Exercise A3.1

Let $a : N \to M$ be a monomorphism, M essential over N, and $b : M \to E$ be a morphism which restricts to a monomorphism on $a(N)$. Then $\ker(b) \cap a(N) = 0$, and as M is essential over N, this implies that $\ker(b) = 0$. Indeed, any non-trivial submodules of M intersect $a(N)$ non-trivially.

Exercise A3.2

(a)

Let $Q_i, i \in I$ be a family of injective R-modules, M, N be R modules, $a : M \to N$ an injective R-linear map, and $b : M \to Q = \prod_{i \in I} Q_i$ a R-linear map. Then if we let $b_i = \pi_i \circ b : M \to Q_i$, we have a morphism $c_i : N \to Q_i$ which agrees with b_i on M since Q_i is in injective. It now follows from the universal property of Q that there is a unique morphism $c : N \to Q$ such that $c_i = \pi_i \circ c$. If we collect all our results, we get that

$$\pi_i \circ c \circ a = \pi_i \circ b,$$

whence $c \circ a = b$ since π_i is an epimorphism, hence Q is injective.

An alternate picture, which emphasizes the component-wise picture of the problem is given as follows. We have the following diagram,

$$\begin{tikzcd} M \arrow[r, hook, "a"] \arrow[d, "b"'] & N \\ Q, & \end{tikzcd}$$

which we can split up for each injective component Q_i of Q as

$$\begin{tikzcd} M \arrow[r, hook, "a"] \arrow[d, "b_i"'] & N \arrow[dl, dashed, "\exists c_i"] \\ Q_i & \end{tikzcd}$$

and we can then let $c : N \to Q$ be the morphism $c : n \mapsto (c_i(n))_{i \in I}$. This shows that we indeed need that Q is the direct product, and the same construction can't be used for a direct sum of injective modules, since $c_i(n)$ can be non-zero for infinitely many $i \in I$. If however we could guarantee that $c_i(n) \neq 0$ for all but finitely many $i \in I$, then this proof would work when Q is the direct product as well, which leads us into the second part of the question.

Suppose that R is Noetherian, and that $Q = \bigoplus_{i\in I} Q_i$ where each Q_i is an injective R-module, and that we have R-modules and morphisms as in the following diagram

$$\begin{tikzcd} I \arrow[r, hook, "a"] \arrow[d, "b"'] & R \\ Q, \end{tikzcd}$$

where I is an ideal in R. Then since R is Noetherian, I is finitely generated by say $(f_1, \dots, f_m)$. Let $i_j \in I$ be the index such that $a(f_j) \in Q_{i_j}$. It then follows that $a(I) \subseteq \bigoplus_{j=1}^{m} Q_{i_j}$, hence we have maps c_i

$$\begin{tikzcd} M \arrow[r, hook, "a"] \arrow[d, "b_i"'] & N \arrow[dl, dashed, "\exists c_i"] \\ Q_i \end{tikzcd}$$

such that $c_i = 0$ whenever $i \neq i_j$ for some $j \in [1..m]$, and

$$\begin{tikzcd} I \arrow[r, hook, "a"] \arrow[d, "b"'] & R \arrow[dl, "c: r\mapsto \sum_{j=1}^{m} c_{i_j}(r)"] \\ Q, \end{tikzcd}$$

is a commutative diagram, which shows that Q is injective by Lemma A.3.4.

For the other direction, suppose that R is a ring, $Q_i, i \in I$ is a family of injective R-modules, and that $Q = \bigoplus_{i\in I} Q_i$ is a non-injective R-module. Then there exist an ideal $I \subset R$ and maps

$$\begin{tikzcd} I \arrow[r, hook, "a"] \arrow[d, "b"'] & R \\ Q, \end{tikzcd}$$

such that b doesn't extend to R. It follows that I must be infinitely generated, as otherwise we could extend b like above, hence R isn't Noetherian.

(b)

Let R be a Noetherian ring and Q be an injective R-module. Let $Q' = \bigoplus_{i\in I} Q_i \subseteq Q$ be a maximal direct sum of indecomposable injective submodules.

Such Q' exists by Zorn's lemma, since $0 \subset Q$ is a direct sum of indecomposable injective submodules, and if $A_0 \subset A_1 \subset A_2 \subset \dots$ is a chain of such sums, then so is $\bigcup A_i$.

Our objective is to show that $Q = Q'$. By part (a), Q' is injective, so $Q = Q' \oplus Q''$ since injective morphisms from injective modules split. But, Q'' is injective as well since $Q'' \cong Q/Q'$ is a quotient of an injective module. Now, Q'' can't contain any indecomposable injective submodule, as this would contradict the maximality of Q'.

Now suppose towards a contradiction that we have some $m \in Q''$, and let $E = E(Rm)$ be the injective envelope of Rm. We claim that E is indecomposable. To see this, suppose that $E = E' \oplus E''$, and that $m \in E'$. Then $Rm \subseteq E'$, so $E'' \cap Rm = 0$ since $E = E' \oplus E''$, whence $E'' = 0$ since E is an essential extension of Rm. We've found an indecomposable injective module $E \subseteq Q''$, a contradiction, hence $Q'' = 0$ and we are done.

Exercise A3.3

We showed in the previous exercise that $E(R/P)$ is an indecomposable injective.

Now suppose that E is an indecomposable injective module, and that P is some associated prime of E. Then $E(R/P)$ is an injective submodule of E, hence a direct summand of E as injective morphisms from injective modules split. Since E is irreducible, it follows that $E = E(R/P)$.

Now let $P \neq Q$ be two prime ideals and suppose towards a contradiction that $E = E(R/P) = E(R/Q)$. Then let $x_P \in \operatorname{ann}(P)$ and $x_Q \in \operatorname{ann}(Q)$. We claim that $M_P = Rx_P, M_Q = Rx_Q$ are two submodules with trivial intersection. To see this, note that e
Then $R/P, R/Q$ are both submodules of E, and we claim that they don't intersect. To see this, let $x \in R/P \cap R/Q$. Then $P \cup Q \subset \operatorname{ann}(x)$
We showed above that when E is an injective indecomposable, then $E = E(R/P)$ for any associated prime P of E.
If we can show that P is the only associated prime of E, then it follows that $Q \neq P$ implies $E(R/Q) \neq E(R/P)$, since other wise R/Q would be a submodule of $E(R/P)$.

So, let Q be an associated prime of $E = E(R/P)$.
We showed in the previous part that this implies that $E = E(R/P) = E(R/Q)$. Our goal is to show that $Q = P$. Let $x \in E$ be such that $\operatorname{ann}(x) = Q$. Since E is an essential extension of R/P, we must have some $rx \in Rx \cap R/P$. But then $rxP = 0$ so $rP \subseteq \operatorname{ann}(x) = Q$.
By symmetric arguments (since $E = E(R/P) = E(R/Q)$), we can find r' such $r'Q \subseteq P$. Then
TODO: Finnish after reading about primary decomposition of modules

Exercise A3.4

(a)

Let R be a Noetherian ring, $P \subset R$ a prime ideal, $I \subset R$ an arbitrary ideal and $\phi : I \to R/P$ some R-linear map. Then if $p \in P, i \in I$, we have $\phi(pi) = p\phi(i) = 0$, so the kernel of ϕ contains PI.

Now let's apply the Artin-Rees lemma. Consider the P-filtration

$$R \subset P \subset P^2 \subset \dots,$$

and the R-submodule I. By the lemma, the filtration

$$I \subset I \cap P \subset I \cap P^2 \subset \dots$$

is P-stable. Hence there exist $d \in \mathbb{N}$ such that $I \cap P^{d+r} = P^r I \cap P^{d+r}$ for all $r \in \mathbb{N}$. In particular, $P^r I \subset PI$ so $I \cap P^d \subset PI \subseteq \ker(\phi)$, hence ϕ factors through $\frac{I}{I \cap P^d} = \frac{P^d + I}{P^d}$.

Now let $E' = \{m \in E : p^k m = 0 \text{ for some } p \in P, k \in \mathbb{N}\}$. Then E'

Exercise A3.8

We need to show to things,

1. That the resulting z is independent of our choice of y.
2. That the resulting z is independent on which representative x we choose from the homology class of x. I.e that any element in $\mathrm{im}(\phi_i'')$ is sent to $\mathrm{im}(\phi_{i-1}')$.

We begin with showing (1), I.e that the given map is a well-defined morphism $F_i'' \to HF_{i-1}'$. Suppose that $y' \in F_i$ is another choice of element in F_i such that $\beta_i(y') = x$. Then $y - y' \in \ker(\beta_i) = \mathrm{im}(\alpha_i)$, so there is some $w \in F_i'$ such that $\alpha_i(w) = y - y'$. We now have that

$$\begin{aligned} z - z' &= \alpha_{i-1}^{-1}\phi_{i-1}(y) - \alpha_{i-1}^{-1}\phi_{i-1}(y') \\ &= \alpha_{i-1}^{-1}\phi_{i-1}(y - y') \\ &= \alpha_{i-1}^{-1}\phi_{i-1}\alpha_i(w) \\ &= \phi_{i-1}'\alpha_{i-1}^{-1}\alpha_{i-1}(w) \\ &= \phi_{i-1}'(w) \\ &\in \mathrm{im}(\phi)'. \end{aligned}$$

To show (2), let $x' \in \mathrm{im}(\phi_i'')$. Then there is $u \in F_{i+1}''$ such that $\phi_i''(u) = x'$. As β_{i+1} is surjective, we have some $v \in F_{i+1}$ such that $\beta_{i+1}(u) = v$. Then by the commutativity of the diagram, $\phi_i(u) \in \beta_i^{-1}(x')$ so we can chose $y = \phi_i(u)$ The $\phi_{i+1}(y) = \phi_{i+1}\phi_i(u) = 0$, and the resulting z is 0 as well since α is a monomorphism.