Exercise 1.1.1

We have $\operatorname{im}(d_{n+1}) = (4)$ and $\ker(d_n) = (2)$. $(4) \subset (2)$ so the maps induce a complex, and the homology at all positive degrees is given by the $\mathbb{Z}$-module quotient $(4)/(2) = \mathbb{Z}/(2)$.

Exercise 1.1.2

Let d, d' be the differential maps in C and D respectively. Let $x \in \ker(d_n)$ be a cycle in C. Then $0 = ud_n(x) = d'_n u(x)$, hence $u(x) \in \ker(d'_n)$ and $u(Z_n(C)) \subseteq Z_n(D)$.

Now let $x \in \operatorname{im}(d_{n+1})$, and $x' \in d_{n+1}^{-1}(x)$. Then $u(x) = ud_{n+1}(x') = d'_{n+1}u(x) \in \operatorname{im}(d'_{n+1})$ and $u(B_n(C)) \subseteq B_n(D)$.

To see that H_n is a functor from $\mathbf{Ch}(\mathbf{mod} - R) \to \mathbf{mod} - R$, Note that

- H_n send complexes to modules (trivial),
- we just showed that H_n send morphism of complexes to morphisms of modules,
- H_n sends the identity morphisms to identity morpshisms (trivial),
- and H_n preserves composition by

$$(u'u)(x + B_n(C)) = u'(u(x + B_n(C))).$$

Exercise 1.1.5

(1) and (2) are just reformulations of each other, we will show that (2) and (3) are equivalent.

(2) $\Rightarrow$ (3): The zero map $0 \to C$ is well-defined for any complex C, and in our particular case, it is a quasi-isomorphism as $H_n = 0$ for all n.

(3) $\Rightarrow$ (2): Only the 0-module is isomorphic to 0, hence $H_n = 0$ for all n

Exercise 1.2.1

For the direct product, note that an element is in the boundary of the product $a \in B_n\left(\prod A_\alpha\right)$ if and only if it's in the boundary of every component $a \in \prod B_n\left(A_\alpha\right)$. The analagous argument applies to cycles as well, and it follows that the homologies are isomorphic.

Note here that the "if" direction only holds since an element in the direct proudct may have non-zero entries for infinitely many indices. Hence the argument can't be immediately carried over for the direct sum.

For the direct sum, an element is in the boundary of the product $a \in B_n\left(\bigoplus A_\alpha\right)$ if and only if it's in the boundary of every component $a \in \prod B_n\left(A_\alpha\right)$, and $a_\alpha = 0$ for all but finitely many indicies, I.e $a \in \bigoplus B_n\left(A_\alpha\right)$. Now all arguments generalize analogously.

Exercise 1.2.2

We first show that the two concepts for "kernel of a map", conicide.

Let $f : A \to B$ be an R-linear map, and $g : C \to A$ be such that $gf = 0$. Then $\mathrm{im}(g) \subseteq \ker(f)$, hence we can factor g so that $g = ig'$ with $g' : G \to \ker(f), i : \ker(f) \to A$. Moreover, g' is uniquely determined by $\ker(f), i$, hence $\ker(f), i$ satisfies the universal properties of kernels in additive categories. Since universal categories are unique up to isomorphism, it follows that the two definitions of the kernel of a map coincide.

Now, if $i : A \to B$ is a monic map, and f is the kernel of i, then $fi = 0$ hence $f = 0$, so $\ker(i) = 0$ by the universal property and i is a monomorphism.

Similarly, if $\ker(i) = 0$, let f be the kernel of i. Then if g is a map such that $gi = 0$, we have that g factors through f, via say $g = g'f$. But then $g = g'f = g'0 = 0$ and i is mono.

The dual statement with cokernels, epis and epimorphisms is analagous.

Exercise 1.2.3

The argument above about kernels coinciding can be applied to every degree simultaneously. All that remains to show is that the functions g', i are morphisms of chain complexes, i.e commute with differentials, but this is easy to see as g' is just g with a narrower codomain, and i is just the identity map with wider codomain.

Exercise 1.2.4

In Exercise 1.2.3, we showed that kernel and cokernels may be applied componentwise to modules of a chain complex in $\mathbf{Ch}(R-\mathrm{mod})$. We first show that we can do this in any abelian category.

Lemma 0.1. Let $\mathcal{A}$ be an abelian category, $(B, d), (C, d') \in \mathbf{Ch}(\mathcal{A})$ be two chain complexes and $f : B \to C$ a morphism of complexes. Let $A_i = \ker(f_i)$, and $j_i : \ker(f_i) \to A_i$ be the injections. Then, as $f_{i-1}d_ij_i = d'_if_ij_i = 0$, we have a unique map $d''_i : A_i \to A_{i-1}$ given by the universal property of the kernel of

f_{i-1} as in the following commutative diagram

$$\begin{tikzcd}
A_i \arrow[r, dashed, "d''_i"] \arrow[d, "j_i"] & A_{i-1} \arrow[d, "j_{i-1}"] \\
B_i \arrow[r, "d_i"] \arrow[d, "f_i"] & B_{i-1} \arrow[d, "f_{i-1}"] \\
C_i \arrow[r, "d'_i"] & C_{i-1}.
\end{tikzcd}$$

Then (A, d'') is a kernel of f.

Proof. Suppose that $(A', d'''), g : A' \to B$ are such that $gf = 0$. Then $g_i f_i = 0$ for all i, and by the universal property of $A_i = \ker(f_i)$, we have unique h_i such that $g_i = j_i h_i$. Then $h : A' \to A$ is a map of chain complexes since

$$j_{i-1}h_{i-1}d'''_i = g_{i-1}d'''_i = d_i g_i = d_i j_i h_i = j_{i-1}d''_i h_i,$$

which since j_{i-1} is mono implies

$$h_{i-1}d'''_i = d''_i h_i.$$

Moreover, h is unique with this property since the h_i are unique. $\square$

The statement and proofs for epimorphisms is dual to the one above.

The statement of the exercise now follows since if

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a sequence in $\mathbf{Ch}(\mathcal{A})$, then kernels and cokernels (hence also images) may be taken component-wise, so $\ker(g) = \operatorname{im}(f)$ if and only if $\ker(g_i) = \operatorname{im}(f_i)$ for all i.

Exercise 1.2.5

Let $x \in \ker(d_{2n})$, with

$$x = (x_{n+i,n-i})_{i\in\mathbb{N}}.$$

with each $x_{r,s} \in C_{r,s}$. Since C is bounded, we have upper and lower bounds N_u, N_l such that we can write

$$x = (x_{n+i,n-i})_{i=N_l}^{N_u}.$$

After relabeling we can assume that $N_l = 0$, and write $N_u = N$.

So, suppose that $x = \sum_{i=0}^{N} x_{n+i,n-i}$ and that $d(x) = 0$. Then

$$\begin{gathered} d^h(x_{n,n}) = 0, \\ d^h(x_{n+i+1,n+i-1}) + d^v(x_{n+i,n-i}) = 0,\ \forall i \in [N-1], \\ d^v x_{n+N,n-N} = 0. \end{gathered}$$

Since the rows are exact, the first condition tells us that there is $y_{n+1,n} \in C_{n+1,n}$ such that $d^h(y_{n+1,n}) = x_{n,n}$. Then

$$\begin{aligned} 0 &= d^h(x_{n+1,n-1}) + d^v(x_{n,n}) \\ &= d^h(x_{n+1,n-1}) + d^v(d^h(y_{n+1,n})) \\ &= d^h(x_{n+1,n-1}) - d^h(d^v(y_{n+1,n})) \\ &= d^h(x_{n+1,n-1} - d^v(y_{n+1,n})), \end{aligned}$$

hence we have $y_{n+2,n-1} \in C_{n+2,n-1}$ such that $d^h(y_{n+2,n-1}) = x_{n+1,n-1} - d^v(y_{n+1,n}))$. Continuing on this way, we can write $y = \sum_{i=0}^{N+1} y_{n+1+i,n-i}$, and get

$$\begin{aligned} d(y) &= \sum_{i=0}^{N+1} d(y_{n+1+i,n-i}) \\ &= \sum_{i=0}^{N+1} d^h(y_{n+1+i,n-i}) + d^v(y_{n+1+i,n-i}) \\ &= \sum_{i=0}^{N+1} x_{n+i,n-i} - d^v(y_{n+i,n-i+1}) + d^v(y_{n+1+i,n-i}) \\ &= \sum_{i=0}^{N+1} x_{n+i,n-i} \\ &= x. \end{aligned}$$

Exercise 1.2.7

(1)

Each component of

$$0 \to Z(C) \to C \xrightarrow{d} B(C)[-1] \to 0$$

is given by

$$0 \to \ker(d_i) \to C_i \xrightarrow{d_i} \operatorname{im}(d_i) \to 0,$$

and is clearly exact. That these component maps commute with differentials to form morphisms of complexes is easy to see.

(2)

Each component of

$$0 \to H(C) \to C/B(C) \xrightarrow{d} Z(C)[-1] \to H(C)[-1] \to 0$$

is given by

$$0 \to \ker(d_i)/\mathrm{im}(d_{i+1}) \to C_i/\mathrm{im}(d_{i+1}) \xrightarrow{d_i} \ker(d_{i-1})/\mathrm{im}(d_i \circ d_{i+1}) \cong \ker(d_{i-1}) \to \ker(d_{i-1})/\mathrm{im}(d_i) \to 0$$

and is clearly exact. That these component maps commute with differentials to form morphisms of complexes is easy to see.

Exercise 1.2.8

I don't know why we need $B[-1]$ and not B in the row $q = 1$ of D. I'll assume this is a mistake, and define D as

$$\begin{tikzcd}
 & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \\
\dots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\
\dots & B_{-2} \arrow[l, "d_B"'] \arrow[d, "f"] & B_{-1} \arrow[l, "d_B"'] \arrow[d, "-f"] & B_0 \arrow[l, "d_B"'] \arrow[d, "f"] & B_1 \arrow[l, "d_B"'] \arrow[d, "-f"] & B_2 \arrow[l, "d_B"'] \arrow[d, "f"] & \dots \arrow[l] \\
\dots & C_{-2} \arrow[l, "d_C"'] \arrow[d] & C_{-1} \arrow[l, "d_C"'] \arrow[d] & C_0 \arrow[l, "d_C"'] \arrow[d] & C_1 \arrow[l, "d_C"'] \arrow[d] & C_2 \arrow[l, "d_C"'] \arrow[d] & \dots \arrow[l] \\
\dots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\
 & \dots & \dots & \dots & \dots & \dots, &
\end{tikzcd}$$

where C is in row $q = 0$. We can then define C' as

$$\begin{tikzcd}
 & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \\
\dots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\
\dots & C_{-2} \arrow[l, "d_C"'] \arrow[d] & C_{-1} \arrow[l, "d_C"'] \arrow[d] & C_0 \arrow[l, "d_C"'] \arrow[d] & C_1 \arrow[l, "d_C"'] \arrow[d] & C_2 \arrow[l, "d_C"'] \arrow[d] & \dots \arrow[l] \\
\dots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\
 & \dots & \dots & \dots & \dots & \dots, &
\end{tikzcd}$$

where C is in row $q = 0$, and B as

$$\begin{array}{ccccccccccccc}
 & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow[d_B]{} & B_{-2} & \xleftarrow[d_B]{} & B_{-1} & \xleftarrow[d_B]{} & B_0 & \xleftarrow[d_B]{} & B_1 & \xleftarrow[d_B]{} & B_2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots, & &
\end{array}$$

where B is in row $q = 0$.

Exercise 1.4.5

(a)

First note that any map is chain homotopic to itself via $s = 0$. Now, let $f, g, h : C \to D$ be chain maps, and s, s' be chain homotopies f to g and g to h respectively. Then $g - f = -sd - ds = -sd + d(-s)$ so $-s$ is a chain homotopy g to f. Moreover,

$$\begin{aligned}
f - h &= (f - g) + (g - h) \\
&= sd + ds + s'd + ds' \\
&= (s + s')d + d(s + s')
\end{aligned}$$

so $s + s'$ is a chain homotopy from f to h, and we've verified that chain homotopy indeed induces an equivalence relation on chain maps.

The category of modules is abelian, so $\operatorname{Hom}(C, D)$ is an abelian group, hence we only need to show that the inherited addition is well-defined in $\operatorname{Hom}_{\mathbf{K}}(C, D)$. Let $f, g : C \to D$ be chain homotopic maps via s and $h : C \to D$ be another map. Then

$$f + h = g + h + sd + ds,$$

hence s is a chain homotopy $f + h$ to $g + h$ and addition is well-defined.

(b)

We have

$$\begin{aligned}
vfu &= v(g + sd + ds)u \\
&= vgu + vsdu + vdsu \\
&= vgu + vsud + dvsu,
\end{aligned}$$

hence vfu is chain homotopic to vgu via $vsu : B \to E$.

It follows that composition is well-defined in $\mathbf{K}$, hence $\mathbf{K}$ is a category. Moreover, addition of morphisms distributes over composition since these operations are inherited from $\mathbf{Ch}$, hence $\mathbf{K}$ is an $\mathbf{Ab}$-category.

(c)

We have

$$\begin{aligned} f_1 + f_2 &= g_1 + g_2 + s_1 d + d s_1 + s_2 d + d s_2 \\ &= g_1 + g_2 + (s_1 + s_2) d + d(s_1 + s_2), \end{aligned}$$

hence the congruence relation of homotopy equivalence is additive.

We now show that additive congruence relations $\sim$ on additive categories C induce additive quotient categories $C/\sim$ and functors $C \to C/\sim$. The hom-sets are abelian group as quotienting by $\sim$ is the same on the hom-sets as taking the quotient by the subgroup of elements that are ~ 0. Moreover, addition distributes over composition on the hom-sets as these operations are inherited from C. Hence $C/\sim$ is an $\mathbf{Ab}$-category

The zero object $z \in \mathbf{ob}(C)$ is also a zero object in $C/\sim$. Similarly, if $A, B \in C/\sim$, then $A, B \in C$ hence we have a product object $A \times_C B \in C$. We will show that this also is a product object in $C/\sim$. Let f_1, f_2 be morphisms $Y \to A, B$ in $C/\sim$, and let f_1', f_2' be representatives of these morphisms in C. Then there exists a unique $f' : Y \to A \times_C B$ such that $f_i' = \pi_i f'$, thus there is a unique map $f : Y \to A$ in the quotient category satisfying the same properties.

The functor is additive induces homomorphisms $\mathrm{Hom} \to \mathrm{Hom}_{\mathbf{K}}$ as $\sim$ is additive.

(d)

No, and to show this we give a counterexample. Let

$$C = \ldots \to 0 \to \mathbb{Z} \to 0 \to \ldots,$$

and

$$D = \ldots \to 0 \to \mathbb{Z}/p\mathbb{Z} \to 0 \to \ldots,$$

and $f : C \to D$ be the chain map induced by the projection $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. Then the kernel of f in $\mathbf{Ch}$ is given by

$$\ker(f) = \ldots \to 0 \to \mathbb{Z} \to 0 \to \ldots,$$

with the injection i induced by the map $\mathbb{Z} \to \mathbb{Z}, n \mapsto pn$. In $\mathbf{K}$ however, i isn't monic. Indeed, let TODO FINNISH LATER

Exercise 1.5.1

We have

$$d(b,c) = (-d(b), d(c) - b),$$

and

$$dsd(b,c) = ds(-d(b), d(c)-b) = d(b-d(c), 0) = (d(d(c))-d(b), d(c)-b) = (-d(b), d(c)-b),$$

hence s is a splitting. Moreover, $\mathrm{cone}(C)$ is split exact as

$$(ds + sd)(b,c) = d(-c, 0) + s(-d(b), d(c) - b) = (d(c), c) + (b - d(c), 0) = (b, c).$$

Exercise 2.2.1

Let P be a projective object in $\mathbf{Ch}$, and consider the surjective morphism $\pi : \mathrm{cone}(P) \to P[-1]$. Surjective morphisms onto projective objects split, hence $P[-1]$ is a direct summand of the complex $\mathrm{cone}(P)$, which is split exact by Exercise 1.5.1. Now, H_n preserves direct sums by Exercise 1.2.1, so $P[-1]$ must be exact as well.

Moreover, as π sends $(p_{n-1}, p_n) \to -p_{n-1}$, we have that the splitting $i : P[-1] \to \mathrm{cone}(P)$ of π is of the form $i : p_{n-1} \mapsto (-p_{n-1}, \phi(p_{n-1}))$ for some map $\phi : P[-1] \to P$.

Now, let d, d' be the differentials on $P, \mathrm{cone}(P)$. Then $-d$ is the differential on $P[-1]$, and as i is a chain map, it commutes with $-d$ and d'. Thus

$$\begin{aligned}
0 &= -d \circ -d \\
&= i \circ -d \circ -d \\
&= d' \circ i \circ -d \\
&= d' \circ (d, \phi \circ -d) \\
&= (-d \circ d, (d \circ \phi \circ -d) - d) \\
&= (0, d \circ -\phi \circ d - d),
\end{aligned}$$

hence $d \circ -\phi \circ d = d$, and $-\phi$ is a splitting, so $P[-1]$ is split exact.

Finally, to see that each P_n is projective, let $g : B \to C$ be a surjective module morphism, and $f : P_n \to C$. Then Let $B., C.$ be the complexes with the n-th entry as B, C and the rest 0.
We skip the other direction.

Exercise 2.2.2

Suppose that $\mathcal{A}$ has enough projectives, and let $(B, d) \in \mathbf{Ch}(\mathcal{A})$, and that $\pi_i : P_i \to B_i$ are surjections onto the components of B from projective objects

P_i. Then as π_{i-1} is a surjection, we have that $d_i \circ \pi_i$ factors through π_{i-1}. I.e there exists $h_i : P_i \to P_{i-1}$ such that the following diagram commutes

$$\begin{array}{ccc} P_i & \xrightarrow{h_i} & P_{i-1} \\ \big\downarrow{\scriptstyle \pi_i} & & \big\downarrow{\scriptstyle \pi_{i-1}} \\ B_i & \xrightarrow{d_i} & B_{i-1}. \end{array}$$

Now, we don't know that whether or not $h \circ h$ is 0 or not, so we can't form a chain complex out of P, h as is. But let's pretend that P, h is a chain complex, and form cone(P) as in Exercise 1.5.1. Then cone(P) is a chain complex, even if P might not be, because

$$\begin{bmatrix} -h & 0 \\ -\mathrm{id} & h \end{bmatrix}^2 = \begin{bmatrix} -h \circ -h & 0 \\ (-h \circ -\mathrm{id}) + (h \circ -\mathrm{id}) & h \circ h \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

LAST EQUALITY IS WRONG SINCE WE DON'T KNOW THAT hh is 0. Moreover, cone(P) is split exact for the same reason as in Exercise 1.5.1. Indeed, let $s : \mathrm{cone}(P)_i \to \mathrm{cone}(P)_{i+1}$ be given by

$$s(b, c) = (-c, 0),$$

and d' be the differential on cone(P). Then

$$\begin{aligned} (sd' + d's)(b, c) &= s(-h(b), h(c) - b) + d'(-c, 0) \\ &= (b - h(c), 0) + (h(c), c) \\ &= (b, c), \end{aligned}$$

hence cone(P) is split exact. As the π_i commute with h_i, the map cone$(P) \to B$ given by $(b_i, c_i) \to (-\pi_i, 0)$ is a chain map, which is also surjective by choice of π_i.

Exercise 2.3.1

Any ideal $J \subset \mathbb{Z}/(m)$ is of the form (d) where $dk = m$. So, let $f : J \to \mathbb{Z}/(m)$ be a $\mathbb{Z}/(m)$-module morphism, where $f(d) = a$. Then, as $0 = f(dk) = kf(d)$, we must have that $k'm = kf(d)$, hence $dk' = f(d)$. Thus we can extend f to $R \to R$ by sending $1 \mapsto k'$, and R is an injective R-module by Baer's Criterion.

Now, consider the R-module $M = \mathbb{Z}/d$ with $dk = m$, and some prime p divides d and m/d. Then we have an injective R-linear map $M \to R$ given by $1 \mapsto k$, which if M was injective, would have a splitting. But the hypothesis assures that this is not the case, hence M is not injective.

Exercise 2.4.2

asd
Let $A \in \mathcal{A}$, and $P. \to A.$ be a projective resolution. Then

$$U(L_i(F))(A) = U(H_i(F(P.))).$$

Now, given any chain complex over an abelian category

$$C. : \ldots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \ldots,$$

we can form short exact sequences

$$0 \to \ker$$

but exact functors commute with taking homologies, and to see this, just split any chain complex into short exact sequences.