

# 1 Micro and macro labor supply elasticity

The elasticity of labor supply is one of the crucial parameters in every macroeconomic model. For example, this elasticity determines the response of hours worked to changes in the tax rate and determines the degree of distortions tax introduce. This elasticity also determines how employment, and hence output, responds to fluctuations in productivity. therefore a key issue in macroeconomics is: how large is this elasticity?

In addition to being a very important issue, it is also well known to be quite controversial. In particular, there is a long-standing controversy driven by the fact that on the one hand, researchers who look at micro data typically estimate relatively small labor supply elasticities, while on the other hand, researchers who use representative agent models to study aggregate outcomes typically employ parameterizations that imply relatively large aggregate labor supply elasticities.

In this section, we first explain how labor economists arrived at the conclusion that the micro labor supply elasticity is small, and then we explain one way in which a small micro elasticity can be reconciled with a large aggregate elasticity.

## 1.1 The micro Frisch elasticity of labor supply

The Frisch elasticity of labor supply measures the percentage change in hours worked due to the percentage change in wages, holding constant the marginal utility of wealth (i.e., the multiplier on the budget constraint  $\lambda_t$ ):

$$\varepsilon_t = \frac{dh_t/h_t}{dw_t/w_t} |_{\lambda_t}$$

It is also called the  $\lambda$ -constant elasticity, or intertemporal elasticity of labor supply. This elasticity measures how hours respond to wage changes abstracting from its effect on wealth. For example, consider a two-period model with no uncertainty,  $r = 0$ , zero initial wealth, and loose borrowing limits which do not bind. Suppose that  $w_t = 1$  at  $t = 1, 2$ . Optimal hours worked would be equal in both periods. Now, change the path of wages so to keep the DPV of wages constant as follows:  $w_1 = 0.5$  and  $w_2 = 1.5$ . The effect of this change in the time path of wages on hours worked is mediated precisely by the size of the Frisch elasticity.

In general, wage changes (i.e., if they not purely transitory) also have wealth effects on labor supply. The Frisch elasticity does not capture the total effect on hours from wage shocks. It captures the component due to intertemporal substitution effects, but not the one due to wealth effects.

### 1.1.1 General expression for Frisch elasticity

Let's derive the general expression for the Frisch elasticity. Households solve:

$$\begin{aligned} & \max_{\{c_t, h_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t) \\ & \text{s.t.} \\ & c_t + a_{t+1} = Ra_t + w_t h_t \\ & a_{t+1} \geq -\underline{a} \end{aligned}$$

The Lagrangean for this problem is

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t, h_t) + \lambda_t [Ra_t + w_t h_t - c_t - a_{t+1}] + \phi_t [a_{t+1} + \underline{a}]\}$$

where  $\lambda_t$  is the multiplier on the time  $t$  budget constraint and  $\phi_t$  is the multiplier on the time  $t$  borrowing constraint. The first order conditions with respect to  $(c_t, h_t, a_{t+1})$  yield

$$u_c = \lambda_t \quad (1)$$

$$-u_h = \lambda_t w_t \quad (2)$$

$$\lambda_t - \phi_t = \beta R E_t [\lambda_{t+1}] \quad (3)$$

Differentiate the intratemporal FOC (2), keeping  $\lambda_t$  constant:

$$\begin{aligned} -u_{hh} dh_t - u_{hc} dc_t &= \lambda_t dw_t \\ -h_t u_{hh} \frac{dh_t}{h_t} - u_{hc} dc_t &= (\lambda_t w_t) \frac{dw_t}{w_t} \end{aligned}$$

Using the intratemporal FOC (2) again to substitute out  $(\lambda_t w_t)$ :

$$\begin{aligned} h_t u_{hh} \frac{dh_t}{h_t} + u_{hc} dc_t &= u_h \frac{dw_t}{w_t} \\ h_t u_{hh} \frac{dh_t}{h_t} + h_t u_{hc} \frac{dc_t}{dh_t} \frac{dh_t}{h_t} &= u_h \frac{dw_t}{w_t} \end{aligned} \quad (4)$$

Now, differentiating the FOC with respect to consumption (1) remembering to keep  $\lambda_t$  constant:

$$\begin{aligned} u_{cc}dc_t + u_{ch}dh_t &= 0 \\ \frac{dc_t}{dh_t} &= -\frac{u_{ch}}{u_{cc}} \end{aligned} \quad (5)$$

and using (5) into (4) one obtains:

$$h_t u_{hh} \frac{dh_t}{h_t} - h_t \frac{u_{hc}^2}{u_{cc}} \frac{dh_t}{h_t} = u_h \frac{dw_t}{w_t}$$

which gives the general expression for the Frisch elasticity :

$$\varepsilon_t \equiv \frac{dh_t/h_t}{dw_t/w_t}|_{\lambda_t} = \frac{u_h}{h_t u_{hh} - h_t \frac{u_{hc}^2}{u_{cc}}}. \quad (6)$$

Note that in this derivation we did not use the Euler equation, so this expression for the Frisch elasticity is independent on how agents behave intertemporally, or whether borrowing limits bind. It only requires agents to be on their intratemporal optimality condition. But it would not work for agents who are at a corner (e.g., optimally choose not to work). For example, for women, who tend to go in and out work much more than men, this derivation is problematic.

To understand the intuition for this expression, ignore the cross derivative,. Then,  $\varepsilon = u_h / (h_t u_{hh})$  which is akin to the inverse of the expression for risk aversion, i.e. akin to the expression for the IES with time-separable preferences. Indeed, the Frisch elasticity measures the willingness to substitute hours worked intertemporally.

### 1.1.2 Example: separable preferences

Consider the following utility function separable in consumption and hours worked:

$$u(c_t, h_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \alpha \frac{h_t^{1+1/\eta}}{1+1/\eta} \quad (7)$$

The Frisch elasticity under these preference specification is

$$\varepsilon_t = \frac{u_h}{h_t \left[ u_{hh} - \frac{u_{hc}^2}{u_{cc}} \right]} = \frac{h_t^{1/\eta}}{h_t \left[ \frac{h_t^{1/\eta-1}}{\eta} - 0 \right]} = \eta,$$

hence it is a constant, independent of the level of hours. In general, this is not the case, and the Frisch elasticity depends on the level of hours worked.

Taking logs and first differences of equation (2) under the utility function (7) yields:

$$\Delta \ln h_t = \eta \Delta \ln w_t + \eta \Delta \ln \lambda_t.$$

One can estimate  $\eta$  through OLS only if it is possible to argue that the last term, which is unobservable (i.e. a residual in the OLS equation) is not correlated with wage growth. Unfortunately, in general it is: when  $\Delta \ln w_t \uparrow \Rightarrow \Delta \ln c_t \uparrow \Rightarrow \Delta \ln \lambda_t \downarrow$  and hence  $\text{cov}(\Delta \ln w_t, \Delta \ln \lambda_t) < 0$  which induces a downward bias in  $\eta$ . Several techniques can be used to estimate  $\eta$  without bias: 1) focusing on anticipated wage changes, 2) using some effective IV method, 3) focusing on wage changes that are clearly transitory in nature, 4) using consumption data to proxy  $\lambda_t$ . In case 1) and 3), the change in wage at date  $t$  has no impact on consumption, and hence on  $\lambda_t$ . Based on these techniques, labor economists have found estimates of  $\eta$  between 0 and 0.4.

Because of these results, some economists have maintained the view that large aggregate labor supply elasticities should be ruled out. Over time, we learned that this view is flawed. It is flawed for several reasons (see the survey by Keane and Rogerson, especially section 2.4). Here we'll pursue one particular direction to show that small micro elasticities can be reconciled with large aggregate elasticities. This direction takes seriously that there are adjustments along the extensive margin, population heterogeneity and aggregation. Most of the structural analyses based on micro data implicitly focus on adjustment along the intensive margin (choice of hours given employment). But adjustment along the extensive margin (work/not work) plays an important role: 2/3 of total fluctuations in aggregate hours are due to fluctuations in employment. In the next of the section we show that a model that, at the micro level, is consistent with this indivisibility in labor supply can, at the macro level, imply large aggregate elasticities.

## 1.2 The indivisible labor model with full insurance

We begin with a static model of labor supply adjustment along the extensive margin in complete markets. This model is due to Hansen (1985) and Rogerson (1988). A common view of labor supply is that individuals face an indivisibility in their choice of hours

worked, i.e. at any date  $t$ ,  $h_{it} \in \{0, \bar{h}\}$ . Then, the commodity set is nonconvex and the Welfare Theorem fails. To deal with this conceptual problem, economists often use lotteries. To be precise, one can introduce a new commodity: instead of choosing hours  $h_{it} \in \{0, \bar{h}\}$ , agents choose a probability of working  $\pi_{it} \in [0, 1]$ . If the outcome of the chosen lottery is good, they are employed with outcome  $s_{it} = e$  and work  $h_{it} = \bar{h}$  hours, if it is bad they remain unemployed ( $s_{it} = n$ ) and do not work ( $h_{it} = 0$ ).

### 1.2.1 Supply and demand of insurance

In what follows, we also assume that there are complete insurance markets. In particular, the market offers insurance claims that pay one unit of consumption contingent on the bad realization of the lottery, i.e. when the outcome is  $s_{it} = u$ , which occurs with probability  $1 - \pi_{it}$ . The insurance market is competitive, i.e. the insurance company asked to price a contract that pays  $q$  units of consumption if unemployment occurs with probability  $1 - \pi$  makes profits

$$pq - (1 - \pi) q,$$

and therefore the zero profit condition implies:  $p(\pi) = 1 - \pi$  which means that the price per unit of consumption insured is  $p(\pi)$  and the contract is actuarially fair.

Now, consider the problem of a household who has already chosen  $\pi_{it}$ . How much insurance would she purchase to insure her unemployment risk? Assume households have separable preferences in consumption and hours worked, i.e.

$$U(c, h) = u(c) - v(h).$$

The household who takes the price function  $p(\pi)$  as given solves:

$$\begin{aligned} & \max_{q_{it}} \pi_{it} [u(c_{it}^e) - v(\bar{h})] + (1 - \pi_{it}) u(c_{it}^u) \\ & \text{s.t.} \\ c_{it}^e &= w_t \bar{h} - p(\pi_{it}) q_{it} \quad \text{if employed} \\ c_{it}^u &= q_{it} - p(\pi_{it}) q_{it} \quad \text{if unemployed} \end{aligned}$$

The FOC with respect to  $q_{it}$  gives:

$$\pi_{it} u_c(c_{it}^e) p(\pi_{it}) = (1 - \pi_{it}) u_c(c_{it}^u) [1 - p(\pi_{it})].$$

Using the solution for the equilibrium price of insurance, we arrive at  $u_c(c_{it}^e) = u_c(c_{it}^u)$  which implies  $c_{it}^e = c_{it}^u$  from strict concavity of  $u$ . Therefore, the agent fully insures herself,  $q_{it} = w_t \bar{h}$  and consumption of individual  $i$  at date  $t$  in every state is  $c_{it} = \pi_{it} w_t \bar{h}$ .

### 1.2.2 Choice of lottery

Now we are ready to determine how households choose their probability of working, i.e., their favorite lottery. This is also a static problem. Note that expected utility is

$$\pi_{it} [u(c_{it}^e) - v(\bar{h})] + (1 - \pi_{it}) u(c_{it}^u) = u(c_t) - \pi_{it} v(\bar{h})$$

where we used the full insurance feature of the consumption allocation. Therefore, the problem can be written as:

$$\begin{aligned} & \max_{\pi_{it}} u(c_{it}) - \pi_{it} v(\bar{h}) \\ & \text{s.t.} \\ & c_{it} = \pi_{it} w_t \bar{h} \end{aligned}$$

with solution

$$u_c(c_{it}) w_t \bar{h} = v(\bar{h}) \rightarrow u_c(\pi_{it} w_t \bar{h}) w_t \bar{h} = v(\bar{h}) \rightarrow \pi_{it}^* = u_c^{-1}\left(\frac{v(\bar{h})}{w_t \bar{h}}\right) \frac{1}{w_t \bar{h}}$$

which implies that  $\pi_{it}^* = \pi_t^*$  is the same across individuals. Therefore also consumption will be the same across all individuals at date  $t$ . Note that if preferences were heterogeneous in the disutility of labor, or if individuals differed in terms of initial wealth endowments, this would not be the case. You may ask: why does the agent bother about choosing a probability of working if, in every state, she gets the same consumption anyway? Because, by choosing optimally  $\pi$ , she can affect her level of consumption.

### 1.2.3 Aggregation

Because every agent makes the same consumption and lottery decision, it is clear that if we had a dynamic RBC model with capital accumulation and productivity shocks, we could define a representative agent —whose chosen allocations equal the aggregate quantities

of the economy— solving the problem:

$$\max_{C_t, H_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [u(C_t) - AH_t] \quad (8)$$

s.t.

$$\begin{aligned} C_t + K_{t+1} &= (1 - \delta) K_t + z_t F(K_t, H_t) \\ z_t &\sim F(z_{t-1}) \end{aligned}$$

where, without loss of generality, we have defined  $A \equiv v(\bar{h})/\bar{h}$  and we have defined aggregate hours  $H_t \equiv \pi_t \bar{h}$ . We have also used the equivalence between competitive equilibrium and social planner allocations, which we can do because, with lotteries and insurance contracts, the consumption set is convex and markets are complete. This is the dynamic model solved by Hansen (JME, 1985), a workhorse of RBC theory.<sup>1</sup>

What is the aggregate Frisch labor supply elasticity in this model? Recall that

$$\eta \equiv \frac{u_H}{Hu_{HH}} = \frac{A}{H \cdot 0} = +\infty.$$

The intuition for the infinite elasticity is that, with full insurance, everyone is ex ante indifferent between working or not, so it is costless to move workers in and out of employment to respond to fluctuations in aggregate productivity and wages. Note though that, ex-post, the unemployed are better off because they get the same consumption and more leisure. So, ex-post, those who win the lottery would not like to work. Implicitly, commitment or perfect enforcement is assumed here, like in every complete market model.

#### 1.2.4 Divisible vs indivisible labor

To further understand the implications of micro-indivisibility for aggregate labor supply, consider a version of problem (8) with log utility over consumption and where the aggregate labor supply elasticity is  $\eta$ . From the intratemporal FOC, we have

$$\frac{z_t}{C_t} F(K_t, H_t) = AH_t^{1/\eta}.$$

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<sup>1</sup>This aggregation result does not depend on agents starting with the same (zero) wealth as in the static model we solved above. If agents differ by initial wealth and have isoelastic utility (and we still have complete markets), we can still define a representative agent even if consumption and lottery decisions differed among agents with different levels of initial wealth.

Assuming a Cobb-Douglas aggregate production function, we obtain

$$\frac{z_t}{C_t} K_t^\alpha = A H_t^{(1/\eta+\alpha)}$$

and taking logs and first differences

$$\Delta \log H_t = \frac{1}{1/\eta + \alpha} \Delta \log z_t + \frac{\alpha}{1/\eta + \alpha} \Delta \log K_t - \frac{1}{1/\eta + \alpha} \Delta \log C_t$$

which shows that, in equilibrium (or equivalently, in the planner's solution), fluctuations in hours due to aggregate shocks are going to be much bigger if  $\eta = \infty$ .

### 1.3 Indivisible labor with imperfect insurance

The starting point is the paper by Chang and Kim (2006). Consider the neoclassical growth model with incomplete markets (as in Aiyagari) in its version with aggregate shocks (as in Krusell-Smith) with two important differences:

1. Individual labor supply is indivisible: individuals can work an indivisible amount of hours  $h_t \in \{0, \bar{h}\}$ , but there are no lotteries and no markets to insure unemployment risk.
2. A household is composed by a couple (husband and wife) who pool income and maximize a household objective function where consumption is a public good for the couple (e.g., the house). The main role of this assumption is to add a source of consumption insurance beyond self-insurance through borrowing and saving.

#### 1.3.1 A static example

Consider an economy where individuals can either work  $h_i \in \{0, \bar{h}\}$  and where there is a distribution of wealth in the population (like in a typical incomplete markets model) which induces a distribution of reservation wages in the population. Call this distribution  $\Phi(w^*)$ , i.e., the fraction of the population who would work at wage  $w$  is  $\Phi(w)$  since they have reservation wage below  $w$ . In other words, the aggregate labor supply curve is

$$H(w) = \int_0^w \bar{h} d\Phi(w^*) = \bar{h} \Phi(w)$$

and the aggregate elasticity at the prevailing wage  $w$  is

$$\eta(w) = \frac{dH/H}{dw/w} = \frac{\Phi'(w)/\Phi(w)}{1/w} = \frac{\Phi'(w)}{\Phi(w)}w$$

which depends on the concentration (the ratio of the density to the pdf) of workers at  $w$ . If the population is concentrated near  $w$ , then the elasticity is very large. Note that in Rogerson's model, for every worker, the reservation wage is the current wage since workers are ex-ante indifferent between working and not working, and hence this elasticity would be infinity.

### 1.3.2 The full-blown dynamic model

**Demographics:** the economy is populated by a measure one of infinitely lived households. A household is a couple of two individuals of gender  $i \in (m, f)$  who pool income together and consume a good which is public within the household (e.g., a house).

**Preferences:** Intra-period utility over consumption and hours worked ( $h_m, h_f$ ) are given by  $u(c, h_m, h_f)$ . Hours are indivisible, i.e.  $h_i \in \{0, \bar{h}\}$ . The employment status of an individual is

$$s_i = \begin{cases} e & \text{if } h_i = \bar{h} \\ n & \text{if } h_i = 0 \end{cases}$$

Therefore the household is in one of four possible employment states:  $(e, e), (e, u), (u, e), (u, u)$ . Couples discount the future at rate  $\beta \in (0, 1)$ .

**Technology:** The economy produces output through an aggregate CRS production technology

$$Y = zF(K, H)$$

where  $H$  is the aggregate of all hours-weighted efficiency units in the population.  $z$  is the aggregate productivity shock which follows a Markov chain.

**Endowments:** Household productivity is the pair  $(\varepsilon_m, \varepsilon_f)$  which follows a joint Markov process independent of  $z$ . A household who chooses not to work can produce at home an amount  $b$  of the final good which can be exchanged in the market.

**Markets:** They are all competitive. The only asset traded is a claim to physical capital. Individuals can borrow up to an exogenously given limit  $\underline{a}$ .

We now turn to the household problem. The individual states are  $(a, \varepsilon_m, \varepsilon_f)$  and the aggregate states are  $(z, \mu)$ , where  $\mu$  is the measure of agents. The law of motion for the distribution is  $\mu' = \Gamma(\mu, z)$ .

Each period, the household observes its own asset holdings  $a$ , the pair of productivity  $(\varepsilon_m, \varepsilon_f)$ , aggregate states  $(z, \mu)$  and makes its own labor supply decision  $(s_m^*, s_f^*)$  by choosing

$$V^*(a, \varepsilon_m, \varepsilon_f; z, \mu) = \max \{V_{ee}(a, \varepsilon_m, \varepsilon_f; z, \mu), V_{en}(a, \varepsilon_m, \varepsilon_f; z, \mu), V_{ne}(a, \varepsilon_m, \varepsilon_f; z, \mu), V_{nn}(a, \varepsilon_m, \varepsilon_f; z, \mu)\}$$

Conditional on the employment decision, the household chooses consumption/saving. For example, the household problem for the  $(e, e)$  couple, in recursive formulation is:

$$\begin{aligned} V_{ee}(a, \varepsilon_m, \varepsilon_f; z, \mu) &= \max_{c, a'} u(c, \bar{h}, \bar{h}) + \beta E[V^*(a', \varepsilon'_m, \varepsilon'_f; z', \mu') | \varepsilon_m, \varepsilon_f, z, \mu] \\ &\text{s.t.} \\ c + a' &= w(\varepsilon_m \bar{h} + \varepsilon_f \bar{h}) + Ra \\ a' &\geq -\underline{a} \end{aligned}$$

The household problem for the  $(n, n)$  couple in recursive formulation is:

$$\begin{aligned} V_{nn}(a, \varepsilon_m, \varepsilon_f; z, \mu) &= \max_{a'} u(c, 0, 0) + \beta E[V^*(a', \varepsilon'_m, \varepsilon'_f; z', \mu') | \varepsilon_m, \varepsilon_f, z, \mu] \\ &\text{s.t.} \\ c + a' &= 2b + Ra \\ a' &\geq -\underline{a} \end{aligned}$$

The solution of the labor supply problem for spouse  $i$  is a policy function

$$s_i^*(a, \varepsilon_m, \varepsilon_f; z, \mu) \in \{e, n\}.$$

We can represent the labor supply decision also as a reservation productivity level (wage)  $\varepsilon_m^*(a, \varepsilon_f; z, \mu)$  such that

$$s_m^* = \begin{cases} e & \text{if } \varepsilon \geq \varepsilon_m^*(a, \varepsilon_f; z, \mu) \\ n & \text{if } \varepsilon < \varepsilon_m^*(a, \varepsilon_f; z, \mu) \end{cases}$$

Intuitively, we have:

$$\frac{\partial \varepsilon_m^*(a, \varepsilon_f; z, \mu)}{\partial a} \geq 0, \frac{\partial \varepsilon_m^*(a, \varepsilon_f; z, \mu)}{\partial \varepsilon_f} \geq 0$$

where the first inequality descends from the fact that leisure is a normal good. The second will hold because there are gains from specialization: the spouse who “produces leisure” for the household must be the one with the lowest productivity.

We omit the definition of recursive competitive equilibrium with aggregate shocks, and the algorithm to compute this equilibrium (but you should work this out on your own).

Chang and Kim specify utility as:

$$u(c, h_m, h_f) = \log c - A_m \frac{h_m^{1+1/\eta}}{1 + 1/\eta} - A_f \frac{h_f^{1+1/\eta}}{1 + 1/\eta}$$

which is consistent with balanced growth (check). Recall  $\eta$  is the Frisch elasticity. In order to be consistent with the micro evidence, they set  $\eta = 0.40$ . This value implies a Frisch consistent with the micro estimates when it is estimated from an artificial panel of individuals generated from the model. The variation in hours in response to wages comes from the fact that they simulate the model at a quarterly frequency but they estimate this regression at an annual frequency (like most of the micro research does).

What is the aggregate Frisch elasticity implied by this model? One way to answer this question is to simulate the heterogeneous-agent incomplete-markets model, generate aggregate time series of  $\{Y, C, I, H\}$  and compare it to a representative agent model with preferences

$$u(C, H) = \ln C - A \frac{H^{1+1/\gamma}}{1 + 1/\gamma}.$$

In terms of amplitude of aggregate fluctuations of  $\{Y, C, I, H\}$ , the heterogeneous agent incomplete-markets model reproduces those of a representative agent model with  $\gamma = 2$ . Therefore, the aggregate Frisch is 5 times as large as the micro Frisch in this model.

This result is reminiscent of Rogerson’s result, but it is not as extreme. Why? Even though there are no contracts that can be explicitly traded to insure unemployment risk, in this economy self-insurance through borrowing/saving and though spousal labor supply provides good hedging against wage risk. So, we are somewhat close to the complete markets case, something that we already knew from Krusell and Smith.