Ch 1

Ex 1.1

We have that
$$p + \mathbf{v}_{pq} + \mathbf{v}_{qr} = q + \mathbf{v}_{qr} = r = p + \mathbf{v}_{pr},$$
so by A3, we see that $\mathbf{v}_{pq} + \mathbf{v}_{qr} = \mathbf{v}_{pr}$.

Ex 1.2

We have that
$$\begin{aligned}\phi \circ \phi^{-1}(u,v) &= \phi\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)\\ &= \frac{u^2+v^2+1}{u^2+v^2+1-u^2-v^2+1}\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}\right)\\ &= \frac{1}{2}(2u,2v)\\ &= (u,v),\end{aligned}$$
and since ϕ is bijective, we have that ϕ^{-1} is its two sided inverse.

Ex 1.3

$x \mapsto x^3$ is bijective on $\mathbb{R}$, and it follows that it induces a chart on $\mathbb{R}$.

Ex 1.4

ϕ_{x+} is injective since x is determined by y up to sign, after which it's fully determined by the requirement $x > 0$ on the codomain. Moreover, its image is open since it's $(-1,1) \times \mathbb{R}$. The other set and function pairs are charts by analogous arguments, and it's easy to see that their union is C.

Ex 1.5

A1 was shown in Ex 1.4. For A2, note that $U_{x+} \cap U_{x-} = \emptyset$, and that $\phi_{x+}(U_{x+} \cap U_{y+}) = (0,1) \times \mathbb{R}$ and similarly for the other combinations. For A3 we have
$$\phi_{x+} \circ \phi_{y+}^{-1}(x,z) = \phi_{x+}\left(x, \sqrt{1-x^2}, z\right) = \left(\sqrt{1-x^2}, z\right)$$
which is infinitely differentiable in both variables on the domain $(0,1) \times \mathbb{R}$. Similar arguments shows smoothness for the other combinations.

Ex 1.7

Suppose that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are atlases. Then $\mathcal{A}_i$ is compatible with itself, since any new chart transition maps are identities. It's easy to see that the relation of being compatible is symmetric, since the order of listing has no effect. Moreover, suppose that $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_2, \mathcal{A}_3$ are compatible, and consider $U_i \in \mathcal{A}_1, V_j \in \mathcal{A}_3$. Then for any $W_k \in \mathcal{A}_3$ we have that $\phi_i \circ \phi_k^{-1}, \phi_k \circ \phi_j^{-1}$ are both infinitely smooth on $\phi_k(W_k \cap U_i), \phi_j(V_j \cap W_k)$ respectively. We can restrict these domains further to obtain infinite smoothness on $\phi_k(V_j \cap W_k \cap U_i), \phi_j(V_j \cap W_k \cap U_i)$, and since $\phi_k \circ \phi_j^{-1}(\phi_j(V_j \cap W_k \cap U_i)) = \phi_k(V_j \cap W_k \cap U_i)$, we see that the composition of the two functions $\phi_i \circ \phi_j^{-1}$ is infinitely smooth at $\phi_j(V_j \cap W_k \cap U_i)$ as well. Since $\mathcal{A}_2$ covers M, we see that $\phi_i \circ \phi_j^{-1}$ is infinitely smooth on all of $\phi_J(V_j \cap U_i)$ and we are done.

Ex 1.13

Let $f : M \to N$ be a smooth map, and V be open in N. Let $U = f^{-1}(V)$, and for each $p \in U$, let $U_p \subseteq M, \phi_p : U_p \to \mathbb{R}^m, V_p \subseteq N, \psi_p : V_p \to \mathbb{R}^n$ be admissible charts such that $p \in U_p$ and $f(U_p) \subseteq V_p$. Then $U = f^{-1}(V) = \bigcup f^{-1}(V \cap V_p)$, hence U is open if every $f^{-1}(V \cap V_p)$ is. Now, as $V \cap V_p$ is open in N we have $B = \psi_p(V \cap V_p)$ open in $\mathbb{R}^n$, hence $A = \phi_p \circ f \circ \psi_p^{-1}(B) = \phi_p(U \cap U_p)$ is open in $\mathbb{R}^n$ (since $\phi_p \circ f \circ \psi_p^{-1}$ is C^∞), whence $U \cap U_p = \phi^{-1}(B)$ is open in M.

Ex 1.14

Let $p \in M_1$ and $(U_1, \phi_1), (U_2, \phi_2)$ be admissible charts on M_1, M_2 respectively such that $p \in U_1, f_{12}(U_1) \subseteq U_2$ and $\phi_1 \circ f_{12} \circ \phi_2^{-1}$ is C^∞. Then let $(V_2, \psi_2), (V_3, \psi_3)$ be admissible charts on M_2, M_3 with $f_{23}(V_2) \in V_3$ and all of the above. Then $U_2 \cap V_2$ is an open set in M_2 containing $f_{12}(p)$, hence $f_{12}^{-1}(U_2 \cap V_2)$ is an open set in M_1 containing p and $f_{23} \circ f_{12} \circ f_{12}^{-1}(U_2 \cap V_2) = f_{23}(U_2 \cap V_2)$ is contained in $f_{23}(V_2)$ which is contained in V_3. The open sets induce charts from the already given charts via restrictions of the coordinate functions.

Finally, since U_2, V_2 are two admissible charts on M_2, we have that $\psi_2 \circ \phi_2^{-1}$ is C^∞, hence

$$\psi_3 \circ f_{23} \circ f_{12} \circ \phi_1^{-1} = (\psi_3 \circ f_{23} \circ \psi_2^{-1}) \circ (\psi_2 \circ \phi_2^{-1}) \circ (\psi_2 \circ f_{12} \circ \phi_1^{-1})$$

is C^∞ as well as it's a composition of three C^∞ functions.