Ch 2

Ex 2.1

We can describe the curve as a sum of two different curves, $\gamma_1(t) = r(t, 1)$ and $\gamma_2(t) = r(\sin(-t), -\cos(-t))$. The first map parameterises a line parallel through the x-axis at height $y = r$. The second map parameterises a circle of radius r, but in the negative direction, and starting at $(0, -r)$ when $t = 0$.

As $(\gamma_2)'_x \leq r$, and $(\gamma_1)'_x = r$, we expect the sum $\gamma = \gamma_1 + \gamma_2$ to have positive x derivative, never "travels left". We also see that γ has vanishing x-derivatives whenever $t = \pi/2 + k\pi$ for integers k. Moreover, γ_1 and γ_2 have the same arclength $|\gamma_1| = |\gamma_2| = r$ over any t-interval, and by thinking about this for a bit, we can convince ourselves that the curve can be drawn by attaching a marker to a wheel of radius r, and then letting that wheel roll on the x-axis.

Both the x- and y-derivatives vanish at $\pi/2 + k\pi$ for $k \in \mathbb{Z}$, hence the curve is not regular (the curve has singularities wherever the "marker touches the ground").

The arclength is given by

$$\begin{aligned}
\sigma(2\pi) &= \int_0^{2\pi} |\gamma'(t)| dt \\
&= \int_0^{2\pi} \sqrt{(r - r\cos(-t))^2 + (r\sin(-t))^2} dt \\
&= \int_0^{2\pi} \sqrt{r^2 - 2r^2\cos(-t) + r^2\cos^2(-t) + r^2\sin^2(-t)} dt \\
&= \int_0^{2\pi} \sqrt{2r^2 - 2r^2\cos(-t)} dt \\
&= r\int_0^{2\pi} \sqrt{2(1 - \cos(t))} dt \\
&= r\int_0^{2\pi} \sqrt{2(2\sin^2(t/2))} dt \\
&= 2r\int_0^{2\pi} \sin(t/2) dt \\
&= 2r(-2\cos(\pi) + 2\cos(0)) \\
&= 8r
\end{aligned}$$

Ex 2.2

We can again decompose γ into a sum of $\gamma_1(t) = 3r(\cos(t), \sin(t))$ and $\gamma_2(t) = r(\cos(-3t), \sin(-3t))$. So we have some small circular motion added to a bigger circular motion. Given the previous exercise, we hypothesise that this is what you get when you attach a marker to the edge of a coin of radius r, and let it roll around another coin of radius $2r$. In this scenario, $\gamma_1(t)$ models exactly the motion of the center of the smaller coin, and in the time it'd take the radius of the center of the smaller coin to make a full lap around the big coin, we'd expect the smaller coin to make three full revolutions about it's own center. Indeed, two revolutions would come from just rolling the length of the diameter of the bigger coin $2\pi(2r)$, and then another revolution would come from "bending" that length around the edge of the bigger coin. Also, the smaller coin would have to spin in the opposite direction, as opposed to its radius. I.e we get exactly γ_1 and γ_2.

As before, we expect singularities whenever the marker touches the rolling surface (the bigger coin), and we can see this analytically as

$$\gamma'(t) = r(-3\sin(t) + 3\sin(-3t), 3\cos(t) - 3\cos(-3t))$$

which vanishes whenever $t = 0+k_2\pi$ for example. Hence the curve is not regular.

The arclength of γ for $t \in [0..2\pi]$ is given by

$$
\begin{aligned}
\sigma(2\pi) &= r\int_0^{2\pi} \sqrt{(-3\sin(t)+3\sin(-3t))^2+(3\cos(t)-3\cos(-3t))^2}dt \\
&= 3r\int_0^{2\pi} \sqrt{(-\sin(t)+\sin(-3t))^2+(\cos(t)-\cos(-3t))^2}dt \\
&= 3r\int_0^{2\pi} \sqrt{\sin^2(t)-2\sin(t)\sin(-3t)+\sin^2(-3t)+\cos^2(t)-2\cos(t)\cos(-3t)+\cos^2(-3t)}dt \\
&= 3r\int_0^{2\pi} \sqrt{2-2\sin(t)\sin(-3t)-2\cos(t)\cos(-3t)}dt \\
&= 3r\int_0^{2\pi} \sqrt{2+2\sin(t)\sin(3t)-2\cos(t)\cos(3t)}dt \\
&= 3r\int_0^{2\pi} \sqrt{2+2\sin(t)(3\sin(t)-4\sin^3(t))-2\cos(t)(4\cos^3(t)-3\cos(t))}dt \\
&= 3r\int_0^{2\pi} \sqrt{2+6\sin^2(t)-8\sin^4(t)-8\cos^4(t)+6\cos^2(t)}dt \\
&= 3r\int_0^{2\pi} \sqrt{2+6-8\sin^4(t)-8\cos^4(t)}dt \\
&= 3r\int_0^{2\pi} \sqrt{8(1-\sin^4(t)-\cos^4(t))}dt \\
&= 6r\int_0^{2\pi} \sqrt{8(2\sin^2(t)\cos^2(t))}dt \\
&= 24r\int_0^{2\pi} |\sin(t)\cos(t)|dt \\
&= 12r\int_0^{2\pi} |\sin(2t)|dt \\
&= 12r2 \\
&= 24r.
\end{aligned}
$$

Ex 2.3

We begin by calculating the curvature κ_1 of γ_1, and we don't want to bother with reparameterisations, hence we use Proposition 2.12. We have that

$$\gamma_1' = ra(-\sin(at), \cos(at)),$$

and

$$\gamma_1'' = -ra^2(\cos(at), \sin(at)),$$

hence

$$\begin{aligned}
\kappa_1 &= \frac{\det[\gamma_1', \gamma_2'']}{|\gamma_1'|^3} \\
&= \frac{r^2 a^3 \sin^2(at) + r^2 a^3 \cos^2(at)}{r^3 a^3} \\
&= \frac{1}{r}.
\end{aligned}$$

As κ_1 is independent of a, it follows that $\kappa_2 = \kappa_1$. Note that $(\gamma_1)_x = (\gamma_2)_x$ and $(\gamma_1)_y = -(\gamma_2)_y$. Hence Φ is just flipping $\mathbb{R}^2$ about the x-axis. Φ is not orientation preserving as it has determinant -1.

Ex 2.4