① $\underset{}{\mathrm{argmin}}\ \frac{1}{2} w' \Sigma w$

s.t $\quad w' \cdot \mathbb{1} = 1$

—— // LAGRANGE // ——

$$L(w, \lambda) = \frac{1}{2} w' \Sigma w + \lambda (1 - w' \mathbb{1})$$

—— // FINDING PARTIAL DERIVATIVES ——

$$\frac{\partial L(w, \lambda)}{\partial w} = \frac{2}{2} \Sigma w^{①} - \lambda \mathbb{1} = 0$$

① Chain rule for vector function.

$Y = X' A X$

$$\frac{\partial y}{\partial x} = d(X') A X + X' A\, d(X)$$

$$= A X + X' A$$

$$= A X + A' X \quad \left\{ \Sigma \text{ is simetric as } A \text{ is simetic} \right.$$

$$= 2 A X //$$

—— // ——

$$\frac{\partial L(w, \lambda)}{\partial \lambda} = 1 - w' \mathbb{1} = 0$$

1 Continuation.

" STEP 1 → SOLVING FOR ω "

$$\Sigma \omega - \lambda \mathbb{1} = 0$$

$$\Sigma \omega = +\lambda \mathbb{1} \quad \times \Sigma^{-1} \text{ on the left side}$$

$$\underbrace{\Sigma^{-1}\Sigma}_{I_m \to \text{Identity Matrix}} \omega = \Sigma^{-1} \lambda \mathbb{1}$$

$$\omega = \lambda \Sigma^{-1} \mathbb{1}$$

" STEP 2 → SOLVING FOR λ "

$$1 - \omega' \mathbb{1} = 0$$

$$\omega' \mathbb{1} = 1$$

$$\mathbb{1}' \cdot \omega = 1$$

FROM STEP 1 "

$$\mathbb{1}' \cdot [\lambda \Sigma^{-1} \mathbb{1}] = 1$$

$$\lambda \underbrace{\mathbb{1}' \Sigma^{-1} \mathbb{1}}_{\text{SCALAR}} = 1$$

$$\lambda = \frac{1}{\mathbb{1}' \Sigma^{-1} \mathbb{1}}$$

" STEB 3 MOVING BACK TO "

$$\omega = \lambda \Sigma^{-1} \mathbb{1}$$

$$\omega = +\lambda \Sigma^{-1} \mathbb{1}$$

$$\boxed{\omega^* = \frac{\Sigma^{-1} \mathbb{1}}{\mathbb{1}' \Sigma^{-1} \mathbb{1}}}$$

② $\underset{}{\operatorname{argmin}} \frac{1}{2} w' \Sigma w$

S.T $\quad r + (\mu - r\mathbb{1})' w = 0{,}1$

// MODELING LAGRANGE //

$$L(w, \lambda) = \frac{1}{2} w' \Sigma w + \lambda (0{,}1 - r - (\mu - r\mathbb{1})' w)$$

// FINDING PARTIAL DERIVATIVES //

EXERCISE 1 was a PROOF OF THIS DERIVATION by CHAIN RULE OVE VECTOR.

$$\frac{\partial L(w, \lambda)}{\partial w} = \frac{2}{2} \Sigma w - \lambda (\mu - r\mathbb{1}) = 0 \quad \text{I}$$

$$\frac{\partial L(w, \lambda)}{\partial \lambda} = 0{,}1 - r - (\mu - r\mathbb{1})' w = 0 \quad \text{II}$$

// SOLVING FOR w //

$$\Sigma w = \lambda (\mu - r\mathbb{1})$$

$\times \Sigma^{-1}$ ON LEFT SIDE

$$\underbrace{\Sigma^{-1} \Sigma}_{\text{IDENTITY MATRIX}} w = \Sigma^{-1} \lambda (\mu - r\mathbb{1})$$

$$I_n w = \Sigma^{-1} \lambda (\mu - r\mathbb{1})$$

$$w = \lambda \Sigma^{-1} (\mu - r\mathbb{1})$$

// confirm if it is a minimum or MAXIMUM

$$\frac{\partial^2 L(w, \lambda)}{\partial w^2} = \Sigma$$

by definition, SIMETRIC MATRIX A DEFINITE POSITIVE

The, $\frac{\partial^2 (w, \lambda)}{\partial w^2} \geq 0$

$\rightarrow w' \Sigma w \geq 0$, $\boxed{\text{IS A MINIMUM}}$ //

2 CONTINUATION ...

II $0{,}1-r-(\mu-r\mathbb{1})'w=0$.

$$-(\mu-r\mathbb{1})'w=-0{,}1+r$$

$$(\mu-r\mathbb{1})'\underline{w}=0{,}1-r.$$

$$(\mu-r\mathbb{1})'\overbrace{\lambda\Sigma^{-1}(\mu-r\mathbb{1})}^{w}=0{,}1-r$$

$$\lambda\,\underbrace{(\mu-r\mathbb{1})'\Sigma^{-1}(\mu-r\mathbb{1})}_{A}=0{,}1-r.$$

$$A=(\mu-r\mathbb{1})'\Sigma^{-1}(\mu-r\mathbb{1})\Rightarrow$$ IT IS A SCALAR, THEN I CAN WORK ON IT AS IT IS A CONSTANT

$$\lambda A=0{,}1-r$$

$$\lambda=\frac{0{,}1-r}{A}\ //$$

— MOVING BACK TO RESULT I —

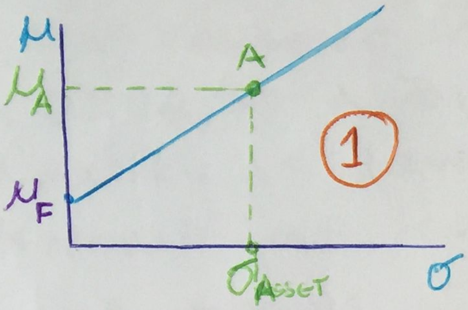
$$w=\lambda\Sigma^{-1}(\mu-r\mathbb{1})$$

$$w=\left(\frac{0{,}1-r}{A}\right)\cdot\Sigma^{-1}(\mu-r\mathbb{1})$$

$\underbrace{\boxed{n}\times n\times n\times\boxed{1}}_{n\times 1}$ The dimention match //

3 Definition of TANGENCY PORTFOLIO

When we have an asset risk free and a risk asset, the graph μ - σ is such as below



(A) is the point where the portfolio is 100% invested in the risk asset

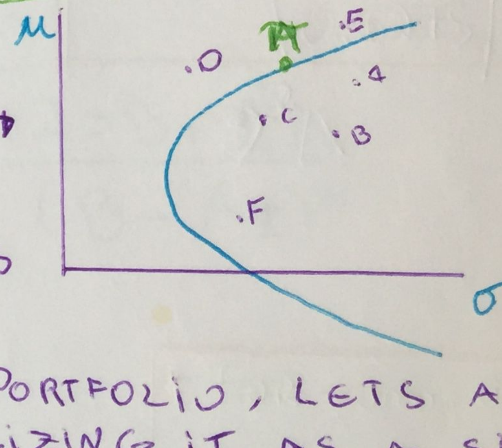
LONG RISK FREE $0 \leq W_A \leq 100\%$

SHORT RISK FREE LEVERAGED ON RISKY ASSET $\longrightarrow W_A > 100\%$

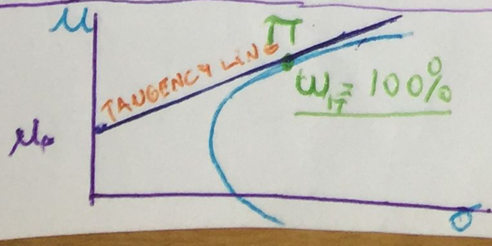
When we have 2 or more assets that carry risk, EACH PAIR of assets will have a correlation that will be in the interval $-1 \leq \rho \leq 1$, the axis σ will follow the equation

$$\sigma_\pi = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \boxed{2\sum_{\substack{i=1 \\ j>1}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j}}$$

The second term will afect the curve, and it will be $\rightarrow$ the portfolio π will be in any point of this curve.



To find the tangency portfolio, lets abstract this portfolio by analizing it as a single risk asset, for that, we assume that we are 100% invested into portfolio. It implies that



the tangency portfolio can be replicated (leveraged), or being a combination of risk free and risky asset (π), in the same way of 1. This is the tangency line portfolio.

3_ Analitical Risk

1_ $N(\mu, \sigma^2 T) \sim N(\mu, \sigma')$

by definition

$$z = \frac{X - \mu}{\sigma'} \Rightarrow z = \frac{X - \mu}{\sigma \sqrt{\tau}}$$

$$X - \mu = z\,\sigma\sqrt{\tau}$$

$$X = \mu + \sigma\sqrt{T}\,\Phi^{-1}(1-C)$$

Since z ~~returns~~ IS the number of std ~~of~~ of a N(0,1), its inverse from the interval of confidence, will return z //

B_ Analytical Risk.

_FIND THE MEAN AND VARIANCE

* $E[X] = \bar{x}$, $E[Y] = \bar{Y}$

$$E[w_x X + (1-w_x)Y] = E[w_x X] + E[(1-w_x)Y]$$
$$= w_x E[X] + (1-w_x) E[Y]$$
$$= w_x \bar{X} + (1-w_x)\bar{Y}$$
$$= w_x \bar{X} + \bar{Y} - w_x \bar{Y}$$
$$= \bar{Y} + w_x(\bar{X} - \bar{Y})$$

the EXPECTATION is THE weigthed sum of the EXPECTATION FROM INDIVIDUAL ASSETS.

* $a + b = 1$ → IT WILL REPRESENT ~~THE~~ WEIGTHS.

$$Var(aX+bY) = E\left[\left(aX + bY - E(aX+bY)\right)^2\right]$$
$$= E\left[\left(aX - E[aX] + bY - E[bY]\right)^2\right]$$
$$= E\left[\left(a(X - E[X]) + b(Y - E[Y])\right)^2\right]$$
$$= E\Big[a^2 \underbrace{(X - E[X])^2}_{Var(X)} + b^2 \underbrace{(Y - E[Y])^2}_{Var[Y]} + 2ab(X-E[X])(Y-E[Y])\Big]$$

$$= a^2 \underbrace{E[(X - E[X])^2]}_{Var[X]} + b^2 \underbrace{E[(Y - E[Y])^2]}_{Var[Y]} + 2ab \underbrace{E[(X-E[X])(Y-E[Y])]}_{cov(X,Y)}$$

$$\rho_{x,y} = \frac{cov[X,Y]}{\sigma_x \sigma_y}$$

$$cov[X,Y] = \rho_{x,y}\sigma_x\sigma_y$$

$$= a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho_{xy}\sigma_x\sigma_y$$

where $a+b = 1$ //

Prove sub-additivity of VaR(N+M) ≤ B.1

$$VaR(\underset{\text{II}}{K+M}) \leq \underbrace{VaR(K)+VaR(M)}_{\text{I}}$$

I $VaR(K)+VaR(M) = \mu_K + \sigma_K\sqrt{T}\cdot\Phi^{-1}(1-c)$

$+ \mu_M + \sigma_M\sqrt{t}\cdot\Phi^{-1}(1-c)$

$$\mu_K+\mu_M+(\sigma_K+\sigma_M)\cdot\Phi^{-1}(1-c)\cdot\sqrt{t}$$

II $VaR(K+M)$

$\mu_\pi = \mu_K + \mu_M$

$\sigma_\pi = \sqrt{\sigma_K^2+\sigma_M^2+2\rho\sigma_A\sigma_B}$

$VaR(K+M) = \mu_K+\mu_M+\sigma_\pi\,\Phi^{-1}(1-c)\cdot\sqrt{t}$

PROOF.

$VaR(K+M) \leq VaR(K)+VaR(M)$

① simplifying Terms.

$$\cancel{\mu_K}+\cancel{\mu_M}+\sqrt{\sigma_K^2+\sigma_M^2+2\rho_{KM}\sigma_K\sigma_M}\,\cancel{\sqrt{t}}\cdot\cancel{\Phi^{-1}(1-c)} \leq$$

$$\cancel{\mu_K}+\cancel{\mu_M}+(\sigma_K+\sigma_M)\cancel{\sqrt{t}}\cdot\cancel{\Phi^{-1}(1-c)}$$

② squaring both sides.

$$\left(\sqrt{\sigma_K^2+\sigma_M^2+2\rho\sigma_K\sigma_M}\right)^2 \leq (\sigma_K+\sigma_K)^2$$

③ simplifying...

$$|\cancel{\sigma_K^2}+\cancel{\sigma_M^2}+2\rho\sigma_K\sigma_M| \leq |\cancel{\sigma_K^2}+\cancel{\sigma_M^2}+2\sigma_K\sigma_M|$$

$$|\cancel{2}\rho_K\cancel{\sigma_K}\cancel{\sigma_M}| \leq |\cancel{2}\cancel{\sigma_K}\cancel{\sigma_M}|$$

$$|\rho_{KM}| \leq 1$$

$$\Longrightarrow -1 \leq \rho_K \leq 1$$

∴ Since the correlation between assets exists., it is proved that

$VaR(K+M) \leq VaR(K)+VaR(M)$

B. Analytical Risk

2 - Show $\Sigma = AA'$

$$\begin{pmatrix} \sigma_1 & \emptyset \\ \rho\sigma_2 & \sqrt{1-\rho^2}\,\sigma_2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & \rho\sigma_2 \\ \emptyset & \sqrt{1-\rho^2}\,\sigma_2 \end{pmatrix} = \overset{\Sigma}{\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}}$$

$A \quad \cdot \quad A'$ $\quad \Sigma$ (covariance matrix)

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & (\rho^2\sigma_2^2 + \sqrt{1-\rho^2}\cdot\sigma_2 \cdot \sqrt{1-\rho^2}\,\sigma_2) \end{pmatrix} = \Sigma$$

(*) $\rho\sigma_2^2 + (1-\rho^2)\cdot\sigma_2^2 = \cancel{\rho\sigma_2^2} + \sigma_2^2 - \cancel{\rho^2\sigma_2^2}$

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2 \end{pmatrix} = \Sigma \quad ///$$

B-2- Write down the results of correlated $Y_1(t)$ and $Y_2(t)$ from $Y = AX$

$$\begin{vmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\,\sigma_2 \end{vmatrix} \cdot \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} Y_1(t) \\ Y_2(t) \end{vmatrix}$$

$$Y_1(t) = \sigma_1 X_1$$

$$Y_2(t) = \rho\sigma_2 X_1 + \sqrt{1-\rho^2}\,\sigma_2 X_2$$

$$\rightarrow Y_2(t) = \frac{Y_1(t)}{\sigma_1} \cdot \rho\sigma_2 + \sqrt{1-\rho^2}\,\sigma_2 X_2.$$

Does the ~~properties~~ keep properties of Brownian motion, if $X_1(t), X_2(t)$ are standard normal?

Note, consider the distribution of $Y_2(t)$ increments its variance

NO. IT WILL NOT KEEP THE PROPERTIES OF BROWINA M. IT WILL NOT BE STATIONARY ANYMORE,

1.) IT WILL NOT BE STATIONARY ANYMORE, To be STATIONARY, The joint probability, mean and variance must be constant. AS $Y(t)_2$ WIL INCREMENT ITS ~~VARIANE~~ VARIANCE, this PROPERTY WILL FAIL.

2.) BROWNION MOTION has a STANDARD NORMAL Distribution $N(0, t-s)$, Since $Y(2)$ will HAVE INCREMENTS ON ITS VARIANE, IT WILL BECAME SUCH AS $N(0, 2\sigma + t-s)$. AS CONSEQUENCE

$$V[W(t) - W(s)] \neq t-s$$

$$W_{t1} - W_{t0} \neq W_{t2} - W_{t1} \neq W_{t3} - W_{t2}$$

To be a BROWINIAN motion, THESE TWO PROPERTIES ABOVE should BE KEPT.