# Problem Set 4

2nd May 2017

## The Aiyagari 1994 Model

Demographics: There are an infinite number of agents with unit measure. Agents are ex-ante homogeneous but ex-post heterogeneous, depending on the history of realization of idiosyncratic shocks.

Preference: Time is discrete. The individuals are infinitely lived and have time-separable preferences over streams of consumption. The expectation is over future sequences of shocks, conditional on the realization at time 0. The individual supplies labor inelastically.

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

The utility function is CRRA:

$$u\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma}$$

Budget constraint: For individual $i$ at time $t$, the budget constraint reads:

$$c_{it} + a_{it+1} = \left(1 + r_t\right) a_{it} + w_t \varepsilon_{it}$$

where $\varepsilon_t$ is a idiosyncratic labor endowment shock.

Liquidity constraint: Every period, agents face the borrowing limit

$$a_{it+1} \geq 0$$

Endowment: Each period, each individual has a idiosyncratic stochastic endowment of efficiency units of labor. The shocks follow a AR(1)/Markov process. Shocks are i.i.d. across individuals. We assume the stochastic process is well-behaved, so there is a unique invariant distribution $\Pi^*\left(\varepsilon\right)$. As a result, the aggregate endowment of efficiency units of labor,

$$H_t = \int \varepsilon_i \Pi\left(\varepsilon_i^*\right) di, \ \forall t$$

is constant over time, i.e. there is no aggregate uncertainty. Note in particular, that $H_t$ is exogenously determined. Thus in all that follows, we normalize $H = 1$.

Technology: There is a continuum of firms which have access to the CRS technology:

$$Y = K^{\alpha} H^{1-\alpha} = K^{\alpha}$$

Capital depreciates at a constant rate $\delta$ so that the law of motion for capital is:

$$K_{t+1} = K_t (1 - \delta) + I_t$$

Market Structure: The final good market (consumption and investment goods), the labor market, and the capital market are all competitive. Markets are incomplete, there is only one risk-free asset (capital, and it is risk-free due to absence of aggregate shocks).

# Questions

1. Calibration: Derive the deterministic equilibrium analytically. To get started, derive the Euler equation and solve for the deterministic equilibrium interest rate. Use the equilibrium expressions to calibrate $\beta, \delta$ and $\alpha$ to target a real (risk-free) interest rate of 4%, a labor share of 64%[1], and a capital-output ratio of 3.

2. Value Function Iteration (VFI): In the same world without uncertainty, take the parameter value $\beta$ calibrated in the previous section. Given an interest rate $r = 4\%$ and a wage rate $w = 1.5$, find the value function and optimal policy function using numerical methods. Set $\sigma = 2$ as a baseline value for relative risk aversion. Set the number of grid points to 500 and use a logarithmically spaced vector for the grid. Follow the algorithm notes and use the Matlab code uploaded on Mondo (covering part of your problem). Present your results by plots.

3. AR(1) discretization: Now we introduce idiosyncratic shocks into the model. Discretize the following AR(1) process into a Markov transition matrix using Martin Floden's code uploaded on Mondo[2]. Choose the number of states $N = 3$[3].

   $$\varepsilon_{it} = (1 - \rho)\,\bar{\varepsilon} + \rho \varepsilon_{it-1} + e_{it}$$

   where

   $$e_{it} \sim \mathcal{N}(0, \theta)$$

[1] With constant return to scale, capital income share is 36%.

[2] Use the AR(1) discretization code as a "black box", you don't need to understand that part.

[3] 7 states are used in Aiyagari 1994.

$\rho = 0.9$, $\theta = 0.03$ and $\bar{\varepsilon} = 1$.
This stochastic process is the same for all agents. However, agents may draw different realizations.

4. Stochastic VFI: Solve the individual optimization problem with uncertainty using stochastic value function iteration. Use the transition matrix defined in the previous question.

5. Stationary Distribution: Given the optimal policy functions solved in the previous question and the law of motion of idiosyncratic risks defined in question 3, find the stationary distribution of agents $x(a, \varepsilon)$. You need to simulate a long series in order to converge to the stationary distribution. Plot your results, in particular the distribution of asset holdings $a$.

6. General Equilibrium Prices: Given the initial guess of $r$ and $w$ in question 2, what is the aggregate capital demand and the aggregate capital supply? Use a bisection method to find the equilibrium interest rate and wage rate. (The code for the bisection method is on Mondo. Use this method to find the solution: $r$, which generates zero value for the excess demand function:$\Phi(r) = K_d(r) - K_s(r)$). Plot the stationary distribution as in the previous question, although this time it is for the market clearing interest rate, $r$.

7. Is the relative variance of log consumption and log income close to that in the data?

8. Comparative Statics: Interpret the changes in the aggregate level of capital and the interest rate when you increase $\sigma$ to 5. Do the same when increasing $\rho$ to 0.95 and changing $\theta$ in order to keep the cross-sectional variance of income in the continuous version of the process unchanged.

9. You have obtained the ergodic distribution of $a$. Compute wealth inequality as measured by Gini and Top decile share to total wealth and report them. How do they compare to empirical estimates of Diaz-Gimenez *et al.* or Piketty?

10. Now change the calibration of $\beta$ to get (in the deterministic equilibrium) a higher risk-free rate, $r = 6\%$, keeping the labour share and the capital-output ratio unchanged. Re-compute the wealth inequality to find out whether Piketty's prediction of higher $(r - g)$ yielding higher wealth inequality hold in the Aiyagari model.