## Ex 2.1.5

### (a)

Picking two non-negative integers $a, b$ such that their sum $a+b \leq m$ is the same as picking three non-negative integers $a+b+c$ such that the sum $a+b+c=m$. To see this, just note that $c$ is completely determined by $a, b$. I.e the amount of monomials $x^a y^b$ with total degree $\leq m$, is the same as the amount of ordered 3-integer partitions of $m$, namely $\binom{m+2}{2}$.

### (b)

The amount of $f, g$-monomials $f(t)^a g(t)^b$ with $a, b \leq m$ is given by $\binom{m+2}{2}$ by the previous exercise. These will all be polynomials in $\mathbb{K}[t]$ of degree $\leq nm$. The $\mathbb{K}$-subspace of $\mathbb{K}[t]$ consisting of polynomials of degree $\leq nm$ has dimension $nm+1$, and since we pick $m$ large enough that $(m+2)(m+1)/2 > nm+1$, it follows that the $f, g$-monomials of degree $\leq m$ for such large enough $m$ are linearly dependent.

### (c)

If we pick $m$ large enough as described in (b), the resulting linear dependence on $f, g$-monomials can be seen as an algebraic dependence $F$ on $f, g$.

### (d)

There are $\binom{k+m}{k}$ $f_1, \ldots, f_k$-monomials of degree $\leq m$, whilst the there are $\binom{nm+k-1}{k-1}$ $t_1, \ldots, t_{k-1}$-monomials of degree $\leq nm$. Given any $n$, for large enough $m$ we have $\binom{k+m}{k} > \binom{nm+k-1}{k-1}$, since the former product has $k$ factors whilst the latter have $k-1$ factors.