Chapter I: Labour Market Theory

Section 3: Models with search and on-the-job search

Literature:

Pierre Cahuc and André Zylberberg: Labour Economics

Chapter 3: Job Search

Dale Mortensen: Wage dispersion

Chapter 2: The Burdett-Mortensen Model

Chapter 3: The Shape of Wage Dispersion

Chapter 4: Wage Dispersion and Worker Flows

3.1 One-sided search models

Definition: One-sided search models

One-sided search models analyse **individuals' optimal search strategy** taking the firm strategy (e.g. wages offered, promotion policies, layoff-policies, ...) as given.

One-sided search models are therefore **not equilibrium search models**.

Research question:

When to accept a job offer?

Framework:

- Discount rate, r.
- Unemployment income (value of leisure) z per period.
- Probability of unemployed worker to meet a firm, λ.
- The expected, discounted life-time utility of an unemployed worker, V_u.
- Wages offered by firms are given by the exogenous wage offer distribution $F(w)$ with $F(\underline{w}) = 0$ and $F(\overline{w}) = 1$.
- Workers are risk neutral and consume their per period wage w, i.e. no savings.
- $V_e(w)$ the expected, discounted life-time utility of a worker employed at wage w.
- Employment ends at rate q (job destruction rate).

Workers have to decide whether to accept or reject a wage offer!

Bellman equation for employed workers:

A worker employed at wage w values employment according to,

$$rV_e(w) = w + q\left[V_u - V_e(w)\right]. \tag{1}$$

The **value of employment** $V_e(w)$ of a forward looking worker equals the wage w plus the expected utility loss $V_u - V_e(w)$ in case of job destruction, which occurs at rate q.

The value of employment $V_e(w)$ **increases with the wage** w, i.e.,

$$\frac{\partial V_e(w)}{\partial w} = \frac{1}{r+q} > 0.$$

Bellman equation for unemployed workers:

A unemployed worker receives unemployment benefits z as long as he stays unemployed, i.e.,

$$rV_u = z + \lambda \int_{\underline{w}}^{\overline{w}} \max \left[V_e(w) - V_u, 0 \right] dF(w) . \tag{2}$$

With probability λ the unemployed worker meets a firm and is offered a wage from the wage distribution $F(w)$.

Given the offered wage w the unemployed worker has to decide whether or not to accept the job at the offered wage.

Decision rule of unemployed workers (optimal search strategy):

An unemployed worker will accept a job, if employment at the wage w paid gives the worker at least as much utility as staying unemployed.

The **reservation wage** makes the worker indifferent between staying unemployed and becoming employed, i.e.,

$$V_u = V_e(R) \tag{3}$$

Unemployed accept the offered wage, if it is no lower than the reservation wage, i.e.,

$$V_e(w) \geq V_u \Leftrightarrow w \geq R, \tag{4}$$

since the value of employment increases with the wage, i.e., $\partial V_e(w)/\partial w > 0$.

The reservation wage R:

Given the decision rule (4), i.e., $V_e(w) \geq V_u \Leftrightarrow w \geq R$, the value of unemployment can be written as follows,

$$\begin{aligned} rV_u &= z + \lambda \int_R^{\overline{w}} \left[V_e(w) - V_u \right] dF(w) \\ &= z + \lambda \int_R^{\overline{w}} \left[V_e(w) - V_e(R) \right] dF(w) . \end{aligned}$$

Using the Bellman equation for employed workers (1) we can substitute $V_e(w) - V_e(R)$, i.e.,

$$rV_u = z + \frac{\lambda}{r+q} \int_R^{\overline{w}} \left[w - R \right] dF(w) .$$

Using the definition of the reservation wage also implies

$$rV_u = rV_e(R) = R + q\left[V_u - V_e(R) \right] = R$$

The reservation wage:

$$R = z + \frac{\lambda}{r+q} \int_R^{\overline{w}} [w - R]\, dF(w)\,. \tag{5}$$

Existence and uniqueness of the reservation wage:

$$\begin{aligned}
\frac{\partial LHS}{\partial R} &= 1 \\
\frac{\partial RHS}{\partial R} &= \frac{\lambda}{r+q} \int_R^{\overline{w}} [-1]\, dF(w) = -\frac{\lambda\,[1 - F(R)]}{r+q}
\end{aligned}$$

Figure 3.1: Reservation wage

Comparative statics of the reservation wage:

$$\Phi\left(R, z, \lambda, r, q\right) = R - z - \frac{\lambda}{r+q}\int_{R}^{\overline{w}}\left[w - R\right] dF\left(w\right)$$

Implicit Function Theorem:

$$\frac{dR}{dx} = -\frac{\Phi'_x}{\Phi'_R} \quad \text{with} \quad \Phi'_R > 0, \text{ for any } x \in \{z, \lambda, r, q\}.$$

Intuition:

- The reservation wage increases with unemployment benefits, i.e., $\Phi'_z < 0$, since the value of being unemployed increases.
- A higher firm contract rate λ also increases the reservation wage, i.e., $\Phi'_\lambda < 0$, since it increases the probability to get (potentially better) offer.
- A higher discount rate r or layoff rate q decreases the reservation wage, i.e. $\Phi'_r > 0$ and $\Phi'_q > 0$, since the expected future utility from getting a better paid job decreases.

Figure 3.2: Comparative statics of the reservation wage

Average unemployment duration:

The probability to exit unemployment (called **hazard rate**) is given by

$$\lambda \left[1 - F\left(R\right)\right],$$

which equals the rate at which an unemployed worker meets a firm λ and the probability that the worker will accept the job offer $1 - F\left(R\right)$.

In the current simple model the firm contact rate λ is assumed to follow a **Poisson process**, (i.e. to be independent of events in the past).

Given the assumption of a Poisson arrival rate **average unemployment duration** T is given by

$$T = \frac{1}{\lambda \left[1 - F\left(R\right)\right]}$$

Intuition:

If an unemployed worker has a 10 percent chance to get an acceptable job offer within a week, then the average unemployment duration equals 10 weeks.

Comparative statics of average unemployment duration:

Average unemployment duration T increases with the **reservation wage** R, i.e.,

$$\frac{\partial T}{\partial R} = \frac{f(R)}{\lambda \left[1 - F(R)\right]^2} > 0,$$

since the probability of meeting an acceptable wage offer decreases.

The effect of an increase in the **job offer arrival rate** λ on the average unemployment duration T is ambiguous, i.e.,

$$\frac{\partial T}{\partial \lambda} = \frac{-1}{\lambda^2 \left[1 - F(R)\right]} + \frac{f(R)}{\lambda \left[1 - F(R)\right]^2} \frac{dR}{d\lambda},$$

- because although the average time to get an acceptable offer decreases,
- the increase in the reservation wage (due to a higher probability of meeting another firm) reduces the probability of accepting a wage offer.

How to estimate the parameter of the model?

Data used:

Individual data on unemployment duration and wages at the beginning of a job.

Empirical difficulties:

- The distribution of rejected wage offers is not observed.
- This makes it impossible to identify the rejection probability $F(R)$.
- Using a family of probability densities (log-normal) for the wage offer distribution to identify the truncated part of $F(w)$ via the estimated parameters of the observed wage offer distribution does not guarantee identification.

3.2 Equilibrium search model - The Diamond Paradox

Definition: Equilibrium search model

Given the optimal **search strategy** of workers, i.e., the decision rule based on the reservation wage R, firms decide on which wages to offer in equilibrium.

The wage offer distribution $F(w)$ is endogenized.

Research question:

What wages are offered by firms?

Framework:

- All firm-worker pair are equally productive, i.e. produce output y.
- The number of firms is given by M.
- The number of workers is normalized to 1.
- Firms profits (flow) are given by,

$$q\Pi = (y - w)\, l(w),$$

profits per worker $(y - w)$ times the number of workers $l(w)$ that will are willing to work for the wage w.

- Since all unemployed workers accept wages that are at least as high as the reservation wage, i.e., $w \geq R$, the number of workers employed in a particular firm is given by

$$l(w) = \begin{cases} \dfrac{1-u}{M} = \dfrac{\lambda}{\lambda + q}\dfrac{1}{M} & \text{for } \; w \geq R, \\ 0 & \text{for } \; w < R. \end{cases}$$

Diamond Paradox:

Any wage offer above the reservation wage, i.e., $w > R$, would decrease profits per worker without increasing the number of employed workers.

It is **profit maximizing to offer a wage equal to the reservation wage**, i.e.,

$$w = R.$$

If all firms offer a wage equal to the reservation wage, then equation (5) implies, i.e.,

$$\begin{aligned} R &= z + \frac{\lambda}{r+q} \int_{R}^{\overline{w}} [w - R] \, dF(w), \\ &= z + \frac{\lambda}{r+q} [R - R] = z, \end{aligned}$$

that the **reservation wage equals unemployment benefits**.

Intuition:

Since firms make take-it-or-leave-it wage offers, firms have all the bargaining power. Thus, firms can extract the whole rent from workers, since labor market frictions prevent between-firm competition (decentralized wage determination).

Diamond Paradox:

If firms pay wages equal to unemployment benefits, there is **no gain from searching for a job**.

If there are only ϵ search costs, unemployed workers have no incentive to search for a job. Thus, workers have no incentive to participate and the **market collapses**.

But, in reality we observe that **identical workers earn different wages** (i.e., $R^2 < 0.4$, in wage equations with a lot of control variables).

How do we get out of the Diamond Paradox?

Albrecht and Axell (1984):

Two types of workers:

- Fraction μ has high value of leisure and consequently high reservation wage.
- Fraction $(1-\mu)$ has low value of leisure (and low reservation wage).
- Firms face **trade-off** between offering high and low wages.

Idea:

High wages attract more workers than low wages, i.e., $l\left(w^h\right) > l\left(w^l\right)$.

Profits per worker a lower at high wage firms, than at low wage firms, i.e.,

$$y - w^h < y - w^l.$$

In equilibrium both firms can make equal profits.

Problem with Albrecht-Axell model:

- High wage firms have no incentive to offer a wage higher than the reservation wage of workers with a high value of leisure.
- An ϵ search cost will prevent the workers with the high value of leisure from searching because the expected pay-off from search is $-\epsilon$ for them.
- Thus, only workers with a low value of leisure will participate in the labor market.
- Then, we are back in the Diamond Paradox.

Alternatives to solve the Diamond Paradox:

- On-the-job search
- Directed search
- Multiple applications

3.3 On-the-job search - Burdett-Mortensen model

Basic idea:

- If workers also search while being employed and not only while being unemployed, firms that pay a higher wages can hire not only unemployed workers but also employed workers that earn less.
- The number of employed workers will therefore increase with the wage offered, such that in equilibrium high wage firms with low profit per worker but large number of employees make the same profit as low wage firms that earn a high profit per worker but have only few workers.

Research question: What wages are offered in equilibrium?

Framework:

- Unemployed workers meet a firm at rate λs_u.
- Employed workers search for a job and meet another firm at rate λs_e, where s_e equals the search intensity of employed workers.
- Firms meet an unemployed worker at rate ηs_u and an employed worker at rate ηs_e.
- M denotes the number of active firms, and normalize the number of workers to unity.
- Firms do not counter an outside offers made by another firm.
- Firms offer all workers the same wage, i.e., they do not condition the wage offer on labor market status (unemployed, employed) nor on the wage earned by worker.
- The demand for a product drops to zero at rate δ. If this happens, the firm has to decide whether to buy a new product idea or to go out of business.

3.3.1 Reservation wages

Unemployed workers have to decide

- which wage offers they accept.

Employed workers have to decide

- which wage offers they accept.

Reservation wage of employed workers:

Employed workers will accept any wage that is above their current wage, i.e.,

current wage = reservation wage

Bellman equation for employed workers:

$$rV_e(w) = w + \lambda s_e \int_w^{\overline{w}} [V_e(x) - V_e(w)]\, dF(x) + q[V_u - V_e(w)] \tag{6}$$

The utility of an employed worker is given by the wage and the potential gains from finding a better paid job and the potential loss of from becoming unemployed.

The value of employment $V_e(w)$ increases with the wage earned, i.e.,

$$\frac{\partial V_e(w)}{\partial w} = \frac{1}{r + q + \lambda s_e [1 - F(w)]} > 0. \tag{7}$$

Bellman equation for unemployed workers:

$$rV_u = z + \lambda s_u \int_{\underline{w}}^{\overline{w}} \max\left[V_e(w) - V_u, 0\right] dF(w). \quad (8)$$

The utility of an unemployed worker is given by unemployment benefits and the potential gains from finding a job.

An unemployed worker accept any wage that gives a higher utility than being unemployed, i.e.,

$$V_e(w) \geq V_u \Leftrightarrow w \geq R,$$

At the **reservation wage** R an unemployed worker is indifferent between employment at the wage R or staying unemployed.

The reservation wage of unemployed workers:

Using integration by parts allows us to rewrite equation (6) as

$$\begin{aligned} rV_e(w) &= w + \lambda s_e \int_w^{\overline{w}} \left[V_e(x) - V_e(w) \right] dF(x) + q\left[V_u - V_e(w) \right] \\ &= w + \lambda s_e \int_w^{\overline{w}} \frac{\left[1 - F(x) \right]}{r + q + \lambda s_e \left[1 - F(x) \right]} dx + q\left[V_u - V_e(w) \right] \end{aligned} \tag{9}$$

Similarly, we can rewrite equation (8) as,

$$\begin{aligned} rV_u &= z + \lambda s_u \int_R^{\overline{w}} \left[V_e(x) - V_e(R) \right] dF(x) \\ &= z + \lambda s_u \int_R^{\overline{w}} \frac{\left[1 - F(x) \right]}{r + q + \lambda s_e \left[1 - F(x) \right]} dx \end{aligned} \tag{10}$$

Integration by parts:

$$\int_{w}^{\overline{w}} \left[V_e(x) - V_e(w)\right] dF(x) = \left[\left[V_e(x) - V_e(w)\right] F(x)\right]_{w}^{\overline{w}} - \int_{w}^{\overline{w}} \frac{\partial V_e(x)}{\partial x} F(x)\, dx$$

Using equation (7) to substitute $\partial V_e(x)/\partial x$ implies

$$\int_{w}^{\overline{w}} \left[V_e(x) - V_e(w)\right] dF(x) = \left[V_e(\overline{w}) - V_e(w)\right] - \int_{w}^{\overline{w}} \frac{F(x)}{r + q + \lambda s_e\left[1 - F(x)\right]} dx$$

Since $V_e(x) = \int \left[\partial V_e(x)/\partial x\right] dx$ we get

$$\int_{w}^{\overline{w}} \left[V_e(x) - V_e(w)\right] dF(x) = \int_{w}^{\overline{w}} \frac{1 - F(x)}{r + q + \lambda s_e\left[1 - F(x)\right]} dx$$

The reservation wage of unemployed workers:

Evaluating equation (9) at $w = R$ and subtracting it from equation (10) implies the following equation for the reservation wage, i.e.

$$R = z + \lambda \left[s_u - s_e\right] \int_R^{\overline{w}} \frac{\left[1 - F(x)\right]}{r + q + \lambda s_e \left[1 - F(x)\right]} dx \tag{11}$$

The reservation wage R is below unemployment benefits z, if the job offer arrival rate for employed workers is higher than for unemployed workers, i.e. $s_u < s_e$.

Intuition:

Having a higher job offer arrival rate while being employed compared to being unemployed, implies faster wage growth for employed workers. Unemployed workers are therefore willing to accept a income cut (from z to R), if the become employed, since this increases the chances to receive higher wage offers in the future.

Comparative statics of the reservation wage:

The comparative statics depend on the sign of $(s_u - s_e)$. Empirical studies have found that the job offer arrival rate for unemployed workers is higher than for employed workers (see e.g. Holzner and Launov, 2010).

Comparative statics, if $s_u > s_e$:

- The reservation wage increases with unemployment benefits z, since the value of being unemployed increases.
- A higher general job arrival rate λ increases the reservation wage, since it reduces the expected duration for getting another, potentially better offer.
- A higher discount rate r or layoff rate q decreases the reservation wage, since the expected future utility from getting a better paid job decreases.

Figure 3.3: Comparative statics of the reservation wage

Reservation wage of employed workers:

Employed workers have the same job finding rate regardless their wage.

Reservation wage:

Employed accept any wage that is higher than their current wage, i.e,

$$w' > w.$$

3.3.2 Steady state conditions

In steady state

- the number of contacts that firms make has to equal the number of contacts that workers make,
- unemployment rate u does not change,
- the wage distribution $G(w)$ is constant.

Matching rate λ in equilibrium:

In equilibrium the number of firm contacts that workers make, i.e., $\lambda s_u u$ contacts by unemployed workers and $\lambda s_e [1-u]$ contacts by employed workers, equals the number of worker contacts that firms make $\eta [s_u u + s_e [1-u]] M$, i.e.,

$$\begin{aligned} \lambda s_u u + s_e [1-u] &= \eta [s_u u + s_e [1-u]] M \\ \lambda &= \eta M \end{aligned}$$

Comparative statics:

- The matching rate of workers increases with the number of active firms, i.e., $\partial\lambda/\partial M > 0$.
- The matching rate of workers decreases with the search intensity of other workers, i.e., $\partial\lambda/\partial s_e < 0$ (congestion externality).

Steady state unemployment rate u:

The unemployment rate increases, if the inflow into unemployment $q\,[1-u]$ is higher than the outflow $\lambda s_u u$, i.e.,

$$\dot{u} = q\,[1-u] - \lambda s_u u,$$

In **steady state** the **unemployment rate** does not change, i.e.,

$$\dot{u} = 0 \Longleftrightarrow u = \frac{q}{q + \lambda s_u} = \frac{q}{q + \eta M s_u} \tag{12}$$

Note:
The unemployment rate decreases, if the number of active firms M increases, because it increases job finding rate of unemployed workers λs_u.

Steady state wage earnings distribution $G(w^-)$:

The fraction of employed workers $(1-u)\,G(w^-)$ that earn a wage less than w is in steady state also determined by equating the in- and outflow, i.e.,

$$\eta M s_u F\left(w^-\right) u = \left[q + \eta M s_e \left[1 - F\left(w^-\right)\right]\right](1-u)\,G\left(w^-\right),$$

where $F(w^-) = F(w) - v(w)$ and $v(w)$ is the mass (significant number) of firms offering the same wage w.

Rearranging implies

$$G\left(w^-\right) = \frac{\eta M s_u F(w^-)}{q + \eta M s_e [1 - F(w^-)]} \frac{u}{1-u}, \tag{13}$$

3.3.3 Equilibrium wage offer and wage earnings distribution

Each firm has to decide on its wage offer given

- the search strategy of unemployed, R, and of employed workers, w,
- the number of active firms M in the labor market,
- and the wages offered by other firms, i.e., the wage offer distribution $F(w)$.

Assume for simplicity that the discount rate is zero, i.e., $r = 0$.
Firms commit to pay a certain wage w to all its workers.

Firms' optimization problem:

Firms' profits are still bounded, since the demand for the product drops to zero at rate δ, i.e., y drops to zero.

If this happens, firms need to acquire a new product idea in the market for innovations or they go out of business.

Firms choose the wage that maximizes **steady state profits**:

$$\delta\Pi(w) = \max_{w}\left[(y-w)\,l(w)\right]$$

Question: How does labour input $l(w)$ change with the wage?

Labour input $l(w)$ in **steady state** depends on the...

- hiring rate, which is given by the contacting rate for unemployed and employed workers ηs_u and ηs_e times the number of that accept a wage w. Note that unemployed workers u and workers earning a wage less than w , i.e., $(1-u)\,G(w^-)$, accept a wage w, where $G(w^-)$ denotes the fraction of workers earning a wage w or below ($G(w)$ equals the wage earnings distribution).

- quitting rate, which is given by $q+\lambda s_e[1-F(w)]$, i.e., the rate at which workers quit into unemployment for an exogenous reason q, and the rate at which workers contact another firm λs_e, times the probability that the offered wage is above the current wage, i.e., $[1-F(w)]$.

In steady state labour input is given by

$$l(w)=\frac{\eta\left[s_u u+s_e(1-u)\,G(w^-)\right]}{q+\lambda s_e[1-F(w)]}.$$

Labour input $l(w)$ in **steady state** is therefore given by

$$\begin{aligned} l(w) &= \frac{\eta\left[s_u u + s_e(1-u)G(w^-)\right]}{q+\lambda s_e\left[1-F(w)\right]}, \\ &= \frac{\eta s_u}{q+\lambda s_e\left[1-F(w)\right]}\left[1+\frac{\lambda s_e F(w^-)}{q+\lambda s_e\left[1-F(w^-)\right]}\right]u, \\ &= \frac{\eta s_u}{q+\eta M s_e\left[1-F(w)\right]}\frac{q+\eta M s_e}{q+\eta M s_e\left[1-F(w^-)\right]}\frac{q}{q+\eta M s_u}. \end{aligned} \tag{14}$$

Properties of labour input at a mass point:

If a mass of firms offer the same wage w, labour input $l(w)$ increases significantly (jumps) at $w + \epsilon$.

Intuition:

- A firm that offers a wage w is not able to hire workers from other firms that offer the same wage w,
- But a firm that offers $w + \varepsilon$ is able to hire all workers at firms that offer w.
- Thus, the hiring rate jumps up significantly, if a firm offers $w + \varepsilon$.
- This increases labour input significantly.

Burdett and Mortensen (1998):

The wage offer distribution $F(w)$ is continuous.

Proof by contradiction:

- Suppose that the wage offer distribution has a mass point.
- This cannot be profit maximizing, since any firm that offers w can offer ε more and obtain a significantly higher labour input.
- Since profits per worker $(y - w)$ stay the same (decrease only marginally), total profits increase.

How can it be optimal to offer different wages in equilibrium?

- Suppose only two firms exist, and they offer the reservation wage R.
- Both firms have an incentive to increase their wage by ε.
- Thus, both firms will increase their wage. But both firms will loose, since they employ the same workers but pay a higher wage.
- An equilibrium is reached, if both firms compete their wages up to the point $\overline{w}$, where one firm is willing to offer the reservation wage R, i.e., gets a higher profit per worker but has a lower labour input compared to the firm offering $\overline{w}$.

The equilibrium therefore requires:

$$\delta\Pi\left(\overline{w}\right) = \delta\Pi\left(R\right),$$

$$\text{or} \quad \left(y - \overline{w}\right) l\left(\overline{w}\right) = \left(y - R\right) l\left(R\right).$$

Labour input $l(w)$:

Since no mass point exists, we get the following equation for labour input (see equation (14)),

$$l(w) = \frac{\eta s_u \left[q + \eta M s_e\right]}{\left[q + \eta M s_e \left[1 - F(w)\right]\right]^2} \frac{q}{q + \eta M s_u}. \tag{15}$$

Comparative statics:

Labour input increases with the wage, because

- the number of workers hired increases with the wage and
- the quitting rate decreases with the wage.

Labour input decreases with the number of active firms, because

- more firms are competing over the same number of workers.

Equilibrium wage offer distribution $F(w)$:

Since all firms are equally productive, **all firms must make equal profits in equilibrium**.

The lowest offered wage equals the reservation wage, i.e., $\underline{w} = R$ and $F(R) = 0$, since no worker would accept a wage below the reservation wage.

$\Pi(w) = \Pi(R)$ for all w on the support of $F(w)$ and

$\Pi(w) < \Pi(R)$ for all w of the support.

Substituting the labour input $l(w)$ implies

$$\frac{y-R}{[q+\eta M s_e]^2} = \frac{y-w}{[q+\eta M s_e[1-F(w)]]^2}$$

Rearranging implies:

$$F(w) = \frac{\lambda s_e + q}{\lambda s_e}\left[1-\sqrt{\frac{y-w}{y-R}}\right]$$

Upper and lower bounds of the wage distribution:

The maximum wage $\overline{w}$ is given by $F(\overline{w}) = 1$,

$$\overline{w} = y - \frac{q^2}{[q + \lambda s_e]^2}(y - R)$$

Reservation wage:

$$R = z + \frac{\lambda [s_u - s_e] \lambda s_e}{[q + \lambda s_e]^2}(y - R) = \frac{[q + \lambda s_e]^2 z + \lambda [s_u - s_e] \lambda s_e y}{[q + \lambda s_e]^2 + \lambda [s_u - s_e] \lambda s_e}$$

The reservation wage equals a weighted average between value of leisure z and the marginal product y.

The reservation wage equals unemployment benefits, if employed workers do not search, i.e., $s_e \to 0$ implies $R \to z$.

Wage earnings distribution $G(w)$:

Given the wage offer distribution $F(w)$, we can use equation (13) to obtain the wage earnings distribution, i.e.,

$$G(w) = \frac{q}{\lambda s_e}\left[\sqrt{\frac{y-R}{y-w}} - 1\right] \quad \text{and} \quad g(w) = \frac{1}{2}\frac{q}{\lambda s_e}\frac{\sqrt{y-R}}{(y-w)^{(1+1/2)}}.$$

Properties:

The density of the wage earnings distribution is an increasing, which is at odds with the empirically observed wage earnings distribution.

Heterogeneity in firm productivity is needed to obtain a realistic wage earnings distribution (Mortensen, 1990; Burdett and Mortensen, 1998; Bontemps, Robin and van den Berg ,1999, 2000).

Figure 3.4: Wage earnings distribution in the simple Burdett-Mortensen model.

Comparative statics of s_e:

- **If employed workers' search less**, i.e. s_e decreases, it becomes less likely that an employed workers meets another firm.
- This reduces the risk of losing a worker, gives firms more market power and reduces the indirect competition between firms. This implies that the **maximum wage offer $\overline{w}$ decreases**.
- But also the **reservation wage R decreases**, since all firms offer lower wages, which decreases the return from searching for a better job.
- Total wages dispersion $\overline{w} - R$ decreases, i.e., $\partial \left[\overline{w} - R\right] / \partial s_e > 0$, since the firm paying the highest wage $\overline{w}$ decreases its wage more than the firm offering the lowest wage R.

Comparative statics of s_e:

- **If employed workers search more**, i.e., s_e increases, the labour market becomes **more competitive**.
- The **maximum wage offer** $\overline{w}$ **increases** up to the marginal product y.
- The **number of workers earning a wage** $w < \overline{w}$ **decreases**, i.e.,

$$\lim_{s_e \to \infty} g(w) = \lim_{s_e \to \infty} \frac{1}{2} \frac{q}{\lambda s_e} \frac{\sqrt{y - R}}{(y - w)^{(1+1/2)}} = 0.$$

- In the limit, i.e., $s_e \to \infty$, **almost all workers earn the marginal product**.

3.3.4 Number of active firms in equilibrium

If the demand for a product drops to zero (which happens at rate δ), **firms can decide whether or not to buy a new product idea** with productivity y.

The decision to enter the market for new products determines the number of active firms, given:

- the fixed cost to invent a new product idea f_e.
- the workers' reservation wage R,
- the wage offer distribution $F(w)$.

The market for new product ideas:

Firms are willing to pay at most the discounted profit $\Pi(w) = \Pi(R)$ for a new product idea (zero profit condition).

Firms buy new product ideas as long as they can afford paying the price f_e, i.e., $\Pi(R) - f_e \geq 0$.

A higher number of active firms decreases the labour force at each firm, since more firms are competing for the same number of workers. The lower labour input decreases profits until $\Pi(R) = f_e$.

Free entry condition:

$$\delta\Pi(R) = \delta f_e \Longleftrightarrow (y - z)\frac{\eta s_u q}{[q + \eta M s_e][q + \eta M s_u]} = \delta f_e. \tag{16}$$

Equilibrium number of active firms:

$$(y-z)\frac{\eta s_u q}{[q+\eta M s_e][q+\eta M s_u]}=\delta f_e.$$

Boundaries: $M=0$ implies $Lhs=(y-z)\,\eta s_u/q$; $M\rightarrow\infty$ implies $Lhs\rightarrow 0$.

An equilibrium exists, i.e., $M^*>0$, **if** $(y-z)\,\eta s_u/q>\delta f_e$.

Comparative statics:

- The number of active firms M^* increases with productivity y (since profits increase) and decreases with unemployment benefits z (since this increases wages).

- The number of active firms M^* decreases with a higher job destruction rate q, since it decreases the average duration of profitable employment.
- The number of active firms M^* also decreases with the cost of inventing a new product idea f_e.

Figure 3.5: Number of active firms

3.3.5 Wage tenure contracts

Firms pay a constant wage for the entire period the worker stays at the firm.

What changes, if wages can vary over time?

Stevens critique:

Firms offer higher wages for two reasons:

(i) A higher wage implies a higher value of employment $V_e(w)$ and increases the rate at which firms hire workers.

(ii) A higher wage reduces the quitting rate to other firms, i.e., $\lambda s_e [1 - F(w)]$.

Idea:

Would another wage contract that offers the same value of employment $V_e(w)$, but allows for wages to change over the employment spell, be able to reduce the quitting rate? Such a contract would be more profitable.

Wage tenure contracts are optimal, i.e., wages that increase over the employment spell are optimal.

Stevens (2004):

If workers are risk neutral, firms offer a step contract $(\underline{w}, y, T)$ with an initial wage equal to the minimum acceptable wage $\underline{w}$, a time to promotion T and the wage after the promotion equal to the marginal product y.

The time to promotion T is used to determine the value of employment $V_e(\underline{w}, y, T)$.

The step contract increase the value of employment from $V_e(\underline{w}, y, T)$ to $V_e(y)$. Since $V_e(y)$ is the maximum value of employment possible, a promoted worker has no incentive to search for another job (which reduces the quitting rate).

If workers are risk neutral, all firms offer a value of employment $V_e(\underline{w}, y, T) = V_u$. Thus, there is no dispersion in $V_e(\underline{w}, y, T)$ and no job-to-job transition (back in the Diamond Paradox).

Burdett and Coles (2003):

If workers are risk averse, firms have an incentive to offer a smooth wage-tenure contract which smooth consumption over time.

Firms offer different V_e, which again generates job-to-job transitions.

3.4 Productivity dispersion and the Burdett-Mortensen model

Idea:

Productivity dispersion can generate a more realistic wage earnings distribution.

Search frictions lead to imperfect labour market competition and allow firms with different productivities to coexist.

New assumption:

- Product ideas y are different in their profitability. The inventor learns about the profitability after investing the fixed cost f_e.

- The profitability is assumed to be randomly drawn from the cumulative distribution $\Gamma(y) \in [0, \overline{y}]$.

- Given the observed profitability y the investor decides, whether it is worth to enter the market given the expected profits a firm with product ideas y can earn in the labour market. This determines the firm with the lowest profitability y^* in the market.

- Those firms that are active in the market, decide on the wage that they are posting.

- All workers search with the same intensity, i.e., $s = s_u = s_e$. This implies $R = z$.

3.4.1 Wage offers distribution

We want to show that:

- more profitable firms offer higher wages,

- the wage offer distribution is identical to the profitability distribution of active firms.

Nash-Equilibrium condition:

No profitable deviation possible:

In equilibrium firms that offer wages that are on the support of the wage offer distribution must make higher profits than firms that offer a wage that is not on the support of the wage offer distribution, i.e.,

$$\begin{aligned} q\Pi(y) &= [y - w(y)]\, l(w(y)) \quad \text{for all} \;\; w(y) \in \mathsf{supp}(F), \\ q\Pi(y) &\geq [y - w(y')]\, l(w(y')) \;\; \text{for all} \;\; w(y') \notin \mathsf{supp}(F). \end{aligned} \tag{17}$$

The least productive firm:

The least productive firm makes zero profit, i.e. $q\Pi(y^*) = 0$.

Which implies that the lowest productivity firm pays a wage equal to its productivity, i.e., $w(y^*) = y^*$.

More productive firms offer higher wages:

The equilibrium condition requires that a highly profitable firm y^h that offers its optimal wage $w\left(y^h\right)$ makes weakly higher profits, i.e.,

$$\begin{aligned}\left[y^h - w\left(y^h\right)\right] l\left(w\left(y^h\right)\right) &\geq \left[y^h - w\left(y^l\right)\right] l\left(w\left(y^l\right)\right) \\ &> \left[y^l - w\left(y^l\right)\right] l\left(w\left(y^l\right)\right) \\ &\geq \left[y^l - w\left(y^h\right)\right] l\left(w\left(y^h\right)\right)\end{aligned}$$

The difference in profitabilities $y^h > y^l$ implies the strict inequality in the middle.

The equilibrium condition for low productivity firms implies the last inequality.

The difference of the first and the last inequality is greater than or equal the difference of the middle inequality, i.e.,

$$\left[y^h - y^l\right] l\left(w\left(y^h\right)\right) \geq \left[y^h - y^l\right] l\left(w\left(y^l\right)\right).$$

$\Longrightarrow$ **Wages increase with profitability**, since higher wages lead to a higher labour input.

Equilibrium wage offer distribution:

Since more profitable firms offer higher wages, it follows that the ranking of wages follows the ranking of profitabilities, i.e., a firm's position in the wage offer distribution equals its position in the profitability distribution of active firms,

$$F\left(w\left(y\right)\right)=\frac{\Gamma\left(y\right)-\Gamma\left(y^{*}\right)}{1-\Gamma\left(y^{*}\right)}\equiv\Gamma_{y^{*}}\left(y\right) \tag{18}$$

Since only firms with a profitability $y > y^{*}$ are active, the profitability distribution is rescaled by the fraction of profitabilities that do not enter the market.

Firms' optimization problem:

Firms' profits:

$$q\Pi(y) = \max_{w} [y - w] \, l(w)$$

FOC for any y:

$$-l(w(y)) + [y - w(y)] \, l'(w(y)) = 0$$

Using condition (18) for the equilibrium wage offer distribution implies that the following first order differential equation has to hold for all y,

$$\frac{\partial w(y)}{\partial y} = [y - w(y)] \frac{\partial l(w(y))}{\partial y} \frac{1}{l(w(y))}, \tag{19}$$

with the terminal condition $w(y^*) = y^*$.

Derivation of equilibrium wage offers:

The differential equation (19) can be solved more easily by defining $T(y) = \ln l(w(y))$. Taking the derivative with respect to y gives,

$$\frac{\partial T(y)}{\partial y} = \frac{\partial l(w(y))}{\partial y} \frac{1}{l(w(y))}$$

Substituting $\partial T(y)/\partial y$ into the differential equation (19) gives,

$$\frac{\partial w(y)}{\partial y} + \frac{\partial T(y)}{\partial y} w(y) = \frac{\partial T(y)}{\partial y} y.$$

Any solution to the above differential equation has to satisfy

$$\begin{aligned} w(y)\, e^{T(y)} &= \int_{y^*}^{y} x \frac{\partial T(x)}{\partial x} e^{T(x)} dx + A, \\ &= y e^{T(y)} - y^* e^{T(y^*)} - \int_{y^*}^{y} e^{T(x)} dx + A, \end{aligned}$$

Equilibrium wage offers:

Since the least profitable firm pays a gross wage y^*, we know that $A = y^* e^{T(y^*)}$.

Substituting $T(y)$ implies the **wage equation**:

$$w(y) = y - \int_{y^*}^{y} \frac{l(w(x))}{l(w(y))} dx \tag{20}$$

The firm with the least profitable has no incentive to pay a wage above the workers' reservation wage, i.e.,

$$w(y^*) = z.$$

Intuition:

Wages increase with profitability, because more profitable firms can gain more from employing more workers.

3.4.2 Number of active firms and the least profitable firm

We want to show that:

- more profitable firms make higher profits,
- unemployment benefits determine the lowest profitability level,
- comparative statics of the number of active firms.

More profitable firms make higher profits:

Rearranging the wage equation (20) implies the following **profit** equation:

$$q\Pi(y) = [y - w(y)]\, l(w(y)) = \int_{y^*}^{y} l(w(x))\, dx. \tag{21}$$

The least profitable firm y^* makes zero profits. The least profitable firm attracts unemployed workers, i.e. $l(w(y^*)) > 0$, but it makes zero profit per worker, since it pays a wage equal to the marginal product, i.e., $w(y^*) = y^*$.

Thus, the **lowest profitability** is determined by the reservation wage of unemployed worker, i.e.

$$y^* = z.$$

The market for new product ideas:

- New products are sold via an auction.
- Firms with the labour force $l\left(w\left(y\right)\right)$ are willing to bid up to $\Pi\left(y\right)$ for a product with profitability y.
- Firms with a different labour force can – according to the equilibrium profit condition (17) – only bid less.
- Thus, firms with a labour force $l\left(w\left(y\right)\right)$ acquire the product with profitability y at the price $\Pi\left(y\right)$.

Number of active firms in equilibrium:

Investors use average profits in the economy to finance new product ideas, i.e.,

$$\int_{z}^{\overline{y}} \Pi\left(y\right) d\Gamma\left(y\right) = f_e.$$

Substituting profits using equation (21) gives the **free entry condition:**

$$\int_{z}^{\overline{y}} \int_{z}^{y} l\left(w\left(x\right)\right) dx d\Gamma\left(y\right) = \delta f_e$$

(Note, that labour input $l\left(w\left(x\right)\right)$ decreases with the number of active firms M.)

Intuition:

Investors finance new product ideas. If the number of active firms is above the equilibrium level, it becomes more difficult for firms to recruit workers, this decreases firms' labour input and profits. Investors have then less funds available to finance new product ideas.

Figure 3.6: Number of active firms, if firms differ in profitability

Comparative statics: Unemployment benefits z:

- Higher unemployment benefits z increase the reservation wage of unemployed workers.

- Thus, the least profitable firm is no longer able to produce profitably. Hence only more profitable firms are active in the market, i.e., y^* increases.

- Since more profitable firms pay higher wages, all wages $w(y)$ increase.

- This reduces total profits in the economy.

- Investors have less funds available to finance new product ideas.

- Thus, the number of active firms M^* in the economy decreases.

- With less firms searching for workers, it becomes more difficult for unemployed workers to find a job, i.e., $\lambda = \eta M$ decreases.

- Thus, unemployment increases.

Comparative statics: Quitting rate q:

- A higher quitting rate reduces the average employment duration of a worker.
- This decreases the labour input $l\left(w\left(y\right)\right)$ of all active firms.
- Lower labour input reduces profits per workers, since firms make positive profits from employing a worker.
- Investors have less funds available to finance new product ideas.
- Thus, the number of active firms M^* in the economy decreases.
- With less firms searching for workers, it becomes more difficult for unemployed workers to find a job, i.e., $\lambda = \eta M$ decreases.
- Thus, unemployment increases for two reasons:
    - More workers are laid off (direct effect of an increase in q).
    - Less unemployed find a job (direct effect through the reduction in λ).

Figure 3.7: Comparative statics, if firms differ in profitability

3.5 Efficiency of the Burdett-Mortensen model

Externalities:

If new firms enter the market, they do not take into account that they make it more difficult for other firms to keep their workers (negative externality), which increases wages and reduces profits.

If new firms enter the market, they do not take into account that they make it easier for workers to find a job, which increases workers' wages and utility.

Question:

Is the number of active firms efficient? (Are firms' profits too high or too low?)

Welfare function with homogenous firms:

The government maximizes aggregate welfare by choosing the optimal number of active firms M in the economy.

Since workers are risk neutral, the **aggregate welfare function** maximizes output per capita (including the value of leisure for unemployed workers) minus the cost necessary to finance new product ideas, i.e.,

$$W = \max_{M} \left[uz + [1-u]\, y - f_e \delta M \right],$$

where the total cost of new product ideas is given by the cost per entry f_e and the number of firms entering the market for new product ideas δM.

FOC:

$$\frac{\partial u}{\partial M} [z-y] - \delta f_e = 0 \quad \text{with} \quad \frac{\partial u}{\partial M} = -\frac{\eta s q}{\left[q + \eta M^S s\right]^2}$$

Efficiency of the decentralized economy with homogenous firms:

The **social planner**'s optimality condition:

$$[y-z]\,\frac{\eta s q}{\left[q+\eta M^S s\right]^2}=\delta f_e$$

In the **decentralized economy**:

$$[y-z]\,\frac{\eta s q}{\left[q+\eta M^* s\right]^2}=\delta f_e.$$

Implication:

The number of active firms is efficient in the decentralized economy, i.e. $M^*=M^S$.

Welfare function with heterogenous firms:

The government maximizes aggregate welfare by choosing the number of active firms M.

Since workers are risk neutral, the **aggregate welfare function** maximizes output per capita (including the value of leisure for unemployed workers) minus the cost necessary to finance firm entry, i.e.,

$$\begin{aligned} W &= \max_M \left[uz + [1-u] \int_z^{\overline{y}} yg\left(w\left(y\right)\right) dy - f_e \delta M \right], \\ &= \max_M \left[uz + M \int_z^{\overline{y}} yl\left(w\left(y\right)\right) d\Gamma_{y^*}\left(y\right) - f_e \delta M \right], \end{aligned}$$

where the total cost of new product ideas is given by the cost per entry f_e and the number of firms entering the market for new product ideas δM.

FOC:

$$z\frac{\partial u}{\partial M}+\int_z^{\overline{y}} yl\left(w\left(y\right)\right)d\Gamma_{y^*}\left(y\right)+M\int_z^{\overline{y}} y\frac{\partial l\left(w\left(y\right)\right)}{\partial M}d\Gamma_{y^*}\left(y\right)-\delta f_e=0$$

Using the fact that $-\partial u/\partial M=l\left(z\right)$ and $\frac{\partial l(w(y))}{\partial M}=-\frac{\partial l(w(y))}{\partial y}\frac{\left[1-\Gamma_{y^*}(y)\right]}{M\gamma_{y^*}(y)}$ implies,

$$\delta f_e=\int_z^{\overline{y}}\left[yl\left(w\left(y\right)\right)-zl\left(z\right)\right]d\Gamma_{y^*}\left(y\right)-\int_z^{\overline{y}} y\frac{\partial l\left(w\left(y\right)\right)}{\partial y}\left[1-\Gamma_{y^*}\left(y\right)\right]dy.$$

Differentiation of $\partial\left[yl\left(w\left(y\right)\right)\right]/\partial y$ and integrating gives,

$$\begin{aligned}\delta f_e &= \int_z^{\overline{y}}\int_z^{y}\left(l\left(w\left(x\right)\right)+x\frac{\partial l\left(w\left(x\right)\right)}{\partial x}\right)dxd\Gamma_{y^*}\left(y\right)-\int_z^{\overline{y}} y\frac{\partial l\left(w\left(y\right)\right)}{\partial y}\left[1-\Gamma_{y^*}\left(y\right)\right]dy\\ &= \int_z^{\overline{y}}\int_z^{y} l\left(w\left(x\right)\right)dxd\Gamma_{y^*}\left(y\right)-\int_z^{\overline{y}}\frac{\partial\int_z^y x\frac{\partial l(w(x))}{\partial x}dx\left[1-\Gamma_{y^*}\left(y\right)\right]}{\partial y}dy,\\ &= \int_z^{\overline{y}}\int_z^{y} l\left(w\left(x\right)\right)dxd\Gamma_{y^*}\left(y\right).\end{aligned}$$

Efficiency of the decentralized economy with heterogenous firms:

The **social planner**'s optimality condition:

$$\int_{z}^{\overline{y}} \int_{z}^{y} l\left(w\left(x\right)\right) dx d\Gamma_{y^*}\left(y\right) = \delta f_e. \tag{22}$$

In the **decentralized economy**:

$$\int_{z}^{\overline{y}} \int_{z}^{y} l\left(w\left(x\right)\right) dx d\Gamma_{y^*}\left(y\right) = \delta f_e.$$

Implication:

The number of active firms is efficient in the decentralized economy, i.e. $M^* = M^S$.

Intuition for efficiency:

- Firms cause no congestion externality to other firms, i.e., their contact rate ηs is constant. The higher number of active firms increases the quitting probability (negative externality).
- The externalities cancel, because the contacting probability of workers increases by the same amount as the quitting probability at firms.
- Indirect competition between firms through on-the-job search reduces the bargaining power of firms and reduces firms' profits. (Note that they have nominally all the bargaining power, since they make take-it-or-leave-it wage offers.)
- The reduction in profits implies that the efficient amount of resources in the economy are used to finance new product ideas (active firms), i.e., the additional gain in output equals the marginal cost of generating a new product idea.

3.4 On-the-job search - Postel-Vinay and Robin (2002)

Basic ideas:

- In the Burdett-Mortensen model firms do not react to outside offers by other firms, i.e. they don't try to keep the worker by competing with the outside firm. This strategy can only be optimal (but need not be optimal), if workers increase their search intensity, if they know that an outside offer leads to a higher wage. Postel-Vinay and Robin (2002) allow firms to react to outside offers.

- In the BM-model firms do not condition their wage offer on the worker's current wage. Thus, some workers accept the wage offer other reject it. This strategy is not optimal. Firms should always condition their wage offer in order to recruit the contacted worker (as long as the wage is not above the marginal product).

- The Burdett-Mortensen model cannot explain within firm wage dispersion of equally skilled workers. Postel-Vinay and Robin (2002) explain within firm wage dispersion based on past contacts with outside firms.

3.4.1 Wage determination

Framework:

- The framework is identical to the BM-framework with heterogenous firms.
- Firms observe a worker's current value of employment and condition their wage offer on it.
- If an employed worker meets another firm, both firms start to Bertrand compete until the least productive firm drops out.

Wage offer strategies:

If a firm with productivity y meets an **unemployed worker**, it offers the worker the same value of employment (only ε more), i.e.,

$$V_e\left(\phi\left(z, y\right), y\right) = V_u\left(z\right). \tag{23}$$

If a firm with productivity y' meets a worker, who is employed at a **firm with productivity** $y < y'$, it offers the worker a wage $\phi\left(y, y'\right)$ such that the incumbent firm cannot outbid the offer, i.e.,

$$V_e\left(\phi\left(y, y'\right), y'\right) = V_e\left(y, y\right). \tag{24}$$

If a firm with productivity y' meets a worker, who is employed at a **firm with productivity** $y \geq y'$, it offers the worker the maximum wage y'. The worker stays at the incumbent firm at the wage $\phi\left(y', y\right)$, i.e.,

$$V_e\left(y', y'\right) = V_e\left(\phi\left(y', y\right), y\right). \tag{25}$$

Bellman equations of unemployed workers:

Value of being unemployed:

$$rV_u = z + \lambda s \int_z^{\overline{y}} \left[V_e \left(\phi \left(z, y \right), y \right) - V_u \right] d\Gamma_{y^*} \left(y \right) \tag{26}$$

Unemployed workers do not gain from searching, since they are offered the value of unemployment,

$$V_e \left(\phi \left(z, y \right), y \right) = V_u \Longrightarrow rV_u = z$$

(Note, that we are back in the Diamond Paradox. Cahuc, Postel-Vinay and Robin (2006) solve this problem by given workers some bargaining power)

Bellman equations of employed workers:

Value of being employed at the wage w at firm with productivity y:

$$\begin{aligned} rV_e(w,y) &= w + \lambda s \int_{q(w,y)}^{y} \left[V_e(\phi(x,y),y) - V_e(w,y)\right] d\Gamma_{y^*}(x) \\ &+ \lambda s \int_{y}^{\overline{y}} \left[V_e(\phi(y,x),x) - V_e(w,y)\right] d\Gamma_{y^*}(x) + q\left[V_u - V_e(w,y)\right] \end{aligned} \tag{27}$$

where the **reservation productivity** $q(w,y)$ is the productivity level at which the outside firm can only offer a wage equal to the workers current wage, i.e.,

$$\phi(q(w,y),y) = w.$$

Thus, workers are only promoted, if workers contact a firm with a productivity $y' \geq q(w,y)$.

If workers contact a firm with productivity $y' \geq y$, they quit to the ouside firm and earn a wage $\phi(y,y')$.

Derivation of equilibrium wages:

Bertrand competition between firms implies according to equations (24) and (25)

$$\begin{aligned} rV_e(w,y) \;=\; & w + \lambda s \int_{q(w,y)}^{y} \left[V_e(x,x) - V_e(w,y)\right] d\Gamma_{y^*}(x) \\ & + \lambda s \left[1 - \Gamma_{y^*}(y)\right]\left[V_e(y,y) - V_e(w,y)\right] + q\left[V_u - V_e(w,y)\right] \end{aligned} \tag{28}$$

Evaluating at $w = y$ implies $q(y,y) = y$ and

$$V_e(y,y) = \frac{y + qV_u}{r + q} \tag{29}$$

Derivation of equilibrium wages:

Substituting equation (29) implies:

$$\begin{aligned}\left[r+q+\lambda s\left[1-\Gamma_{y^*}\left(q\left(w,y\right)\right)\right]\right]V_e\left(w,y\right) &= w+\lambda s\int_{q(w,y)}^{y}\frac{x+qV_u}{r+q}d\Gamma_{y^*}\left(x\right) \qquad (30)\\ &\quad +\lambda s\left[1-\Gamma_{y^*}\left(y\right)\right]\frac{y+qV_u}{r+q}+qV_u\end{aligned}$$

By definition, the reservation productivity $q\left(w,y\right)$ is the type of incumbent employer form which a type y firm can poach a worker with wage w. Using equation (24) implies,

$$\begin{aligned}V_e\left(w,y\right) &= V_e\left(q\left(w,y\right),q\left(w,y\right)\right) \qquad (31)\\ &= \frac{q\left(w,y\right)+qV_u}{r+q}\end{aligned}$$

where the last equality follows from equation (29).

Derivation of equilibrium wages:

Substituting $V_e(w,y)$ from equation (31) into equation (30) implies,

$$[r+q+\lambda s[1-\Gamma_{y^*}(q(w,y))]]\frac{q(w,y)}{r+q}=w+\lambda s\int_{q(w,y)}^{y}\frac{x}{r+q}d\Gamma_{y^*}(x)+\lambda s[1-\Gamma_{y^*}(y)]\frac{y}{r+q}.$$

Integration by parts implies

$$\int_{q(w,y)}^{y}\frac{x}{r+q}d\Gamma_{y^*}(x)=\frac{y\Gamma_{y^*}(y)-q(w,y)\Gamma_{y^*}(q(w,y))}{r+q}-\int_{q(w,y)}^{y}\frac{1}{r+q}\Gamma_{y^*}(x)\,dx$$

Substitution implies

$$\begin{aligned} q(w,y) &= w+\frac{\lambda s}{r+q}[y-q(w,y)]-\frac{\lambda s}{r+q}\int_{q(w,y)}^{y}\Gamma_{y^*}(x)\,dx \\ &= w+\frac{\lambda s}{r+q}\int_{q(w,y)}^{y}[1-\Gamma_{y^*}(x)]\,dx. \end{aligned}$$

Wages earned by unemployed worker:

- The **reservation productivity** of unemployed workers equals unemployment benefits, i.e., $q(w, y) = z$.
- The **reservation wage** of an unemployed worker depends on the productivity y of the contracted firm, i.e. $w = \phi(z, y)$.
- Unemployed workers are always offered their reservation wage, i.e., they receive an offer that makes them just indifferent between accepting and rejecting.

$$\phi(z, y) = z - \frac{\lambda s}{r+q} \int_z^y \left[1 - \Gamma_{y^*}(x)\right] dx \tag{32}$$

Note: An unemployed worker is willing to accept a wage below unemployment benefits, since – once employed – she can expect wage gains, if she meets another firm (since both firms will enter Bertrand competition).

Wages earned by workers that meet a more productive firm:

- The **reservation productivity** of a worker, who is currently employed at a firm with productivity y, and who meets a more productive firm with productivity $y' > y$, is the productivity of the current employer (since the current employer is willing to bid up to the marginal product), i.e., $q(w, y) = y$.
- The **wage** of a worker depends on the productivity y' of the contracted firm and the maximum bid of the incumbent firm (which equals the productivity y of the incumbent firm), i.e. $w = \phi(y, y')$.

$$\phi(y, y') = y - \frac{\lambda s}{r+q} \int_{y}^{y'} \left[1 - \Gamma_{y^*}(x)\right] dx. \tag{33}$$

Note:

A worker is willing to accept a wage below the marginal product of her old employer, since she can expect wage gains, if she meets another firm.

If workers meet a more productive firm, workers might accept wage cuts in turn for higher expected wage growth later in their career.

Wages earned by workers that meet a less productive firm:

- The **reservation productivity** of a worker, who is currently employed at a firm with productivity y, and who meets a less productive firm with productivity $y' \leq y$, is the productivity of the outside firm (since the outside firm is willing to bid up to the marginal product), i.e., $q(w, y') = y'$.

- The **wage** of a worker depends on the maximum bid of the outside firm (which equals its productivity y') and the productivity of the incumbent firm, i.e. $w = \phi(y, y')$.

$$\phi(y', y) = y' - \frac{\lambda s}{r+q} \int_{y'}^{y} \left[1 - \Gamma_{y^*}(x)\right] dx. \tag{34}$$

Note:

A worker, who is contacted by a less productive firm, is now rewarded for her patience and is promoted in order to prevent the worker from quitting to the less productive firm.

3.4.2 Labour input

Framework:

- Denote by $L(w|y)$ the number of workers at a type-y firm that earn a wage w or less.

- Denote by $L(y) = L(y|y)$ the total number of workers at type-y firm.

Labour input $L(w|y)$:

Inflow:

- Unemployed workers $\eta s u$, since all unemployed workers are offered acceptable wages.
- Employed workers $\eta s \int_{y^*}^{q(w,y)} L(x)\, d\Gamma_{y^*}(x)$, since all workers at firms with a productivity $y \leq q(w,y)$ accept a job at a wage w or less at a type-y firm.

Outflow:

- Workers exit at rate $q + \eta M s \left[1 - \Gamma_{y^*}(q(w,y))\right]$, because they are either promoted, if they meet a firm with productivity $y' \in [q(w,y), y]$, or they move to another firm, if they meet a firm with productivity $y' \in [y, \overline{y}]$.

In steady state:

$$L(w|y) = \frac{\eta s u + \eta s \int_{y^*}^{q(w,y)} L(x)\, d\Gamma_{y^*}(x)}{q + \eta M s \left[1 - \Gamma_{y^*}(q(w,y))\right]} \tag{35}$$

Number of workers employed at firms with productivity $x \leq y$:

Number of workers employed at firms with productivity $x \leq y$ is denoted by

$$\int_{y^*}^{y'} L(x)\, d\Gamma_{y^*}(x).$$

Inflow:

- All unemployed workers that meet a firm with productivity $x \leq y$, i.e., $\eta s u \Gamma_{y^*}(y')$

Outflow:

- Workers exit at rate $q + \eta M s \left[1 - \Gamma_{y^*}(y')\right]$, because they move to another firm, if other firm has a productivity $x \in [y', \overline{y}]$.

In steady state:

$$\int_{y^*}^{y'} L(x)\, d\Gamma_{y^*}(x) = \frac{\eta s u \Gamma_{y^*}(y')}{q + \eta M s \left[1 - \Gamma_{y^*}(y')\right]} \tag{36}$$

Steady state labour input $L\left(w|y\right)$**:**

Substituting $\int_{y^*}^{q(w,y)} L\left(x\right) d\Gamma_{y^*}\left(x\right)$ using equation (36) implies

$$L\left(w|y\right) = \frac{\eta s\left[q+\eta M s\right]}{\left[q+\eta M s\left[1-\Gamma_{y^*}\left(q\left(w,y\right)\right)\right]\right]^2}\frac{q}{q+\eta M s} \tag{37}$$

Differentiating equation (36) gives the **number of workers employed at a firm with productivity** y', i.e.,

$$L\left(y'\right) = \frac{\eta s\left[q+\eta M s\right]}{\left[q+\eta M s\left[1-\Gamma_{y^*}\left(y'\right)\right]\right]^2}\frac{q}{q+\eta M s} \tag{38}$$

Implication:

The number of workers employed at a type-y firm at a wage w or less equals the number of workers employed at a firm with the reservation productivity $q(w,y)$, i.e.,

$$L(w|y) = L(q(w,y)). \tag{39}$$

Intuition:

- Workers enter at the same rate, either from unemployment or from firms with a lower productivity.
- Workers that enter the more productive firm $y > q(w,y)$ from firms with a productivity $y' \in [q(w,y), y]$, and earn wages $w' > w$.
- Worker exit at the same rate.
  - If they are employed at the firm with productivity $q(w,y)$, they always change to more productive firms.
  - If they are employed at the type-y firm, they are either promoted to a wage $w' > w$ or they change to a more productive firm.

Comparing labour inputs with the Burdett-Mortensen model:

- All unemployed workers accept a job offer, since all firms regardless whether in the BM-model or the PVR-model offer at least the reservation wage.
- Workers in the BM-model and in the PVR-model only change employers, if the employer is more productive.
- The least productive firm in both models has a productivity equal to unemployment benefits, i.e., $y^* = z$.

$\Longrightarrow$ A firm with productivity y has the same labour input in the BM-model and the PVR-model (for a given number of active firms M).

Comparing equation (15) for the BM-model and (38) for the PVR-model shows this formally, i.e.,

$$l\left(w\left(y\right)\right) = L\left(y\right) = \frac{\eta s\left[q + \eta M s\right]}{\left[q + \eta M s\left[1 - \Gamma_{y^*}\left(y\right)\right]\right]^2} \frac{q}{q + \eta M s}.$$

3.4.3 Profits and number of active firms

Idea:

- Firms in the PVR-model extract the whole rent from unemployed worker, since they make state-dependent wage offers.

- Firms in the PVR-model have to Bertrand compete, if one of their workers meets another firm.

Profits of a firm with productivity y:

A firm with productivity y pays wages in the range $w \in [\phi(z, y), y]$,

- where $\phi(z, y)$ equals the wage paid to unemployed worker and to worker at the lowest productivity firm,
- and y equals the wage, if the employed worker meets another firm with productivity y.

Profits of a firm with productivity y are therefore given by,

$$\delta \Pi(y) = \int_{\phi(z,y)}^{y} [y - w] \, dL(w|y) \tag{40}$$

Derivation of profits:

Integration by parts implies

$$\delta\Pi\left(y\right)=\left[\left[y-w\right]L\left(w|y\right)\right]_{\phi(z,y)}^{y}+\int_{\phi(z,y)}^{y}L\left(w|y\right)dw$$

Changing the variable of integration from the wage to productivity gives

$$\int_{\phi(z,y)}^{y}L\left(w|y\right)dw=\int_{z}^{y}L\left(\phi\left(x,y\right)|y\right)\frac{\partial\phi\left(x,y\right)}{\partial x}dx$$

where equation (34) implies

$$\frac{\partial\phi\left(x,y\right)}{\partial x}=1+\frac{\eta Ms\left[1-\Gamma_{y^{*}}\left(x\right)\right]}{r+q}.$$

Furthermore, the equality of $L\left(w|y\right)=L\left(q\left(w,y\right)\right)$ implies

$$L\left(\phi\left(x,y\right)|y\right)=L\left(q\left(\phi\left(x,y\right),y\right)\right)=L\left(x\right)$$

and the last equality is implied by the definition of the reservation productivity $q\left(\phi\left(x,y\right),y\right)$ and the wage $\phi\left(x,y\right)$.

Comparing profits with the Burdett-Mortensen model:

The profit of a firm in the **PVR-model** is given by

$$\delta\Pi^{PVR}(y) = \int_z^y L(x)\left[1 + \frac{\lambda s\left[1 - \Gamma_{y^*}(x)\right]}{r+q}\right]dx$$

The profit of a firm in the **BM-model** is given by

$$\delta\Pi^{BM}(y) = \int_z^y l(w(x))\,dx$$

$\Longrightarrow$ Thus, firms in the PVR-model earn more profits than firms in the BM-model, i.e., $\Pi^{PVR}(y) > \Pi^{BM}(y)$.

Intuition:

Firms in the PVR-model earn higher profits than firms in the BM-model, since their **wage policy is not constrained**, i.e., they can adapt their wage offer according to the worker's current value of employment.

Comparing number of active firms with the BM-model:

Investors have **more funds available in the PVR-model**, since all firms earn more profits in the PVR-model than in the BM-model, i.e.,

$$\int_{z}^{\overline{y}} \Pi^{PVR}(y)\, d\Gamma(y) > \int_{z}^{\overline{y}} \Pi^{BM}(y)\, d\Gamma(y).$$

Thus, the **number of active firms** in the PVR-model is inefficiently compared with the social optimum (22), i.e.,

$$M^{PVR} > M^{BM} = M^{S}.$$

Implication:

The **social planner** would to tax profits in the PVR-model such that they are equal to the profits in the BM-model.

3.5 Optimal wage policy

Basic idea:

- Firms in the PVR-model can adapt their wage offer according to the worker's outside option.
- This is the reason why firms in the PVR-model make higher profits than firms in the BM-model.
- Thus, making conditional wage offer and reacting to outside offers of workers seems to be an optimal wage policy.
- However, promoting workers in response to outside offers implies that workers have a high incentive to search for an outside offer.
- This moral hazard problem might reduce the profit of PVR-firms.

Postel-Vinay and Robin (2004):

Firms can decide between two wage policies:

- Reacting to outside offers (i.e., enter Bertrand competition like in PVR)
- Not reacting to outside offers (like in BM).

Trade-off:

Reacting to outside offers implies that firms can keep a worker, if he is contacted by a less productive firm.

Worker's job finding rate λs increases, since they increase their search intensity s.

Result:

High productivity firms find it optimal to react to outside offers, since they are less likely to loose a worker to a more productive firm.

Low productivity firms find it optimal not to react to outside offers, since they cannot win a Bertrand game against a more productive firm, but they can reduce the job finding rate by not reacting to outside offers.

Holzner (2011):

- Like in PVR (2004), but firms can offer wage tenure contracts.
- Wage tenure contracts are designed to reduce the quitting probability of workers.
- Wage tenure contracts start at the lowest acceptable wage $\underline{w}$ (workers are credit constrained).
- PVR-firms need a lower $\underline{w}$ than BM-firms in order to extract the whole rent from workers. This can be see by comparing the reservation wage of unemployed workers in the PVR- and the BM-model, i.e., $\phi(z, y) << z$.
- A relatively high acceptable wage, i.e., $\underline{w} \to z$, reduces PVR-firms profits and ensures that BM- and PVR-firms can coexist.

3.6 Explaining wage dispersion

Basic idea:

- On-the-job search models imply that wages increase with experience (PVR and BM) and tenure (PVR, and Burdett and Coles, 2003).
- This wage increase occurs even in the absence of human capital accumulation (on-the-job learning).
- Thus, any Mincer wage equation that measures the return to human capital accumulation by the experience-coefficient is biased.
- Extensions of the BM, PVR and the Burdett and Coles (2003) model allow for a decomposition of the wage and an identification of the impact of on-the-job learning and on-the-job search.

Carrillo-Tudela (2010):

Includes human capital accumulation and different skill levels into the Burdett-Mortensen model with heterogenous firms.

A **worker's productivity** is given by

$$h = ae^{\rho x}$$

where a equals a worker's ability, x experience and ρx human capital due to experience x.

A firm with productivity y will offer "piece rate p" such that a worker's wage is given by $w = hpy$

A worker's **log-wage** is then given by

$$\log w = \log a + \rho x + \log y + \log p$$

where

$$p = y - \int_z^y \frac{l\left(p\left(x\right)\right)}{l\left(p\left(y\right)\right)} dx$$

Bagger, Fontaine, Postel-Vinay and Robin (2011):

Includes human capital accumulation and different skill levels into the Postel-Vinay and Robin model with worker bargaining power.

A **worker's productivity** is also given by

$$h = ae^{\rho x}$$

A firm with productivity y will offer "piece rates p", i.e., $w = hpy$.

A worker's piece rate p also depends on the productivity y' of the firms that was involved in the last Bertrand game with the current employer.

A worker's **log-wage** is then given by

$$\log w = \log a + \rho x + \log y + \log p$$

where

$$\log p = -(1-\gamma)[\log y - \log y'] - \lambda s (1-\gamma)^2 \int_{\log y'}^{\log y} \frac{[1 - \Gamma_{y^*}(x)]}{r + q + \lambda s \gamma [1 - \Gamma_{y^*}(x)]} dx$$

Results:

Carrillo-Tudela (2010):

Productivity differences explain around 60% of wage variation, on-the-job search frictions around 25% and the joint effect of on-the-job search and human capital accumulation around 15%.

Bagger, Fontaine, Postel-Vinay and Robin (2011):

Wage increases due to promotions in a job (within-job-search effect) are more important than wage increases due to job changes (between-job-search effect).

Human capital accumulation is the primary source of wage growth in the early working life.