Questão 1

$$f(x,y) = x^2 + xy + y^2 - 3x$$

$$\nabla f(x,y) = (2x+y-3,\ x+2y) = (0,0) \iff \begin{cases} 2x+y-3=0 \\ x+2y = 0 \end{cases}$$

$$\iff \begin{cases} 2x+y=3 & (1) \\ x=-2y & (2) \end{cases}$$

Substituindo (2) em (1)

$$2(-2y)+y=3 \implies -4y+y=3$$
$$\implies -3y=3, \text{ ou seja, } y=-1$$

Substituindo $y=-1$ em (2) temos que $x=-2(-1)=2$

$\therefore$ $(2,-1)$ é o único ponto crítico de f

Por outro lado

$$f_{xx}(x,y)=2 \qquad f_{xy}(x,y)=1$$
$$f_{yx}(x,y)=1 \qquad f_{yy}(x,y)=2$$

são funções contínuas em $\mathbb{R}^2$. Então, podemos usar o Teste da Derivada Segunda:

$$D(A_f) = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 4-1=3$$

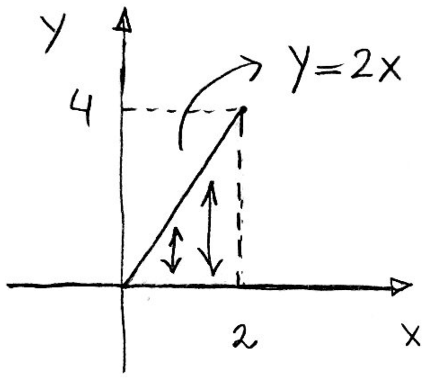
Pto crítico	$f(x,y)$	$D(A_f)$	$f_{xx}(y)$	Conclusão
$(2,-1)$	$\underbrace{(2)^2+2(-1)+(-1)^2-3(2)}_{4-2+1-6=-3}$	$3>0$	$2>0$	MÍN

$\therefore$ $(2,-1)$ é um pto de MÍN global

Questão 2

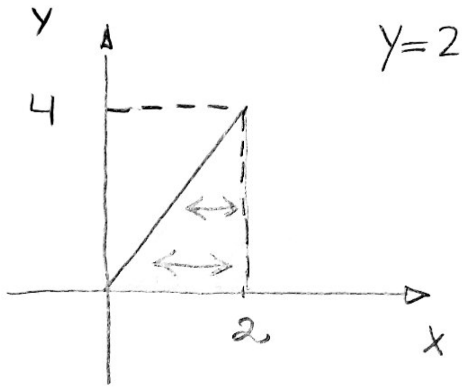
$$\int_0^2 \int_0^{2x} f(x,y)\,dy\,dx = \int_0^2 \Bigg[\underbrace{\int_{y=0}^{y=2x} f(x,y)\,dy}_{\text{"Variação vertical"}} \Bigg] dx = \iint_D f(x,y)\,dA$$

$$D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2 \text{ e } 0 \leq y \leq 2x \} \longleftarrow \text{Tipo I}$$



Tipo I

$y = 2x \Rightarrow x = \dfrac{y}{2}$



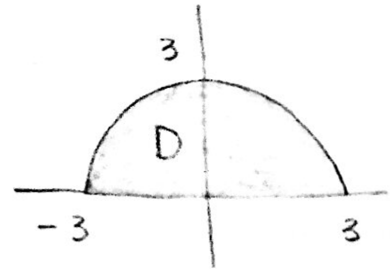
Tipo II

$$\Rightarrow D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 4 \text{ e } \tfrac{y}{2} \leq x \leq 2 \}$$

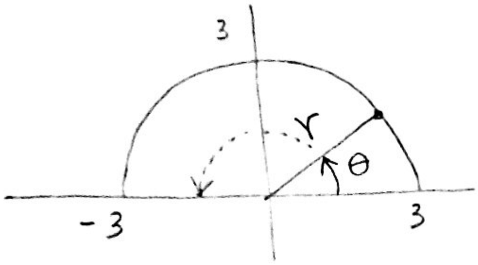
logo

$$\boxed{\int_0^2 \int_0^{2x} f(x,y)\,dy\,dx = \int_0^4 \int_{y/2}^{2} f(x,y)\,dx\,dy}$$

Questão 3



$$\iint_D x\, dA \underset{\substack{\uparrow \\ \text{Usando} \\ \text{coordenadas polares} \\ x = r\cos\theta \\ y = r \operatorname{sen}\theta}}{=} \iint_Q (r\cos\theta)\, \underset{\substack{\uparrow \\ \text{Jacobiano}}}{r}\, dr\, d\theta$$



$$Q = \{ (r,\theta) \mid 0 \leq r \leq 3 \text{ e } 0 \leq \theta \leq \pi \}$$

$$= \int_0^{\pi}\int_0^{3} r^2 \cos\theta\, dr\, d\theta = \int_0^{\pi} (\cos\theta)\, \frac{r^3}{3}\Bigg|_{r=0}^{r=3} d\theta$$

$$= \int_0^{\pi} \left(\frac{3^3}{3} - \frac{0^3}{3} \right) \cos\theta\, d\theta = 9 \int_0^{\pi} \cos\theta\, d\theta$$

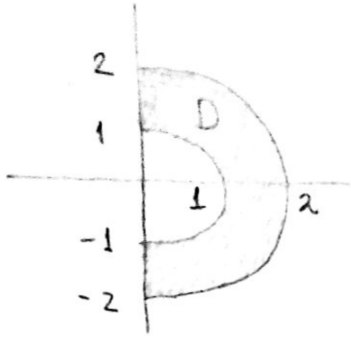
$$= 9 \operatorname{sen}\theta \Big|_{\theta=0}^{\theta=\pi} = 9(\operatorname{sen}\pi - \operatorname{sen} 0) = 0$$

OBS 1: $\iint_D x\, dA$ NÃO é a área da região D

OBS 2: Como $D = \{ (x,y) \in \mathbb{R}^2 \mid -3 \leq x \leq 3 \text{ e } 0 \leq y \leq \sqrt{9-x^2} \}$ é uma região tipo I, então

$$\iint_D x\, dA = \int_{-3}^{3}\int_0^{\sqrt{9-x^2}} x\, dy\, dx = \int_{-3}^{3} x\left[\sqrt{9-x^2} - 0\right] dx = -\frac{1}{3}(9-x^2)^{3/2}\Bigg|_{x=-3}^{x=3} = 0$$

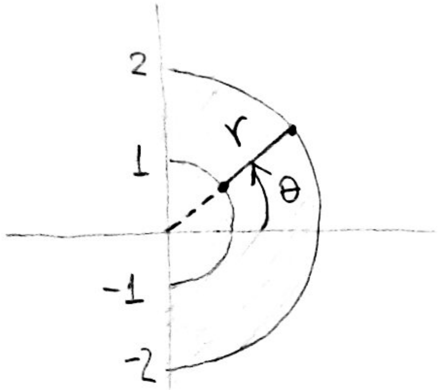
Questão 3



$$\iint_X x\, dA \underset{\substack{\uparrow \\ \text{Coordenadas} \\ \text{Polares}}}{=} \iint_Q (r\cos\theta)\underset{\substack{\uparrow \\ \text{Jacobiano}}}{r}\, dr\, d\theta$$

$x = r\cos\theta$

$y = r\operatorname{sen}\theta$



$$Q = \left\{ (r,\theta) \,\middle|\, 1 \leq r \leq 2 \ \text{ e } \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\}$$

ou

$$Q = \left\{ (r,\theta) \,\middle|\, 1 \leq r \leq 2 \ \text{ e } \ \frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2} \right\}$$

$$= \int_{-\pi/2}^{\pi/2} \int_1^2 r^2 \cos\theta\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^3}{3} \Bigg|_{r=1}^{r=2} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{2^3}{3} - \frac{1^3}{3} \right) \cos\theta\, d\theta = \frac{7}{3} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{7}{3} \operatorname{sen}\theta \Bigg|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} = \frac{7}{3}\left(\operatorname{sen}\left(\frac{\pi}{2}\right) - \operatorname{sen}\left(-\frac{\pi}{2}\right) \right) = \frac{7}{3}\left(1 - (-1)\right) = \frac{14}{3}$$

<u>OBS1</u>: $\iint x\, dA$ NÃO é a área da região D

OBS 2 Sejam

$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1 \text{ e } 0 \leq x \leq \sqrt{1-y^2}\}$$

$$D_2 = \{(x,y) \in \mathbb{R}^2 \mid -2 \leq y \leq 2 \text{ e } 0 \leq x \leq \sqrt{4-y^2}\}$$

ambas regiõs Tipo II. Então

$$D_1 \cup D = D_2$$

Logo

$$\iint_{D_2} x\,dA = \iint_{D_1 \cup D} x\,dA = \iint_{D_1} x\,dA + \iint_{D} x\,dA$$

$$\therefore \iint_{D} x\,dA = \iint_{D_2} x\,dA - \iint_{D_1} x\,dA$$

$$\iint_{D_2} x\,dA = \int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} x\,dx\,dy = \int_{-2}^{2} \frac{1}{2}(4-y^2)\,dy = \left(2y - \frac{1}{6}y^3\right)\Bigg|_{y=-2}^{y=2}$$

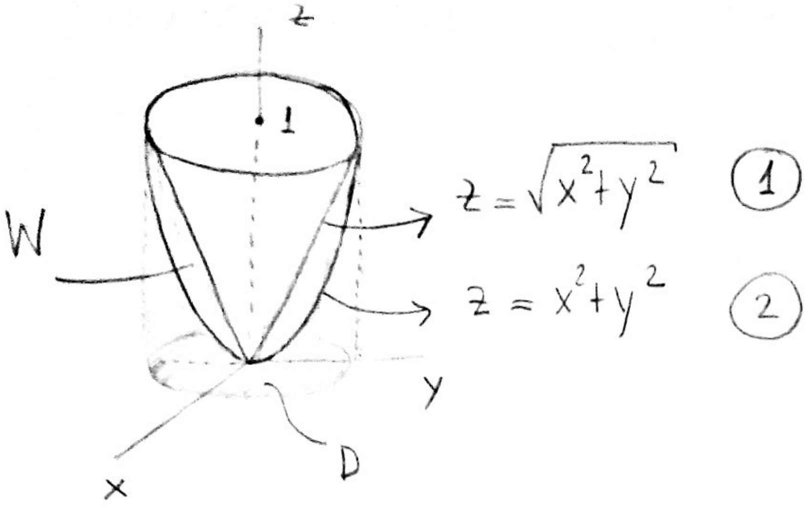
$$= \left(2.2 - \frac{1}{6}.8\right) - \left[2(-2) - \frac{1}{6}(-8)\right] = 8 - \frac{8}{3} = \frac{16}{3}$$

$$\iint_{D_1} x\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} x\,dx\,dy = \int_{-1}^{1} \frac{1}{2}(1-y^2)\,dy = \left(\frac{1}{2}y - \frac{1}{6}y^3\right)\Bigg|_{y=-1}^{y=1}$$

$$= \left(\frac{1}{2} - \frac{1}{6}\right) - \left(-\frac{1}{2} + \frac{1}{6}\right) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\therefore \iint_{D} x\,dA = \frac{16}{3} - \frac{2}{3} = 14/3$$

Questão 4



Substituindo (2) em (1)

$$z=\sqrt{z} \quad \text{ou} \quad z^2=z$$

Como

$$0=z^2-z=z(z-1)$$

$$\Rightarrow \begin{cases} z=0 \\ z=1 \end{cases}$$

Substituindo $z=1$ em (2)

$$x^2+y^2=1$$

Observemos que

$$W=\{(x,y,z)\in\mathbb{R}^2 \mid (x,y)\in D \ \text{e} \ (x^2+y^2)\leq z\leq\sqrt{x^2+y^2}\}$$

(Região sólida tipo I)

$$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1\}$$

$$\Rightarrow Vol(W)=\iiint_W dV=\iint_D\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dA$$

$$=\iint_D\left[\sqrt{x^2+y^2}-(x^2+y^2)\right]dA$$

Usando Coordenadas Polares

$x=r\cos\theta$
$y=r\,\mathrm{sen}\theta$

Jacobiano

$$=\iint_Q[r-r^2]\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^1(r^2-r^3)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(\frac{1}{3}r^3-\frac{1}{4}r^4\right)\Big|_{r=0}^{r=1}d\theta=\left(\frac{1}{3}-\frac{1}{4}\right)\cdot 2\pi=\frac{\pi}{6}$$

$$\therefore \boxed{Vol(W) = \frac{\pi}{6}}$$

OBS 1: Usando coordenadas cilíndricas temos que

Cone: $z = \sqrt{r}$

parabolóide: $z = r^2$

coordenadas cilíndricas

Jacobiano

$$\Rightarrow Vol(W) = \iiint_W dV = \iiint_{\tilde{Q}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{r}} r\,dz\,dr\,d\theta, \text{ onde}$$

$$\tilde{Q} = \{(z,r,\theta) \mid r^2 \leq z \leq \sqrt{r},\ 0 \leq r \leq 1 \text{ e } 0 \leq \theta \leq 2\pi\}$$

OBS 2: ERRO FREQUENTE

$$Vol(W) = Vol(\text{Parabolóide}) - Vol(\text{Cone})$$

$$= \iint_D (x^2+y^2)\,dA - \iint_D \sqrt{x^2+y^2}\,dA$$

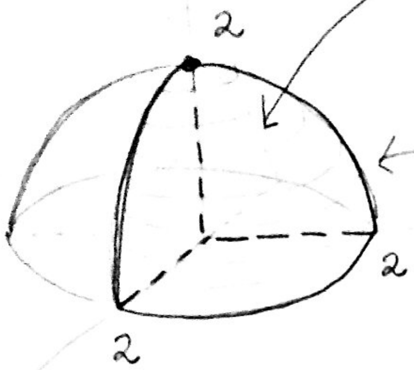
Note que

$$\iint_D (x^2+y^2)\,dA = \iint_D \int_0^{x^2+y^2} dz\,dA \equiv$$ Volume abaixo do parabolóide e acima do $z=0$

Idem para $\iint_D \sqrt{x^2+y^2}\,dA$

Questão 5

$$W = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0 \leq z \leq \sqrt{4-x^2-y^2} \\ x \geq 0 \\ y \geq 0 \end{array} \right\}$$

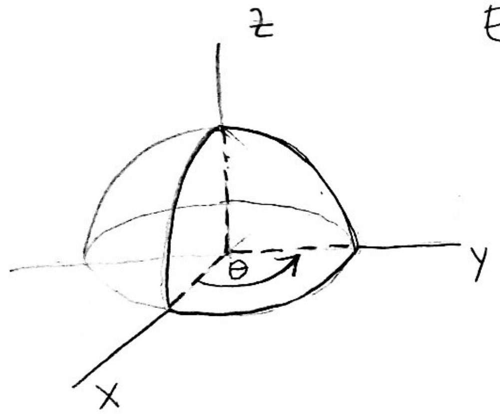


$\sqrt{4-x^2-y^2}$

Primeiro octante em $\mathbb{R}^3$: $x \geq 0$, $y \geq 0$, $z \geq 0$

$$\iiint_W z\sqrt{x^2+y^2+z^2}\, dV = \iiint_Q (\rho\cos\phi)\cdot\rho\cdot\underbrace{(\rho^2 \operatorname{sen}\phi)}_{\text{Jacobiano}}\, d\rho\, d\theta\, d\phi$$

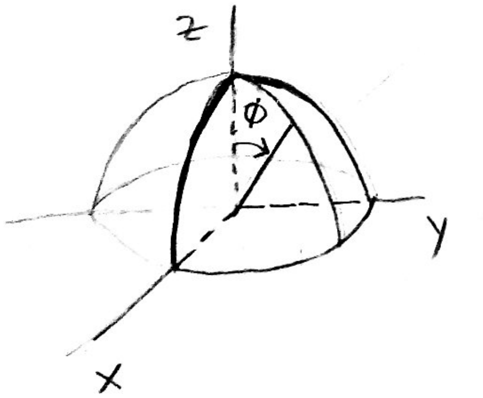
Coordenadas Esféricas



$$Q = \left\{ (\rho,\theta,\phi) \;\middle|\; \begin{array}{l} 0 \leq \rho \leq 2 \\ 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \phi \leq \frac{\pi}{2} \end{array} \right\}$$

$0 \leq \theta \leq \frac{\pi}{2}$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho^4 \cos\phi \operatorname{sen}\phi\, d\rho\, d\theta\, d\phi$$



$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{5}\rho^5 \Big|_{\rho=0}^{\rho=2} \cos\phi \operatorname{sen}\phi\, d\theta\, d\phi$$

$$= \frac{32}{5}\int_0^{\pi/2}\int_0^{\pi/2} \cos\phi \operatorname{sen}\phi\, d\theta\, d\phi$$

$0 \leq \phi \leq \frac{\pi}{2}$

$$= \frac{32}{5}\cdot\frac{\pi}{2}\int_0^{\pi/2} \cos\phi \operatorname{sen}\phi\, d\phi = \frac{16}{5}\pi\left(\frac{1}{2}\operatorname{sen}^2\phi\right)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}$$

$$= \frac{16}{5}\pi \cdot \frac{1}{2}\left[\operatorname{sen}^2\left(\frac{\pi}{2}\right) - \operatorname{sen}^2(0)\right]$$

$$= \frac{8}{5}\pi\left(1^2 - 0^2\right)$$

$$= \frac{8}{5}\pi$$