomials vanishing on X. Thus, $V(I(X))$ is minimal among the varieties containing X, which is the same as saying that $V(I(X))$ is the closure of X in the Zariski topology.

Ex 2.19

(a)

Let X be a non-empty topology such that it can't be written as a finite union of non-empty connected closed sets. Then in particular, X isn't connected, so we can write $X = X_1 \cup X_1'$ where $X_1 \cap X_1' = \emptyset$ and X_1, X_1' are non-empty closed sets. But then atleast one of these sets must be disconnected, say X_1, and we can write $X_1 = X_2 \cup X_2'$ like before. Continuing this way yields an infinite chain

$$X_1 \supsetneq X_2 \supsetneq X_3 \supsetneq \dots,$$

and X can't be Noetherian.

(b)

Let $X = \bigcup_{1 \leq i \leq r} X_i$. Then given any infinite strict decreasing chain of closed subsetes Y_i we get r infinite (possibly non-strict) decreasing chains from $Y_i \cap X_j$ for each $1 \leq j \leq r$. Taking the pairwise union of all r chains yields the original chain Y_i, and it follows that atleast one of the chains must contain infinitely many strict inclusions, and by removing duplicates from this chain, we get an infinite strictly decreasing chain of closed subsets. In other words, if X is non-Noetherian, then one of the X_i must be non-Noetherian.

Ex 2.20

(a)

$Y \cap A$ closed in the subspace topology by definition implies that there is some closed set Z in X such that $Z \cap A = Y$. Then $Y \subseteq Z$. Since $\overline{Y}$ is the intersection of all closeds set in X which contain Y, we have $\overline{Y} \subseteq Z$. It follows that $\overline{Y} \cap A \subseteq Z \cap A = Y$, but we also have $Y \subset A, Y \subset \overline{Y}$, so $\overline{Y} \cap A = Y$.

(b)

If $\overline{A} = U_1 \cup U_2$ with U_1, U_2 closed in the $\overline{A}$ subspace topology, then there are closed subsets X_i such that $U_i = X_i \cap \overline{A}$ and $A = (X_1 \cap A) \cup (X_2 \cap A)$. Moreover, if $X_i \cap A = A$, then we'd have $\overline{A} \subseteq X_i$, and $\overline{A} = U_i$, so the union $A = (X_1 \cap A) \cup (X_2 \cap A)$ is a non-trivial decomposition.

For the other direction, first note that $\overline{U_1} \cup \overline{U_2} \supseteq \overline{U_1 \cup U_2}$ since $\overline{U_1} \cup \overline{U_2}$ is a closed set which covers both U_1 and U_2.

If $A = U_1 \cup U_2$ with U_1, U_2 closed in the A subspace topology, then there are closed subsets X_i such that $U_i = X_i \cap A$. Then $X_i \cap \overline{A}$ must cover $\overline{U_i}$, and

$$\overline{A} \subseteq \overline{U_1} \cup \overline{U_2} \subseteq (X_1 \cap \overline{A}) \cup (X_2 \cap \overline{A}),$$

but it's clear that the inclusions must hold in the other direction as well since everything on the RHS is intersected with the LHS, and we see that $\overline{A}$ is reducible. Moreover, the two sets on the RHS are non-empty as the $U_i = X_i \cap A \subseteq X_i \cap \overline{A}$ are.

Ex 2.21

(a)

Assume that the cover U_i of X consists only of open connected sets and that they all pairwise intersect each other. Let X_1, X_2 be two closed proper subets of X such that $X_1 \cup X_2 = X$. Then X_1, X_2 are also both open as they complement each other. Now consider some U_1. If U_1 intersects both X_1 and X_2, we have that $U_1 \cap X_i$ is closed, since $U_1 \cap X_1 = (U_1^c \cup X_2)^c$, so $U_1 = (U_1 \cap X_1) \cup (U_1 \cap X_2)$ is a decomposition of U_1 into closed sets, whence $U_1 \cap X_1$ must intersect $U_2 \cap X_2$ by hypothesis of the U_i being connected. It follows that $X_1 \cap X_2 \neq \emptyset$ in this case.

Now consider the case where U_1 only intersects one of the X_i, say X_1. Let U_2 be a set which intersects X_2 (such exist since the U_i cover X). Then as U_1 and U_2 intersect each other, it follows that U_2 must intersect both X_1, X_2, and we can conclude like before that X_1 must intersect X_2 in this case as well.

(b)

Let X, U_i be as above but with the further stipulation that each U_i is irreducible. Assume towards a contradiction that X is reducible and let $X = X_1' \cup \ldots \cup X_r'$ where each X_i is a closed proper subset of X and no X_i is contained in the union of all $X_j, j \neq i$. Let $X_1 = X_1', X_2 = X_2 \cup \ldots X_r'$. Then $X = X_1 \cup X_2$ is a decomposition of X into two closed proper subsets.

Like above, if a U_i intersects both X_1, X_2 we get a decomposition of $U_i = (U_i \cap X_1) \cup (U_i \cap X_2)$ into non-empty closed proper subsets, which is imposible by our assumption that each U_i is irreducible. Thus all U_i must be contained in either X_1 or X_2 but not both. This is imposible since they all pairwise intersect each other (and cover both X_1, X_2).

Ex 2.22

(a)

If $f(X) = U_1 \cup U_2$ is a decomposition of $f(X)$ into non-intersecting closed proper subsets, then $X = f^{-1}(U_1) \cup f^{-1}(U_2)$ is the same for X. So $f(X)$ disconnected implies X disconnected.

(b)

Like above.

Ex 2.23

(a)

First note that $I(\overline{X}) = I(V(I(X)) = I(X)$, so we'll prove that $I(Y_1 \setminus Y_2) = I(Y_1) : I(Y_2)$.

Let $f \in I(Y_1 \setminus Y_2)$. Then f vanishes on Y_1, and we have that $fg \in I(Y_1)$ for all $g \in \mathbb{K}[\boldsymbol{x}]$, and in particular, for all $g \in I(Y_2)$, so $f \in I(Y_1) : I(Y_2)$.

Now let $f \in I(Y_1) : I(Y_2)$. Then for all $g \in I(Y_2)$, we have that $fg \in I(Y_1)$. Now, since Y_2 is a subvariety, it's the zero locust of $I(Y_2)$, and for all points $a \in Y_1 \setminus Y_2$, there is some polynomial g_a such that $g_a(a) \neq 0$. But $fg_a \in I(Y_1)$, so it must be that $f(a) = 0$, and we see that $f \in I(Y_1 \setminus Y_2)$.

(b)

Using the Nullstellensatz, part (a), and the fact that the J_i are radical, we get

$$\begin{aligned}
\overline{V(J_1) \setminus V(J_2)} &= V(I(\overline{V(J_1) \setminus V(J_2)})) \\
&= V(I(V(J_1)) : I(V(J_2))) \\
&= V(\sqrt{J_1} : \sqrt{J_2}) \\
&= V(J_1 : J_2).
\end{aligned}$$

Ex 2.24

Let $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ be irreducible affine varieties and assume towards a contradiction that $X \times Y = U_1 \cup U_2$ is a decomposition into closed proper subsets. Let π_X, π_Y be the projections onto X, Y repectively. First note that

$$\begin{aligned}
U_1 \cup U_2 &= X \times Y \\
&= \pi_X(X) \times \pi_Y(Y) \\
&= \pi_X(U_1 \cup U_2) \times \pi_Y(U_1 \cup U_2) \\
&= (\pi_X(U_1) \cup \pi_X(U_2)) \times (\pi_Y(U_1) \cup \pi_Y(U_2)) \\
&= (\pi_X(U_1) \times \pi_Y(U_1)) \cup (\pi_X(U_1) \times \pi_Y(U_2)) \\
&\quad \cup (\pi_X(U_2) \times \pi_Y(U_1)) \cup (\pi_X(U_2) \times \pi_Y(U_2)),
\end{aligned}$$

from where it will follow that either $Y = \pi_Y(U_1) \cup \pi_Y(U_2)$, or $X = \pi_X(U_1) \cup \pi_X(U_2)$ are non-trivial decompositions into subsets. We will show that $\pi_X(U_i), \pi_Y(U_i)$ are closed, which will contradict X, Y being irreducible.

Let $f \in I(U_1) \subseteq \mathbb{K}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m]$ and fix the last m indeterminates of f to some point $y \in \pi_Y(U_1)$. Call the new polynomial $g_{f,y}$. Then for any $x \in \pi_X(U_1)$ we have $g_{f,y}(x) = f(x, y)$ and since $(x, y) \in U_1$ and $f \in I(U_1)$, we see that $g_{f,y}$ vanishes on $\pi_X(U_1)$. We claim that

$$\pi_X(U_1) = \bigcap_{f \in I(U_1), y \in \pi_Y(U_1)} V\left(g_{f,y}\right).$$

We've already shown one inclusion, for the other direction, suppose that $x \in X$ such that all $g_{f,y}$ vanish on x. Then $f(x,y) = 0$ for all $y \in Y, f \in I(U_1)$, which in turn means that $x \times Y \subset V(I(U_1)) = U_1$ so $x \in \pi_X(U_1)$. It follows that $\pi_X(U_1)$ is closed and we are done.

Ex 2.30

If U_i is some strict descending chain of closed irreducible sets in A, then by Ex 2.20 (a), we have that $\overline{U}_i$ is a strict descending chain of closed sets in X, which are irreducible by Ex 2.20 (b).

Ex 2.33

First, let's identify said matrices with $\mathbb{A}^6$ according to

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}.$$

The rank of a matrix is at most 1 if and only all of its minors of order ≥ 2 are zero. For our matrix above, this yields the following set of equations

$$\begin{aligned} a_1a_5 &= a_2a_4 \\ a_1a_6 &= a_3a_4 \\ a_2a_6 &= a_3a_5. \end{aligned}$$

If we let X denote our set in $\mathbb{A}^6$ identified with the given set of matrices, then it follows from the discussion above that

$$X = V(J = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5)).$$

is a variety. To show that it's irreducible, note that we can surjectively parameterize X with t_1, t_2, t_3, u according to

$$\begin{gathered} a_1 = t_1, a_2 = t_2, a_3 = t_3, \\ a_4 = ut_1, a_5 = ut_2, a_6 = ut_3. \end{gathered}$$

Irreducibility follows from the following lemma.

Lemma 0.1. Let $X \subseteq \mathbb{A}^n$ be a variety that is parameterized by $a_i = f_i(y_1, y_2, \ldots, y_m)$. Then $I(X)$ is prime and X is irreducible.

Proof. $I(X)$ can be identified with the ideal of algebraic dependencies on the f_i. I.e the kernel of $\mathbb{K}[\boldsymbol{x}] \to \mathbb{K}[f_1, f_2, \ldots, f_n]$. This kernel is prime since the image of the map is a subring of $\mathbb{K}[y_1, y_2, \ldots, y_m]$ which is an integral domain. $\square$

Ex 2.34

(a)

Let $n \in \mathbb{N}$ be such that $\dim X \geq n$. Then there exist some chain

$$Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n$$

of closed irreducible subsets. Since Y_i is irreducible, any two non-empty open sets in Y_i must intersect. Let $V_i = Y_i \setminus Y_{i-1}$. Then each V_i is open and non-empty in Y_i, since Y_{i-1} is closed in Y_i.

Now, pick some U which intersects Y_0. Since $U \cap Y_1$ is open and non-empty in Y_1, it must intersect V_1, whence $U \cap Y_0 \subsetneq U \cap Y_1$. Moving on, $U \cap Y_2$ is open in Y_2, hence it must intersect V_2 and it follows that

$$U \cap Y_0 \subsetneq U \cap Y_1 \subsetneq U \cap Y_2.$$

This procedure can be repeated to show that

$$U \cap Y_0 \subsetneq U \cap Y_1 \subsetneq U \cap Y_2 \subsetneq \ldots \subsetneq U \cap Y_n \subset U$$

is an n-long chain of strict inclusions in Y. Moreover, each of the $U \cap Y_i$ is irreducible in U since a reduction of $Y_i \cap U = T_1 \cup T_2$ yields a reduction

$$Y_i = (Y_i \setminus (U \setminus T_1)) \cup (Y_i \setminus (U \setminus T_2)).$$

Hence $\dim X \leq \sup_{i \in I}(\dim U_i)$. The reversed inequality is immediate from Ex 2.30.

(b)

Let X be an irreducible affine variety and

$$Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset X$$

a chain of irreducible subvarieties. Then if U is non-empty and open in X, we can translate the chain in affine space $T_{\boldsymbol{c}}(a_1, a_2, \ldots, a_n) = (a_1 + c_1, \ldots, a_n + c_n)$ (this is a homeomorphism) such that $T_{\boldsymbol{c}}(Y_0)$ intersects U. We can then repeat the argument of (a) to show that

$$T_{\boldsymbol{c}}(Y_0) \cap U \subsetneq T_{\boldsymbol{c}}(Y_1) \cap U \subsetneq \ldots \subsetneq T_{\boldsymbol{c}}(Y_n) \cap U \subset U$$

is a chain of irreducible closed subvarieties in U.

This is not the case in arbitrary topological spaces. Consider for example the space $\mathbb{N}$ given the topology where the non-trivial closed subsets are of the form $[1..n]$ and $[1..\infty]$. Then $\{0\}$ is open in $\mathbb{N}$ but has dimension 0.

Ex 2.35

First suppose that we have some chain

$$\{a\} \subset Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset X$$

of irreducible closed subsets Y_i. Then $Y_n = \bigcup_{i=1}^{r} (Y_n \cap X_i)$, is a decomposition into closed sets, and since Y_n is irreducible, we need $Y_n \subseteq X_i$ for some i, whence $\max(\dim X_i : a \in X_i) \geq \operatorname{codim}\{a\}$.

For the other direction, let X_i be the irreducible variety containing a with maximal dimension. Then by Prop 2.28 (b), we have $\operatorname{codim}_{X_i}\{a\} = \dim X_i = \max(\dim X_j : a \in X_j)$.

Ex 2.36

(a)

X being Noetherian is equivalent to Let X be Noetherian and $U_i, i \in I$ be an open cover of X. Then X admits a decomposition into irreducible subsets $X = X_1 \cup X_2 \cup \ldots \cup X_r$. Let $U_i, i \in I_1$ be a subcover which covers X_1 where all $U_i, i \in I_1$ intersect X_1, and no U_i is contained in some other U_j. Then the $U_i, i \in I_1$ must pairwise intersect each other. Use the axiom of choice to pick a sequence of unique indices $i_1, i_2, \ldots$ in I_1, and construct the chain

$$U_{i_1} \supseteq U_{i_1} \cap U_{i_2} (\supseteq U_{i_1} \cap U_{i_2}) \cap U_{i_3} \supseteq \ldots,$$

Each intersection is non-empty, since it's an intersection non-empty as X_1 is irreducible. Moreover, since X is Noetherian, this must either be a finite chain so that I_1 is finite, or it must stabilize, so that

$$U_{i_n} \subset \bigcap_{j=1}^{n-1} U_{i_j}$$

for some n. This contradicts how I_1 was constructed.

(b)

Quick aside: I revisited an old solution to this problem after finding the reference to this problem in the start of Chapter 5. The old solution was wrong, so I've re-solved this problem using tools from Chapters 1 through 4. I also had trouble solving it the second time around as well, and after looking at this link, https://math.stackexchange.com/a/119349/887520 , I decided to try to do it using Noether Normalization. This was very difficult for me (took 2 days almost to show that the morphism f below is surjective). So, this is all just a heads up to say that this solution is a bit messy and uses a few theorems and lemmas from all over the place.

Let X be an irreducible subvariety in $\mathbb{C}[\boldsymbol{x}]$ of dimension $d \geq 1$. It follows that the Krull dimension of $A(X)$ is d. The Noether normalization lemma tells us that $A(X)$ is integral and finitely generated over some free polynomial algebra $\mathbb{C}[y_1, y_2, \ldots, y_s]$. We will show that $d = s$. First note that $d \geq s$, since we have a chain of length s of prime ideals $(0) \subsetneq (y_1) \subsetneq y_1, y_2) \subsetneq \ldots \subsetneq (y_1, y_2, \ldots, y_s)$ which can be extended to a chain of the same length in $A(X)$ by the Going Up Theorem. Now suppose that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_d$ is a chain of prime ideals in $A(X)$ of length d. Then any ideal $\mathfrak{p}_i \cap \mathbb{K}[y_1, y_2, \ldots, y_s]$ is prime, after which Corollary 5.9 in Atiyah-Macdonald tells us that the chain contracts to a chain of strict inclusions of prime ideals in $\mathbb{C}[y_1, y_2, \ldots, y_s]$. Exercise 11.7 in Atiyah-Macdonald tells us that this ring has dimension s, so $s \geq d$, and we have $s = d$.

Now consider the inclusion $f^* : \mathbb{C}[y_1, y_2, \ldots, y_d] \to A(X)$ where the y_i's are algebraically independent polynomials in $A(X)$. We can identify $\mathbb{C}[y_1, y_2, \ldots, y_d]$ with a free $\mathbb{C}$-algebra in d variables, and doing so induces a morphism of varieties. $f : X \to \mathbb{A}^d$. We claim that f is surjective. To show this, we will first construct f explicitly.

By the proof of Corollary 4.8, we have that $f = (\phi_1, \phi_2, \ldots, \phi_d)$ where $\phi_i = f^*(y_i)$, but f^* is just the inclusion, so $f = (y_1, y_2, \ldots, y_d)$. Now let $a \in \mathbb{A}^d$ and consider $J = (y_1 - a_1, y_2 - a_2, \ldots, y_d - a_d)$ as an ideal in $A(X)$. We want to show that there is some $b \in V(J)$, since doing so would imply that $y_i(b) - a_i = 0 \Leftrightarrow y_i(b) = a_i \Leftrightarrow f(b) = a$.

First of, note that $A(X)$ is a finitely generated $\mathbb{C}[y_1, y_2, \ldots, y_d]$-module by the Noether normalization lemma. We will consider $J' = (y_1 - a_1, y_2 - a_2, \ldots, y_d - a_d)$ as an ideal in $\mathbb{C}[y_1, y_2, \ldots, y_d]$. Note that $J \neq J'$, these two ideals have the same generators but different parent rings! Now let's show that $V(J) \neq \emptyset$.

Suppose towards a contradiction that $V(J) = \emptyset$. Then $\sqrt{J} = I(V(J)) = (1) \Rightarrow J = (1)$, so we can write

$$1 = \sum_{i=1}^{d} h_i(\boldsymbol{x}) y_i(\boldsymbol{x})$$

for some set of polynomials $h_i \in A(X)$. It follows that $J'A(X) = A(X)$. Now Nakayama's lemma grants us some $r \in \mathbb{C}[y_1, y_2, \ldots, y_d]$ such that $rA(X) = 0$ and $r - 1 \in J'$. It follows that $r(a) = 1$ so $r \neq 0$. But the y_i are algebraically independent, so $y_i r = 0$ is a contradiction. Hence $V(J) \neq \emptyset$ and we've shown that f is surjective.

f is a polynomial mapping, hence it's continuous in the classical topology. So we have a surjective continuous map $f : X \to \mathbb{A}^d = \mathbb{C}^d$. We have that $\mathbb{C}^d$ isn't compact (for $d \geq 1$), hence X isn't either (since X compact $\Rightarrow f(X) = \mathbb{C}^d$ compact, a contradiction, https://math.stackexchange.com/questions/26514/continuous-image-of-compact-sets-are-compact)

Ex 2.40

(a)

With $J = (x_1x_4 - x_2x_3)$, we recognize $V(J)$ as the set of 2 by 2 matrices with rank ≤ 1. This variety is parameterizeable with

$$a_1 = t_1, a_2 = t_2, a_3 = ct_1, a_4 = ct_2,$$

whence

$$R = \mathbb{K}[x_1, x_2, x_3, x_4]/J \cong \mathbb{K}[t_1, t_2, ct_1, ct_2] \subset \mathbb{K}[t_1, t_2, c]$$

so R is an integral domain. Moreover, R isn't a field, so J is a prime ideal which isn't maximal, whence it can at most have height $\dim(\mathbb{K}[x_1, x_2, x_3, x_4]) - 1 = 3$, and R can at most have dimension 3. But in $\mathbb{K}[t_1, t_2, ct_1, ct_2]$ we have the prime ideals

$$(0) \subsetneq (t_1, t_2) \subsetneq (t_1, t_2, ct_1, ct_2)$$

So R has dimension 3. We quickly verify that all the ideals are prime. (0) is prime since we're in a domain. (t_1, t_2, ct_1, ct_2) is prime since we get a field when we quotient by it. (t_1, t_2) is prime since any polynomial in $\mathbb{K}[t_1, t_2, ct_1, ct_2]$ has terms where the sum of the degrees of t_1, t_2 is greater than the degree of c, so $Q = \mathbb{K}[t_1, t_2, ct_1, ct_2]/(t_1, t_2)$ consists of all polynomials in $\mathbb{K}[t_1, t_2, ct_1, ct_2]$ with terms where the degree of c is equal to the sum of the degrees of t_1, t_2. Hence $Q \cong \mathbb{K}[ct_1, ct_2]$ is an integral domain. Note that (t_1) (or (t_2)) isn't a prime ideal, since it contains ct_1t_2 but neither ct_1 nor t_2 (or neither ct_2 nor t_1).

(b)

$x_1 | x_2x_3 = x_1x_4$, but $x_1 \not| x_2$, $x_1 \not| x_3$ (any element in J has deg ≥ 2, so x_1 won't divide any representatives of $x_2 + J$ or $x_3 + J$ since they all contain a term of x_2, or x_3 respectively).

(c)

Again, follows from $x_1 + J \not| x_2 + J$, and similar.

(d)

Under the isomorphic map $x_1 \mapsto t_1, x_2 \mapsto t_2, x_3 \to ct_1, x_4 \mapsto ct_2$, we see that (x_1, x_2) is the contraction of the prime ideal (t_1, t_2), which has height 1, so (x_1, x_2) has height 1 also.

Ex 3.12

(a)

Suppose that $A(X)$ is a UFD, $Y \subset X$ is an irreducible subvariety, and that $\text{codim}_X Y \geq 2$. We will begin by showing that there are two elements $f_1, f_2 \in$

$I(Y)$ which are prime in $A(X)$.

Since Y is irreducible, $I(Y)$ is prime, and as $A(X)$ is a UFD, we must have some element $f_1 \in I(Y)$ which is prime in $A(X)$. Moreover, $I(Y)$ has height at least 2, so it's strictly bigger than (f_1) which has height 1 by Prop 2.28 (c), hence $I(Y)$ must contain some element m where $f_1 \not| m$. But again, $I(Y)$ is prime, so it contains some prime factor f_2 of m.

It follows that $(f_i) \subset I(Y) \Rightarrow V(f_i) \supset Y \Rightarrow D(f_i) \subset U$. Hence we can write $\phi = h_1/f_1^{k_1}$ on $D(f_1)$, and $\phi = h_2/f_2^{k_2}$ on $D(f_2)$ with $f_i \not| h_i$. On $D(f_1) \cap D(f_2)$ we have

$$h_1/f_1^{k_1} = h_2/f_2^{k_2} \Leftrightarrow h_1 f_2^{k_2} = h_2 f_1^{k_1},$$

but $V(h_1 f_2^{k_2} - h_2 f_1^{k_1})$ is closed, so $h_1 f_2^{k_2} = h_2 f_1^{k_1}$ must hold on $\overline{D(f_1) \cap D(f_2)}$. Our next claim is that $\overline{D(f_1) \cap D(f_2)}$ is all of $A(X)$. To see this, note that that $A(X)$ is a UFD, hence an integral domain, whence X is irreducible and any open subspace is dense. Thus $h_1 f_2^{k_2} = h_2 f_1^{k_1}$ on all of $A(X)$, and since $f_2^{k_1} | h_2 f_1^{k_1}$, but f_2 divides neither h_2 nor f_1, we have $k_1 = 0$, and $\phi = h_1 = h_2$ on $A(X)$.

For the other direction, we assume that $\operatorname{codim}_X(Y) = 1$, since if the codimension is 0 we'd have $Y = X$. Then Prop 2.38 $I(Y) = (f)$, with f a non-unit, so $Y = V(f) \Rightarrow U = D(f)$ and Corollary 3.10 tells us that $\mathcal{O}_X(U) = \mathcal{O}_X(D(f)) \cong A_f(X)$ which isn't isomorphic to $A(X)$ when f isn't a unit.

(b)

Consider Exercise 2.40 and Example 3.3. We know that (x_1, x_2) is a prime ideal of height 1 in $A(X)$, so $Y = V(x_1, x_2)$ has codimension 1 in X, but yet the example shows that $\mathcal{O}_X(X \setminus U)$ isn't $A(X)$.

Ex 3.20

We show that the corresponding localized coordinate rings are isomorphic, after which Lemma 3.19 gives the desired isomorphisms. Let $\mathbb{K}[\boldsymbol{x}]_{I(a)} \to A(X)_{I(a)}$ be given by

$$\phi\left(g/f\right) = \frac{g + I(X)}{f + I(X)}.$$

ϕ is well defined since $f \in \mathbb{K}[\boldsymbol{x}] \setminus I(a)$, and $a \in X \Rightarrow I(a) \supset I(X)$, so $f + I(X)$ is non-zero. Suppose that $\phi(g/f) = 0$. Then $(h + I(X))(g + I(X)) = hg + I(X) = 0$ for some $h + I(X) \in A(X) \setminus I(a)$, which in turn implies that $hg \in I(X)$. Hence $hg/1 \in I(X)\mathbb{K}[\boldsymbol{x}]_{I(a)}$, but $h(a) \neq 0$, so h is a unit in $\mathbb{K}[\boldsymbol{x}]_{I(a)}$, so $hh^{-1}g/1 = g/1 \in I(X)\mathbb{K}[\boldsymbol{x}]_{I(a)}$, whence $\ker(\phi) \subseteq I(X)\mathbb{K}[\boldsymbol{x}]_{I(a)}$.

Now suppose that $g \in I(X)$, then $\phi(g/1) = 0$, so all the generators of $I(X)\mathbb{K}[\boldsymbol{x}]_{I(a)}$ lie in $\ker(\phi)$ and we have equality of the two ideals. The desired result now follows from the first homomorphism theorem and Lemma 3.19.

Ex 3.21

(a)

Let $\epsilon \neq 0$ and define $f_\epsilon : \mathbb{R} \to \mathbb{R}$ as follows,

$$f_\epsilon(x) = \begin{cases} 0 \text{ for } x \in (a-\epsilon, a+\epsilon), \\ x-(a+\epsilon) \text{ for } x \in [a+\epsilon), \\ x-(a-\epsilon) \text{ for } x \in (a-\epsilon]. \end{cases}$$

Then f_ϵ is continuous, and agrees with the zero function on $(a-\epsilon, a+\epsilon)$, so they reside in the same germ. Since we can pick ϵ arbitrarily small, it follows that the functions in a given germ all agree on a only. Now let I denote the ideal of all germs which are 0 at a. We claim that $\mathcal{F}_a/I$ is isomorphic to $\mathbb{R}$. To see this, note that $f(a) = g(a)$ if and only if $f(a) - g(a) = 0$ and $f(a) - g(a) \in I$. Hence every equivalence class of $\mathcal{F}_a/I$ may be identified with the value which the stalks admit at a, and $\mathcal{F}_a/I \cong I$ is a field and I is maximal. Moreover, it's the only maximal ideal since it contains all non-units, indeed if $f(a) \neq 0$, then we can pick a small enough neighbourhood of a in which f is invertible.

(b)

The open subsets of $\mathbb{R}$ are all infinite, so if two polynomials agree on an open subset of $\mathbb{R}$ they must be equal. It follows that any stalk is isomorphic to the polynomial ring which is not local.

Ex 3.22

(a)

Let $a \in U$. Then since $(U, \phi) \sim (U, \psi)$, we have by definition some open $V_a \subset U$ containing a such that $\phi\big|_{V_a} = \psi\big|_{V_a}$. The $V_a : a \in U$ form an open cover of U, and $\phi = \psi$ on U by the gluing property.

(b)

The vanishing set of their difference is closed in X, and since their difference vanishes on some open subset $V_a \subset X$, it vanishes on $\overline{V_a} = X$.

(c)

Yes, consider for example the f_ϵ and the zero function from the solution to Ex 3.21 (a) with $U = \mathbb{R}$.

Ex 3.23

Let $\phi : A(X)_{I(Y)} \to \mathcal{O}_{X,Y}$ be given by

$$\phi(g/f) = \overline{(D(f), g/f)}.$$

ϕ is well defined, for if $g/f = g'/f'$, then we have $h \in A(X) \setminus I(Y)$ such that $h(gf' - g'f) = 0$. Hence g/f and g'/f' agree as regular functions on the open set $U = D(f) \cap D(f') \cap D(h) = D(ff'h)$. We have that U is contained in both $D(f), D(f')$. Moreover, U intersects Y as $f, f', h \notin I(Y)$ and $I(Y)$ is prime since Y is irreducible so $ff'h \notin I(Y)$, so there must be some point $y \in Y$ where all f, f', h are non-zero and $y \in U \cap Y$. We can conclude that $\overline{(D(f), g/f)} = \overline{(D(f'), g'/f')}$.

Now suppose that $g/f \in \ker(\phi)$. Then we have that $0/1$ and g/f agree as functions on some neighbourhood V which intersects Y. We can define g/f on all of $D(f)$. Vanishing sets of regular functions are closed, so g/f vanishes on the closure of V in $D(f)$, but Y is irreducible, so $\overline{D(f)} = Y$ and $D(f)$ is irreducible as well, whence g/f vanishes on all of $D(f)$. It follows that $f(1 \cdot g - 0 \cdot f) = 0$ as polynomial functions on X, hence $g/f = 0/1$ in $A(X)_{I(Y)}$.

Finally, to see that ϕ is surjective since, let $\overline{(U, \phi)} \in \mathcal{O}_{X,Y}$, $y \in U$ and V_y be an open subset of U containing y such that ϕ is given by g/f on V_y. Then $y \in D(f)$, so $D(f) \cap Y \neq \emptyset$, and $\phi(g/f) = \overline{(D(f), g/f)}$ is a stalk at Y which is equal to $\overline{(U, \phi)}$ since they agree on V_y (note that g/f is an element of the domain $A(X)_{I(Y)}$ since $f \notin I(Y)$ as $f(y) \neq 0$).

Ex 3.24

If $a \in V$, then $a \in U \cap V$, so any representative (V, ϕ) of a germ in the original stalk can be restricted down to U. The other direction is immediate since open subsets in U are open in X (by virtue of U being open). Passing up and down through the restriction doesn't change equivalence class since any regular function defined on V agrees with itself on $V \cap U$. So the restriction of stalks is bijective.

Since the original restriction maps in the sheaves are homomorphisms, it follows that if $(V, f), (V', f')$ are representatives of germs in $\mathcal{F}_a$, that their product

$$(V \cap V', f|_{V \cap V'} f'|_{V \cap V'})$$

restrict to the restrictions of their product,

$$(V \cap V' \cap U, (f|_{V \cap V'} f'|_{V \cap V'})\,|_U) = (V \cap V' \cap U, f|_{V \cap V' \cap U} f'|_{V \cap V' \cap U}).$$

The same is true for sums and any other algebraic operation, so we see that the restriction of the stalk is an isomorphism.

Ex 4.12

A general affine conic be given by

$$f(x,y) = a_5x^2 + a_4xy + a_3y^2 + a_2x + a_1y + a_0,$$

where the a_i are such that (f) is prime.

We will show that $A = \mathbb{K}[x,y]/(f)$ is isomorphic exactly one of either $B_1 = \mathbb{K}[x,y]/(g_1)$ or $B_2 = \mathbb{K}[x,y]/(g_2)$ where $g_1 = x^2 - y$ and $g_2 = xy - 1$.

First we show that $B_1 \ncong B_2$. To see this, note that $B_1 \cong \mathbb{K}[x]$ whilst $B_2 \cong \mathbb{K}[y, 1/y]$. We see that any element in B_2 is a sum of units, whilst the only units in B_1 are scalars, so the two algebras can't be isomorphic.

Now consider f again. We will show that we can obtain either g_1 or g_2 from f by a linear change of variables.

case $a_5 = a_3 = 0$**.** Since f is quadric, we have $a_4 \neq 0$ in this case, so we can write

$$f(x,y) = a_4\left(xy + \frac{a_2}{a_4}x + \frac{a_1}{a_4}y\right) + a_0 = a_4\left(x + \frac{a_1}{a_4}\right)\left(y + \frac{a_2}{a_4}\right) - \frac{a_1a_2}{a_4} + a_0.$$

Since f is irreducible, we can assume that $\frac{a_1a_2}{a_4} - a_0 \neq 0$, and after relabeling we have

$$f = (ax+b)(y+c) - 1$$

where we were able to rescale f so that $\frac{a_1a_2}{a_4} - a_0 = 1$ since we only care about $V(f)$. We now have that $f(x,y) = g_2(ax+b, y+c)$, so $\phi : \mathbb{K}[\boldsymbol{x}] \to \mathbb{K}[\boldsymbol{x}]$ where $\phi(x,y) = (ax+b, y+c)$ takes $I(g_2)$ to $I(f)$, hence it induces a well defined homomorphism $\overline{\phi} : A(V(g_2)) \to A(V(f))$, where $\overline{\phi}(h(x,y) + I(g_2)) = h(ax+b, y+c) + I(f)$, which is invertible with $\overline{\phi}^{-1}(h(x,y) + I(f)) = h\left(\frac{x-b}{a}, y-c\right) + I(g_2)$. We've shown that $A(V(f)) \cong A(V(g_2))$ whence $V(f) \cong V(g_2)$ by Corollary 4.8.

case $a_5 \neq 0, a_4 \neq 0, a_3 = 0$. By permuting variables, this case covers when $a_3 \neq 0, a_4 \neq 0, a_5 = 0$ as well. We care only about $V(f)$ so we can assume that $a_5 = 1$ and

$$\begin{aligned} f(x,y) &= x^2 + a_4xy + a_2x + a_1y + a_0. \\ &= \left(x + a_4y + a_2 - \frac{a_1}{a_4}\right)\left(x + \frac{a_1}{a_4}\right) + a_0 - \frac{a_2a_1}{a_4} + \frac{a_1^2}{a_4^2}, \end{aligned}$$

and $a_0 - \frac{a_2a_1}{a_4} + \frac{a_1^2}{a_4^2} \neq 0$ since f is reducible so after relabeling and division we get

$$af(x,y) = (ax + by + c)(x+d) - 1,$$

which just like in the previous case gives us an isomorphism between the $A(V(g_2))$ and $A(V(af)) = A(V(f))$ given by

$$\phi(h(x,y) + (g_2)) = h(ax + by + c, x + d) + (f)$$

and

$$\phi^{-1}(h(x,y) + (f)) = h\left(y - d, \frac{x}{b} + \frac{by}{a} - \frac{db}{a} - \frac{c}{b}\right) + (g_2).$$

case $a_5 \neq 0, a_4 = a_3 = 0$**.** By permuting variables, this case covers when $a_3 \neq 0, a_4 = a_5 = 0$ as well. We care only about $V(f)$ so we can assume that $a_5 = 1$ and

$$f(x,y) = x^2 + a_2 x + a_1 y + a_0,$$

which gives us an isomorphism between the $A(V(g_1))$ and $A(V(f))$ given by

$$\phi(h(x,y) + (g_1)) = h(x, a_2 x + a_1 y + a_0) + (f)$$

and

$$\phi^{-1}(h(x,y) + (f)) = h\left(x, \frac{y}{a_1} - \frac{a_2}{a_1}x - \frac{a_0}{a_1}\right) + (g_1).$$

case $a_5 \neq 0, a_3 \neq 0$**.** We care only about $V(f)$ so we can assume that $a_5 = 1$ and

$$\begin{aligned} f(x,y) &= x^2 + a_4 xy + a_3 y^2 + a_2 x + a_1 y + a_0. \\ &= (x + Hy + P)(x + Ty + Q) + a_0 - QP \end{aligned}$$

where

$$H = \frac{a_4}{2} + \sqrt{\frac{a_4^2}{4} - a_3},\ T = \frac{a_4}{2} - \sqrt{\frac{a_4^2}{4} - a_3},$$

and

$$Q = \left(1 - \frac{H}{T}\right)^{-1}\left(a_2 - \frac{1}{T}a_1\right) P = \left(1 - \frac{T}{H}\right)^{-1}\left(a_2 - \frac{1}{H}a_1\right)$$

when $a_4^2 \neq 4a_3$. We get an isomorphism between $A(V(g_2))$ and $A(V(f))$ like in previous cases but we skip the final calculations. This leaves one more case.

case $a_5 \neq 0$**, and** $a_4^2 = 4a_3$**.** We can assume that $a_5 = 1$ and get

$$\begin{aligned} f(x,y) &= a_4 x^2 + a_4^2 xy + a_3 y^2 + a_2 x + a_1 y + a_0 \\ &= (x + \sqrt{a_3}y)^2 + (a_4 - 2\sqrt{a_3})y + a_2 x + a_1 y + a_0 \\ &= (x + \sqrt{a_3}y)^2 + a_2 x + a_1 y + a_0, \end{aligned}$$

where we picked the square root of a_3 which yields cancelation as desired. We see that $A(V(f))$ is isomorphic to $A(V(g_1))$ in this case, and also that this case generalizes the $a_4 = a_3 = 0$ case.

Ex 4.13

Let **Aff** denote the category of affine varieties on, and **Coord** denote the category of coordinate rings.

Lemma 0.2. Let $F : \mathbf{Aff} \to \mathbf{Coord}$ denote the map which assigns each affine variety it's coordinate ring. Then F is an invertible contravariant functor, and $\mathbf{Aff^{op}} \cong \mathbf{Coord}$ as categories.

Proof. Remark 1.16 tells us that F is bijective on the objects, and Corollary 4.8 tells us that $f \mapsto f^*$ is bijective on the arrows. What remains to show is that $f \mapsto f^*$ preserves composition in a contravariant way. This is straightforward to verify. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be affine varieties and morphisms between them, and $h \in A(Z)$ be some regular function on Z. Then

$$(g \circ f)^*(h) = h \circ (g \circ f) = (h \circ g) \circ f = f^*(g^*(h)) = (f^* \circ g^*)(h)$$

□

The previous lemma tells us that epimorphisms (right cancelative maps), are mapped to monomorphisms (left cancelative) and vice versa. The problem though is that epimorphisms/monomorphisms need not be surjective/injective homomorphisms. The converse is true though, as we will soon see in the part (a).

(a)

Suppose that f is surjective. Then f is right cancelative in the category of **Set**, hence it's right cancelative and an epimorphism in the category of **Aff**. It follows that f^* is a monomorphism in the category **Coord**. Now let $g \in \ker(f^*)$, and consider the algebra homomorphism $\phi : \mathbb{K}[x] \to A(Y)$ given by $\phi(x) = g$. Then $f^* \circ \phi = f^* \circ 0$, which since f^* is a monomorphism yields that ϕ is the zero map, so $g = 0$ in $A(Y)$ and f^* is injective.

The converse is not true unfortunately, and this has to do with cases where the image of f is dense in Y, but not surjective. Consider for example $X = Y = \mathbb{A}^1$, and $f : X \to Y$ given by $f(x) = x^2$. Then f isn't surjective, but $\mathrm{im}(f)$ is dense in Y, so any $g_1, g_2 \in A(Y)$ that agree on $\mathrm{im}(f)$ agree on all of Y. Hence $f^*(g_1) = f^*(g_2) \Rightarrow g_1 = g_2$ and f^* is injective.

(b)

No, Example 4.9 exhibits a bijective morphism f of varieties $f : \mathbb{A}^1 \to V(x_1^2 - x_2^3)$, but the corresponding algebra homomorphism $f^* : A(V(x_1^2 - x_2^3)) \to \mathbb{K}[t]$ is not surjective.

(c)

Yes, isomorphisms $\mathbb{A}^1 \to \mathbb{A}^1$ correspond to polynomial maps which are invertible by polynomial maps, and these are always linear (in the univariate case).

(d)

No! Consider for example $f(x, y) = (x, x^2+y)$. This map is regular and bijective, and it's inverse is given by $f^{-1}(x, y) = (x, y - x^2)$, which is also regular and bijective.

Ex 4.19

(a) $\cong$ (c). An isomorphism $f : \mathbb{A}^1 \setminus \{1\} \to V(x_2 - x_1^2, x_3 - x_1^3) \setminus \{0\}$ is given by

$$f(x) = \left((x-1), (x-1)^2, (x-1)^3\right),$$

where the inverse is

$$f^{-1}(x_1, x_2, x_3) = x_1 + 1.$$

(d) $\ncong$ (a). We have that $V(x_1x_2)$ is reducible so $A(V(x_1x_2))$ is not an integral domain. Meanwhile, $\mathbb{A}^1 \setminus \{1\} \cong D(x-1)$ has coordinate ring $\mathbb{K}[x]_{x-1}$ which is an integral domain.

(a) $\ncong$ (e). We just saw that the coordinate ring of $\mathbb{A}^1 \setminus \{1\}$ is local. This is not the case for $V(x_2^2 - x_1^3 - x_1^2)$ since every point on the (infinite) variety induces a maximal ideal.
(a) $\ncong$ (f). We just saw that the coordinate ring of $\mathbb{A}^1 \setminus \{1\}$ is local. This is not the case for $V(x_2^2 - x_1^2 - 1)$ since every point on the (infinite) variety induces a maximal ideal.
(b) $\cong$ (d). We have $V(x_1^2 + x_2^2) = V((x_1 - ix_2)(x_1 + ix_2))$, and if $a, b \in \mathbb{A}^2$ is such that $ab = 0$, then

$$ab = 0 \Leftrightarrow \left(\frac{a+b}{2} + i\frac{a-b}{2i}\right)\left(\frac{a+b}{2} - i\frac{a-b}{2i}\right) = 0,$$

so the isomorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ $f(x_1, x_2) = \left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2i}\right)$ sends $V(x_1x_2)$ to $V((x_1 + ix_2)(x_1 - ix_2))$.

(d) $\ncong$ (e). We claim that $f = -x_2^2 + x_1^3 + x_1^2$ is irreducible. To see this, note that any factors would need to be of the form

$$(ax_1^2 + bx_1 + cx_2 + d)(Bx_1 + Cx_2 + D) =$$
$$aBx_1^3 + aCx_1^2x_2 + (aD + bB)x_1^2 + (bC + Bc)x_1x_2 + cCx_2^2 + (b + B)x_1 + (c + C)x_2 + dD$$

We get

$$aB = 1, cC = -1, aC = 0$$

which is impossible. It follows that $A(V(f))$ is an integral domain whilst $A(V(xy))$ isn't so the two varieties are not isomorphic.

(d) $\ncong$ (f). We claim that $x_1^2 - x_2^2 - 1$ is irreducible. To see this, note that any factors would need to be of the form

$$(ax_1+bx_2+c)(Ax_1+Bx_2+C) = aAx_1^2+bBx_2^2+cC+(aB+Ab)x_1x_2+(aC+Ac)x_1+(bC+bC)x_2.$$

so we need $a, b, c, A, B, C \in \mathbb{C}$ such that

$$\begin{aligned} aA &= 1 \\ bB &= -1 \\ cC &= -1 \\ aB &= -Ab \\ aC &= -Ac \\ bC &= -Bc. \end{aligned}$$

If we fix $a = 1$ we get $A = 1$, then $B = -b$, $c = -C$, after which we get $b = 1, B = -1$ or $b = -1, B = 1$. Assume the first case, then $bC = -Bc$ turns into $C = c$, contradicting our previous equation $c = -C$ since $c \neq 0$ due to $cC = -1$. It follows that $A(V(x_1^2 - x_2^2 - 1))$ is an integral domain which $A(V(x_1x_2))$ is not.

(e) $\ncong$ (f).
Let $X = V(x_1^2 - x_2^2 - 1), Y = V(x_2^2 - x_1^3 - x_1^2)$. When graphing the projection of Y onto $\mathbb{R}$, we see that the resulting curve intersects itself at the origin. Meanwhile X is isomorphic to the unit circle by $(x_1, x_2) \mapsto (x_1, ix_2)$, which doesn't intersect itself. Thus we guess that investigating the behaviour of $A(X)$ near the origin might lead to a proof that the two varieties aren't isomorphic. We give it a try by looking at the completion of $A(X)$ by the ideal $I((0,0)) = (x, y)$. Proposition 10.13 in Atiyah MacDonald tells us that

$$\widehat{A(Y)} \cong \mathbb{K}[[x_1, x_2]] \otimes_{\mathbb{K}[x_1, x_2]} A(Y) \cong \mathbb{K}[[x_1, x_2]]/I(Y).$$

We now claim that f_Y factors in $\mathbb{K}[[x_1, x_2]]$, whence $\widehat{A(Y)}$ isn't a domain. We will use Hensel's Lemma to show this, more specifically, the variant given in Theorem 7.3 in Eisenbud. We use the lemma to find a square root of $x_1 + 1$ by searching for a solution of $g(z) = X^2 - (x_1 + 1) = 0$ in $\widehat{A(Y)}[z]$. We have that $g'(X)^2 = 4X^2$, and

$$g(1) = 1 - x_1 - 1 = x_1 \equiv 0 \bmod g'(1)(x_1, x_2) = (x_1, x_2),$$

whence applying the lemma gives us a root b of g in $\widehat{A(Y)}$, so that $x_1 + 1 = b^2$ and $f_Y = x_2^2 - x_1^2(x_1 + 1) = (x_2 - x_1b)(x_2 + x_1b)$.

Since X is isomorphic to the unit circle, we will just say that X is the unit circle from now on. We will now show that every completion of $A(X)$ is a domain, after which the non-isomorphism will follow. Consider the completion of $A(X)$ at the point p on X (I.e the maximal ideal $I(p)$). This is isomorphic to the completion of $A(I(f_X(x_1+p_1, x_2+p_2)))$ at the origin, where $f_X = x_1^2 + x_2^2 - 1$ (since we're just "moving" the circle such that p winds up at the origin). Denote the new polynomial by $f_{X,p}$. Then

$$\begin{aligned} f_{X,p} &= (x_1+p_1)^2 + (x_2+p_2)^2 - 1 \\ &= x_1^2 + 2x_1p_2 + p_1^2 + x_2^2 + 2x_2p_2 + p_2^2 - 1 \\ &= x_1^2 + 2x_1p_2 + x_2^2 + 2x_2p_2 \end{aligned}$$

since p is on the circle and $p_1^2 + p_2^2 = 1$. Furthermore,

$$f_{X,p} = x_1^2 + 2x_1p_2 + x_2^2 + 2x_2p_2 \qquad =$$

We are done if we can show that $f_{X,p}$ is irreducible in $\mathbb{K}[[x_1, x_2]]$ for all p. TODO Todo todo finnish

Conclusion: We have the following equivalence classes

$$\begin{gathered} (a) \cong (c), \\ (b) \cong (d), \\ (e), \\ (f), \end{gathered}$$

Ex 5.7

(a)

I was stuck here for a long time, so we'll solve this exercise in a perhaps overly detailed manner.

Let $X_1 = X_2 = \mathbb{A}^1$, and $U_i \subset X_i$ be $D(x)$, I.e $\mathbb{A}^1 \setminus 0$. Let $\mathbb{P}^1$ be X_1 glued with X_2 via the isomorphism $\phi : U_1 \to U_2$, where $\phi : x \mapsto 1/x$. Let i_1, i_2 be the injection maps from X_1, X_2 to $\mathbb{P}^1$.

Let Y be an prevariety and $f : Y \to \mathbb{P}^1$ be a map. Let $Y_1 = f^{-1}(i_1(X_1)), Y_2 = f^{-1}(i_2(X_2))$. We claim that f is a morphism precisely when both

$$\begin{gathered} f_1 : Y_1 \to \mathbb{A}^1, f_1 : y \mapsto i_1^{-1}(f(y)), \\ f_2 : Y_2 \to \mathbb{A}^1, f_2 : y \mapsto i_2^{-1}(f(y)). \end{gathered}$$

First of, we have that f is a morphism if and only if both

$$\begin{gathered} \hat{f}_1 : Y_1 \to \mathbb{P}^1, \hat{f}_1 : y \mapsto f(y), \\ \hat{f}_2 : Y_2 \to \mathbb{P}^1, \hat{f}_2 : y \mapsto f(y), \end{gathered}$$

are morphism by Remark 4.5 (b) and Lemma 4.6. Also, the injections i_1, i_2 are both isomorphisms onto their images (they are both their own inverse), hence each $\hat{f}_k$ is a morphism exactly when $f_k = i_k^{-1} \circ \hat{f}_k$ is a morphism.

Now, set $Y = \mathbb{A}^1 \setminus 0 = D(x)$. Then Y_1, Y_2 are both open in $\mathbb{A}^1$, and all such sets are distinguished. Thus we can suppose $Y_1 = D(h)$ and we can write

$$f_1 = \frac{g}{h^k}.$$

After canceling common factors and extracting all powers of x we are left with an expression of the form

$$f = x^m \frac{G}{H}$$

where $m \in \mathbb{Z}$ and $(G, H) = 1, x \not| G$, $x \not| H$. But i_1^{-1} is the identity, so $f = f_1$ on $D(h_1)$ and $Hf - x^m G = 0$ here. But $D(h_1)$ is dense in Y, so $Hf - x^m G = 0$ on all of $D(x)$ and

$$f = x^m \frac{G}{H}$$

Dividing by H might seem suspicious here, but this results in a well-defined function since $(G, H) = 1$. Indeed, we can send any root r of H to $r \mapsto i_2(0) =$ "∞", after which chasing the definitions yields that $Y_1 = D(xH), Y_2 = D(xG)$ and $f_1 = f, f_2 = 1/f$ which are both well-defined morphisms on their corresponding domains.

If $m = 0$, then $f(0) \in \mathbb{A}^1 \setminus 0 = i_1(X_1) \cap i_2(X_2)$ and we can trivially extend the domain of f to all of $\mathbb{A}^1$. If $m > 0$, then $f(0) = 0$ and we can extend f_1 by sending $f_1 : 0 \mapsto 0$, whilst we can leave f_2 untouched as $0 \notin i_2(X_2)$ and $0 \notin Y_2$. If $m < 0$ we have $f(0) \notin i_1(X_1)$, so we only need to worry about f_2. Since $i_2^{-1} : x \mapsto 1/x$, we have that $f_2 = x^{-m}\frac{H}{G}$ and we can simply extend f_2 by sending $f_2 : 0 \mapsto 0$ again.

(b)

In part (a) we had that every regular function from $D(x) \to \mathbb{P}^1$ could be written as $x^m \frac{G}{H}$ coprime, and as G, H are univariate polynomials, they'll never be simultaneously zero. This is not the case for coprime polynomial in two variables.

Consider the regular function $f : \mathbb{A}^2 \setminus 0 = D(x) \cup D(y) \to \mathbb{P}^1$ given by

$$f(x, y) = \frac{y}{x}$$

Then f is regular because (using the notation of part (a) with $Y = D(x) \cup D(y)$), $Y_1 = D(x)$, $Y_2 = D(y)$ and $f_1 = \frac{y}{x}, f_2 = \frac{x}{y}$ are both quotient of polynomials with non-vanishing denominators on their domains.

Now suppose towards a contradiction that $\hat{f}$ is an extension of f to the affine plane and let $\hat{f}_k = i_k^{-1} \circ \hat{f}\big|_{\hat{f}^{-1}(i_k(X_k))}$ as before. Then $(0,0)$ must lie in one or both of $\hat{f}^{-1}(i_1(X_1))$ or $\hat{f}^{-1}(i_2(X_2))$, and we assume the index 1 case. Now let U_0 be some open neighbourhood of $(0,0)$ such that $\hat{f}_1$ is a rational function on U_0. We have $\hat{f}_1 = y/x$ on $D(x) \cup D(y)$, hence on $U_0 \setminus (0,0)$, and there is no rational function on any open set U containing 0 which is y/x on $U \setminus (0,0)$, so we arrive in a contradiction.

(c)

Let $f : \mathbb{P}^1 \to \mathbb{A}^1$ and $f_1 = f \circ i_1, f_2 = f \circ i_2$. Then f_1, f_2 are both morphisms $\mathbb{A}^1 \to \mathbb{A}^1$ by the glueing construction of $\mathbb{P}^1$. On $\mathbb{A}^1 \setminus \{0\}$ we have that $f_1(x) = f_2(1/x)$, again by the construction of $\mathbb{P}^1$. Since f_1, f_2 are both in $A(\mathbb{A}^1) = \mathbb{K}[x]$, this is possible only if $f_1 = f_2 \in \mathbb{K}$.

Ex 5.8

We will solve these exercises using homogeneous coordinates. That is, we will write points $i_1(x) = [x : 1]$ and $i_2(x) = [1 : x]$ with the equivalence $[x_1 : x_2] = \lambda[x_1 : x_2]$ for all $\lambda \in \mathbb{K} \setminus 0$. Then i_1 is injective since $i_1(x_1) = i_1(x_2)$ implies $[x_1 : 1] = [x_2 : 1]$ which is the case only when $x_1 = x_2$. Moreover if $x \neq 0$,

$$i_1(x) = [x : 1] = 1/x[x : 1] = [1 : 1/x] = i_2(1/x)$$

and we see that the equivalence classes $[x_1 : x_2]$ correspond exactly to the points on $\mathbb{P}^1$.

(a)

We begin by retracing some of the steps of Exercise 5.7 (a) with this new notation to show the following lemma.

Lemma 0.3. Let $f : \mathbb{A}^1 \to \mathbb{P}^1$ be a morphism. Then f must be of the form $f(x) = [p(x) : q(x)]$, where $p, q \in \mathbb{K}[\boldsymbol{x}]$ are polynomials with no root in common.

Proof. Let $Y_1 = f^{-1}(i_1(\mathbb{A}^1)), Y_2 = f^{-1}(i_2(\mathbb{A}^1))$ and $f_1 = i_1^{-1} \circ f\big|_{Y_1}$, $f_2 = i_2^{-1} \circ f\big|_{Y_2}$. Then both f_k are morphisms $\mathbb{A}^1 \supseteq Y_k \to \mathbb{A}^1$, hence they can be written as

$$f_k = \frac{g_k}{h_k}$$

with $(g_k, h_k) = 1$. But then

$$f\big|_{i_1(Y_1)} = i_1 \circ f_1 = [g_1(x)/h_1(x) : 1] = [g_1(x) : h_1(x)],$$

and as $\mathbb{P}^1$ is irreducible and Y_1 is dense in $\mathbb{P}^1$, we have

$$f = [g_1(x) : h_1(x)]$$

on all of $\mathbb{P}^1$ and we are done. Note that it also follows that $g_1 = h_2, g_2 = h_1$. $\square$

Now let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an automorphism. From the lemma it follows that we can write

$$f\big|_{i_1(\mathbb{A}^1)}([x:1]) = [p_1(x) : q_1(x)],$$
$$f\big|_{i_2(\mathbb{A}^1)}([1:x]) = [p_2(x) : q_2(x)],$$

and as these expressions agree on the intersection $i_1(\mathbb{A}^1) \cap i_2(\mathbb{A}^2) = \{[x_1 : x_2] = [x_1/x_2 : 1] : x_1, x_2 \in \mathbb{K} \setminus 0\}$, we have

$$[p_1(x_1/x_2) : q_1(x_1/x_2)] = [p_2(x_1/x_2) : q_2(x_1/x_2)]$$

which in turn yields

$$p_1(x)q_2(x) = p_2(x)q_1(x)$$

whenever $x \notin V(q_1, q_2)$, but if $x \in V(q_1, q_2)$ then both sides are 0 so we have

$$p_1(x)q_2(x) = p_2(x)q_1(x)$$

everywhere and $(p_k, g_k) = 1$ yields $p_1 = cp_2, q_1 = cq_2$. If we now relabel $p = p_1, q = q_1$, we get

$$f([x_1 : x_2]) = [p(x_1/x_2) : q(x_1/x_2)]$$

when $x_2 \neq 0$ and

$$f([x_1 : x_2]) = [cp(x_2/x_1) : cq(x_2/x_1)] = [p(x_2/x_1) : q(x_2/x_1)]$$

when $x_1 \neq 0$. Moreover, since f is injective, we have that f is injective on $i_1(\mathbb{A}^1)$, hence p, q must have degree 1 and we are done.

(b)

Ex 5.9

For any ringed spaces X, Y, we always have that for any morphism $f : X \to Y$ the pull-back $f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is an algebra homomorphism. This is required by the definition of a morphism of a ringed space. Thus we will only focus on the other direction for both subproblems.

(a)

Let $f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ be an algebra homomorphism. Let $y_1, y_2, \ldots, y_n$ be the coordinate functions on Y, and $\phi_i = f^*(y_i)$. We claim that

$$f = (\phi_1, \phi_2, \ldots, \phi_n)$$

is a morphism $X \to Y$.

Let U_i be a cover of X by affine varieties. It follow from the definition of a sheaf that f^* followed by restriction to any U_i is an algebra homomorphism $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(U_i)$ and we write this as

$$f_i^* = ?\big|_{U_i} \circ f^*.$$

From the proof of Corollary 4.8, we know that f_i^* is the pullback of $f\big|_{U_i}$, hence $f\big|_{U_i}$ is a morphism $U_i \to X$ and f is a morphism by Lemma 4.6.

(b)

No, for example Exercise 5.7 (c) tells us that $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \cong \mathbb{K}$, so there is only one non-zero $\mathbb{K}$-algebra homomorphism $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \to \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) = \mathbb{K}[x]$ and that is the injection $\mathbb{K} \to \mathbb{K}[x], k \mapsto k$. Meanwhile, from the construction of $\mathbb{P}^1$ we already have two non-trivial homomorphism i_1, i_2 for $\mathbb{A}^1 \to \mathbb{P}^1$.

Ex 5.11

Let $Y \subseteq X$ be a closed subset and $U \subseteq X$ be an open affine subset. Let $i : U \to \mathbb{A}^n$ be an embedding into affine space. Then $i(U \cap Y)$ is closed in $\mathbb{A}^n$, hence a zero set of some polynomials $V(f_1, f_2, \ldots, f_m)$. In other words, $U \cap Y$ is isomorphic to $V(f_1, f_2, \ldots, f_m)$ via i as ringed spaces. Hence $U \cap Y$ is an affine open set of X.

It follows that if U_i is a finite cover of X by affine open sets, then $Y \cap U_i$ is a finite cover of Y by affine open sets, and Y is a prevariety.

Moreover, the structure sheaf on Y obtained from gluing the $U_i \cap Y$ together agrees with that of Construction 5.10 (b). Let $U \subset Y$ be an open set, $\mathcal{O}_y$ be the structure sheaf from Construction 5.10 (b) and $\mathcal{O}'_Y$ be the structure sheaf obtained from glueing $U_i \cap Y$.

We have $f \in \mathcal{O}'_Y(U)$ precisely when f is regular on all $U_i \cap Y$. Now let $a \in U$ and i be such that $a \in U_i$. Then let F be any element in the preimage of f under the quotient homomorphism $A(U_i) \to A(U_i \cap Y) = A(U_i) + I(Y)$. Then f and F agree on $U_i \cap Y$, so $f \in \mathcal{O}_Y(U)$ by Construction 5.10 (b).

Now let $f \in \mathcal{O}_Y(U)$. Then given a U_i and $a \in U_i \cap Y$, we have that there is some open $V_a \in X$ containing a such that there is some $g \in \mathcal{O}_X(V_a)$ and $f\big|_{V_a \cap U} = g\big|_{V_a \cap U}$. But then $f\big|_{V_a \cap U} \in \mathcal{O}_X(V_a \cap U)$, so given any $a \in U_i \cap U$, we can find V_a such that $f\big|_{V_a \cap U}$ is regular, hence f is regular on $U_i \cap U$ and finally on U by Lemma 4.6.

Ex 5.21

Lemma 0.4. Let V be a prevariety obtained by glueing the affine open sets U_i. Then Z is closed in V if and only if every $Z \cap U_i$ is closed.

Proof. We have that Z is closed if and only if $V \setminus Z$ is open, which happens if and only if every $U_i \cap (V \setminus Z) = U_i \setminus Z$ is open, and this is the case if and only if every $U_i \cap Z$ is closed. $\square$

Let i_1, i_2 be the injections into $\mathbb{P}^1$ and define $X_1 = i_1(\mathbb{A}^1), X_2 = i_2(\mathbb{A}^1)$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ can be obtained by glueing the patches $X_i \times X_j,\ i,j \in \{1,2\}$. The diagonal $\Delta_{\mathbb{P}^1}$ intersects $X_1 \times X_1$ and $X_2 \times X_2$ as $\Delta_{\mathbb{A}^1}$ and is thus closed there. It intersects $X_1 \times X_2, X_2 \times X_2$ as $V(xy-1)$ and is closed there as well. Hence $\Delta_{\mathbb{P}^1}$ is closed in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1$ is separated.

Ex 5.22

(a)

Let $\pi_{xx} : (X \times Y) \times (X \times Y) \to X \times X$ be the projection morphism onto the two X coordinates. Then the inverse image of the diagonal of X,

$$\pi_{xx}^{-1}(\Delta(X)) = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y) : x_1 = x_2\}$$

is closed in $(X \times Y) \times (X \times Y)$, and similarly, we have

$$\pi_{yy}^{-1}(\Delta(Y)) = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y) : y_1 = y_2\}$$

closed as well. Intersecting the two yields $\Delta_{X \times Y}$ and we are done.

(b)

Let U_i, V_i be finite open affine coverings of X, Y respectively. Then $U_i \times V_j$ is a finite open covering of $X \times Y$. Every U_i, V_j is irreducible, since if $A \cup B = U_i$ are two closed non-trivial sets in U_i, then $U_i \setminus A, U_i \setminus B$ are open in U_i and in X, whence $X \setminus (U_i \setminus A) = A \cup (X \setminus U_i)$ and $B \cup (X \setminus U_i)$ are two closed non-trivial sets. Their union is

$$(A \cup (X \setminus U_i)) \cup (B \cup (X \setminus U_i)) = (A \cup B) \cup (X \setminus U_i) = U_i \cup (X \setminus U_i) = X.$$

We now have that every $U_i \times V_j$ is irreducible by Exercise 2.24. If we can show that every $U_i \times V_j$ intersects every $U_r \times V_s$, then we can apply Exercise 2.21 which yields $X \times Y$ irreducible.

Every U_i, U_r and V_j, V_s intersect since they are open sets in the irreducible spaces X, Y, and it follows immediately that $U_i \times V_j$ and $U_r \times V_s$ intersect.

Ex 5.23

(a)

Let U, V be affine open sets in the variety X, and $\pi_U : U \times V \to X, \pi_V : U \times V \to X$ be the projections (followed by inclusions into X). Then it follows from Proposition 5.20 (b) that $\{(u,v) \in U \times V : \pi_U(u) = \pi_V(v)\} = \Delta_{U \cap V}$ is closed in $U \times V$.

Since both U, V are affine, we have that $U \times V$ is affine as well. Let $e : U \times V \xrightarrow{\sim} Z \subset \mathbb{A}^n$ be an embedding onto some Zariski closed set in affine space. Then e sends closed sets to closed sets. Hence $e(\Delta_{U \cap V})$ is closed in $\mathbb{A}^n$, so $\Delta_{U \cap V}$ is an affine variety.

Finally, $f : U \cap V \to \Delta_{U \cap V}, f : x \mapsto (x, x)$ is an isomorphism with inverse $f^{-1} : (x, x) \mapsto x$, and $U \cap V$ is affine as well.

(b)

Like above, we have that $X \cap Y \cong \Delta_{X \cap Y}$, so it's enough to consider the diagonal. Also like above, the diagonal $\Delta_{X \cap Y}$ is closed in $X \times Y$, and $X \times Y$ is affine so embeddable into some Zariski closed set in affine space, hence $\Delta_{X \cap Y}$ is as well. From now on we identify all of our varieties as embedded into affine space this way (that is we may assume that $\Delta_{X \cap Y}, X \times Y \subset \mathbb{A}^{2n}$ are Zariski closed).

Now, let U be some irreducible component of $\Delta_{X \cap Y}$. We have $X \times Y$ irreducible by Exercise 5.22, so we can apply Proposition 2.28 (b) to get

$$\begin{aligned}\dim \Delta_{X \cap Y} &= \dim X \times Y - \operatorname{codim}_{X \times Y}(U) \\ &= \dim X + \dim Y - \operatorname{codim}_{X \times Y}(U).\end{aligned}$$

Hence we are done if we can show that $\operatorname{codim}_{X \times Y}(U) \leq n$. This follows from Lemma XYZ which says that $\operatorname{codim}_{X \times Y}(U)$ is the same as the height of $I(U)$ in $A(X \times Y)$. Since U is an irreducible component of $\Delta_{X \times Y}$, we have that $I(U)$ is a minimal prime ideal of $I(\Delta_{X \times Y})$, hence $I(U)$ and $I(\Delta_{X \times Y})$ have the same height ?
but we claim that the height of $I(U)$
Now, $I(U)$ is a minimal ideal of $I(\Delta_{X,Y})$
, and since $U \subseteq \Delta_{X \cap Y}$, we have $I(U) \supseteq I(\Delta_{X \cap Y})$
$\cong A(X) \otimes A(Y)$.
TODO todo Todo : Finnish!

Ex 5.24

We will look locally enough where X is affine, and reduce this problem to the affine case which we already solved in Exercise 2.34 (b).

Let X be a variety, $U \subseteq X$ a dense open subset and

$$Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset X$$

a chain of irreducible closed subsets. Then let Z be a affine open set which intersects Y_0. Since U is dense, we have that U intersects Z as well. Let

$e : Z \to \mathbb{A}^n$ be an embedding into affine space such that $e(Z)$ is the zero locust of some set of polynomials. All our sets intersect Z, and we write $Z' = e(Z), Y_i' = e(Z \cap Y_i), U' = e(Z \cap U)$.

We can use the construction from Exercise 2.34 (a) to show that $Y_{i-1}' \subsetneq Y_i'$ are strict inclusions. In brief, we set $V_i = Y_{i-1} \setminus Y_i$, which is non-empty and open in Y_i and intersects $Z \cap Y_i$ since Y_i is irreducible, hence $\emptyset \neq e(Z \cap V_i) \subset Y_i' \setminus Y_{i-1}'$.

Now we claim that U' is dense in Z', which is equivalent to U' intersecting every open set of Z' (and this is the property we will need anyhow). Let V' be open in Z' and $V = e^{-1}(V)$. Then V is open in Z, and by the definition of the subspace topology, we have some open V'' in X such that $V'' \cap Z = V$. But Z is open, so V is open in X, and V intersects U since U is dense in X.

To recap, we now have a strict chain of closed irreducible subsets in affine space

$$Y_0' \subsetneq Y_1' \subsetneq \ldots \subsetneq Y_n' \subset Z'$$

and a dense open set $U' \subseteq Z'$, and we are free to use the translation of Exercise 2.34 (b) to conclude that $\dim U' = \dim U \geq n$.

The other inequality is Exercise 2.30.

Ex 6.14

Write $a = [a_0 : a_1 : \ldots : a_n]$ and $e_{i,j} = a_j x_i - a_i x_j$. We claim that $I_p(\{a\}) = (e_{i,j} : 0 \leq i < j \leq n)$. First note that $e_{i,j}(a) = a_j a_i - a_i a_j = 0$ for all i, j. Moreover, suppose that $b = [b_0 : b_1 : \ldots : b_n]$ is such that $e_{i,j}(b) = 0$ for all i, j. Then $a_i b_j = a_j b_i$ for all i, j, which is exactly the same thing as saying all of the minors of order 2 of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ b_0 & b_1 & \ldots & b_n \end{pmatrix}$$

vanish, which in turn happens if and only if the matrix has at most rank 1, I.e $a = b$ in $\mathbb{P}^n$. We've shown that $V((e_{i,j} : 0 \leq i < j \leq n)) = \{a\}$ and we are done.

Ex 6.29

In affine space, a line L_1 and a point a not on this line, span a plane P_1, and any line which intersect both L_1, a lies on this plane. If we then pick a line L_2 which doesn't intersect on this plane, we see that no line which intersects both L_1, a, also intersects L_2. Any lines L_2 which doesn't intersect P_1 is parallel to L_1, and since parallel lines in general don't intersect in $\mathbb{A}^3$, we see that it's impossible to find a third line L which intersects L_1, L_2, a when L_1, L_2 are parallel.

There is one more case where it's impossible to find such L. Let q be the point where L_2 meets P_1. If q, a spans a line which is parallel to L_1, then it's again impossible to find L which intersects L_1, L_2, a, since such a line would have to contain both q, a, whence it wont meet L_1.

This latter case correspond to when the plane P_2 containing L_2 and a is parallel to L_1.

Projective space is more flexible as we have points at infinity, and parallel lines intersect at infinity. Here if either P_1, L_2 or P_2, L_1 are parallel, then there is a line L through a in each corresponding plane P_i which intersects the other line L_j $i \neq j$ at infinity.

We solve the problem in $\mathbb{A}^4$. Here the lines L_1, L_2 corresponds to planes through the origin spanned by the vectors u_1, u_2 and v_1, v_2 respectively. a corresponds to a line through the origin spanned by a vector, which we will denote by a as well. The planes P_1, P_2 now correspond to three dimensional hypersurfaces spanned by a, u_1, u_2 and a, v_1, v_2 respectively. Their intersection is a plane which contains a, spanned by say a, w. Then the intersection plane meets the hypersurface corresponding to L_1 in at least one point, parameterised by say $sa + tw = s'a + t'u_1 + r'u_2$, and we have that $(s + s')a + tw = t'u_1 + r'u_2$, so the plane $\langle a, w\rangle$ meets the plane of $\langle u_1, u_2\rangle$. The same is of course true of the plane $\langle v_1, v_2\rangle$, hence the line L in $\mathbb{P}^3$ corresponding to $\langle a, w\rangle$ will meet all of a, L_1, L_2.

This line is unique, since the plane $\langle a, w\rangle$ is unique. Indeed, since $L_1 \neq L_2$, we have that the intersection $\langle a, u_1, u_2\rangle \cap \langle a, v_1, v_2\rangle$ is exactly a plane.

Ex 6.30

(a)

The only if part is immediate.

Let R be a homogeneous ring such that for any homogeneous $f, g \in R$, we have $fg = 0$ implies $f = 0$ or $g = 0$, and suppose that $p, q \in R$ are such that $pq = 0$. Write $r = \deg p, s = \deg q$, and let p_r, q_s be the leading homogeneous components. Since a graded ring is the direct sum of it's homogeneous components, every homogeneous component of pq must be zero, in particular the leading component $(pq)_{rs} = p_r q_s$. Our assumption on R then tells us that either $p_r = 0$ or $q_s = 0$, and since these are the leading components, $p = 0$ or $q = 0$, and R is an integral domain.

(b)

Suppose $f, g \in S(X)$ are such that $fg = 0$. Then $X = V(fg) = V(f) \cup V(g)$ and if X is irreducible, we must have either $V(f) = X$ or $V(g) = X$, whence $f = 0$ or $g = 0$ and $S(X)$ must be an integral domain.

Similarly, if X is reducible, let $X = U \cup V$ be a non-trivial decomposition and $f \in I_p(U), g \in I_p(V)$ be non-zero polynomials. Then fg vanishes on both U and V hence on X and $fg = 0$, so $S(X)$ isn't an integral domain.

Ex 6.31

(a)

By Lemma 6.18, any strictly increasing chain of varieties in $\mathbb{P}^n$ corresponds to a chain of cones in $\mathbb{A}^{n+1}$, to which we can append a point immediately after the empty variety a la

$$X_0 = \emptyset \subsetneq \{a\} \subsetneq C(X_1) \subsetneq C(X_2) \subsetneq \ldots \subsetneq C(X_m),$$

hence $\dim C(X) \geq 1 + \dim X$.

For the other direction, we take the opportunity to explore homogeneous coordinate ring and pass to the algebraic side. We provide a sequence of lemmas to this end.

Lemma 0.5. Let X be a projective variety. Then

$$S(X) = A(C(X)).$$

Proof. It follows from Remark 6.17 that $I_p(X) = I_a(C(X))$, after which the equality of the lemma follows from the definitions of homogeneous and affine coordinate rings. $\square$

Lemma 0.6. Let X be a non-empty irreducible projective variety. Then $S(X)$ contains a homogeneous prime ideal J of height 1.

Proof. Let x_i be a non-zero coordinate on $C(X)$. Then (x_i) is a homogeneous ideal in $A(C(X))$. Moreover, (x_i) is prime, as it's the image of a prime ideal under the surjective ring homomorphism $\pi : \mathbb{K}[\boldsymbol{x}] \to A(C(X)), \pi : x \mapsto x + I_a(C(X))$. Finally, (x_i) is its own minimal ideal, and therefore has height at least one by Krull's Principal Ideal theorem, and height exactly one since X is irreducible, and $S(X) = A(C(X))$ is an integral domain so (0) is prime there. $\square$

Corollary 0.7. If X is a projective variety, then $1 + \dim X \leq \dim C(X)$.

Proof. The respective coordinate rings are equal, and by the lemma, $A(C(X))$ contains a maximal chain of prime ideals which are all homogeneous, and since all of these but possibly the irrelevant ideals correspond to irreducible projective varieties, the corollary follows. $\square$

(b)

We can reduce to the case where X, Y are pure dimensional by simply ignoring irreducible components of non-maximal dimension. It follows from part (a) that $\dim C(X) + \dim C(X) \geq n+2$, whence Exercise 5.23 (a) yields

$$\dim C(X) \cap C(Y) \geq \dim C(X) + \dim C(Y) - (n+1) \geq 1.$$

Now, if $x \in C(X) \cap C(Y)$ then $\lambda x \in C(X)$ and $\lambda x \in C(Y)$ for all λ, hence $x \in X \cap Y$ and $x \in C(X \cap Y)$. The other inclusion follows similarly, hence $C(X \cap Y) = C(X) \cap C(Y)$ has dimension at least 1, so it's at least a line in affine space, and least a point in projective space.

Ex 6.36

In the real plane, our surface is given by the graph $x_2 = \pm\sqrt{\frac{1}{x_1} + x_1^2}$ when $x_1 \in \mathbb{R}\backslash(-1, 0]$. For large x_1, we have two asymptotes where $x_2 \to x_1$ and $x_2 \to -x_1$. Hence we expect two points at infinity, namely $a = [0:1:1], b = [0:-1:1]$.

When x_1 goes to 0 from the positive side we see that $x_2 \to \pm\frac{1}{\sqrt{x_1}} \to \pm\infty$. So we might also expect the points at infinity $c = [0:0:1], d = [0:0:-1]$. However, they are not in the closure as we can see bellow. Perhaps because we don't have any points on the curve with $x_1 = 0$?

We see that a, b are the only points at infinity, since $\overline{X} = V_p(x_1^3 - x_1x_2^2 + x_0^3)$, which when intersected with $V_p(x_0)$ becomes $V_p(x_1^2 - x_2^2) = V_p((x_1 - x_2)(x_2 + x_1))$, and consists of a, b exactly.

Ex 7.3

(a)

Let $\mathbb{P}^1$ be the projective line as introduced in Example 5.5 (a), and $\mathbb{P}^{1'}$ be the projective line as given in Definition 6.1, using Notation 6.2, with structure sheaf as in Definition 7.2. We will show that the two are isomorphic.

As a set, we have that $\mathbb{P}^1 = X_0 \coprod X_1 / \sim$ where $X_0 = X_1 = \mathbb{A}^1$ and $x_0 \sim x_1$ for $x_0 \in X_0, x_1 \in X_1$ whenever $x_0, x_1 \neq 0$ and $x_0 = 1/x_1$.

We claim that the function $f : \mathbb{P}^1 \to \mathbb{P}^{1'}$ which sends $f(x_0) = [x_0 : 1], x_0 \in X_0$ and $f(x_1) = [1 : x_0], x_1 \in X_1$ is well-defined and bijective. To show that it's well-defined, let $x_0 \sim x_1$. Then $x_0, x_1 \neq 0$ and

$$f(x_1) = [1 : x_1] = 1/x_1[1 : x_1] = [1/x_1 : 1] = [x_0 : 1] = f(x_0).$$

Now suppose that $f(x_0) = f(x_1)$. This is impossible whenever either $x_0 = 0$ or $x_1 = 0$, so we have $x_0, x_1 \neq 0$. Then

$$[1 : x_1] = [x_0 : 1] = [1 : 1/x_0]$$

hence $x_1 = 1/x_0$ and $x_0 \sim x_1$. Finally, let $x = [x_0 : x_1] \in \mathbb{P}^{1'}$. If $x_0 = 0$, then $x = f(0_0)$, otherwise $x = f(x_1/x_0)$ with $x_1/x_0 \in X_1$. Hence f is a bijective function.

Now suppose that $Z \subseteq \mathbb{P}^1$ is a closed set. Then $Z_0 = Z \cap X_0, Z_1 = Z \cap X_1$ are both closed in $\mathbb{A}^1$. Let $J_0 = I(Z_0), J_1 = I(Z_1)$. Then let $g \in J_1$ and $x_1 \in Z_1$. We have that $g^h([1 : x_1]) = g(x_1) = 0$, hence g^h vanishes on $f(Z_1)$. Similarly, now let $g^h \in J_1^h$ and $[x_0 : x_1] \in V_p(J_0^h)$. First of, every term but the leading term of g^h has a factor x_0, hence $g^h([0 : 1]) \neq 0$ and $x_0 \neq 0$. Thus we can assume $x_0 = 1$. Now, $(g^h)^i = g$, so $g(x_1) = g^{hi}(x_1) = g^h([1 : x_1]) = 0$. It follows that $f(Z_k) = V_p(J_k^h)$ for $k = 0, 1$, and $f(Z) = V_p(J_0^h) \cup V_p(J_1)^h$ is closed.

Now let $Z \subseteq \mathbb{P}^{1'}$ be closed, and suppose $Z = V_p(J)$. Then let $g \in J$, $[1 : x_1] \in Z \cap U_0$. Before we move on, we need some new notation. Let $g^{i_0} = g(1, x_1)$ and $g^{i_1} = g(x_0, 1)$ be the dehomogenizations onto each coordinate. Then $0 = g([1 : x_1]) = g^{i_0}(x_1)$ so $f^{-1}(Z \cap U_0) \subseteq V_a(J^i)$. Now suppose that $x_1 \in V_a(J^{i_0})$. Then again, $g([1 : x_1]) = g^{i_0}(x_1) = 0$, so $f(x_1) \in Z \cap U_0$ and $f^{-1}(Z \cap U_0) = V_a(J^{i_0})$. We get

$$V_a(J^{i_0}) \cup V_a(J^{i_1}) = f^{-1}(Z \cap U_0) \cup f^{-1}(Z \cap U_1) = f^{-1}(Z).$$

Now, let i_0, i_1 be the injections $X_1, X_2 \to \mathbb{P}^1$. Then, $i_1^{-1}(V_a(J^{i_0}))$ is closed in X_1, as it i_1^{-1} leaves the x_1 coordinates unchanged and it's still the vanishing set of J^{i_0}. It follows from the lemma bellow that $i_1^{-1}(V_a(J^{i_1}))$ is closed as well.

Lemma 0.8. Let X_0, X_1, i_0, i_1 be as in the gluing construction of $\mathbb{P}^1$, and $Z \subseteq X_0$ be a closed set. Then $i_1^{-1}(i_0(X_0))$ is closed in X_1.

Proof. Let $f \in V(X_0)$. Then define $\hat{f}(x) = x^{\deg(f)} f(1/x) \in \mathbb{K}[x]$. We have that $V(\hat{f}) \setminus \{0\} = V(\hat{f}(x)) \cap i_1^{-1}(i_0(X_0)) = i_1^{-1}(i_0(V(f)))$, and $f(0) \neq 0$ since f has a constant term which is equal to the leading coefficient of f. It follows that $V(\hat{f}) = i_1^{-1}(i_0(V(f)))$, and by picking generators of X_0, hatting them and intersecting their vanishing sets, we see that $i_1^{-1}(i_0(X_0))$ is closed in X_1. $\square$

It follows that both $V_a(J^{i_k}), k = 1, 2$ are closed in both $X_k, k = 1, 2$, and $f^{-1}(Z)$ is closed in $\mathbb{P}^1$. Hence $f : \mathbb{P}^1 \to \mathbb{P}^{1'}$ is continuous.

It remains to show that f, f^{-1} are morphisms of ringed spaces. So, suppose that g is regular on $\mathbb{P}^{1'}$. Then g is locally the quotient of two homogeneous polynomials of the same degree, and

$$i_0^*(f^*(g)) = g(x_0, 1/x_0)$$

is also locally a quotient of polynomials, hence regular on X_0. Similarly, for X_1 as well. As $f^*(g)$ is regular on both X_0 and X_1, it's regular on $\mathbb{P}^1$.

Now suppose that g is regular on $\mathbb{P}^1$. This, by definition, is the case exactly when both $g_0 = i_0^*(g), g_1 = i_1^*(g)$ are regular. Let $f_0 = f \circ i_0, f_1 = f \circ i_1$. Then

$\mathrm{im}(f_0) = U_1$, (where $U_1 = \{[x_0 : 1] : x_0 \in \mathbb{A}^1\}$).

We have that

$$g \circ f^{-1}\big|_{U_1} = g \circ i_0 \circ i_0^{-1} \circ f^{-1}\big|_{U_1} = g_0 \circ f_0^{-1},$$

so by the gluing property of sheaves, we'll be done if we can show that $g_0 \circ f_0^{-1} : U_1 \to \mathbb{A}^1$ is regular. Locally, we can write $g_0 = p_0/q_0$ as a quotient of polynomials, which in turn is the restriction to U_1 of function which is locally a quotient of homogeneous polynomials $\mathbb{K}[\boldsymbol{x}][x_1, x_2]$ of the same degree via

$$g_0 \circ f_0^{-1}([x_0 : 1]) = g_0(x_0) = \frac{p_0(x_0)}{q_0(x_0)} = \left(\frac{p_0^h}{q_0^h} x_1^{\deg q_0 - \deg p_0}\right)([x_0 : 1]),$$

and we are done.

(b)

Let $\mathcal{O}_X$ be the structure sheaf defined as a closed subvariety via Construction 5.10 (b), and $\mathcal{O}'_X$ defined as in Definition 7.1. Let $U \subseteq X$ be an open set and suppose $f \in \mathcal{O}_X(U)$. Then given some $a \in U$, by Construction 5.10 (b), we have some V open in $\mathbb{P}^n$ containing a, and $g \in \mathcal{O}_{\mathbb{P}^n}(V)$ such that $f\big|_V = g\big|_U$, whence f can be written as a quotient of two homogeneous polynomials on $U \cap V \ni a$. Hence $f \in \mathcal{O}'_X(U)$.

Now suppose that $f \in \mathcal{O}'_X(U)$, and let $a \in U$. Then we have some open $U_a \subseteq U$ containing a such that f can be written as a quotient of two homogeneous polynomials of the same degree on U_a. Hence $f\big|_{U_a} \in \mathcal{O}_{\mathbb{P}^n}(U_a)$, and f satisfies Construction 5.10 b with $V = U_a, \Psi = f\big|_{U_a}$ for all $a \in U$.

Fleshing out the proof of Lemma 7.4

First note that f is continuous, since if $V(g_1, g_2, \ldots, g_m)$ is a closed set in $\mathbb{P}^m$, then

$$f^{-1}(V(g_1, g_2, \ldots, g_m)) = V(g_1 \circ f, g_2 \circ f, \ldots, g_m \circ f)$$

is closed in X, since compositions of quotients of homogeneous polynomials of the same degree are again quotients of homogeneous polynomials of the same degree. It follows that the U_i are open in $\mathbb{P}^n$.

The next problem we deal with is how to apply Proposition 4.7 in the end of the proof. For this we need U_i affine, and I can't see why this is known. Instead, define $V'_i \subset \mathbb{P}^n$ to be an affine cover of $\mathbb{P}^n$, just as we defined V_i to be an affine cover of $\mathbb{P}^m$. Then $U_i \cap V'_j$ is closed in every V'_j, hence affine, and we can apply Proposition 4.7 here and lift back up to U_i with Lemma 4.6 whence $f\big|_{U_i}$ is a morphism.

Ex 7.6

By Example 7.5 (c), we have that $X = V_p(x_0^2) \subset \mathbb{P}^2$ is isomorphic to $Y = \mathbb{P}^1$, meanwhile $S(Y) = \mathbb{K}[x] \ncong \mathbb{K}[x_0, x_1]/(x_0^2) = S(X)$ since $S(X)$ has a non-trivial element $\overline{x_0}$ of order 2, whilst $S(Y)$ doesn't.

Ex 7.8

We solve Exercise 7.8 before 7.7, since we use 7.8 to solve 7.7.

Let U_i be the standard affine cover of $\mathbb{P}^n$, let $V_{i,j}$ be an affine cover of $f(U_i)$ for all i, and finally let $\hat{U}_{i,j} = f^{-1}(V_{i,j})$. Then $\hat{U}_{i,j}$ is an open cover of the affine open set U_i, hence all $\hat{U}_{i,j}$ are affine as well. Regular functions on $\hat{U}_{i,j}$ are quotients of polynomials in the $x_k, k \neq i$, and it follows from Proposition 4.XX that the restriction of f to $\hat{U}_{i,j}$ is locally quotients such of polynomials in each coordinate. We can pass from the affine embedding of $\hat{U}_{i,j}$ to the embedding in $\mathbb{P}^n$ via tacking on the $x_k = 1$ coordinate to the tuple. When we do this, quotients of polynomials in $x_k, k \neq i$ turn into their homogenised versions via multiplying each term by a power of x_k such that every term has the same degree. Since the numerators are non-zero, and $\mathbb{P}^n$ is invariant to multiplication, we can clear all denominators by multiplication with their product, and what remains is a function which is locally a homogeneous polynomial in each coordinate.

I.e we have that for every $a \in \mathbb{P}^n$, there is some open $V_a \ni a$ such that we can write $f\big|_{V_a} = (f_1^a, f_2^a, \ldots, f_m^a)$ with all f_i^a homogeneous. We have $\mathbb{P}^n$ compact, so we can pick a finite subcover $V_a, a \in A$ with $|A|$ finite. Since only the zero polynomial is zero on any open set, we have that f admits the same representation as homogeneous polynomials $f_i^a = f_i^b$ on V_a, V_b whenever $V_a \cap V_b \neq \emptyset$. But since $\mathbb{P}^n$ is connected and $|A|$ is finite, we can use induction to conclude that $f_i^a = f_i^b$ for all $a, b \in A$, and we can write $f = (f_1, f_2, \ldots, f_m)$ on all of $\mathbb{P}^n$.

Ex 7.7

(a)

The statement of the exercise is false. Suppose that $n = m = 2$, and $X = V(x_0)$, and let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be given by $f([x_0 : x_1 : x_2]) = [1 : 1 : 1]$. As $[1 : 1 : 1] \notin X$, we have $f^{-1}(X) = \emptyset$.

We need the additional statement that $X \cap \mathrm{im}(f) \neq \emptyset$, or equivalently that $f^{-1}(X) \neq \emptyset$.

We use Exercise 7.8 to solve this.

By Remark 6.33, we know that $I(X) = (g)$ for some homogeneous polynomial

g. Then

$$f^{-1}(X) = f^{-1}(V_p(g)) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0) = V_p(g \circ f).$$

Let $h = g \circ f$. Then h is a homogeneous polynomial since f is (by Exercise 7.8). Moreover, as $f^{-1}(X) = V_p(h) \neq \emptyset$, we see that h isn't a constant.

The irreducible components of $V_p(h)$ correspond to the principal ideals of the prime factors of h. Any such ideal is prime, and Krull's Principal Ideal Theorem tells us that their heights are at most one, hence the corresponding varieties have dimension at least $n - 1$.

(b)

This will follow from Exercise 7.8 after we prove the following lemma.

Lemma 0.9. Let $n > m$ and $f_1, f_2, \dots, f_m \in \mathbb{K}[x_1, x_2, \dots, x_n]$ be m polynomials in n variables. Then $V_a(f_1, f_2, \dots, f_n)$ is either empty or infinite.

Proof. Let $J = I_a(V_a(f_1, f_2, \dots, f_m)) = (f_1, f_2, \dots, f_m)$. Suppose that the f_i share some root. Then J is a proper ideal, and applying Krull's height theorem yields that every minimal ideal of J has height at most m. Minimal prime ideals of J correspond to irreducible components of $V(J)$, so Lemma 2.27 (b) tells us that the dimension of any irreducible component of $V_a(f_1, f_2, \dots, f_m)$ is at least $n - m \geq 1$, whence $V_a(f_1, f_2, \dots, f_m)$ is infinite as all finite affine sets have dimension 0. □

Now, Exercise 7.8 tells us that any morphism $f : \mathbb{P}^n \to \mathbb{P}^m$ is of the form $f = [f_1 : f_2 : \dots : f_m]$ where all f_i have the same degree d. Suppose towards a contradiction that such a morphism where $d \geq 1$ exist. Then since the f_i are homogeneous, we have $f_i(0) = 0$ for all f_i. Hence the lemma tells us that the f_i will always have a common non-zero root as well, which would yield an ill-defined morphism f since $[0 : 0 : \dots : 0]$ isn't a well defined projective point. Hence $\deg f_i = 0$ and f must be constant.

(c)

We have non-constant automorphisms $\mathbb{P}^{n+m} \to \mathbb{P}^{n+m}$ by Example 7.5 (a). There are no non-constant morphisms $\mathbb{P}^{n+m} \to \mathbb{P}^n, \mathbb{P}^{n+m} \to \mathbb{P}^m$ by Exercise 7.7 (b), hence no non-constant morphisms $\mathbb{P}^{n+m} \to \mathbb{P}^n \times \mathbb{P}^m$.

Ex 7.14

Let $a = [a_0 : a_1 : a_2], b = [b_0 : b_1 : b_2]$ be two points on the cubic curve X. A line L through a, b can be parameterised as

$$L = \{[a_0 s + b_0 t : a_1 s + b_1 t : a_2 s + b_2 t] : [s : t] \in \mathbb{P}^1\}.$$

Let $h_{a,b} : \mathbb{P}^1 \to L$ be the parameterisation above. We know that X is the vanishing set of a homogeneous cubic in three variables. Name it g. Then let $g_{a,b} = g \circ h_{a,b} \in \mathbb{K}[s,t]$. This is again a homogeneous cubic, but in two variables. We already know that $g_{a,b}$ vanishes on $[0 : 1], [1 : 0]$, hence it must lie in the ideals $(s), (t)$ and it follows that we can write $g_{a,b} = st(As+Bt)$ where not both A and B are zero. We now see that the remaining root of $g_{a,b}$, namely $[B : -A]$, is given by homogeneous polynomials of the same degree in a, b,

$$\begin{aligned}
[B : -A] =& [2B : -2A] \\
=& [g_{a,b}(1,1) - g_{a,b}(-1,1) : -g_{a,b}(1,1) + g_{a,b}(1,-1)] \\
=& [g_{a,b}(1,1) + g_{a,b}(1,-1) : -g_{a,b}(1,1) + g_{a,b}(1,-1)] \\
=& [g(a_0+b_0, a_1+b_1, a_2+b_2) + g(a_0-b_0, a_1-b_1, a_2-b_2) \\
& : -g(a_0+b_0, a_1+b_1, a_2+b_2) + g(a_0-b_0, a_1-b_1, a_2-b_2)].
\end{aligned}$$

As $a \neq b$, we have $h_{a,b}$ bijective, and it follows that the remaining root of g is given by $f(a,b) = h_{a,b}([B : -A])$, which we can write as

$$f(a,b) = [f_0(a,b) : f_1(a,b) : f_2(a,b)]$$

where

$$\begin{aligned}
f_0([a_0 : a_1 : a_2], [b_0 : b_1 : b_2]) =& a_0(g(a_0+b_0, a_1+b_1, a_2+b_2) + g(a_0-b_0, a_1-b_1, a_2-b_2)) \\
& + b_0(-g(a_0+b_0, a_1+b_1, a_2+b_2) + g(a_0-b_0, a_1-b_1, a_2-b_2)) \\
=& (a_0-b_0)g(a_0+b_0, a_1+b_1, a_2+b_2) \\
& + (a_0+b_0)g(a_0-b_0, a_1-b_1, a_2-b_2),
\end{aligned}$$

and similarly

$$\begin{aligned}
f_1([a_0 : a_1 : a_2], [b_0 : b_1 : b_2]) =& (a_1-b_1)g(a_0+b_0, a_1+b_1, a_2+b_2) \\
& + (a_1+b_1)g(a_0-b_0, a_1-b_1, a_2-b_2), \\
f_2([a_0 : a_1 : a_2], [b_0 : b_1 : b_2]) =& (a_2-b_2)g(a_0+b_0, a_1+b_1, a_2+b_2) \\
& + (a_2+b_2)g(a_0-b_0, a_1-b_1, a_2-b_2).
\end{aligned}$$

Now, the f_i will only simultaneously vanish on Δ_X, since we have that at least one of A, B non-zero, and $f_i = a_iB - b_iA$, so $f_0(a,b) = f_1(a,b) = f_2(a,b) = 0$ would imply $B[a_0 : a_1 : a_2] = A[b_0 : b_1 : b_2]$ hence $a = b$. It follows that f is a well-defined function on $(X \times X) \setminus \Delta_X$. It's a morphism by the following lemma.

Lemma 0.10. Let $X \subseteq \mathbb{P}^m \times \mathbb{P}^n$ be a subvariety, and $f : X \to \mathbb{P}^k$ be a function $f = [f_1 : f_2 : \ldots : f_k]$ in which each f_i is a homogeneous polynomial in the coordinates of $[x_1 : x_2 : \ldots : x_m] \in \mathbb{P}^m$ and $[y_1 : y_2 : \ldots : y_n] \in \mathbb{P}^n$, $f_i(x_1, x_2, \ldots, x_m, y_1, y_2 \ldots y_n)$ such that all f_i have degree d. Then f is a morphism of varieties.

Proof. Later TODO Todo todo ☐

Now consider the set $V_1 \subset (X \times X) \setminus \Delta_X$ given by (a, b) such that $f(a, b) = a$. Then V_1 is closed by Prop 5.20 (b), since it's the subset where $\pi_1 = f$. Similarly, if V_2 is the set of (a, b) where $f(a, b) = b$, it's closed as well. It now follows that U is open since

$$U = (X \times X) \setminus (\Delta_X \cup V_1 \cup V_2).$$

Ex 7.15

We proceed as suggested in the hint. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $Y = \mathbb{P}^1 \times \{[1:0]\} \subset X$ be a hypersurface. Let $f : X \to \mathbb{P}^3$, $f([x_0, x_1] : [y_0 : y_1]) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$ be the Segre embedding of X. Then Y is mapped to $f(Y) = \{[x_0 : 0 : x_1 : 0] : [x_0 : x_1] \in \mathbb{P}^1\}$. In $\mathbb{P}^3$, we have that $f(X) = V_p(z_0z_3 - z_1z_2)$ and $f(Y) = (V_p)_{f(X)}(z_1, z_3)$. Moreover, $(I_p)_{f(X)}(f(Y)) = (\overline{z}_1, \overline{z}_3)$ can't be generated by fewer polynomials. Indeed, suppose towards a contradiction that we have some homogeneous $\overline{g} \in S(f(X))$ such that $(I_p)_{f(X)}(f(Y)) = (\overline{g})$. Then $\overline{g} | \overline{z}_1$ and $\overline{g} | \overline{z}_3$. But both $\overline{z}_1, \overline{z}_3$ are irreducible, as they are degree 1 elements in the graded ring $S(f(X))$, hence no such $\overline{g}$ exists.

Ex 7.28

(a)

Let $f : \mathbb{P}^n \to X \subset \mathbb{P}^N$ be the degree d embedding of $\mathbb{P}^n$, and let $z_{k_0,k_1,\ldots,k_n} = z_{\boldsymbol{k}}$ be the coordinate of $f(x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n})$ where $\sum k_i = d$. Let J be the ideal generated of all polynomials of the form $z_{\boldsymbol{k}} z_{\boldsymbol{k}'} - z_{\boldsymbol{r}} z_{\boldsymbol{r}'}$ where $\boldsymbol{k} + \boldsymbol{k}' = \boldsymbol{r} + \boldsymbol{r}'$, and write $X' = V(J)$. Then $X \subseteq X'$ by definition of the Veronese embedding.

Now suppose that $x \in X'$. Then some coordinate of x is non-zero, and we can assume that $x_{d,0,0,\ldots,0} = 1$. Let $\boldsymbol{e}_i$ be the $n+1$-tuple with a 1 at index i and 0 everywhere else, and consider $y \in \mathbb{P}^n$ where $y_0 = 1$ and $y_i = x_{(d-1)\boldsymbol{e}_0 + \boldsymbol{e}_i} = x_{d-1,0,\ldots,0,1,0,\ldots 0}$. Then

$$\begin{aligned}
f(y)_{\boldsymbol{k}} &= y_0^{k_0} \prod_{i=1}^n y_i^{k_i} \\
&= x_{d\boldsymbol{e}_0} \prod_{i=1}^n x_{(d-1)\boldsymbol{e}_0 + \boldsymbol{e}_i}^{k_i} \\
&= x_{d\boldsymbol{e}_0} \prod_{i=1}^n x_{(d-k_i)\boldsymbol{e}_0 + k_i\boldsymbol{e}_i} x_{d\boldsymbol{e}_0}^{k_i - 1} \\
&= x_{d\boldsymbol{e}_0} \prod_{i=1}^n x_{(d-k_i)\boldsymbol{e}_0 + k_i\boldsymbol{e}_i} \\
&= x_{\boldsymbol{k}} x_{d\boldsymbol{e}_0}^n \\
&= x_{\boldsymbol{k}}
\end{aligned}$$

hence $f(y) = x$ and $x \in \operatorname{im}(f) = X$, $X = X'$.

(b)

Let $Y \subset \mathbb{P}^n$ be a projective variety, and $g_1, g_2, \ldots, g_m$ be homogeneous polynomials of the same degree $2d$ which generate an ideal J such that $V(J) = Y$. (such f_i exist by Remark 6.XX). Then let $f : \mathbb{P}^n \to X \subset \mathbb{P}^N$ be the degree d Veronese embedding, and $Y' = f(Y)$.

Let I_d denote $n+1$-tuples whose sum is d. Since each g_i is homogeneous of degree $2d$, it follows that we can write

$$g_i = \sum_{\boldsymbol{k}, \boldsymbol{r} \in I_d} a_{i,\boldsymbol{k},\boldsymbol{r}} x^{\boldsymbol{k}} x^{\boldsymbol{r}}.$$

Note that this description is not unique, indeed there are multiple tuples $\boldsymbol{k}, \boldsymbol{r}$ which sum to the same tuple, which means that descriptions of the form above are unique only up to the sum of all $a_{i,\boldsymbol{k},\boldsymbol{r}}$ with the same $\boldsymbol{k} + \boldsymbol{r}$. We shall not worry about this though, and simply pick some set of $a_{i,\boldsymbol{k},\boldsymbol{r}}$ such that the equation above holds. Now let

$$h_i = \sum_{\boldsymbol{k}, \boldsymbol{r} \in I_d} a_{i,\boldsymbol{k},\boldsymbol{r}} z_{\boldsymbol{k}} z_{\boldsymbol{r}} \in S(X).$$

Then since $h_i = g_i \circ f$, it follows that $y \in Y$ precisely when $f(y) \in V_X(h_1, h_2, \ldots, h_m)$. We showed in part (a) that X is the zero locust of quadratic forms in $\mathbb{P}^N$, and since each h_i is quadratic as well, it now follows that Y' also is the zero locust of quadratic forms in $\mathbb{P}^N$.

Ex 7.30

(a)

Let $f = ax_0^2 + bx_0x_1 + cx_0x_2 + dx_1^2 + ex_1x_2 + fx_2^2$ be a quadratic form on $\mathbb{P}^2$. Scaling the polynomial f by a non-zero constant does not change it's vanishing set, hence each quadric curve in $\mathbb{P}^2$ defined by a polynomial f like above can be identified with the point $[a : b : c : d : e : f] \in \mathbb{P}^5$. More over, this correspondence is bijective, since if $V(f), V(g)$ are identified with the same point in $\mathbb{P}^5$, we have $f = \lambda g$ hence $V(f) = V(g)$.

Under this correspondence, the set of conics in $\mathbb{P}^2$ are identified with the subset of points $[a : b : c : d : e : f] \in \mathbb{P}^5$ such that $f = ax_0^2 + bx_0x_1 + cx_0x_2 + dx_1^2 + ex_1x_2 + fx_2^2$ is an irreducible polynomial. This is the U described in the exercise. We will show that the complement of U, $V = \mathbb{P}^5 \setminus U$ is closed.

Suppose instead that $f = ax_0^2 + bx_0x_1 + cx_0x_2 + dx_1^2 + ex_1x_2 + fx_2^2$ is reducible. Homogeneous polynomials factor into homogeneous polynomials, and it follows that we can write

$$f = (Ax_0 + Bx_1 + Cx_2)(Dx_0 + Ex_1 + Fx_2)$$

whence

$$\begin{aligned}
a &= AD \\
b &= AE + BD \\
c &= AF + CD \\
d &= BE \\
e &= BF + CE \\
f &= CF,
\end{aligned}$$

which gives us a parameterisation of V in terms of $([A : B : C], [D : E : F]) \in \mathbb{P}^2 \times \mathbb{P}^2$.

Now let $s : \mathbb{P}^2 \times \mathbb{P}^2 \to X \subset \mathbb{P}^8$ be the Segre embedding, and $g : X \to \mathbb{P}^5$ be the map

$$\begin{aligned}
&g([z_{0,0} : z_{0,1} : z_{0,2} : z_{1,0} : z_{1,1} : z_{1,2} : z_{2,0} : z_{2,1} : z_{2,2}]) \\
&= [z_{0,0} : z_{0,1} + z_{1,0} : z_{0,2} + z_{2,0} : z_{1,1} : z_{1,2} + z_{2,1} : z_{2,2}].
\end{aligned}$$

g is a linear automorphism followed by a projection, hence it's a closed morphism by Proposition 7.17. Since s is an isomorphism as well, we have that $g \circ s$ is a closed morphism and $\mathrm{im}(g \circ s)$ is closed. But $\mathrm{im}(g \circ s)$ is parameterised in exactly the same way as V, hence V is closed and U is open.

(b)

Let $A = [A_0 : A_1 : A_2]$ be a point in $\mathbb{P}^2$. It follows immediately from our solution to (a), that the points $[a : b : c : d : e : f] \in U$ which correspond to conics which pass through A are precisely those points which satisfy the linear equation $aA_0^2 + bA_0A_1 + cA_0A_2 + dA_1^2 + eA_1A_2 + fA_2^2 = 0$.

(c)

We introduce some new language to deal with this exercise.

Definition 0.11. A set of $k > n$ points $p_i \in \mathbb{P}^n$ are said to be in general position if no hyperplane of $\mathbb{P}^n$ contains a subset of $n + 1$ points p_i. I.e if no linear homogeneous polynomial $f \in S(\mathbb{P}^n)$ annihilates all p_i.

So, in our case, we have 5 points $p_1, p_2, p_3, p_4, p_5 \in \mathbb{P}^2$ which are in general position.

Lemma 0.12. Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the veronese embedding. Then any set of points in general position in $\mathbb{P}^n$ are mapped to a set of points in general position in $\mathbb{P}^N$ under v_d.

Proof. Let $p_i, i \in [1..k] \in \mathbb{P}^N, k > N$ be a set of points, and suppose that $f \in S(\mathbb{P}^N)_1$ is a linear homogeneous polynomial such that $p_i \in V_p(f)$ for all i. Then suppose that $p_i \in U_0$, then we have

$$(v^d)^{-1}(p_i) = \left[1 : \frac{x_1^N}{x_0 x_1^{N-1}} : \frac{x_2^N}{x_0 x_2^{N-1}} : \ldots : \frac{x_n^N}{x_0 x_n^{N-1}}\right]$$

□

Lemma 0.13. Let $P_i \in \mathbb{P}^n$ be a set of $k > n$ points. Then the points are in general position if and only if their corresponding vectors in $\mathbb{A}^{n+1}$ are linearly independent.

Let $A_1, A_2, A_3, A_4, A_5 \in \mathbb{P}^2$ be 5 points, such that no three of them lie on the same line in $\mathbb{P}^2$, and let f_1, f_2, f_3, f_4, f_5 be their corresponding linear functionals $\mathbb{A}^6 \to \mathbb{A}$ as in (b). Then our first claim is that the f_i are linearly independent. We assume this for now, and show how to use it to solve the exercise.

Suppose that $0 = \sum b_i f_i$ is a non-trivial linear independence. Let $F : \mathbb{P}^2 \to \mathbb{P}^5$ be the map from (b) which sends a point $A_i = [A_{i,0} : A_{i,1} : A_{i,2}]$ to $F(A_i) = [A_{i,0}^2 : A_{i,0}A_{i,1} : A_{i,0}A_{i,2} : A_{i,1}^2 : A_{i,1}A_{i,2} : A_{i,2}^2]$. Note that F coincides with the degree 2 Veronese embedding. Now, as no combination of more than 2 A_i lie on the same line
Suppose that $f_1 \in (f_2, f_3, f_4, f_5)$. Then since all f_i are linear, it follows that f_1 can be written as a $\mathbb{K}$-linear combination of the other f_i. Hence f_1, f_2, f_3, f_4, f_5 linearly independent implies that $J = (f_1, f_2, f_3, f_4, f_5)$ is a minimal generating set. Now inductively define S_i, J_i by $S_0 = S(\mathbb{P}^5)$, $J_0 = (0)$ and $S_i = S_{i-1}/J_i$, $J_i = (f_i) + J_{i-1}$. I.e each S_i is obtained from S_{i-1} via taking the quotient by f_i. Then $S_0 = \mathbb{K}[x_1, x_2, \ldots, x_6]$, and if S_i is isomorphic to a polynomial ring in $6 - i$ variables, then S_{i+1} is isomorphic to a polynomial ring in $6 - (i + 1)$ variables since it's obtained from S_i via a non-trivial linear relation. It follows that each J_i is prime, and that $J = J_5 = (f_1, f_2, f_3, f_4, f_5)$ is a prime ideal of height 5. Hence $V_p(J) = \{a\}$ for some point $a \in \mathbb{P}^5$. Points $a \in V_p(J)$ correspond bijectively with conics that pass through A_1, A_2, A_3, A_4, A_5 by part (b), thus there is exactly one conic which passes through the given points so long as the linear homogeneous polynomials which correspond to the point are linearly independent. We will now show that this is the case whenever no three A_i lie on a line.

TODO Todo todo
Suppose towards a contradiction that $\sum b_i f_i = 0, b_i \in \mathbb{K}$ is a non-trivial linear dependence, and assume that $b_1 \neq 0$. Then we have $f_1 = \sum_{i=2}^n c_i f_i$, with $c_i = -b_i/b_1$. Write the point A_i as $A_i = [A_{i,0} : A_{i,1} : A_{i,2}]$. Then for all i we have $A_{1,i} = \sum$
Then each point A_i corresponds to a line $L_i \in \mathbb{A}^3$ which passes through the origin, and no three lines lie in the same plane.

Let $F : \mathbb{P}^2 \to \mathbb{P}^5$ be the map from (b) which sends a point $A_i = [A_{i,0} : A_{i,1} : A_{i,2}]$ to $F(A_i) = [A_{i,0}^2 : A_{i,0}A_{i,1} : A_{i,0}A_{i,2} : A_{i,1}^2 : A_{i,1}A_{i,2} : A_{i,2}^2]$. Note that F coincides with the degree 2 Veronese embedding. A linear dependence on the f_i corresponds to a linear relation on $\mathrm{im}(F)$,
but we showed in Exercise 7.28 that the Veronese embedding is given by

$$\mathrm{im}(F) = V(z_{0,1}z_{1,0} - z_{1,1}z_{0,0}, z_{0,2}z_{2,0} - z_{2,2}z_{0,0}, z_{1,2}z_{2,1} - z_{2,2}z_{1,1},)$$

and none of the polynomials inside t
Using our notation in from part (a), we can break down g into a linear automorphism followed by a projection like this

$$g = \pi_{0,1,2,4,5,8} \circ ([z_{0,0} : z_{0,1}+z_{1,0} : z_{0,2}+z_{2,0} : z_{1,0} : z_{1,1} : z_{1,2}+z_{2,1} : z_{2,0} : z_{2,1} : z_{2,2}]),$$

and we see that $\dim(V) = 6$
We just need to find point $A, B, C \in \mathbb{P}^2$ that define linearly independent function $f_A, f_B, f_C : U \to$.

Ex 7.31

The projection f from $a = [0 : 0 : 1 : 0]$ to $L = V(y_2)$ is given by $f : \mathbb{P}^3 \backslash \{a\} \to \mathbb{P}^2$ where $f([y_0, y_1, y_2, y_3]) = [y_0 : y_1 : y_3]$.

(a)

$f(X)$ may be parameterised as $f(X) = \{[x_0^3 : x_0^2 x_1 : x_1^3] : [x_0 : x_1] \in \mathbb{P}^1\}$, so $f(X) = g(\mathbb{P}^1)$ where $g([x_0 : x_1]) = [x_0^3 : x_0^2 x_1 : x_1^3]$. From our parameterisation of f, it follows that

$$f(X) \subseteq V(y_0^2 y_2 - y_1^3).$$

Moreover, suppose $a = [a_0 : a_1 : a_2] \in V(y_0^2 y_2 - y_1^3)$. We will show that a lies in the image of g, hence in $f(X)$. First consider the case $a_0 = 1$. Then $a_2 = a_1^3$ and we have

$$g([1 : a_1]) = [1 : a_1 : a_1^3] = [a_0 : a_1 : a_2] = a.$$

Now consider the case when $a_0 = 0$. Then $a_1 = 0$ as well, and a_2 must be non-zero, so we can assume $a_2 = 1$. Then

$$g([0 : 1]) = [0 : 0 : 1] = [a_0 : a_1 : a_2] = a,$$

hence $f(X) = V(y_0^2 y_2 - y_1^3)$. Note that for both cases, the element in the pre-image is uniquely determined, hence g is injective (thus bijective onto its image).

(b)

Note that g from part (a) is an isomorphism (the degree 3 Veronese embedding) composed with f. Hence f is an isomorphism if and only if g is. We claim that g isn't an isomorphism, since if it were, it would be an isomorphism on the affine subset $U_1 = \{[x_0 : x_1] \in \mathbb{P}^1 : x_1 \neq 0\} \cong \mathbb{A}^1$, but on this set g restricts to $g([x_0 : 1]) = [x_0^3 : x_0^2 : 1]$, so we can identify $g\big|_{U_1}$ with $\hat{g} : \mathbb{A}^1 \to \mathbb{A}^2$ where $\hat{g}(t) = (t^3, t^2)$, and we saw in Example 4.9 that this map isn't an isomorphism.

Ex 8.22

To be extra clear for my own sake, and get things right, we will follow construction 8.12 step by step. Let $U \in G(2,4)$ be a plane in $\mathbb{K}^4$. Then U is spanned by two basis vectors, say

$$v_1 = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4, \tag{1}$$
$$v_2 = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4. \tag{2}$$

The correspondence of construction 8.12 first sends U to

$$\begin{aligned} v_1 \wedge v_2 =& (a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \wedge (b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4) \\ =& (a_1b_2 - a_2b_1)e_1 \wedge e_2 + (a_1b_3 - a_3b_1)e_1 \wedge e_3 + (a_1b_4 - a_4b_1)e_1 \wedge e_4 \\ &+ (a_2b_3 - a_3b_2)e_2 \wedge e_3 + (a_2b_4 - a_4b_2)e_2 \wedge e_4 + (a_3b_4 - a_4b_3)e_3 \wedge e_4, \end{aligned}$$

and then to

$$[a_1b_2 - a_2b_1 : a_1b_3 - a_3b_1 : a_1b_4 - a_4b_1 : a_2b_3 - a_3b_2 : a_2b_4 - a_4b_2 : a_3b_4 - a_4b_3]$$

in $\mathbb{P}^5$. Thus the correspondence sends the minor of columns i, j in the matrix $[v_1v_2]^T$ to the coordinate $x_{i,j}$ in

$$[x_{1,2} : x_{1,3} : x_{1,4} : x_{2,3} : x_{2,4} : x_{3,4}] \in \mathbb{P}^5.$$

(a)

It's immediate from the correspondence above that $G(2,4) \subseteq V(f = x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3})$. To see that this containment is an equality, note that $\dim G(2,4) = 2(4-2) = 4$ and as f is prime, $\dim V(f) = \dim \mathbb{P}^5 - 1 = 4$ hence the varieties are the same as they are both irreducible.

(b)

A line L' intersects L in $\mathbb{P}^3$ exactly when the intersection of the corresponding planes $U, U' \in \mathbb{K}^4$ intersect in a line. This happens if and only if the basis vectors of the two spaces, v_1, v_2, v_1', v_2', are linearly dependent, and the determinant of the matrix $M(L, L') = [v_1\, v_2\, v_1'\, v_2']^T$ is 0. The determinant expansion by minors of $\det(M(L, L'))$ along the top two rows shows that $\det(M(L, L')) = 0$

can be expressed as a linear relation among the 2×2 minors contained in the bottom two rows, with coefficients in the top to rows. Since the coordinates of the point corresponding to U' in $\mathbb{P}^5$ are given by it's 2×2 minors, this is exactly what it means for the embedding of $G(2,4)$ in $\mathbb{P}^5$ to be the variety of a linear homogeneous polynomial.

It follows that the set of lines intersecting four general lines is given by the vanishing set of four general linear polynomials in $\mathbb{P}^5$, which would have dimension 1, and hence should only consist of a single line in general.

Ex 8.23

(a)

Let $v_1, \ldots, v_k$ be an orthogonal basis for the k-dimensional subspace $U \subseteq \mathbb{K}^n$. Then the distance $d_U(a)$ from U to a point $a = (a_1, \ldots, a_n) \in \mathbb{K}^n$ is given by $|a - a_{\|U}|$ where $a_{\|U}$ is the projection of a onto U as

$$a_{\|U} = \sum_{i=1}^{k} \frac{\langle a, v_i \rangle}{|v_i|^2} v_i.$$

After squaring and clearing denominators, we have that

$$f = d_U(a)^2 \prod_{i=1}^{k} |v_i|^4 = \sum_{i=1}^{k} (|v_i|^2 a_i - \langle a, v_i \rangle)^2$$

is a polynomial in a, U which is zero precisely when $d_U(a) = 0$, I.e when $a \in L$. It follows that our variety is given by $V_{G(k,n),\mathbb{K}}(f)$, however, we've thus far treated the embeddings in of the elements U in $\mathbb{A}^n$. It's clear though that $V(f)$ is a cone, since $a \in L \Rightarrow ka \in kL, \forall k \in \mathbb{K}$ so the variety is a projective variety as well.

(b)

This is really unintuitive to me so I'll begin with an example. Consider the ambient space $\mathbb{P}^3$, and the two varieties $X = V_p(x_1^2 + x_2^2 - x_0^2, x_3 - x_0)$ $Y = V_p(x_1^2 + x_2^2 - x_0^2, x_3 + x_0)$. Then on the affine patch U_0 where $x_0 = 1$, we have that X is the circle at height 1, and Y is the circle at height -1.

Now consider the geometry of our setup in $\mathbb{R}^3$. We have two circles parallel with the x_1x_2-plane at different heights $1, -1$, and take all the lines joining them. Then if we fix some point $x \in X$, the join of x with Y is a sort of hollow skew double sided cone, and when we rotate x around the circle X, the union of all skew cones trace a volume, i.e 3-dimensional space, which isn't all of $\mathbb{R}^3$, which is strange? There is one thing missing from this picture though, we're not considering the entire complex circles. Thus, taking all of the complex points into account on the circles as well, we should expect that the resulting join is all of $\mathbb{P}^3$. Let's verify this on the affine patch.

Let $p = [1 : a : b : c] \in U_0$ be some fixed arbitrary point on our affine patch. Then let $q_X = [1 : X_1 : X_2 : 1] \in X \cap U_0$ and $q_Y = [1 : Y_1 : Y_2 : -1] \in Y \cap U_0$ be affine points on our varieties. We will vary the points q_X, q_Y such that the line through both points meets p as well. The lines L_X, L_Y through p, q_X and p, q_Y are given by

$$\begin{aligned} L_X &= \{[1 : ta + (1-t)X_1 : tb + (1-t)X_2 : tc + 1 - t] : t \in \mathbb{C}\}, \\ L_Y &= \{[1 : sa + (1-s)Y_1 : sb + (1-s)Y_2 : sc - 1 + s] : s \in \mathbb{C}\}. \end{aligned}$$

We will now find q_X, q_Y such that these lines coincide. First, we can use the last coordinate to express s in terms of t. We need $tc + 1 - t = sc - 1 + s$, hence $s = \frac{tc+2-t}{c+1}$, and the two lines can be jointly parameterised according to

$$\begin{aligned} &L_X = \{[1 : ta + (1-t)X_1 : tb + (1-t)X_2 : tc + 1 - t] : t \in \mathbb{C}\}, \\ &L_Y = \left\{ \left[1 : \frac{tc + 2 - t}{c+1}(a - Y_1) + Y_1 : \frac{tc + 2 - t}{c+1}(b - Y_2) + Y_2 : tc + 1 - t\right] : t \in \mathbb{C} \right\}. \end{aligned}$$

We see that these equations linearly determine each Y_1, Y_2 in terms of X_1, X_2 respectively. Name this correspondence $Y_1 = f_1(X_1), Y_2 = f_2(X_2)$. Then X_1, X_2 need to satisfy $X_1^2 + X_1^2 = 1$ and $f_1(X_1)^2 + f_2(X_2)^2 = 1$. This system has a solution in the affine plane since the circle is reducible into linear factors (over $\mathbb{K}$ algebraically closed, but not over $\mathbb{R}$!), hence we can substitute one variable for the other, and then solve the remaining polynomial. It follows that p lies on the join of X, Y - hence we've verified that the join is all of $\mathbb{P}^3$ (or at least that it contains U_0).

Now, let's solve the actual exercise. TODO Todo todo - maybe solve, spent a lot of time on example.

Ex 9.9

Let $f \in \mathbb{K}[x_0, x_1, \ldots, x_n]$ be some irreducible homogeneous polynomial of degree 2. Our first claim is that there exists a suitable change of coordinates such that f can be written as $f = x_0 x_1 + x_2^2 + x_3^2 + \ldots + x_m^2$ where $2 \leq m \leq n$.

To see this, first note that any bilinear form

$$f = \sum_{i \leq j} c_{i,j} x_i x_j$$

can be written as

$$f = \begin{bmatrix} x_0 & x_1 & \ldots & x_n \end{bmatrix} A \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where A is a symmetric matrix with $A_{i,j} = A_{j,i} = c_{i,j}/2$ for $i \neq j$ and $A_{i,i} = c_{i,i}$. We can perform a change of coordinates which diagonalises this matrix, and then by rescaling the coordinates, we can do so in such a way that the first two elements on the diagonal are 1/2, and the remaining are either 1 or 0 (f is irreducible so the first two diagonal entries must be non-zero). After we've diagonalised it we can perform a row swap to obtain a matrix of the form

$$\hat{A} = \begin{bmatrix} 0 & 1/2 & 0 & 0 & \dots & 0 \\ 1/2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 \text{ or } 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 \text{ or } 0 & \dots & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 1 \text{ or } 0, \end{bmatrix}$$

which has a corresponding quadratic form with the desired shape.

So, suppose $f = x_0x_1 + x_2^2 + x_3^2 + \ldots + x_m^2$ is an irreducible quadratic form, and consider $X = V_p(f)$. Then the projection $\pi : X \to \mathbb{A}^{n-1}$ given by $\pi([a_0 : a_1 : \ldots : a_n]) = (a_1, \ldots, a_n)$, and it's invertible on $\mathbb{A}^{n-1} \setminus 0$ with inverse

$$\pi^{-1}(a_1, \ldots, a_n) = ([-(a_2^2 + a_3^2 + \ldots + a_n^2) : a_1^2 : a_1a_2 : \ldots : a_1a_n]),$$

hence $V_p(f)$ is birational to $\mathbb{P}^{n-1}$.

An example of a irreducible quadric surface which isn't isomorphic to some projective space, consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. This is given by the variety $V_p(z_{0,0}z_{1,1} - z_{0,1}z_{1,0})$, hence is an irreducible quadric hypersurface of $\mathbb{P}^3$, but $\mathbb{P}^1 \times \mathbb{P}^1$ isn't isomorphic to $\mathbb{P}^2$ by Exercise 7.7 (c).

Ex 9.18

We follow along with Example 9.15. By Lemma 9.14, we have that the blow-up of $\mathbb{A}^3$ along the x_3 axis satisfies

$$\widetilde{\mathbb{A}^3} \subseteq \{((a_1, a_2, a_3), [b_1 : b_2]) \in \mathbb{A}^3 \times \mathbb{P}^1 : b_1a_2 = b_2a_1\} =: Y.$$

The affine patch $U_0 \subset Y$ where $b_1 = 1$, is given by $V_{\mathbb{A}^3 \times \mathbb{P}^1}(a_2 - a_1b_2)$. Hence there is an isomorphism $f : \mathbb{A}^3 \to U_0$ given by

$$f : (a_1, b_2, a_3) \mapsto ((a_1, a_1b_2, a_3), [1 : b_2]),$$

and we see that $\mathbb{A}^3$ is birationally equivalent to Y, hence $\dim Y = 3$. Moreover, Y is covered by the U_i, which are all isomorphic to affine space hence irreducible, and as they intersect, Y is irreducible as well by Exercise 2.21 (a). As $\widetilde{\mathbb{A}^3} \subseteq Y$ and both varieties have dimension 3, it follows that they are the same.

The blow-up $\pi : \widetilde{\mathbb{A}^3} \to \mathbb{A}^3$ maps $U = \mathbb{A}^3 \setminus V(x_1, x_2)$ to U, and the exceptional set is given by

$$\pi^{-1}(V(x_1, x_2)) = \{((0, 0, a_3), [b_1 : b_2]) \in \mathbb{A}^3 \times \mathbb{P}^1\} \cong \mathbb{A}^1 \times \mathbb{P}^1.$$

For two lines $L_1 = t(a_1, a_2, a_3), L_2 = t(b_1, b_2, b_3)$ through $V(x_1, x_2)$ to intersect in the blow-up, we need them to intersect in $\mathbb{A}^3$, and that their projections onto the first two coordinates coincide, $t(a_1, a_2) = t(b_1, b_2)$, since these are the lines which get sent to the projective axis.

The geometric interpretation of this is that the exceptional set parameterizes the projected directions onto the x_1, x_2-plane. So in some sense, the blow-up unravels all planes parallel x_1, x_2-plane where each plane is unraveled in a way analogous to the blow-up of $\mathbb{A}^2$ at the origin. Note however that the unraveling happens out into an extra fourth dimension, and has nothing to do with the x_3 coordinate.

Ex 9.19

Suppose that $f_1, \ldots, f_s$ is a generating set for $I(Y_1)$ and that $g_1, \ldots, g_r$ is a generating set for $I(Y_2)$. The exceptional set of the blow-up of X is given by $\pi^{-1}(Y_1 \cap Y_2)$, and it follows $\widetilde{Y_i}$ is the closure of the embedding of the $Y_i \setminus Y_j, j \neq i$. The embeddings of these open sets are given by

$$\begin{aligned}\pi^{-1}(Y_1 \setminus Y_2) &= \{(a, [f_1(a) : f_2(a) : \ldots : f_s(a) : g_1(a) : g_2(a) : \ldots : g_r(a)]) : a \in Y_1 \setminus Y_2\} \\ &= \{(a, [0 : 0 : \ldots : 0 : g_1(a) : g_2(a) : \ldots : g_r(a)]) : a \in Y_1 \setminus Y_2\},\end{aligned}$$

and

$$\begin{aligned}\pi^{-1}(Y_2 \setminus Y_1) &= \{(a, [f_1(a) : f_2(a) : \ldots : f_s(a) : g_1(a) : g_2(a) : \ldots : g_r(a)]) : a \in Y_2 \setminus Y_1\} \\ &= \{(a, [f_1(a) : f_2(a) : \ldots : f_s(a) : 0 : 0 : \ldots : 0]) : a \in Y_2 \setminus Y_1\}.\end{aligned}$$

We then see that the first s coordinates of the projective part of $\pi^1(Y_1 \setminus Y_2)$ are 0, hence the same is true of the closure $\widetilde{Y_1}$. Meanwhile, the last r coordinates of the projective part of $\pi^1(Y_2 \setminus Y_1)$ are 0, hence the same is true of the closure $\widetilde{Y_2}$. It follows that the two blow-ups are disjoint.

Ex 9.22

(a)

For a given polynomial $f \in \mathbb{K}[\boldsymbol{x}]$, let

$$g_f(x_1, y_2, \ldots, y_n) = f(x_1, x_1 y_2, \ldots, x_1 y_n)/x_1^{\min \deg f}.$$

Then g_f is always a polynomial in $\mathbb{K}[x_1, y_2, \ldots, y_n]$, as every term of $f(x_1, x_1 y_2, \ldots, x_1 y_n)$ is divisible by $x_1^{\min \deg f}$. Thus given some ideal $J \in \mathbb{K}[\boldsymbol{x}]$, we can define the ideal

$J_g \in \mathbb{K}[x_1, y_2, \dots, y_n]$ as $J_g = (g_f : f \in J)$. Moreover, $f(x_1, x_1y_2, \dots, x_1y_n) = 0$ implies that either $g_f(x_1, y_2, \dots y_n) = 0$ or $x_1 = 0$.

Now, first consider the blow-up of an irreducible variety $X = V(J)$ at the origin. Then $\widetilde{X}$ is irreducible as well. We've already seen that the affine patch of $\widetilde{\mathbb{A}^n}$ given by $V_1 = \{(x, [y]) \subset \widetilde{\mathbb{A}^n} : y_1 = 1\}$ can be parameterised as

$$V_1 = \{((x_1, x_1y_2, \dots, x_1y_n), [1 : y_2 : \dots : y_n]) \in \widetilde{\mathbb{A}^n}\},$$

and if we consider $\widetilde{X} \cap V_1$ with this parameterisation, we see that

$$\begin{aligned}\widetilde{X} \cap V_1 &\subseteq \{((x_1, x_1y_2, \dots, x_1y_n), [1 : y_2 : \dots : y_n]) \in \widetilde{\mathbb{A}^n} : f(x_1, x_1y_2, \dots, x_1y_n) = 0, f \in J\} \\ &\subseteq \{((x_1, x_1y_2, \dots, x_1y_n), [1 : y_2 : \dots : y_n]) \in \widetilde{\mathbb{A}^n} : x_1g_f(x_1, y_2, \dots, y_n) = 0, f \in J\} \\ &= V_{V_1}(x_1J_g) \\ &= V_{V_1}(J_g) \cup V_{V_1}(x_1).\end{aligned}$$

But since $\widetilde{X}$ is irreducible, and $\widetilde{X} \cap V_1$ is open in $\widetilde{X}$, we have $\widetilde{X} \cap V_1$ irreducible as well. This means that either $\widetilde{X} \cap V_1 \subseteq V_{V_1}(J_g)$ or $\widetilde{X} \cap V_1 \subseteq V_{V_1}(x_1)$. If $X \not\subset V(x_1)$ then clearly the first case must hold. If $X \subseteq V(x_1)$, then $x_1 \in J$ and $1 \in J_g$ so $V_{V_1}(J_g) = \emptyset$, but we also have that $y_1 = 0$ for all $(x, [y]) \in \pi^{-1}(X_1 \backslash \{0\})$, hence $\widetilde{X} \cap V_1 = \emptyset$ as well. In both cases $\widetilde{X} \cap V_1 \subseteq V_{V_1}(J_g)$.

Meanwhile, we have an injective morphism

$$\phi : V_{V_1}(J_g) \to X,\ ((x_1, x_1y_2, \dots, x_1y_n), [1 : y_2 : \dots : y_n]) \mapsto (x_1, y_2, \dots, y_n),$$

hence $\dim V_{V_1}(J_g) \leq \dim X = \dim \widetilde{X} = \dim \widetilde{X} \cap V_1$ (where the last equality follows from Exercise 5.24). To recap, we have two irreducible varieties $\widetilde{X} \cap V_1$, $V_{V_1}(J_g)$, both of which are closed in V_1, such that $\widetilde{X} \cap V_1 \subseteq V_{V_1}(J_g)$ and $\dim V_{V_1}(J_g) = \dim \widetilde{X} \cap V_1$. It follows that $\widetilde{X} \cap V = V_{V_1}(J_g)$ and we are done with the case when X is irreducible.

Now consider the case when $X = X_1 \cup \dots \cup X_r = V(J_1) \cup \dots \cup V(J_r)$ is the irreducible decomposition of X. Then $\widetilde{X} = \widetilde{X_1} \cup \dots \cup \widetilde{X_r}$, hence

$$\begin{aligned}\widetilde{X} \cap V_1 &= \bigcup_{i=1}^{r} (\widetilde{X}_r \cap V_1) \\ &= \bigcup_{i=1}^{r} V_{V_1}((J_i)_g) \\ &= V_{V_1}\left(\prod_{i=1}^{r} (J_i)_g\right),\end{aligned}$$

so we are done if we can show that $(JJ')_g = J_gJ'_g$. But this is seen easily since both ideals are generated by all products $g_fg_{f'}$ with $f \in J$ and $f' \in J'$.

(b)

First of, we have

$$
\begin{aligned}
g_f^{\mathrm{in}}(x_1, y_2, \ldots, y_n) &= f^{\mathrm{in}}(x_1, x_1y_2, \ldots, x_1y_n)/x_1^{\deg(f^{\mathrm{in}})} \\
&= f^{\mathrm{in}}(1, y_2, \ldots, y_n),
\end{aligned}
$$

so f^{in} agrees with g_f^{in} on V_1 if we evaluate f^{in} on the projective part of V_1 (i.e the y-coordinates). Moreover, on $\pi^{-1}(0)$, $g_f = g_f^{\mathrm{in}}$ since all terms but the initial term in g_f is divisible by x_1. Combining these to facts shows that $f^{\mathrm{in}} = g_f$ on the projective part of E_1, where $E_1 = (0 \times \mathbb{P}^{n-1}) \cap V_1$ is the first affine patch of the exceptional set of $\mathbb{A}^n$ blown up at the origin. In other words, the first affine patch of the exceptional set of the blow-up of X is given by

$$
\begin{aligned}
\pi^{-1}(0) \cap V_1 &= V_{E_1}(J_g) \\
&= 0 \times V_{A_1}(J^{\mathrm{in}})
\end{aligned}
$$

where A_1 is the first affine patch of $\mathbb{P}^{n-1}$, and gluing together all such patches yields

$$\pi^{-1}(0) = 0 \times V_p(J^{\mathrm{in}}).$$

(c)

Let $hf \in (f)^{\mathrm{in}}$. Then $hf^{\mathrm{in}} = h^{\mathrm{in}}f^{\mathrm{in}} \in (f^{\mathrm{in}})$ hence $(f)^{\mathrm{in}} = (f^{\mathrm{in}})$.

For a counter example when J isn't principal. Consider $J = (f_1 = x + y, f_2 = x - y)$ in lex order with $y < x$. Then $f_1^{\mathrm{in}} = y, f_2^{\mathrm{in}} = -y$, so $(f_1^{\mathrm{in}}, f_2^{\mathrm{in}}) = (y)$. Meanwhile $x \in J$ so $x \in J^{\mathrm{in}}$.

Ex 9.25

(a)

We already know what the blow-up of $\mathbb{A}^2$ at the ideal (x_1, x_2) is from Example 9.15. Denote this blow-up by Y. Our aim will be to show that the blow up of the plane at $J = (f_1 = x_1^2, f_2 = x_1x_2, f_3 = x_2^2)$, which we will denote by $\widetilde{\mathbb{A}^2}$, is isomorphic to Y. Let $X = V(J)$. Then by Lemma 9.14,

$$
\begin{aligned}
\widetilde{\mathbb{A}^2} &\subseteq \{((x_1, x_2), [y_1, y_2, y_3]) \in \mathbb{A}^2 \times \mathbb{P}^2 : y_1f_2(x_1, x_2) = f_1(x_1, x_2)y_2, y_1f_3(x_1, x_2) = f_1(x_1, x_2)y_3, y_2f_3(x_1, x_2) = . \\
&= \{((x_1, x_2), [y_1, y_2, y_3]) \in \mathbb{A}^2 \times \mathbb{P}^2 : y_1x_1x_2 = x_1^2y_2, y_1x_2^2 = x_1^2y_3, y_2x_2^2 = x_1x_2y_3\} \\
&:= Y''.
\end{aligned}
$$

Moreover, on $\pi^{-1}(\mathbb{A}^2 \setminus \{0\})$ we have $y_1y_3 = y^2$, hence this equation holds on its closure $\widetilde{\mathbb{A}^2}$ as well, and we have

$$\widetilde{X} \subseteq V_{Y''}(y_1y_3 = y_2^2) := Y'.$$

We will be done if we can show that $Y \cong Y'$, as from this it would follow that Y' is an irreducible variety of dimension 2 containing $\widetilde{\mathbb{A}^2}$, hence $\widetilde{\mathbb{A}^2} = Y' \cong Y$.

Consider the injective morphism

$$\begin{aligned}&\phi : Y \to Y',\\ &\phi : (x, [y_1 : y_2]) \mapsto (x, [y_1^2 : y_1 y_2 : y_2^2]),\end{aligned}$$

and let U_i be the affine patch $y_i = 1$ of Y'. Then we can invert ϕ on these patches as

$$\begin{aligned}&\phi^{-1} : Y' \to Y,\\ &\phi^{-1} : (x, [y_1 = 1 : y_2 : y_3]) \mapsto (x, [y_1 : y_2]),\\ &\phi^{-1} : (x, [y_1 : y_2 = 1 : y_3]) \mapsto (x, [y_1 : y_2]),\\ &\phi^{-1} : (x, [y_1 : y_2 : y_3 = 1]) \mapsto (x, [y_2 : y_3]).\end{aligned}$$

Note that ϕ^{-1} is injective on all of Y' as well, since $y_1 y_3 = y_2^2$ on Y' hence any two y_i, y_j determine the third y_k uniquely up to sign, but this signage can be ignored above as in the image of ϕ^{-1} we have $[y_i : y_j] = [-y_i : -y_j]$. Hence $Y' \cong Y$ and we are done.

(b)

Let's try the blow-up $\mathbb{A}^2$ at (x_1^2, x_2). By Lemma 9.14, we have

$$\begin{aligned}\widetilde{\mathbb{A}^2} &\subseteq \{((x_1, x_2), [y_1, y_2]) \in \mathbb{A}^2 \times \mathbb{P}^1 : x_1^2 y_2 = y_1 x_2.\}\\ &=: Y\end{aligned}$$

On the affine patch U_1 where $y_1 \neq 0$ this becomes equation $x_2 = y_2 x_1^2$, hence we have an isomorphism

$$\phi_1 : \mathbb{A}^2 \to U_1, (x_1, y_2) \mapsto ((x_1, x_1^2 y_2), [1 : y_2]).$$

Similarly, on the affine patch U_2 the equation becomes $x_1^2 = y_1 x_2$, and we have an isomorphism

$$\phi_2 : V_{\mathbb{A}^3}(x_1^2 - x_2 x_3) \to U_2, (x_1, x_2, x_3) \mapsto ((x_1, x_2), [x_3 : 1]).$$

Hence Y can be covered by irreducible patches of dimension 2, and must itself be an irreducible variety of dimension 2, whence $\widetilde{\mathbb{A}^2} = Y$.

Now, let's consider the tangent cone of $U_2 = V_{\mathbb{A}^3}(x_1^2 - x_2 x_3)$ at the origin. As $x_1^2 - x_2 x_3$ is homogeneous, this is simply $V(x_1^2 - x_2 x_3)$.

Meanwhile, let $\widehat{\mathbb{A}^2}$ be the blow-up of $\mathbb{A}^2$ at the origin. As $\widehat{\mathbb{A}^2}$ can be covered by patches isomorphic to $\mathbb{A}^2$, it follows that the tangent cone at any point is

$C(\mathbb{P}^1) = \mathbb{A}^2$.

Now, as the coordinate ring of the tangent cone of U_1 is isomorphic to $\mathbb{K}[\boldsymbol{x}]/(x_1^2 - x_2x_3)$, hence not a UFD, it can't be isomorphic to any tangent cone of $\widehat{\mathbb{A}^2}$ as they all have coordinate rings $\mathbb{K}[x_1, x_2]$, and it now follows from the hint that $\widetilde{\mathbb{A}^2} \not\cong \widehat{\mathbb{A}^2}$.

Ex 9.25

(a)

We will use the answer from here: https://math.stackexchange.com/questions/4001996/fundamental-points-of-cremona-plane-transformation, but it uses limits, and so we will motivate limits with a lemma first.

Lemma 0.14. Let X, Y be varieties, $f : X \to Y$ a morphism, and $a_i \in X, i \in \mathbb{N}$ a sequence such that $\lim_{i\to\infty} a_i = a$ in the classical topology on U. Then

$$\lim_{i\to\infty} f(a_i) = f(a).$$

Proof. We can assume that X, Y are affine by restricting f to open affine sets containing $a, f(a)$. Then since Zariski closed subsets of affine varieties are closed in the classical topology, it follows that f is continuous in the classical topology, after which our statement follows as well. $\square$

Now assume that there exists an extension $\hat{f}$ of f to all of $\mathbb{P}^2$. Now consider two different sequences on $\mathbb{P}^2$, one on the variety $V(x_1)$ given by $r_i = [1 : 0 : \lambda_i]$ and one on $V(x_2)$ given by $s_i[1 : \lambda_i : 0]$ where $\lambda_i = 1/i \to 0$. On the first sequence, f is given by

$$f(r_i) = f([1 : 0 : \lambda_i]) = [0 : \lambda_i : 0] = [0 : 1 : 0],$$

whilst, at the second sequence, $f(s_i) = [0 : 0 : \lambda_i] = [0 : 0 : 1]$. We now arrive at the contradiction

$$[0 : 1 : 0] = \lim_{i\to\infty} \hat{f}(r_i) = \hat{f}([1 : 0 : 0]) = \lim_{i\to\infty} \hat{f}(s_i) = [0 : 0 : 1].$$

(b)

TODO Todo todo LATER!

Ex 10.6

Let $I_a, J_{f(a)}$ be the maximal ideals of $\mathcal{O}_{X,a}, \mathcal{O}_{Y,f(a)}$ respectively. By Corollary 10.5, it will suffice to show that f induces a linear map $g^* : J_{f(a)}/J_{f(a)}^2 \to I_a/I_a^2$ (note that the arrow goes in the opposite direction as it's a morphism in the dual space of the tangent space).

We construct g^* as

$$g^* : \left(U_{f(a)}, \phi\right) + J_{f(a)}^2 \mapsto \left(f^{-1}\left(U_{f(a)}\right), f^*\phi\right) + I_a^2.$$

First of, g is well defined, as $f^{-1}(U_{f(a)})$ is open in X since f is a morphism, $a \in f^{-1}(U_{f(a)})$, and $f^*\phi \in \mathcal{O}_X(f^{-1}(U_{f(a)}))$. Linearity of g^* follows by linearity of f^*.

Ex 10.13

(a)

First suppose that $f = \boldsymbol{x}^{\boldsymbol{\alpha}}$ is a monomial. If $\alpha_i \geq 1$, we have

$$\frac{\partial f}{\partial x_i} = \alpha_i \boldsymbol{x}^{(\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n)},$$

hence

$$x_i \frac{\partial f}{\partial x_i} = \alpha_i \boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha_i f.$$

If instead $\alpha_i = 0$ then we just have

$$\frac{\partial f}{\partial x_i} = 0.$$

It now follows that

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^n \alpha_i f = f \sum_{i=1}^n \alpha_i = df.$$

The result now follow for general homogeneous polynomials since

$$f \mapsto \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}$$

is a linear operator.

(b)

Assume WLOG that $a_0 = 1$ and consider the affine patch of $\mathbb{A}^n$ denoted U_0. Let $f_i'(x_1, \ldots, x_n) = f_i(1, x_2, \ldots, x_n)$ denote the dehomogenization of f. Then X is smooth if and only if the affine Jacobi criterion is holds for $X' = V(I') = V(f_1', \ldots f_n')$ at $a' = (a_1, \ldots, a_n)$. We differentiate, and note that

$$\frac{\partial f_i'}{\partial x_j}(a') = \frac{\partial f_i}{\partial x_j}(a),$$

for $j \neq 0$ hence X is smooth at a if and only if $\left(\frac{\partial f_i}{x_j}\right)_{i,j\geq 1}$ has rank $n - \operatorname{codim}_X\{a\}$. Note however that this is an $r \times n$ matrix, and does not include the derivatives with respect to x_0. To deal with this, just note that since $a_0 = 1$, it follows from part (a) that

$$\frac{\partial f_i}{\partial x_0}(a) = df(a) - \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(a) = -\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(a),$$

hence the omitted 0-th column is linearly dependent with the other columns and does not change the rank.

Ex 10.17

Let $\widetilde{X_k}$ be the blow-up of X_k at the origin. The exceptional set whenever $k > 0$ is given by $\{0\} \times V_p(x_2^2) = \{((0,0),[1:0])\}$. Let's investigate $\widetilde{X_k}$ around this point, and consider the affine patch U_1 of $\widetilde{X_k}$. As in the Example 10.16, Exercise 9.22 (a) tells us that this patch of the blow-up is given by

$$V\left(\frac{(x_1y_2)^2 - x_1^{2k+1}}{x_1^2}\right) = V(y_2^2 - x_1^{2(k-1)}) \cong X_{k-1}.$$

Hence, the k-th blow-up of X_k at the origin is the first blow-up which is smooth. Since isomorphic varieties have isomorphic blow-ups, it follows that $X_k \ncong X_l$ when $x \neq l$.

Ex 10.18

Let $f : \mathbb{P}^1 \to X \subset \mathbb{P}^3$ be the degree 3 Veronese embedding given by

$$f : [x_0 : x_1] \mapsto [y_0 : y_1 : y_2 : y_3] = [x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3].$$

Then by Exercise 7.28,

$$X \subseteq X' = V_p(f_1 = y_0y_3 - y_1y_2, f_2 = y_0y_2^3 - y_1^3y_3),$$

and as both X, X' are irreducible varieties of codimension 2 in $\mathbb{P}^3$, we have $X = X'$ (we needed the dimensional argument to see that $X = X'$, since we haven't motivated that f_1, f_2 generate the ideal of all the polynomials of the form $z_{\boldsymbol{k}}z_{\boldsymbol{k'}} - z_{\boldsymbol{r}}z_{\boldsymbol{r'}}$ as given in the solution of Exercise 7.28).

The Jacobian of f_1, f_2 is now given as

$$\begin{pmatrix} y_3 & -y_2 & -y_1 & y_1 \\ y_2^3 & -3y_1^2y_3 & 3y_0y_2^2 & -y_1^3 \end{pmatrix}$$

which when evaluated at any point $a \in X$, has rank at least $3 - \operatorname{codim}_X(\{a\}) = 3 - 1 = 2$ since the upper row never vanishes, and the lower row only vanishes when $a_1 = a_2 = 0$, which can't happen when $a \in X$.

Ex 10.22

(a)

Suppose that $X \subseteq \mathbb{P}^N$ with $\dim X = n$. The answer is trivial if there is exist an automorphism ϕ on $\mathbb{P}^N$ such that at most $2n + 1$ coordinates on $\phi(X)$ are non-zero, so assume no such ϕ exist. I.e, that for every $\phi \in \mathrm{Aut}(\mathbb{P}^N)$, and $i \in [0..N]$, we have that the affine patch U_i of $\mathbb{P}^N$ intersects X.

We claim that X must be smooth (?).

It now follows that
Then $S(X)$ contains a chain of n radical homogeneous ideals not equal to the projective ideal. Let
TODO Todo todo: Finnish!!!

(b)

Ex 10.23

(a)

We prove the contrapositive. Suppose that $X = X_1 \cup \ldots \cup X_r \subset \mathbb{P}^n$ is the irreducible decomposition of the hypersurface X, and that $r \geq 2$. Then by Exercise 6.31 (b), the two irreducible components X_1, X_2 meet in some point $p \in X_1 \cap X_2$, which by Remark 10.10 (b) then must be a singular point.

So, in general, any smooth variety $X \subset \mathbb{P}^n$ of pure dimension $\geq n/2$ is irreducible.

(b)

Let $N_d = \binom{n+d}{n} - 1$, and $X_d \subseteq \mathbb{P}^{N_d}$ be the set of points corresponding to homogeneous degree d polynomials in $n+1$ variables which carve out smooth, hence irreducible, varieties of $\mathbb{P}^n$. We are asked to show that this is a dense open set, but since projective space is irreducible, it will be enough to show that it's an open non-empty set.

To see that X is non-empty just consider some Fermat hypersurface from Example 10.21. It remains only to show that X is open.

From Exercise 10.13 (a), it follows that a hypersurface $V_p(f)$ contains a if all partial derivatives of f vanish at a. Hence the hypersurface is smooth if and only if the partial derivatives never simultaneously vanish. We can now model the rest of our proof on the proof of Proposition 7.17. Namely,

$$
\begin{aligned}
V_p(f'_{x_0}, \ldots, f'_{x_n}) = \emptyset \Leftrightarrow \sqrt{(f'_{x_0}, \ldots, f'_{x_n})} = (1) \text{ or } \sqrt{(f'_{x_0}, \ldots, f'_{x_n})} = (x_0, \ldots, x_n) \\
\Leftrightarrow \mathbb{K}[\boldsymbol{x}]_k = (f'_{x_0}, \ldots, f'_{x_n})_k \text{ for some } k \in \mathbb{N}, k \geq d-1,
\end{aligned}
$$

which in turn happens if and only if the $\mathbb{K}$-linear map

$$
F : (\mathbb{K}[\boldsymbol{x}]_{k-d-1})^{n+1} \to \mathbb{K}[\boldsymbol{x}]_k,\ (h_0, \ldots, h_n) \mapsto h_0 f'_{x_0} + \ldots + h_n f'_{x_n}
$$

is surjective, I.e has rank $\binom{n+k}{k}$. But this happens if and only if some minor of size $\binom{n+k}{k}$ of F is non-zero. The minors of F are polynomials in the coefficients of the f'_{x_i}, and in turn f. Hence this is an open condition on the coefficients of f and X is open.

Ex 10.24

(a)

It means that the gradient of f is the same at a and b. This means that the tangent space of f at a is equal to the tangent space of f at b.

(b)

Since f is a quadratic form, all partial derivatives of f are linear forms. Hence the parameterisation of the dual curve F is a linear map. Moreover, as these linear forms never simultaneously vanish at X by assumption, the kernel of this map interpreted in $\mathbb{A}^3$ intersects the cone $C(X)$ trivially. But then the kernel of the map must be trivial, since $C(X)$ is a 3-dimensional surface in $\mathbb{A}^3$ (as X is irreducible, it's not a line or a point in $\mathbb{P}^2$), and can't be contained in any subspace of $\mathbb{A}^3$ of vector space dimension less than 3. I.e the linear map is an isomorphism, and the curve parameterised by F differs from the curve determined by f by a change of coordinates only, hence they must both be irreducible of degree 2.

To do part (c), we'll need to verify that the dual of the dual is indeed the original curve, and to do that, we will write a Macaulay2 script which explicitly calculates the implicit form of F for a generic conic f. We know that F is a conic, hence an implicit form for F amounts to finding a linear relation among products of order 2 of the partial derivatives of f. I.e, if we write

$$
f = ax_1^2 + bx_1x_2 + cx_1x_3 + dx_2^2 + ex_2x_3 + rx_3^2,
$$

and let $g_i = f'_{x_i}$, then we're looking for linear relations among

$$\begin{aligned}
g_1^2 &= 4a^2x_1^2 + 4abx_1x_2 + 4acx_1x_3 + b^2x_2^2 + 2bcx_2x_3 + c^2x^3 \\
g_1g_2 &= 2abx_1^2 + (4ad + b^2)x_1x_2 + (2ae + cb)x_1x_3 + 2bdx_2^2 + (be + 2cd)x_2x_3 + cex^3 \\
g_1g_3 &= 2acx_1^2 + (2ae + bc)x_1x_2 + (4ar + c^2)x_1x_3 + bex_2^2 + (2br + ce)x_2x_3 + 2crx^3 \\
g_2^2 &= b^2x_1^2 + 4bdx_1x_2 + 2bex_1x_3 + 4d^2x_2^2 + 4dex_2x_3 + e^2x^3 \\
g_2g_3 &= bcx_1^2 + (be + 2dc)x_1x_2 + (2br + ec)x_1x_3 + 2dex_2^2 + (4dr + e^2)x_2x_3 + 2erx^3 \\
g_3^2 &= c^2x_1^2 + 2cex_1x_2 + 4crx_1x_3 + e^2x_2^2 + 4erx_2x_3 + 4r^2x^3,
\end{aligned}$$

Now, we're only concerned with $F(X)$, I.e we only need a quadratic form in the g_i which vanishes when $(x_1, x_2, x_3) \in X$. We can therefor reduce everything modulo f. Here is a script which does this.

```
clearAll

unorderedPairs = l ->
        flatten ((0..<#l) / (i -> (toList (i..<#l) / (j -> (l_i, l_j)))))
degreeTwoMonomials = l -> unorderedPairs l / times

dualConic = f -> (
        R := ring f;
        baseR := baseRing R;
        Rq := R/ideal(f);

        -- extract cofficeints of quadratic forms of
        -- partial derivatives % ideal(f) into a matrix
        g := flatten entries sub(jacobian f, Rq);
        gg := degreeTwoMonomials g;
        mons := degreeTwoMonomials gens R;
        monsQ := mons / (r -> r_Rq);
        ggCoeffs := gg / (x -> coefficients(x, Monomials=>monsQ)) / last;
        M := sub(fold((v1, v2) -> v1 | v2, ggCoeffs), baseR);

        -- Recast that matrix as a map from the free module baseR^6,
        -- to the quotient module baseR^6 modulo the relations
        -- induced by f
        fCoeffsVec := last coefficients(f, Monomials=>mons);
        fCoeffsVecInBaseR := sub(fCoeffsVec, baseR);
        M = map(coker fCoeffsVecInBaseR, baseR^6, M);

        -- The kernel of M will be the baseR-linear dependecies
        -- among the partial derivatives
        k := ker M;
        cs := (entries k_0) / (c -> sub(c, R));
        sum apply(mons, cs, times)
```

```
)

Rgeneric = QQ[a,b,c,d,e,r]
R = Rgeneric[x_0 .. x_2]

f = a * x_0^2 + b * x_0 * x_1 + c * x_0 * x_2
        + d * x_1^2 + e * x_1 * x_2 + r * x_2^2
F = dualConic f

<< "f:␣" << f << endl << endl;
<< "F:␣" << F << endl << endl;
<< "dualConic␣F:␣" << dualConic F << endl << endl;
<< "F(jacboian␣f)␣%␣f:␣"
        << F(toSequence flatten entries jacobian f) % f << endl;
<< "f(jacboian␣F)␣%␣F:␣"
        << f(toSequence flatten entries jacobian F) % F << endl;
```

The script produces the following output (with some manual wrapping added for readability).

```
i0 : load "dual-conic.m2"


      2               2                       2
f: a*x  + b*x x  + d*x  + c*x x  + e*x x  + r*x
      0      0 1      1      0 2      1 2      2

       2         2                           2         2
F: (- e  + 4d*r)x  + (2c*e - 4b*r)x x  + (- c  + 4a*r)x  + ...
                 0                 0 1                 1

                                                     2         2
 ... + (- 4c*d + 2b*e)x x  + (2b*c - 4a*e)x x  + (- b  + 4a*d)x
                       0 2                 1 2                 2

                2               2                       2
dualConic F: a*x  + b*x x  + d*x  + c*x x  + e*x x  + r*x
                0      0 1      1      0 2      1 2      2

F(jacboian f) % f: 0
f(jacboian F) % F: 0
```

We see in this output that the dual curve of the dual curve is indeed the original curve.

(c)

A line L_i in $\mathbb{P}^2$ is determined by a linear form like $L_i = V_p(X_i x_0 + Y_i x_1 + Z_i x_2)$. Given 5 lines L_i, let $p_i = [X_i : Y_i : Z_i]$. The L_i are in general position precisely when the p_i are. From Exercise 7.30 (c) we know that any 5 points of $\mathbb{P}^2$ in general position uniquely determine a conic, so let F be the conic determined by the p_i. Let f be the dual of F. We showed in part (b) that this implies that the dual of f is F. This means that F can be parameterised as $F = [f'_{x_0} : f'_{x_1} : f'_{x_2}]$, and passes through the points p_i for some inputs $a^i = [a^i_0 : a^i_1 : a^i_2]$. In other words,

$$[f'_{x_0}(a^i_0) : f'_{x_1}(a^i_1) : f'_{x_2}(a^i_2)] = p_i = [X_i : Y_i : Z_i],$$

hence the tangent space of f at p_i is L_i and we are done.

We end with a remark that that emphasises that the tangent space of f is indeed equal to L_i at a_i, and not just parallel to it. Consider f as a function on affine space. Then since f is a cone, the tangent plane at any point will intersect the origin of $\mathbb{A}^3$. Similarly, if we interpret L_i in affine space, it will be a plane intersecting the origin. Hence both are spaces determined by an implicit equation of the form $Ax_0 + Bx_1 + Cx_2 = 0$. I.e there is no non-zero "m" value in the equation as $Ax_0 + Bx_1 + Cx_2 = m$. Maybe this is obvious, I just needed to write it out for myself.

Ex 12.14

(a)

For this, note that if $\mathrm{Spec}(R)$ is an irreducible scheme, then any two distinct distinguished open sets $D(f) \cap D(g) \neq \emptyset$ intersect, and so for every $f \neq g \in R$, there is a prime ideal that does not contain f or g, whence it does not contain fg. In particular, we have $fg \neq 0$ for $f, g \in R$ with f, g distinct since every prime ideal contains 0.

Thus for $\mathrm{Spec}(R)$ to be irreducible whilst R is not an integral domain, we need nilpotency. Let's try $R = k[x]/x^2$. Then $\mathrm{Spec}(R) = \{(x)\}$ is irreducible whilst R is not a ring.

(b)

Let $R = \mathbb{Z}, S = \mathbb{Q}$. Then $\mathbb{Q}$ is a field and so $\dim \mathbb{Q} = 0$, whilst $\dim \mathbb{Z} = 1$.

(c)

The maximal ideal $(x_1, x_2 - 1)$ contains

$$(x_2 - 1)^2 + 2(x_2 - 1) + x_1^2 = x_2^2 - 2x_2 + 1 + 2x_2 - 2 + x_1^2 = x_2^2 + x_1^2 - 1$$

and so $(x_1, x_2 - 1) \in \operatorname{Spec}(R)$ whilst $R/(x_1, x_2 - 1) = \mathbb{R}$, and so the residue field is $\mathbb{R}$ as well.

(d)

We're asked to find a ring with exactly two prime ideals where one is contained in the other. Let $R = k[x]_{(x)}$. Then $\operatorname{Spec}(R) = \{(0), (x)\}$ and so R will do.

Ex 12.15

(a)

Every open set of the spectrum of any ring is a union of distinguished open sets, so it will suffice to show that $D(f)$ contains a closed point for every $f \in A(X)$ whenever $D(f) \neq \emptyset$. In other words, that f is not contained in every maximal ideal whenever f is not contained in every prime ideal. So, let $f \in R$ be such that f is contained in every maximal ideal of $A(X)$. Then by remark 1.14.(b), we have $f \in \sqrt{0}$ and so f is contained in every prime ideal.

(b)

For a counterexample, let $R = k[x]_{(x)}$. Then $\operatorname{Spec}(R) = \{(0), (x)\}$ and (x) is the only closed point, the closure of which is not all of the spectrum, hence not dense.

Ex 12.21

First, a lemma.

Lemma 0.15. Let R be a ring. Then R is the product of two rings $R \cong R_1 \times R_2$ if and only if there are two nilpotent elements $e_1, e_2 \in R$ which sum to $e_1 + e_2 = 1$ and multiply to 0 $e_1 e_2 = 0$. If this is the case, these nilpotents can be picked as $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Proof. Suppose first that $R = R_1 \times R_2$. Then clearly $(1, 0)$ and $(0, 1)$ are nilpotent elements of R which satisfy the imposed conditions.

Suppose now that $e_1, e_2 \in R$ are nilpotent and that $e_1 + e_2 = 1$. Then $e_1 R$ is a ring with identity e_1, since $e_1(e_1 f) = e_1 f$, $e_1(f + g) = e_1 f + e_1 g$ and $e_1(f) e_1(g) = e_1(fg)$. The same is true of $e_2 R$. Now consider the ring morphism $\phi : e_1 R \times e_2 R \to R$ which sends $\phi : (e_1 f, e_2 g) \mapsto e_1 f + e_2 g$. It is indeed a ring homomorphism as $\phi(e_1, e_2) = e_1 + e_2 = 1$,

$$\begin{aligned} \phi((e_1 f_1, e_2 g_1) + (e_1 f_2, e_2 g_2)) &= e_1 f_1 + e_2 g_1 + e_1 f_2 + e_2 g_2 \\ &= \phi((e_1 f_1, e_2 g_1) + \phi((e_1 f_2, e_2 g_2)), \end{aligned}$$

and

$$\begin{aligned}\phi(e_1f_1, e_2g_1)\phi(e_1f_2, e_2g_2) &= (e_1f_1 + e_2g_1)(e_1f_2 + e_2g_2)\\ &= (e_1f_1f_2 + e_2g_1g_2)\\ &= \phi((e_1f_1, e_2g_1)e_1f_2, e_2g_2)).\end{aligned}$$

Moreover, it's surjective as all $f \in R$ can be written as $f = e_1f + e_2f$, and it's injective since $e_1f + e_2g = 0$ implies that $0 = e_1e_1f + e_1e_2g = e_1f$ and the same for $e_2g = 0$. Hence $R \cong e_1R \times e_2R$. □

We will show that $\operatorname{Spec}(R)$ is disconnected if and only if R contains two nilpotents $e_1, e_2 \in R$ as in the lemma above.

Suppose that R contains two such nilpotents e_1, e_2, and let $\mathfrak{p} \in \operatorname{Spec}(R)$. If $e_1 \notin \mathfrak{p}$, then $e_2 \in \mathfrak{p}$ as $e_1e_2 = 0$ and $\mathfrak{p}$ is prime. Moreover, no prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ can contain both e_1 and e_2 as $e_1 + e_2 = 1$. We've shown that every prime ideal of R contains either e_1 or e_2 but never both, which translates to $D(e_1) \cap D(e_2) = \emptyset$ and $D(e_1) \cup D(e_2) = \operatorname{Spec}(R)$. Thus $\operatorname{Spec}(R)$ is disconnected.

Now suppose that $\operatorname{Spec}(R)$ is disconnected. We can suppose that R is reduced, as the topology of the spectrum remains unchanged when we quotient out by the nilradical. Let $U_1 \cup U_2 = \operatorname{Spec}(R)$ be two open sets which cover $\operatorname{Spec}(R)$. Then U_1, U_2 are both open and closed, and so $U_i = V(J_i)$ for radical ideals $J_1, J_2 \subset R$. Then every prime ideal contains either J_1 or J_2 but never both. Hence $J_1J_2 \subset \sqrt{0} = 0$, and $1 \in J_1 + J_2$. Let $e_1 \in J_1, e_2 \in J_2$ be such that $e_1 + e_2 = 1$. Then $e_1e_2 \in \sqrt{0} = 0$ and so $e_1e_2 = 0$. It follows that e_1, e_2 are nilpotent as $e_1e_1 = e_1e_1 + e_1e_2 = e_1(e_1 + e_2) = e_1$.

Ex 12.42

If R is a ring with only one prime ideal $\mathfrak{m}$, then this must be a maximal ideal and $\mathfrak{m} = \sqrt{0}$. Hence every element in R is either nilpotent or a unit.

(a)

Suppose that $\operatorname{Spec}(R)$ is a fat double point. Then R has a K-basis $1, x$, and every element in R can be written as $a + bx$ with $a, b \in \mathbb{K}$. As $\mathbb{K}$ is algebraically closed, R is not a field since it would have finite degree over $\mathbb{K}$, and so x is not a unit. Hence x is nilpotent and $x^2 = 0$. It follows that $R \cong K[x]/x^2$.

(b)

Consider the two rings $R_1 = K[x, y]/(x^2, xy, y^2)$ and $R_2 = K[x, y]/(x^3)$. As vector spaces we have

$$R_1 = \operatorname{Span}(1, x, y),\ R_2 = \operatorname{Span}(1, x, x^2).$$

Moreover, the prime ideals of R_1 are the prime ideals containing (x^2, xy, y^2) which necessarily contain both x and y, so $\text{Spec}(R_1) = \{(x, y)\}$, whilst similarly $\text{Spec}(R_2) = \{(x)\}$. It follows that both $\text{Spec}(R_1)$ and $\text{Spec}(R_2)$ are triple points, but they are not isomorphic since R_2 has nilpotent elements of order 3, whilst R_1 does not.

Ex 12.43

By picking some affine open neighbourhood of P, we can assume that $X = \text{Spec}(k[x_1, \ldots, x_n]/I) = \text{Spec}(A)$, and after translating, we can assume that $P = (x_1, \ldots, x_n)$. Any morphism $D \to X$ which sends (x) to $(x_1, \ldots, x_n)$ is equivalent to a $\mathbb{K}$-algebra morphism $\phi : A \to k[x]/(x^2)$ such that $\phi^{-1}((x)) = (x_1, \ldots, x_n)$. In other words, $\phi(x_i)$ must lie in (x) and so there are scalars $c_i \in k$ such that $\phi(x_i) = c_i x$.

We now claim that an assignment of scalars c_i such that the induced morphism is well-defined, is exactly equivalent to an element of $T_P X$. To see this, note that we need $\phi(f) = 0$ for all $f \in I$, and as $\phi(x_i^r) = \phi(x_i)^r \in (x)^r = 0$ for $r > 1$, we have that $\phi(f) = f_1(c_1, \ldots, c_n)$ where f_1 is the linear part of f. Hence, an assignment of scalars c_i induces a well-defined morphism if and only if $(c_1, \ldots, c_n) \in V(f_1 : f \in I) = T_P X$.

Ex 13.6

Let us first recall what a morphism $\phi : U \to \mathbb{K}^{n+1}$ is.

Lemma 0.16. Let $U \subset \mathbb{P}^n$ be an open subset and $\phi : U \to \mathbb{K}^{n+1}$ be a morphism of varieties. Then ϕ is of the form $\phi = (\phi_1, \ldots, \phi_n)$ with $\phi_i \in \mathcal{O}_{\mathbb{P}^n}(U)$, and any function which can be written as this is a morphism $U \to \mathbb{K}^{n+1}$.

Proof. First let $\phi : U \to \mathbb{K}^{n+1} = \mathbb{A}^{n+1}$ be a morphism. By definition, ϕ induces a $\mathbb{K}$-algebra morphism

$$\phi^* : \mathcal{O}_{\mathbb{A}^{n+1}}(V) \to \mathcal{O}_{\mathbb{P}^n}(\phi^{-1}(V))$$

for any open set $V \subset \mathbb{A}^{n+1}$. In particular,

$$\phi^*(x_i) \in \mathcal{O}_{\mathbb{P}^n}(\phi^{-1}(\mathbb{A}^{n+1})) = \mathcal{O}_{\mathbb{P}^n}(U),$$

is regular. But $\phi^*(x_i)$ is just the i-th coordinate of ϕ, and so ϕ can be written in the desired way with $\phi_i = \phi^*(x_i)$.

Now suppose that $\phi_i \in \mathcal{O}_{\mathbb{P}^n}(U)$ for $i \in [0..n]$ and let $\phi = (\phi_0, \ldots, \phi_n)$. Then let V be open in $\mathbb{A}^n$ and $f \in \mathcal{O}_{\mathbb{A}^{n+1}}(V)$. Then let $a \in \phi^{-1}(V)$, and $V_{\phi(a)} \subset V$ be an open neighbourhood about a such that $f = g/h$ on $\phi(a)$ with $g, h \in \mathbb{K}[\boldsymbol{x}]$. Then on $U_a = \phi^{-1}(V_\phi(a))$, we have that

$$\phi^*(f) = g(\phi)/h(\phi).$$

Now, as $\mathcal{O}_{\mathbb{P}^n}(U_a)$ is closed a ring, polynomials in regular function are regular functions whence $g(\phi), h(\phi) \in \mathcal{O}_{\mathbb{P}^n}(U_a)$. Moreover, as h doesn't vanish on $V_{\phi(a)} = \phi(U_a)$, whence $h \circ \phi$ doesn't vanish on U_a, and so $1/h(\phi) \in \mathcal{O}_{\mathbb{P}^n}(U_a)$. We've shown that $\phi^*(f)$ is a regular function, and that ϕ induces a $\mathbb{K}$-algebra morphism

$$\phi^* : \mathcal{O}_{\mathbb{A}^{n+1}}(V) \to \mathcal{O}_{\mathbb{P}^n}(\phi^{-1}(V)).$$

$\square$

Now let $f/g \in \mathcal{O}_{\mathbb{P}^n}(-1)(U)$ with $f, g \in \mathbb{K}[\boldsymbol{x}]$ and $\deg(f) = d, \deg(g) = d+1$. Then let $\phi_i = x_i f/g$ and $\phi = (\phi_0, \ldots, \phi_n)$. Then each ϕ_i is a quotient of degree $d+1$ homogeneous polynomials and ϕ is a morphism $U \to \mathbb{K}^{n+1}$. Moreover, given any point point $a = [a_0 : \ldots : a_n] \in U$ we have

$$\phi(a) = (a_0 f(a)/g(a), \ldots, a_n f(a)/g(a)) = f(a)/g(a)(a_0, \ldots, a_n)$$

which lies in the cone of a, so $\phi : \mathcal{F}(U)$.

Denote this mapping $f/g \to \phi$ by $\Theta : \mathcal{O}_{\mathbb{P}^n}(-1)(U) \to \mathcal{F}(U)$. Then Θ is a $\mathcal{O}_{\mathbb{P}^n}(U)$-module morphism, as if $\psi, \in \mathcal{O}_{\mathbb{P}^n}(U)$, we have

$$\begin{aligned}
\Theta(\psi f/g) &= (x_0 \psi f/g, \ldots, x_n \psi f/g) \\
&= \psi \cdot (x_0 f/g, \ldots, x_n f/g) \\
&= \psi \Theta(f/g)
\end{aligned}$$

and

$$\begin{aligned}
\Theta(f/g + F/G) &= (x_0(f/g + F/G), \ldots, x_n(f/g + F/G)) \\
&= (x_0 f/g + x_0 F/G, \ldots, x_n f/g + x_n F/G) \\
&= (x_0 f/g, \ldots, x_n f/g) + (x_0 f/g, \ldots, x_n F/G) \\
&= \Theta(f/g) + \Theta(F/G).
\end{aligned}$$

It's easy to see that Θ is injective, since all x_i never simultaneously vanish on $\mathbb{P}^n$. Moreover, as Θ doesn't depend on U, it's clear that Θ commutes with restriction and thus defines a morphism of $\o_{\mathbb{P}^n}$-modules $\o_{\mathbb{P}^n}(-1) \to \mathcal{F}$.

To see that Θ is surjective, let $\phi = (\phi_0, \ldots, \phi_n) \in \mathcal{F}(U)$. Then for every $a = [a_0 : \ldots : a_n] \in U$ we have

$$\phi(a) = h(a)(a_0, \ldots, a_n) \in \mathbb{A}^{n+1}$$

for some function $h : U \to \mathbb{K}$. It follows that we can write

$$\phi = h \cdot (x_0, \ldots, x_1)$$

and $\phi_i = h \cdot x_i$ on U. It follows that $h = \phi_i / x_i$ on every affine patch $x_i \neq 0$, hence lies in $\mathcal{O}_{\mathbb{P}^n}(-1)(U \cap x_i \neq 0)$, and gluability yields $h \in \mathcal{O}_{\mathbb{P}^n}(-1)$, whence $\phi = \Theta(h)$.

Ex 13.8

Suppose that $f : \mathcal{F} \to \mathcal{G}$ has inverse $g : \mathcal{G} \to \mathcal{F}$, and let $P \in X$. We claim that the inverse to f_P is given by g_P. Indeed,

$$f_P(g_P(\overline{(U,\phi)})) = f_P(\overline{(U, g_U(\phi))}) = \overline{(U, f_U(g_U(\phi)))} = \overline{(U,\phi)},$$

shows that g_P is a right inverse, and the proof that it is a left inverse is identical.

Now suppose that $f : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules such that for each $P \in X$, the stalk local morphism $f_P : \mathcal{F}_P \to \mathcal{G}_P$ has inverse $g_P : \mathcal{G}_P \to \mathcal{F}_P$. Our first objective is to show that the g_P glue together to a morphism $g : \mathcal{G} \to \mathcal{F}$. To do this, we define the notion of compatible germs.

Definition 0.17. Let $\mathcal{F}$ be a sheaf on X, and suppose that $\{s_P\}_{P\in U}$ be a set of germs for each point in some open set $U \subset X$. We say that the germs form a compatible set of germs if there for all $P \in U$, exist a representative

$$(\psi_P, U_P) \in s_P$$

such that the germ of ψ_P at all points $Q \in U_P$ is equal to s_Q.

Lemma 0.18. Let $\{s_P\}_{P\in U}$ be a set of germs of a sheaf $\mathcal{F}$ on an open set $U \subset X$. Then they are compatible if and only if there exist a cover U_i of U and sections $f_i \in \mathcal{F}(U_i)$ such that the germ of f_i at any $Q \in U_i$ is equal to s_Q.

Proof. Suppose that the s_P are compatible. Then there exist representatives (ψ_P, U_P) for each point such that the germ of ψ_P at $Q \in U_P$ is s_Q. As the U_P cover U, and $\psi_P \in \mathcal{F}(U_P)$, the conditions of the lemma are satisfied.

Now suppose that we have a cover U_i of U and sections $f_i \in \mathcal{F}(U_i)$ such that the germ of f_i at any $Q \in U_i$ is equal to s_Q. Then for any $P \in U$, let U_P be any open set U_i containing P, and let $\psi_P = f_i$. Then the germ of ψ_P at any $Q \in U_P$ is again s_Q, hence the set of germs are compatible. $\square$

Lemma 0.19. Let $\{s_P\}_{P\in U}$ be a set of germs. Then they are compatible if and only if there exist some $f \in \mathcal{F}(U)$ such that the germ of f at p is s_P. Moreover, f is uniquely determined by $\{s_P\}_{P\in U}$ in this way.

Proof. It's clear that any $f \in \mathcal{F}(U)$ give rise to a compatible set of germs since the conditions of the previous lemma are satisfied with the cover $\{U\}$ of U.

Now suppose that the s_P form a compatible set of germs. Then let $U_i, i \in I$ be a cover of U such that we have $f_i \in \mathcal{F}(U_i)$ for all $i \in I$ that agree with the germs s_P. Then for any intersection $U_{ij} = U_i \cap U_j$, the germs of f_i and f_j agree at all points of U_{ij}. Thus for each point $P \in U_{ij}$, we have some neighbourhood U_P where $f_i\big|_{U_P} = f_j\big|_{U_P}$. As the U_P cover U_{ij}, it follows by the identity axiom that $f_i\big|_{U_{ij}} = f_j\big|_{U_{ij}}$. Thus the f_i may be glued to a single $f \in \mathcal{F}(U)$ and we are done.

To see that f must be unique, suppose that f' is another element such that the germ of f' at $P \in X$ is s_P. Then for every point $P \in U$, there is some open set $U_P \subset U$ containing P where f and f' agree. As the U_P cover U, $f = f'$ on U by the identity axiom. $\square$

With the notion of compatible germs in hand, we proceed in constructing an inverse of f.

Let $U \subset X$ be open and $\phi \in \mathcal{G}(U)$. We will show that $\psi_P = g_P(\phi_P)$ for $P \in U$ forms a compatible set of germs, and after this we will show that the map $g : \mathcal{G}(U) \to \mathcal{F}(U)$ assigning $g(\phi) = \psi$ like this is a morphism inverse to f.

Let (ψ_{U_P}, U_P) be a representative of ψ_P. Then as

$$\phi_P = f_P(\psi_P) = f_P(\overline{(\psi_{U_P}, U_P)}) = \overline{(f(\psi_{U_P}), U_P)},$$

there is some neighbourhood W_P of P where $f(\psi_{U_P})$ and ϕ agree. Now let $Q \in W_P$. Then

$$\phi_Q = (f(\psi_{U_P}))_Q = f_Q((\psi_{U_P})_Q)$$

and

$$\psi_Q = g_Q(\phi_Q) = g_Q(f_Q((\psi_{U_P})_Q)) = (\psi_{U_P})_Q.$$

Hence the ψ_P form a compatible set of germs, since for all $P \in U$ we have an open neighbourhood W_P and representative ψ_{U_P} such that the germs of ψ_{U_P} at points $Q \in W_P$ are equal to ψ_Q.

Let ψ be the unique element obtained from gluing together the compatible set of germs and define $g_U : \mathcal{G}(U) \to \mathcal{F}(U)$ such that $g_U(\phi) = \psi$. Then $f_U(g_U(\phi))$ is indeed equal to ϕ since their germs agree at all points

$$f_U(g_U(\phi))_P = f_P(g_P(\phi_P)) = \phi_P.$$

Similarly, $g_U(f_U(\psi))_P = g_P(f_P(\psi_P)) = \psi_P$. Thus g_U is an inverse function of f_U for every open set $U \subset X$, and as the inverse of a module morphism is again a module morphism,, f_U is an isomorphism for all open sets $U \subset X$.

Ex 13.18

By Lemma 13.17 (b), we may identify $\mathcal{G}$ with its sheafification. After doing this, define $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ as follows,

$$f_U : (\phi_P)_{P \in U} \mapsto (f'_P(\phi_P))_{P \in U}$$

These maps are $\mathcal{O}_X(U)$-module maps as the f'_P are all $\mathcal{O}_X(U)$-module maps and addition/scalar multiplication in the sheafifications are defined point wise. Moreover, the f_U clearly commute with restriction, hence they define a morphism of sheaves $\mathcal{F} \to \mathcal{G}$. Finally,

$$f \circ \theta(\phi) = (f'(\phi)_P)_{P \in U} = f'(\phi)$$

and so it provides the necessary factorization.

To see that it is the only map which factors f' in the desired way, we show that a morphism from a presheaf to a sheaf is determined on the level of stalks. Indeed, suppose that $U \subset X$ is open and $\phi \in \mathcal{F}'(U)$. Then we showed in the previous exercise that the $(f'(\phi))_P, P \in U$ uniquely glue to $f'(\phi)$, but $(f'(\phi))_P = f'_P(\phi_P)$, hence f' is determined by the f'_P. So, suppose that $F : \mathcal{F} \to \mathcal{G}$ is another map factoring f' through θ. Then $(F - f) \circ \theta = 0$, hence $(F - f)_P \circ \theta_P = 0$ for all $P \in X$. But as θ_P is an isomorphism by Lemma 13.17, and in particular surjective, it follows that $(F - f)_P = 0$ for all $P \in X$. Now, since a morphism of sheaves is determined at the stalk-level, it follows that $F = f$.

Ex 13.20

We identify $\mathcal{F}$ and $\mathcal{G}$ with their sheafifications. It's now obvious that f commutes with taking the germ at a point p, in the sense that applying f and the taking the germ at p is the same as taking the germ at p and then applying f_p,

$$f((s_q)_{q\in U})\big|_p = (f_q(s_q))_{p\in U}\big|_p = f_p(s_p) = f_p\left((s_q)_{p\in U}\big|_p\right).$$

(a)

The desired result follows immediately from the commutativity above, since $(s_q)_{q\in U}\big|_p \in \ker(f)_p$ is equivalent to the LHS in the equality chain being 0, and $(s_q)_{q\in U}\big|_p \in \ker(f_p)$ is equivalent to the RHS being 0.

(b)

$\mathrm{im}'(f)$ is the set $(f(s))_{q\in U}$ for $s \in \mathcal{F}(U)$, and then definition of sheafification dictates that $\mathrm{im}(f)(U)$ is the set $((f(s))_q)_{q\in U} = (f_q(s_q))_{q\in U}$ for $s \in \mathcal{F}(U)$. Hence $\mathrm{im}(f)(U)_p$ is given by

$$\{(f_q(s_q))_{q\in U}\big|_p : s \in \mathcal{F}(U)\} = \{f_p(s_p) : s \in \mathcal{F}(U)\} = \mathrm{im}(f_p)(U).$$

Ex 13.24

Note that $\mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(0)$ by Construction 13.4.(b), so any section over any open set is a quotient of homogeneous polynomials. Moreover, it follows by exercise 13.26 that it will suffice to show that the sequence is exact over some fixed open set $U \subset \mathbb{P}^1$.

The only reasonable choice for f is

$$f : \frac{a}{b} \mapsto \left(\frac{x_0 a}{b}, \frac{x_1 a}{b}\right),$$

after which it makes sense to guess that $d = 2$ and

$$g : \left(\frac{a}{b}, \frac{a'}{b'}\right) \mapsto \frac{x_0 a'}{b'} - \frac{x_1 a}{b}.$$

We verify that this induces an exact sequence. It's immediate that $g \circ f = 0$, and that f is injective. To see that g is surjective, let $c/d \in \mathcal{O}_{\mathbb{P}^1}(2)$. Then as c is homogeneous of positive degree, we can write $c = x_0 c_0 - x_1 c_1$, and we see that

$$g\left(\frac{c_0}{d}, \frac{c_1}{d}\right) = \frac{c_0 x_0}{d} - \frac{c_1 x_1}{d} = c/d$$

so g is surjective. Finally, $\left(\frac{a}{b}, \frac{a'}{b'}\right) \in \ker(g)$ if and only if

$$0 = \frac{x_0 a'}{b'} - \frac{x_1 a}{b} = \frac{x_0 a' b - x_1 a b'}{b' b}$$

which since x_0 and x_1 are relatively prime, happens if and only $a'b = x_1 c_1$ and $ab' = x_0 c_0$ for some homogeneous polynomials c_1, c_0. But then

$$x_0 x_1 c_1 = x_0 a' b = x_1 a b' = x_0 x_1 c_2$$

so $c_1 = c_2$ and we denote both of them by c. Now we have

$$f\left(\frac{c}{b'b}\right) = \left(\frac{x_0 c}{b'b}, \frac{x_1 c}{b'b}\right) = \left(\frac{ab'}{b'b}, \frac{a'b}{b'b}\right) = \left(\frac{a}{b}, \frac{a'}{b'}\right)$$

and we see that $\ker(g) = \operatorname{im}(f)$.

Ex 13.25

Let $V \subset U$ be open in $\mathbb{P}^n$ and define the maps

$$\phi_U : \mathcal{O}_{\mathbb{P}^n}(-d)(U) \to \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}\big|_U}\left(\mathcal{O}_{\mathbb{P}^n}(d)\big|_U, \mathcal{O}_{\mathbb{P}^n}\big|_U\right)$$

to be given by

$$\left(\phi_U\left(\frac{a}{b}\right)\right)_V : \frac{c}{d} \mapsto \frac{ac}{bd}$$

where

$$\deg(b) = \deg(a) + d,\ \deg(c) = \deg(d) + d,$$

and b is non-zero on $U \supset V$, d is non-zero on V. It's immediate that each $\phi_U(a/b)_V$ is a $\mathcal{O}_{\mathbb{P}^n}(V)$-module morphism, and that they commute with restriction. Hence $\phi_U(a/b)$ is a well-defined morphism of sheaves $\mathcal{O}_{\mathbb{P}^n}(d)\big|_U \to \mathcal{O}_{\mathbb{P}^n}\big|_U$. We will now verify that ϕ is a morphism of sheaves

$$\phi : \mathcal{O}_{\mathbb{P}^n}(-d) \to \left(U \mapsto \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}\big|_U}\left(\mathcal{O}_{\mathbb{P}^n}(d)\big|_U, \mathcal{O}_{\mathbb{P}^n}\big|_U\right)\right).$$

It's easy to see that ϕ commutes with restriction, each ϕ_U is an $\mathcal{O}_{\mathbb{P}^n}(U)$-module morphism since

$$\begin{aligned}\left(\frac{e}{f}\phi_U\left(\frac{a}{b}\right)+\phi_U\left(\frac{g}{h}\right)\right)_V\left(\frac{c}{d}\right) &= \frac{e}{f}\frac{ac}{bd}+\frac{gc}{hd}\\ &= \left(\frac{e}{f}\frac{a}{b}+\frac{g}{h}\right)\frac{c}{d}\\ &= \left(\phi_U\left(\frac{e}{f}\frac{a}{b}+\frac{g}{h}\right)\right)_V\left(\frac{c}{d}\right).\end{aligned}$$

Now we shall define a morphism

$$\psi_U : \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}\big|_U}\left(\mathcal{O}_{\mathbb{P}^n}(d)\big|_U, \mathcal{O}_{\mathbb{P}^n}\big|_U\right) \to \mathcal{O}_{\mathbb{P}^n}(-d)(U)$$

by picking an element $f \in \mathbb{K}[x_0, \ldots, x_n]_d$ which is non-zero on U, and then defining

$$\psi_U : \rho \mapsto \frac{\rho(f)}{f}.$$

To see that that ψ_U doesn't depend on our choice of f, let $g \in \mathbb{K}[x_0, \ldots, x_n]_d$ be another element which is non-zero on U. Then, using the fact that ρ is a $\mathcal{O}_{\mathbb{P}^n}(U)$-module morphism we get

$$\frac{\rho(f)}{f} = \frac{g}{g}\frac{\rho(f)}{f} = \frac{\rho(fg)}{fg} = \frac{f}{f}\frac{\rho(g)}{g} = \frac{\rho(g)}{g},$$

and thus the ψ_U are set theoretically well-defined. They clearly commute with restrictions, and to see that they are $\mathcal{O}_{\mathbb{P}^n}(U)$-module morphisms, note that

$$\psi_U\left(\frac{a}{b}\rho+\eta\right) = \frac{\frac{a}{b}\rho(f)+\eta(f)}{f} = \frac{a\rho(f)+b\eta(f)}{bf} = \frac{a\rho(f)}{bf}+\frac{\eta(f)}{f} = \frac{a}{b}\psi_U(\rho)+\psi_U(\eta).$$

Finally, ϕ_U and ψ_U are indeed inverse each other as

$$\begin{aligned}(\phi_U(\psi_U(\rho)))\left(\frac{a}{b}\right) &= \left(\phi_U\left(\frac{\rho(f)}{f}\right)\right)\left(\frac{a}{b}\right)\\ &= \frac{\rho(f)a}{fb}\\ &= \rho\left(\frac{fa}{fb}\right)\\ &= \rho\left(\frac{a}{b}\right)\end{aligned}$$

where we used the fact that ρ is an $\mathcal{O}_{\mathbb{P}^n}(U)$-module morphism and $\deg(a) = \deg(b)+d = \deg(b)+\deg(f)$, hence we may move $\frac{a}{fb}$ inside the function application. The other direction is given by

$$\psi_U\left(\phi_U\left(\frac{a}{b}\right)\right) = \psi_U\left(\frac{c}{d}\mapsto\frac{ac}{bd}\right) = \frac{fa}{fb} = \frac{a}{b}.$$

Ex 13.26

(a)

If the sequence is exact on sections for all open $U \subset X$, we have that $\mathrm{im}'(f_i) \cong \ker(f_{i-1})$ for all i, hence the $\mathrm{im}'(f_i)$ are sheaves and

$$\mathrm{im}'(f_i) = \mathrm{im}(f_i) \cong \ker(f_{i-1}),$$

whence the sequence of sheaves is exact.

(b)

A morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves is exact if $\ker(f) = 0$, and as $\ker(f)(U) = \ker(f_U)$, this happens if and only if every $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subset X$. It follows that $\mathrm{im}'(f)(U) \cong \mathcal{F}(U)$, hence $\mathrm{im}'(f)$ is a sheaf.

As $\mathrm{im}'(\mathcal{F}_1 \to \mathcal{F}_2) = \mathrm{im}(\mathcal{F}_1 \to \mathcal{F}_2)$, it's immediate that the sequence is exact on sections over U.

Ex 14.11

(a)

Let $f : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-1)$ be a morphism of $\mathcal{O}_{\mathbb{P}^1}$-sheaves. Then as the global sections of $\mathcal{O}_{\mathbb{P}^1}(-1)$ is 0, it follows that f maps all global sections of $\mathcal{O}_{\mathbb{P}^1}$, I.e k, to 0.

On the affine open set U_0, a morphism of $\mathcal{O}_{\mathbb{P}^1}\big|_{U_0}$-sheaves

$$\mathcal{O}_{\mathbb{P}^1}\big|_{U_0} \to \mathcal{O}_{\mathbb{P}_1}(-1)\big|_{U_0} \cong \mathcal{O}_{\mathbb{P}^1}\big|_{U_0}$$

corresponds to a morphism of $k[x_1]$-modules $k[x_1] \to k[x_1]$, and such morphisms are precisely given by multiplication with with polynomials in $k[x_1]$. As f vanishes on k, it follows that $f_{U_0} = 0$. The same argument on U_1 shows that $f = 0$.

(b)

Let $p = [1 : 0]$ and $f : \mathcal{O}_{\mathbb{P}^1}(-1) \to k_p$ be a morphism of $\mathcal{O}_{\mathbb{P}^1}$-sheaves. As $[1 : 0] \notin U_1$, we have that k_p, and then also f is equal to 0 on U_1.

On U_0, multiplication by x_0 gives an isomorphism $\mathcal{O}_{\mathbb{P}^1}(-1)\big|_{U_0} \to \mathcal{O}_{\mathbb{P}^1}\big|_{U_0}$. From there, a morphism of $\mathcal{O}_{\mathbb{P}^1}\big|_{U_0}$-sheaves $\mathcal{O}_{\mathbb{P}^1}\big|_{U_0} \to k_p$ is equivalent to a morphism of $k[x_1]$-modules $k[x_1] \to k[x_1]/(x_1)$ which are all given by multiplication by a polynomial $g \in k[x_1]$ followed by evaluation at 0. Following all these morphisms, we see that

$$f : \frac{a(x_0, x_1)}{b(x_0, x_1)} \mapsto \frac{x_0 a(x_0, x_1)}{b(x_0, x_1)} \mapsto \frac{x_0 g(x_0, x_1) a(x_0, x_1)}{x_0^{\deg(g)} b(x_0, x_1)} \mapsto \frac{g(1,0)a(1,0)}{b(1,0)},$$

and by letting $c = g(1,0) \in k$, we see that f is given by

$$f : \frac{a(x_0,x_1)}{b(x_0,x_1)} \mapsto \frac{ca(p)}{b(p)}.$$

Now let $a(x_0,x_1)/x_0^{\deg(a)+1} \in \mathcal{O}_{\mathbb{P}^1}(-1)(U_0)$ for some arbitrary homogeneous polynomial $a(x_0,x_1)$. Then we know that

$$f_{U_0}\left(\frac{a(x_0,x_1)}{x_0^{\deg(a)}+1}\right) = ca(1,0),$$

and this will hold after restricting to $U_0 \cap U_1$ as well,

$$f_{U_0\cap U_1}\left(\left.\frac{a(x_0,x_1)}{x_0^{\deg(a)}+1}\right|_{U_0\cap U_1}\right) = ca(1,0).$$

But $x_0^{\deg(a)+1}/x_1^{\deg(a)+1} \in \mathcal{O}_{\mathbb{P}^1(U_0\cap U_1)}$, hence

$$\begin{aligned}\frac{x_0^{\deg(a)+1}}{x_1^{\deg(a)+1}}ca(1,0) &= \frac{x_0^{\deg(a)+1}}{x_1^{\deg(a)+1}}f_{U_0\cap U_1}\left(\left.\frac{a(x_0,x_1)}{x_0^{\deg(a)}+1}\right|_{U_0\cap U_2}\right)\\ &= f_{U_0\cap U_1}\left(\left.\frac{a(x_0,x_1)}{x_1^{\deg(a)}+1}\right|_{U_0\cap U_2}\right),\end{aligned}$$

and as $\frac{a}{x_1^{\deg(a)}+1} \in \mathcal{O}_{\mathbb{P}^1}(-1)(U_1)$ we see that

$$f_{U_1}\left(\frac{a(x_0,x_1)}{x_1^{\deg(a)}+1}\right) = 0,$$

whence

$$f_{U_0\cap U_1}\left(\left.\frac{a(x_0,x_1)}{x_1^{\deg(a)}+1}\right|_{U_0\cap U_2}\right) = 0.$$

We've shown that follows that $\frac{x_0^{\deg(a)+1}}{x_1^{\deg(a)+1}}ca(1,0) = 0$, and as a was an arbitrary homogeneous polynomial, $c = 0$ whence $f = 0$.

(c)

Let $f : k_p \to \mathcal{O}_{\mathbb{P}^1}$ be a morphism of $\mathcal{O}_{\mathbb{P}^1}$-sheaves. Any morphism of sheaves is determined on stalks, and as $(k_p)_q = 0$ whenever $p \neq q$, f is determined on the stalk p. The only morphism here is the 0-morphism since any $\mathcal{O}_{\mathbb{P}^1,p}$-linear morphism would have to satisfy

$$\frac{a(x_0,x_1)}{b(x_0,x_1)}f_p(c) = f_p\left(\frac{a(x_0,x_1)}{b(x_0,x_1)}\cdot c\right) = f_p\left(\frac{a(1,0)}{b(1,0)}c\right) = \frac{a(1,0)}{b(1,0)}f_p(c)$$

which implies that $f_p(c) = 0$.

Ex 14.12

As $X \subset \mathbb{P}^n$ is irreducible, $I(X)$ is prime, and as X has codimension 1, $I(X)$ has height 1, hence $I(X) = (f)$ for some homogeneous polynomial by the principal ideal theorem. We will now prove that the following sequence is exact,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^n} \longrightarrow X \longrightarrow 0 \ .$$

By Lemma 13.21.(b), it's enough to see that the restrictions to an affine cover are exact, which by Remark 14.5.(a) and Lemma 14.7.(b) is equivalent to checking that the sequences of modules which are induced by the restriction to the affine patches are exact. On the patch U_0, we obtain the sequence of $\mathbb{K}\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right]$-modules

$$0 \longrightarrow \mathbb{K}\left[x_0, \ldots, x_n, \frac{1}{x_0}\right]_{-d} \xrightarrow{\cdot f} \mathbb{K}\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right] \longrightarrow \frac{\mathbb{K}\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right]}{\left(f/x_0^d\right)} \longrightarrow 0 \ ,$$

which is clearly exact.