Ch 1

1.1

Suppose that ab maps to 1 in A/I. Then there is some nilpotent element r with $r^n = 0$ such that $ab = 1 - r$. It follows that

$$ab(1 + r + r^2 + \ldots + r^{n-1}) = (1 - r)(1 + r + r^2 + \ldots + r^{n-1}) = 1 - r^n = 1$$

hence a has inverse $b(1 + r + \ldots + r^{n-1})$.

1.2

Suppose that $I_1 \times \ldots \times I_n$ is a prime ideal of $A_1 \times \ldots \times A_n$. Then

$$(A_1 \times \ldots \times A_n)/(I_1 \times \ldots \times I_n) \cong A_1/I_1 \times \ldots \times A_n/I_n$$

is a domain, and as non-trivial cartesian products always have zero divisors, we must have $A_i = I_i$ for all but one $i = j$. It follows that

$$(A_1 \times \ldots \times A_n)/(I_1 \times \ldots \times I_n) \cong A_j/I_j$$

is a domain and so I_j is prime as well.

The other direction is imediate.

1.3

(a)

Let $\mathfrak{m}$ be maximal in B. Then

$$A/f^{-1}(\mathfrak{m}) \cong B/\mathfrak{m}$$

is a field so $\mathfrak{m}$ is maximal in A. Hence if $a \in \mathrm{rad}(A) \subset f^{-1}(\mathfrak{m})$ then $f(a) \in \mathfrak{m}$, and as $\mathfrak{m}$ is an arbitrary maximal ideal of B, $f(a) \in \mathrm{rad}(B)$.

An example where the inclusion is strict is given by $\mathbb{Z} \to \mathbb{Z}/p^2$, as $\mathrm{rad}(\mathbb{Z}) = \emptyset$ whilst $\mathrm{rad}(\mathbb{Z}/p^2) = (p)$.

(b)

Suppose A is semi-local and $b \in \mathrm{rad}(B)$ and $a = f^{-1}(b)$. Then a is in all maximal ideals of A which contain $\ker(f)$. Now let $\mathfrak{m}'$ be a maximal ideal of A' which does not contain $\ker(f)$. Then $\ker(f) + \mathfrak{m}' = (1)$ and so there is some $m' \in \mathfrak{m}'$ such that $m' = 1 + c$ with $c \in \ker(f)$. Let M be the set of maximal ideals of A containing $\ker(f)$ and M' be the set of maximal ideals of A not containing M. It follows by the proof of Theorem 1.3 that

$$\mathrm{rad}(A) = \left(\bigcap_{\mathfrak{m} \in M} \mathfrak{m}\right) \left(\prod \mathfrak{m}' \in M' \mathfrak{m}'\right)$$

hence

$$a' = a(1+c_1)(1+c_2)\dots(1+c_k) \in \mathrm{rad}(A)$$

for some $c_i \in \ker(f)$ and $f(a') = b$.

1.4

Suppose A is a UFD and $a \in A$ is irreducible. Then a must have only one prime factor by irreducibility, and so a is prime thus (a) is prime. Now suppose towards a contradiction

$$(a_0) \subseteq (a_1) \subset \dots$$

is an infinite ascending chain of ideals of A. Suppose $a_0 = p_0 p_1 \dots p_k$ is a factorisation of a_0 into prime elements with repetitions allowed. Then as $a_0 \in (a_1)$, every prime which divides a_1 divides a_0, and so the prime factors of a_1 is some subset of the p_i. Strict inclusions correspond to omision of prime factors, which clearly can only happen finitely many times.

For the other direction, suppose that A is a domain where irreducible elements are prime, and where principal ideals satisfy the ascending chain condition. Let $a \in A$, and S be the set of prinicipal ideals containing A. By hypothesis, S contain a maximal element (it is non-empty as it contains (a)), and so let $M \subset S$ be the set of maximal elements of S. Let $(m) \in M$. Any principal ideal containing (m) also must contain a, thus (m) is maximal among all principal ideals whence m is irreducible, and also prime by hypothesis. Let $a_0 = a$, and $(m_0) \in M$ and let $a_1 = m_0/a_0$ if a_1 isn't a unit, let $(m_1) \in M \setminus (m_0)$. Then as $m_0 a_1 = c m_1$, and $m_1 \not| m_0$ since $(m_0) \not| (m_1)$, we have $m_1 | a_1$, and we can write $a_2 = a_1/m_1$. Continuing this way yields an ascending chain

$$(a_0) \subsetneq (a_1) \subsetneq (a_2) \subsetneq \dots$$

which must terminate at some a_k by hypothesis. It follows that

$$a = m_0 m_1 \dots m_k$$

is a prime factorisation of a.

1.5

Let $P = \bigcap_{\lambda \in \Lambda} P_\lambda$. Then P is an ideal, and if $x \notin P, y \notin P$, then there is some P_x, P_y such that $x \notin P_x, y \notin P_y$. As the family is totally ordered, we have either $P_x \subset P_y$ or $P_y \subset P_x$, and we can assume the former. Then $y \notin P_x$, hence $xy \notin P_x$ by primality, whence $xy \notin P$.

Minimal primes of A containing I are precisely minimal primes of A/I subject to no restrictions. We showed above that chains of prime ideals in any ring have lower bounds. Hence A/I has minimal primes by Zorn's Lemma.

1.6