Ch 1

Ex 1.3.9

(a)

We have $V(P) = V(P_1) \cup V(P_2)$, and since the variety of a prime ideal is irreducible, we must have some $V(P_i) = V(P)$, hence $P = \sqrt{P} = I(V(P)) = I(V(P_i)) = \sqrt{P_i} = P_i$.

(b)

Let $g \in (I_1 \cap I_2) : f$. Then $gf \in I_1 \cap I_2$, hence $gf \in I_1$ and $gf \in I_2$, so $g \in (I_1 : f) \cap (I_2 : f)$.

Now suppose $g \in (I_1 : f) \cap (I_2 : f)$. Then $gf \in I_1$ and $gf \in I_2$, hence $gf \in I_1 \cap I_2$, so $g \in (I_1 \cap I_2 : f)$.

(c)

Let $f \in \sqrt{I_1} \cap \sqrt{I_2}$. Then we have n_1, n_2 such that $f^{n_i} \in I_i$. Suppose that $n_1 \geq n_2$. Then $f^{n_1} \in I_1 \cap I_2$ and $f \in \sqrt{I_1 \cap I_2}$.

Now let $f \in \sqrt{I_1 \cap I_2}$. Then we have n such that $f^n \in I_i$ and $f \in \sqrt{I_1} \cap \sqrt{I_2}$.

Ex 1.3.11

Since R is Noetherian we must have some maximal $J = \operatorname{ann}(m) \in S$. We claim that J must be prime. To see this, let $fg \in J$ and suppose that $g \notin J$. Then $fgm = 0$ hence $f \in \operatorname{ann}(gm) \supseteq \operatorname{ann}(m) = J$. But $\operatorname{ann}(gm) \in S$ hence $\operatorname{ann}(gm) = \operatorname{ann}(m)$ and $f \in J$.

Ex 1.3.12

(a)

Let $I = \bigcap Q_i$ be a irredundant primary decomposition. Then $\sqrt{I} = \bigcap \sqrt{Q_i} = \bigcap P_i = \bigcap_{P_i : P_i \text{is a minimal ideal of } I}$.

(b)

Let

$$\begin{aligned} I &= (y(y+z), x(x-z), (x+z)(x-z)) \\ &= (y(y+z), x(x-z), z(x-z)). \end{aligned}$$

We will try to find a primary decomposition for I by first finding the set of primes among $\sqrt{I : f}$ for $f \in R$. We see that

$$\begin{aligned} I : (x-z) &= (x, z, y(y+z)) = (x, z, y^2) \\ I : x &= (x-z, y(y+z), z(x-z)) = (x-z, y(y+z)) \\ I : z &= (x-z, y(y+z), z(x-z)) = (x-z, y(y+z)) \\ I : (y+z) &= (y, x(x-z), z(x-z)) \\ I : y &= (y+z, x(x-z), z(x-z)), \end{aligned}$$

and

$$\begin{aligned} \sqrt{I : (x-z)} &= (x, z, y) \\ \sqrt{I : x} &= (x-z, y(y+z)) \\ \sqrt{I : z} &= (x-z, y(y+z)) \\ \sqrt{I : (y+z)} &= (y, x(x-z), z(x-z)) = (y, x-z) \\ \sqrt{I : y} &= (y+z, x(x-z), z(x-z)) = (y+z, (x-z)), \end{aligned}$$

out of which the primes are $(x, z, y), (y, x-z)$ and $(y+z, x-z)$. We can now guess that

$$I'(y+z, x-z) \cap (y, x-z) \cap (xz, x^2, z^2, y(y+z))$$

is a primary decomposition of I. The first two ideals of the decomposition are prime, hence primary. To see that $J = (xz, x^2, z^2, y(y+z))$ is primary, note that x, y, z generate $\mathbb{K}[x, y, z]/J$ and are all nilpotent (y is nilpotent as $y^2 = -yz$ and $y^4 = (-yz)^2 = z^2(-y)^2 = 0$). Hence every element, and in particular every zero divisor in $\mathbb{K}[x, y, z]/J$ is nilpotent and J is primary. To see that $I = I'$, note that

$$\begin{aligned} I' &= (y+z, x-z) \cap (y, x-z) \cap (xz, x^2, z^2, y(y+z)) \\ &= (y(y+z), x-z) \cap (xz, x^2, z^2, y^2, y(y+z)) \\ &= (y(y+z), x-z) \cap (xz, x(x-z), z(x-z), y(y+z)) \\ &= (y(y+z), x-z) \cap ((xz, y(y+z)) \cup (x(x-z), y(y+z)) \cup (z(x-z), y(y+z))) \\ &= (y(y+z), xz(x-z)) \cup (y(y+z), x(x-z)) \cup (y(y+z), z(x-z)) \\ &= (y(y+z), x(x-z), z(x-z)) \\ &= I \end{aligned}$$

(c)

Let's calculate $J = (xz - y^2, xw - yz) : (x, y)$. We have

$$\begin{aligned} J &= (xz - y^2, xw - yz) : (x, y) \\ &= (xz - y^2, xw - yz) : x \cap (xz - y^2, xw - yz) : y \\ &= \big((xz - y^2, xw - yz) \cup (z^2 - wy)\big) \cap \big((xz - y^2, xw - yz) \cup (wy - z^2)\big) \\ &= (xz - y^2, xw - yz, z^2 - wy). \end{aligned}$$

As both J and (x, y) are prime,

Ex 1.4.2

Let $U = \mathbb{A}^n \setminus Z$ be Zariski open. Then Z is the vanishing set of an ideal and since $\mathbb{K}[\boldsymbol{x}]$ is Noetherian such ideals are finitely generated. We get

$$\begin{aligned} U &= \mathbb{A}^n \setminus Z \\ &= \mathbb{A}^n \setminus V(f_1, f_2, \ldots, f_m) \\ &= \mathbb{A}^n \setminus \left(\bigcap_{i \in [m]} V(f_i) \right) \\ &= \bigcup_{i \in [m]} (\mathbb{A}^n \setminus V(f_i)) \\ &= \bigcup_{i \in [m]} U_{f_i}, \end{aligned}$$

so not only can every open set be written as union of distinguished open sets, but even more as a finite union of distinguished open sets.

Ex 1.4.4

We use the fact that the ring of regular functions on all of X is equal to $A(X)$. This is shown in the text immediately after the Nullstellensatz.

Let $f : X \to Y$ be an isomorphism of affine varieties where f_i are the components of f, and define $f^* : A(Y) \to A(X)$ as $f^*(g) = g \circ f$. It follows from the rules of composition that f^* is a $\mathbb{K}$-algebra homomorphism. Now suppose that $g \in \ker(f^*)$. Then g vanishes on all of $\operatorname{im}(f)$, but f is an isomorphism so $\operatorname{im}(f) = Y$ hence $g = 0$ and f^* is injective. A similar argument applied to f^{-1} shows that $(f^{-1})^*$ is an injective homomorphism $A(X) \to A(Y)$, and with two injective homomorphisms in opposing directions, it follows that $A(X) \cong A(Y)$.

We tackle the other direction, and let $f^* : A(Y) \to A(X)$ be a $\mathbb{K}$-algebra homomorphism. Then let $f_i = f^*(y_i)$ where y_i is the i-th coordinate function on Y, and $f = (f_1, f_2, \ldots, f_m)$. Then $f : X \to \mathbb{A}^m$, and we can construct $F^* : \mathbb{K}[\boldsymbol{x}] \to A(X)$ via $F^*(g) = g \circ f$. Now let $\pi_Y : \mathbb{K}[\boldsymbol{x}] \to A(Y)$ be the projection onto the quotient. Then we have $(F^* - (f^* \circ \pi_Y)) : \mathbb{K}[\boldsymbol{x}] \to A(X)$ and given some coordinate function $x_i \in \mathbb{K}[\boldsymbol{x}]$, we have

$$(F^* - (f^* \circ \pi_Y))(x_i) = F^*(x_i) - f^*(x_i + I(Y)) = (x_i \circ f) - f^*(y_i) = f_i - f_i = 0,$$

and since these x_i generate $\mathbb{K}[\boldsymbol{x}]$, we have $F^* = f^* \circ \pi_Y$. Hence $\ker(F^*) = I(Y)$ and it follows that $\operatorname{im}(f) = Y$ since the elements in $\ker(F^*)$ are precisely those which vanish on $\mathrm{im}(f)$. I.e we can improve and write $f : X \to Y$, and we claim that $f^*(g) = g \circ f$. To see this, we just do a similar calculation as above,

$$f^*(y_i) - (y_i \circ f) = f_i - f_i = 0.$$

Now, f must be surjective, as $\ker(f^*)$ is the set of polynomial function in $A(Y)$ which vanish on $\mathrm{im}(f)$, whence $\mathrm{im}(f) = Y$ since $\ker(f^*) = \emptyset$. We can construct a surjective function in the other direction in a similar manner, and it follows that $X \cong Y$.

Ex 2.2.1

Note that the exercise starts counting from x_0. We will start counting at x_1, and thus expect a results with $n-1$ substituted wherever we find n. Now, $\dim(\mathbb{K}[\boldsymbol{x}]_i)$ is given by the amount of monomials of degree i in n variables. I.e the amount of ordered n-integer partitions of i, which is $\binom{n-1+i}{i}$ (amount of ways to place $n-1$ separators among i elements).

Ex 2.2.5

Let $c_i(n)$ be the coefficient of t^i in the expansion of $(1 + t + t^2 + \ldots)^n$. Then $c_i(n)$ is given by the number of ordered ways we can choose n non-negative powers of t such that the sum of their powers is i. This is again the amount of ordered n-integer partitions of i, hence $\binom{n-1+i}{i}$ as above, and we see that $\mathrm{HF}(\mathbb{K}[\boldsymbol{x}], i) = c_i(n)$. Hence $\mathrm{HS}(\mathbb{K}[\boldsymbol{x}], t) = (\mathrm{HS}(\mathbb{K}[x], t))^n$ and we are done.

Ex 2.2.6

Let $R = \mathbb{K}[x, y, z]/(x^2, y^3, z^4)$. Note that the generators are nilpotent in R, and therefore so is all of R, and R is Artinian. In particular, all terms of degree greater or equal to $2+3+4 = 9$ vanish by the pigeonhole principle. Instead of computing the dimension of each homogeneous component of R by hand, we generalize and inductively compute the Hilbert series of $R_n = \mathbb{K}[\boldsymbol{x}]/I_n$ where $I_n = (x_1^2, x_2^3, \ldots, x_n^{n+1})$. First of, we have

$$\begin{aligned}
(R_1)_0 &= \mathbb{K} \\
(R_1)_1 &= \mathbb{K}x \\
(R_1)_2 &= 0 \\
(R_1)_3 &= 0 \\
&\vdots
\end{aligned}$$

Now, note that R_k is $R_{k-1}[x_k + (x_k^{k+1})]$. It follows that

$$(R_k)_i = (R_{k-1})_i \oplus x_k (R_{k-1})_{i-1} \oplus x_k^2 (R_{k-1})_{i-2} \oplus \ldots \oplus x_k^{k-1} (R_{k-1})_{i-(k-1)}, \oplus x_k^k (R_{k-1})_{i-k},$$

whence

$$\mathrm{HF}(R_k, i) = \sum_{j=0}^{\min(k,i)} \mathrm{HF}(R_{k-1}, i-j).$$

I haven't figured out how to find a closed form expression for $\mathrm{HF}(R_k, i)$.

If there are $k_i \in \mathbb{N}_{>0}$ for each $i \in [n]$ such that $x_i^{k_i} \in I$, since then every generator in the quotient is nilpotent, and it follows that the whole quotient ring is as well. For the other direction, suppose I is an ideal such that $\mathbb{K}[\boldsymbol{x}]/I$ is Artinian. Then the strictly descending sequence $(x_i + I) \supsetneq (x_i^2 + I) \supsetneq (x_i^3 + I)$ must stabilize after some k_i and $x_i^{k_i} + I = x_i^{k_i+1} + I$ implies $x_i^{k_i}(x_i - 1) \in I$ so $x_i^{k_i} \in I$ as it's a monomial ideal, hence homogeneous ideal, and such ideals contain all terms of the polynomials they contain.

Ex 2.3.1

(a)

We begin by calculating $\ker(\phi)$ and $\mathrm{im}(\phi)$. By looking at the matrix, we see that $[1,1,1] \subseteq \ker(\phi)$ and that $[1,-1,0] = \phi([1,0,0])$, $[-1,0,1] = \phi([0,0,1])$, so by a dimensionality argument it follows that

$$\ker(\phi) = \{[k,k,k] : k \in \mathbb{K}\},\ \mathrm{im}(\phi) = \{[k_1 - k_2, -k_1, k_2] : k_1, k_2 \in \mathbb{K}\}.$$

Thus

$$H_1(V) = \ker(\phi) = \mathbb{K}[1,1,1]$$

and

$$H_2(V) = V_0/\mathrm{im}(\phi) = [\mathbb{K},\mathbb{K},\mathbb{K}]/(\mathbb{K}[1,-1,0] \oplus \mathbb{K}[-1,0,1]).$$

(b)

Using the Rank-Nullity Theorem yields

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \dim(H_i(V)) &= \sum_{i=0}^{n} (-1)^i \left(\dim(\ker(\phi_i)) - \dim(\mathrm{im}(\phi_{i+1}))\right) \\
&= \sum_{i=0}^{n} (-1)^i \left(\dim(V_i) - \dim(\mathrm{im}(\phi_i)) - \dim(\mathrm{im}(\phi_i + 1))\right) \\
&= \sum_{i=0}^{n} (-1)^i \dim(V_i),
\end{aligned}$$

where we assume that the sequence is padded with zero spaces like $0 \to \ldots \to 0$ whence $\dim(\mathrm{im}(\phi_0)) = \dim(\mathrm{im}(\phi_{n+1}) = 0$.

Ex 2.3.4

For the base case, assume that $\Delta P(i) = c$ is constant. Then $P(i) = cP(i-1)$, and it follows by induction that $P(n) = P(0)c^n$.

For the inductive step, suppose that $\Delta P(i)$ has degree s and is of the form $a_s i^s + \dots a_0$. Then we follow the hint and let $h(i) = a_s s! \binom{i}{s+1}$. We then have

$$\begin{aligned}
\Delta h &= a_s s! \frac{i!}{(i-s-1)!(s+1)!} - a_s s! \frac{(i-1)!}{(i-s-2)!(s+1)!} \\
&= a_s s! \left(\frac{i!}{(i-s-1)!(s+1)!} - \frac{(i-s-1)(i-1)!}{(i-s-1)!(s+1)!} \right) \\
&= a_s s! \frac{i! - (i-s-1)(i-1)!}{(i-s-1)!(s+1)!} \\
&= a_s s! \frac{(s+1)(i-1)!}{(i-s-1)!(s+1)!} \\
&= a_s \frac{(i-1)!}{(i-s-1)!} \\
&= a_s \prod_{k=1}^{s+1} (i-k),
\end{aligned}$$

hence $\operatorname{lt}(\Delta h(i)) = a_s i^s$, and $\Delta P(i) - \Delta(h(i))$ has degree $s-1$. Applying the inductive assumption, it follows that $P(i) - h(i)$ is a polynomial with rational coefficients, and since $h(i)$ is constructed as such, $P(i)$ is as well, and we are done.

Ex 2.3.5

(a)

We can assume that $p_n \neq 0$ since at least one coordinate must be non-zero. Now consider the graded and linear change of coordinates $f_p : \mathbb{K}[\boldsymbol{x}] \to \mathbb{K}[\boldsymbol{x}]$ where $f_p : x_i \mapsto x_i + \frac{p_i x_n}{p_n}$ which satisfies $f_p(0 : 0 : \dots : 0 : 1) = p$. Precomposition with f_p induces a graded isomorphism $f_p^* : R/I(p) \cong R/I(0 : 0 : \dots : 0 : 1)$, hence we may just as well assume that $p = (0 : 0 : \dots : 0 : 1)$ from the beginning.

It now follows that $R/I(p) = \mathbb{K}[\boldsymbol{x}]/(x_0 : x_1 : \cdots : x_{n-1}) \cong \mathbb{K}[\boldsymbol{x}]$ hence $\operatorname{HF}(R/I(p), i) = 1$ and $\operatorname{HP}(R/I(p), i) = 1$.

(b)

Let $J = I(p_1) \cap I(p_2)$. Then $\ker(\phi) = J \oplus J$, and we see that the sequence is exact when the function $I(p_1) \cap I(p_1) \to I(p_1) \oplus I(p_2)$ is given by $i \mapsto i \oplus i$.

(c)

We have that $I(p_1) + I(p_2) = I(p_1 \cap p_2) = I(\emptyset) = (1) = R$, hence $R/(I(p_1) + I(p_2)) = 0$. All homomorphisms from (b) are graded, and since the Hilbert polynomial is additive on graded exact sequences (since the rank of a vector space is by the Rank-Nullity theorem), we have

$$\mathrm{HP}(I(p_1)\cap I(p_2)) = \mathrm{HP}(I(p_1)\oplus I(p_2)) - \mathrm{HP}(I(p_1)+I(p_2)) = \mathrm{HP}(I(p_1)\oplus I(p_2)) - \mathrm{HP}(R).$$

From (a), it follows that $\mathrm{HP}(I(p_1)) = \mathrm{HP}(R) - \mathrm{HP}(R/I(p_1)) = \mathrm{HP}(R) - 1$, hence $\mathrm{HP}(I(p_1) \oplus I(p_2)) = 2\mathrm{HP}(R) - 2$ and t follows that

$$\begin{aligned}
\mathrm{HP}(R/(I(p_1 \cup p_2)) &= \mathrm{HP}(R) - \mathrm{HP}(I(p_1) \cap I(p_2)) \\
&= \mathrm{HP}(R) - (\mathrm{HP}(I(p_1) \oplus I(p_2)) - \mathrm{HP}(R)) \\
&= \mathrm{HP}(R) - (2\mathrm{HP}(R) - 2 - \mathrm{HP}(R)) \\
&= 2.
\end{aligned}$$

Repeating this argument inductively, suppose that $p_1, \dots, p_m \in \mathbb{P}^n$ are m projective points, and that $\mathrm{HP}(R/I(p_1, p_2 \dots, p_{m-1})) = m - 1$. Then let $J = I(p_1, p_2, \dots, p_{m-1})$ and consider the exact sequence

$$0 \longrightarrow J \cap I(p_n) \xrightarrow{\psi} J \oplus I(p_n) \xrightarrow{\phi} J + I(p_n) \longrightarrow 0$$

where ψ is the diagonal injection as above and $\phi(f, g) = f - g$. Then like before, $\{p_1, p_2, \dots, p_{n-1}\} \cap \{p_n\} = \emptyset$, hence $J + I(p_n) = (1) = R$. By the inductive assumption, $\mathrm{HP}(R/J) = n - 1$ so

$$\mathrm{HP}(J \oplus I(p_n)) = \mathrm{HP}(R) - (n - 1) + (\mathrm{HP}(R) - 1) = 2\mathrm{HP}(R) - n,$$

and by the additivity of Hilbert series,

$$\mathrm{HP}(R/(J\cap I(p_n))) = \mathrm{HP}(R) - \mathrm{HP}(J\oplus I(p_n)) = \mathrm{HP}(R) - (2\mathrm{HP}(R) - n - \mathrm{HP}(R)) = n$$

Ex 2.3.11

A homogeneous degree two polynomial in $\mathbb{K}[x, y, z]$ is of the form

$$f = a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2,$$

with at least one a_i non-zero, and it's zero set (a conic curve) is invariant to multiplication by $\mathbb{K}^*$. Hence we can identify such a conic curve with the point $(a_0 : a_1 : a_2 : a_3 : a_4 : a_5) \in \mathbb{P}^5$.

The Jacobian of such a conic is given by

$$Jf = \begin{pmatrix} 2a_0x + a_1y + a_2z \\ a_1x + 2a_3y + a_4z \\ a_2x + a_4y + 2a_5z \end{pmatrix}^T$$

Hence the conic is smooth whenever

$$Y_f = V\left(f, 2a_0x + a_1y + a_2z, a_1x + 2a_3y + a_4z, a_2x + a_4y + 2a_5z\right)$$

is empty. However, if we look closely, we see that $f \in (2a_0x + a_1y + a_2z, a_1x + 2a_3y + a_4z, a_2x + a_4y + 2a_5z)$ as

$$\begin{aligned}
&x(2a_0x + a_1y + a_2z) + y(a_1x + 2a_3y + a_4z) + z(a_2x + a_4y + 2a_5z)\\
=&2a_0x^2 + a_1xy + a_2xz + a_1xy + 2a_3y^2 + a_4yz + a_2xz + a_4yz + 2a_5z^2\\
=&2a_0x^2 + 2a_1xy + 2a_2xz + 2a_3y^2 + 2a_4yz + 2a_5z^2\\
=&2f,
\end{aligned}$$

and it follows that

$$Y_f = V\left(2a_0x + a_1y + a_2z, a_1x + 2a_3y + a_4z, a_2x + a_4y + 2a_5z\right).$$

But then Y_f is exactly $\ker A$ where A is the matrix

$$A = \begin{pmatrix} 2a_0 & a_1 & a_2 \\ a_1 & 2a_3 & a_4 \\ a_2 & a_4 & 2a_5 \end{pmatrix}^T$$

The characteristic polynomial of A is given by

$$\begin{aligned}
p_A = \det(A - \lambda I) = &- x^3\\
&+ (2a_0 + 2a_3 + 2a_5)\,x^2\\
&+ \left(a_1^2 + a_2^2 - 4a_0a_3 + a_4^2 - 4a_0a_5 - 4a_3a_5\right)x\\
&- 2a_2^2a_3 + 2a_1a_2a_4 - 2a_0a_4^2 - 2a_2^2a_5 + 8a_0a_3a_5,
\end{aligned}$$

and since a symmetric matrix has the same rank as number of non-zero eigenvalues, we see that $\ker A = Y_f$ contains homogeneous points precisely when

$$0 = p_A(0) = 2a_2^2a_3 + 2a_1a_2a_4 - 2a_0a_4^2 - 2a_2^2a_5 + 8a_0a_3a_5.$$

This is a closed condition on the points in $\mathbb{P}^5$, and the open subset formed by the complement is the set of points which correspond to smooth conics.

Ex 3.1.3

Let $(R, \mathfrak{m})$ be a local ring, and P a f.g projective R-module. Let $f_1, \ldots, f_m$ be a generating set for P which is minimal with respect to cardinality, and let $\phi : R^m \to P, \phi : e_i \mapsto f_i$ be a surjective homomorphism onto P. Suppose $(a_1, \ldots, a_m) \in \ker\phi$. Then $\sum a_if_i = 0$, and it follows that no a_i is a unit, as otherwise we could remove the corresponding f_i from our generating set and end up with a smaller set. But as R is local, any non-unit lies in $\mathfrak{m}$, hence $\ker(\phi) \subseteq \mathfrak{m}R^m$. As P is projective, we have $R^m = P \oplus \ker(\phi)$, and $\mathfrak{m}R^m = \mathfrak{m}P \oplus \mathfrak{m}\ker(\phi)$, so $\ker(\phi) \subseteq \mathfrak{m}P \oplus \mathfrak{m}\ker(\phi)$, and $\ker(\phi) \cap P = \emptyset$ implies that $\ker(\phi) \subseteq \mathfrak{m}\ker(\phi)$. Hence $\ker(\phi) = 0$ by Nakayama, and $P = R^m$.

Ex 3.2.1

Consider the free resolution

$$0 \longrightarrow R(-d-e) \xrightarrow{\phi} R(-d) \oplus R(-e) \xrightarrow{\psi} R \longrightarrow R/(f,g) \longrightarrow 0,$$

where we will discuss the components of the sequence in the following paragraph.

Let $e_1 = (1,0), e_2 = (0,1)$ be generators for $R(-e) \oplus R(-d)$. Then $\deg e_1 = d$ and $\deg e_2 = e$. We define the map ψ as given by $\psi(h_1, h_2) = fh_1 + gh_2$. This is a degree 0 map. Suppose $(h_1, h_2) \in \ker(\phi)$. Then $fh_1 = -gh_2$, and after dividing by the greatest common denominator of h_1, h_2, we can suppose that h_1, h_2 are coprime as well. As R is a UFD, and h_1, h_2 and f, g are pairwise coprime, it must be the case that $h_1 = f$ and $h_2 = -g$. It's now easy to see that $\ker(\phi) = ((-g, f))$, and as $\ker(\phi)$ is a principal ideal generated by a degree $d + e$ element, it's isomorphic to $R(-d-e)$ as a graded R-module. Hence the sequence is a graded exact sequence, and it follows that

$$\begin{aligned}
\mathrm{HP}(R/(f,g), i) &= \mathrm{HP}(R(-d-e), i) - \mathrm{HP}(R(-d), i) - \mathrm{HP}(R(-e), i) + \mathrm{HP}(R, i) \\
&= \mathrm{HP}(R, i-d-e) - \mathrm{HP}(R, i-d) - \mathrm{HP}(R, i-e) + \mathrm{HP}(R, i) \\
&= \binom{i-d-e+2}{2} - \binom{i-d+2}{2} - \binom{i-e+2}{2} + \binom{i+2}{2} \\
&= \frac{1}{2}\big((i-d-e+2)(i-d-e+1) \\
&- (i-d+2)(i-d+1) \\
&- (i-e+2)(i-e+1) \\
&+ (i+2)(i+1)\big) \\
&= \frac{1}{2}\big((i^2 - 2id - 2ie + 3id^2 + 2de - 3d + e^2 - 3e + 2) \\
&- (i^2 - 2id + 3id^2 - 3d + 2) \\
&- (i^2 - 2ie + 3ie^2 - 3e + 2) \\
&+ (i^2 + 3i + 2)\big) \\
&= \frac{1}{2}(2de) \\
&= de.
\end{aligned}$$

Now, since f, g share no non-trivial divisor, it follows that the intersection of their vanishing sets is finite, and Bezout's Theorem now follows from our result above combined with Exercise 2.3.5 (c).

Ex 3.3.1

Consider the sequence

$$0 \longrightarrow R(-d)/(I : f) \xrightarrow{\phi} R/I \xrightarrow{\psi} R/(I, f) \longrightarrow 0,$$

where ψ is the canonical quotient map and $\phi : g + (I : f) \mapsto fg + I$. First note that g is well defined, since if $g_1 + (I : f) = g_2 + (I : f)$, then $(g_1 - g_2)f \in I$, hence $fg_1 + I = fg_2 + I$. Moreover, this sequence is exact. It's immediate from the definitions that $\mathrm{im}(\phi) \subseteq \ker(\psi)$, whilst the other inclusion follows from the fact that $\ker(\psi)$ consists of $g + I \in R/I$ such that $g \in (I, f)$, I.e $g + I = hf + I = \phi(h + (I : f))$ for some $h \in R$. The sequence is clearly graded as well, and we are done.

Resolution of a Regular Sequence

From now on, elements in product/sum sets will be denoted with separating ; as opposed to ,, so $(x; y) \in X \times Y$ isn't confused with the ideal (x, y) generated by x, y.

The text in Ch 3.3 asks us to compute the resolution of the ideal generated by some regular sequence, so let's do that.

Let $R = \mathbb{K}[x, y, z, w]$. Then $f_1 = x - w$ is a non-zero divisor on R. $f_2 = z^3 - w^3$ is a non-zero divisor on $R/(f_1)$. Finally, $f_3 = y^2 + yx + x^2$ is a non-zero divisor on $R/(f_1, f_2, f_3)$, and we pick this to be our module M.

The first step of our sequence will be

$$R(-1) \oplus R(-3) \oplus R(-2) \xrightarrow{\psi} R/(f_1, f_2, f_3) \longrightarrow 0,$$

where $\psi : e_i \mapsto f_i$. Now suppose $g = (g_1; g_2; g_3) \in \ker(\phi)$. Then our hypothesis that f_1, f_2, f_3 is regular implies that $g_3 \in (f_1, f_2)$, so we can write $g_3 = A_1 f_1 + A_2 f_2$. But then

$$\begin{aligned} 0 &= (A_1 f_1 + A_2 f_2) f_3 + g_2 f_2 + g_1 f_1 \\ &= (A_2 f_3 + g_2) f_2 + (A_1 f_3 + g_1) f_1, \end{aligned}$$

and again our hypothesis implies that f_1, f_2 are coprime so $f_1 | (A_2 f_3 + g_2)$ and $f_2 | (A_1 f_3 + g_1)$. This gives $g_2 \in (f_1, f_3)$ and $g_1 \in (f_2, f_3)$. We claim that $\ker(\psi)$ is given by

$$I = \left((f_2; -f_1; 0), (f_3; 0; -f_1), (0; f_3; -f_2) \right).$$

It's easy to see that $I \subseteq \ker(\psi)$. For the other inclusion, we will argue via the rank of each graded piece. By the Rank-Nullity Theorem, we expect

$$\dim(R(-1) \oplus R(-3) \oplus R(-2))_i = \dim(\ker(\phi))_i + \dim(f_1, f_2, f_3)_i,$$

and after simplifying

$$\begin{aligned} \dim_{\mathbb{K}}(\ker(\phi))_i &= \binom{3+i-1}{i} + \binom{3+i-3}{i} + \binom{3+i-2}{i} - \dim_{\mathbb{K}}(f_1, f_2, f_3)_i \\ &= \binom{i+2}{i} + \binom{i+1}{i} + \binom{i}{i} - \dim_{\mathbb{K}}(f_1, f_2, f_3)_i. \end{aligned}$$

Thus the hard part lies in computing $\dim_{\mathbb{K}}(f_1, f_2, f_3)_i$.
we already showed that $(g_1; g_2; g_3) \in \ker(\psi)$ means that we can write

$$\begin{aligned} g_1 &= A_2 f_2 + A_3 f_3 \\ g_2 &= B_1 f_1 + B_3 f_3 \\ g_3 &= C_1 f_1 + C_2 f_2, \end{aligned}$$

hence

$$\begin{aligned} (g_1; g_2; g_3) &= (A_2 f_2 + A_3 f_3; B_1 f_1 + B_3 f_3; C_1 f_1 + C_2 f_2) \\ &= (A_2 f_2; B_1 f_1; 0) + (A_3 f_3; 0; C_1 f_1) + (0; B_3 f_3; C_2 f_2), \end{aligned}$$

and the fact that $(g_1; g_2; g_3) \in \ker(\psi)$ means that

$$\begin{aligned} 0 &= A_2 f_2 f_1 + B_1 f_1 f_2 + A_3 f_3 f_1 + C_1 f_1 f_3 + B_3 f_3 f_2 + C_2 f_2 f_3 \\ &= (A_2 + B_1) f_2 f_1 + (A_3 + C_1) f_3 f_1 + (B_3 + C_2) f_3 f_2. \end{aligned}$$

If we now consider points on $V(f_1)$, the above equation turns into $(B_3 + C_2) f_3 f_2 = 0$, hence $B_3 + C_2 \in (f_1)$ since our hypothesis implies that f_1 is coprime to both f_2 and f_3. TODO Todo todo!!! maybe finnish

Ex 3.3.1

We will solve the exercise in a slightly more abstract setting. Suppose we have a graded free resolution of M as

$$G : 0 \longrightarrow G_m \xrightarrow{\xi_m} \dots \xrightarrow{\xi_2} G_1 \xrightarrow{\xi_1} G_0 \xrightarrow{\xi_0} M \longrightarrow 0,$$

and graded module monomorphisms $f : N = M(-d) \to M$, $\phi_i : F_i = G_i(-d) \to G_i$ of degree d such that the following diagram commutes

$$\begin{tikzcd} & & & & & 0 \arrow[d] & \\ F : 0 \arrow[r] & F_m \arrow[r, "\psi_m"] \arrow[d, "\phi_m"] & \dots \arrow[r, "\psi_2"] & F_1 \arrow[r, "\psi_1"] \arrow[d, "\phi_1"] & F_0 \arrow[r, "\psi_0"] \arrow[d, "\phi_0"] & N \arrow[r] \arrow[d, "f"] & 0 \\ G : 0 \arrow[r] & G_m \arrow[r, "\xi_m"] & \dots \arrow[r, "\xi_2"] & G_1 \arrow[r, "\xi_1"] & G_0 \arrow[r, "\xi_0"] & M \arrow[r] \arrow[d, "g"] & 0 \\ & & & & & \operatorname{coker}(f) \arrow[d] & \\ & & & & & 0, & \end{tikzcd}$$

where the ψ_i are the same as the maps ξ_i but between the regraded modules F_i. We will find a graded free resolution to $\operatorname{coker}(f)$.

First note that both g, ξ_0 are surjective, hence their composition is as well. Thus we let $H_0 = G_0$ and $\partial_0 : H_0 \to \operatorname{coker}(f)$ be given by $\partial_0 = g \circ \xi_0$. We want to define the next module in the sequence as $H_1 = F_0 \oplus G_1$. Before we determine the function $\partial_1 : H_1 \to H_0$, let's investigate $\ker(\partial_0)$. First note that $\operatorname{im}(\xi_1) \subseteq \ker(\partial_0)$ by exactness of G. Moreover, we have that

$$\ker(\partial_0) = \xi_0^{-1}(\ker(g)) = \xi_0^{-1}(\operatorname{im}(f)),$$

and as ψ_0 is surjective, we have $\operatorname{im}(f \circ \psi_0) = \operatorname{im}(f)$, now commutativity of the diagram yields

$$\ker(\partial_0) = \xi_0^{-1}(\operatorname{im}(f \circ \psi_0)) = \xi_0^{-1}(\operatorname{im}(\xi_0 \circ \phi_0)).$$

Thus $a \in \ker(\partial_0)$ if and only if $a - \phi_0(b) \in \ker(\xi_0) = \operatorname{im}(\xi_1)$ for some $b \in F_0$. But

$$\partial_0 \circ \phi_0 = g \circ \xi_0 \circ \phi_0 = g \circ f \circ \psi_0 = 0,$$

hence $\phi_0(b) \in \ker(\xi_0)$ for all $b \in F_0$, so we need $a \in \ker(\xi_0) = \operatorname{im}(\xi_1)$ as well. Using all our results thus far, we get

$$\begin{aligned} \ker(\partial_0) &= \{\phi(a') + \xi_1(b') : (a', b') \in F_0 \oplus G_1\} \\ &= \operatorname{im}([\phi_0, \xi_1]), \end{aligned}$$

and we define $\partial_1 = [\phi_0, \xi_1]$.

Now we define $H_2 = F_1 \oplus G_2$ and try to determine the map $\partial_2 : H_2 \to H_1$ such that $\operatorname{im}(\partial_2) = \ker(\partial_1)$. To do so, we start with an element $c \in F_1$. By commutativity of our diagram, $(a = \psi_1(c), b = -\phi_1(c))$ is a pair in $\ker(\partial_1)$. As ϕ_0 is injective, it follows that there only is one $a \in F_0$ for every $b \in G_1$ such that $(a, b) \in \ker(\partial_1)$, namely $a = -\phi_0^{-1}(\xi_1(b))$. Fixing a instead, we see that any b which lies in $\xi_1^{-1}(-\phi_0(a))$ works. I.e, if $a = \psi_1(c)$, then any b such that $b - \phi_1(c) \in \ker(\xi_1) = \operatorname{im}(\xi_2)$. Hence

$$\ker(\partial_1) = \{(\psi_1(c), -\phi_1(c) + \xi_2(d)) : (c, d) \in (F_1 \oplus G_2)\} = \operatorname{im}\left(\begin{bmatrix} \psi_1 & 0 \\ -\phi_1 & \xi_2 \end{bmatrix}\right),$$

and we can set

$$\partial_2 = \begin{bmatrix} \psi_1 & 0 \\ -\phi_1 & \xi_2 \end{bmatrix}.$$

The remaining steps follow by similar arguments. TODO Todo todo: Maybe finish/solve the actual exercise instead

Ex 4.1.10

(a)

Let $G' = G = \{f_1 = x^2 + y, f_2 = xy + x\}$. Then

$$\begin{aligned} S(f_1, f_2) &= yf_1 - xf_2 \\ &= yx^2 + y^2 - yx^2 - x^2 \\ &= -x^2 + y^2, \end{aligned}$$

and $-x^2 + y^2$ reduces to $y^2 + y$ modulo G' so we add $G' := G' \cup \{f_3 = y^2 + y\}$. We now have

$$\begin{aligned} S(f_1, f_3) &= y^2 f_1 - x^2 f_3 \\ &= y^2x^2 + y^3 - x^2y^2 - x^2y \\ &= -x^2y + y^3, \end{aligned}$$

which reduces to 0 modulo G' according to

$$\begin{aligned} (-x^2y + y^3) - (-yf_1 + yf_3) &= (-x^2y + y^3) - (x^2y - y^2 + y^3 + y^2) \\ &= 0. \end{aligned}$$

We now compute

$$\begin{aligned} S(f_2, f_3) &= yf_2 - xf_3 \\ &= xy^2 + xy - xy^2 - xy \\ &= 0, \end{aligned}$$

and we are done, with a SAGBI basis for I given by $G' = \{x^2+y, xy+x, y^2+y\}$

(b)

Let $G' = G = \{f_1 = x + y + z, f_2 = xy + xz + yz, f_3 = xyz\}$. Then

$$\begin{aligned} S(f_1, f_2) &= yf_1 - f_2 \\ &= xy + y^2 + yz - xy - xz - yz \\ &= y^2 - xz, \end{aligned}$$

which has a leading term indivisible by any of the leading terms of G', hence we add $G' := G' \cup \{f_4 = y^2 - xz\}$. Since the leading term of f_1 divides that of f_2, and $S(f_1, f_2)$ reduces to 0 by our new set G' containing f_4, we can conclude that f_2 is redundant and update $G' := G' \setminus \{f_2\}$. We now compute

$$\begin{aligned} S(f_1, f_3) &= yzf_1 - f_2 \\ &= xyz + y^2z + yz^2 - xyz \\ &= y^2z + yz^2, \end{aligned}$$

which reduces to $-z^3$ according to

$$\begin{aligned}(y^2z + yz^2) - (zf_4 + z^2f_1) &= (y^2z + yz^2) - (y^2z - xz^2 + xz^2 + yz^2 + z^3) \\ &= -z^3,\end{aligned}$$

hence we add $G' := G' \cup \{f_5 = z^3\}$. Again $\mathrm{lm}(f_1)|\mathrm{lm}(f_3)$, hence f_3 is redundant after adding f_5, and we update $G' := G' \setminus \{f_3\}$. We now compute

$$\begin{aligned}S(f_1, f_4) &= y^2f_1 - xf_3 \\ &= xy^2 + y^3 + y^2z - xy^2 + x^2z \\ &= y^3 + x^2z + y^2z\end{aligned}$$

which reduces to 0 according to

$$\begin{aligned}(y^3 + x^2z + y^2z) - (yf_4 + xzf_1 + zf_4) &= (y^3 + x^2z + y^2z) - (y^3 - xyz + x^2z + xyz + xz^2 + y^2z - xz^2) \\ &= 0.\end{aligned}$$

We now compute

$$\begin{aligned}S(f_1, f_5) &= z^3f_1 - xf_5 \\ &= xz^3 + yz^3 + z^4 - xz^3 \\ &= yz^3 + z^4\end{aligned}$$

which reduces to 0 according to

$$\begin{aligned}(yz^3 + z^4) - (yf_5 + zf_5) &= (yz^3 + z^4) - (yz^3 + z^4) \\ &= 0.\end{aligned}$$

We now compute

$$\begin{aligned}S(f_4, f_5) &= z^3f_4 - y^2f_5 \\ &= y^2z^3 + xz^4 - y^2z^3 \\ &= xz^4\end{aligned}$$

which reduces to 0 according to

$$(xz^4) - (xzf_5) = 0,$$

and we are done with the Groebner basis $G = (x + y + z, y^2 - xz, z^3)$.

Ex 4.1.11

Such a change of variables may be obtained as $L : \mathbb{Q}^n \to \mathbb{Q}^n$ where $L(x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $L(x_i) = x_i$ for $i \neq n$. Then L is invertible, and as $\mathbb{Q}$ is infinite, we can pick $a_1, a_2, \ldots, a_n$ such that each $L(q_n)$ is unique for all $q \in X$.

For the other part of the question, let $p_i, i \neq n \in \mathbb{K}[x_n]$ be the Lagrange polynomial of nodes (q_n, q_i) for $q \in X$. Then $p_i(q) = q_i$, so $x_i - p_i(x_n)$ will vanish on all of X. Let $p_n = \prod_{q \in X}(x_n - q_n)$. It's clear that every g_i lies in $I(X)$, where $g_i = x_i - p_i(x_n), i \neq n$ and $g_n = p_n$. Moreover, no polynomial in $I(X)$ has a leading monomial smaller than x_n^d. To see this, note that if the leading monomial of some $f \in I(X)$ is a power x_n^k, then since we are using lex order, f contains no variables $x_1, \ldots, x_{n-1}$. It follows that f must be of degree at least d since it's a univariate polynomial in x_n vanishing at d different points. It's now easy to see that the g_i form a Groebner basis since $\mathrm{lm}(g_i) = x_i$ when $i \neq n$.

Ex 4.2.6

(a)

I generated by a subset of the variables $\Rightarrow R/I$ is a free polynomial ring in the remaining variables, hence a domain $\Rightarrow I$ is prime.

I is prime $\Rightarrow I$ is irreducible and radical $\Rightarrow I$ is generated by pure powers of variables as it's irreducible, but then the exponents of those powers are all 1 since I is radical.

(b)

The radical of an ideal I is the intersection of all prime ideals containing I. Since prime ideals are generated by pure powers, it follows that the radical of I is generated by squarefree monomials. So, I radical $\Rightarrow I$ generated by squarefree monomials.

Now suppose that I is generated by squarefree monomials. Then I is an intersection of prime ideals, hence it's own radical.

(c)

This is just the definition of a primary ideal applied to the minimal generators of I (note that xm a minimal generator $\Rightarrow x \notin I$), so the $\Rightarrow$ direction is immediate.

Suppose that I has the described property and that $ab \in I$. Then every variable in every monomial of a exists to some power in I. We can then pick N large enough so that every term of the expansion of a^N contains a power of a variable which lies in I, hence a^N lies in I.

Ex 4.4.4

The Lex order used in Exercise 4.1.11 implies that p_n (using notation of the exercise) is a Groebner basis for the discrete variety projected onto the x_n axis.

Of course, $\mathbb{K}[x_n]$ is a PID, so the Groebner basis is just single polynomial p_n, and furthermore this polynomial is just a product of the linear forms vanishing on the x_n coordinates of the variety.

For the exercise, we needed the q_n to be different as otherwise we couldn't construct the Lagrange polynomials. It's unclear to me how to interpret this geometrically. If the q_n coordinates of the points in X weren't all different, then p_n wouldn't have degree d since the projection onto x_n would consist of $< d$ points - but I'm not sure what this means.

Ex 4.4.6

The map is invertible by projection onto the first two coordinates, $\phi^{-1} = \pi : (a, b, c, d, e) \mapsto (a, b)$. Hence the two varieties are isomorphic, whence we see that $X = \phi(\mathbb{A}^2)$ is an irreducible variety of dimension 2.

Thus we're expecting $I(\Gamma_\phi)_5 = I(X)$ to be generated by 3 irreducible polynomials. A good guess seems to be $I(X) = (t_0^2 - t_2, t_1^2 - t_4, t_0 t_1 - t_3)$. Moreover, if we choose the Lex ordering $t_4 > t_3 > t_2 > t_1 > t_0$, then this might have a good chance of being a Groebner basis as well, since no element in I can involve only t_0 and t_1 (Since p, q are algebraically independent).

We compute a Groebner basis with Macaulay2 and see that our guesses are indeed correct.

Ex 5.1.4

We assume that ∂ is defined to be linear. We then have

$$\begin{aligned}\partial\partial\{v_1, \dots, v_n\} &= \partial \sum_{i=1}^{n} (-1)^{i+1} \{v_1, \dots, \hat{v_i}, \dots, v_n\} \\ &= \sum_{i=1}^{n} (-1)^{i+1} \partial \{v_1, \dots, \hat{v_i}, \dots, v_n\} \\ &= \sum_{i=1}^{n} (-1)^{i+1} \left(\sum_{j<i} (-1)^{j+1} \{v_1, \dots, \hat{v_j}, \dots, \hat{v_i} \dots, v_n\} + \sum_{j>i} (-1)^{j} \{v_1, \dots, \hat{v_j}, \dots, \hat{v_i} \dots, v_n\} \right).\end{aligned}$$

In this sum, a term $\{v_1, \dots, \hat{v_r}, \dots, \hat{v_s} \dots, v_n\}$ appears twice, one time when $i = r, j = s$ with sign $(-1)^{r+1}(-1)^{s}$, and one time when $i = s, j = r$ with sign $(-1)^{s+1}(-1)^{r+1}$, and these two terms cancel each other. Hence the whole sum is just 0.

Ex 5.1.15

First, we define what a connected component of a simplicial complex Δ of V is. By considering the geometric realisation $|\Delta|$, it seems sensible to say that two faces $\tau, \sigma \in \Delta$ are incident whenever $\tau \cap \sigma \neq \emptyset$, and that two faces $\tau, \sigma \in \Delta$ lie on the same connected component if there is a sequence of incident faces

$$\tau = \tau_0, \ldots, \tau_k = \sigma \in \Delta,\ \tau_i \cap \tau_{i+1} \neq \emptyset.$$

This is clearly an equivalence relation, and we will now show that it coincides with the hint.

Suppose that v_i, v_j lie in the same connected component. Then there is some sequence of incident faces

$$\{v_i\} = \tau_0, \ldots, \tau_k = \{v_j\}.$$

Let $\sigma_i = \tau_i \cap \tau_{i+1}$. Then $\sigma_i \in \Delta$ by the definition of a simplicial complex. Let $w_i \in \sigma_i$ be a representative element drawn from every σ_i. Then for $0 < i < k$, both w_i and w_{i-1} must lie in τ_i, since both σ_i, σ_{i-1} are subsets of τ_i. Now let $c_i = \{\sigma_i, \sigma_{i-1}\}$. Since $c_i \subset \tau_i \in \Delta$ we have $c_i \in \Delta$ as well, and the c_i form (one of) the sequence of incident edges

$$\{v_i\} = \tau_0, c_1 \ldots, c_{k-1}, \tau_k = \{v_j\}$$

which we wanted to find.

Now, the rank of $H_0(\Delta)$ is given as $\ker(\partial_0)/\mathrm{im}(\partial_1)$ where $\partial_i : C_i \to C_{i-1}$, so let's try to understand these two modules first. We begin with $\mathrm{im}(\partial_1)$, which is a submodule of C_0 and is given as $\{\partial m : m \in C_1\}$. Let $\Delta_i \subset \Delta$ be the subset of oriented i-simplices. Then by definition, $m \in C_1$ precisely when

$$m = \sum_{\tau_i \in C_1} r_i \tau_i.$$

so C_1 is the set of formal R-linear combinations of edges (two element sets) in Δ. Meanwhile, C_0 is the set of formal R-linear combinations of vertices. As C_{-1} is defined to be 0, $\ker(\partial_0)$ is all of C_0. Hence H_0 is the cokernel of ∂_1.

Our claim is now that $v + \mathrm{im}(\partial_1) = u + \mathrm{im}(\partial_1)$ for $v, u \in V$ if and only if v and u are connected. Let

$$C_1 \ni m = \sum_{v_i, v_j : (i,j) \in I_m} r_{i,j} \{v_i, v_j\}.$$

Then

$$\partial(m) = \sum_{v_i, v_j : (i,j) \in I_m} r_{i,j} \partial(\{v_i, v_j\}) = \sum_{(i,j) \in I_m} r_{i,j}(v_i - v_j).$$

Now suppose that $v + \mathrm{im}(\partial_1) = u + \mathrm{im}(\partial_1)$. Then $v - u \in \mathrm{im}(\partial_1)$, and we have some $m \in C_1$ such that $v - u = \sum_{(i,j) \in I_m} r_{i,j}(v_i - v_j)$. In the sum of $\partial(m)$, only incident edges can contribute to cancelation of terms in the image. Also, we can never cancel all but one terms form a single connected component, since cancelation among two edges in the boundary either cancels all four vertices, or just two of them. Hence that v, u are the only vertices which remain after cancelation of the sum $\sum_{(i,j)\in I_m} r_{i,j}(v_i - v_j)$, implies that they must belong to the same connected component. Similarly, if v is connected to u via som sequence of edges $\{v_i, v_{i+1}\}$, the the boundary of this sum of edges telsecopes and is reduced down to $v - u$, hence $v + \mathrm{im}(\partial_1) = u + \mathrm{im}(\partial_1)$.

It follows that the dimension of H_0 is the number of connected components of Δ.

Ex 5.1.17

First of all, ∂_2 has full rank, hence $H_0 = \ker(\partial_2) = 0$.

Similarly, we can quickly see that ∂_1 has atleast rank 3 since it's a "staircase with three steps", and since $\mathrm{im}(\partial_2)$ has dimension 2 and lies in $\ker(\partial_1)$ which has dimension at most $5 - 3 = 2$, they must be equal, thus $H_1 = 0$. Moreover, this proves that $\ker(\partial_2)$ is 2 dimensional, a result we will need later.

Finally, H_0 has rank 1 by Exercise 5.1.15.

If we remove C_2 from the sequence, then $H_1 = \ker(\partial_1)$ which we've already showed is two dimensional.

Ex 5.1.17

(a)

First note that the orientation of phases doesn't really matter to us when computing the homologies. We just have to be carefull to be consistent with what orderings we use for a given basis element, but that order can be whatever we wan't. We use the following Macaulay2 code to compute the Betti numbers

```
clearAll

-- The abstract simplicial complex over V generated by
-- faces. That is, we expect faces to be a subset
-- of the powerset of V, and return the smallest simplicial
-- complex containing faces according to Definition 5.1.3

simplicialComplex = (V, faces) -> (
        orderedFaces := faces / sort;
        maxSize := max (orderedFaces / length);
        singletons := V / (s -> {s});
        C := {singletons};
        subFaces := unique flatten (orderedFaces / subsets);
```

```
        for i in 2..maxSize do(
                C = append(C, select(subFaces, f -> #f == i));
        );
        C
)

indexOf = (l, e) -> (
        i := position(l, x -> x === e);
        if instance(i, Nothing) then -1 else i
);

boundary = (v, C, R) -> (
        img := C_(#v - 2);
        boundaryVec := vector{#img : 0_R};
        sign := 1_R;
        for i in 0..<#v do (
                bv := select(v, e -> (e =!= v_i));
                k := indexOf(img, bv);
                addVec := vector toList join((k):0_R, 1:(sign), (#img - (k + 1)):0_R);
                boundaryVec = boundaryVec + addVec;
                sign = -1_R * sign;
        );
        boundaryVec
)

simplicalComplexMaps = (C, R) -> (
        maps := {};
        for i in 1..<#C do (
                vecs := C_i / (face -> boundary(face, C, R));
                maps = append(maps, matrix(vecs));
        );
        maps
)


V = toList (v_0..v_9)

torus = {
        {v_0,v_5,v_3},
        {v_0,v_1,v_5},
        {v_1,v_5,v_2},
        {v_2,v_6,v_5},
        {v_2,v_0,v_6},
        {v_0,v_6,v_3},
        {v_4,v_3,v_5},
        {v_4,v_5,v_8},
```

```
        {v_7,v_5,v_8},
        {v_7,v_5,v_6},
        {v_7,v_9,v_6},
        {v_7,v_8,v_9},
        {v_6,v_9,v_3},
        {v_9,v_4,v_3},
        {v_4,v_0,v_8},
        {v_1,v_0,v_8},
        {v_9,v_1,v_8},
        {v_9,v_1,v_2},
        {v_9,v_0,v_2},
        {v_9,v_0,v_4}
}

torusC = simplicialComplex(V, torus)
torusMapsQQ = simplicalComplexMaps(torusC, QQ)
torusMapsZZ2 = simplicalComplexMaps(torusC, ZZ/2)
ccTorusQQ = chainComplex torusMapsQQ
ccTorusZZ2 = chainComplex torusMapsZZ2
homTorusQQ = prune HH ccTorusQQ
homTorusZZ2 = prune HH ccTorusZZ2

<< "Torus over QQ, rank H_0: " << rank homTorusQQ_0 << endl;
<< "Torus over QQ, rank H_1: " << rank homTorusQQ_1 << endl;
<< "Torus over QQ, rank H_2: " << rank homTorusQQ_2 << endl;

<< "Torus over ZZ/2, rank H_0: " << rank homTorusZZ2_0 << endl;
<< "Torus over ZZ/2, rank H_1: " << rank homTorusZZ2_1 << endl;
<< "Torus over ZZ/2, rank H_2: " << rank homTorusZZ2_2 << endl;


klein = {
        {v_0,v_5,v_3},
        {v_0,v_1,v_5},
        {v_1,v_5,v_2},
        {v_2,v_6,v_5},
        {v_2,v_0,v_6},
        {v_0,v_6,v_4},
        {v_4,v_3,v_5},
        {v_4,v_5,v_8},
        {v_7,v_5,v_8},
        {v_7,v_5,v_6},
        {v_7,v_9,v_6},
        {v_7,v_8,v_9},
        {v_6,v_9,v_4},
        {v_9,v_4,v_3},
```

```
        {v_4,v_0,v_8},
        {v_1,v_0,v_8},
        {v_9,v_1,v_8},
        {v_9,v_1,v_2},
        {v_9,v_0,v_2},
        {v_9,v_0,v_3}
}

kleinC = simplicialComplex(V, klein)
kleinMapsQQ = simplicalComplexMaps(kleinC, QQ)
kleinMapsZZ2 = simplicalComplexMaps(kleinC, ZZ/2)
ccKleinQQ = chainComplex kleinMapsQQ
ccKleinZZ2 = chainComplex kleinMapsZZ2
homKleinQQ = prune HH ccKleinQQ
homKleinZZ2 = prune HH ccKleinZZ2

<< "Klein␣over␣QQ,␣rank␣H_0:␣" << rank homKleinQQ_0 << endl;
<< "Klein␣over␣QQ,␣rank␣H_1:␣" << rank homKleinQQ_1 << endl;
<< "Klein␣over␣QQ,␣rank␣H_2:␣" << rank homKleinQQ_2 << endl;

<< "Klein␣over␣ZZ/2,␣rank␣H_0:␣" << rank homKleinZZ2_0 << endl;
<< "Klein␣over␣ZZ/2,␣rank␣H_1:␣" << rank homKleinZZ2_1 << endl;
<< "Klein␣over␣ZZ/2,␣rank␣H_2:␣" << rank homKleinZZ2_2 << endl;
```

It produces the following output.

```
ii9 : load "simplicial-complex.m2"
Torus over QQ, rank H_0: 1
Torus over QQ, rank H_1: 2
Torus over QQ, rank H_2: 1
Torus over ZZ/2, rank H_0: 1
Torus over ZZ/2, rank H_1: 2
Torus over ZZ/2, rank H_2: 1
Klein over QQ, rank H_0: 1
Klein over QQ, rank H_1: 1
Klein over QQ, rank H_2: 0
Klein over ZZ/2, rank H_0: 1
Klein over ZZ/2, rank H_1: 2
Klein over ZZ/2, rank H_2: 1
```

Ex 5.2.2

The definition of a simplicial complex requires that if $\tau \in \Delta$, then $\sigma \in \Delta$ whenever $\sigma \subset \tau$.

Ex 5.2.3

(a)

We informally show how to turn a simplicial 3-polytope P into a planar graph. Pick any triangular face, and call it U. Call the remaining part of the polytope U'. Then U and U' glue together to form P at the edges of the triangular face we picked. Both U, U' form surfaces that are isomorphic to a disc, I.e neither surfaces has any kinds of holes. Now stretch the triangle U so that the remaining nodes of the polytope all lie inside it. That U' is two dimensional without holes implies that the graph inside the vertices of U is planar (after stretching), hence the whole graph is planar.

The result is now just a reformulation of Euler's characteristic for planar graphs. For planar graphs stemming from simplicial polytopes, every face is a triangle. Hence the planar graph is a collection of triangles with glued together at some of their edges. For one triangle, we have 3 vertices, 3, edges, and 2 faces (inner and outer), hence $f_0 - f_1 + f_2 = 3$. If we inductively add a node to a triangular planar graph, we always add 1 edge, 1 face, and 2 edges, which maintains Euler's characteristic. We can also add an edge between two existing nodes. This adds a face and an edge, which again maintains the euler characteristic.

(b)

Since the boundary of a 3-polyope is isomorphic to the sphere, every edge on the boundary graph of the polyope must be incident to two faces, and as our polytope is simplicial, every face is incident to three edges. It follows
We do the same induction as in part (a). For a triangular graph of 3 nodes, we have 3 edges and 2 faces (inner and outer). Hence the given identity holds. If we add a node to a triangular graph with $f_1 = m, f_2 = n, 2m = 3n$, we add 1 face and 2 edges, and $2m + 1 = 3n + 2$

Ex 6.1.4

If $sM = 0$ for some $s \in S$, then for $m/s' \in M_S$ we have $m/s' = sm/ss' = 0/ss'$, hence $M_S = 0$.

Now suppose $M_S = 0$. Then for every $m/s' \in M_S$, there is some $s \in S$ such that $sm = 0$. Let $m_1, \dots, m_k$ be generators of M, and let $s_i \in S$ be such that $s_i m_i = 0$. Then let $s = \prod s_i$. It follows that $sm_i = 0$ for all i, whence $sM = 0$.

Ex 6.1.5

By the previous exercise, P is in the support of M if and only if $s \in S = R \setminus P$ and $\operatorname{ann}(M)$ have empty intersection, I.e iff $\operatorname{ann}(M) \subseteq P$.