Ch 1

Ex 1.1

Let $n \in \mathbb{Z}$ be such that $x^n = 0$. Now consider repeating the conjugate rule according to, $(1-x)(1+x) = 1-x^2$, $(1-x)(1+x)(1+x^2) = 1-x^4$, and on until we get

$$(1+x)(1-x)\prod_{k=1}^{n}(1+x^{2^k}) = 1 - x^{2^{n+1}} = 1 - 0 \cdot x^{2^{n+1}-n} = 1$$

so $1+x$ is a unit.

The same proof for an arbitrary unit u yields

$$(u+x)(u-x)\prod_{k=1}^{n}(u+x^{2^k}) = u^{2^{n+1}} - x^{2^{n+1}} = u^{2^{n+1}},$$

which is also whence $u+x$ is.

Ex 1.2

i)

For the $\Leftarrow$ direction, let k be an integer such that all $a_i^k = 0$ for $i > 0$. Then every term in every coefficient $(f-a_0)^{kn}$ must contain at least k factors of some $\neq a_0$ coefficient of f. Hence $(f-a_0)^{kn} = 0$ and $f - a_0$ is nilpotent. But a_0 is a unit, so f is a unit by Ex 1.1.

For the $\Rightarrow$ direction, we assume that f is a unit and prove the given statement using induction.The case when $r = 0$ follow directly from the fact that the leading coefficient of the product is the product of the leading coefficients of the terms.

Now assume that $a_n^{r+1}b_{m-r}$ is true for all $r < k$, and consider the $n+m-k$-th coefficient, c_{n+m-k}, of $g = a_n^k f f^{-1}$. It's given as

$$c_{n+m-k} = a_n^k \sum_{i+j=n+m-k} a_i b_j,$$

but all terms with $j > m-k$ are 0 since then $a_n^k b_j = 0$, so the only term which remains is $a_n^{k+1}b_{n-k} = 0$, and $a_n^{r+1}b_{m-r}$ is true for all $r \in [0..m]$.

But then it must be that $a_n^{m+1} = 0$ since b_0 is a unit, so a_n is nilpotent. Then we also have that $a_n x^n$ is nilpotent, and $f - a_n x^n$ is a unit by Ex 1.1. Continuing reduction of f like this, term by term this way until we get to the constant term proves the desired result.

ii)

The $\Leftarrow$ direction is analogous to the previous subproblem, just that we don't need to subtract by a_0.

For the $\Rightarrow$ direction, we know that $f+1$ is a unit by Ex 1.1, which in turn leads to all a_i for $i>0$ being nilpotent. Now, since nilpotent elements form an ideal, $a_i x^i$ is nilpotent for every $i>0$, and we can subtract them all from f and remain in the nilradical, hence a_0 is nilpotent as well.

iii)

Let $g = b_0 + \ldots + b_m x^m$ be a non-zero polynomial of minimal degree m which annihilates f. Then $a_n b_m = 0$. But $a_n g$ annihilates f as well, and $\deg(a_n g) < m$, so $a_n g = 0$. Now we have $(f - a_n x^n) g = 0$ as well, so $a_{n-1} b_m = 0$ and $a_{n-1} g = 0$. This process can be continued all the way down, showing that $b_m f = 0$.

iv)

First, $(c_1, \ldots, c_l) \subseteq (a_1, \ldots, a_n)$ which immediately gives us the $\Rightarrow$ direction.

For the $\Leftarrow$ direction, let A_i be a solution to $\sum a_i A_i = 1$, and B_i similarly. Then we have

$$\begin{aligned} 1 &= \left(\sum a_i A_i\right)\left(\sum b_i B_i\right) \\ &= \sum_{a_i} \sum_{b_i} a_i A_i b_i B_i \\ &= \sum_{k=0}^{n+m} \sum_{i=0}^{n+m} a_i A_i b_{k-i} B_{k-i} \\ &= \sum_{k=0}^{n+m} \sum_{c_k} \sum_{i=0}^{n+m} A_i B_{k-i}, \end{aligned}$$

and we are done.

Ex 1.3

i)

The $\Leftarrow$ direction of Ex 1.2.i) made no assumptions on the dimension of the polynomials.

For the other direction, assume that the statement is true for $n < k$ and consider $A[x_1, \ldots, x_k]$. This ring can be considered as single variable polynomial ring with scalars in $A[x_1, \ldots, x_{k-1}]$. By Ex 1.2.i), we then have f nilpotent $\Rightarrow c_i(f)$ nilpotent for $i > 0$ and a unit for $i = 0$. This immediately yields all coefficients for terms with factors of x_n being nilpotent. The remainder of the polynomial is a unit and the statements regarding these coefficients follows by induction.

ii)

Again, we get the $\Leftarrow$ direction from Ex 1.2.ii).
We'll use the same induction as above for the $\Rightarrow$ direction. By Ex 1.2.i), we then have $f \in A[x_1, \dots, x_{k-1}][x_k]$ nilpotent $\Rightarrow c_i(f)$ nilpotent for all i, and the coefficients are polynomials of degree $k-1$ so their coefficients are nilpotent by induction.

iii)

Ex 1.3.iii) made no assumptions of degree on the polynomial ring. Replace deg with total degree, and x^n with $\mathrm{lm}(f)$.

iv)

Again, if we organise terms by total degree, we get the same proof as in Ex 1.2.iv).

Ex 1.4

Let f be in the Jacobson radical. Then $1 - fg$ is a unit for all $g \in A[x]$. We will show that f is nilpotent. We can pick $g = x$, and $h = 1 - fx$ is a unit by Ex 1.1. Then all coefficients of non-constant terms of h are nilpotent by Ex 1.2.i). But these are precisely the coefficients of f, so f is nilpotent by Ex 1.2.ii)

Ex 1.6

Lemma 0.1. The Jacobson radical does not contain any non-zero idempotents.

Proof. Let e be in the Jacobson radical such that $ee = e$. Then $1 - e$ and $1 + e$ are both units by Prop 1.7 (?). Let their inverses be named u, u'. Then uu' is the inverse of the product $(1-e)(1+e) = 1-e^2 = 1-e$. We have $uu'(1-e) = 1$, but we also have $u(1-e) = 1$, so $u' = 1$, $1 - e = 1$ and $e = 0$. $\square$

Ex 1.7

Let P be a prime ideal, and consider $x + P \in A/P$. Then $x^n + P = x + P$ and $x^n - x + P = P$, so $(x^{n-1} - 1 + P)(x + P) = P$. But A/P is a domain, so either $x + P = P$ or $x^{n-1} - 1 + P = P$. Ignoring the trivial case, we get that $x^{n-1} + P = 1 + P$, so $x + P$ has the inverse $x^{n-2} + P$, and A/P is a field, whence P is maximal.

Ex 1.8

Let P be the set of prime ideals in A, and consider a chain C

$$\mathfrak{p}_0 \supseteq \mathfrak{p}_1 \supseteq \dots$$

of ideals $\mathfrak{p}_i \in P$. We claim that $\mathfrak{p} = \bigcap_{\mathfrak{p}_i \in C} \mathfrak{p}_i$ is a prime ideal. Indeed, given $ab \in \mathfrak{p}$ we have that $ab \in \mathfrak{p}_i$ for all $\mathfrak{p}_i \in C$, and if there is some $n \in \mathbb{N}$ such that $a \notin \mathfrak{p}_n$, then $n \notin \mathfrak{p}_i$ for all $i \geq n$ since $i \geq n \Rightarrow \mathfrak{p}_i \subseteq \mathfrak{p}_n$. It follows that $b \in \mathfrak{p}_i$ for all $i \geq n$. Then $b \in \mathfrak{p}_i$ for all $\mathfrak{p}_i \in C$, as otherwise, the argument above could be repeated with b instead, leading to a contradiction. It follows that $\mathfrak{p} \in P$, and $\mathfrak{p}$ is an upper bound to C, whence P admits minimal elements by Zorn's lemma.

Ex 1.9

The $\Rightarrow$ direction is immediate from Prop 1.14.

Let I be an ideal which is the intersection of prime ideals in A, and let $x^n \in I$ for some $x \in A, n \in \mathbb{Z}$. Any prime ideal which contains x^n necessarily contains x, so all prime ideals which intersect to I contain x, whence I do as well.

Ex 1.10

i) $\Rightarrow$ ii) Given i), Prop 1.8 tells us that $\text{nilrad}(A)$ is prime. Let $x \in A$ be a non-unit. Let I be the maximal ideal containing x by Cor 1.5. Maximal ideals are prime, so I is prime. Thus $I = \text{nilrad}(A)$ and $x \in \text{nilrad}(A)$, whence x is nilpotent.

ii) $\Rightarrow$ iii) follows from the fact that all nilpotent elements are factored out (Prop 1.7) and only units remain in $A/\text{nilrad}(A)$.

iii) $\Rightarrow$ i) Given iii), we know that $\text{nilrad}(A)$ is maximal, but it's also the intersection of all prime ideals. Thus all prime ideals equal $\text{nilrad}(A)$.

Ex 1.12

The only unit which is idempotent is 1, since if $e, e' \in A$ such that $ee = e,\ ee' = 1$, then $1 = ee' = eee' = e$.

Let A, m be a local ring and suppose towards a contradiction that $e \in A$ is an idempotent not equal to $0, 1$. Then $e \in m$ by Cor 1.5, and since m is the only maximal ideal, it's the Jacobson radical, which by Prop 1.9 means that $1 - ge$ is a unit for all $g \in A$. In particular, $1 - e$ is a unit, but it's also a zero divisor since $e(1 - e) = 0$, a contradiction.

Ex 1.13

We show that $\mathfrak{a} \neq (1)$. First, if $|\Sigma| = 1$ and f is the sole member of Σ, we have that $\deg(f) > 0$ since f is irreducible, so $\mathfrak{a} = (f)$ doesn't contain the constants.

Now assume that $|\Sigma| > 1$, and let $f, g \in \Sigma$. Then $f(x_g) \notin \Sigma$ since $k[x_g] \cap \Sigma = (g(x_g)) \not\ni f(x_g)$ since $f \neq g$ and f is irreducible.

Ex 1.14

We will show that Σ admits maximal elements by an application of Zorn's lemma, and to do this, all we need to do is verify that if

$$\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \dots$$

is a chain in Σ, then $\mathfrak{a} = \bigcup \mathfrak{a}_i$ is an ideal. Let $a, b \in \mathfrak{a}, r \in A$. Then $a \in \mathfrak{a}_i$ for some i and $ra \in \mathfrak{a}_i \subseteq \mathfrak{a}$. Even more, since $i < j \Rightarrow \mathfrak{a}_i \subseteq \mathfrak{a}_j$, then $a \in \mathfrak{a}_j$ for all $j > i$. It follows that there is some $\mathfrak{a}_k$ which contains both a, b, whence it contains $a + b$, and we see that $a + b \in \mathfrak{a}$.

Now, let $\mathfrak{p} \in \Sigma$ be a maximal element, and consider $a, b \in A$ such that $ab \in \mathfrak{p}$. Then ab is a zero divisor and $abc = 0$ for some $c \in A$. But then a, b are both zero divisors as well and $(a), (b) \in \Sigma$. Now assume towards a contradiction that $a, b \notin \mathfrak{p}$. It follows that $(a) + \mathfrak{p} = (b) + \mathfrak{p}(1)$ since $\mathfrak{p}$ is maximal and $(a) + \mathfrak{p}, (b) + \mathfrak{p}$ both consist soleley of zero divisors. Multiplying the two ideal sums yields

$$(1) = (1)^2 = ((a) + \mathfrak{p})((b) + \mathfrak{p}) = (ab) + (a)\mathfrak{p} + (b)\mathfrak{p} + \mathfrak{p}^2 \subseteq \mathfrak{p}$$

which contradicts $\mathfrak{p}$ being a proper ideal.

Ex 1.15

i)

Since any ideal which contains E is required to contain a and vice verca, we have $V(a) = V(E)$. Moreover, if $P \supset a$ is a prime ideal, then $\text{nilrad}(a) \subset P$ and $V(a) \subseteq V(r(a))$. The other inclusion is immediate.

ii)

Any ideal contains 0 so $V(0)$ is the set of prime ideals. Only (1) contains 1 so $V(1) = \emptyset$.

iii)

$x \in V(\bigcup E_i)$ means that x is a prime ideal which contain all E_i. In other words, $x \in V(E_i)$ for all E_i, so $x \in \bigcap V(E_i)$.

$y \in \bigcap V(E_i)$ means that y is a prime ideal which contain some E_i all $V(E_i)$, and just like above, we see that this is a simple rewording.

iv)

By and Ex 1.1.13 i) we have

$$V(a \cap b) = V(r(a \cap b)) = V(r(ab)) = V(ab),$$

Now, if $x \in V(ab)$ then x is a prime ideal which contains ab. With the following lemma, this means that x contains either a or b, whence $x \in V(a) \cup V(b)$.

Lemma 0.2. Let P be a prime ideal such that $IJ \in P$ for two ideals I, J. Then either $I \in P$ or $J \in P$.

Proof. Assume that $J \notin P$. Then there is some $x \in J \setminus P$. For all $i \in I$ we have $xi \in P$, and since P is a prime ideal, $i \in P$. □

Ex 1.17

Let U be open in X. Then $U = V(E)^c$ for some $E \subset A$. Now,

$$U = V(E)^c = \left(V\left(\bigcup_{f \in E} f\right)\right)^c = \left(\bigcap_{f \in E} V(f)\right)^c = \bigcup_{f \in E} V(f)^c = \bigcup_{f \in E} X_f.$$

i)

We have that any $a \in X_f \cap X_g$ is a prime ideal which doesn't contain f, nor g. Such an ideal can't contain fg by the lemma in 1.15.iv) so $x \in X_{fg}$. Likewise, $b \in X_{fg}$ means that b is a prime ideal which doesn't contain fg, whence it clearly can't contain either f or g.

ii)

$X_f = \emptyset \Leftrightarrow V(f) = X$ which means that f is in all prime ideals, so $f \in \text{nilrad}(A)$.

iii)

Every non-unit is contained in a maximal ideal, which is prime, so if $X_f = X$, then f must be a unit.

iv)

$X_f = X_g \Leftrightarrow V(f) = V(g) \Leftrightarrow V(r(f)) = V(r(g))$ by Ex 1.15.i)

Ex 1.18

Lemma 0.3. Every proper ideal I is contained in some maximal ideal.

Proof. Let I be a proper ideal. Let Σ_I be the set of ideals which contain I. Then Σ_I contains I and is not empty. Order Σ_I by inclusion and apply Zorn's lemma like in the proof of Theorem 1.3 to show that Σ_I contains a maximal element. $\square$

i)

If x is a closed point, then $x = V(E)$ for some $E \subset A$, which means that p_x is the only prime ideal which covers E. Since $E \subset p_x$, we know there is some maximal ideal which covers E, which in turn would be prime, so it must be p_x.

If p_x is maximal, then $V(x) = \{p_x\}$.

ii)

$\overline{\{x\}}$ is the intersection of all closed sets containing x, which by Ex 1.15 is

$$\bigcap_{E:x\in V(E)} V(E) = V\left(\bigcup_{E:x\in V(E)} E\right).$$

Now, E is such that $x \in V(E)$ precisely when x contains E, so $\bigcup_{E:x\in V(E)} E = x$ and $\overline{\{x\}} = V(x) = V(p_x)$

iii)

Let $y \in \overline{\{x\}}$. By ii) we then have $y \in V(p_x)$ which by definition means that $x \subseteq y$.

Let $x \subseteq y$. By definition we have $y \in V(p_x)$ which by ii) means that $y \in \overline{\{x\}}$.

iv)

Let $x \neq y$. Then we must have either $x \not\subset y$ or $y \not\subset x$. Assume $x \not\subset y$. Then $y \notin V(p_x)$, but $x \in V(p_x)$.

Ex 1.19

Let $a, b \in A \setminus \text{nilrad}(A)$. Then $X_a, X_b \neq \emptyset$ since if $V(a) = \text{spec}(A)$, we'd have a in every prime ideal, and therefore a in the nilradical. If $\text{spec}(A)$ is irreducible, we now get that $X_a \cap X_b$ is non-empty. Since X_f is the set of prime ideals which doesn't contain f, $X_a \cap X_b \neq \emptyset$ means that there is some prime ideal p which contains neither a nor b. Then p can't contain ab on account of being prime. Since the nilradical is the intersection of all prime ideals, it can't contain ab

either.

Now let A be a ring with prime nilradical. Let $a, b \in A$ be such that X_a, X_b are non-empty. Then neither a, b can be in the nilradical, which in turn means that ab isn't either and X_{ab} is non-empty. I.e there is some prime ideal p which doesn't contain ab, but then p doesn't contain a nor b, whence $p \in X_a \cap X_b$. Since the X_f form a basis, and we just showed that any two non-empty basis elements intersect, we conclude that $\text{spec}(A)$ is irreducible.

Ex 1.20

i)

Let U, V be two open sets in $\overline{Y}$. Since U is the neighbourhood of some $x \in Y$, we have that $U \cap Y \neq \emptyset$, and the same for V. But $U \cap Y$, $V \cap Y$ are open in Y and must therefore intersect by the hypothesis. Thus U, V intersects as well and $\overline{Y}$ is irreducible.

ii)

Let Y be an irreducible subspace and Σ be the set of irreducible subspaces of X which contain Y. Let

$$Y_1 \subset Y_2 \subset Y_3 \subset \dots$$

be some chain in Σ, and denote their union $M = \bigcup Y_i$. Then M is an open set, and given $U, V \in M$, there must be some least Y_i which contains both, so $U \cap V \neq \emptyset$. Hence the chain has a maximal element, whence σ does as well by Zorn's lemma.

iii)

They are clearly closed by i). They cover X since for any $y \in X$ we have that the subspace $\{y\}$ is irreducible. If the space is Hausdorff, then these are precisely the irreducible components of X, since if Z is a subspace of X and $x, y \in Z$, then we have $U_x, U_y \in X$ which don't intersect, and $U_x \cap U_y \cap Z$ is empty as well.

iv)

Let p_x be a prime ideal, and $Y = V(p_x)$ be given the subspace topology. Consider two non-empty basis elements $X_a \cap Y, X_b \cap Y$. If $X_a \cap Y = (V(a))^c \cap Y$ is non empty, this means that there is some ideal in $V(p_x)$ which doesn't contain $a \in Y$. Let J_a, J_b be those ideals. Both a, b contain p_x since they lie in $Y = V(p_x)$. So we have two ideals $J_a \in X_a \cap Y$, $J_a \in X_a \cap Y$, and neither of them contain p_x, so the two basis elements intersect and $V(p_x)$ is irreducible.

Now let $V(E)$ be some closed irreducible set which contains $V(p_x)$ for a minimal prime ideal p_x. We have $p_x \in V(p_x) \subseteq V(E)$. Since $V(E)$ is irreducible, x

is dense in $V(E)$, and by Ex 1.18.ii), $\overline{\{x\}} = V(p_x)$ so $V(E) = V(p_x)$ is an irreducible component.

Ex 1.21

i)

To be explicit, X_f is the set of prime ideals in A which don't contain f. The map φ^{*-1} sends ideals in A to ideals in B which contain $\ker\varphi$ by Prop 1.1. Any ideal p in A which doesn't contain f must clearly be mapped to an ideal $\varphi(p)$ which doesn't contain $\varphi(f)$, so $\varphi^{*-1}(X_f) \subset Y_{\varphi(f)}$ (Note that if $p \in X_f$ is such that $\varphi(p)$ isn't prime, then φ^* doesn't map anything to X_f and $\varphi^{*-1}(X_f) = \emptyset$). If $p \in Y_{\varphi(f)}$, then $\varphi^*(p)$ is a prime ideal in X and clearly $f \not\in \varphi^*(p)$, so $\varphi^{*-1}(X_f) = Y_{\varphi(f)}$.

ii)

Let $b \in \varphi^{*-1}(V(a))$. Then b is a prime ideal in B such that $\varphi^{-1}(b)$ is a prime ideal in A containing a. Then $\varphi\varphi^{-1}b = b$ contains φa. But if b is a prime ideal containing $\varphi(a)$, then it necessarily contains a^e, the smallest ideal generated by $\varphi(a)$, so $b \in V(a^e)$.

Let $b \in V(a^e)$. Then b is a prime ideal containing the extension a^e of $\varphi(a)$. Since $a^e \supseteq \varphi(a)$ we have $\varphi^{-1}(b) \supseteq \varphi^{-1}(a^e) \supseteq \varphi^{-1}(\varphi(a)) = a$ so $b \in \varphi^{*-1}(V(a))$.

Ex 1.28

It's shown in the exercise statement that any regular function $\varphi : X \to Y$ can be used to induce a map $\overline{\varphi} : P(Y) \to P(X)$. We first show that this map is a K-algebra homomorphism.

Let $f, g \in P(Y),\ a, b \in K$. Then

$$\varphi(af + bg) = (af + bg) \circ \varphi = a(f \circ \varphi) + b(g \circ \varphi) = a\overline{\varphi}(f) + b\overline{\varphi}(g)$$
$$\varphi(fg) = fg \circ \varphi = (f \circ \varphi)(g \circ \varphi) = \overline{\varphi}(f)\overline{\varphi}(g),$$

and the constant polynomial functions on $P(Y)$ ignore their arguments, whence $\overline{\varphi}$ restricts to the identity on K.

To see that the correspondence is injective, let $\varphi, \psi : X \to Y$ be two regular maps such that $\overline{\varphi} = \overline{\psi}$. Let $x \in X$ be an arbitrary point on the variety, and write $a = \varphi(x), b = \psi(x)$. Then $a, b \in Y$. Since $\overline{\varphi} = \overline{\psi}$, we have that for all $g \in P(Y)$, $g(a) = g(b)$, which in implies that $a = b$, whence $\varphi(x) = \psi(x)$ for all $x \in X$, and finally $\varphi = \psi$.

We now show that it is surjective. Let $\mu : P(Y) \to P(X)$ be a K-algebra homomorphism. Let $\xi_i \in P(Y)$ be the i-th coordinate function in $P(Y)$ and $\hat{\xi}_i = \mu(\xi_i) \in P(X)$, and pick arbitrary representatives $f_i \in \hat{\xi}_i$ for every i. We claim that $\varphi = (f_1, f_2, \dots f_n)\big|_X$ is such that $\overline{\varphi} = \mu$.

We begin with showing that φ is a regular map $X \to k^m$. Let $x \in X$. Then

$$\begin{aligned}\varphi(x) &= (f_1(x), f_2(x), \dots f_n(x)) \\ &= (\mu(\xi_1)(x), \mu(\xi_2)(x), \dots, \mu(\xi_n)(x)) \in k^m.\end{aligned}$$

It follows from arguments in previous parts of the exercise that $\overline{\varphi}$ is a K-algebra homomorphism $X \to k[t_1, t_2, \dots t_n]$. To see that $\overline{\varphi} = \mu$, note that in $P(X)$ we have

$$\overline{\varphi}(\xi_i) = \xi_i \circ \varphi = f_i = \hat{\xi}_i = \mu(\xi_i),$$

and any homomorphism is determined by where it maps generators. It follows that $\mathrm{im}(\varphi) = \mathrm{im}(\mu) \subseteq Y$ and φ can be identified with a regular map $X \to Y$.

Ch 2

Ex 1

Let $a \otimes b \in \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$. Then as m, n are co-prime, we have $a \in (m, n) = \mathbb{Z}$, and there exist $k_1, k_2 \in \mathbb{N}$ where $a = k_1 m + k_1 n$. It follows that

$$a \otimes b = (k_1 m + k_1 n) \otimes b = k_1 m \otimes b + k_1 n \otimes b = k_1 n \otimes b = k_1 \otimes (nb) = 0.$$

Ex 2

We follow the given advise and tensor the exact sequence

$$\mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$$

with M to get the sequence

$$M \otimes \mathfrak{a} \longrightarrow M \otimes A \longrightarrow M \otimes A/\mathfrak{a} \longrightarrow 0,$$

which is also exact by Prop 2.18. Now consider the map $f : M \otimes A \to M/\mathfrak{a}M$ given by $m \otimes a \to am + \mathfrak{a}M$. This map has kernel $M \otimes \mathfrak{a}$, so the sequence

$$M \otimes \mathfrak{a} \longrightarrow M \otimes A \xrightarrow{f} M/\mathfrak{a}M \longrightarrow 0,$$

is exact as well, and the two modules are isomorphic.

Ex 3

Let m be the maximal ideal in A, $k = A/m$, and let $M_k = k \otimes M$ be the k-module obtained by extension of scalars. By Ex 2, we have $M_k \cong M/mM$. If we had $M_k = 0$, we'd have $M = 0$, since $M/mM = M_k = 0 \Rightarrow M = mM$ whence $M = 0$ follows by Nakayama. The same is true of N_k.

We've shown that it's enough to prove the statement with M_k, N_k, since that would yield the following chain of implications

$$M \otimes_A N = 0 \Rightarrow M_k \otimes_k N_k = 0 \xRightarrow{\text{remains to prove}} N_k = 0 \vee M_k = 0 \Rightarrow N = 0 \vee M = 0.$$

Now, M_k, N_k are k-vector spaces, and given two non-zero vector spaces, we can pick one basis element in each e_m, e_n, and define the non-zero bilinear map $(m, n) = (m \cdot e_m)(n \cdot e_n)$. So their tensor product must be non-zero by the universal property.

Ex 4

First assume that M is flat, and consider two A-modules N', N and an injective map $f : N' \to N$. Since M is flat, we have that $1_M \otimes f$ is injective. It follows that the restriction $1_{M'} \otimes f$ must be injective as well, and M_i is flat by Prop 2.19.

For the other direction, we need two lemmas.

Lemma 0.4. Let M_i be a set of A-modules indexed by the (potentially infinite) set J, and N another A-module. Then

$$N \otimes \left(\bigoplus_{i \in J} M_i \right) \cong \bigoplus_{i \in J} N \otimes M_i$$

Proof. Let

$$f : N \times \left(\bigoplus_{i \in J} M_i \right) \to \bigoplus_{i \in J} N \otimes M_i$$

be given by $f\left(n, \sum_{i \in S} m_i\right) = \sum_{i \in S} (n \otimes m_i)$. This map is A-bilinear and as such, induces an A-linear map

$$f' : N \otimes \left(\bigoplus_{i \in J} M_i \right) \to \bigoplus_{i \in J} N \otimes M_i$$

where $f'\left(n \otimes \sum_{i \in S} m_i\right) = \sum_{i \in S} (n \otimes m_i)$.

Now define $g_i : N \times M_i \to N \otimes M$ by $g_i(n, m_i) = n \otimes m_i$ Then g_i is A-bilinear and induces an A-linear map $g_i' : N \otimes M_i \to N \otimes M$. Now define the map

$$h : \bigoplus_{i \in J} N \otimes M_i \to N \otimes \left(\bigoplus_{i \in J} M_i \right)$$

where

$$f'\left(\sum_{i\in S}(n\otimes m_i)\right)=\sum_{i\in S}g_i'(n\otimes m_i)=\sum_{i\in S}n\otimes m_i=n\otimes\left(\sum_{i\in S}m_i\right).$$

This function is well-defined, as S always is a finite set. The two maps f', h are inverse each other, and the two modules are isomorphic. $\square$

The second lemma is given as follows

Lemma 0.5. Let J be a potentially infinite indexing set, and

$$\dots \longrightarrow M_i' \xrightarrow{f_i} M_i \xrightarrow{h_i} M_i'' \longrightarrow \dots\,, \quad i\in J$$

be a set of exact sequences. Then

$$\dots \longrightarrow M' \xrightarrow{f} M \xrightarrow{h} M'' \longrightarrow \dots$$

is exact where $M=\bigoplus_{i\in J}M_i$, and M', M'' are defined analogously.

Proof. Let $m\in\ker(h)$. Then we can write m as a finite sum $m=\sum_{i\in S}m_i$, and its image is given as $0=\sum_{i\in S}h(m_i)$. But since h is defined as the sum of h_i on the relevant coordinates, we have $0=\sum_{i\in S}h_i(m_i)$, and each $h_i(m_i)\in M_i''$ so if their sum is zero we must have that for every $i\in S$, $h_i(m_i)=0$ and we have that $m_i\in\ker f_i=\operatorname{im}(f_i)$ whence $m\in\operatorname{im}(f)$.

The other inclusion can be shown analogously. $\square$

Now, as every M_i is flat, we can direct sum all the corresponding tensored exact sequences, and use distributivity in reverse to yield an exact sequence of M tensored with the module N.

Ex 5

First of, A is a flat A-module since $A\otimes_A M\cong M$. Moreover, $A[x]$ is the direct sum of the A-modules, $(x^i) : i\in\mathbb{N}$, and each such (x^i) is generated by one element and therefore isomorphic to A. Since all of the direct summands of $A[x]$ are flat, it follows that $A[x]$ itself is flat.

Ex 6

Let $\hat{f} : A[x]\times M\to M[x]$ be given by $\hat{f}(a(x),m)=a(x)m$. This is a A-bilinear mapping, and as such introduces a A-linear map $f : A[x]\otimes M\to M[x]$ given by $f(a(x)\otimes m)=a(x)m$. Even more, this map is $A[x]$-linear since $a_1(x)f(a_2(x)\otimes m)=a_1(x)a_2(x)m=f(a_1(x)a_2(x)\otimes m)$.

Now consider the map $g : M[x]\to A[x]\otimes M$ given by the A-linear extension of $g(mx^i)=x^i\otimes m$. This map is also $A[x]$ linear since $g(a(x)mx^i)=a(x)x^i\otimes m$.

We have that $g\circ f=\mathrm{id}$ and the two modules are isomorphic.

Ex 7

$\mathfrak{p}[x]$ being prime is equivalent to $A[x]/\mathfrak{p}[x]$ not having zero divisors. To show this, we will first show that $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$. To see this, consider the map $f(ax^i + \mathfrak{p}[x]) = (a+\mathfrak{p})x^i$, which we extend by the ring axioms to all of $A[x]/\mathfrak{p}[x]$. First note that it's well-defined on the elements above, as if $ax^i - bx^i \in \mathfrak{p}[x]$, then $a-b \in \mathfrak{p}$, and $f(ax^i+\mathfrak{p}[x]) = f(bx^i+\mathfrak{p}[x])$. It's also easy to see that we can extend f to a ring homomorphism. Moreover, it's injective since if $a(x) \in \ker(f)$ only when every coefficient of a lies in $\mathfrak{p}$, whence $a(x) \in \mathfrak{p}[x]$. It's surjective since the $(a+\mathfrak{p})x^i$ generate the image for all $i \in \mathbb{N}, a \in A$.

It's easy to see that $(A/\mathfrak{p})[x]$ is an integral domain, since the lowest degree term in any product $f(x)g(x)$ is the product of the lowest degree terms in $f(x), g(x)$. It follows that $A[x]/\mathfrak{p}[x]$ is an integral domain and we are done.

For the second question, the answer is no. Let $n \notin \mathfrak{m}$, then $nx \notin \mathfrak{m}[x]$ so $nx + \mathfrak{m}[x]$ is a non-trivial element in $A[x]/\mathfrak{m}[x]$ without an inverse. Indeed we could not have $f(x)nx = 1 + m(x)$ for some $m(x) \in \mathfrak{m}[x]$, since $f(x)nx$ doesn't have a constant term, and $m(x)$ can't have -1 as a constant term since $-1 \notin \mathfrak{m}$.

Ex 8

i)

Given some exact sequence

$$0 \longrightarrow W' \longrightarrow W \longrightarrow W'' \longrightarrow 0,$$

we have that

$$0 \longrightarrow M \otimes W' \longrightarrow M \otimes W \longrightarrow M \otimes W'' \longrightarrow 0,$$

is exact, after which it follows that

$$0 \longrightarrow N \otimes (M \otimes W') \longrightarrow N \otimes (M \otimes W) \longrightarrow N \otimes (M \otimes W'') \longrightarrow 0,$$

is exact, and our desired result follows from Prop 2.14.

ii)

First of, by Prop 2.14 we have that $N \cong B \otimes_B N$, and given some A-module M, Prop 2.15 yields $M \otimes_A (B \otimes_B N) \cong (M \otimes_A B) \otimes_B N$, after which A-flatness follows by A-flatness of B followed by B-flatness of N, just like in the previous sub-exercise.

Ex 9

Let m_i' finitely generate M' and m_i'' finitely generate M''. Then let $M_i = g^{-1}(m_i'')$. For each M_i pick one representative $m_i \in M_i$. Now, we have that $\mathrm{im}(f) = \ker(g)$ is finitely generated as well, and we let n_i be a finite generating set for $\ker(g)$. Our claim is that the m_i and n_i together generate M.

To see this, consider any element $m \in M$. Then we can write $g(m) = \sum_{i \in I} m_i'' a_i$ for some finite index set I and a set of scalars $a_i \in A$. It follows that

$$g(m) = g\left(\sum_{i \in I} a_i m_i\right),$$

and $m - \sum_{i \in I} a_i m_i \in \ker(g)$. We can then write

$$m - \sum_{i \in I} a_i m_i = \sum_{i \in J} b_i n_i,$$

and finally,

$$m = \sum_{i \in I} a_i m_i + \sum_{i \in J} b_i n_i.$$

Ex 10

If the induced homomorphism is surjective, we have that for every $n \in N$, there is some $m \in M$ such that $n - u(m) \in \mathfrak{a}N$. In other words, $u(M) + \mathfrak{a}N = N$, and we have $u(M) = N$ by Corollary 2.7.

Ex 14

I couldn't find a quetion in the exercise description.

Ex 15

For the first statement, let $m \in M$. Then $m = \mu(m')$ for some $M' \in C$. Since C is the direct sum of M_i we have

$$m' = \sum_{i \in J} a_i m_i$$

where $m_i \in M_i,\ a_i \in A$ and $J \subseteq I$ is a finte index set. It follows that

$$m = \mu\left(\sum_{i \in J} a_i m_i\right) = \sum_{i \in J} \mu(a_i m_i) = \sum_{i \in J} \mu_i(a_i m_i).$$

Since J is finite, we can inductively invoke the defining property of a directed set to conclude that there exist some $k \in I$ such that $i \leq k$ for all $i \in J$. By the defining relations of M, we have that

$$\sum_{i\in J}\mu_i(a_im_i) = \sum_{i\in J}\mu_k(\mu_{ik}(a_im_i)) = \mu_k\left(\sum_{i\in J}\mu_{ik}(a_im_i)\right),$$

and we see that $m = \mu_k(m_k)$ where $m_k = \sum_{i\in J}\mu_{ik}(a_im_i)$

For the second part of the question, note that $\mu_i(m_i) = 0$ implies that $m_i \in D$ (since μ_i can be factored as an injection followed by a quotient of D), but we also have $m_i - \mu_{ij}(m_i) \in D$ for all $j \geq i$. It follows that $\mu_{ij}(m_i) \in D$ which means that $\mu_{ij}(m_i) = 0$ in M and M_j for all $j \geq i$ (and there is always at least one such j with $j = i$).

Ex 16

We begin by showing that the direct limit exhibits the given property. Let N, α_i be given as in the exercise description. Then let $\alpha : M \to N$ be given as $\alpha(m) = \alpha_i(m_i)$ where $i \in I,\ m_i \in M_i$ are such that $m = \mu_i(m_i)$. We begin by showing that this map is well definied. If $\mu_j(m_j) = \mu_i(m_i)$, then there exists $k \in I$ where $i \leq k,\ j \leq k$. Using this k we get $\mu_j(m_j) = \mu_k(\mu_{jk}(m_j))$ and $\mu_i(m_i) = \mu_k(\mu_{ik}(m_i))$, so $\mu_k(\mu_{ik}(m_i) - \mu_{jk}(m_j)) = 0$, and by Ex 2.15, there exist $l \in I, l \geq k$ such that $\mu_{kl}(\mu_{ik}(m_i) - \mu_{jk}(m_j)) = 0$. We now have

$$\begin{aligned}
\alpha_i(m_i) &= \alpha_k(\mu_{ik}(m_i)) \\
&= \alpha_l(\mu_{kl}(\mu_{ik}(m_i))) \\
&= \alpha_l(\mu_{kl}(\mu_{jk}(m_j))) \\
&= \alpha_k(\mu_{jk}(m_j)) \\
&= \alpha_j(m_j),
\end{aligned}$$

which shows that α is well defined. It's clear that $\alpha_i = \alpha \circ \mu_i$, To see that α is a homomorphism, let $m, m \in m$, where $m = \mu_i(m_i),\ n = \mu_j(n_j)$, and let $k \geq i, j$, $m_k = \mu_{ik}(m_i), n_k = \mu_{jk}(n_k)$. Then $\alpha(m) = \alpha_i(m_i) = \alpha_k(\mu_{ik}(m_i)) = \alpha_k(m_k)$, and similarly $\alpha(n) = \alpha_k(n_k)$, whence homomorphism properties of α follows by homomorphism properties from α_k. Finaly, to see that the homomorphism is unique, let $\beta : M \to N$ be an arbitrary map which factors the α_i. Then for any $m \in M$ we have that ther exist $i \in I, m_i \in M_i$ such that $m = \mu_i(m_i)$ and we see that

$$\beta(m) = \beta(\mu_i(m_i)) = \alpha_i(m_i) = \alpha(\mu_i(m_i)) = \alpha(m)$$

so $\beta = \alpha$.

We now show that the universal property uniquely determines the direct limit up to isomorphism. Let M, μ_i, and M', μ_i' be two modules and families of homomorphisms which exhibit the universal property. Then both modules can take the place of N in the definition of the universal property, and invoking the universal property for each module, we see that there exists two homomorphisms

$$\alpha : M \to M',\ \mu_i' = \alpha \circ \mu_i,$$
$$\alpha' : M' \to M,\ \mu_i = \alpha' \circ \mu_i'.$$

But then $\varphi = \alpha' \circ \alpha$ is a homorphism $M \to M$ such that $\varphi \circ \mu_i = \mu_i$, and by the universal property, there can only be one such map. We conlcude that φ is the identity map and $M \cong M'$.

Ch 3

Ex 1

Let $s \in S$ be such that $sm = 0$ for all $m \in M$. Then $(m, 1) = (0, 1)$ since $s(m - 0) = 0$, and it follows that $S^{-1}M = 0$.
For the other direction, let M be an A-module finitely generated by x_i, and S be such that $S^{-1}M = 0$. We know that for every $m/s \in S^{-1}M$ there is $u \in S$ such that $um = 0$, in particular, let $u_i \in S$ be such that $u_i x_i = 0$. Then the product of all u_i annihilates all generators, and therefor all of M.

Ex 2

For any $a \in \mathfrak{a}$, $x \in A$, we have that $-ax \in \mathfrak{a}$ so $1 - ax \in S$ is a unit in $S^{-1}A$, which in turn means a is in the Jacobson radical of $S^{-1}A$.

Now let M be a finitely generated A-module and $\mathfrak{a} \subseteq A$ be an ideal such that $\mathfrak{a}M = M$. Now let S be as above. We have that $(S^{-1}\mathfrak{a})(S^{-1}A) = S^{-1}(\mathfrak{a}A) = S^{-1}A$. Now we can apply Nakayama's lemma in and see that $S^{-1}A = 0$. But by the previous exercise, this happens only when there is $s \in S$ s.t $sA = 0$, and it's true for any $s \in S$ that $s \equiv 1 \mod \mathfrak{a}$, whence we are done.

Ex 3

Let $f : A \to S^{-1}A$, $g : S^{-1}A \to U^{-1}(S^{-1}A)$ be the canonical maps and consider $g \circ f$. Then if $st \in ST$, we have

$$g(f(st)) = g(uf(s)) = g(u)g(f(s))$$

a unit since $g(u), f(s)$ are units. Moreover, $g(f(a)) = 0$ implies that there exists $u \in U$ s.t. $uf(a) = 0$. But then $0 = uf(a) = f(t)f(a) = f(at)$ so there exists $s \in S$ s.t $sta = 0$. Finally, every element in $U^{-1}(S^{-1}A)$ is of the form

$$\begin{aligned} g(f(a)f^{-1}(s))g(u)^{-1} &= g(f(a)f^{-1}(s))g(f(t))^{-1} \\ &= g(f(a))g(f(s))^{-1}g(f(t))^{-1} \\ &= g(f(a))g(f(st))^{-1}, \end{aligned}$$

and Corollary 3.2 delivers our desired result.

Ex 4

We first give an account of how these modules are defined. We have that A acts on B via $ab = f(a)b$, $a \in A, b \in B$. Similarly, $s \in S$ acts on B via f, and $S^{-1}A$ acts on B also via f. So, the elements in $S^{-1}B$ are of the form $b/f(s)$, just like the elements in $T^{-1}B$ which are all of the form $b/t = b/f(s)$. It's immediate that the two modules are isomorphic, they are the same module!

Ex 5

Let A be such that for each prime ideal $\mathfrak{p} \subseteq A$, we have that $A_{\mathfrak{p}}$ has no nilpotents. Assume towards a contradiction that A has some nilpotent element $a^r = 0$. Then the map $e_r : A \to A^r$ has a non-trivial kernel $a \in \ker(e_r)$. It follows from Proposition 3.9 that there must be some prime ideal $\mathfrak{p} \subseteq A$ such that $(e_r)_{\mathfrak{p}}$ has non-trivial kernel, but this is contradiction to our hypothesis, so A must have an empty radical.

For the second question, the answer is no. Let k be a field and consider the ring $R = k \times k$. This is not a domain since it has the zero divisors $(1,0) \cdot (0,1) = 0$. R admits only two ideals, $((1,0)), ((0,1))$. Let $I = ((1,0))$ and consider R_I. We have that $(a,b)/(c,d) = (1,0)(a,b)/(c,0) = (b,0)/(c,0) = (bc^{-1},0)/(1,0)$ for all $a,b,c,d \in k$, so in other words $R_I \cong R$, and R_I is a field, and a domain in particular.

Ex 7

i)

$\Rightarrow$ **direction**:
First of all, we show that if $a \in A - S$, then $(a/1)$ is a non-trivial ideal in $S^{-1}A$. To see this, first note that all units lie in S by virtue of S beeing saturated and containing 1. So a is not a unit in A. If $a/1$ were a unit in $S^{-1}A$, then we'd have b/c such that $(a/1)(b/c) = (ab)/c = 1/1$, which in turn means that $(ab - c)s = 0$ for some $s \in S$. But since $c \in S$, this would mean that $ab \in S$, which is imposible since S is saturated and doesn't contain a.

Let $\mathfrak{p}_a = \mathfrak{m}^c_{a/1}$ where $\mathfrak{m}_{a/1}$ is the maximal ideal in $S^{-1}A$ containing the principal ideal $(a/1)$. Such exist by Corollary 1.3 and our argument in the previous paragraph. It now follows from Prop 3.11 iv) that $\mathfrak{p}_a$ is prime. Moreover it contains a since $f(a)$ is contained in $\mathfrak{m}_{a/1}$. Then $A - S = \bigcup_{a \in A-s} \mathfrak{m}_a$, and we see that $A - S$ is a union of prime ideals.

$\Leftarrow$ **direction**: Let S be a subset of A such that $A - S$ is a union of ideals $A - S = \bigcup_{\mathfrak{a}_i, i \in J} \mathfrak{a}_i$. Consider two elements $a, b \in A$ such that $ab \in S$. Then ab doesn't lie in any $\mathfrak{a}_i$. But then neither a nor b can lie in any $\mathfrak{a}_i$ since that would imply $ab \in \mathfrak{a}_i$.

ii)

Let $\overline{S}$ be the intersection of all saturated multiplicative subsets containing S. Then $\overline{S} = \{a \in A : \exists b \in A, ab \in S\}$, since if $ab \in A$ then a must be contained in any saturated multiplicative set containing a, and if $ab \in \overline{A}$, then $abc = a(bc) = b(ac) \in A$ for some $c \in A$, whence it follows that $a, b \in \overline{A}$.

From i) we know that $\overline{S}$ is a union of prime ideals which don't meet $\overline{S} \supseteq S$. It remains to show that any prime ideal which doesn't meet S also won't meet $\overline{S}$. To do so, consider a prime ideal $\mathfrak{p}$ which meets $\overline{S}$ then it must meet S, since if $a \in \mathfrak{p} \cap \overline{S}$, then either $a \in S$, or $a \notin S$, whence there must be some $b \in A$ such that $ab \in S$ (by our constructive definition of $\overline{S}$), but $ab \in \mathfrak{p}$ as well.

For the second quetion, let $\mathfrak{p}$ be prime ideal which dosen't meet $1 + \mathfrak{a}$. Then $\mathfrak{p} + \mathfrak{a}$ is a non-trivial ideal, as otherwise we'd be able to write $1 = xp + ya$, whence $xp = 1 - ya \in 1 + \mathfrak{a}$. It follows that there exist some maximal ideal $\mathfrak{m}$ which contains $\mathfrak{p} + \mathfrak{a}$, so every ideal which doesn't meet $1 + \mathfrak{a}$ is contained in a maximal ideal containing $\mathfrak{a}$. As no proper ideal containing $\overline{a}$ can meet $\overline{a} + 1$, we deduce that $\overline{1 + \mathfrak{a}}$ is the union of all maximal ideals containing $\overline{a}$.

Ex 8

i) $\Rightarrow$ ii): φ bijective means we can construct its inverse $\psi = \varphi^{-1}$, and as $t/1$ is a unit in $T^{-1}(A)$, we see that $\psi(t/1) = t/1$ is a unit in $S^{-1}A$.

ii) $\Rightarrow$ iii): Let $u/1$ be the inverse of $t/1$, then $ut/1 \in S$

iii) $\Rightarrow$ iv): Let $xt \in S$, then $x, t \in \overline{S}$ by definition.

iv) $\Rightarrow$ v): we have that $S \subseteq T$ and $T \subseteq \overline{S}$, so $\overline{T} = \overline{S}$. The saturation of a multiplicatively closed set is the complement of all prime ideals not meeting the set (Ex 7.ii)), so if $\mathfrak{p} \cap S = 0$, then $\mathfrak{p} \subseteq \overline{S}^c = \overline{T}^c \Rightarrow \mathfrak{p} \cap T = 0$. The other implication follows analogously.

v) $\Rightarrow$ i): Let $a/s \in S^{-1}A$ be such that $a/s = 0$ as an element in T. Then there exists some $t \in T$ such that $ta = 0$. But then since t is a unit in $T^{-1}A$, we either have $T^{-1}A = 0$, or $a = 0$. Now, every prime ideal which meets S also meets T, so if $S^{-1}A$ contains non-trivial prime ideals, then so does $T^{-1}A$ by Prop 3.11.iv), and since the only rings which don't contain prime ideals are either the zero ring or fields, we see that $T^{-1}A = 0$ exactly when $S^{-1}A = 0$. It follows that φ is bijective.

Ex 21

Just to recap, spec(A) is the set of all prime ideals in A, and the closed sets are of the form $V(E)$ for arbitrary $E \subseteq A$ where $V(E)$ is the set of all prime ideals containing E. $V(E) = V(r(\mathfrak{q}))$ where $\mathfrak{a}$ is the smallest ideal containing E. The basic open sets are of the form X_f, where X_f is the complement of $V(f)$, I.e the set of all prime ideals not containing f.

i)

Let $X = \text{spec}(A), Y = \text{spec}(S^{-1}A)$. Let $f \in A$ and $\mathfrak{p} \in \varphi^{*-1}(X_f)$. Then $\mathfrak{p}$ is a prime ideal in $S^{-1}B$, and $\varphi(f) \notin \mathfrak{p}$, since otherwise we'd have $f \in \varphi^{-1}(\mathfrak{p}) = \varphi^*(\mathfrak{p}) \in X_f$. Hence $Y_{\varphi(f)} \supseteq \varphi^{*-1}(X_f)$. For the other inclusion, suppose $\mathfrak{p}$ is a prime ideal in Y not containing $\varphi(f)$. Then $\varphi^{-1}(\mathfrak{p}) = \varphi^*(\mathfrak{p})$ doesn't contain $\varphi^{-1}(\varphi(f)) = f$. We've shown that φ^* is continuous. It's immediate from Prop 3.11.iv) that φ^* is injective, so φ^* is a homomorphism onto its image.

ii)

We draw the setup in diagrams. First, we have homomorphisms between the rings as follows

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s_a"'] & B \arrow[d, "s_b"] \\ S^{-1}A \arrow[r, "S^{-1}f"] & S^{-1}B, \end{tikzcd}$$

where we define $S^{-1}f$ so the diagram commutes $a/s \mapsto f(a)/f(s)$. This induces the following diagram on the respective spectra

$$\begin{tikzcd} S^{-1}Y \arrow[r, "S^{-1}f^*"] \arrow[d, "i_y"'] & S^{-1}X \arrow[d, "i_x"] \\ Y \arrow[r, "f^*"] & X, \end{tikzcd}$$

and our task is to show that the diagram commutes, I.e that $S^{-1}f^*$ is the restriction of f^*, and that $f^{*-1}(S^{-1}X) = S^{-1}Y$. We will show that spec is a contravariant functor, from which it will follow that the diagram commutes. We showed in i) that spec maps morphisms f in the $\mathbf{CRing}^{\mathbf{op}}$ to morphisms f^* in **Top**. Now let f, g be two composable ring homomorphisms. Then $(f \circ g)^* = (f \circ g)^{-1} = g^{-1} \circ f^{-1} = g^* \circ f^*$, so the functor preserves composition in a contravariant we've shown the first statement.

Our setup differs slightly from that in the exercise in the sense that we've not identified the spectra of the fraction rings with their inclusions in to the spectra of the original rings. So, when the exercise asks us to show that $f^{*-1}(S^{-1}X) = S^{-1}Y$, in our setup, this translates to showing that $f^{*-1}(i_x(S^{-1}X)) = i_y(S^{-1}Y)$.

Let $\mathfrak{p} \in f^{*-1}(i_x(S^{-1}X))$. Then $f^*(\mathfrak{p}) \subseteq i_x(S^{-1}X)$. In the original rings, i_x corresponds to contracting ideals from $S^{-1}A$ back to A, so $f^*(\mathfrak{p})$ is a prime ideal in A which doesn't touch S. Moreover, $\mathfrak{p}$ is a prime ideal in B since it lies in the inverse image of f^*, and it doesn't meet $f(S)$ since $f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$ doesn't meet S. It follows that $\mathfrak{p} \in i_y(S^{-1}Y)$ and we are done (the other inclusion is trivial).

iii)

We have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q_a"'] & B \arrow[d, "q_b"] \\ A/\mathfrak{a} \arrow[r, "\overline{f}"] & B/\mathfrak{b}, \end{tikzcd}$$

and the spec functor induces the following commutative diagram

$$\begin{tikzcd} \operatorname{spec}(B/\mathfrak{b}) \arrow[r, "\overline{f}^*"] \arrow[d, "i_y"'] & \operatorname{spec}(A/\mathfrak{a}) \arrow[d, "i_x"] \\ Y \arrow[r, "f^*"] & X, \end{tikzcd}$$

which since i_y, i_x are injections, tells us that $\overline{f}^*$ is the restriction of f^* to $V(\mathfrak{b}) = i_x(\operatorname{spec}(B/\mathfrak{b}))$.

(the following is not part of iii), but used in iv)) Just like in i), we also show that $f^{*-1}(i_x(\operatorname{spec}(A/\mathfrak{a}))) = i_y(\operatorname{spec}(B/\mathfrak{b}))$. Let $\mathfrak{p} = f^{*-1}(i_x(\operatorname{spec}(A/\mathfrak{a})))$. Then $f^*(\mathfrak{p}) \in i_x(\operatorname{spec}(A/\mathfrak{a}))$, so $f^*(\mathfrak{p})$ is an ideal in A which contains $\mathfrak{a}$. Moreover, $\mathfrak{p}$ is a prime ideal in B since it lies in the inverse image of f^*, and it doesn't meet $\mathfrak{b} = \mathfrak{a}^e$ since $f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$ doesnt meet $\mathfrak{a} = f^{-1}(\mathfrak{a}^e)$. It follows that $\mathfrak{p} \in i_y(\operatorname{spec}(B/\mathfrak{b}))$.

iv)

We first draw two diagrams of our situation. We have the following diagram of rings

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l_a"'] & B \arrow[d, "l_b"] \\ A_{\mathfrak{p}} \arrow[r, "\overline{\overline{f}}"] \arrow[d, "q_a"'] & (f(A)-\mathfrak{p})^{-1}B \arrow[d, "q_a"'] \\ k(\mathfrak{p}) \arrow[r, "\overline{f}"] & (f(A)-\mathfrak{p})^{-1}B/(\mathfrak{p}B), \end{tikzcd}$$

which after passing through spec turns into

$$\begin{tikzcd}
X & & Y \arrow[ll, "f^*"'] \\
\operatorname{spec}(A_{\mathfrak{p}}) \arrow[u, hook, "i_a"] & & \operatorname{spec}((f(A)-\mathfrak{p})^{-1}B) \arrow[ll, "\overline{\overline{f^*}}"'] \arrow[u, hook, "i_b"'] \\
\operatorname{spec}(k(\mathfrak{p})) \arrow[u, hook, "i'_a"] & & \operatorname{spec}((f(A)-\mathfrak{p})^{-1}B)/(\mathfrak{p}B). \arrow[ll, "\overline{f^*}"'] \arrow[u, hook, "i'_b"']
\end{tikzcd}$$

We have that $\operatorname{spec}(k(\mathfrak{p})) = \{(0)\}$, and after the two inclusions i'_a, i_a, the zero ideal in $\operatorname{spec}(k(\mathfrak{p}))$ is mapped to $\mathfrak{p}$ in A. From ii) and iii) it also follows that $f^{*-1}(i_a(i'_a(\operatorname{spec}(k(\mathfrak{p}))))) = i_b(i'_b(\operatorname{spec}((f(A)-\mathfrak{p})^{-1}B/(\mathfrak{p}B))))$. Combining these two facts tells us that $f^{*-1}(\mathfrak{p}) = i_b(i'_b(\operatorname{spec}((f(A)-\mathfrak{p})^{-1}B/(\mathfrak{p}B))))$, so the two spaces are homeomorphic in a natural way.

It remains to show that $\operatorname{spec}((f(A)-\mathfrak{p})^{-1}B/(\mathfrak{p}B)) = \operatorname{spec}(k(\mathfrak{p}) \otimes_A B)$. We will show that the corresponding rings are isomorphic. To see this, let

$$\varphi : (f(A) - f(\mathfrak{p}))^{-1}B \to k(\mathfrak{p}) \otimes_A B$$

be given by $\varphi(b/f(a)) = \frac{1+\mathfrak{p}}{a+\mathfrak{p}} \otimes_A b$. Then φ is well-defined, since if $b_1/f(a_1) = b_2/f(a_2)$, then there exists $s \in A - \mathfrak{p}$ such that $f(s)(b_1 f(a_2) - b_2 f(a_1)) = 0$, whence

$$\begin{aligned}
\varphi(b_1/f(a_1)) - \varphi(b_2/f(a_2)) &= \frac{1+\mathfrak{p}}{a_1+\mathfrak{p}} \otimes_A b_1 - \frac{1+\mathfrak{p}}{a_2+\mathfrak{p}} \otimes_A b_2 \\
&= \frac{sa_2+\mathfrak{p}}{sa_1a_2+\mathfrak{p}} \otimes_A b_1 - \frac{sa_1+\mathfrak{p}}{sa_1a_2+\mathfrak{p}} \otimes_A b_2 \\
&= \frac{1\mathfrak{p}}{sa_1a_2+\mathfrak{p}} \otimes_A f(s)f(a_2)b_1 - \frac{1+\mathfrak{p}}{sa_1a_2+\mathfrak{p}} \otimes_A f(s)f(a_1)b_2 \\
&= \frac{1+\mathfrak{p}}{sa_1a_2+\mathfrak{p}} \otimes_A f(s)(f(a_2)b_1 - f(a_1)b_2) \\
&= 0.
\end{aligned}$$

Moreover, φ is surjective since for any $\frac{a_1+\mathfrak{p}}{a_2+\mathfrak{p}} \otimes_A b$, we have

$$\frac{a_1+\mathfrak{p}}{a_2+\mathfrak{p}} \otimes_A b = \frac{1+\mathfrak{p}}{a_2+\mathfrak{p}} \otimes_A f(a_1)b = f(f(a_1)b/f(a_2)).$$

Finally, the kernel of φ are all $b/f(a)$ such that b can be written as $b = b'f(a')$ where $a' \in \mathfrak{p}$, in other words, $\ker(\varphi) = (f(A)-\mathfrak{p})^{-1}Bf(\mathfrak{p})$, and the isomorphism follows from the First Homomorphism Theorem.

Ch 4

Ex 4

To see that $\mathfrak{m}$ is maximal, note that $\mathbb{Z}[t]/\mathfrak{m} \cong \mathbb{Z}/2\mathbb{Z}$ is a field. To see that $\mathfrak{q}$ is $\mathfrak{m}$-primary, note that $2 \in r(\mathfrak{q})$ and $t \in r(\mathfrak{q})$, and since $\mathfrak{m}$ is maximal $r(\mathfrak{q}) = \mathfrak{m}$. Also, $t \notin \mathfrak{m}^k$ when $k > 1$, and $2 \notin \mathfrak{q}$, so $\mathfrak{m}^k \neq \mathfrak{q}$ for any $k \geq 1$.

Ex 5

$\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a primary decomposition since all of the ideals being intersected are primary ($\mathfrak{m}^2$ is so by Prop 4.2, and the others since they're prime).

To see that it's reduced, the first criterion is satisfied since the two prime ideals and $r(\mathfrak{m}^2) = \mathfrak{m}$ are all pairwise distinct, and the second criterion is fulfilled since

$$\begin{aligned} z^2 &\in \mathfrak{p}_2 \cap \mathfrak{m}^2 \setminus \mathfrak{p}_1, \\ y^2 &\in \mathfrak{p}_1 \cap \mathfrak{m}^2 \setminus \mathfrak{p}_2, \\ x &\in \mathfrak{m}^2 \cap \mathfrak{p}_1 \setminus \mathfrak{p}_2. \end{aligned}$$

To see that the decomposition is equal to $\mathfrak{a}$, note that

$$\begin{aligned} \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2 &= (x, y) \cap (x, z) \cap (x, y, z)^2 \\ &= (x, yz) \cap (x^2, xy, xz, y^2, yz, z^2) \\ &= (x^2, xy, xz, yz) \\ &= (x, y)(x, z) \\ &= \mathfrak{a}. \end{aligned}$$

We have that $\mathfrak{p}_1, \mathfrak{p}_2$ are minimal since they are both contained in $\mathfrak{m}$, but don't contain each other. $\mathfrak{m}$ is embedded.

Ex 7

i)

Let $f \in \mathfrak{a}^e$. Then we can write

$$f = \sum_{i=0}^{n} h_i(x) a_i$$

for some set of elements $a_i \in \mathfrak{a}$ and polynomials $h_i(x) \in A[x]$. But we have $h_i(x)a_i \in \mathfrak{a}[x]$ for each i, so $f \in \mathfrak{a}[x]$.

For the other inclusion, note that the monomials $ax^k, a \in \mathfrak{a}, k \in \mathbb{N}$ generate $\mathfrak{a}[x]$ as an A-module, and each $ax^k \in \mathfrak{a}^e$ for each $a \in \mathfrak{a}, k \in \mathbb{N}$, so $\mathfrak{a}[x] \subseteq \mathfrak{a}^e$.

ii)

Let $\varphi : A[x] \to (A/\mathfrak{p})[x]$ be the homorphism which sends $a_n x^n + \ldots + a_1 x + a_0$ to $\overline{a_n} x^n + \ldots + \overline{a_1} x + \overline{a_0}$ where $\overline{a} = a + \mathfrak{p}$. Then it's easy to see that $\ker(\varphi) = \mathfrak{p}[x]$, so $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$. The fact that $\mathfrak{p}[x]$ is prime now follows from the fact that the polynomial ring over an integral domain is an integral domain (seen easily by looking at degree 0 terms during polynomial multiplication).

iii)

That $r(\mathfrak{q}[x]) = \mathfrak{p}[x]$ follows from the following lemma.

Lemma 0.6. Let $\mathfrak{a}$ be an ideal in A. Then $r(\mathfrak{a}[x]) = (r(\mathfrak{a}))[x]$.

Proof. Since $\mathfrak{a} \subset \mathfrak{a}[x]$, we have $r(\mathfrak{a}) \subset r(\mathfrak{a}[x])$, and as $(r(\mathfrak{a}))[x]$ is the extension of $r(\mathfrak{a})$ (hence the minimal ideal of $A[x]$ containing $r(\mathfrak{a})$, we have $(r(\mathfrak{a}))[x] \subseteq r(\mathfrak{a}[x])$.

For the other inclusion, let $g \in A[x]$ be a polynomial of degree d such that $g^k \in \mathfrak{a}[x]$. Let a_i be the i-th coefficient of g and b_i be the i-th coefficient of g^k. Then

$$b_{ik} = a_i^k + H$$

where H is a sum of terms each having a factor of some a_j for $j < i$. Since $b_0 = a_0^k$ we have that $a_0 \in r(\mathfrak{a})$, and it now follows from induction that $a_i \in r(\mathfrak{a})$ for all $i \leq d$. We see that $g \in (r(\mathfrak{a}))[x]$ and we are done. $\square$

It remains to show that $\mathfrak{q}[x]$ is a primary ideal. This is equivalent to $A[x]/\mathfrak{q}[x] \cong (A/\mathfrak{q})[x]$ beeing non-zero, and having all zero-divisors being nilpotent. We know it's non-zero since $a \notin \mathfrak{a} \Rightarrow a \notin \mathfrak{q}[x]$.

Let $f \in (A/\mathfrak{q})[x]$ be a zero divisor. By Ex 1.2 iii), there exists some $a \in A/\mathfrak{q}$ such that $af = 0$. But then every coefficient of f must be a zero-divisor. Since $\mathfrak{q}$ is primary, and the coefficients lie in $A/\mathfrak{q}$, they must then all be nilpotent. Ex 1.2 ii) now tells us that f is nilpotent.

iv)

It follows from iii) that the primary decomposition of $\mathfrak{q}[x]$ is as given.

Non-containment follows from $q \in \left(\bigcap_{j \neq i} \mathfrak{q}_j\right) \setminus \mathfrak{q}_i \Rightarrow qx \in \left(\bigcap_{j \neq i} \mathfrak{q}_j[x]\right) \setminus \mathfrak{q}_i[x]$.

Similarly, we have that the prime ideals $\mathfrak{p}_i[x]$ belonging to the $\mathfrak{q}_i[x]$ are all different from one another by virtue of the $\mathfrak{p}_i$ being different from one another.

v)

We have that the minimal prime ideals belonging to $\mathfrak{a}[x]$ are the minimal elements among the set of prime ideals containing $\mathfrak{a}[x]$.

It follows from ii) that a minimal prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$ extends to a prime ideal $\mathfrak{p}[x]$ containing $\mathfrak{a}[x]$. We are done if we can show that all prime ideals containing $\mathfrak{a}[x]$ are extensions of prime ideals containing $\mathfrak{a}$.

Let $\mathfrak{b}$ be a prime ideal containing $\mathfrak{a}[x]$. Then $\mathfrak{b}^c = \mathfrak{b}/(x)$ is a prime ideal in A containing $\mathfrak{a}$. We also have $\mathfrak{b} \supseteq \mathfrak{b}^{ce} = (\mathfrak{b}/(x))[x]$. But clearly, $(\mathfrak{b}/(x))[x] \supseteq \mathfrak{b}$, so $\mathfrak{b}$ is an extension of a prime ideal in A which contains $\mathfrak{a}$.

Ex 8

First note that $\mathfrak{p}_i$ is maximal in $A_i = k[x_1, x_2, \ldots, x_i]$ since $A_i/\mathfrak{p}_i \cong A$ is a field. It follows that the powers of $\mathfrak{p}_i$ are $\mathfrak{p}_i$-primary. Ex 7 tells us that $\mathfrak{p}_i$ is prime in A_{i+1}, and that it's powers are $\mathfrak{p}_i$-primary in A_{i+1}. Repeating this argument inductively shows that this holds for all $A_j, j \geq i$.

Ex 9

Let $x \in A$ be a zero divisor such that $xa = 0$ for some $a \neq 0$. Then $x \in (0 : a)$, so any prime ideal which contains $(0 : a)$ must also contain x. There is at least one prime ideal containing $(0 : a)$ since $1 \cdot a \neq 0$ and any non-trivial ideal is contained in a maximal ideal, whence there is a minimal prime ideal which contains $(0 : a) \supseteq \{x\}$.

For the other inclusion, let $x \in \mathfrak{p} \in D(A)$. Then there is some $a \in A$ such that $\mathfrak{p}$ is minimal among the set of prime ideals which contain $(0 : a)$. Note that all elements in $(0 : a)$ are zero divisors, so the set $\mathfrak{p}' = \mathfrak{p} \cap D$ where D are the zero divisors in A, contains $(0 : a)$ as well. Moreover, it's an ideal since D is an ideal. We will show that it's prime as well. If $bc \in \mathfrak{p}'$ then either b or c is in $\mathfrak{p}$. Say that $b \in \mathfrak{p}$, we also have that b is a zero divisor, since bc is, so $b \in D$ also. By minimality of $\mathfrak{p}$, it follows that $\mathfrak{p}' = \mathfrak{p}$ and all elements in $\mathfrak{p}$ are zero divisors.

For the second part of the question, let $\mathfrak{p} \in D(S^{-1}A)$. Then $\mathfrak{p} \in \operatorname{spec}(S^{-1}A)$ by definition. Let $a/s \in S^{-1}A$ be such that $\mathfrak{p}$ is a minimal prime ideal containing $(0 : a/s) = (0 : a/1)$. Then Prop 3.11 iv) tells us that $\mathfrak{p}^c$ is a prime ideal. Since function inverses preserve inclusion, we have that $\mathfrak{p}^c$ contains $(0 : a/1)^c = (0 : a)^{ec} \supseteq (0 : a)$. Even more, if $\mathfrak{q}$ is a prime ideal such that $(0 : a) \subseteq \mathfrak{q} \subseteq \mathfrak{p}^c$, then $(0 : a/1) \subseteq \mathfrak{q}^e \subseteq \mathfrak{p}^{ce} = \mathfrak{p}$, and $\mathfrak{q}^e$ is prime by Prop 3.11 iv). But since $\mathfrak{p}$ is the minimal prime ideal containing, we have that $\mathfrak{q}^e = \mathfrak{p}$ whence we see that $\mathfrak{p}^c$ is a minimal ideal containing $(0 : a)$. In summary $\mathfrak{p}^c \in D(A) \cap \operatorname{spec}(S^{-1}A)$.

For the other inclusion, let $\mathfrak{p} \in D(A) \cap \operatorname{spec}(S^{-1}A)$. Then $\mathfrak{p}$ is a prime ideal in A, which doesn't meet S, and is a minimal prime ideal containing some $(0 : a)$. Then clearly, $\mathfrak{p}^e$ is a minimal prime ideal containing $(0 : a)^e = (0 : a/1)$ whence $\mathfrak{p}^e \in D(S^{-1}A)$.

For the third part of the question, Prop 4.5 tells us that each $\mathfrak{p}_i$ are the prime ideals occuring in the set $r(0 : a)$ for some $a \in A$. Furthermore, any prime ideal containing $(0 : a)$ must also contain $r(0 : a)$, so if $\mathfrak{p}_i = r(0 : a)$ is prime, then $\mathfrak{p}_i$ is minimal among the prime ideals containing $(0 : a)$. Also, if $r(0 : a)$, let $\mathfrak{p}$ be the minimal prime ideal containing $r(0 : a)$. The proof of Prop 4.5 tells us that $r(0 : a) = \bigcap \mathfrak{p}_i$, and thus $\mathfrak{p} \supseteq \bigcap \mathfrak{p}_i$, whence Prop 1.11 tells us that $\mathfrak{p}$ contains some $\mathfrak{p}_j$, but $\mathfrak{p}_j \supset (0 : a)$, so $\mathfrak{p} = \mathfrak{p}_j$ by minimality of $\mathfrak{p}$.

Ex 10

i)

Let $x \in S_{\mathfrak{p}}(0)$. Then there exists $a \in A - \mathfrak{p}$ such that $xa = 0$, so x is a zero divisor, and $x \in \mathfrak{p}$ since $0 \in \mathfrak{p}$ but $a \notin \mathfrak{p}$.

ii)

We have the following chain of equivalences

$$\begin{aligned}
\mathfrak{p} \text{ is a minimal prime ideal} &\xLeftrightarrow{\text{Cor 3.13}} A_{\mathfrak{p}} \text{ is a ring with only one prime ideal} \\
&\xLeftrightarrow{\text{Prop 1.8}} \mathfrak{p}^e = \operatorname{nilrad}(A_{\mathfrak{p}}) \\
&\Leftrightarrow \mathfrak{p} = r(S_{\mathfrak{p}}(0))
\end{aligned}$$

iii)

Let $x \in S_{\mathfrak{p}}(0)$. Then there exists $a \in A - \mathfrak{p}$ such that $xa = 0$. But then $a \in A - \mathfrak{p}' \supseteq A - \mathfrak{p}$, so $x \in S_{\mathfrak{p}'}(0)$.

iv)

Let $x \in \bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0)$. Then there exists an $a_{\mathfrak{p}} \in A - \mathfrak{p}$ for all $\mathfrak{p} \in D(A)$ such that $xa_{\mathfrak{p}} = 0$. In other words, the anihilator of x is not contained in any of the $\mathfrak{p} \in D(A)$. But any non-trivial ideal is contained in some prime ideal, so any non-trivial annihilator is contained in some ideal in $D(A)$. It follows that $(0 : x) = (1)$ and $x = 0$.

Ex 13

i)

First we show that $r(\mathfrak{p}^{(n)}) = \mathfrak{p}$. Let $x^m \in \mathfrak{p}^{(n)}$. Then there exists some $a \in A - \mathfrak{p}$ and $p \in \mathfrak{p}^n$ such $a(x^m - p) = 0 \Leftrightarrow ax^m = ap$. As $ax^m = ap \in \mathfrak{p}$, but $a \notin \mathfrak{p}$, we get $x \in \mathfrak{p}$.

Now we show that the ideal is primary, we have the following chain of equalities $(\mathfrak{p}^{(n)})^e = (\mathfrak{p}^n)^{ece} = (\mathfrak{p}^n)^e = (\mathfrak{p}^e)^n$, which shows that $(\mathfrak{p}^{(n)})^e$ is $\mathfrak{p}^e$-primary in A_p, since it's a power of a maximal ideal. We can now use Prop 4.8 ii) to conclude that $\mathfrak{p}^{(n)}$ is primary.

ii)

Let $\mathfrak{a}$ be the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$.
Let $\mathfrak{p}^n = \bigcap \mathfrak{q}_i$ be a minimal primary composition. Then

Ch 5

Ex 1

Let $V(E) \in \operatorname{spec}(B)$ be a closed set, and I be the radical ideal $I = \bigcap_{\mathfrak{p} \in V(E)}$ such that $V(E) = V(I)$. We will show that $f^*(V(E)) = f^*(V(I)) = V(f^{-1}(I))$. It's immedieate that $f^*(V(I)) \subseteq V(f^{-1}(I))$ since every prime ideal in $f^*(V(I))$ will contain $f^{-1}(I)$.

For the other inclusion, let $\mathfrak{p} \in V(f^{-1}(I))$ be a prime ideal in A. Then $f(\mathfrak{p})$ is a prime ideal in $f(A)$, and Theorem 5.10 tells us that there exists some prime ideal $\mathfrak{q}$ in B such that $f(\mathfrak{p}) = \mathfrak{q} \cap f(A)$. By applying f^{-1} we see that $f^{-1}(f(\mathfrak{p})) = f^*(\mathfrak{q})$. Now we claim that $\mathfrak{p} = f^{-1}(f(\mathfrak{p}))$. To see this, note that if $a \in A$ is such that $f(a) = f(p)$ for some $p \in \mathfrak{p}$, then $a - p \in \ker(f)$, but $\mathfrak{p}$ contains $\ker(f)$ since $\mathfrak{p}$ contains $f^{-1}(I)$. It follows that $a \in \mathfrak{p}$ and $\mathfrak{p} = f^{-1}(f(\mathfrak{p})) = f^*(\mathfrak{q}) \in f^*(V(I))$.

Ex 8

i)

The hint gives the whole solution, but we write it out in full anyway. Let $\overline{K}$ be the algebraic closure of the field of fractions of B. Then we can factor f, g in $\overline{K}[x]$ as

$$f = \prod (x - \chi_i),$$
$$g = \prod (x - \eta_i).$$

Then the χ_i, η_i are roots of fg (which is monic since f, g are), and they are all integral over C. C is integrally closed, so the χ_i, η_i all lie in C, whence the coefficients of the polynomials f, g do as well, so $f, g \in C[x]$.

ii)

We no longer have a full field of fractions, and we can't construct a field which contains B. What we can do, is to create a bigger ring $B^+ \supset B$, such that the polynomials f, g factor into linear factors over B^+, after which the remainder of the proof would come along just like above.

Let y_1 be a new formal indeterminate, and consider $B_1 = B[y_1]/f(y_1)$. In this ring, we have that $f(y_1) = 0$, so $x - y_1$ is a factor of $f = (x - y_1)f_1$ (since the polynomials are monic, the division algorithm still works in the non-field B). Now repeat this process starting with B_1, f_1, and continue this way until f is completely factored into linear factors, and then do the same for g. The final result is a ring B^+ containing B, where f, g splits into linear factors, from which we may proceed like in part i).

Ex 9

We proceed as advised by the hint. Let $f(x) \in B[x]$ be integral over $A[x]$ such that

$$f^n + f^{n-1}g_{n-1} + \dots g_0 = 0.$$

A small side-step, note that we can't apply Ex 8.ii) to

$$f(f^{n-1} + f^{n-2}g_{n-1} + \dots g_1) = -g_0 \in A[x],$$

since the polynomials $f, (f^{n-1} + f^{n-2}g_{n-1} + \dots g_1)$ need not be monic. To circumvent this issue, let $f_1 = f - x^r$ for some $r \geq \max(n, \deg(g_i), \deg(f))$. Then f_1 is monic and

$$(f_1 + x^r)^n + (f + x^r)^{n-1}g_{n-1} + \dots g_0 = 0.$$

Expanding the expression above yields

$$f_1^n + f_1^{n-1}h_{n-1} + \dots h_0 = 0.$$

where $h_0 = \sum_{i=0}^{n} x^{nr-ir}g_{n-i} \in A[x]$ with $g_n = 1$. We can now apply Ex 8.ii) to

$$f_1(f_1^{n-1} + f_1^{n-2}h_{n-1} + \dots h_1) = -h_0 \in A[x],$$

since f_1 is monic, and $(f_1^{n-1} + f_1^{n-2}h_{n-1} + \dots h_1)$ is monic as well since we picked r large enough. It follows that $f_1 \in C[x]$, so $f \in C[x]$ as well and we are done.

Ex 16

We pick up from the following task in the exercise text: "Show that x_n is integral over the ring $A' = k[x'_1, x'_2, \dots, x'_{n-1}]$."

It follows from the construction of the λ_i, x'_i that $f(x'_1+\lambda_1 x_n, x'_2+\lambda_2 x_n, \dots, x'_{n-1}+\lambda_{n-1}x_n, x_n) = 0$. Moreover, this polynomial has a leading coefficient $\lambda = H(\lambda_1, \lambda_2, \dots \lambda_{n-1}, 1) \neq 0 \in k$ when considered as a polynomial in $k[x'_1, x'_2, \dots, x_{n-1'}][x_n]$, and can therefore easily be made monic, whence x_n is algebraic over $k[x'_1, x'_2, \dots x'_{n-1}]$.

Ex 17

By reverse inclusion, maximal ideals correspond to minimal varieties, and by the proof in the exercise text, any proper ideals correspond to non-empty varieties. So maximal (proper by definition) correspond to minimal non-empty varieties. Points are non-empty minimal sets, and they are all varieties since they induce ideals of the form given in the text. It follows that all maximal ideals are of the given form as well.

Ex 31

Let $R = \{x \in K^* : v(x) \geq 0\} \cup \{0\}$. Then R is a sub ring of K since if $a, b \in R$, then

- $v(a+b) \geq \min(v(a), v(b)) \geq 0 \Rightarrow a+b \in R$
- $v(ab) = v(a) + v(b) \geq 0 \Rightarrow ab \in R$
- $v(1) = v(1^m) = mv(1)$ for all $m \in \mathbb{N}$ so $v(1) = 0$ and $1 \in R$. We also have $v(-1) = v(1)/2 = 0$.
- $v(-a) = v(-1 \cdot a) = v(-1) + v(a) = v(a) \geq 0$ so $-a \in R$.

It's an integral domain since it's a sub ring of a field. Finally, it's an evaluation ring, since $0 = v(1) = v(aa^{-1}) = v(a) + v(a^{-1})$ so if $v(a) < 0$ then $v(a^{-1}) > 0$ and vice versa (if $v(a) = v(a^{-1}) = 0$ then both $a, a^{-1} \in R$, which is allowed).

Ch 6

Ex 3

By Prop 6.3 we have M/N_i Noetherian, Cor 6.4 gives $(M/N_1) \oplus (M/N_2)$ Noetherian, and we are done after noting that $M/(N_1 \cap N_2) \cong (M/N_1) \oplus (M/N_2)$ (was an early exercise or proposition or something, follows by considering the kernel of $m \mapsto (m + N_1, m + N_2)$).

Ex 8

No, we can still have strict infinite ascending chains of ideals which aren't prime. But to aid us in our search for a counterexample, we'll explore if there is anything useful which must hold in such a scenario first.

Let A be a ring such that $\text{spec}(A)$ is Noetherian. Let

$$I_0 \subseteq I_1 \subseteq \dots$$

be a ascending chain of ideals. Then $V(I_i)$ is an ascending chain of closed subsets in $\text{spec}(A)$, hence it must be stationairy after some index $i = k$. From the definition of the spectrum toplogy, it follows that all ideals I_i from the point $i = k$ and on, are covered by the same prime ideals. Hence they have the same radical. Thus every ideal in the tail of the chain

$$I_k \subseteq I_{k+1} \subseteq \dots$$

lies in $r(I_k)$, and has $r(I_k)$ as its radical. Passing to $A' = A/I_k$ and writing $J_i = I_{k+i} + I_k \in A/I_k$, we get an infnite ascending chain

$$J_0 \subseteq J_1 \subseteq \dots$$

which is contained in the nilradical of A'. We will use this to craft a counter example.

Let k be a algebraically closed field of characteristic 0. Let $B = k[x_1, x_2, \dots,]$ be the polynomial ring in infinitely many indeterminates. Let $A = B/(x_1^2, x_2^2, \dots,)$ be the quotient ring where we've modded out all powers of x_i. Then

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \dots$$

is a strictly ascending infinite chain of ideals. However any prime ideal of A must contain the nilradical, which contains every x_i. But $(x_1, x_2, \dots)$ is maximal since we get k if we quotient by it. It follows that A only has one prime ideal, whence $\text{spec}(A)$ is Noetherian.

Ch 7

Ex 2

Suppose f is nilpotent and $f^k = 0$. Then the the 0-th coefficient of f^k is $a_0^k = 0$ so a_0 is nilpotent. Denothe the m-th coefficient of f^k by c_m. Then $c_{km} = a_m^k + E(k,m)$ where $E(k,m)$ is an sum where each term has a factor a_j^i for some $j < m, i \geq k$. By induction we have that $E(k,m) = 0$, so $a_m^k = 0$ for all m.

Now assume that all a_i are nilpotent. We need the following lemma before we show that f is nilpotent. Then $a_i \in r(0)$ by definition, and Corollary 7.14 tells us that there is some exponent such that $r(0)^k = 0$. It follows that any product of k elements from a_i is 0. But the coefficients of f^k are sums of k-element products of a_i, so $f^k = 0$.

Ex 11

No, but before we produce a counterexample, we'll try and find some properties which such a counterexample must exhibit.

Let A be a ring which is locally Noetherian. Let

$$I_1 \subsetneq I_2 \subsetneq \dots$$

be an infinite strictly ascending chain. Let $\mathfrak{p}$ be a prime ideal. Then the chain of I_i^e is stationary in $A_\mathfrak{p}$ by hypothesis. Let k be the index from which all the I_i extend to the same ideal in $A_\mathfrak{p}$. Since the original chain is strictly ascending, there exists $x \in I_k \setminus I_{k+1}$. Since $x/1 \in I_{k+1}^e$, there exist some $y \in I_{k+1}$ such that $x/s = y/t$ for $s, t \in A - \mathfrak{p}$ and this is the case if and only if we have $u \in A - \mathfrak{p}$ such that $u(xt - ys) \in A - \mathfrak{p}$. If the ideal $x, y \in \mathfrak{p}$, we'd have that $u(xt - ys) \in \mathfrak{p}$ a contradiction. So there can be no ideal which contains the entire chain, since such an ideal would be contained in a maximal (I.e prime) ideal. In particular, the union of the chain can't be an ideal.

Even more, every ideal in the chain has to be contained in infinitely many maximal ideals. Suppose towards a contradiciton I_r is some ideal only contained in the finitely many maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$. Then there has to be some index $r_1 > r$ such that $I_{r_1} \not\subset \mathfrak{m}_1$, as otherwise $\mathfrak{m}_1$ would contain the whole chain. Let $R = \max(r_i)$. Since I_R is an ideal, it's contained in some maximal ideal $\mathfrak{m}_R$ which isn't equal to any other $\mathfrak{m}_{r_i}$ since none of them contain I_R. But then since $I_R \supset I_r$, we have that $\mathfrak{m}_R \supset I_r$, a contradiction.

Let k be an algebraically closed field of characteristic 0 and $A = \prod_{i=1}^{\infty} k$. First, note that

$$((1,0,0,0,\dots,)) \subsetneq ((1,1,0,0,\dots,)) \subsetneq ((1,1,1,0,\dots,)) \subsetneq ((1,1,1,1,0,\dots,))$$

is an infinite chain of strictly increasing ideals in A, whence A isn't Noetherian.

The maximal ideals coincide with the prime ideals in A and are precisely the $\mathfrak{m}_i$ where

$$\mathfrak{m}_\mathfrak{i} = ((1,1,1,\dots,1,0,1,1,1,\dots))$$

where the 0 is at index i. To see this, first note that since k is a field, any ideal in A can be identified with the indices where it has non-zero elements. Then suppose $\mathfrak{a}$ is an ideal in A which is zero at more than one index. Say at least indices i and j, $i \neq j$. Then $A/\mathfrak{a}$ contains the zero divisors $(1_i + \mathfrak{a})(1_j + \mathfrak{a}) = 0$ (where 1_i is the element of all zeros except for a 1 at index i). So any prime ideal is of the form $\mathfrak{m}_i$. The $\mathfrak{m}_i$ are maximal since they induce fields as quotient rings.

Let's dig a little deeper and investigate what the induced localizations look like. Let $f : A \to A_{\mathfrak{m}_1}$ be the canonical injection and consider some elements $x, y \in A$. Then $f(x) = f(y)$ if and only if there are $u, s, t, \in A \setminus \mathfrak{m}_1$ such that $u(xt - ys) = 0$. Let $p : A \to k$ denote the projection onto the first entry. I.e $p(a, b, c, \dots,) = a$. Let $i : k \to A$ denote the corresponding injection. The elements in $A \setminus \mathfrak{m}_1$ are precisely those which have a non-zero entry in the first index, and if $p(x) =$

$X, p(y) = Y$ where $X, Y \neq 0$, then $i(1)(xi(X^{-1}) - yi(Y^{-1})) = 0$. If both $X = Y = 0$, then picking $u = s = t = i(1)$ also results in $u(xt - ys) = 0$. If only one of X, Y are non-zero, then we can't pick $u, s, t \in A \setminus \mathfrak{m}_1$ such that $u(xt - ys) = 1$. So in other words, $f(x) = f(y)$ precisely when either both $x, y \in \mathfrak{m}_1$ or both $x, y \notin \mathfrak{m}_1$. So each $A_{\mathfrak{m}_i}$ is a ring of two elements, hence isomorphic to $\mathbb{F}_2$, whence they are all clearly Noetherian.

Ex 14

We flesh out the steps in the hint to show that $r(\mathfrak{a}) \supseteq I(V(\mathfrak{a}))$.

Let $f \in A \backslash r(\mathfrak{a})$. Then since $r(\mathfrak{a})$ is the intersection of all prime ideals containing $\mathfrak{a}$, there must be some prime ideal $\mathfrak{p}$ which does not contain f. Let $\overline{f}$ be the image of f in $B = A/\mathfrak{p}$. Let $C = B_f = B[1/\overline{f}]$ and let $\mathfrak{m}$ be a maximal ideal of C. C as a k-algebra is finitely generated by $1/\overline{f}, \overline{t_1}, \overline{t_2}, \ldots, \overline{t_n}$, and Corollary 7.9 can be applied to yield $C/\mathfrak{m} \cong k$. Let $\varphi : A \to k$ be given by the composition

$$A \xrightarrow{a} B = A/\mathfrak{p} \xrightarrow{b} C = B_f \xrightarrow{c} C/\mathfrak{m} \xrightarrow{d} k.$$

Then $(\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_n))$ defines a point in k^n. Moreover, consider any $g \in \mathfrak{a}$. We have that $\varphi(g(t_1, t_2, \ldots, t_n)) = 0$ since $a(g(t_1, t_2, \ldots, t_n)) = 0$. But all of the a, b, c, d are homomorphisms and commute with algebraic operations (which is what polynomials really are), so $g(\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_n)) = 0$ and $(\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_n)) \in V(\mathfrak{a})$. We also have $\varphi(f(t_1, t_2, \ldots, t_n)) \neq 0$, since $b(a(f(t_1, t_2, \ldots, t_n)))$ is a unit, and $k \neq 0$, so $x = (\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_n))$ is a point in $V(\mathfrak{a})$ such that $f(x) \neq 0$ whence $f \notin I(V(\mathfrak{a}))$.

Ch 9

Ex 2

First of, every coefficient in fg is a sum of terms of the form $a_i b_j$ with $a_i \in c(f), b_j \in c(g)$ so $c(fg) \subseteq c(f)c(g)$.

(the arguments in this paragraph uses lots of results from Prop 9.2 and the proof of it without reference.) For the other inclusion, first consider the case when A is a discrete valuation ring. Let x generate the maximal ideal $\mathfrak{m}$ in A, and let v be the valuation on A where $v(a) = k \Leftrightarrow (a) = (x^k)$. Then $v(a + b) = \min(v(a) + v(b))$, and there exist a_i, b_j such that $v(a_i b_j) = v(c(f)c(g))$. But $a_i b_j$ is a term in the sum which is the $i+j$-th coefficient of fg, so $v(c(fg)) \leq v(a_i b_j)$, and it follows that $c(fg) \supseteq (x^{v(a_i b_j)}) = c(f)c(g)$, whence $c(fg) = c(f)c(g)$.

We give a quick lemma before we generalize to Dedekind domains.

Lemma 0.7. Let A be a Dedekind domain, $\mathfrak{m}$ a prime ideal and $\mathfrak{a}$ an arbitrary ideal. Then the exponent of $\mathfrak{m}$ in the prime ideal factorisation of $\mathfrak{a}$ is precisely the exponent of $\mathfrak{a}^e = (\mathfrak{m}^e)^r$ in the localization $A_\mathfrak{m}$ (where we allow $r = 0$).

Proof. Let $\mathfrak{m}$ be a primary ideal belonging to $\mathfrak{a}$, and $r, \mathfrak{b}$ be such that $\mathfrak{a} = \mathfrak{m}^r \mathfrak{b}$ and r is maximal. It follows that $\mathfrak{b}$ isn't contained in $\mathfrak{m}$. Then by 3.11.ii) and 3.11.iv), we have

$$\begin{aligned}(A - \mathfrak{m})^{-1}\mathfrak{a} &= ((A - \mathfrak{m})^{-1}\mathfrak{m})^r (A - \mathfrak{m})^{-1}\mathfrak{b} \\ &= ((A - \mathfrak{m})^{-1}\mathfrak{m})^r\end{aligned}$$

□

Now let A be a Dedekind domain. Since Gauss's lemma holds in discrete valuation rings, we see that $c(f)c(g)$ extends to the same ideal as $c(fg)$ in every localization. It follows that they have the same prime factorization, whence they must be equal.

Ex 9

First we show that the problem statement is equivalent to the given sequence being exact. This follows from the fact that $x \in A$ is a solution to the equation system induced by $\varphi(x) = (x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n)$, and if the sequence is exact, then any elements in $\bigoplus_{i=1}^n A/\mathfrak{a}_i$ are in in the image of φ precisely when they are in the kernel of ψ. $(x_1 + \mathfrak{a}_1, \ldots, x_n + \mathfrak{a}_n)$ is in the kernel of ψ exactly when $x_i - x_j + \mathfrak{a}_i + \mathfrak{a}_j = 0$, and we see that the two formulations of the problem are equivalent.

Prop 3.8 tells us that exactness is a local property (it also tells us that it's enough to localize at maximal ideals, so we may skip localizing at (0)), and since all localizations of a Dedekind domain are discrete valuation rings, we show that the sequence is exact when A is a DVR.

Since A is a DVR, we may write $\mathfrak{a}_i = (a)^{k_i}$ where a generates the maximal ideal of A. It also follows that $\mathfrak{a}_i + \mathfrak{a}_j = (a)^{k_{ij}}$ where $k_{ij} = \max(k_i, k_j)$. Now, since all x_i are in the equivalence class of x in their corresponding quotient rings, we can write $\varphi(x) = (x + (a)^{k_1}, \ldots, x + (a)^{k_n})$. Then we see that $\psi(\varphi(x))$ consists of entries $x - x + (a)^{k_i} + (a)^{k_j} = x - x + (a)^{k_{ij}} = (a)^{k_{ij}}$, so $\mathrm{im}(\varphi) \subseteq \ker(\psi)$. Now let $x_i \in A$ and $x' = (x_1 + (a)^{k_1}, \ldots, x_n + a^{k_n})$ be such that $x' \in \ker(\psi)$. Then $x_i - x_j \in (a)^{k_{ij}}$ for every $i \neq j$. Let $s \in [1..n]$ be such that k_s is maximal. Then every single x_i is congruent to x_s modulo (a^{k_s}), and since every other $(a)^{k_i} \supseteq (a)^{k_s}$, they are all congruent to x_s modulo every $(a)^{k_i}$. Pick some representative $x \in x_s + (a)^{k_s}$. Then $\varphi(x) = x'$, and we see that $\mathrm{im}(\varphi) = \ker(\psi)$.

Ch 10

Ex 9

We follow the hint. Let $\varphi : A[x] \to (A/\mathfrak{m})[x]$ be the natural homomorphism and g_0, h_0 be elements in the preimages of $\overline{g}, \overline{h}$. Then $\varphi(g_0 h_0) = \overline{f}$ by definition.

Hence $g_0h_0 - f \in \ker(\varphi) = \mathfrak{m}A[x]$. Now inductively assume that we have g_{k-1}, h_{k-1} in the preimages of $\overline{g}, \overline{h}$ such that $g_{k-1}h_{k-1} - f \in \mathfrak{m}^{k-1}A[x]$. We will construct g_k, h_k with analogous properties and the same degrees. We will let $p, q \in A[x]$ be generic polynomials and set $g_k = g_{k-1} + p, h_k = h_{k-1} + q$. We then want to set p, q in such a way that

$$\begin{aligned} f - g_k h_k &\in \mathfrak{m}^k A[x], \\ g_k - g_{k-1} &\in \mathfrak{m}^{k-1} A[x], \\ h_k - h_{k-1} &\in \mathfrak{m}^{k-1} A[x], \\ \deg(p) &\leq \deg(g_{k-1}), \\ \deg(q) &\leq \deg(h_{k-1}). \end{aligned}$$

If we can do this, then it follows that the i-th coefficients of (g_j) forms a Cauchy sequence in A, and since A is $\mathfrak{m}$-adically complete, we have that there is some $g \in A[x]$ which is sent to (g_j) by the natural map $A[x] \to \hat{A}[x]$. The first condition implies that $(g_j)(h_j) = f$ in $\hat{A}[x]$, and it follows that $gh = f$. We now turn our efforts to finding p, q which fulfil the requirements above.

Since $f - g_{k-1}h_{k-1} \in \mathfrak{m}^{k-1}A[x]$, we can write $f - g_{k-1}h_{k-1} = \sum c_i x^i$ where $c_i \in \mathfrak{m}^{k-1}A[x]$. Our requirements can now be rewritten into

$$\begin{aligned} pq + ph_k + qg_k - \sum c_i x^i &\in \mathfrak{m}^k A[x], \\ p &\in \mathfrak{m}^{k-1} A[x], \\ q &\in \mathfrak{m}^{k-1} A[x]. \end{aligned}$$

Since $\varphi(g_{k-1}), \varphi(h_{k-1})$ are monic, they generate the trivial ideal (1) in $(A/\mathfrak{m})[x]$, and it follows that there exist $a_i, b_i \in A[x]$ such that $a_i g_{k-1} + b_i h_{k-1} = x_i + r_i$ where $r_i \in \mathfrak{m}A[x]$. Moreover, since $A/\mathfrak{m}$ is a field, it follows that $\deg(a_i) \leq \deg(h_{k-1}), \deg(b_i) \leq \deg(g_{k-1})$. If we let $p = \sum b_i c_i, q = \sum a_i c_i$ we get that both $p, q \in \mathfrak{m}^{k-1}A[x]$, the degrees are sufficiently low, and

$$\begin{aligned} pq + ph_k + qg_k - \sum c_i x^i &= pq + \sum c_i (b_i h_k + a_i g_k - x_i) \\ &= pq + \sum c_i r_i \\ &\in \mathfrak{m}^{(k-1)^2} A[x] + \mathfrak{m}^{k-1}\mathfrak{m}A[x] \\ &= \mathfrak{m}^k A[x], \end{aligned}$$

whence we are done.

Ch 11

Ex 6

We begin with showing the inequality $1 + \dim A \leq \dim A[x]$. Let

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n$$

be a strictly increasing chain of prime ideals in A. Then let $\mathfrak{q}_i = \mathfrak{p}_i(x)$ be the ideal of polynomials in $A[x]$ with coefficients in $\mathfrak{p}_i$. Then every $\mathfrak{q}_i$ is a prime ideal, since $(A/\mathfrak{p}_i)[x] \cong A[x]/\mathfrak{q}_i$, and $A/\mathfrak{p}_i$ is a domain whence $(A/\mathfrak{p}_i)[x] = A[x]/\mathfrak{q}_i$ is aswell. We can define $\mathfrak{q}_{n+1} = (x) + \mathfrak{q}_n$ to be the ideal of polynomials where the constant coefficient is in $\mathfrak{p}_n$. This is prime, since it's the preimage of the prime ideal $\mathfrak{p}_n$ under the quotient map by (x). Moreover, $\mathfrak{q}_{n+1} \supsetneq \mathfrak{q}_n$ and we have a strictly increasing chain of length $n+1$ in $A[x]$. It follows that $1 + \dim A \leq \dim A[x]$.

We use the hint to show the second inequality. Let $\mathfrak{p}$ be a prime ideal in A. By Ex 3.21.iv), we have that the spectrum of the fiber of $\mathfrak{p}$, I.e the prime ideals in $A[x]$ which contract to $\mathfrak{p}$, is homeomorphic to the spectrum of $k(\mathfrak{p}) \otimes_A A(x)$ which in turn is isomorphic to $k(\mathfrak{p})[x]$, a field. Since $k(\mathfrak{p})[x]$ has two ideals, $(0), (x)$, it follows that there are exactly two prime ideals in $A[x]$ which contract back to $\mathfrak{p}$. It's easy to verify that these are given by $\mathfrak{p}A[x]$ (Ex 4.7.ii)), and $\mathfrak{p}+(x)$.

Any chain of strict inclusions in $A[x]$ must contract back to a chain of (not necessarily strict) inclusions in A. It then follows that after removing duplicates in the contracted chain, it can never be more than half as short (modulo odd length chains in $A[x]$, for which we have to round up the length after halving it), and the second inequality follows.

Ex 7

Let $\mathfrak{p}$ be a prime ideal of height m in A. The hint claims that we can pick $a_1, a_2, \ldots, a_m \in A$ such that $\mathfrak{p}$ is a minimal ideal of $(a_1, a_2, \ldots, a_m)$. To see why, note that $\dim(A_\mathfrak{p}) = m$, so by the Dimension Theorem of Noetherian Local Rings, we have that there is some $\mathfrak{p}^e$-primary ideal $\mathfrak{q}$ in $A_\mathfrak{p}$ generated by m elements $\mathfrak{q} = (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_m)$. When we contract back to A, this ideal is $\mathfrak{p}$-primary since inverse images of primary ideals are primary, and by Prop 1.8 & 3.11.iv) we have

$$\begin{aligned} r(\mathfrak{q}^c) &= r(\mathfrak{q})^c \\ &= \mathfrak{p}^{ec} \\ &= \mathfrak{p}. \end{aligned}$$

Moreover, given $a_i \in f^{-1}(\overline{a}_i)$, we have that $\mathfrak{q}^c = (a_1, a_2, \ldots, a_m)$ since

$$a_i \in f^{-1}(\overline{a}_i) \subset f^{-1}(\mathfrak{q}) = \mathfrak{q}^c \subseteq \mathfrak{p},$$

so no element of $A - \mathfrak{p}$ is a zero-divisor in $A/(a_1, a_2, \ldots, a_m)$ whence $(a_1, a_2, \ldots, a_m)$ is a contracted ideal. We've shown the that the first statement proposed in the hint holds. Denote $\mathfrak{a} = \mathfrak{q}^c$.

Continuing along with the hint, Ex 4.7.iv) tells us that $\mathfrak{p}A[x]$ is a minimal ideal of $\mathfrak{a}A[x]$, so the height of $\mathfrak{p}A[x]$ is $\leq m$ by Corollary 11.16 (the a_i generate $\mathfrak{a}A[x]$

as well). It follows that $\mathfrak{p}A[x]$ has height m. Suppose that we picked $\mathfrak{p}$ to be a maximal ideal with maximal height in A. Then the only prime ideal in $A[x]$ containing $\mathfrak{p}A[x]$ is $\mathfrak{p}+(x)$, since we saw in Exercise 6 that this is the only other prime ideal in $A[x]$ which contracts to $\mathfrak{p}$, and $A[x]/(\mathfrak{p}+(x)) = A/\mathfrak{p}$ is a field, so $\mathfrak{p}+(x)$ is maximal. It follows that the first bound in Exercise 6 is tight in the Noetherian case and we are done.