

1 Some Applications of Bewley Models

1.1 Precautionary Savings

The previous graph allows to examine the amount of savings that would occur in the economy with full insurance. Under full insurance, the Euler equation implies $(1 + r)\beta = 1$, hence the supply of capital is infinitely elastic at $r = 1/\beta - 1$. The point where this horizontal line crosses $K(r)$ represents the stock of capital of the neoclassical deterministic growth model, K^{FI} . The magnitude $(K^* - K^{FI})$ is the amount of aggregate capital accumulated for self-insurance. To express this comparison in terms of saving rates, note that with Cobb-Douglas production function with capital share α ,

$$r + \delta = \alpha \left(\frac{Y}{K} \right) = \alpha \delta \left(\frac{Y}{\delta K} \right) = \frac{\alpha \delta}{s} \Rightarrow s = \frac{\alpha \delta}{r + \delta},$$

hence there is a one-to-one mapping between the equilibrium interest rate and the saving rate: differences between r^* and $1/\beta - 1$ translate directly into the precautionary saving rate.

1.1.1 How much saving for self-insurance in the US?

Aiyagari (1994) calibrates the model to replicate certain key facts of the U.S. economy.¹ He finds that with log utility and *iid* shocks, the precautionary saving rate is approximately zero, i.e. $r^* \simeq 1/\beta - 1$ (probably a lower bound). The reason is that agents with low risk aversion who face income shocks with low persistence are not too concerned about the missing insurance markets and do accumulate little capital for self-insurance. With *CRRA* utility, risk-aversion parameter equal to 5, and autocorrelation of the income shocks of 0.9, the size of precautionary savings 14% of aggregate output (probably an upper bound). These households are much more concerned about self-insurance since with high persistence income can remain low for a long time, plus for them consumption fluctuations are very costly.

These are both extreme parameterizations. A reasonable estimate, in a model economy calibrated to the U.S. with risk aversion around 2, the precautionary saving rate would be

¹One attractive feature of this class of neoclassical growth models with idiosyncratic earnings risk is that they can be calibrated exactly as in the same way as the representative agent model, except for the stochastic process that governs earnings risk. This process is parametrized by using empirical microeconomic studies on earnings dynamics based on panel data surveys, like the Panel Study of Income Dynamics (PSID).

5% of aggregate income, i.e. roughly $1/4$ of total aggregate savings. Note the importance of equilibrium considerations here: in the income fluctuation problem, one can always set r close enough to $1/\beta - 1$ in order to generate any desired amount of savings (even $+\infty$). The equilibrium model imposes more discipline because, given demand for capital, there is only one value of the interest rate consistent with the consumption/saving decisions of the households.²

1.1.2 Comparative Statics on Precautionary Saving

The Figure depicting the equilibrium is helpful to perform some comparative statics on the equilibrium. We are interested in the effects on the borrowing constraint b , risk aversion, the persistence of the shock and the variance of the shock.

Borrowing limit— Suppose we increase b , i.e. we slacken the liquidity constraint and increase the maximum amount that can be borrowed by the individual. Graphically, the asset supply curve $A(r)$ shifts upward, with $K(r)$ constant, which leads to a rise in the interest rate and a reduction of precautionary savings. The interest rate increases because more individuals have negative wealth, so the supply of capital falls at any given r . Note here an important point: agents can both save and borrow for self-insurance. The availability of a generous borrowing limit reduces the need for precautionary saving.

Risk aversion— The $K(r)$ curve is unaffected by changes in preferences. If we raise risk aversion, the $A(r)$ curve shifts downward: individuals are more concerned about consumption smoothing, so they cumulate higher buffer-stock savings: for any given r , $A(r)$ is larger. This leads to a lower equilibrium interest rate.

Changes in the income process— Suppose we increase the variance of the uninsurable income shock. $K(r)$ is unchanged, but the supply of capital $A(r)$ would go up (curve shifts down), as individuals cumulate more savings to cope with the higher uninsurable uncertainty of their income. Increasing the persistence of the shock has a similar effect.

²In particular, the logic that the closer is $1+r$ to $1/\beta$, the higher are savings is reversed in equilibrium because $(1+r)\beta = 1$ is the complete-markets benchmark which has the lowest saving rate (no saving for self-insurance).

1.2 Understanding Wealth Inequality

The Bewley models contain a theory of consumption and wealth inequality. In the model, agents are ex-ante equal and become different as time goes by due to variation in the realization of their income shocks. It is a theory of inequality based largely on luck. In response to shocks, they choose optimally how much to consume and how much to save. Hence, different paths of shocks induce different levels of consumption and wealth across agents. Note that, in the absence of endogenous labor supply, the model has nothing to say on earnings (wages times hours worked) inequality.

For a full description of facts on earnings and wealth inequality, one should refer to a recent paper by Budria, Diaz-Gimenez, Quadrini and Rios-Rull (2002). The following table, reproduced from the paper above, provides the key statistics for the US (from the *Survey of Consumer Finances*, 1998). The key fact to observe is that wealth is much more unequally distributed than earnings. Both distributions are skewed (mean>median), but the wealth distribution much more so that the earnings distribution: the top 1% of the wealth distribution owns 30% of the US wealth.

U.S. SCF (1998). Values of earnings and wealth are in thousands of 1998 \$											
	Mean	$\frac{Mean}{Median}$	Gini	CV	Q1	Q3	Q5	Top 1%	Top 5%	Bottom 5%	<i>Share of top 1%</i>
Earnings	21.1	1.57	0.61	2.65	-0.7	20.7	101.9	491	136	0.0	7.5%
Wealth	47.4	4.03	0.80	6.53	-2.3	51	770	5,988	1,150	-4.7	31%

After the model is calibrated and simulated, we can use it as a measurement tool for following question: how far can we go in explaining wealth inequality when idiosyncratic earnings shocks are the only source of heterogeneity among households? The typical answer is that the standard model generates too much asset holdings at the bottom and too little at the top: the Gini generated by the model economy is around 0.4 –much smaller than the data value 0.8.

This standard model needs to be modified to 1) introduce an extra incentive for the rich to accumulate capital, 2) reduce the incentives for the poor to save for self-insurance purposes.

Inequality at the bottom– Modelling carefully the welfare state goes a long way in generating the right amount of asset holdings at the bottom. Once we introduce public

insurance schemes (e.g., social security, housing benefits, child benefits, unemployment insurance), the incentives for private self-insurance are much reduced because some of these benefits are means-tested. See, for example, Hubbard, Skinner and Zeldes (1995).

Inequality at the top— To improve the quantitative explanation of inequality at the top, several alternatives have been pursued. 1) Quadrini (2000) explores the role of entrepreneurship. Implicitly, entrepreneurs have a higher return on their investment, hence a stronger incentive to accumulate. Empirically, a large fraction of wealth at the top is held by entrepreneurs. 2) Krusell and Smith (1997) study heterogeneity in discount factors. A Markov process regulates transitions between two levels of patience $\{\beta_L, \beta_H\}$. In the β_H state, households are more patient and save more. Small differences in β lead to a jump in the wealth Gini. 3) De Nardi (2003) studies the role of bequest. If (rich) households have a stronger bequest motive than poor households, this represents an additional reason to save. 4) Castaneda, Diaz-Jimenez and Rios-Rull (2003) add to an otherwise standard income process a very high realization of earnings (roughly 200 times larger than the mean) which occurs with a very low probability. They argue that income data are top-coded, so one does not observe these realizations in the data, even though they exist.

1.3 A Bewley Model with Entrepreneurship

We have argued that, empirically, a large fraction of wealth at the top is held by entrepreneurs. Here we want to write down a model where we have both entrepreneurs (individuals owning their own business) and workers (individuals working for someone else) and argue that it can replicate the upper tail in the wealth distribution. We follow Quadrini (2000) and Kitao (2008).

Demographics— Agents are infinitely lived. Every period agents choose an occupation. Entrepreneurs run their own business, and workers supply labor in the market. Entrepreneurs can manage one project which combines her managerial ability, capital and labor.

Endowments— Each agent is endowed with labor productivity $\varepsilon \in E$ and entrepreneurial ability $\theta \in \Theta$. The joint process is regulated by a Markov chain $\Gamma_{\varepsilon, \theta}$.

Preferences— They are standard, time-separable preferences defined over streams of

consumption, with discount factor β

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Technology– There are two production sectors. The corporate sector is CRS with production $Y = F(K, H)$. Capital is rented by the corporate firms at the risk-free rate r and depreciates at rate δ . The non-corporate sector is composed by many entrepreneurs who run their own project according to the production function $y = f(k, n, \theta) = \theta k^{\nu_1} n^{\nu_2}$, with $\nu_1 + \nu_2 < 1$. This formulation mirrors the classic span of control model developed by Lucas (1978), i.e. entrepreneurial ability determines the size of the firm. Also, note that $(1 - \nu_1 - \nu_2)$ is the share of output retained by the entrepreneurs as rents or profits from his managerial skills.

Financial markets– Workers cannot borrow to finance consumption. Households with savings are indifferent between lending directly to the corporate sector and lending to the banking sector. Both sectors pay an interest rate r . The banking sector allocates funds to entrepreneurs. The banking sector is competitive and has an operation cost ϕ units of the final good for any unit of capital intermediated. Therefore, entrepreneurs can borrow to finance their project at rate $r_d = r + \phi$. Entrepreneurs can borrow only up to a fraction d of their assets. Hence, they can invest in their project up to $(1 + d)a$.

Problems of the worker and entrepreneur– The problem of the worker is written as

$$\begin{aligned} V^w(a, \varepsilon, \theta) &= \max_{c, a', i} u(c) + \beta \{i E[V^w(a', \varepsilon', \theta')] + (1 - i) E[V^e(a', \varepsilon', \theta')]\} \\ &\quad s.t. \\ c + a' &= w\varepsilon + Ra \\ a' &\geq 0 \end{aligned}$$

and the problem of the entrepreneur is written as:

$$\begin{aligned} V^e(a, \varepsilon, \theta) &= \max_{c, a', i} u(c) + \beta i \{E[V^w(a', \varepsilon', \theta')] + (1 - i) E[V^e(a', \varepsilon', \theta')]\} \\ &\quad s.t. \\ c + a' &= \pi(a, \varepsilon, \theta) \\ a' &\geq 0 \end{aligned}$$

where $\pi(a, \varepsilon, \theta)$ are profits after payments of factors of production and loans and are determined as:

$$\begin{aligned} \pi(a, \varepsilon, \theta) &= \max_{k, n} f(k, n, \theta) + (1 - \delta)k - (1 + \tilde{r})(k - a) - w \max\{n - \varepsilon, 0\} \\ &\quad s.t. \\ k &\leq (1 + d)a \\ \tilde{r} &= \begin{cases} r & k \leq a \\ r + \phi & \text{if } k > a \end{cases} \end{aligned}$$

Note that, if $a \geq k$, then the entrepreneur has some excess saving that she invests at the risk free rate while, if $k > a$, then she borrows the difference at rate $r + \phi$. Another important implication of the way we wrote the problem of the entrepreneur is that an entrepreneur cannot sell her efficiency units in the labor market, which amplifies the risk of entrepreneurship, in case θ turns out to be very low. A limit of this model is that entrepreneurs never make an operating loss within a period, as they can always choose $k = n = 0$ and earn the risk free rate on saving a .³

Why are entrepreneurs richer than workers? Individuals with high entrepreneurial ability have access to a saving/investment technology with higher return than workers (who save at rate r) thanks to this decreasing return technology f , and therefore they accumulate wealth faster.

1.4 Role of Redistributive Taxation

In this model-economy where some of the earnings risk is uninsurable because of market incompleteness, there could be scope for public insurance, i.e. government intervention through taxation and redistribution from the rich-lucky to the poor-unlucky.

In a model with exogenous labor supply, suppose that the government (as in a Ramsey-style optimal taxation problem) chooses a labor income tax τ and a lump-sum subsidy t in order to maximize the ex-ante welfare of the households. Clearly, the optimal tax rate in the Aiyagari economy would be $\tau = 1$. The government would tax away all income and redistribute equally across all agents. This policy would achieve the first-best because

³If there was a fixed entry cost of becoming entrepreneurs and a lower operating cost that must be paid every period, entrepreneurs may choose to stay in business even though, for one or more periods, $\pi < 0$. This will happen if θ is mean reverting.

taxation entails no distortions and no loss of efficiency and, at the same time, generates full insurance.

A more interesting economy is one with an endogenous margin of labor/leisure choice where there is a *trade-off between insurance and efficiency*. To evaluate this trade-off, we develop a variation of the benchmark Aiyagari economy with endogenous labor supply which follows Floden and Linde (2001).

Model— First of all, period utility is given by $u(c, l)$ i.e., we introduce leisure $l \in (0, 1)$ in order to have a margin where distortions matter. This means that we will have an optimal policy for labor supply $h(a, \varepsilon)$. Notice also that the agents might be using their elastic labor supply to self-insure. Take an agent who is liquidity constrained and has a low realization of the productivity shock: to keep his consumption high, he could intensify his labor supply.

The new budget constraint reads

$$c + a' = (1 + r) a + (1 - \tau) w \varepsilon h + t,$$

where τ is a flat earnings tax, and t is the lump-sum transfer of the government. Note that the tax scheme is progressive because the least productive agents pay less taxes but get the same transfer. In other words, the average tax rate faced by a household $(\tau - t/w\varepsilon h)$ is increasing in labor income $w\varepsilon h$.

With leisure, the new equilibrium condition in the labor market becomes

$$N = \int_{A \times E} \varepsilon h(a, \varepsilon) d\lambda^*$$

where N is aggregate labor demand and the RHS is aggregate labor supply.

The government budget constraint (balanced in equilibrium) reads

$$T = \tau w N,$$

where T denotes aggregate transfers (and in equilibrium it equals t).

The definition of the recursive competitive equilibrium for this economy is very similar to the benchmark case (the chief differences are the existence of a decision rule for leisure and of the balanced budget condition of the government).

Computation— Suppose the tax rate τ is a parameter of the problem. We guess $\{r^0, N^0\}$. Given r^0 , if the production function is CRS, we get w^0 . From the government

budget constraint and from our guess of H^0 , we recover t and we have all the inputs needed for solving the household problem. The rest of the computation algorithm is the same as before, with the caveat that we need to check two equilibrium conditions, asset market clearing

$$K^0(r^0, N^0) = \int_{A \times E} a d\lambda_{(r^0, N^0)}$$

and labor market clearing:

$$N^0 = \int_{A \times E} \varepsilon h(a, \varepsilon; r^0, N^0) d\lambda_{(r^0, N^0)}$$

Question— Floden-Linde ask the following question: what is the level of government redistribution that maximizes welfare? What welfare gains does such redistribution imply for individuals compared to the pure laissez-faire, no-redistribution benchmark?

In an economy with heterogeneous agents there is not a unique welfare function, it all depends on what weights are assigned to each type. Floden-Linde assume an equal-weight social welfare function, i.e. they solve

$$\max_{\tau} W(\tau) = \int_{A \times E} V^*(a, \varepsilon; \tau) d\lambda_{\tau}^*,$$

where $V^*(a, \varepsilon; \tau)$ is the value function associated to the competitive equilibrium indexed by τ . It is a version of the Ramsey taxation problem with incomplete markets.

Intuitively, for low levels of redistribution, welfare is low because individuals have a large amount of undesired consumption fluctuations; for very high levels of taxes, consumption insurance is very high but at the same time heavy distortions on labor supply are imposed. So, there will be an interior level of τ , call it τ^* , that maximizes welfare.

Results— When the model is calibrated to the U.S. economy, they find that $\tau_{US}^* = .27$. The welfare gain from this level of redistribution increase annual consumption by 5.6% per year, compared to the no redistribution-case where $\tau = 0$.

Floden and Linde examine also the case of Sweden, a country that traditionally has heavy government intervention and generous welfare programs. They calibrate the same model to Sweden. The major difference is the wage process: shocks are much less variable and less persistent than the U.S., so wage fluctuation in Sweden are more insurable through precautionary savings. The key reason, perhaps, is that unions and other wage

compressing institutions reduce wage volatility already before taxes. It shouldn't come as a surprise then that they find an "optimal" tax rate $\tau_{Sweden}^* = .03$, i.e., very low. Essentially, labor endowment fluctuations in Sweden are small and individuals can largely self-insure against them. The amount of actual government transfers in Sweden is much larger than 3%, so in this sense there is "too much" public insurance in Sweden: as the optimum amount is exceeded the tax-induced distortions from actual redistribution can be quite costly.

Finally, keep in mind that the authors only considered a flat tax on labor income. Often governments use capital income tax for redistribution, which is much more distortionary, but also more progressive since the rich have a higher capital-income to labor-income ratio in this model.

As far as transfers are concerned, it would be more efficient to condition the transfer on ε (i.e. agent with low ε would receive more), but Floden-Linde assume that ε is private information, hence unobservable to the government, so the government cannot condition on ε . A more sophisticated approach to optimal taxation (called the Mirleesian approach) would ask the question of what is the optimal tax scheme that maximizes welfare, given this private information constraint. In other words, the planner would not tie its hands to tax with a flat tax and redistribute lump sum.

1.5 Optimal Quantity of Debt

Aiyagari and Mc Grattan (1998) study the quantity of government debt that maximizes welfare (same social welfare function as Floden and Linde) in the U.S. economy. The economy is like the one in Floden and Linde, except for the government sector.

Every period the government has two type of outlays: transfers T , and interest payments on the stock of existing public (one-period) debt B . These outlays need to be financed by distortionary taxes on labor income at rate τ . The government budget constraint reads

$$T + (1 + r) B = B' + \tau w H,$$

where in the stationary equilibrium $B' = B$. Here capital letters denote aggregate quantities.

Government debt is an additional risk-free asset and, by no arbitrage, it must carry the same rate of return as capital in equilibrium. Debt has a number of negative and positive

effects on the equilibrium. On the negative side, first of all, debt is costly because financing interest payments on debt requires distortionary taxes. Second, public debt crowds-out productive capital because some of the savings are shifted away from productive capital into unproductive debt. Note that the equilibrium condition in the asset market is now

$$K(r) + B = A(r) \Rightarrow K(r) = \tilde{A}(r) \equiv A(r) - B.$$

Note that, in the graphical representation of the stationary equilibrium, the aggregate demand is still $K(r)$. The aggregate supply shifts to the left by an amount B (it's as if the effective borrowing constraint shifts), so the interest rate rises unambiguously.

FIGURE

The rise in the equilibrium interest rate means that government debt has an advantage as well. An increase in debt is effectively like introducing a looser borrowing constraint: the government enhances liquidity by providing additional means for consumption smoothing, besides claims to physical capital. Thus, increases in debt raise the return on assets, and make assets cheaper to hold. Recall that the closer the equilibrium interest rate to β , the more efficient the economy. Put differently, with incomplete markets agents are forced to hold assets for self-insurance which is costly because it reduces their consumption. The higher the equilibrium interest rate, the lower this cost. Note however that this argument holds if the borrowing limit is exogenous. It would be greatly weakened if we had assumed that agents can borrow up to the natural borrowing limit, since a higher r has no effect on an exogenous borrowing constraint, but it tightens the natural borrowing limit.

After calibrating their model, Aiyagari and McGrattan conclude that the optimal quantity of debt is very close to the actual one for the U.S. economy, i.e., around 2/3 of GDP.

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