Problem 1

If I is an injective ideal, the S.E.S

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

splits, and $R = I \oplus R/I$.

(1)

By the splitting above, I is a direct summand of the free R-module R, hence projective

(2)

By the splitting above, R/I is a direct summand of the free R-module R, hence projective and in particular, flat.

Problem 2

Let π^*M, π^*N be the R-modules obtained from M, N via restriction of scalars along $\pi : R \to R/I$. We are asked to show that the functor is an isomorphism on hom sets,

$$\pi^* : \mathrm{Hom}_{R/I}(M, N) \xrightarrow{\cong} \mathrm{Hom}_R(\pi^*M, \pi^*N).$$

Let $\phi : \pi^*M \to \pi^*N$ be an R-module morphism. As π^*M, π^*N are equal to M, N as sets, we have that ϕ induces a funtion of sets $\overline{\phi} : M \to N$ via

$$\overline{\phi} : m \to \phi(m).$$

Furthermore, $\overline{\phi}$ is R/I-linear as

$$\overline{\phi}([r]m) = \phi(\pi(r)m) = \pi(r)\phi(m) = [r]\overline{\phi}(m).$$

As both $\bar{\cdot}$ and π^* do nothing to change the underlying functions, they just change categories, it's immediate that they are mutually inverse isomorphisms of abelian groups.

Problem 3

The middle sequence is exact so it's homologies are 0, and as the outer two sequences have $d = 0$, taking homologies doesn't change the underlying modules. The only thing that remains to calculate is the connecting morphism. But we don't even need to calculate it as the long exact sequence looks like

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\partial_n} \mathbb{Z}/2 \longrightarrow 0 \longrightarrow \ldots,$$

hence ∂_n is an isomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$, and there is only one such isomorphism, the identity morphism.

Problem 4

Both Hom and $\otimes$ are addititive funtors in either argument and so the derived functors Tor and Ext are as well.

(1)

We have,

$$\mathrm{Ext}^n_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \cong \mathrm{Ext}^n_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \oplus \mathrm{Ext}^n_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}),$$

but $\mathbb{Z}/3\mathbb{Z}$ is free hence projective, and so $\mathrm{Ext}^n_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, M) = 0$ for all $n > 0$ and any $\mathbb{Z}/3\mathbb{Z}$-module M. As $\mathrm{Ext}^0_R(M, N) \cong \mathrm{Hom}_R(M, N)$ we have that

$$\mathrm{Ext}^0_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$$

hence

$$\mathrm{Ext}^n_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) = \begin{cases} \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \text{ if } n = 0, \\ 0 \text{ else.} \end{cases}$$

(2)

We have

$$\mathrm{Tor}^{\mathbb{Z}}_n(\mathbb{Z} \oplus (\mathbb{Z}/4), \mathbb{Q}) \cong \mathrm{Tor}^{\mathbb{Z}}_n(\mathbb{Z}, \mathbb{Q}) \oplus \mathrm{Tor}^{\mathbb{Z}}_n(\mathbb{Z}/4, \mathbb{Q}).$$

As $\mathbb{Z}$ is projective,

$$\mathrm{Tor}^{\mathbb{Z}}_n(\mathbb{Z}, \mathbb{Q}) \cong \begin{cases} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \text{ if } n = 0, \\ 0 \text{ else.} \end{cases}$$

Meanwhile, a (deleted) projective resolution $P_\bullet \to \mathbb{Z}/4$ is given by

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{4} \mathbb{Z} \longrightarrow 0,$$

and tensoring with $\mathbb{Q}$ we get

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \xrightarrow{4} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Now, if $a/b \in \mathbb{Q}/\mathbb{Z}$, then $a/4b \in \mathbb{Q}/\mathbb{Z}$ whence multplication by $4 : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is surjective. It's not injective however as $4a/b \in \mathbb{Z} \Leftrightarrow b|4$, hence it has kernel generated by $1/4\mathbb{Q}/\mathbb{Z}$. We get

$$\mathrm{Tor}^{\mathbb{Z}}_0(\mathbb{Z}/4, \mathbb{Q}/\mathbb{Z}) \cong \frac{\ker(0_{\mathbb{Q}/\mathbb{Z}})}{\mathrm{im}(4_{\mathbb{Q}/\mathbb{Z}})} = 0,$$

and

$$\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/4, \mathbb{Q}/\mathbb{Z}) \cong \frac{\ker(4_{\mathbb{Q}/\mathbb{Z}})}{\mathrm{im}(0_{\mathbb{Q}/\mathbb{Z}})} = 1/4\mathbb{Q}/\mathbb{Z},$$

Thus

$$\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z} \oplus (\mathbb{Z}/4), \mathbb{Q}) \cong \begin{cases} \mathbb{Q} \text{ if } n = 0, \\ 1/4\mathbb{Q}/\mathbb{Z} \text{ if } n = 1, \\ 0 \text{ else.} \end{cases}$$

(3)

A (deleted) projective resolution of $\mathbb{Q}[x,y]/(x)$ is given by

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Q}[x,y] \xrightarrow{x} \mathbb{Q}[x,y] \xrightarrow{0} 0$$

afterwhich applying $\operatorname{Hom}(\_, \mathbb{Q}[x,y]/(y))$ yields

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Q}[x,y]}(\mathbb{Q}[x,y], \mathbb{Q}[x,y]/(y)) \xrightarrow{x^*} \operatorname{Hom}_{\mathbb{Q}[x,y]}(\mathbb{Q}[x,y], \mathbb{Q}[x,y]/(y)) \xrightarrow{0} 0 \longrightarrow \dots$$

Now, for any ring R and R-module M we have that the functor $\operatorname{Hom}_R(R, \_)$ naturaly isomorphic to the identity functor via

$$\operatorname{Hom}_R(R, M) \ni \phi \mapsto \phi(1) \in M,$$

and

$$\operatorname{Hom}(\operatorname{Hom}_R(R, M), \operatorname{Hom}_R(R, N)) \ni f \mapsto (m \mapsto (f(1 \mapsto m))(1) \in \operatorname{Hom}(M, N).$$

Applying this to our resolution yields

$$0 \longrightarrow \mathbb{Q}[x,y]/(y) \xrightarrow{x} \mathbb{Q}[x,y]/(y) \xrightarrow{0} 0 \longrightarrow \dots,$$

afterwhich we can read homology groups according to

$$\operatorname{Ext}_{\mathbb{Q}[x,y]}(\mathbb{Q}[x,y]/(x), \mathbb{Q}[x,y]/(y)) \cong \begin{cases} 0 \text{ if } i = 1, \\ (\mathbb{Q}[x,y]/(y))/(x\mathbb{Q}[x,y]/(y)) \cong \mathbb{Q} \text{ if } i = 1, \\ 0 \text{ else.} \end{cases}$$

Problem 5

The short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

induces a long exact sequence

$$0 \to \operatorname{Hom}_R(R/I, N) \to \operatorname{Hom}_R(R, N) \to \operatorname{Hom}_R(I, N) \to \operatorname{Ext}^1_R(R/I, N) \cong 0 \to \dots,$$

and in particular the restriction map $\operatorname{Hom}_R(R, N) \to \operatorname{Hom}_R(I, N)$ is surjective. As this is true for all left ideals I, it follows from Baer's Criterion that N is injective.

Problem 6

(1)

If $Q_\bullet \xrightarrow{g_0} M$ is a projective resolution of M, it follows that

$$Q_1 \longrightarrow Q_0 \xrightarrow{f_n g_0} P_n \longrightarrow \ldots \longrightarrow P_0$$

is a projective resolution of N, afterwhich the desired result is immediate.

(2)

Let M be an arbitrary R-module and $P_\bullet \to M$ a projective resolution. Then by part (1), $L_{k+n}(M) = L_k(P_n) = 0$ for all $n > 0$.

Problem 7

(1)

As we're in a PID, multiplication by r is injective, and so the only non zero homology is given by

$$H_0(K(r)) = R/rR.$$

(2)

By the Kunneth formula, we have a split exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(K(r)) \otimes H_j(C) \longrightarrow H_n(K(r) \otimes C) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}_1(H_i(K(r), H_j(C)) \longrightarrow 0$$

Furthermore, as $H_i(K(r)) = 0$ for $i \neq 0$, the sequence becomes

$$0 \longrightarrow R/rR \otimes H_n(C) \longrightarrow H_n(K(r) \otimes C) \longrightarrow \mathrm{Tor}_1(R/rR, H_n(C)) \longrightarrow 0$$

Now whenever R is a pid and M is an R-module,

$$\mathrm{Tor}_1^R(R/rR, M) = M_r = \{m \in M : rm = 0\},$$

and as r is injective on $H_n(C)$ by hypothesis, $\mathrm{Tor}_1^R(R/rR, M = 0)$. Finally our desired result now follows from

$$R/rR \otimes H_n(C) \cong H_n(C)/rH_n(C).$$

Problem 8

(1)

We can show something even stronger, that if

$$\mathrm{Hom}(C, D) \xrightarrow{g^*} \mathrm{Hom}(B, D) \xrightarrow{f^*} \mathrm{Hom}(A, D)$$

is exact for every R-module D (a weaker hypothesis as we do not require g^* injective), then

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. To see this, let $D = \mathrm{coker}(f)$

(2)

As F is a left-adjoint, F preserves colimits, and in particular cokernels. The desired result now follows from the followin lemma.

Lemma 0.0.1. Let F be an additve functor which preserves cokernels. Then F is right exact.

Proof. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. Then $\mathrm{coker}(g) = 0$ and so $\mathrm{coker}(F(g)) = F(\mathrm{coker}(g)) = 0$ whence $F(g)$ is surjective. Moreover, $F(f)F(g) = F(fg) = F(0) = 0$. It remains to show that $\ker(F(g)) \subseteq \mathrm{im}(F(f)) = \ker(\mathrm{coker}(F(f))$. But $\mathrm{coker}(F(f)) = F(\mathrm{coker}(f)) = F(g)$ and the the lemma statement follows. $\square$