List 1

Exercise 1

(c)

Let V, T be a k-vector space. Then we can consider V as a $k[t]$ module by letting t act on V via $t^k v \mapsto T^{\circ k}(v)$ for $v \in V$, where $T^{\circ k}$ denotes k repeated applications.

Now let M be a $k[t]$ module. Then we can consider M as a k-space, and multiplication by t, $\cdot t : m \mapsto tm$ as a k-linear map.

These two procedures are clearly inverse each other, and so $k[t]$ modules are the same as pairs of V, T of a k-space and an endomorphism on it.

(d)

Let M be a $k[G]$-module. Then for each g in G, define $\rho_g \in \mathrm{Aut}_k(M)$ by $\rho_g : m \mapsto gm$. Note that ρ_g has an inverse $\rho_{g^{-1}}$, and so really is an automorphism. Then the map $g \mapsto \rho_g$ is a group homomorphism, as $\rho_e = I$ and

$$\rho_{gf}(m) = gfm = g\rho_f(m) = \rho_g \rho_f(m).$$

Now let $\rho : G \to \mathrm{Aut}_k(V)$ be a G-representation over some k-space V. Then we can turn V into a $k(G)$-module by letting $g \in G$ act on V via $gv \mapsto \rho(g)(v)$, and extending algebraically. This is well defined as if $g, f \in M$, then

$$\rho(gf)(v) = gfv = \rho(g) \circ \rho(f)(v),$$

and as each $\rho(g)$ lives in $\mathrm{Aut}_k(V)$, everything distributes in the right way.
These two procedures (functors) are inverse each other and so yada yada...

Exercise 2

(a)

A $k[x]$-module is the same as a vector space V together with an endomorphism T. A $k[x]/(x^n)$-module should then such a pair V, T where $T^{\circ n} = 0$.

(b)

First, of $k[x, x^{-1}] \cong k[x, y]/(xy)$, so a $k[x, x^{-1}]$-module ought to be a vector space V together with two mutually inverse automorphisms T, T^{-1}.

Exercise 6

(c)

Let $\phi : V \to W$ be a $k[t]$-linear map. Then for $v \in V$ we have

$$\phi(T(v)) = \phi(tv) = t\phi(v) = W(\phi(v)).$$

Now let $\phi : V \to W$ be a k-linear map which commutes with T, W. Then

$$\phi(tv) = \phi(T(v)) = W(\phi(v)) = t\phi(v),$$

and ϕ is a $k[t]$-linear map as well.

Exercise 9

$$\begin{aligned}(\phi + \psi)(am + n) &= \phi(am + n) + \psi(am + n) \\ &= a\phi(m) + \phi(n) + a\psi(m) + \psi(n) \\ &= a(\phi + \psi)(m) + (\phi + \psi)(n)\end{aligned}$$

Exercise 10

Any $\mathbb{Z}$-map ϕ out of $\mathbb{Z}$ is determined by where it sends 1, since $\phi(n) = n\phi(1)$.

(a)

We can send 1 anywhere, as $\phi_k : n \mapsto kn$ is an $\mathbb{Z}$-map for all $k \in \mathbb{Z}$. Indeed,

$$\phi_k(am + n) = kam + kn = a\phi_k(m) + \phi_k(n).$$

Moreover, $(\phi_k + \phi_l)(m) = km + lm = (k + l)m = \phi_{k+l}(m)$, and $\phi_k + \phi_l = 0 \Leftrightarrow k + l = 0$ so $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$.

(b)

We can send 1 anywhere in $Z/(m)$, as $\phi_k : n \mapsto kn + (m)$ is an $\mathbb{Z}$-map for all $k \in \mathbb{Z}$. Indeed,

$$\phi_k(ax + n) = kax + kn + (m) = a\phi_k(x) + \phi_k(n).$$

Moreover, $(\phi_k + \phi_l)(x) = kx + lx + (m) = (k + l)x + (m) = \phi_{k+l}(x)$, and $\phi_k + \phi_l = 0 \Leftrightarrow k + l \in (m)$, and so $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(m)) \cong \mathbb{Z}/(m)$.

(c)

Let $\phi : A \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)$ be given by $\phi(a) : k \mapsto ka$. Then $\phi(a)$ is a $\mathbb{Z}$-map since $\phi(a)(lm + n) = lma + na = l\phi(a)(m) + \phi(a)(n)$ and ϕ is group homomorphism as $\phi(a + b)(k) = ka + kb = \phi(a)(k) + \phi(b)(k)$ and $\phi(0)(k) = 0k = 0$ is the zero-map. Moreover, ϕ is injective, since $\phi(a) = 0$ gives that $0 = \phi(a)(1) = 1a = a$. Finally, ϕ is surjective since if $\psi : \mathbb{Z} \to A$, then $\psi(n) = n\psi(1) = n\phi(\psi(1))$. We've shown that $A \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)$.

(d)

Let $\phi : \mathbb{Z}/(m) \to \mathbb{Z}$ be a $\mathbb{Z}$-map. Then $m\phi(1) = \phi(m) = 0$, hence $\phi(1) = 0$ since $\mathbb{Z}$ is a domain, and $\phi = 0$.

(f)

Let $\phi : \mathbb{Q} \to \mathbb{Z}$ be a $\mathbb{Z}$-map. Then $b\phi(a/b) = \phi(a)$, and so $b|\phi(a)$. But $a/b = ca/cb$ for any c, and so $c|\phi(a)$ for all $c \in \mathbb{Z}$. It follows that $\phi(a) = 0$ and so $\phi = 0$.

Exercise 18

(b)

Let M be an R/I module, and let R act on M by

$$rm = (r + I)m.$$

Then any $i \in I$ annihilates all of M since $im = (0 + I)m = 0$.

Now let M be an R module which is annihilated by I. Then let R/I act on M by

$$(r + I)m = rm.$$

This is well defined, since if $r + I = r' + I$, we have $r - r' \in I$ and so

$$(r + I)m = rm = rm - rm + r'm = r'm = (r' + I)m.$$

(c)

Any $m + IM \in M/IM$ is annihilated by I. Hence M/IM is an R/I-module by the part (b).

(d)

Suppose $m_1, \dots m_n$ generate M, and let $m + IM \in M/IM$. Then we can write m as an R-linear combination of the m_i,

$$m = \sum_{i=1}^{n} r_i m_i.$$

Moreover,

$$m + IM = \sum_{i=1}^{n} r_i(m_i + IM)$$

and since r_i acts as $r_i + I$ on M/IM,

$$m + IM = \sum_{i=1}^{n} r_i(m_i + IM) = \sum_{i=1}^{n} (r_i + I)(m_i + IM)$$

and so the $m_i + IM$ generates M/IM as a R/I-module.

To see that the size of the genarating set may decrease, note that we may have some $m_i \in IM$. For example if $R = k[x]$, $M = R/(x^2 - 1)$, $I = (x)$. Then 1 and x generate M, but $M/IM = k$ is generated by 1 as a $R/I = k$ module.

(e)

Suppose that $M = \bigoplus_{i=1}^{n} Re_i$ is a free module with basis $e_1, \dots, e_n$. Then we showed in the part (d) that M/IM is generated by $e_i + IM$ as an R/I module. We now show that the $e_i + IM$ doesn't satisfy any non-trivial R/I-linear relation. We have

$$0 = \sum_{i=1}^{n} (r_i + I)(e_i + IM)$$

if and only if

$$im = \sum_{j=1}^{n} r_j e_j$$

for some $i \in I, m \in M$. But then we can write

$$m = \sum_{j=1}^{n} r'_j e_j$$

and as the e_i are a basis, it follows that $ir'_j - r_j = 0$ for all j. Hence $r_j \in I$, and $r_j + I = 0$.

Exercise 19

It follows from Exercise 18.(e) that M/IM has a basis of cardinality $|J|$ whenever J is a basis for M. Furthermore as R/I is a vector space, every basis of M/IM has the same cardinality, and so every basis of M has the same cardinality.

Exercise 20

(a)

First of, as R is a PID we in particular have that every ideal is finitely generated so R is Noetherian. It follows that if we let S be the set of ideals of the form $f(N)$ for R-linear maps $f : M \to R$, then S has a maximal element $u(N)$.

(b)

Let $u(N) = (a_1)$. If $a_1 = 0$, we have that N is in the kernel of every morphism $M \to R$. As M is free, we can suppose $M = \bigoplus_{i \in I} RE_i$, and we have projections $\pi_i : M \to R$ onto the i-th cordinate for each $i \in I$. Since N is in the kernel of every π_i, it follows that no element of N has any non-zero coordinate. Hence $N = 0$.

(c)

Let $e_1' = r_1x_1 + \ldots + r_nx_n$. Then let $b_i = u(x_i)$. We then have

$$a_1 = u(e_1') = u(r_1x_1 + \ldots + r_nx_n) = r_1b_1 + \ldots + r_nb_n.$$

Let r be the generator of $(r_1, \ldots, r_n)$ and b_i' be such that $r = r_1b_1' + \ldots r_nb_n'$. Then define the R-map $u' : x_i \mapsto b_i'$. Then $u'(e_1') = r$, hence $u'(N) \supseteq (r) \supseteq (a_1) = u(N)$, whence maximality of $u(N)$ yields $r = a_1$. The desired result now follows as $\pi_i(e_1') = r_i \in (a_1)$.

(d)

Follows immediately from our solution above.

(e)

We have $a_1u(e_1) = u(a_1e_1) = u(e_1') = a_1$, hence $u(e_1) = 1$ as R is a domain. Let $M' = \ker(u)$, and define $\phi : M \to M' \oplus R$ by $\phi : x \mapsto (x - u(x)e_1, u(x))$. Then indeed $u(x - u(x)e_1) = u(x) - u(x) = 0$ so ϕ is well-defined. Now let $x \in \ker(\phi)$. Then by looking at the second coordinate of ϕ, we see that $x \in \ker(u)$, whence we get that $\phi(x) = (x, 0)$ so $x = 0$, and ϕ is injective. Now let $x, r \in M' \oplus R$. Then $\phi(x+re_1) = (x+re_1 - u(x+re_1), u(x+re_1)) = (x, r)$ and so ϕ is surjective as well, hence an isomorphism.

(f)

Suppose first that the rank of M is 1. We claim that e_1 generates M. To see this, note that $e_1' = r_1x_1$ and from part (c) it follows that $r = r_1 = a_1$ in this case so $e_1 = x_1$ and e_1 is a basis for M. Moreover, $e_1' = a_1e_1$ is a basis for N since

Now suppose that M is of rank n, and the statement holds for all modules of rank $< n$. Then let e_1, a_1, u be as above. As $M \cong \ker(u) \oplus R$, there exists a basis $e_2, e_3, \ldots, e_n$ for $\ker(u)$ such that $e_2, \ldots, e_m$ is a basis for $N \cap \ker(u)$.

List 2

Exercise 2

(a), (b), (c)

Let $m \in \ker(f)$. Then $f(m) = 0$ and so by injectivity of f'' and commutativity, we get $m'' = 0$. It follows by exactness that there is $m' \in M'$ in the preimage of m. As $f(m) = 0$, again by commutativity and the fact that $M' \to N' \to N$ are all injective, we have that $m' = 0$. It follows now that m is 0 by injectivity.

Here we used injectivity of f' and f''.

Now let $n \in N$. Then let $m'' = (f'')^{-1}(n'')$ and m_0 be some element in the preimage of m''. Then $f(m_0) - n$ is mapped to 0 along $N \to N''$, and so there is some element $n'_1 \in N'$ in the preimage of $f(m_0) - n$ by exactness. As f' is surjective, we have $m'_1 \in (f')^{-1}(n'_1)$. Now have that $f(m_1) = f(m_0) - n$ by commutativity, and so $f(m_1 - m_0) = n$.

Here we used surjectivity of f' and f''

(d)

Let $n'' \in N''$. Then by exactness, we have $n \in N$ which maps to n'', and by surjectivity of f we have $m \in M$ which maps to n. It follows that $f''(m'') = n''$ by commutativity.

Here we only used surjectivity of f.

(e)

We give a counter example

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "6"] \arrow[d, "3"] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & \mathbb{Z}/6\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

(f)

Let $m' \in \ker(f')$. Then $f(m) = 0$, and since $m' \mapsto m \mapsto f(m)$ are all injective, we have $m' = 0$.

Here we used injectivity of f only.

(g)

We give a counter example.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "6"] \arrow[d, "3"] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & \mathbb{Z}/6\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Exercise 3

(a)

We give a counter example.

$$\begin{tikzcd}
\mathbb{Z}/8\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
\mathbb{Z}/4\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

(b)

This is a stronger statement than Exercise 2 (g), which we gave a counter example for. I.e it is false.

Exercise 10

(a)

Both statements are equivalent to $f_2 = 0$.

(b)

If f_1 is surjective, $\ker(f_2) = M_2$ and so $f_2 = 0$. If f_4 is injective, $\mathrm{im}(f_3) = 0$, and so $f_3 = 0$. Then $M_3 = \ker(f_3) = \mathrm{im}(f_2) = 0$.

Exercise 13

(a)

Let $\hat{G} = \{\hat{g}_1, \ldots, \hat{g}_m\}$ be a generating set for M'', and G be some set of choices $g_i \in \pi^{-1}(\hat{g}_i)$ for each $\hat{g}?i$. Also let $F = \{f_1, \ldots, f_n\}$ be the set of generators for M' injected into M. Then let $m \in M$. Let m'' be the image of m in M''. Then we can write m'' as a linear combination of the $\hat{g}_i$, and pulling this back to M we get a linear combination of g_i which maps to the same element as m. It follows that $m - \sum a_i g_i$ is in the kernel, and thus can be written as a linear combination of f_i, after which we can see that m is a linear combination of elements from F and G.

(b)

M'' is generated by the image of the generators of M.

(c)

We have that M'' is always finitely generated whenever M is. Hence the statement is equivalent to saying that all submodules of any finitely generated module M are finitely generated. This is true over Noetherian rings, hence we need to consider some non-Noetherian R to construct a counterexample. Moreover, If R is non-Noetherian, then it will have some infinitely generated ideal, which we can take as our submodule. A counterexample is given by

$$0 \longrightarrow (x_1, x_2, \ldots) \longrightarrow k[x_1, x_2, \ldots] \longrightarrow k \longrightarrow 0 \ .$$

Exercise 19

Being projective is the same thing as being a direct summand of a free module, and over PID's, submodules of free modules are free.

Exercise 21

Let

Exercise 22

Let P be a projective $k[t]$-module. Then P is a direct summand of some free $k[t]$-module $F = \bigoplus_{i \in I} k[t]$, and P is naturally graded. Now, t acts as a degree 1 map on F, and therefore on P as well. Our result now follows from the fact that no finite dimensional subspace of F can have a degree 1 endomorphism.

List 3

Exercise 1

(a)

Let

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

be an exact sequence. We will show that

$$0 \longrightarrow \mathrm{Hom}(N'', M) \xrightarrow{g^*} \mathrm{Hom}(N, M) \xrightarrow{f^*} \mathrm{Hom}(N', M)$$

is exact. First, as g is surjective, it follows that $hg = 0 \Rightarrow h = 0$ and so g^* is injective. Moreover, $f^* g^* = (gf)^* = 0$ since the original sequence is exact. Finally, if $h \in \ker(f^*)$, then $hf = 0$, and as g is the cokernel of f, it follows that we have a lift $h'' \in \mathrm{Hom}(N'', M)$ such that $h'' g = g^*(h'') = h$. Hence $\mathrm{im}(g^*) = \ker(f^*)$ and the sequence is exact.

(b)

$\mathrm{Hom}(\_, M)$ being right exact is exactly what it means for M to be injective, so for a counterexample, we need to pick some non-injective module M, and we'll chose the $\mathbb{Z}$-module $\mathbb{Z}$. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and the identity morphism $\mathrm{id} \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$. There is no other morphism $f \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ such that $2f = \mathrm{id}$.

Exercise 3

(a)

This follows from the fact that left-adjoint functors commute with colimits,

$$\begin{aligned} \left(\bigoplus_{i\in I} Re_i\right) \otimes_R \left(\bigoplus_{j\in J} Re_j\right) &= \bigoplus_{i\in I}\left(Re_i \otimes_R \bigoplus_{j\in J} Re_j\right) \\ &= \bigoplus_{i\in I}\bigoplus_{j\in J} Re_i \otimes_R Re_j \\ &= \bigoplus_{(i,j)\in I\times J} R(e_i \otimes_R e_j). \end{aligned}$$

(b)

Suppose that $E \oplus M = F$ and $E' \oplus M' = F'$ with F, F' free. Then

$$\begin{aligned} F \otimes F' &= (E \oplus M) \otimes (E' \oplus M') \\ &= E \otimes E' \oplus E \otimes M' \oplus M \otimes E' \oplus M \otimes M' \end{aligned}$$

and so $E \otimes E'$ is a direct summand of the free module $F \otimes F'$, hence projective.

Exercise 5

By the structure theorem, $\mathbb{Z}/6 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/2$ and we see that $\mathbb{Z}/3$ is a direct summand of the free $\mathbb{Z}/6$-module $\mathbb{Z}/6$.

Exercise 9

(a)

We name the functions in the diagram,

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] & P \arrow[r, "g"] \arrow[d, dashed, "\exists! p"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \arrow[r, "f'"] & P' \arrow[r, "g'"] & M' \arrow[r] & 0. \end{tikzcd}$$

We have existence of p by the fact that P is projective and g' surjective. Now $g'pf = gf = 0$ by commutativity and exactness of the top row, and so $\mathrm{im}(pf) \subset \mathrm{im}(f')$ by exactness of the bottom row. As f' is injective, it has an inverse on its image $(f')^{-1} : \mathrm{im}(f) \to K'$, giving us a map $k : K \to K'$ by $k = (f')^{-1}pf$ and we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "k"] & P \arrow[r, "g"] \arrow[d, "p"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \arrow[r, "f'"] & P' \arrow[r, "g'"] & M' \arrow[r] & 0. \end{tikzcd}$$

(b)

We will show that the following sequence is exact,

$$0 \longrightarrow K \xrightarrow{\begin{bmatrix} f \\ k \end{bmatrix}} P \oplus K' \xrightarrow{\begin{bmatrix} p & -f' \end{bmatrix}} P' \longrightarrow 0 \ .$$

The composition of the two middle maps is given by

$$\begin{bmatrix} p & -f' \end{bmatrix}\begin{bmatrix} f \\ k \end{bmatrix} = pf - f'k$$

which is 0 by commutativity of the diagram from part (a). Moreover, $\begin{bmatrix} f \\ k \end{bmatrix}$ is injective as f is.

Let $(a, b) \in \ker\begin{bmatrix} p & -f' \end{bmatrix}$. Then $p(a) = f'(b)$. By injectivity of f, we have some $c \in K$ such that $f(c) = a$. Then $f'(k(c)) = f'(b)$ by commutativity, and $b = k(c)$ by injectivity of f'. It follows that $(a, b) = (f(c), k(c)) \in \mathrm{im}(\begin{bmatrix} f \\ k \end{bmatrix})$ and we have exactness at $P \oplus K'$.

Finally, to see that $\begin{bmatrix} p & -f' \end{bmatrix}$ is surjective, let $a' \in P'$. Then let $a \in g^{-1}g'(a')$. By commutativity, $g'p(a) = g'(a')$, hence $p(a) - a' \in \ker(g') = \mathrm{im}(f')$ and we have some $b \in K'$ such that $f'(b) = p(a) - a'$. It follows that

$$\begin{bmatrix} p & -f' \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = p(a) - f'(b) = p(a) - p(a) + a' = a'$$

and we see that the map is surjective.

(c)

As P' is projective, the sequence from (b) splits and $K \oplus P' \cong K' \oplus P$.

Exercise 10

Let $F'' \to F' \to M \to 0$ be a finite presentation of M. Then, let $K' = \mathrm{im}(F'' \to F')$. By Exercise 9, we have the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "k"] & F \arrow[r, "g"] \arrow[d, "p"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \arrow[r, "f'"] & F' \arrow[r, "g'"] & M' \arrow[r] & 0, \end{tikzcd}$$

and $F \oplus K' \cong F' \oplus K$. As F, F' and K' all are finitely generate, it follows that K must be finitely generated.

We use this to give a module which is not finitely presented. Let $R = k[x_1, x_2, \ldots]$ and $M = k$. Then we have the exact sequence

$$0 \longrightarrow (x_1, x_2, \ldots) \longrightarrow k[x_1, x_2, \ldots] \longrightarrow k \longrightarrow 0.$$

and as $k[x_1, x_2, \ldots]$ is free and of finite rank 1, whilst $(x_1, x_2, \ldots)$ is not finitely generated, it follows that k cannot be finitely presented.

List 4

Exercise 1

If $\mathrm{id}, \mathrm{id}' : A \to A$ are to identity morphisms then $\mathrm{id} = \mathrm{id} \circ \mathrm{id}' = \mathrm{id}'$.

Exercise 2

Suppose both $g, g' : B \to A$ are two-sided inverses of $f : A \to B$. Then

$$g = g \circ \mathrm{id} = g \circ f \circ g' = \mathrm{id} \circ g' = g'$$

and the two morphisms are equal. To see that one-sided inverses need not be unique, consider $f : \mathbb{Z} \to \mathbb{Z}^2$ with $f : a \to (a, 0)$ and $g, g' : \mathbb{Z}^2 \to \mathbb{Z}$ with $g : (a, b) \to a + b$ and $g' : (a, b) \to a$.

Exercise 3

As natural transformations compose, composition is well-defined in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Moreover, the identity on a functor F is given by $(\mathrm{id}_F)_X = \mathrm{id}_{F(X)}$.

Now suppose that $\eta : F \to G$ is an isomorphism of the functors $F, G : \mathcal{C} \to \mathcal{D}$ with inverse $\xi : G \to F$. Then in particular,

$$\xi_X \circ \eta_X = (\mathrm{id}_F)_X = \mathrm{id}_{F(X)},$$

and

$$\eta_X \circ \xi_X = (\mathrm{id}_G)_X = \mathrm{id}_{G(X)},$$

so $\eta_X : F(X) \to G(X)$ and $\xi_X : G(X) \to F(X)$ are mutually inverse each other, hence isomorphisms.

Suppose instead that $\eta : F \to G$ is a natural isomorphism. Then let ξ be a family of maps for each object $X \in \mathcal{C}$ such that $\xi_X : G(X) \to F(X)$ is the two-sided inverse of $\eta_X : F(X) \to G(X)$. Then for any morphism $f : X \to Y$, consider the diagram below.

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(X)"] & G(X) \arrow[r, "\xi_X"] \arrow[d, "G(X)"] & F(X) \arrow[d, "F(X)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \arrow[r, "\xi_Y"] & F(Y) \end{tikzcd}$$

The left square commutes and the composition of the horizontal maps are identity maps, hence the outermost rectangle defined by these composition commute as well. It follows that

$$F(X) \circ \xi_X \circ \eta_X = \xi_Y \circ G(X) \circ \eta_X,$$

and as η_X is an isomorphism, the right square commutes and ξ defines a natural transformation inverse to η.

Exercise 4

Define $\eta_X : \mathrm{Hom}_R(M \otimes N, X) \to \mathrm{Hom}_S(N, \mathrm{Hom}_R(M, X))$ by

$$\eta_X(\phi) : n \mapsto (m \mapsto \phi(m \otimes n)).$$

Then $\eta_X(\phi)(n)$ is an R-module morphism as

$$\begin{aligned} \eta_X(\phi)(n)(rm + m') &= \phi((rm + m') \otimes n) \\ &= \phi(r(m \otimes n) + m' \otimes n)) \\ &= r\phi(m \otimes n) + \phi(m' \otimes n') \\ &= r\eta_X(\phi)(n)(m) + \eta_X(\phi)(n)(m'), \end{aligned}$$

and $\eta_X(\phi)$ is an S-module morphism as

$$\begin{aligned} \eta_X(\phi)(sn + n')(m) &= \phi(m \otimes (sn + n')) \\ &= \phi(m \otimes sn + m \otimes n') \\ &= \phi(ms \otimes n) + \phi(m \otimes n') \\ &= \eta_X(\phi)(n)(ms) + \eta_X(\phi)(n')(m) \\ &= (s\eta_X(\phi)(n))(m) + \eta_X(\phi)(n')(m). \end{aligned}$$

Naturality of η_X follows by the fact that both $F(X)$ and $G(X)$ ultimately send elements into X, and postcomposing with and $X \xrightarrow{f} Y$ can be done before or after η_X and the result will be the same.

I won't do the rest.

Exercise 5

Denote the natural isomorphism by η and let $f_A = \eta(\mathrm{id}_A) : B \to A$ and $f_B = \eta^{-1}(\mathrm{id}_B) : A \to B$. Then we have the following commutative diagram by naturality along $f_B : A \to B$.

$$\begin{tikzcd}
\mathrm{Hom}(A, A) \arrow[r] \arrow[d, "(f_B)_*"] & \mathrm{Hom}(B, A) \arrow[d, "(f_B)_*"] \\
\mathrm{Hom}(A, B) \arrow[r] & \mathrm{Hom}(B, B)
\end{tikzcd}$$

By looking at where id_A is sent, we see that $f_B \circ \eta(\mathrm{id}_A) = f_B \circ f_A$ is equal to $\eta \circ f_B \circ \mathrm{id}_A = \eta \circ f_B = \mathrm{id}_B$, I.e that $f_B f_A = \mathrm{id}_B$. If we now follow the same procedure in the naturality diagram along $f_A : B \to A$, we get that $f_A f_B = \mathrm{id}_A$, hence the two maps are isomorphisms between A and B.

Exercise 8

Suppose that L is another R-module such that

$$\begin{tikzcd}
L \arrow[r, "\phi"] \arrow[d] & M \arrow[d, "f"] \\
0 \arrow[r] & N
\end{tikzcd} .$$

Then $f \circ \phi = 0$ by commutativity, and so ϕ factors through the kernel of f, and we have

$$\begin{tikzcd}
L \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right] & & \\
& \ker(f) \arrow[r] \arrow[d] & M \arrow[d, "f"] \\
& 0 \arrow[r] & N
\end{tikzcd}$$

whence we have verified that $\ker(f)$ is the colimit of the cospan.

List 5

Exercise 1

Let $X \xrightarrow{f} Y$, $Y \xrightarrow{g_1} Z$ and $Y \xrightarrow{g_2} Z$ be maps of sets where f is surjective and $g_1 f = g_2 f$. Then let $y \in Y$. As f is surjective, there is $x \in X$ such that $f(x) = y$. We then have $g_1(y) = g_1(f(x)) = g_2(f(x)) = g_2(y)$, and as y was arbitrary, $g_1 = g_2$ and f is epic.

For the other direction, suppose instead that f is epic, and let $y \in Y$. Suppose towards a contradiction that $y \notin \mathrm{im}(f)$. Then we may construct two maps

$g_1, g_2 : Y \to \{1, 2\}$ which agree on all elements on Y, except for that $g_i(y) = i$. Then $g_1 \neq g_2$ but $g_1 f = g_2 f$, contradicting the fact that f is an epimorphism.

Exercise 3

The category of free $\mathbb{Z}/4\mathbb{Z}$-modules is an **Ab**-category since it is immediate from the definition of an **Ab**-category that any full subcategory of an **Ab**-category remains an **Ab**-category.

It is additive since finite direct sums of free modules remain free.

Finally, it is not abeliean since for example, the kernel of a morphism of free $\mathbb{Z}/4\mathbb{Z}$-modules need not be free. Indeed, consider the map

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z}.$$

In the category of ordinary (not necessarily free) $\mathbb{Z}/4\mathbb{Z}$-modules, this map has kernel $2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ which is not a free $\mathbb{Z}/4\mathbb{Z}$-module, and we will now show that it has no kernel in the category of free $\mathbb{Z}/4\mathbb{Z}$-modules. Suppose towards a contradiction that $f : M \to \mathbb{Z}/4\mathbb{Z}$ is the kernel of our map. Then $2f = 0$ and $\mathrm{im}(f) \subset \{0, 2\}$. We will show that any kernel is a monomorphism, and that monomorphisms in our category are injective, whence it follows that $|M| \leq 2$ hence $M = 0$. This is a contradiction since the sequence

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z}$$

is exact and doesn't factor through 0. Let's prove the results we need.

Lemma 0.0.1. Equalizers are monic in any category.

Proof. Let $e : E \to X$ be the equalizer of $f, f' : X \to Y$. Then suppose that $g, g' : Z \to E$ are such that $eg = eg'$. As $feg = f'eg = f'eg'$, there exist a unique map $Z \to E$ which which factorize $eg = eg'$ through e. Thus uniqueness forces $g = g'$ and e is monic. □

Corollary 0.0.2. Kernels are always monic

Proof. The kernel of any map $f : X \to Y$ is the equalizer of f and 0. □

Lemma 0.0.3. Monic morphisms in the category of free $\mathbb{Z}/4\mathbb{Z}$-modules are injective.

Proof. Let $f : M \to N$ be a monic morphism of free $\mathbb{Z}/4\mathbb{Z}$-modules where $M = \bigoplus_{i \in I} \mathbb{Z}/4\mathbb{Z}e_i$ and $N = \bigoplus_{j \in J} \mathbb{Z}/4\mathbb{Z}u_j$. Suppose that $f(x) = f(y)$. Then let $g_1, g_2 : \mathbb{Z}/4\mathbb{Z} \to M$ be the morphisms which sends $g_1(1) = x$ and $g_2(1) = y$. Then $f(g_1(1)) = f(g_2(1))$ and as morphisms of free modules are determined by where they send generators, $fg_1 = fg_2$. As f is monic, it follows that $g_1 = g_2$ hence $x = y$ and f is injective. □

Exercise 5

(1)

Suppose that $f : X \to Y$ is a monomorphism. Then $f \circ \ker(f) = 0 = f \circ 0$ so $\ker(f) = 0$ as f is mono.

(2)

Suppose that $f : X \to Y$ is both a mono- and epimorphism. As we're in an abelian, and in particular additive category, it follows from part (1) that the kernel and cokernel of f is 0. Moreover, as we're in an abelian category, f is the kernel of it's cokernel $Y \to 0$, and the following sequence is exact,

$$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0.$$

As $0 \circ \mathrm{id}_Y = 0$, id_Y factors through $\ker 0 = f$ via some $g : Y \to X$ and we have a right-inverse $fg = \mathrm{id}_Y$. Moreover, this is also a left-inverse since $fgf = (fg)f = f$ and as f is monic, $gf = \mathrm{id}_X$.

Exercise 6

We prove this our own way. First we'll state and prove some lemmas that help us in being explicit.

Lemma 0.0.4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, $\eta : F \Rightarrow G$ be a natural transformation, and $H : \mathcal{B} \to \mathcal{C}$ be a functor. Then we have a natural transformation

$$\xi : F \circ H \Rightarrow G \circ H$$

where

$$\xi_X = \eta_{H(X)}.$$

Similarly, if $L : \mathcal{D} \to \mathcal{E}$ is a functor, then we have a natural transformation

$$\chi : L \circ F \Rightarrow L \circ G$$

where

$$\chi_X = L(\eta_X).$$

Proof. For the first statement, let $f : X \to Y$ in $\mathcal{B}$. Then $H(f) : H(X) \to H(Y)$ in $\mathcal{C}$ and we have $\eta_{H(X)}, \eta_{H(Y)}$ such that the following square commutes,

$$\begin{tikzcd}
F \circ H(X) \arrow[r, "\eta_{H(X)}"] \arrow[d, "F \circ H(f)"] & G \circ H(X) \arrow[d, "G \circ H(f)"] \\
F \circ H(Y) \arrow[r, "\eta_{H(Y)}"] & G \circ H(Y)
\end{tikzcd}.$$

For the second statement, let $f : X \to Y$ be in $\mathcal{C}$. Then we have morphisms η_X, η_Y making the following square commute

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y), \end{array}$$

and as functors preserve commutative diagrams, we get

$$\begin{array}{ccc} L \circ F(X) & \xrightarrow{L(\eta_X)} & L \circ G(X) \\ \downarrow{\scriptstyle L\circ F(f)} & & \downarrow{\scriptstyle L\circ G(f)} \\ L \circ F(Y) & \xrightarrow{L(\eta_Y)} & L \circ G(Y). \end{array}$$

□

Lemma 0.0.5. Let $X \in \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then

$$F \circ \Delta_X = \Delta_{F(X)}.$$

Proof. Δ_X sends all objects to X and all morphisms to id_X. $F \circ \Delta_X$ sends all objects to $F(X)$ and all morphisms to $\mathrm{id}_{F(X)}$. The same is true of $\Delta_{F(X)}$. □

Lemma 0.0.6. Let $I, \mathcal{C}$ and $\mathcal{D}$ be categories. Furthermore, let

$$F : I \to \mathcal{C},\, G : I \to \mathcal{D}$$

be functors and

$$L : \mathcal{C} \to \mathcal{D},\, R : \mathcal{D} \to \mathcal{C},$$

be left and right adjoint functors via the adjugant Φ. Suppose there is a natural transformation

$$\eta : L \circ F \to G.$$

Then there is a natural transformation

$$\xi : F \to R \circ G$$

given by

$$\xi_X = \Phi_{F(X),G(X)}(\eta_X).$$

Similarly, if such ξ exists it implies existence of η where

$$\eta_X = \Phi^{-1}_{F(X),G(X)}(\chi_X).$$

Proof. Let $f : X \to Y$ in I. Then we have a commutative diagram

$$\begin{array}{ccc} L \circ F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle L\circ F(f)} & & \downarrow{\scriptstyle G(f)} \\ L \circ F(Y) & \xrightarrow{\eta_Y} & G(Y). \end{array}$$

Let $\Phi_{X,Y}$ denote the adjugant. Then we get morphisms

$$\begin{tikzcd} F(X) \arrow[r, "\Phi_{F(X),G(X)}(\eta_X)"] \arrow[d, "F(f)"] & R \circ G(X) \arrow[d, "R\circ G(f)"] \\ F(Y) \arrow[r, "\Phi_{F(Y),G(Y)}(\eta_Y)"] & R \circ G(Y), \end{tikzcd}$$

and naturality of adjunction tells us that the square commutes. The other direction follows in the same way. $\square$

Now let's state and prove the theorem.

Theorem 0.0.7. Let $F : I \to \mathcal{C}$ be a functor, and $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$ be a left/right-adjoint functor pair. Then $L(\mathrm{colim}(F)) = \mathrm{colim}(L \circ F)$.

Proof. By the definition of colimits, we have a natural transformation $\tau : F \Rightarrow \Delta_{\mathrm{colim}(F)}$, and it follows that we have a natural transformation $L \circ F \Rightarrow L\circ\Delta_{\mathrm{colim}(F)} = \Delta_{L(\mathrm{colim}(F))}$. This is true whether L is a left adjoint or not. What remains to be shown is that $L(\mathrm{colim}(F))$ is initial among all objects $K \in \mathcal{D}$ with natural transformations $L \circ F \Rightarrow \Delta_K$.

Suppose that $K \in \mathcal{D}$ is an object such that there is a natural transformation $\eta : L \circ F \Rightarrow \Delta_K$. Then we have a natural transformation

$$\xi : F \to \Delta_{R(K)}$$

given by

$$\xi_X = \Phi_{F(X),K}(\eta_X).$$

As $\mathrm{colim}(F)$ is the initial object in $\mathcal{C}$ with respect to this property, it follows that we have a unique morphism $h : \mathrm{colim}(F) \to R(K)$ such that for any $X \in I$ we have the following commutative diagram

$$\begin{tikzcd} F(X) \arrow[rr, "\tau_X"] \arrow[dr, "\xi_X"'] & & \mathrm{colim}(F) \arrow[dl, "h"] \\ & R(K). & \end{tikzcd}$$

By naturality of adjoints we get the following commutative diagram

$$\begin{tikzcd} L \circ F(X) \arrow[rr, "L(\tau_X)"] \arrow[dr, "\Phi^{-1}_{F(X),K}(\xi_X)"] & & L(\mathrm{colim}(F)) \arrow[dl, "\Phi^{-1}_{F(X),K}(h)"] \\ & K & \end{tikzcd}$$

and as

$$\xi_X = \Phi_{F(X),K}(\eta_X),$$

we have

$$\Phi^{-1}_{F(X),K}(\xi_X) = \eta_X,$$

and the previous diagram can be simplified to

$$\begin{tikzcd}
L \circ F(X) \arrow[rr, "L(\tau_X)"] \arrow[dr, "\eta_X"'] & & L(\mathrm{colim}(F)) \arrow[dl, "\Phi^{-1}_{F(X),K}(h)"] \\
& K &
\end{tikzcd}$$

whence we see that $L(\tau) : L \circ F \Rightarrow \Delta_{L(\mathrm{colim}(F))}$ is initial among all constant functors with natural transformations from $L \circ F$, so $L(\mathrm{colim}(F))$ is the colimit of $L \circ F$. $\square$

Exercise 8

$\mathbb{Z}$ is a PID, so flat $\mathbb{Z}$-modules are the same thing as torsionfree $\mathbb{Z}$-modules. $\mathbb{Q}$ is a torsionfree $\mathbb{Z}$-module whilst $\mathbb{Q}/\mathbb{Z}$ isn't.

Exercise 9

(1)

Let $f : A \to B$ be an injective R-module morphism. Then as M is flat $f \otimes \mathrm{id}_M$ is injective, and as N is flat

$$(f \otimes \mathrm{id}_M) \otimes \mathrm{id}_N = f \otimes (\mathrm{id}_M \otimes \mathrm{id}_N) = f \otimes \mathrm{id}_{M \otimes N}$$

is injective, so $M \otimes N$ is flat.

(2)

We begin with a lemma.

Lemma 0.0.8. Let A, M, N be $\mathbb{R}$-modules, and $a \in A, \phi \in \mathrm{Hom}(M, N)$. Then

$$a \otimes \phi = 0$$

if and only if

$$a \otimes \phi(m) = 0$$

for all $m \in M$.

Proof. An element $a \otimes \phi$ in $A \otimes \mathrm{Hom}(M, N)$ is 0 if and only if every bilinear map out of $A \times \mathrm{Hom}(M, N)$ vanishes at (a, ϕ), so let's define some bilinear maps out of $A \times \mathrm{Hom}(M, N)$. For any $m \in M$, define the map

$$f_m : A \times \mathrm{Hom}(M, N) \to A \otimes N$$

by

$$f_m : a \times \phi \mapsto a \otimes \phi(m).$$

This map is indeed bilinear as

$$f_m(a + ra', \phi) = (a + ra') \otimes \phi = a \otimes \phi + r(a' \otimes \phi)$$

and

$$f_m(a, \phi + r\phi') = a \otimes (\phi + r\phi')(m) = a \otimes (\phi(m) + r\phi'(m)) = a \otimes \phi(m) + r(a \otimes \phi'(m)).$$

The first direction now follows,

$$a \otimes \phi = 0 \Rightarrow 0 = f_m(a, \phi) = a \otimes \phi(m).$$

Can't prove the other direction for now. $\square$

Let $f : A \to B$ be an injective R-module morphism, and $a \in A, \phi \in \operatorname{Hom}(M, N)$ be such that $f(a) \otimes \phi = 0$. Then $f(a) \otimes \phi(m) = 0$ for all $m \in M$, and in particular

$$(f \otimes \mathrm{id}_N) : a \otimes \phi(m) \mapsto f(a) \otimes \phi(m) = 0$$

and flatness of N implies that $a \otimes \phi(m) = 0$ for all $m \in M$, whence $a \otimes \phi = 0$, and $f \otimes \mathrm{id}_{\operatorname{Hom}(M,N)}$ is injective.

List 6

Exercise 2

A submodule of a torsionfree module must be torsionfree.

Exercise 3

Let $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ be given by $f(1) = 2$. Then f is injective whilst

$$(f \otimes \mathrm{id}_{(2)})(1 \otimes 2) = (2 \otimes 2) = (1 \otimes 4) = 0.$$

Note that here the fact that we're tensoring by the submodule (2) instead of the full module $\mathbb{Z}/4\mathbb{Z}$ means that $(1 \otimes 2)$ is non-zero since we can't shift the 2 to the left.

Exercise 6

(1)

No, there is no way to factor id through 2 in the diagram

$$\begin{tikzcd} (2) \arrow[r, hook, "2"] \arrow[d, "\mathrm{id}"] & \mathbb{Z}/4\mathbb{Z} \\ (2) & \end{tikzcd}$$

(2)

There is no way to factor the inclusion through multiplication by x in the following diagram,

$$\begin{tikzcd} \mathbb{Z}[x] \arrow[r, hook, "\cdot x"] \arrow[d, hook, "i"] & \mathbb{Z}[x] \\ \mathbb{Q}[x]. & \end{tikzcd}$$

(3)

Yes, R is a PID since $\mathbb{Q}$ is a field, and $\mathbb{Q}(x)$ is divisible.

Exercise 8

Suppose that $f_\bullet : C_\bullet \to D_\bullet$ is an epimorphism in $\mathbf{Ch}(\mathcal{A})$. Then let $g, g' : D_n \to X$ be two morphisms in $\mathcal{A}$ such that $gf_n = g'f_n$. These g, g' can be turned into chain maps $g_\bullet, g'_\bullet : D_\bullet \to X$ via

$$\begin{tikzcd} \dots \arrow[r] & D_{n+1} \arrow[r] \arrow[d, "0"] & D_n \arrow[r] \arrow[d, "g"] & D_{n-1} \arrow[r] \arrow[d, "0"] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

As $gf_n = g'f_n$ it immediately follows that $g_\bullet f_\bullet = g'_\bullet f_\bullet$, and since $f_\bullet$ is an epimorphism, it follows that $g_\bullet = g'_\bullet$ whence $g = g'$ and f_n is an epimorphism.

Now suppose instead that $f_\bullet : C_\bullet \to D_\bullet$ is a chain map such that each f_n is an epimorphism. Then let $g_\bullet, g'_\bullet : D_\bullet \to X_\bullet$ be maps such that $g_\bullet f_\bullet = g'_\bullet f_\bullet$. Two chain maps are equal if and only if they are equal componentwise, and so for each n, we have $g_n f_n = g'_n f_n$. As f_n is an epimorphism, $g_n = g'_n$ and as this holds for every n, $g_\bullet = g'_\bullet$ and $f_\bullet$ is an epimorphism.

Exercise 9

The long exact sequence becomes

$$\dots \longrightarrow 0 \longrightarrow H_n(C_\bullet) \longrightarrow 0 \longrightarrow \dots$$

whence $H_n(C_\bullet) = 0$.

Exercise 11

First we prove exactness at C'_n. An element $c' \in C'_n$ is in the kernel of (f'_n, i_n) if and only if it's in $\ker(f'_n) \cap \ker(i_n)$. Suppose c' is such an element. By exactness of the top row, there is a $c'' \in C''_{n+1}$ which maps to c'. As $f'_n(c') = 0$, commutativity yields that $\delta'_{n+1} f''_{n+1}(c'') = 0$, and so exactness tells us that there is an element $d \in D_{n+1}$ which maps to $f''_{n+1}(c'') \in D''_{n+1}$. We now have that $c' = \partial_n (f''_{n+1})^{-1} q_{n+1}(d)$ and so

$$\ker(c') \subseteq \mathrm{im}(\partial_n (f''_{n+1})^{-1} q_{n+1}).$$

The other inclusion is immediate by exactness and commutativity.

For exactness at D_n, let $d \in \ker(\partial_n (f''_{n+1})^{-1} q_{n+1})$. Then by exactness, there is $c \in C_n$ such that

$$p_n(c) = (f''_{n+1}) q_{n+1}(d).$$

It follows that $f_n(c) - d \in \ker(q_n + 1) = \mathrm{im}(j_n)$, and so there is $d' \in D'_n$ such that $j_n(d') = f_n(c) - d$, hence $d = j_n(-d') - f_n(-c)$ and the kernel lies in the image. The other direction is immediate by commutativity and exactness.

Finally, let's show exactness at $D'_n \oplus C_n$. Let $(d', c) \in D'_n \oplus C_n$ be such that $j_n(d') = f_n(c)$. Then

$$p_n(c) = (f'')_n^{-1} q_n j_n(d'_n) = 0$$

and there is $c' \in C'_n$ which maps to c. Commutativity then yields that $f'_n(c') - d' \in \ker(j_n) = \mathrm{im}(\delta_{n+1})$, and so we have $d'' \in D''_{n+1}$ which maps to $f'_n(c') - d'$. But then if we let $\widetilde{c'} = \partial_{n+1}(f''_{n+1})^{-1}(d'')$ we have

$$f'_n(c' - \widetilde{c'}) = f'_n(c) - f'_n(\partial_{n+1}(f''_{n+1})^{-1}(d'')) = \delta_{n+1}(d'') + d' f'_n(\partial_{n+1}(f''_{n+1})^{-1}(d'')) = d'$$

and we are done (as the other inclusion is trivial).

Exercise 12

First of, we have long exact sequences

$$\dots \longrightarrow H_{n+1}(C) \longrightarrow H_{n+1}(C'') \xrightarrow{\delta^C_{n+1}} H_n(C') \longrightarrow H_n(C) \longrightarrow \dots$$

$$\dots \longrightarrow H_{n+1}(D) \longrightarrow H_{n+1}(D'') \xrightarrow{\delta^D_{n+1}} H_n(D') \longrightarrow H_n(D) \longrightarrow \dots,$$

and by a remark in Section 4.1 of the lecture notes, the passage to long exact sequences is natural in the sense that we have a chain map

$$\begin{tikzcd}
\dots \arrow[r] & H_{n+1}(C) \arrow[r] \arrow[d, "H_{n+1}(f)"] & H_{n+1}(C'') \arrow[r, "\delta^C_{n+1}"] \arrow[d, "H_{n+1}(f'')"] & H_n(C') \arrow[r] \arrow[d, "H_n(f')"] & H_n(C) \arrow[r] \arrow[d, "H_n(f)"] & \dots \\
\dots \arrow[r] & H_{n+1}(D) \arrow[r] & H_{n+1}(D'') \arrow[r, "\delta^D_{n+1}"] & H_n(D') \arrow[r] & H_n(D) \arrow[r] & \dots,
\end{tikzcd}$$

and as it is assumed that $H_n(f'')$ is an isomorphism for every n, we can apply our results from Exercise 11 from which the desired statement follows immediately.

Exercise 13

List 7

Exercise 1

We have Let $P_\bullet \to N$ be a projective resolution of N. Then

$$\begin{aligned}
\mathrm{Tor}_n^R(M,N) &= H(M \otimes_R P_n) \\
&= H(P_n \otimes_R M) \\
&= \mathrm{Tor}_n^R(N,M)
\end{aligned}$$

where we used balancing of Tor for the last equality.

Exercise 4

As R is a domain, a projective resolution of $R/(a)$ is given by

$$0 \longrightarrow R \xrightarrow{\_\cdot a} R \xrightarrow{\pi} R/(a) \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\,_R\mathbf{Mod}}(\_, M)$ to the deleted resolution yields

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\,_R\mathbf{Mod}}(R,M) \arrow[r, "(_\cdot a)^*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\,_R\mathbf{Mod}}(R,M) \arrow[r] \arrow[d, "\cong"] & 0 \\
& M \arrow[r, "_\cdot a"] & M &
\end{tikzcd}$$

So we have

$$\begin{aligned}
\mathrm{Ext}_R^0(R,M) &= \{m \in M : am = 0\} \\
\mathrm{Ext}_R^1(R,M) &= M/aM
\end{aligned}$$

and $\mathrm{Ext}_R^n(R,M) = 0$ for $n > 1$.

Exercise 5

If M is flat, then $\_ \otimes_R M$ is an exact functor and it's n-th derived functor (I.e $\mathrm{Tor}_n^R(M, \_)$) is 0 for all $n > 0$.

Now suppose that $\mathrm{Tor}_n^R(M, \_) = 0$ for all $n > 0$. Let $f : A \to B$ be an injective morphism of R-modules. Then we have an short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow \mathrm{coker}(f) \longrightarrow 0,$$

and a long exact sequence

$$\ldots \longrightarrow \mathrm{Tor}_1^R(M, \mathrm{coker}(f)) = 0 \longrightarrow \mathrm{Tor}_0^R(M,A) \longrightarrow \mathrm{Tor}_0^R(M,B) \longrightarrow \mathrm{Tor}_0^R(M, \mathrm{coker}(f)) \longrightarrow 0,$$

hence an injection

$$M \otimes_R A = \mathrm{Tor}_0^R(M, A) \xrightarrow{\mathrm{Tor}_0^R(M)(f)} \mathrm{Tor}_0^R(M, B) = M \otimes_R B.$$

Now, choosing projective resolutions $P_\bullet \to A, Q_\bullet \to B$ yields the following commutative diagram

$$\begin{tikzcd}
M \otimes_R P_1 \arrow[r] \arrow[d, "\mathrm{id}_M \otimes \phi_1"] & M \otimes_R P_0 \arrow[r] \arrow[d, "\mathrm{id}_M \otimes \phi_0"] & M \otimes_R A \arrow[r] \arrow[d, "\mathrm{id}_M \otimes f"] & 0 \\
M \otimes_R Q_1 \arrow[r] & M \otimes_R Q_0 \arrow[r] & M \otimes_R B \arrow[r] & 0,
\end{tikzcd}$$

where ϕ_0, ϕ_1 are the maps granted by the comparison theorem. As $M \otimes_R$ is right exact, the rows are exact and we may insert cokernels as follows

$$\begin{tikzcd}
M \otimes_R P_1 \arrow[r] \arrow[d, "\mathrm{id}_M \otimes \phi_1"] & M \otimes_R P_0 \arrow[r] \arrow[d, "\mathrm{id}_M \otimes \phi_0"] & \mathrm{coker}(\mathrm{id}_M \otimes d_1^P) \arrow[r, "\cong"] & M \otimes_R A \arrow[d, "\mathrm{id}_M \otimes f"] \\
M \otimes_R Q_1 \arrow[r] & M \otimes_R Q_0 \arrow[r] & \mathrm{coker}(\mathrm{id}_M \otimes d_1^Q) \arrow[r, "\cong"] & M \otimes_R B.
\end{tikzcd}$$

But $\mathrm{coker}(\mathrm{id}_M \otimes d_1^P)$ is exactly $\mathrm{Tor}_0^R(M, A)$, and the same for B. Moreover, $\mathrm{Tor}_0^R(M)(f)$ is the map induced by $\mathrm{id}_M \otimes \phi_0$ on the cokernels, and so we have the following two commutative diagrams

$$\begin{tikzcd}
M \otimes_R P_0 \arrow[r] \arrow[d, "\mathrm{id}_M \otimes \phi_0"] & \mathrm{coker}(\mathrm{id}_M \otimes d_1^P) \arrow[r, "\cong"] \arrow[d, "\mathrm{Tor}_0^R(M)(f)"] & M \otimes_R A \\
M \otimes_R Q_0 \arrow[r] & \mathrm{coker}(\mathrm{id}_M \otimes d_1^Q), \arrow[r, "\cong"] & M \otimes_R B.
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \otimes_R P_0 \arrow[r] \arrow[d, "\mathrm{id}_M \otimes \phi_0"] & \mathrm{coker}(\mathrm{id}_M \otimes d_1^P) \arrow[r, "\cong"] & M \otimes_R A \arrow[d, "\mathrm{id}_M \otimes f"] \\
M \otimes_R Q_0 \arrow[r] & \mathrm{coker}(\mathrm{id}_M \otimes d_1^Q) \arrow[r, "\cong"] & M \otimes_R B,
\end{tikzcd}$$

whence surjectivity onto the cokernels yields the following commutative diagram

$$\begin{tikzcd}
\mathrm{coker}(\mathrm{id}_M \otimes d_1^P) \arrow[r, "\cong"] \arrow[d, "\mathrm{Tor}_0^R(M)(f)"] & M \otimes_R A \arrow[d, "\mathrm{id}_M \otimes f"] \\
\mathrm{coker}(\mathrm{id}_M \otimes d_1^Q) \arrow[r, "\cong"] & M \otimes_R B.
\end{tikzcd}$$

Finally, injectivity of $\mathrm{Tor}_0^R(M)(f)$ now yields injectivity of $\mathrm{id}_M \otimes f$ whence f is flat.

Exercise 6

If M is projective, then $\mathrm{Hom}(M, \_)$ is an exact functor, and so all n-th derived functors (I.e $\mathrm{Ext}^n_R(M, \_)$) for $n > 0$ are 0.

Now suppose that $\mathrm{Ext}^n_R(M, \_) = 0$ for all $n > 0$, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence. As $\mathrm{Ext}^n(M, \_)$ vanishes for non-zero indices, the long exact sequence becomes

$$0 \longrightarrow \mathrm{Ext}^0_R(M, A) \xrightarrow{\mathrm{Ext}^0_R(M,f)} \mathrm{Ext}^0_R(M, B) \xrightarrow{\mathrm{Ext}^0_R(M,g)} \mathrm{Ext}^0_R(M, C) \longrightarrow 0$$

and $\mathrm{Ext}^0_R(M, \_)$ is an exact functor. As $\mathrm{Hom}_{\,_R\mathbf{Mod}}(M, \_)$ is left exact, $\mathrm{Ext}^0_R(M, \_)$ is naturally isomorphic to $\mathrm{Hom}_{\,_R\mathbf{Mod}}(M, \_)$, and $\mathrm{Hom}_{\,_R\mathbf{Mod}}(M, \_)$ is then also exact, whence M is projective.

Exercise 7

Three steps

1. Every element of $\mathrm{colim} M_i$ comes from some M_i. In other words there is a surjection $\bigoplus M_i \to \mathrm{colim} M_i$: We have maps $\phi_j : M_j \to \mathrm{colim} M_i$ for all $j \in I$, hence there is a unique map $\phi : \bigoplus M_i \to \mathrm{colim} M_i$ that factors all these ϕ_j. Suppose now that $X \in {}_R\mathbf{Mod}$ and $f, g : \mathrm{colim} M_i \to X$ are such that $f\phi = g\phi$. Then $f\phi, g\phi$ are two cocones of $i \mapsto M_i$, and as they are the same, there is a unique map $\mathrm{colim} M_i \to X$ which they factor through. Hence both f and g must be equal to this unique map, and in particular they are equal to each other.

2. Show that colim is left exact, in other words preserves preserves kernels and

3. Show that colim is surjective. Automatic since colimits commute with colimits (?), and coker is a colimit.

Exercise 8

We begin with a lemma.

Lemma 0.0.9. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{U} \mathcal{C}$ be additive functors of abelian categories, and suppose that U is exact. Then

$$L_n(U \circ F) \cong U(L_n(F)).$$

Proof. □

Let M be an $\mathbb{R}$-module, I be a filtered category and $A : I \to {}_R\mathbf{Mod}$ be an I-shaped diagram. We are asked to show that

$$\operatorname{Tor}_n^R(M, \operatorname{colim}(A)) \cong \operatorname{colim}(\operatorname{Tor}_n^R(M, \_) \circ A).$$

Breaking the left hand side into factors we get

$$\begin{aligned} L_n(M \otimes_R \_)(\operatorname{colim}(A)) &= L_n(M \otimes_R \_) \circ \operatorname{colim}(A) \\ &= H_n \circ \mathbf{K}(M \otimes_R \_) \circ P \circ \operatorname{colim}(A). \end{aligned}$$

We begin by showing that $P \circ \operatorname{colim}(A) \cong \operatorname{colim}(P \circ A)$ where $P : {}_R\mathbf{Mod} \to \mathbf{K}({}_R\mathbf{Mod})$ takes modules to (deleted) projective resolutions in the homotopy category, and $\operatorname{colim} : {}_R\mathbf{Mod}^I \to {}_R\mathbf{Mod}$ takes I-shaped diagrams in ${}_R\mathbf{Mod}$ to their colimit.

Exercise 10

(1)

We will show that the localization

$$\phi : M \times K \to S^{-1}M,$$

with $S = R^*$ and

$$\phi : (m, a/b) \to am/b$$

satisfies the universal property of the tensor product of M and K.

ϕ is R-bilinear and so factors through the tensor product $\phi(m, a/b) = h(m \otimes_R a/b)$. We claim that h is an isomorphism. For surjectivity, we have $m/s = h(m \otimes_R 1/s)$. For injectivity, $am/b = 0$ if and only if $sam - 0b = 0$ for some $s \in S = R^*$. But if $sam = 0$ then $m \otimes_r a/b = sam \otimes_r 1/sb = 0$, so h is injective, hence an isomorphism.

As shown above, an element $m/s \in M$ is zero if and only if there is some $s' \in S$ such that m is s' torsion. Hence $M \otimes_R K = S^{-1}M$ is 0 if and only if M is $S = R^*$-torsion.

(2)

Let $P_\bullet \to M$ be a projective resolution of M. Then

$$\begin{aligned} \operatorname{Tor}_i^R(M, N) \otimes K &= H_i(P_\bullet \otimes N) \otimes K \\ &= \frac{\ker(d_i^P \otimes \mathrm{id}_N)}{\operatorname{im}(d_{i+1}^P \otimes \mathrm{id}_N)} \otimes K, \end{aligned}$$

and as K is torsionfree, K is flat, and the short exact sequence

$$0 \longrightarrow \operatorname{im}(d_{i+1}^P \otimes \mathrm{id}_N) \longrightarrow \ker(d_i^P \otimes \mathrm{id}_N) \longrightarrow \frac{\ker(d_i^P \otimes \mathrm{id}_N)}{\operatorname{im}(d_{i+1}^P \otimes \mathrm{id}_N)} \longrightarrow 0$$

remains eact after being tensored with K so

$$0 \longrightarrow \mathrm{im}(d_{i+1}^P \otimes \mathrm{id}_N) \otimes K \longrightarrow \ker(d_i^P \otimes \mathrm{id}_N) \otimes K \longrightarrow \frac{\ker(d_i^P \otimes \mathrm{id}_N)}{\mathrm{im}(d_{i+1}^P \otimes \mathrm{id}_N)} \otimes K \longrightarrow 0$$

from which it follows that

$$\frac{\ker(d_i^P \otimes \mathrm{id}_N)}{\mathrm{im}(d_{i+1}^P \otimes \mathrm{id}_N)} \otimes K \cong \frac{\ker(d_i^P \otimes \mathrm{id}_N) \otimes K}{\mathrm{im}(d_{i+1}^P \otimes \mathrm{id}_N) \otimes K}.$$

Finally, again as K is flat it $\_ \otimes_R K$ is exact and commutes with kernels and cokernels, whence

$$\frac{\ker(d_i^P \otimes \mathrm{id}_N) \otimes K}{\mathrm{im}(d_{i+1}^P \otimes \mathrm{id}_N) \otimes K} \cong \frac{\ker(d_i^P \otimes \mathrm{id}_N \otimes \mathrm{id}_K)}{\mathrm{im}(d_{i+1}^P \otimes \mathrm{id}_N \otimes \mathrm{id}_K)}.$$

From our calculations, it follows that

$$\mathrm{Tor}_i^R(M, N) \otimes K \cong \mathrm{Tor}_i^R(M, N \otimes K),$$

but $N \otimes K$ is torsionfree, hence

$$\mathrm{Tor}_i^R(M, N) \otimes K \cong \mathrm{Tor}_i^R(M, N \otimes K) = 0,$$

and $\mathrm{Tor}_i^R(M, N)$ is torsion.

List 8

Exercise 1