0.1 Self-Similar Models

Both Arnaud and Nagai start from the characteristic temperature for an isothermal sphere of mass M_{500}

$$kT_{500} = \mu m_p \frac{GM_{500}}{2R_{500}} \tag{1}$$

where M_{500} is given by

$$M_{500} = 500\rho_c(z)\frac{4\pi}{3}{R_{500}}^3 \tag{2}$$

and

$$\rho_c(z) = \frac{3H(z)^2}{8\pi G}. \tag{3}$$

The characteristic gas density is

$$\rho_{g,500} = 500 f_B \rho_c(z) \tag{4}$$

and the electron density is

$$n_{e,500} = \frac{\rho_{g,500}}{\mu_e m_p} \tag{5}$$

Putting it all together, we have

$$\begin{aligned} P_{500} = n_{e,500} \times kT_{500} &= \frac{\rho_{g,500}}{\mu_e m_p} \times \mu m_p \frac{GM_{500}}{2R_{500}} \\ &= \frac{\mu}{\mu_e}\rho_{g,500} \times \frac{GM_{500}}{2R_{500}} \\ &= \frac{\mu}{\mu_e} 500 f_B \rho_c(z) \times \frac{GM_{500}}{2R_{500}} \end{aligned} \tag{6}$$

From Eq.2 we have

$$R_{500} = \left(\frac{3}{4\pi}\frac{M_{500}}{500\rho_c(z)}\right)^{1/3} \tag{7}$$

whence

$$P_{500} = \frac{\mu}{\mu_e} 500 f_B \rho_c(z) \times \frac{GM_{500}}{2R_{500}}$$

$$
\begin{aligned}
&= \frac{\mu}{\mu_e} 500 f_B \rho_c(z) \times \frac{G M_{500}}{2} \left(\frac{4\pi}{3} \frac{500\rho_c(z)}{M_{500}} \right)^{1/3} \\
&= \frac{\mu}{\mu_e} f_B G \frac{3}{8\pi} \left(\frac{4\pi}{3} 500 \rho_c(z) \right)^{4/3} M_{500}^{2/3} \\
&= \frac{\mu}{\mu_e} f_B G \frac{3}{8\pi} \left(\frac{4\pi}{3} 500 \frac{3H(z)^2}{8\pi G} \right)^{4/3} M_{500}^{2/3} \\
&= \frac{\mu}{\mu_e} f_B \frac{3}{8\pi} \left(\frac{500 G^{-1/4} H(z)^2}{2} \right)^{4/3} M_{500}^{2/3} \qquad (8)
\end{aligned}
$$

Substituting $f_B = 0.175, \mu = 0.59, \mu_e = 1.14, G = 6.67384 \times 10^{-8}\mathrm{cm}^3/(\mathrm{g\,s}^2)$, $h(z) \equiv H(z)/H_0$, we have

$$
\begin{aligned}
P_{500} &= \frac{\mu}{\mu_e} f_B \frac{3}{8\pi} \left(\frac{500 G^{-1/4} H(z)^2}{2} \right)^{4/3} M_{500}^{2/3} \\
&= (4197.54)\, H(z)^{8/3} M_{500}^{2/3} \\
&= (4197.54)\, h(z)^{8/3} H_0^{8/3} M_{500}^{2/3} \\
&= (4197.54)\, (70\ \mathrm{km/s/Mpc})^{8/3} \, (3 \times 10^{14} M_\odot)^{2/3} \, h(z)^{8/3} h_{70}^{8/3} \left[\frac{M_{500}}{3 \times 10^{14} M_\odot} \right]^{2/3} \\
&= 1.65 \times 10^{-3}\, h(z)^{8/3} h_{70}^{8/3} \left[\frac{M_{500}}{3 \times 10^{14} M_\odot} \right]^{2/3} \mathrm{keV\,cm^{-3}} \qquad (9)
\end{aligned}
$$

This is to be compared to Eq 13 of A10, with which it agrees exactly. I therefore conclude that there are no arithmetic errors in the derivation of Arnaud's P500/M500 relationship.

Note that if we instead normalize M_{500} to $1 \times 10^{15} M_\odot$, and take $h_{70} = 1$ or $h = 0.7$, as Nagai et al do, we have:

$$
P_{500} = 5.89 \times 10^{-12}\, h(z)^{8/3} \left[\frac{M_{500}}{1 \times 10^{15} M_\odot} \right]^{2/3} \mathrm{erg\,cm^{-3}} \qquad (10)
$$

This is to be compared to Eq 3 of Nagai et al, with $h = 0.7$, which yields

$$
P_{500} = 1.14 \times 10^{-11}\, h(z)^{8/3} \left[\frac{M_{500}}{1 \times 10^{15} M_\odot} \right]^{2/3} \mathrm{erg\,cm^{-3}} \qquad (11)
$$

(Note that $h(z)$ is equivalent to the $E(z)$ of Nagai et al. And N07 actually take $h = 0.72$, but I've used $h = 0.7$ for simplicity) The ratio of these two expressions is very close to 2. I conclude that there is really a factor of 2 difference in the pressure predicted by N07 and A10 for a given M_{500}. This means that when using the A10 normalization (instead of the N07), we require a cluster of twice the M_{500} to reproduce the same SZ signal.

The claim in N07 is that they start from the same equations I used to derive Eq. 9, but if I've slipped a factor of 2 I don't know where it is.