Macroeconomic Theory

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Contents

Chapter 1

Overview and Summary

Broadly, these notes will first provide a quick warm-up for dynamic general equilibrium models before we will discuss the two workhorses of modern macroeconomics, the neoclassical growth model with infinitely lived consumers and the Overlapping Generations (OLG) model. The focus of these notes is as much on developing a coherent language for formulating dynamic macroeconomic model and on the techniques to analyze them, as is it on the application of these models for basic questions in economic growth and business cycle research.

In chapter 2 I will first present a simple dynamic pure exchange economy with two infinitely lived consumers engaging in intertemporal trade. In this model the connection between competitive equilibria and Pareto optimal equilibria can be easily demonstrated. Furthermore it will be demonstrated how this connection can exploited to compute equilibria by solving a particular social planners problem, an approach developed first by Negishi (1960) and discussed nicely by Kehoe (1989). Furthermore I will show the equivalence of equilibria in a market arrangement in which households trade dated consumption goods at the beginning of time, and equilibria under a market structure where trade takes place sequentially and in every period households exchange the current period consumption good and a simple financial asset.

This baseline model with then enriched, in chapter 3, by production (and simplified by dropping one of the two agents), to give rise to the neoclassical growth model. This model will first be presented in discrete time to discuss discrete-time dynamic programming techniques; both theoretical as well as computational in nature. The main reference will be Stokey et al., chapters 2-4. On the substantive side, I will argue how this model can be mapped into the data and what it implies for economic growth in the short and in the long run. On the methodological side I will give the general mathematical treatment of discrete time dynamic programming in chapters **??** (where I discuss the required mathematical concepts) and chapter 5 (where the main general results in the theory of dynamic programming will be summarized).

In chapter 6 I will introduce models with risk. After setting up the appropriate notation I will first discuss a pure exchange version of the stochastic model and show how this model (essentially a version of Lucas' (1978) asset pricing model) can be used to develop a simple theory of asset pricing. I will then turn to a stochastic version of the model with production in order to develop the Real Business Cycle (RBC) theory of business cycles. Cooley and Prescott (1995) are a good reference for this application.

Chapter 7 presents a general discussion of the two welfare theorems in economies with infinite-dimensional commodity spaces, as is typical for macroeconomic applications in which the economy extends forever and thus there are an (countably) infinite number of dated consumption goods being traded. We will heavily draw on Stokey et al., chapter 15's discussion of Debreu (1954) for this purpose.

The next two topics are logical extensions of the preceding material. In chapter 8 we discuss the OLG model, due to Samuelson (1958) and Diamond (1965). The first main focus in this module will be the theoretical results that distinguish the OLG model from the standard Arrow-Debreu model of general equilibrium: in the OLG model equilibria may not be Pareto optimal, fiat money may have positive value, for a given economy there may be a continuum of equilibria (and the core of the economy may be empty). All this could not happen in the standard Arrow-Debreu model. References that explain these differences in detail include Geanakoplos (1989) and Kehoe (1989). Our discussion of these issues will largely consist of examples. One reason to develop the OLG model was the uncomfortable assumption of infinitely lived agents in the standard neoclassical growth model. Barro (1974) demonstrated under which conditions (operative bequest motives) an OLG economy will be equivalent to an economy with infinitely lived consumers. One main contribution of Barro was to provide a formal justification for the assumption of infinite lives. As we will see this methodological contribution has profound consequences for the macroeconomic effects of government debt, reviving the Ricardian Equivalence proposition. As a prelude we will briefly discuss Diamond's (1965) analysis of government debt in an OLG model.

In the final module of these notes, chapter 9, we will discuss the neoclassical growth model in continuous time to develop continuous time optimization techniques. After having learned the technique we will review the main developments in growth theory and see how the various growth models fare when being contrasted with the main empirical findings from the Summers-Heston panel data set. We will briefly discuss the Solow model and its empirical implications (using the article by Mankiw et al. (1992) and Romer, chapter 2), then continue with the Ramsey model (Intriligator, chapter 14 and 16, Blanchard and Fischer, chapter 2). In this model growth comes about by introducing exogenous technological progress. We will then review the main contributions of endogenous growth theory, first by discussing the early models based on externalities (Romer (1986), Lucas (1988)), then models that explicitly try to model technological progress (Romer (1990).[1]

[1]Previous versions of these notes contained a chapter on models with heterogeneous households. This material has been merged into my manuscript *An Introduction to Macroeconomics with Household Heterogeneity* and significantly expanded there.

Chapter 2

A Simple Dynamic Economy

2.1 General Principles for Specifying a Model

An economic model consists of different types of entities that take decisions subject to constraints. When writing down a model it is therefore crucial to clearly state what the agents of the model are, which decisions they take, what constraints they have and what information they possess when making their decisions. Typically a model has (at most) three types of decision-makers

1. Households: We have to specify households' **preferences** over **commodities** and their **endowments** of these commodities. Households are assumed to optimize their preferences over a constraint set that specifies which combination of commodities a household can choose from. This set usually depends on initial household endowments and on market prices.

2. Firms: We have to specify the **production technology** available to firms, describing how commodities (inputs) can be transformed into other commodities (outputs). Firms are assumed to maximize (expected) profits, subject to their production plans being technologically feasible.

3. Government: We have to specify what **policy** instruments (taxes, money supply etc.) the government controls. When discussing government policy from a positive point of view we will take government polices as given (of course requiring the government budget constraint(s)

to be satisfied), when discussing government policy from a normative point of view we will endow the government, as households and firms, with an objective function. The government will then maximize this objective function by choosing policy, subject to the policies satisfying the government budget constraint(s)).

In addition to specifying preferences, endowments, technology and policy, we have to specify what **information** agents possess when making decisions. This will become clearer once we discuss models with risk. Finally we have to be precise about how agents interact with each other. Most of economics focuses on market interaction between agents; this will be also the case in this course. Therefore we have to specify our **equilibrium concept,** by making assumptions about how agents perceive their power to affect market prices. In this course we will focus on competitive equilibria, by assuming that all agents in the model (apart from possibly the government) take market prices as given and beyond their control when making their decisions. An alternative assumption would be to allow for market power of firms or households, which induces strategic interactions between agents in the model. Equilibria involving strategic interaction have to be analyzed using methods from modern game theory.

To summarize, a description of any model in this course should always contain the specification of the elements in bold letters: what commodities are traded, preferences over and endowments of these commodities, technology, government policies, the information structure and the equilibrium concept.

2.2 An Example Economy

Time is discrete and indexed by $t = 0, 1, 2, \ldots$ There are 2 individuals that live forever in this pure exchange economy.[1] There are no firms, and the government is absent as well. In each period the two agents consume a nonstorable consumption good. Hence there are (countably) infinite number of commodities, namely consumption in periods $t = 0, 1, 2, \ldots$

[1] One may wonder how credible the assumption is that households take prices as given in an economy with two households. To address this concern, let there be instead two classes of households with equal size. Within each class, there are many households (if you want to be really safe, a continuum) that are all identical and described as in the main text. This economy has the same equilibria as the one described in the main text.

Definition 1 *An allocation is a sequence* $(c^1, c^2) = \{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ *of consumption in each period for each individual.*

Individuals have preferences over consumption allocations that can be represented by the utility function

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \tag{2.1}$$

with $\beta \in (0, 1)$.

This utility function satisfies some assumptions that we will often require in this course. These are further discussed in the appendix to this chapter. Note that both agents are assumed to have the same time discount factor β.

Agents have deterministic endowment streams $e^i = \{e_t^i\}_{t=0}^{\infty}$ of the consumption goods given by

$$\begin{aligned} e_t^1 &= \begin{cases} 2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases} \\ e_t^2 &= \begin{cases} 0 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd} \end{cases} \end{aligned}$$

There is no risk in this model and both agents know their endowment pattern perfectly in advance. All information is public, i.e. all agents know everything. At period 0, before endowments are received and consumption takes place, the two agents meet at a central market place and trade all commodities, i.e. trade consumption for all future dates. Let p_t denote the price, in period 0, of one unit of consumption to be delivered in period t, in terms of an abstract unit of account. We will see later that prices are only determined up to a constant, so we can always normalize the price of one commodity to 1 and make it the numeraire. Both agents are assumed to behave competitively in that they take the sequence of prices $\{p_t\}_{t=0}^{\infty}$ as given and beyond their control when making their consumption decisions.

After trade has occurred agents possess pieces of paper (one may call them contracts) stating

> in period 212 I, agent 1, will deliver 0.25 units of the consumption good to agent 2 (and will eat the remaining 1.75 units)
>
> in period 2525 I, agent 1, will receive one unit of the consumption good from agent 2 (and eat it).

and so forth. In all future periods the only thing that happens is that agents meet (at the market place again) and deliveries of the consumption goods they agreed upon in period 0 takes place. Again, all trade takes place in period 0 and agents are committed in future periods to what they have agreed upon in period 0. There is perfect enforcement of these contracts signed in period 0.[2]

2.2.1 Definition of Competitive Equilibrium

Given a sequence of prices $\{p_t\}_{t=0}^{\infty}$ households solve the following optimization problem

$$\begin{aligned} &\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \\ &\text{s.t.} \\ \sum_{t=0}^{\infty} p_t c_t^i &\leq \sum_{t=0}^{\infty} p_t e_t^i \\ c_t^i &\geq 0 \text{ for all } t \end{aligned}$$

Note that the budget constraint can be rewritten as

$$\sum_{t=0}^{\infty} p_t(e_t^i - c_t^i) \geq 0$$

The quantity $e_t^i - c_t^i$ is the net trade of consumption of agent i for period t which may be positive or negative.

For arbitrary prices $\{p_t\}_{t=0}^{\infty}$ it may be the case that total consumption in the economy desired by both agents, $c_t^1 + c_t^2$ at these prices does not equal total endowments $e_t^1 + e_t^2 \equiv 2$. We will call equilibrium a situation in which prices are "right" in the sense that they induce agents to choose consumption so that total consumption equals total endowment in each period. More precisely, we have the following definition

Definition 2 *A (competitive) Arrow-Debreu equilibrium are prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *and allocations* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ *such that*

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[2] A market structure in which agents trade only at period 0 will be called an Arrow-Debreu market structure. We will show below that this market structure is equivalent to a market structure in which trade in consumption and a particular asset takes place in each period, a market structure that we will call sequential markets.

1. *Given* $\{\hat{p}_t\}_{t=0}^{\infty}$, *for* $i = 1, 2$, $\{\hat{c}_t^i\}_{t=0}^{\infty}$ *solves*

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \tag{2.2}$$

s.t.

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i \tag{2.3}$$

$$c_t^i \geq 0 \text{ for all } t \tag{2.4}$$

2.

$$\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 \text{ for all } t \tag{2.5}$$

The elements of an equilibrium are allocations and prices. Note that we do not allow free disposal of goods, as the market clearing condition is stated as an equality.[3] Also note the ˆ's in the appropriate places: the consumption allocation has to satisfy the budget constraint (2.3) only at equilibrium prices and it is the equilibrium consumption allocation that satisfies the goods market clearing condition (2.5). Since in this course we will usually talk about competitive equilibria, we will henceforth take the adjective "competitive" as being understood.

2.2.2 Solving for the Equilibrium

For arbitrary prices $\{p_t\}_{t=0}^{\infty}$ let's first solve the consumer problem. Attach the Lagrange multiplier λ_i to the budget constraint. The first order necessary conditions for c_t^i and c_{t+1}^i are then

$$\frac{\beta^t}{c_t^i} = \lambda_i p_t \tag{2.6}$$

$$\frac{\beta^{t+1}}{c_{t+1}^i} = \lambda_i p_{t+1} \tag{2.7}$$

[3] Different people have different tastes as to whether one should allow free disposal or not. Personally I think that if one wishes to allow free disposal, one should specify this as part of technology (i.e. introduce a firm that has available a technology that uses positive inputs to produce zero output; obviously for such a firm to be operative in equilibrium it has to be the case that the price of the inputs are non-positive -think about goods that are actually bads such as pollution).

and hence

$$p_{t+1}c_{t+1}^i = \beta p_t c_t^i \text{ for all } t \tag{2.8}$$

for $i = 1, 2$.

Equations (2.8), together with the budget constraint can be solved for the optimal sequence of consumption of household i as a function of the infinite sequence of prices (and of the endowments, of course)

$$c_t^i = c_t^i\left(\{p_t\}_{t=0}^{\infty}\right)$$

In order to solve for the equilibrium prices $\{p_t\}_{t=0}^{\infty}$ one then uses the goods market clearing conditions (2.5)

$$c_t^1\left(\{p_t\}_{t=0}^{\infty}\right) + c_t^2\left(\{p_t\}_{t=0}^{\infty}\right) = e_t^1 + e_t^2 \text{ for all } t$$

This is a system of infinite equations (for each t one) in an infinite number of unknowns $\{p_t\}_{t=0}^{\infty}$ which is in general hard to solve. Below we will discuss Negishi's method that often proves helpful in solving for equilibria by reducing the number of equations and unknowns to a smaller number.

For our particular simple example economy, however, we can solve for the equilibrium directly. Sum (2.8) across agents to obtain

$$p_{t+1}\left(c_{t+1}^1 + c_{t+1}^2\right) = \beta p_t(c_t^1 + c_t^2)$$

Using the goods market clearing condition we find that

$$p_{t+1}\left(e_{t+1}^1 + e_{t+1}^2\right) = \beta p_t(e_t^1 + e_t^2)$$

and hence

$$p_{t+1} = \beta p_t$$

and therefore equilibrium prices are of the form

$$p_t = \beta^t p_0$$

Without loss of generality we can set $p_0 = 1$, i.e. make consumption at period 0 the numeraire.[4] Then equilibrium prices have to satisfy

$$\hat{p}_t = \beta^t$$

[4]Note that multiplying all prices by $\mu > 0$ does not change the budget constraints of agents, so that if prices $\{p_t\}_{t=0}^{\infty}$ and allocations $(\{c_t^i\}_{t=0}^{\infty})_{i\in 1,2}$ are an AD equilibrium, so are prices $\{\mu p_t\}_{t=0}^{\infty}$ and allocations $(\{c_t^i\}_{t=0}^{\infty})_{i=1,2}$

so that, since $\beta < 1$, the period 0 price for period t consumption is lower than the period 0 price for period 0 consumption. This fact just reflects the impatience of both agents.

Using (2.8) we have that $c_{t+1}^i = c_t^i = c_0^i$ for all t, i.e. consumption is constant across time for both agents. This reflects the agent's desire to smooth consumption over time, a consequence of the strict concavity of the period utility function. Now observe that the budget constraint of both agents will hold with equality since agents' period utility function is strictly increasing. The left hand side of the budget constraint becomes

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i = c_0^i \sum_{t=0}^{\infty} \beta^t = \frac{c_0^i}{1-\beta}$$

for $i = 1, 2$.

The two agents differ only along one dimension: agent 1 is rich first, which, given that prices are declining over time, is an advantage. For agent 1 the right hand side of the budget constraint becomes

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^1 = 2 \sum_{t=0}^{\infty} \beta^{2t} = \frac{2}{1-\beta^2}$$

and for agent 2 it becomes

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^2 = 2\beta \sum_{t=0}^{\infty} \beta^{2t} = \frac{2\beta}{1-\beta^2}$$

The equilibrium allocation is then given by

$$\begin{aligned}
\hat{c}_t^1 &= \hat{c}_0^1 = (1-\beta)\frac{2}{1-\beta^2} = \frac{2}{1+\beta} > 1 \\
\hat{c}_t^2 &= \hat{c}_0^2 = (1-\beta)\frac{2\beta}{1-\beta^2} = \frac{2\beta}{1+\beta} < 1
\end{aligned}$$

which obviously satisfies

$$\hat{c}_t^1 + \hat{c}_t^2 = 2 = \hat{e}_t^1 + \hat{e}_t^2 \text{ for all } t$$

Therefore the mere fact that the first agent is rich first makes her consume more in *every* period. Note that there is substantial trade going on; in each even period the first agent delivers $2 - \frac{2}{1+\beta} = \frac{2\beta}{1+\beta}$ to the second agent and in all odd periods the second agent delivers $2 - \frac{2\beta}{1+\beta}$ to the first agent. Also note that this trade is mutually beneficial, because without trade both agents receive lifetime utility

$$u(e_t^i) = -\infty$$

whereas with trade they obtain

$$\begin{aligned}
u(\hat{c}^1) &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2}{1+\beta}\right) = \frac{\ln\left(\frac{2}{1+\beta}\right)}{1-\beta} > 0 \\
u(\hat{c}^2) &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2\beta}{1+\beta}\right) = \frac{\ln\left(\frac{2\beta}{1+\beta}\right)}{1-\beta} < 0
\end{aligned}$$

In the next section we will show that not only are both agents better off in the competitive equilibrium than by just eating their endowment, but that, in a sense to be made precise, the equilibrium consumption allocation is socially optimal.

2.2.3 Pareto Optimality and the First Welfare Theorem

In this section we will demonstrate that for this economy a competitive equilibrium is socially optimal. To do this we first have to define what socially optimal means. Our notion of optimality will be Pareto efficiency (also sometimes referred to as Pareto optimality). Loosely speaking, an allocation is Pareto efficient if it is feasible and if there is no other feasible allocation that makes no household worse off and at least one household strictly better off. Let us now make this precise.

Definition 3 *An allocation* $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ *is feasible if*

1. $$c_t^i \geq 0 \text{ for all } t, \text{ for } i = 1, 2$$

2. $$c_t^1 + c_t^2 = e_t^1 + e_t^2 \text{ for all } t$$

Feasibility requires that consumption is nonnegative and satisfies the resource constraint for all periods $t = 0, 1, \ldots$

Definition 4 *An allocation* $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ *is Pareto efficient if it is feasible and if there is no other feasible allocation* $\{(\tilde{c}_t^1, \tilde{c}_t^2)\}_{t=0}^{\infty}$ *such that*

$$\begin{aligned} u(\tilde{c}^i) &\geq u(c^i) \textit{ for both } i = 1, 2 \\ u(\tilde{c}^i) &> u(c^i) \textit{ for at least one } i = 1, 2 \end{aligned}$$

Note that Pareto efficiency has nothing to do with fairness in any sense: an allocation in which agent 1 consumes everything in every period and agent 2 starves is Pareto efficient, since we can only make agent 2 better off by making agent 1 worse off.

We now prove that every competitive equilibrium allocation for the economy described above is Pareto efficient. Note that we have solved for one equilibrium above; this does not rule out that there is more than one equilibrium. One can, in fact, show that for this economy the competitive equilibrium is unique, but we will not pursue this here.

Proposition 5 *Let* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ *be a competitive equilibrium allocation. Then* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ *is Pareto efficient.*

Proof. The proof will be by contradiction; we will assume that $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ is not Pareto efficient and derive a contradiction to this assumption.

So suppose that $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ is not Pareto efficient. Then by the definition of Pareto efficiency there exists another feasible allocation $(\{\tilde{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ such that

$$\begin{aligned} u(\tilde{c}^i) &\geq u(\hat{c}^i) \text{ for both } i = 1, 2 \\ u(\tilde{c}^i) &> u(\hat{c}^i) \text{ for at least one } i = 1, 2 \end{aligned}$$

Without loss of generality assume that the strict inequality holds for $i = 1$.

Step 1: Show that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$

where $\{\hat{p}_t\}_{t=0}^{\infty}$ are the equilibrium prices associated with $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$. If not, i.e. if

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$

then for agent 1 the $\tilde{}$-allocation is better (remember $u(\tilde{c}^1) > u(\hat{c}^1)$ is assumed) and not more expensive, which cannot be the case since $\{\hat{c}_t^1\}_{t=0}^{\infty}$ is part of a competitive equilibrium, i.e. maximizes agent 1's utility given equilibrium prices. Hence

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 \tag{2.9}$$

Step 2: Show that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

If not, then

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 < \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

But then there exists a $\delta > 0$ such that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + \delta \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

Remember that we normalized $\hat{p}_0 = 1$. Now define a new allocation for agent 2, by

$$\begin{aligned} \check{c}_t^2 &= \tilde{c}_t^2 \text{ for all } t \geq 1 \\ \check{c}_0^2 &= \tilde{c}_0^2 + \delta \text{ for } t = 0 \end{aligned}$$

Obviously

$$\sum_{t=0}^{\infty} \hat{p}_t \check{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + \delta \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

and

$$u(\check{c}^2) > u(\tilde{c}^2) \geq u(\hat{c}^2)$$

which can't be the case since $\{\hat{c}_t^2\}_{t=0}^{\infty}$ is part of a competitive equilibrium, i.e. maximizes agent 2's utility given equilibrium prices. Hence

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2 \tag{2.10}$$

Step 3: Now sum equations (2.9) and (2.10) to obtain

$$\sum_{t=0}^{\infty} \hat{p}_t(\tilde{c}_t^1 + \tilde{c}_t^2) > \sum_{t=0}^{\infty} \hat{p}_t(\hat{c}_t^1 + \hat{c}_t^2)$$

But since both allocations are feasible (the allocation $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ because it is an equilibrium allocation, the allocation $(\{\tilde{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ by assumption) we have that

$$\tilde{c}_t^1 + \tilde{c}_t^2 = e_t^1 + e_t^2 = \hat{c}_t^1 + \hat{c}_t^2 \text{ for all } t$$

and thus

$$\sum_{t=0}^{\infty} \hat{p}_t(e_t^1 + e_t^2) > \sum_{t=0}^{\infty} \hat{p}_t(e_t^1 + e_t^2),$$

our desired contradiction. ■

2.2.4 Negishi's (1960) Method to Compute Equilibria

In the example economy considered in this section it was straightforward to compute the competitive equilibrium by hand. This is usually not the case for dynamic general equilibrium models. Now we describe a method to compute equilibria for economies in which the welfare theorem(s) hold. The main idea is to compute Pareto-optimal allocations by solving an appropriate social planners problem. This social planner problem is a simple optimization problem which does not involve any prices (still infinite-dimensional, though) and hence much easier to tackle in general than a full-blown equilibrium analysis which consists of several optimization problems (one for each consumer) plus market clearing and involves allocations *and* prices. If the first welfare theorem holds then we know that competitive equilibrium allocations are Pareto optimal; by solving for all Pareto optimal allocations we have then solved for all potential equilibrium allocations. Negishi's method provides an algorithm to compute all Pareto optimal allocations and to isolate those who are in fact competitive equilibrium allocations.

We will repeatedly apply this trick in this course: solve a simple social planners problem and use the welfare theorems to argue that we have solved for the allocations of competitive equilibria. Then find equilibrium prices that support these allocations. The news is even better: usually we can read off the prices as Lagrange multipliers from the appropriate constraints of the social planners problem. In later parts of the course we will discuss economies in which the welfare theorems do not hold. We will see that these economies are much harder to analyze exactly because there is no simple optimization problem that completely characterizes the (set of) equilibria of these economies.

Consider the following social planner problem

$$\begin{aligned} & \max_{\{(c_t^1,c_t^2)\}_{t=0}^\infty} \alpha^1 u(c^1)+\alpha^2 u(c^2) \qquad (2.11) \\ = & \max_{\{(c_t^1,c_t^2)\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \left[\alpha^1 \ln(c_t^1)+\alpha^2 \ln(c_t^2)\right] \\ & \text{s.t.} \\ c_t^i \geq & \ 0 \text{ for all } i, \text{ all } t \\ c_t^1+c_t^2 = & \ e_t^1+e_t^2 \equiv 2 \text{ for all } t \end{aligned}$$

for a Pareto weights $\alpha^i \geq 0$. The social planner maximizes the weighted sum of utilities of the two agents, subject to the allocation being feasible. The weights α^i indicate how important agent i's utility is to the planner. Note that the solution to this problem will depend on the Pareto weights, i.e. the optimal consumption choices are functions of $\alpha=(\alpha^1,\alpha^2)$

$$\{(c_t^1,c_t^2)\}_{t=0}^\infty = \{(c_t^1(\alpha),c_t^2(\alpha))\}_{t=0}^\infty$$

We have the following

Proposition 6 *Any allocation $\{(c_t^1,c_t^2)\}_{t=0}^\infty$ that solves the solves the social planners problem* (2.11) *for some vector of Pareto weights $\alpha > 0$ is Pareto efficient.*

Proposition 7 *Conversely, any Pareto efficient allocation $\{(c_t^1,c_t^2)\}_{t=0}^\infty$ is the solution to the social planners problem* (2.11) *for some vector of Pareto weights $\alpha \geq 0, \alpha \neq 0$.*

Proof. Omitted (but a good exercise, or consult MasColell et al., proposition 16.E.2). Note that the second proposition requires the lifetime utility possibility set to be convex, which follows from the period utility function of both households being strictly concave. ■

This proposition states that we can characterize the set of all Pareto efficient allocations by varying the α's. As will become apparent below, all that matters are the relative weights α^1/α^2, and we will demonstrate that, by choosing a particular α^1/α^2, the associated efficient allocation for that vector of welfare weights α turns out to be the competitive equilibrium allocation.

Now let us solve the planners problem for arbitrary $\alpha \geq 0$.[5] Attach Lagrange multipliers $\frac{\mu_t}{2}$ to the resource constraints (and ignore the non-negativity constraints on c_t^i since they never bind, due to the period utility function satisfying the Inada conditions). The reason why we divide the Lagrange multipliers by 2 will become apparent in a moment.

The first order necessary conditions are

$$\begin{aligned}\frac{\alpha^1\beta^t}{c_t^1} &= \frac{\mu_t}{2}\\ \frac{\alpha^2\beta^t}{c_t^2} &= \frac{\mu_t}{2}\end{aligned}$$

Combining yields

$$\frac{c_t^1}{c_t^2} = \frac{\alpha^1}{\alpha^2} \tag{2.12}$$

$$c_t^1 = \frac{\alpha^1}{\alpha^2}c_t^2 \tag{2.13}$$

i.e. the ratio of consumption between the two agents equals the ratio of the Pareto weights in every period t. A higher Pareto weight for agent 1 results in this agent receiving more consumption in every period, relative to agent 2.

Using the resource constraint in conjunction with (2.13) yields

$$\begin{aligned}c_t^1 + c_t^2 &= 2\\ \frac{\alpha^1}{\alpha^2}c_t^2 + c_t^2 &= 2\\ c_t^2 &= 2/(1+\alpha^1/\alpha^2) = c_t^2(\alpha)\\ c_t^1 &= 2/(1+\alpha^2/\alpha^1) = c_t^1(\alpha)\end{aligned}$$

i.e. the social planner divides the total resources in every period according to the Pareto weights. Note that the division is the same in every period, independent of the agents' endowments in that particular period. The Lagrange

[5]Note that for $\alpha^i = 0$ the solution to the problem is trivial. For $\alpha^1 = 0$ we have $c_t^1 = 0$ and $c_t^2 = 2$ and for $\alpha^2 = 0$ we have the reverse.

multipliers are given by

$$\mu_t = (\alpha^1 + \alpha^2)\beta^t \tag{2.14}$$

Note that if we wouldn't have done the initial division by 2 we would have to carry the $\frac{1}{2}$ around from now on; the results below wouldn't change at all, though). Also, as it is clear from the above discussion what matter for allocations are the relative Pareto weights, without loss of generality we can normalize their sum to one, and henceforth $\alpha^1 + \alpha^2 = 1$. The Lagrange multipliers are then given by

$$\mu_t = \beta^t \tag{2.15}$$

Hence, and to summarize, for this economy the set of Pareto efficient allocations is given by

$$PO = \{\{(c_t^1, c_t^2)\}_{t=0}^{\infty} : c_t^1 = 2/(1+\alpha^2/\alpha^1) \text{ and } c_t^2 = 2/(1+\alpha^1/\alpha^2) \text{ for some } \alpha^1/\alpha^2 \in [0, \infty)\}$$

How does this help us in finding the competitive equilibrium for this economy? Starting from the set PO we would like to construct an equilibrium (and in fact, Negishi used the argument that follows to construct an existence proof of equilibrium). Now consider the list of requirements that equilibrium allocations have to satisfy. First, equilibrium allocations have to satisfy market clearing. But every feasible allocation does so. The other requirement is that the equilibrium allocations have to be optimal, subject to not violating the household budget constraint at equilibrium prices. The first order necessary conditions of the household maximization problem read as

$$\frac{\beta^t}{c_t^i} = \lambda_i p_t$$

Compare this to the first order necessary conditions from the social planners problem:

$$\frac{\alpha^i \beta^t}{c_t^i} = \frac{\mu_t}{2}.$$

This suggests that if we were to construct an equilibrium with allocations from the set PO, it will have prices given by $p_t = \mu_t = \beta^t$. This is also intuitive: the Lagrange multiplier μ_t measures how binding the resource constraint in period t is in the social optimum. It is thus a measure of scarcity. But intuitively, this is exactly the role of a competitive equilibrium price p_t, namely to signal to households how scarce consumption in any given period t is. Note that the close connection between Lagrange multipliers on resource constraints in a social planner problem and prices in competitive equilibrium is not specific to this model, but rather emerges in general whenever the first welfare theorem applies and thus equilibrium allocations are Pareto efficient.

Now, if Pareto weights α^i and Lagrange multipliers of the budget constraints are related by $\lambda_i = \frac{1}{2\alpha^i}$ the first order conditions from the competitive equilibrium and from the social planner problem coincide. Recall that α^i determines how much household i gets to consume in an efficient allocation. The Lagrange multiplier λ_i measures how tight the budget constraint is for household i in a competitive equilibrium. Thus it is perhaps not surprising that the determination of the "right" α comes from assuring that an efficient allocation $(c_t^1(\alpha), c_t^2(\alpha))$, at the appropriate prices $\{p_t\} = \{\mu_t\}$, satisfies the budget constraints of each household (which is the one remaining part of the equilibrium requirements we have not checked yet).

To make this formal, define the transfer functions $t^i(\alpha)$, $i = 1, 2$ by

$$t^i(\alpha) = \sum_t \mu_t \left[c_t^i(\alpha) - e_t^i\right]$$

The number $t^i(\alpha)$ is the amount of the numeraire good (we pick the period 0 consumption good) that agent i would need as transfer in order to be able to afford the Pareto efficient allocation indexed by α in a competitive equilibrium with equilibrium prices $\{p_t\} = \{\mu_t\}$. Thus the "right" $\alpha = (\alpha^1, \alpha^2)$ is the one that satisfies, for all $i = 1, 2$

$$t^i(\alpha) = 0. \tag{2.16}$$

Using the Lagrange multipliers in (2.15) prior to normalization we note that

$$t^i(\alpha) = \sum_t \mu_t \left[c_t^i(\alpha) - e_t^i\right] = \sum_{t=0}^{\infty} (\alpha^1 + \alpha^2)\beta^t \left[c_t^i(\alpha) - e_t^i\right]$$

and thus $t^i(\theta\alpha) = \theta t^i(\alpha)$ for any $\theta > 0$, since $c_t^i(\alpha)$ only depends on α^1/α^2. That is, the t^i as functions of the Pareto weights are homogeneous of degree one. Thus equations (2.16) only pin down the desired α up to a scalar (whenever $t^i(\alpha) = 0$ for all i, so will be $t^i(\theta\alpha) = \theta t^i(\alpha)$) and it is indeed innocuous so normalize α such that $(\alpha^1 + \alpha^2) = 1$. But then one may wonder how both equations (2.16) can be satisfied simultaneously if we only have one degree of freedom? Fortunately the transfer functions sum to zero for all

α, since

$$\sum_{i=1}^{2} t^i(\alpha) = \sum_{i=1}^{2} \sum_t \mu_t \left[c_t^i(\alpha) - e_t^i\right] = \sum_t \mu_t \sum_{i=1}^{2} \left[c_t^i(\alpha) - e_t^i\right] = 0$$

where the last equality uses the resource constraint. Thus, effectively we only have one equation (say $t^1(\alpha) = 0$) and one unknown α^1/α^2 to solve.[6]

Doing so yields

$$\begin{aligned} t^1(\alpha) &= \sum_{t=0}^{\infty} (\alpha^1 + \alpha^2)\beta^t \left[c_t^1(\alpha) - e_t^1\right] \\ &= \sum_{t=0}^{\infty} \beta^t \left[2/(1 + \alpha^2/\alpha^1) - e_t^1\right] \\ &= \frac{2}{(1-\beta)(1+\alpha^2/\alpha^1)} - \frac{2}{1-\beta^2} = 0 \end{aligned}$$

and thus

$$\begin{aligned} \frac{1}{1+\alpha^2/\alpha^1} &= \frac{1}{1+\beta} \\ \alpha^2/\alpha^1 &= \beta \end{aligned}$$

and the corresponding consumption allocations are

$$\begin{aligned} c_t^1 &= \frac{2}{1+\alpha^2/\alpha^1} = \frac{2}{1+\beta} \\ c_t^2 &= \frac{2}{1+\alpha^1/\alpha^2} = \frac{2\beta}{1+\beta} \end{aligned}$$

Hence we have solved for the equilibrium allocations; equilibrium prices are given by the Lagrange multipliers $\mu_t = \beta^t$ (note that without the normalization by $\frac{1}{2}$ at the beginning we would have found the same allocations and equilibrium prices $p_t = \frac{\beta^t}{2}$ which, given that equilibrium prices are homogeneous of degree 0, is perfectly fine, too). To summarize, to compute competitive equilibria using Negishi's method one does the following

[6]More generally, with N different households there are N Pareto weights. With the normalization $\sum_i \alpha^i = 1$ and the fact that $\sum_i t^i(\alpha) = 0$ this yields a system of $N-1$ equations in $N-1$ unknowns (the relative Pareto weights α^i/α^1).

1. Solve the social planners problem for Pareto efficient allocations indexed by the Pareto weights $\alpha = (\alpha^1, \alpha^2)$.

2. Compute transfers, indexed by α, necessary to make the efficient allocation affordable. As prices use Lagrange multipliers on the resource constraints in the planners' problem.

3. Find the normalized Pareto weight(s) $\hat{\alpha}$ that makes the transfer functions 0.

4. The Pareto efficient allocations corresponding to $\hat{\alpha}$ are equilibrium allocations; the supporting equilibrium prices are (multiples of) the Lagrange multipliers from the planning problem

Remember from above that to solve for the equilibrium directly in general involves solving an infinite number of equations in an infinite number of unknowns. The Negishi method reduces the computation of equilibrium to a finite number of equations in a finite number of unknowns in step 3 above. For an economy with two agents, it is just one equation in one unknown, for an economy with N agents it is a system of $N-1$ equations in $N-1$ unknowns. This is why the Negishi method (and methods relying on solving appropriate social planners problems in general) often significantly simplifies solving for competitive equilibria.

2.2.5 Sequential Markets Equilibrium

The market structure of Arrow-Debreu equilibrium in which all agents meet only once, at the beginning of time, to trade claims to future consumption may seem empirically implausible. In this section we show that the same allocations as in an Arrow-Debreu equilibrium would arise if we let agents trade consumption and one-period bonds in each period. We will call a market structure in which markets for consumption and assets open in each period Sequential Markets and the corresponding equilibrium Sequential Markets (SM) equilibrium.[7]

Let r_{t+1} denote the interest rate on one period bonds from period t to period $t+1$. A one period bond is a promise (contract) to pay 1 unit of

[7]In the simple model we consider in this section the restriction of assets traded to one-period riskless bonds is without loss of generality. In more complicated economies (e.g. with risk) it would not be. We will come back to this issue in later chapters.

the consumption good in period $t+1$ in exchange for $\frac{1}{1+r_{t+1}}$ units of the consumption good in period t. We can interpret $q_t \equiv \frac{1}{1+r_{t+1}}$ as the relative price of one unit of the consumption good in period $t+1$ in terms of the period t consumption good. Let a_{t+1}^i denote the amount of such bonds purchased by agent i in period t and carried over to period $t+1$. If $a_{t+1}^i < 0$ we can interpret this as the agent taking out a one-period loan at interest rate (between t and $t+1$) given by r_{t+1}. Household i's budget constraint in period t reads as

$$c_t^i + \frac{a_{t+1}^i}{(1+r_{t+1})} \leq e_t^i + a_t^i \tag{2.17}$$

or

$$c_t^i + q_t a_{t+1}^i \leq e_t^i + a_t^i$$

Agents start out their life with initial bond holdings a_0^i (remember that period 0 bonds are claims to period 0 consumption). Mostly we will focus on the situation in which $a_0^i = 0$ for all i, but sometimes we want to start an agent off with initial wealth ($a_0^i > 0$) or initial debt ($a_0^i < 0$). Since there is no government and only two agents in this economy the initial condition is required to satisfy $\sum_{i=1}^2 a_0^i = 0$.

We then have the following definition

Definition 8 *A Sequential Markets equilibrium is allocations* $\left\{\left(\hat{c}_t^i, \hat{a}_{t+1}^i\right)_{i=1,2}\right\}_{t=0}^\infty$, *interest rates* $\{\hat{r}_{t+1}\}_{t=0}^\infty$ *such that*

1. *For* $i = 1, 2$, *given interest rates* $\{\hat{r}_{t+1}\}_{t=0}^\infty$ $\{\hat{c}_t^i, \hat{a}_{t+1}^i\}_{t=0}^\infty$ *solves*

$$\max_{\{c_t^i, a_{t+1}^i\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \ln(c_t^i) \tag{2.18}$$

$$s.t.$$

$$c_t^i + \frac{a_{t+1}^i}{(1+\hat{r}_{t+1})} \leq e_t^i + a_t^i \tag{2.19}$$

$$c_t^i \geq 0 \text{ for all } t \tag{2.20}$$

$$a_{t+1}^i \geq -\bar{A}^i \tag{2.21}$$

2. For all $t \geq 0$

$$\begin{aligned}
\sum_{i=1}^{2} \hat{c}_t^i &= \sum_{i=1}^{2} e_t^i \\
\sum_{i=1}^{2} \hat{a}_{t+1}^i &= 0
\end{aligned}$$

The constraint (2.21) on borrowing is necessary to guarantee existence of equilibrium. Suppose that agents would not face any constraint as to how much they can borrow, i.e. suppose the constraint (2.21) were absent. Suppose there would exist a SM-equilibrium $\{(\hat{c}_t^i, \hat{a}_{t+1}^i)_{i=1,2}\}_{t=1}^{\infty}, \{\hat{r}_{t+1}\}_{t=0}^{\infty}$. Without constraint on borrowing agent i could always do better by setting

$$\begin{aligned}
c_0^i &= \hat{c}_0^i + \frac{\varepsilon}{1+\hat{r}_1} \\
c_t^i &= \hat{c}_t^i \text{ for all } t > 0 \\
a_1^i &= \hat{a}_1^i - \varepsilon \\
a_2^i &= \hat{a}_2^i - (1+\hat{r}_2)\varepsilon \\
a_{t+1}^i &= \hat{a}_{t+1}^i - \prod_{\tau=1}^{t}(1+\hat{r}_{\tau+1})\varepsilon
\end{aligned}$$

i.e. by borrowing $\varepsilon > 0$ more in period 0, consuming it and then rolling over the additional debt forever, by borrowing more and more. Such a scheme is often called a Ponzi scheme. Hence without a limit on borrowing no SM equilibrium can exist because agents would run Ponzi schemes and augment their consumption without bound. Note that the $\varepsilon > 0$ in the above argument was arbitrarily large.

In this section we are interested in specifying a borrowing limit that prevents Ponzi schemes, yet is high enough so that households are never constrained in the amount they can borrow (by this we mean that a household, knowing that it can not run a Ponzi scheme, would always find it optimal to choose $a_{t+1}^i > -\bar{A}^i$). In later chapters we will analyze economies in which agents face borrowing constraints that are binding in certain situations. Not only are SM equilibria for these economies quite different from the ones to be studied here, but also the equivalence between SM equilibria and AD equilibria will break down when the borrowing constraints are occasionally binding.

We are now ready to state the equivalence theorem relating AD equilibria and SM equilibria. Assume that $a_0^i = 0$ for all $i = 1, 2$. Furthermore assume that the endowment stream $\{e_t^i\}_{t=0}^{\infty}$ is bounded.

Proposition 9 *Let allocations* $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ *and prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *form an Arrow-Debreu equilibrium with*

$$\frac{\hat{p}_{t+1}}{\hat{p}_t} \leq \xi < 1 \text{ for all } t. \tag{2.22}$$

Then there exist $\left(\bar{A}^i\right)_{i=1,2}$ *and a corresponding sequential markets equilibrium with allocations* $\{\left(\tilde{c}_t^i, \tilde{a}_{t+1}^i\right)_{i=1,2}\}_{t=0}^{\infty}$ *and interest rates* $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ *such that*

$$\tilde{c}_t^i = \hat{c}_t^i \text{ for all } i, \text{ all } t$$

Reversely, let allocations $\{\left(\hat{c}_t^i, \hat{a}_{t+1}^i\right)_{i=1,2}\}_{t=0}^{\infty}$ *and interest rates* $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ *form a sequential markets equilibrium. Suppose that it satisfies*

$$\begin{aligned} \hat{a}_{t+1}^i &> -\bar{A}^i \text{ for all } i, \text{ all } t \\ \hat{r}_{t+1} &\geq \varepsilon > 0 \text{ for all } t \end{aligned} \tag{2.23}$$

for some ε. *Then there exists a corresponding Arrow-Debreu equilibrium* $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$, $\{\tilde{p}_t\}_{t=0}^{\infty}$ *such that*

$$\hat{c}_t^i = \tilde{c}_t^i \text{ for all } i, \text{ all } t.$$

That is, the set of equilibrium allocations under the AD and SM market structures coincide.[8]

Proof. We first show that any consumption allocation that satisfies the sequence of SM budget constraints is also in the AD budget set (step 1). From this in fairly directly follows that AD equilibria can be made into SM equilibria. The only complication is that we need to make sure that we can find a large enough borrowing limit $\bar{A}^i$ such that the asset holdings required to implement the AD consumption allocation as a SM equilibrium do not

[8] The assumption on $\frac{\hat{p}_{t+1}}{\hat{p}_t}$ and on $\hat{r}_{t+1}$ can be completely relaxed if one introduces borrowing constraints of slightly different form in the SM equilibrium to prevent Ponzi schemes. See Wright (*Journal of Economic Theory*, 1987). They are required here since I insisted on making the $\bar{A}^i$ a fixed number.

violate the no Ponzi constraint. This is shown in step 2. Finally, in step 3 we argue that an SM equilibrium can be made into an AD equilibrium.

Step 1: The key to the proof is to show the equivalence of the budget sets for the Arrow-Debreu and the sequential markets structure. This step will then be used in the arguments below. Normalize $\hat{p}_0 = 1$ (as we can always do) and relate equilibrium prices and interest rates by

$$1 + \hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}} \tag{2.24}$$

Now look at the sequence of sequential markets budget constraints and assume that they hold with equality (which they do in equilibrium since lifetime utility is strictly increasing in each of the consumption goods)

$$\begin{aligned} c_0^i + \frac{a_1^i}{1+\hat{r}_1} &= e_0^i && (2.25) \\ c_1^i + \frac{a_2^i}{1+\hat{r}_2} &= e_1^i + a_1^i && (2.26) \\ &\vdots \\ c_t^i + \frac{a_{t+1}^i}{1+\hat{r}_{t+1}} &= e_t^i + a_t^i && (2.27) \end{aligned}$$

Substituting for a_1^i from (2.26) in (2.25) one gets

$$c_0^i + \frac{c_1^i}{1+\hat{r}_1} + \frac{a_2^i}{(1+\hat{r}_1)(1+\hat{r}_2)} = e_0^i + \frac{e_1^i}{(1+\hat{r}_1)}$$

and, repeating this exercise, yields[9]

$$\sum_{t=0}^{T} \frac{c_t^i}{\prod_{j=1}^{t}(1+\hat{r}_j)} + \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = \sum_{t=0}^{T} \frac{e_t^i}{\prod_{j=1}^{t}(1+\hat{r}_j)}$$

Now note that (using the normalization $\hat{p}_0 = 1$)

$$\prod_{j=1}^{t}(1+\hat{r}_j) = \frac{\hat{p}_0}{\hat{p}_1} * \frac{\hat{p}_1}{\hat{p}_2} \cdots * \frac{\hat{p}_{t-1}}{\hat{p}_t} = \frac{1}{\hat{p}_t} \tag{2.28}$$

[9] We define

$$\prod_{j=1}^{0}(1+\hat{r}_j) = 1$$

Taking limits with respect to t on both sides gives, using (2.28)

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i + \lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$

Given our assumptions on the equilibrium interest rates in (2.23) we have

$$\lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} \geq \lim_{T\to\infty} \frac{-\bar{A}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = 0 \tag{2.29}$$

and since $\lim_{T\to\infty} \prod_{j=1}^{T+1}(1+\hat{r}_j) = \infty$ (due to the assumption that $\hat{r}_{t+1} \geq \varepsilon > 0$ for all t), we have

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i.$$

Thus any allocation that satisfies the SM budget constraints and the no Ponzi conditions satisfies the AD budget constraint when AD prices and SM interest rates are related by (2.24).

Step 2: Now suppose we have an AD-equilibrium $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$, $\{\hat{p}_t\}_{t=0}^{\infty}$. We want to show that there exist a SM equilibrium with same consumption allocation, i.e.

$$\tilde{c}_t^i = \hat{c}_t^i \text{ for all } i, \text{ all } t$$

Obviously $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ satisfies market clearing. Define asset holdings as

$$\tilde{a}_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - e_{t+\tau}^i\right)}{\hat{p}_{t+1}}. \tag{2.30}$$

Note that the consumption and asset allocation so constructed satisfies the SM budget constraints since, recalling $1+\tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$ we have, plugging in from (2.30):

$$\begin{aligned}
\hat{c}_t^i + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - e_{t+\tau}^i\right)}{\hat{p}_{t+1}(1+\tilde{r}_{t+1})} &= e_t^i + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t-1+\tau}\left(\hat{c}_{t-1+\tau}^i - e_{t-1+\tau}^i\right)}{\hat{p}_t} \\
\hat{c}_t^i + \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - e_{t+\tau}^i\right)}{\hat{p}_t} &= e_t^i + \sum_{\tau=0}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - e_{t+\tau}^i\right)}{\hat{p}_t} \\
\hat{c}_t^i &= e_t^i + \frac{\hat{p}_t\left(\hat{c}_t^i - e_t^i\right)}{\hat{p}_t} = \hat{c}_t^i.
\end{aligned}$$

Next we show that we can find a borrowing limit $\bar{A}^i$ large enough so that the no Ponzi condition is never violated with asset levels given by (2.30). Note that (since by assumption $\frac{\hat{p}_{t+\tau}}{\hat{p}_{t+1}} \leq \xi^{\tau-1}$) we have

$$\tilde{a}^i_{t+1} \geq -\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau} e^i_{t+\tau}}{\hat{p}_{t+1}} \geq -\sum_{\tau=1}^{\infty} \xi^{\tau-1} e^i_{t+\tau} > -\infty \tag{2.31}$$

so that we can take

$$\bar{A}^i = 1 + \sup_t \sum_{\tau=1}^{\infty} \xi^{\tau-1} e^i_{t+\tau} < \infty \tag{2.32}$$

where the last inequality follows from the fact that $\xi < 1$ and the assumption that the *endowment stream is bounded*.[10] This borrowing limit $\bar{A}^i$ is so high that agent i, knowing that she can't run a Ponzi scheme, will never hit it.

[10]One way to deal with potentially growing endowment streams is to state the No Ponzi condition as

$$\lim_{T\to\infty} \frac{a^i_{T+1}}{\prod_{j=1}^{T+1}(1+\tilde{r}_j)} \geq 0.$$

This common way of stating the constraint involves equilibrium interest rates however and thus can only be checked once the equilibrium has actually been found and thus I did not want to introduce it as the benchmark No Ponzi condition. But it is helpful in environments with growing endowments.

Note that equation (2.29) in the first step of the proof still holds and thus step 1 of the proof goes through even with this No-Ponzi condition. The implied asset holdings in equation (2.30) satisfy this No Ponzi condition even when endowments are growing at a positive rate $1+g$ since

$$\begin{aligned}
& \lim_{T\to\infty} \frac{\tilde{a}^i_{t+1}}{\prod_{j=1}^{t+1}(1+\tilde{r}_j)} = \lim_{T\to\infty} \frac{\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}^i_{t+\tau} - e^i_{t+\tau}\right)}{\hat{p}_{t+1}}}{\prod_{j=1}^{t+1}(1+\tilde{r}_j)} \\
= \quad & \lim_{T\to\infty} \sum_{\tau=1}^{\infty} \hat{p}_{t+\tau}\left(\hat{c}^i_{t+\tau} - e^i_{t+\tau}\right) \geq -\lim_{t\to\infty} \sum_{\tau=1}^{\infty} \hat{p}_{t+\tau} e^i_{t+\tau} \\
= \quad & -\lim_{t\to\infty} \hat{p}_{t+1} e^i_{t+1} \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}}{\hat{p}_{t+1}} \cdot \frac{e^i_{t+\tau}}{e^i_{t+1}} \geq -\lim_{t\to\infty} \hat{p}_{t+1} e^i_{t+1} \sum_{\tau=1}^{\infty} \left[\xi(1+g)\right]^{\tau-1} \\
= \quad & \frac{-\lim_{t\to\infty} \hat{p}_{t+1} e^i_{t+1}}{1-\xi(1+g)} = 0
\end{aligned}$$

as long as $\xi(1+g) < 1$, that is, as long as prices fall faster than endowments grow (so that $\lim_{t\to\infty} \hat{p}_{t+1} e^i_{t+1} = 0$), or equivalently, as long as interest rates are larger, than growth rates $\tilde{r}_{t+1} > g$.

It remains to argue that $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ maximizes lifetime utility, subject to the sequential markets budget constraints and the borrowing constraints defined by $\bar{A}^i$. Take any other allocation satisfying the SM budget constraints, at interest rates given by (2.24). In step 1. we showed that then this allocation would also satisfy the AD budget constraint and thus could have been chosen at AD equilibrium prices. If this alternative allocation would yield higher lifetime utility than the allocation $\{\tilde{c}_t^i = \hat{c}_t^i\}_{t=0}^{\infty}$ it would have been chosen as part of an AD-equilibrium, which it wasn't. Hence $\{\tilde{c}_t^i\}_{t=0}^{\infty}$ must be optimal within the set of allocations satisfying the SM budget constraints at interest rates $1+\tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$.

Step 3: Now suppose $\{\left(\hat{c}_t^i, \hat{a}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$ and $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ form a sequential markets equilibrium satisfying

$$
\begin{aligned}
\hat{a}_{t+1}^i &> -\bar{A}^i \text{ for all } i, \text{ all } t \\
\hat{r}_{t+1} &> 0 \text{ for all } t
\end{aligned}
$$

We want to show that there exists a corresponding Arrow-Debreu equilibrium $\{(\tilde{c}_t^i)_{i\in I}\}_{t=0}^{\infty}$, $\{\tilde{p}_t\}_{t=0}^{\infty}$ with

$$\hat{c}_t^i = \tilde{c}_t^i \text{ for all } i, \text{ all } t$$

Again obviously $\{(\tilde{c}_t^i)_{i\in I}\}_{t=0}^{\infty}$ satisfies market clearing and, as shown in step 1, the AD budget constraint. It remains to be shown that it maximizes utility within the set of allocations satisfying the AD budget constraint, for prices $\tilde{p}_0 = 1$ and $\tilde{p}_{t+1} = \frac{\tilde{p}_t}{1+\hat{r}_{t+1}}$. For any other allocation satisfying the AD budget constraint we could construct asset holdings (from equation 2.30) such that this allocation together with the asset holdings satisfies the SM-budget constraints. The only complication is that in the SM household maximization problem there is an additional constraint, the no-Ponzi constraints. Thus the set over which we maximize in the AD case is larger, since the borrowing constraints are absent in the AD formulation, and we need to rule out that allocations that would violate the SM no Ponzi conditions are optimal choices in the AD household problem, at the equilibrium prices. However, by assumption the no Ponzi conditions are not binding at the SM equilibrium allocation, that is $\hat{a}_{t+1}^i > -\bar{A}^i$ for all t. But for maximization problems with concave objective and convex constraint set (such as the SM household maximization problem) if in the presence of the additional constraints $\hat{a}_{t+1}^i \geq -\bar{A}^i$ for a maximizing choice these constraints are not binding, then this maximizer is also a maximizer of the relaxed problem with the constraint removed. Hence $\{\tilde{c}_t^i\}_{t=0}^{\infty}$ is optimal for household i within the set of allocations satisfying only the AD budget constraint. ■

This proposition shows that the sequential markets and the Arrow-Debreu market structures lead to identical equilibria, provided that we choose the no Ponzi conditions appropriately (e.g. equal to the ones in (2.32)) and that the equilibrium interest rates are sufficiently high. Usually the analysis of our economies is easier to carry out using AD language, but the SM formulation has more empirical appeal. The preceding theorem shows that we can have the best of both worlds.

For our example economy we find that the equilibrium interest rates in the SM formulation are given by

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}} = \frac{1}{\beta}$$

or

$$r_{t+1} = r = \frac{1}{\beta} - 1 = \rho$$

i.e. the interest rate is constant and equal to the subjective time discount rate $\rho = \frac{1}{\beta} - 1$.

The proof of the proposition also gives us insights into how households use financial markets in order to smooth consumpion. Consider the sequential markets budget constraint of household 1, evaluated at $c_t^i = \frac{2}{1+\beta}$ and $\frac{1}{1+r_{t+1}} = \beta$:

$$\frac{2}{1+\beta} + \beta a_{t+1}^1 = e_t^1 + a_t^1.$$

Now take $t = 0$ and use the initial condition $a_0^i = 0$ to obtain

$$\begin{aligned} \frac{2}{1+\beta} + \beta a_1^1 &= 2 \\ a_1^1 &= \frac{2}{1+\beta} \end{aligned}$$

and thus from the market clearing condition

$$a_1^2 = -\frac{2}{1+\beta}.$$

Repeating the same argument for $t = 1$ yields

$$\begin{aligned} \frac{2}{1+\beta} + \beta a_2^1 &= 0 + \frac{2}{1+\beta} \\ a_2^1 &= a_2^2 = 0, \end{aligned}$$

and so forth. That is, in every even period the income rich household $i = 1$ uses part of her endowment to buy bonds (she is a saver), and in every odd period she uses the proceeds of these bonds to finance her consumption in these periods of low (zero!) endowment. Household $i = 2$ does exactly the reverse, he borrows in even periods to consume despite having zero income, and in odd periods repays the loan fully with part of his income and uses the remaining part for consumption. This way, by making good use of the bond market, both households achieve smooth consumption despite their very non-smooth endowment profiles.

2.3 Appendix: Some Facts about Utility Functions

The utility function

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \tag{2.33}$$

described in the main text satisfies the following assumptions that we will often require in our models.

2.3.1 Time Separability

The utility function in (2.33) has the property that total utility from a consumption allocation c^i equals the discounted sum of period (or instantaneous) utility $U(c_t^i) = \ln(c_t^i)$. Period utility $U(c_t^i)$ at time t only depends on consumption in period t and not on consumption in other periods. This formulation rules out, among other things, habit persistence, where the period utility from consumption c_t^i would also depend on past consumption levels $c_{t-\tau}^i$, for $\tau > 0$.

2.3.2 Time Discounting

The fact that $\beta < 1$ indicates that agents are impatient. The same amount of consumption yields less utility if it comes at a later time in an agents' life. The parameter β is often referred to as (subjective) time discount factor. The subjective time discount rate ρ is defined by $\beta = \frac{1}{1+\rho}$ and is often, as we have seen above, intimately related to the equilibrium interest rate in the economy (because the interest rate is nothing else but the market time discount rate).

2.3.3 Standard Properties of the Period Utility Function

The instantaneous utility function or felicity function $U(c) = \ln(c)$ is continuous, twice continuously differentiable, strictly increasing (i.e. $U'(c) > 0$) and strictly concave (i.e. $U''(c) < 0$) and satisfies the Inada conditions

$$\begin{aligned}
\lim_{c \searrow 0} U'(c) &= +\infty \\
\lim_{c \nearrow +\infty} U'(c) &= 0
\end{aligned}$$

These assumptions imply that more consumption is always better, but an additional unit of consumption yields less and less additional utility. The Inada conditions indicate that the first unit of consumption yields a lot of additional utility but that as consumption goes to infinity, an additional unit is (almost) worthless. The Inada conditions will guarantee that an agent always chooses $c_t \in (0, \infty)$ for all t, and thus that corner solutions for consumption, $c_t = 0$, can be ignored in the analysis of our models.

2.3.4 Constant Relative Risk Aversion (CRRA) Utility

The felicity function $U(c) = \ln(c)$ is a member of the class of Constant Relative Risk Aversion (CRRA) utility functions. These functions have the general form

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \tag{2.34}$$

where $\sigma \geq 0$ is a parameter. For $\sigma \to 1$, this utility function converges to $U(c) = \ln(c)$, which can be easily shown taking the limit in (2.34) and applying l'Hopital's rule. CRRA utility functions have a number of important properties. First, they satisfy the properties in the previous subsection.

Constant Coefficient of Relative Risk Aversion

Define as $\sigma(c) = -\frac{U''(c)c}{U'(c)}$ the (Arrow-Pratt) coefficient of relative risk aversion. Hence $\sigma(c)$ indicates a household's attitude towards risk, with higher $\sigma(c)$ representing higher risk aversion, in a quantitatively meaningful way. The (relative) risk premium measures the household's willingness to pay (and thus reduce safe consumption $\bar{c}$) to avoid a proportional consumption gamble in which a household can win, but also lose, a fraction of $\bar{c}$. See figure 2.3.4 for a depiction of the risk premium.

Arrow-Pratt's theorem states that this risk premium is proportional (up to a first order approximation) to the coefficient of relative risk aversion $\sigma(\bar{c})$. This coefficient is thus a quantitative measure of the willingness to pay to avoid consumption gambles. Typically this willingness depends on the level of consumption $\bar{c}$, but for a CRRA utility function it does not, in that $\sigma(c)$ is constant for all c, and equal to the parameter σ. For $U(c) = \ln(c)$ it is not only constant, but equal to $\sigma(c) = \sigma = 1$. This explains the name of this class of period utility functions.

Intertemporal Elasticity of Substitution

Define the intertemporal elasticity of substitution (IES) as $ies_t(c_{t+1}, c_t)$ as

$$ies_t(c_{t+1}, c_t) = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\left[\frac{\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{d\left(\frac{c_{t+1}}{c_t}\right)}}{\frac{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}{\frac{c_{t+1}}{c_t}}}\right]^{-1}$$

that is, as the inverse of the percentage change in the marginal rate of substitution between consumption at t and $t+1$ in response to a percentage change in the consumption ratio $\frac{c_{t+1}}{c_t}$. For the CRRA utility function note that

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}$$

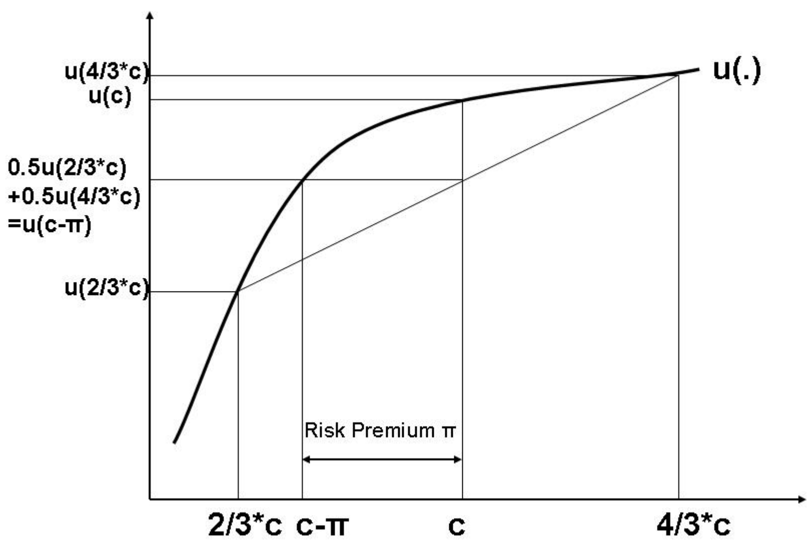

Figure 2.1: Illustration of Risk Aversion

and thus

$$ies_t(c_{t+1}, c_t) = -\left[\frac{-\sigma\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma-1}}{\frac{\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}}{\frac{c_{t+1}}{c_t}}}\right]^{-1} = \frac{1}{\sigma}$$

and the intertemporal elasticity of substitution is constant, independent of the level or growth rate of consumption, and equal to $1/\sigma$. A simple plot of the indifference map should convince you that the IES measures the curvature of the utility function. If $\sigma = 0$ consumption in two adjacent periods are perfect substitutes and the IES equals $ies = \infty$. If $\sigma \to \infty$ the utility function converges to a Leontieff utility function, consumption in adjacent periods are prefect complements and $ies = 0$.

The IES also has a nice behavioral interpretation. From the first order conditions of the household problem we obtain

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{p_{t+1}}{p_t} = \frac{1}{1 + r_{t+1}} \tag{2.35}$$

and thus the IES can alternatively be written as (in fact, some economists define the IES that way)

$$ies_t(c_{t+1}, c_t) = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{1}{1+r_{t+1}}\right)}{\frac{1}{1+r_{t+1}}}\right]}$$

that is, the IES measures the percentage change in the consumption growth rate in response to a percentage change in the gross real interest rate, the intertemporal price of consumption.

Note that for the CRRA utility function the Euler equation reads as

$$(1 + r_{t+1})\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} = 1.$$

Taking logs on both sides and rearranging one obtains

$$\ln(1 + r_{t+1}) + \log(\beta) = \sigma\left[\ln(c_{t+1}) - \ln(c_t)\right]$$

or

$$\ln(c_{t+1}) - \ln(c_t) = \frac{1}{\sigma}\ln(\beta) + \frac{1}{\sigma}\ln(1 + r_{t+1}). \tag{2.36}$$

This equation forms the basis of all estimates of the IES; with time series data on consumption growth and real interest rates the IES $\frac{1}{\sigma}$ can be estimated from a regression of the former on the later.[11]

Note that with CRRA utility the attitude of a household towards risk (atemporal consumption gambles) measured by risk aversion σ and the attitude towards consumption smoothing over time measured by the intertemporal elasticity of substitution $1/\sigma$ are determined by the same parameter, and varying risk aversion necessarily implies varying the IES as well. In many applications, and especially in consumption-based asset pricing theory this turns out to be an undesirable restriction. A generalization of CRRA utility by Epstein and Zin (1989, 1991(often also called recursive utility) introduces a (time non-separable) utility function in which two parameters govern risk aversion and intertemporal elasticity of substitution separately.

2.3.5 Homotheticity and Balanced Growth

Finally, define the marginal rate of substitution between consumption at any two dates t and $t+s$ as

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}}$$

The lifetime utility function u is said to be homothetic if $MRS(c_{t+s}, c_t) = MRS(\lambda c_{t+s}, \lambda c_t)$ for all $\lambda > 0$ and c.

It is easy to verify that for a period utility function U of CRRA variety the lifetime utility function u is homothetic, since

$$MRS(c_{t+s}, c_t) = \frac{\beta^{t+s}\left(c_{t+s}\right)^{-\sigma}}{\beta^t\left(c_t\right)^{-\sigma}} = \frac{\beta^{t+s}\left(\lambda c_{t+s}\right)^{-\sigma}}{\beta^t\left(\lambda c_t\right)^{-\sigma}} = MRS(\lambda c_{t+s}, \lambda c_t) \tag{2.37}$$

With homothetic lifetime utility, if an agent's lifetime income doubles, optimal consumption choices will double in *each* period (income expansion paths

[11]Note that in order to interpret (2.36) as a regression one needs a theory where the error term comes from. In models with risk this error term can be linked to expectational errors, and (2.36) with error term arises as a first order approximation to the stochastic version of the Euler equation.

are linear).[12] It also means that consumption allocations are independent of the units of measurement employed.

This property of the utility function is crucial for the existence of a balanced growth in models with growth in endowments (or technological progress in production models). Define a balanced growth path as a situation in which consumption grows at a constant rate, $c_t = (1+g)^t c_0$ and the real interest rate is constant over time, $r_{t+1} = r$ for all t.

Plugging in for a balanced growth path, equation (2.35) yields, for all t

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) = \frac{1}{1+r}.$$

But for this equation to hold *for all* t we require that

$$MRS(c_{t+1}, c_t) = MRS((1+g)^t c_1, (1+g)^t c_0) = MRS(c_1, c_0)$$

and thus that u is homothetic (where $\lambda = (1+g)^t$ in equation (2.37)). Thus homothetic lifetime utility is a necessary condition for the existence of a balanced growth path in growth models. Above we showed that CRRA period utility implies homotheticity of lifetime utility u. Without proof here we state that CRRA utility is the *only* period utility function such that lifetime utility is homothetic. Thus (at least in the class of time separable lifetime utility functions) CRRA period utility is a necessary condition for the existence of a balanced growth path, which in part explains why this utility function is used in a wide range of macroeconomic applications.

[12] In the absense of borrowing constraints and other frictions that we will discuss later.

Chapter 3

The Neoclassical Growth Model in Discrete Time

3.1 Setup of the Model

The neoclassical growth model is arguably the single most important workhorse in modern macroeconomics. It is widely used in growth theory, business cycle theory and quantitative applications in public finance.

Time is discrete and indexed by $t = 0, 1, 2, \ldots$ In each period there are three goods that are traded, labor services n_t, capital services k_t and a final output good y_t that can be either consumed, c_t or invested, i_t. As usual for a complete description of the economy we have to specify technology, preferences, endowments and the information structure. Later, when looking at an equilibrium of this economy we have to specify the equilibrium concept that we intend to use.

1. Technology: The final output good is produced using as inputs labor and capital services, according to the aggregate production function F

   $$y_t = F(k_t, n_t)$$

   Note that I do not allow free disposal. If I want to allow free disposal, I will specify this explicitly by defining an separate free disposal technology. Output can be consumed or invested

   $$y_t = i_t + c_t$$

Investment augments the capital stock which depreciates at a constant rate δ over time

$$k_{t+1} = (1-\delta)k_t + i_t.$$

In what follows we make the assumption that the capital stock depreciates independent of whether it is used in production or not (which then conveniently implies that it will never be optimal -for the social planner or the actors in the competitive equilibrium- to not use the capital stock fully in production). We can rewrite the last equation as

$$i_t = k_{t+1} - k_t + \delta k_t$$

i.e. gross investment i_t equals net investment $k_{t+1} - k_t$ plus depreciation δk_t. We will require that $k_{t+1} \geq 0$, but not that $i_t \geq 0$. This assumes that the existing capital stock can be dis-invested and eaten. Note that I have been a bit sloppy: strictly speaking the capital stock and capital services generated from this stock are different things. We will assume (once we specify the ownership structure of this economy in order to define an equilibrium) that households own the capital stock and make the investment decision. They will rent out capital to the firms. We denote both the capital *stock* and the *flow* of capital services by k_t. Implicitly this assumes that there is some technology that transforms one unit of the capital stock at period t into one unit of capital services at period t. We will ignore this subtlety for the moment.

2. Preferences: There is a large number of identical, infinitely lived households. Since all households are identical and we will restrict ourselves to type-identical allocations[1] we can, without loss of generality assume that there is a single representative household. Preferences of each household are assumed to be representable by a time-separable utility function:

$$u\left(\{c_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

3. Endowments: Each household has two types of endowments. At period 0 each household is born with endowments $\bar{k}_0$ of initial capital. Furthermore each household is endowed with one unit of productive time in each period, to be devoted either to leisure or to work.

[1]Identical households receive the same allocation by assumption.

4. Information: There is no risk in this economy and we assume that households and firms have perfect foresight.

5. Equilibrium: We postpone the discussion of the equilibrium concept to a later point as we will first be concerned with an optimal growth problem where we solve for Pareto optimal allocations.

3.2 Optimal Growth: Pareto Optimal Allocations

Consider the problem of a social planner that wants to maximize the utility of the representative agent, subject to the technological constraints of the economy. Note that, as long as we restrict our attention to type-identical allocations, an allocation that maximizes the utility of the representative agent, subject to the technology constraint is a Pareto efficient allocation and every Pareto efficient allocation solves the social planner problem below. Just as a reference we have the following definitions

Definition 10 *An allocation* $\{c_t, k_t, n_t\}_{t=0}^{\infty}$ *is feasible*[2] *if for all* $t \geq 0$

$$
\begin{aligned}
F(k_t, n_t) &= c_t + k_{t+1} - (1-\delta)k_t \\
c_t &\geq 0, k_t \geq 0, 0 \leq n_t \leq 1 \\
k_0 &\leq \bar{k}_0
\end{aligned}
$$

Definition 11 *An allocation* $\{c_t, k_t, n_t\}_{t=0}^{\infty}$ *is Pareto efficient if it is feasible and there is no other feasible allocation* $\{\hat{c}_t, \hat{k}_t, \hat{n}_t\}_{t=0}^{\infty}$ *such that*

$$
\sum_{t=0}^{\infty} \beta^t U(\hat{c}_t) > \sum_{t=0}^{\infty} \beta^t U(c_t)
$$

Note that in this definition I have used the fact that all households are identical.

---

[2]Strictly speaking the resource constraint for period $t = 0$ should read as

$$
F(k_0, n_0) = c_0 + k_1 - (1-\delta)\bar{k}_0
$$

but given our assumption that the capital stock depreciates independent of whether it will be used in production or not, the distinction between k_0 and $\bar{k}_0$ will be immaterial in what follows.

3.2.1 Social Planner Problem in Sequential Formulation

The problem of the planner is

$$\begin{aligned} w(\bar{k}_0) &= \max_{\{c_t,k_t,n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\ s.t. \quad F(k_t,n_t) &= c_t + k_{t+1} - (1-\delta)k_t \\ c_t &\geq 0, k_t \geq 0, 0 \leq n_t \leq 1 \\ k_0 &\leq \bar{k}_0 \end{aligned}$$

The function $w(\bar{k}_0)$ has the following interpretation: it gives the *total lifetime* utility of the representative household if the social planner chooses $\{c_t,k_t,n_t\}_{t=0}^{\infty}$ *optimally* and the initial capital stock in the economy is $\bar{k}_0$. Under the assumptions made below the function w is strictly increasing, since a higher initial capital stock yields higher production in the initial period and hence enables more consumption or capital accumulation (or both) in the initial period.

We now make the following assumptions on preferences and technology.

Assumption 1: U is continuously differentiable, strictly increasing, strictly concave and bounded. It satisfies the Inada conditions $\lim_{c\searrow 0} U'(c) = \infty$ and $\lim_{c\to\infty} U'(c) = 0$. The discount factor β satisfies $\beta \in (0,1)$

Assumption 2: F is continuously differentiable and homogenous of degree 1, strictly increasing and strictly concave. Furthermore $F(0,n) = F(k,0) = 0$ for all $k,n > 0$. Also F satisfies the Inada conditions $\lim_{k\searrow 0} F_k(k,1) = \infty$ and $\lim_{k\to\infty} F_k(k,1) = 0$. Also $\delta \in [0,1]$

From these assumptions two immediate consequences for optimal allocations are that $n_t = 1$ for all t since households do not value leisure in their utility function. Also, since the production function is strictly increasing in capital, $k_0 = \bar{k}_0$. To simplify notation we define $f(k) = F(k,1) + (1-\delta)k$, for all k. The function f gives the total amount of the final good available for consumption or investment (again remember that the capital stock can be eaten). From assumption 2 the following properties of f follow more or less directly: f is continuously differentiable, strictly increasing and strictly concave, $f(0) = 0$, $f'(k) > 0$ for all k, $\lim_{k\searrow 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 1-\delta$.

Using the implications of the assumptions, and substituting for $c_t =$

$f(k_t) - k_{t+1}$ we can rewrite the social planner's problem as

$$\begin{aligned} w(\bar{k}_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\ 0 &\leq k_{t+1} \leq f(k_t) \\ k_0 &= \bar{k}_0 > 0 \text{ given} \end{aligned} \tag{3.1}$$

The only choice that the planner faces is the choice between letting the consumer eat today versus investing in the capital stock so that the consumer can eat more tomorrow. Let the optimal sequence of capital stocks be denoted by $\{k_{t+1}^*\}_{t=0}^{\infty}$. The two questions that we face when looking at this problem are

1. Why do we want to solve such a hypothetical problem of an even more hypothetical social planner. The answer to this questions is that, by solving this problem, we will have solved for competitive equilibrium allocations of our model (of course we first have to define what a competitive equilibrium is). The theoretical justification underlying this result are the two welfare theorems, which hold in this model and in many others, too. We will give a loose justification of the theorems a bit later, and postpone a rigorous treatment of the two welfare theorems in infinite dimensional spaces until chapter 7 of these notes.

2. How do we solve this problem?[3] The answer is: dynamic programming. The problem above is an infinite-dimensional optimization problem, i.e. we have to find an optimal infinite sequence $(k_1, k_2, \ldots)$ solving the problem above. The idea of dynamic programing is to find a simpler maximization problem by exploiting the stationarity of the economic environment and then to demonstrate that the solution to the simpler maximization problem solves the original maximization problem.

To make the second point more concrete, note that we can rewrite the

[3]Just a caveat: infinite-dimensional maximization problems may not have a solution even if the u and f are well-behaved. So the function w may not always be well-defined. In our examples, with the assumptions that we made, everything is fine, however.

problem above as

$$
\begin{aligned}
w(k_0) &= \max_{\substack{\{k_{t+1}\}_{t=0}^{\infty} \text{ s.t.} \\ 0\leq k_{t+1}\leq f(k_t),\ k_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t U(f(k_t)-k_{t+1}) \\
&= \max_{\substack{\{k_{t+1}\}_{t=0}^{\infty} \text{ s.t.} \\ 0\leq k_{t+1}\leq f(k_t),\ k_0 \text{ given}}} \left\{ U(f(k_0)-k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t)-k_{t+1}) \right\} \\
&= \max_{\substack{k_1 \text{ s.t.} \\ 0\leq k_1\leq f(k_0),\ k_0 \text{ given}}} \left\{ U(f(k_0)-k_1) + \beta \left[\max_{\substack{\{k_{t+1}\}_{t=1}^{\infty} \\ 0\leq k_{t+1}\leq f(k_t),\ k_1 \text{ given}}} \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t)-k_{t+1}) \right] \right\} \\
&= \max_{\substack{k_1 \text{ s.t.} \\ 0\leq k_1\leq f(k_0),\ k_0 \text{ given}}} \left\{ U(f(k_0)-k_1) + \beta \left[\max_{\substack{\{k_{t+2}\}_{t=0}^{\infty} \\ 0\leq k_{t+2}\leq f(k_{t+1}),\ k_1 \text{ given}}} \sum_{t=0}^{\infty} \beta^{t} U(f(k_{t+1})-k_{t+2}) \right] \right\}
\end{aligned}
$$

Looking at the maximization problem inside the [ ]-brackets and comparing to the original problem (3.1) we see that the [ ]-problem is that of a social planner that, given initial capital stock k_1, maximizes lifetime utility of the representative agent from period 1 onwards. But agents don't age in our model, the technology or the utility functions doesn't change over time; this suggests that the optimal value of the problem in [ ]-brackets is equal to $w(k_1)$ and hence the problem can be rewritten as

$$
w(k_0) = \max_{\substack{0\leq k_1\leq f(k_0) \\ k_0 \text{ given}}} \{U(f(k_0)-k_1) + \beta w(k_1)\}
$$

Again two questions arise:

2.1 Under which conditions is this suggestive discussion formally correct? We will come back to this in chapters 4-5. Specifically, moving from the 2^{nd} to the 3^{rd} line of the above argument we replaced the maximization over the entire sequence with two nested maximization problems, one with respect to $\{k_{t+1}\}_{t=1}^{\infty}$, conditional on k_1, and one with respect to k_1. This is not an innocuous move.

2.2 Is this progress? Of course, the maximization problem is much easier since, instead of maximizing over infinite sequences we maximize over just one number, k_1. But we can't really solve the maximization problem, because the function $w(.)$ appears on the right side, and we don't know this function. The next section shows ways to overcome this problem.

3.2.2 Recursive Formulation of Social Planner Problem

The above formulation of the social planners problem with a function on the left and right side of the maximization problem is called recursive formulation. Now we want to study this recursive formulation of the planners problem. Since the function $w(.)$ is associated with the sequential formulation of the planner problem, let us change notation and denote by $v(.)$ the corresponding function for the recursive formulation of the problem.

Remember the interpretation of $v(k)$: it is the discounted lifetime utility of the representative agent from the current period onwards if the social planner is given capital stock k at the beginning of the current period and allocates consumption across time optimally for the household. This function v (the so-called value function) solves the following recursion

$$v(k) = \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta v(k')\} \tag{3.2}$$

Note again that v and w are two very different functions; v is the value function for the recursive formulation of the planners problem and w is the corresponding function for the sequential problem. Of course below we want to establish that $v = w$, but this is something that we have to prove rather than something that we can assume to hold! The capital stock k that the planner brings into the current period, result of past decisions, completely determines what allocations are feasible from today onwards. Therefore it is called the "state variable": it completely summarizes the state of the economy today (i.e. all future options that the planner has). The variable k' is decided (or controlled) today by the social planner; it is therefore called the "control variable", because it can be controlled today by the planner.[4]

Equation (3.2) is a functional equation (the so-called Bellman equation): its solution is a function, rather than a number or a vector. Fortunately the mathematical theory of functional equations is well-developed, so we can draw on some fairly general results. The functional equation posits that the discounted lifetime utility of the representative agent is given by the utility that this agent receives today, $U(f(k) - k')$, plus the discounted lifetime utility from tomorrow onwards, $\beta v(k')$. So this formulation makes clear the planners trade-off: consumption (and hence utility) today, versus a higher capital stock to work with (and hence higher discounted future utility) from

[4] These terms come from control theory, a field in applied mathematics.

tomorrow onwards. Hence, for a given k this maximization problem is much easier to solve than the problem of picking an infinite sequence of capital stocks $\{k_{t+1}\}_{t=0}^{\infty}$ from before. The only problem is that we have to do this maximization for every possible capital stock k, and this posits theoretical as well as computational problems. However, it will turn out that the functional equation is much easier to solve than the sequential problem (3.1) (apart from some very special cases). By solving the functional equation we mean finding a value function v solving (3.2) and an optimal policy function $k' = g(k)$ that describes the optimal k' from the maximization part in (3.2), as a function of k, i.e. for each possible value that k can take. Again we face several questions associated with equation (3.2):

1. Under what condition does a solution to the functional equation (3.2) exist and, if it exists, is it unique?

2. Is there a reliable algorithm that computes the solution (by reliable we mean that it always converges to the correct solution, independent of the initial guess for v?

3. Under what conditions can we solve (3.2) and be sure to have solved (3.1), i.e. under what conditions do we have $v = w$ and equivalence between the optimal sequential allocation $\{k_{t+1}\}_{t=0}^{\infty}$ and allocations generated by the optimal recursive policy $g(k)$

4. Can we say something about the qualitative features of v and g?

The answers to these questions will be given in the next two chapters. The answers to 1. and 2. will come from the Contraction Mapping Theorem, to be discussed in Section 4.3. The answer to the third question makes up what Richard Bellman called the Principle of Optimality and is discussed in Section 5.1. Finally, under more restrictive assumptions we can characterize the solution to the functional equation (v, g) more precisely. This will be done in Section 5.2. In the remaining parts of this section we will look at specific examples where we can solve the functional equation by hand. Then we will talk about competitive equilibria and the way we can construct prices so that Pareto optimal allocations, together with these prices, form a competitive equilibrium. This will be our versions of the first and second welfare theorem for the neoclassical growth model.

3.2.3 An Example

Consider the following example. Let the period utility function be given by $U(c) = \ln(c)$ and the aggregate production function be given by $F(k, n) = k^{\alpha} n^{1-\alpha}$ and assume full depreciation, i.e. $\delta = 1$. Then $f(k) = k^{\alpha}$ and the functional equation becomes

$$v(k) = \max_{0 \leq k' \leq k^{\alpha}} \{\ln(k^{\alpha} - k') + \beta v(k')\}$$

Remember that the solution to this functional equation is an entire function $v(.)$. Now we will discuss several methods to solve this functional equation.

Guess and Verify (or Method of Undetermined Coefficients)

We will guess a particular functional form of a solution and then verify that the solution has in fact this form (note that this does not rule out that the functional equation has other solutions). This method works well for the example at hand, but not so well for most other examples that we are concerned with. Let us guess

$$v(k) = A + B\ln(k)$$

where A and B are unknown coefficients that are to be determined. The method consists of three steps:

1. Solve the maximization problem on the right hand side, given the guess for v, i.e. solve

   $$\max_{0 \leq k' \leq k^{\alpha}} \{\ln(k^{a} - k') + \beta(A + B\ln(k'))\}$$

   Obviously the constraints on k' never bind and the objective function is strictly concave and the constraint set is compact in k', for any given k. Thus, the first order condition for k' is sufficient for the unique solution. The FOC yields

   $$\begin{aligned} \frac{1}{k^{\alpha} - k'} &= \frac{\beta B}{k'} \\ k' &= \frac{\beta B k^{\alpha}}{1 + \beta B} \end{aligned} \tag{3.3}$$

2. Evaluate the right hand side at the optimal solution $k' = \frac{\beta B k^\alpha}{1+\beta B}$. This yields

$$\begin{aligned}
\text{RHS} &= \ln\left(k^a - k'\right) + \beta\left(A + B\ln(k')\right) \\
&= \ln\left(\frac{k^\alpha}{1+\beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B k^\alpha}{1+\beta B}\right) \\
&= -\ln(1+\beta B) + \alpha\ln(k) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B\ln(k)
\end{aligned}$$

3. In order for our guess to solve the functional equation, the left hand side of the functional equation, which we have guessed to equal LHS= $A + B\ln(k)$ must equal the right hand side, which we just found, for *all possible* values of k. If we can find coefficients A, B for which this is true, we have found a solution to the functional equation. Equating LHS and RHS yields

$$\begin{aligned}
A + B\ln(k) &= -\ln(1+\beta B) + \alpha\ln(k) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B\ln(\\
(B - \alpha(1+\beta B))\ln(k) &= -A - \ln(1+\beta B) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) \qquad (3
\end{aligned}$$

But this equation has to hold for *every* capital stock k. The right hand side of (3.4) does not depend on k but the left hand side does. Hence the right hand side is a constant, and the only way to make the left hand side a constant is to make $B - \alpha(1+\beta B) = 0$. Solving this for B yields $B = \frac{\alpha}{1-\alpha\beta}$. Since the left hand side of (3.4) equals to 0 for $B = \frac{\alpha}{1-\alpha\beta}$, the right hand side better is, too. Therefore the constant A has to satisfy

$$\begin{aligned}
0 &= -A - \ln(1+\beta B) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) \\
&= -A - \ln\left(\frac{1}{1-\alpha\beta}\right) + \beta A + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)
\end{aligned}$$

Solving this mess for A yields

$$A = \frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right]$$

We can also determine the optimal policy function $k' = g(k)$ by plugging in $B = \frac{\alpha}{1-\alpha\beta}$ into (3.3):

$$\begin{aligned} g(k) &= \frac{\beta B k^{\alpha}}{1+\beta B} \\ &= \alpha\beta k^{\alpha} \end{aligned}$$

Hence our guess was correct: the function $v^*(k) = A + B\ln(k)$, with A, B as determined above, solves the functional equation, with associated policy function $g(k) = \alpha\beta k^{\alpha}$.

Note that for this specific example the optimal policy of the social planner is to save a constant fraction $\alpha\beta$ of total output k^{α} as capital stock for tomorrow and and let the household consume a constant fraction $(1-\alpha\beta)$ of total output today. The fact that these fractions do not depend on the level of k is very unique to this example and not a property of the model in general. Also note that there may be other solutions to the functional equation; we have just constructed one (actually, for the specific example there are no others, but this needs some proving). Finally, it is straightforward to construct a sequence $\{k_{t+1}\}_{t=0}^{\infty}$ from our policy function g that will turn out to solve the sequential problem (3.1) (of course for the specific functional forms used in the example): start from $k_0 = \bar{k}_0$ and then recursively

$$\begin{aligned} k_1 &= g(k_0) = \alpha\beta k_0^{\alpha} \\ k_2 &= g(k_1) = \alpha\beta k_1^{\alpha} = (\alpha\beta)^{1+\alpha} k_0^{\alpha^2} \\ &\vdots \\ k_t &= (\alpha\beta)^{\sum_{j=0}^{t-1}\alpha^j} k_0^{\alpha^t} \end{aligned}$$

Obviously, since $0 < \alpha < 1$ we have that

$$\lim_{t\to\infty} k_t = (\alpha\beta)^{\frac{1}{1-\alpha}} = k^*$$

for all initial conditions $k_0 > 0$. Not surprisingly, k^* is the unique solution to the equation $g(k) = k$.

Value Function Iteration: Analytical Approach

In the last section we started with a clever guess, parameterized it and used the method of undetermined coefficients (guess and verify) to solve for the solution v^* of the functional equation. For just about any other than the log-utility, Cobb-Douglas production function, $\delta = 1$ case this method would not work; even your most ingenious guesses would fail when trying to be verified.

Consider instead the following iterative procedure for our previous example

1. Guess an arbitrary function $v_0(k)$. For concreteness let's take $v_0(k) = 0$ for all

2. Proceed by solving

$$v_1(k) = \max_{0 \leq k' \leq k^\alpha} \{\ln(k^\alpha - k') + \beta v_0(k')\}$$

   Note that we can solve the maximization problem on the right hand side since we know v_0 (since we have guessed it). In particular, since $v_0(k') = 0$ for all k' we have as optimal solution to this problem

$$k' = g_1(k) = 0 \text{ for all } k$$

   Plugging this back in we get

$$v_1(k) = \ln(k^\alpha - 0) + \beta v_0(0) = \ln k^\alpha = \alpha \ln k$$

3. Now we can solve

$$v_2(k) = \max_{0 \leq k' \leq k^\alpha} \{\ln(k^\alpha - k') + \beta v_1(k')\}$$

   since we know v_1 and so forth.

4. By iterating on the recursion

$$v_{n+1}(k) = \max_{0 \leq k' \leq k^\alpha} \{\ln(k^\alpha - k') + \beta v_n(k')\}$$

   we obtain a sequence of value functions $\{v_n\}_{n=0}^\infty$ and policy functions $\{g_n\}_{n=1}^\infty$. Hopefully these sequences will converge to the solution v^* and associated policy g^* of the functional equation. In fact, below we will state and prove a very important theorem asserting exactly that (under certain conditions) this iterative procedure converges for any initial guess v_0, and converges to the unique correct solution v^*.

In the homework I let you carry out the first few iterations of this procedure. Note however, that, in order to find the solution v^* exactly you would have to carry out step 3. above a lot of times (in fact, infinitely many times), which is, of course, infeasible. Therefore one has to implement this procedure numerically on a computer.

Value Function Iteration: Numerical Approach

Even a computer can carry out only a finite number of calculation and can only store finite-dimensional objects. Hence the best we can hope for is a numerical approximation of the true value function. The functional equation above is defined for all $k \geq 0$ (in fact there is an upper bound, but let's ignore this for now). Because computer storage space is finite, we will approximate the value function for a finite number of points only.[5] For the sake of the argument suppose that k and k' can only take values in $\mathcal{K} = \{0.04, 0.08, 0.12, 0.16, 0.2\}$. Note that the value function v_n then consists of 5 numbers, $(v_n(0.04), v_n(0.08), v_n(0.12), v_n(0.16), v_n(0.2))$

Now let us implement the above algorithm numerically. First we have to pick concrete values for the parameters α and β. Let us pick $\alpha = 0.3$ and $\beta = 0.6$.

1. Make the initial guess $v_0(k) = 0$ for all $k \in \mathcal{K}$

2. Solve
$$v_1(k) = \max_{\substack{0 \leq k' \leq k^{0.3} \\ k' \in \mathcal{K}}} \left\{ \ln\left(k^{0.3} - k'\right) + 0.6 * 0 \right\}$$
This obviously yields as optimal policy $k'(k) = g_1(k) = 0.04$ for all $k \in \mathcal{K}$ (note that since $k' \in \mathcal{K}$ is required, $k' = 0$ is not allowed). Plugging this back in yields
$$\begin{aligned}
v_1(0.04) &= \ln(0.04^{0.3} - 0.04) = -1.077 \\
v_1(0.08) &= \ln(0.08^{0.3} - 0.04) = -0.847 \\
v_1(0.12) &= \ln(0.12^{0.3} - 0.04) = -0.715 \\
v_1(0.16) &= \ln(0.16^{0.3} - 0.04) = -0.622 \\
v_1(0.2) &= \ln(0.2^{0.3} - 0.04) = -0.55
\end{aligned}$$

[5] In this course I will only discuss so-called finite state-space methods, i.e. methods in which the state variable (and the control variable) can take only a finite number of values. For a general treatment of computational methods in economics see the textbooks by Judd (1998), Miranda and Fackler (2002) or Heer and Maussner (2009).

3. Let's do one more step by hand

$$v_2(k) = \left\{ \max_{\substack{0 \leq k' \leq k^{0.3} \\ k' \in \mathcal{K}}} \ln\left(k^{0.3} - k'\right) + 0.6 v_1(k') \right\}$$

Start with $k = 0.04$:

$$v_2(0.04) = \max_{\substack{0 \leq k' \leq 0.04^{0.3} \\ k' \in \mathcal{K}}} \left\{ \ln\left(0.04^{0.3} - k'\right) + 0.6 v_1(k') \right\}$$

Since $0.04^{0.3} = 0.381$ all $k' \in \mathcal{K}$ are possible. If the planner chooses $k' = 0.04$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.04\right) + 0.6 * (-1.077) = -1.723$$

If he chooses $k' = 0.08$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.08\right) + 0.6 * (-0.847) = -1.710$$

If he chooses $k' = 0.12$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.12\right) + 0.6 * (-0.715) = -1.773$$

If $k' = 0.16$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.16\right) + 0.6 * (-0.622) = -1.884$$

Finally, if $k' = 0.2$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.2\right) + 0.6 * (-0.55) = -2.041$$

Hence for $k = 0.04$ the optimal choice is $k'(0.04) = g_2(0.04) = 0.08$ and $v_2(0.04) = -1.710$. This we have to do for all $k \in \mathcal{K}$. One can already see that this is quite tedious by hand, but also that a computer can do this quite rapidly. Table 1 below shows the value of

$$\left(k^{0.3} - k'\right) + 0.6 v_1(k')$$

for different values of k and k'. A $*$ in the column for k' that this k' is the optimal choice for capital tomorrow, for the particular capital stock k today

Table 1

k \ k'	0.04	0.08	0.12	0.16	0.2
0.04	-1.7227	-1.7097^*	-1.7731	-1.8838	-2.0407
0.08	-1.4929	-1.4530^*	-1.4822	-1.5482	-1.6439
0.12	-1.3606	-1.3081^*	-1.3219	-1.3689	-1.4405
0.16	-1.2676	-1.2072^*	-1.2117	-1.2474	-1.3052
0.2	-1.1959	-1.1298	-1.1279^*	-1.1560	-1.2045

Hence the value function v_2 and policy function g_2 are given by

Table 2

k	$v_2(k)$	$g_2(k)$
0.04	-1.7097	0.08
0.08	-1.4530	0.08
0.12	-1.3081	0.08
0.16	-1.2072	0.08
0.2	-1.1279	0.12

In Figure 3.2.3 we plot the true value function v^* (remember that for this example we know to find v^* analytically) and selected iterations from the numerical value function iteration procedure. In Figure 3.2.3 we have the corresponding policy functions.

We see from Figure 3.2.3 that the numerical approximations of the value function converge rapidly to the true value function. After 20 iterations the approximation and the truth are nearly indistinguishable with the naked eye (and they are not distinguishable in the plot above). Looking at the policy functions we see from Figure 2 that the approximating policy function do not converge to the truth (more iterations don't help, and the step 10 and fully converged policy functions lie exactly on top of each either). This is due to the fact that the analytically correct value function was found by allowing $k' = g(k)$ to take any value in the real line, whereas for the approximations we restricted $k' = g_n(k)$ to lie in $\mathcal{K}$. The function g_{10} approximates the true policy function as good as possible, subject to this restriction. Therefore the approximating value function will not converge exactly to the truth, either. The fact that the value function approximations come much closer is due to the fact that the utility and production function induce "curvature" into

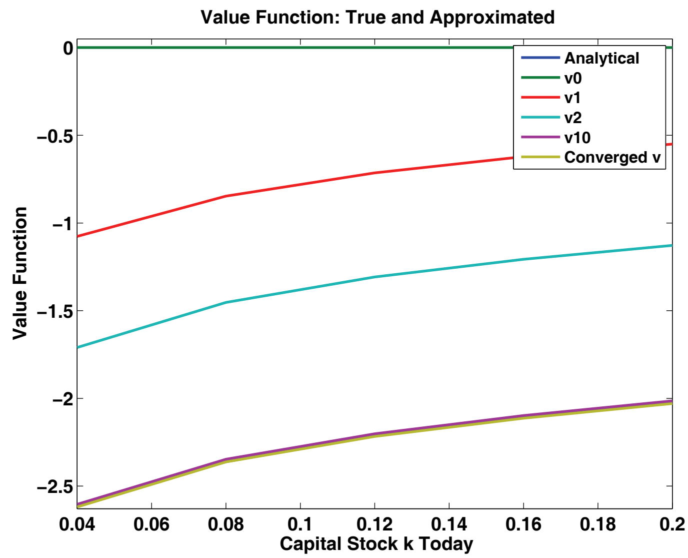

Figure 3.1: True and Approximated Value Function

the value function, something that we may make more precise later. Also note that we we plot the true value and policy function only on $\mathcal{K}$, with MATLAB interpolating between the points in $\mathcal{K}$, so that the true value and policy functions in the plots look piecewise linear.

3.2.4 The Euler Equation Approach and Transversality Conditions

We now relate our example studied above with recursive techniques to the traditional approach of solving optimization problems. Note that this approach also, as the guess and verify method, will only work in very simple examples, but not in general, whereas the recursive numerical approach works for a wide range of parameterizations of the neoclassical growth model. First let us look at a finite horizon social planners problem and then at the related infinite horizon problem

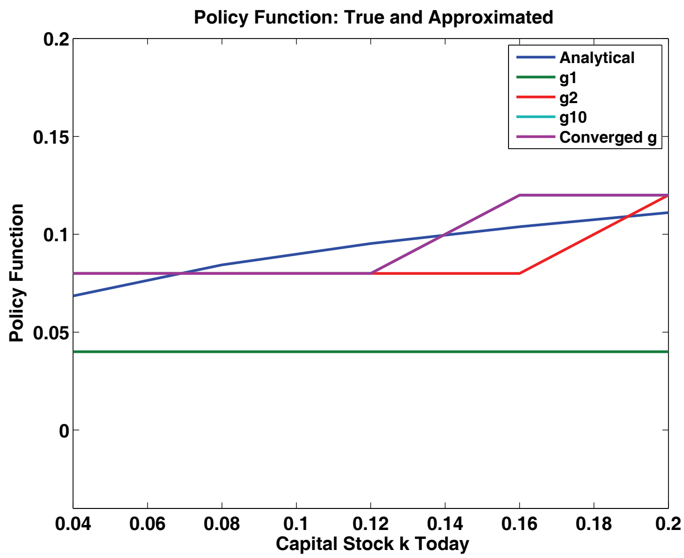

Figure 3.2: True and Approximated Policy Function

The Finite Horizon Case

Let us consider the social planner problem for a situation in which the representative consumer lives for $T < \infty$ periods, after which she dies for sure and the economy is over. The social planner problem for this case is given by

$$\begin{aligned} w^T(\bar{k}_0) &= \max_{\{k_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1}) \\ 0 &\leq k_{t+1} \leq f(k_t) \\ k_0 &= \bar{k}_0 > 0 \text{ given} \end{aligned}$$

Obviously, since the world goes under after period T, $k_{T+1} = 0$. Also, given our Inada assumptions on the utility function the constraints on k_{t+1} will never be binding and we will disregard them henceforth. The first thing we note is that, since we have a finite-dimensional maximization problem and since the set constraining the choices of $\{k_{t+1}\}_{t=0}^T$ is closed and bounded, by the Bolzano-Weierstrass theorem a solution to the maximization problem exists, so that $w^T(\bar{k}_0)$ is well-defined. Furthermore, since the constraint set is convex and we assumed that U is strictly concave (and the finite sum of strictly concave functions is strictly concave), the solution to the maximization problem is unique and the first order conditions are not only necessary, but also sufficient.

Forming the Lagrangian yields

$$L = U(f(k_0)-k_1)+\ldots+\beta^t U(f(k_t)-k_{t+1})+\beta^{t+1}U(f(k_{t+1})-k_{t+2})+\ldots+\beta^T U(f(k_T)-k_{T+1})$$

and hence we can find the first order conditions as

$$\frac{\partial L}{\partial k_{t+1}} = -\beta^t U'(f(k_t)-k_{t+1})+\beta^{t+1}U'(f(k_{t+1})-k_{t+2})f'(k_{t+1}) = 0 \qquad \text{for all } t = 0,\ldots,T-1$$

or

$$\begin{array}{ccccc} \underbrace{U'(f(k_t)-k_{t+1})} & = & \underbrace{\beta U'(f(k_{t+1})-k_{t+2})} & \underbrace{f'(k_{t+1})} & \text{for all } t=0,\ldots,T-1 \\ \begin{array}{c}\text{Utility cost}\\ \text{for saving}\\ \text{1 unit more}\\ \text{capital for } t+1\end{array} & = & \begin{array}{c}\text{Discounted}\\ \text{add. utility}\\ \text{from one more}\\ \text{unit of cons.}\end{array} & \begin{array}{c}\text{Add. production}\\ \text{possible with}\\ \text{one more unit}\\ \text{of capital in } t+1\end{array} & \end{array} \tag{3.5}$$

The interpretation of the optimality condition is easiest with a variational argument. Suppose the social planner in period t contemplates whether to save one more unit of capital for tomorrow. One more unit saved reduces consumption by one unit, at utility cost of $U'(f(k_t) - k_{t+1})$. On the other hand, there is one more unit of capital for production to produce with tomorrow, yielding additional output $f'(k_{t+1})$. Each additional unit of production, when used for consumption, is worth $U'(f(k_{t+1}) - k_{t+2})$ utils tomorrow, and hence $\beta U'(f(k_{t+1}) - k_{t+2})$ utils today. At the optimum the net benefit of such a variation in allocations must be zero, and the result is the first order condition above.

This first order condition some times is called an Euler equation (supposedly because it is loosely linked to optimality conditions in continuous time calculus of variations, developed by Euler). Equations (3.5) is second order difference equation, a system of T equations in the $T+1$ unknowns $\{k_{t+1}\}_{t=0}^T$ (with k_0 predetermined). However, we have the terminal condition $k_{T+1} = 0$ and hence, under appropriate conditions, can solve for the optimal $\{k_{t+1}\}_{t=0}^T$ uniquely. We can demonstrate this for our example from above.

Again let $U(c) = \ln(c)$ and $f(k) = k^{\alpha}$. Then (3.5) becomes

$$\begin{aligned} \frac{1}{k_t^{\alpha} - k_{t+1}} &= \frac{\beta \alpha k_{t+1}^{\alpha-1}}{k_{t+1}^{\alpha} - k_{t+12}} \\ k_{t+1}^{\alpha} - k_{t+2} &= \alpha\beta k_{t+1}^{\alpha-1}(k_t^{\alpha} - k_{t+1}) \end{aligned} \tag{3.6}$$

with $k_0 > 0$ given and $k_{T+1} = 0$. A little trick will make our life easier. Define $z_t = \frac{k_{t+1}}{k_t^{\alpha}}$. The variable z_t is the fraction of output in period t that is saved as capital for tomorrow, so we can interpret z_t as the saving rate of the social planner. Dividing both sides of (3.6) by k_{t+1}^{α} we get

$$\begin{aligned} 1 - z_{t+1} &= \frac{\alpha\beta(k_t^{\alpha} - k_{t+1})}{k_{t+1}} = \alpha\beta\left(\frac{1}{z_t} - 1\right) \\ z_{t+1} &= 1 + \alpha\beta - \frac{\alpha\beta}{z_t} \end{aligned}$$

This is a first order difference equation. Since we have the boundary condition $k_{T+1} = 0$, this implies $z_T = 0$, so we can solve this equation backwards. Rewriting yields

$$z_t = \frac{\alpha\beta}{1 + \alpha\beta - z_{t+1}} \tag{3.7}$$

We can now recursively solve backwards for the entire sequence $\{z_t\}_{t=0}^{T}$, given that we know $z_T = 0$. We obtain as general formula (verify this by plugging it into the first order difference equation (3.7) above)

$$z_t = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}}$$

and hence

$$\begin{aligned} k_{t+1} &= \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^{\alpha} \\ c_t &= \frac{1 - \alpha\beta}{1 - (\alpha\beta)^{T-t+1}} k_t^{\alpha} \end{aligned}$$

One can also solve for the discounted future utility at time zero from the above optimal allocation. Taking logs of the above equations yields

$$\begin{aligned} \log(c_t) &= \log(1 - \alpha\beta) - \log\left(1 - (\alpha\beta)^{T-t+1}\right) + \alpha \log(k_t) \\ \log(k_{t+1}) &= \log(\alpha\beta) + \log\left(\frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}}\right) + \alpha \log(k_t). \end{aligned}$$

Iterating on time in the second equation delivers

$$\begin{aligned}
\log(k_t) &= \log(\alpha\beta) + \log\left(\frac{1-(\alpha\beta)^{T-(t-1)}}{1-(\alpha\beta)^{T-(t-1)+1}}\right) + \alpha\log(k_{t-1}) \\
&= \log(\alpha\beta)\sum_{j=0}^{t-1}\alpha^j + \sum_{j=1}^{t}\alpha^{j-1}\log\left(\frac{1-(\alpha\beta)^{T-(t-j)}}{1-(\alpha\beta)^{T-(t-j)+1}}\right) + \alpha^t\log(k_0)
\end{aligned}$$

and thus

$$\log(c_t) = \log(1-\alpha\beta) - \log\left(1-(\alpha\beta)^{T-t+1}\right) + \alpha\log(\alpha\beta)\sum_{j=0}^{t-1}\alpha^j + \sum_{j=1}^{t}\alpha^j\log\left(\frac{1-(\alpha\beta)^{T-(t-j)}}{1-(\alpha\beta)^{T-(t-j)+1}}\right)$$

Thus

$$\begin{aligned}
w^T(k_0) &= \alpha\log(k_0)\sum_{t=0}^{T}(\alpha\beta)^t + \sum_{t=0}^{T}\beta^t\log(1-\alpha\beta) - \sum_{t=0}^{T}\beta^t\log\left(1-(\alpha\beta)^{T-t+1}\right) + \alpha\log(\alpha\beta) \\
&\quad + \sum_{t=0}^{T}\beta^t\sum_{j=1}^{t}\alpha^j\log\left(\frac{1-(\alpha\beta)^{T-(t-j)}}{1-(\alpha\beta)^{T-(t-j)+1}}\right)
\end{aligned}$$

Note that the optimal policies and the discounted future utility are functions of the time horizon T that the social planner faces.

Taking the limit yields[6]

$$\begin{aligned}
\lim_{T\to\infty} w^T(k_0) &= \alpha\log(k_0)\lim_{T\to\infty}\sum_{t=0}^{T}(\alpha\beta)^t + \lim_{T\to\infty}\sum_{t=0}^{T}\beta^t\log(1-\alpha\beta) + \alpha\log(\alpha\beta)\lim_{T\to\infty}\sum_{t=0}^{T}\beta^t\sum_{j=0}^{t-1} \\
&= \frac{\alpha}{1-\alpha\beta}\log(k_0) + \frac{\log(1-\alpha\beta)}{1-\beta} + \alpha\log(\alpha\beta)\lim_{T\to\infty}\sum_{t=0}^{T}\beta^t\frac{1-\alpha^t}{1-\alpha} \\
&= \frac{\alpha}{1-\alpha\beta}\log(k_0) + \frac{\log(1-\alpha\beta)}{1-\beta} + \alpha\log(\alpha\beta)\left[\frac{1}{(1-\alpha)(1-\beta)} - \frac{1}{(1-\alpha)(1-\alpha\right. \\
&= \frac{\alpha}{1-\alpha\beta}\log(k_0) + \frac{\log(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\log(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}
\end{aligned}$$

[6] It is actually easier to first compute

$$\lim_{T\to\infty}\log(c_t) = \log(1-\alpha\beta) + \alpha\log(\alpha\beta)\sum_{j=0}^{t-1}\alpha^j + \alpha^{t+1}\log(k_0)$$

and then take the infinite discounted sum.

We observe that for this specific example

$$\begin{aligned} & \lim_{T\to\infty} \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} k_t^\alpha \\ = & \ \alpha\beta k_t^\alpha \end{aligned}$$

and

$$\lim_{t\to\infty} w^T(k_0) = \frac{\alpha}{1-\alpha\beta}\log(k_0) + \frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right] = w(k_0)$$

So is it the case that the optimal policy for the social planners problem with infinite time horizon always is the limit of the optimal policies for the $T-$horizon planning problem (and the same is true for the value of the planning problem)? Our results from the guess and verify method seem to indicate this, and for this example this is indeed true, but it is not true in general. We can't in general interchange maximization and limit-taking: the limit of the finite maximization problems is often but not always equal to maximization of the problem in which time goes to infinity.

In order to prepare for the discussion of the infinite horizon case let us analyze the first order difference equation

$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$$

graphically. On the y-axis of Figure **??** we draw z_{t+1} against z_t on the x-axis. Since $k_{t+1} \geq 0$, we have that $z_t \geq 0$ for all t. Furthermore, as z_t approaches 0 from above, z_{t+1} approaches $-\infty$. As z_t approaches $+\infty$, z_{t+1} approaches $1 + \alpha\beta$ from below asymptotically. The graph intersects the x-axis at $z^0 = \frac{\alpha\beta}{1+\alpha\beta}$. The difference equation has two steady states where $z_{t+1} = z_t = z$. This can be seen by

$$\begin{aligned} z &= 1 + \alpha\beta - \frac{\alpha\beta}{z} \\ z^2 - (1+\alpha\beta)z + \alpha\beta &= 0 \\ (z-1)(z-\alpha\beta) &= 0 \\ z &= 1 \text{ or } z = \alpha\beta \end{aligned}$$

From Figure **??** we can also determine graphically the sequence of optimal policies $\{z_t\}_{t=0}^T$. We start with $z_T = 0$ on the y-axis, go to the $z_{t+1} = 1+\alpha\beta-$

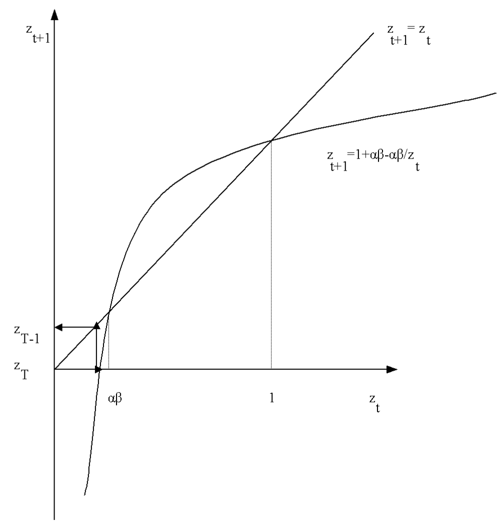

Figure 3.3: Dynamics of Savings Rate

$\frac{\alpha\beta}{z_t}$ curve to determine z_{T-1} and mirror it against the 45-degree line to obtain z_{T-1} on the y-axis. Repeating the argument one obtains the entire $\{z_t\}_{t=0}^{T}$ sequence, and hence the entire $\{k_{t+1}\}_{t=0}^{T}$ sequence. Note that going with t backwards to zero, the z_t's approach $\alpha\beta$. Hence for large T and for small t (the optimal policies for a finite time horizon problem with long horizon, for the early periods) come close to the optimal infinite time horizon policies solved for with the guess and verify method.

In general, absent the assumptions of log-utility and full depreciation we cannot reduce the second order difference equation implied by the Euler equation

$$U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1}) - k_{t+2})f'(k_{t+1})$$

with initial condition k_0 and terminal condition $k_{T+1} = 0$ to a first order difference equation. Instead, we need to solve the system of T Euler equations for the unknown variables $k_1, k_2, ..., k_T$ numerically. One option is to simply feed it into a multi-dimensional nonlinear equation solver. Alternatively consider the following *Shooting Algorithm*:

1. Guess $\hat{k}_T$. This implies $\hat{c}_T = f(\hat{k}_T)$

2. For each $t = 1, ...T-1$, and given $\hat{k}_t, \hat{k}_{t+1}$ from the previous step solve $\hat{k}_{t-1}$ from the Euler equation

$$U'(f(\hat{k}_t) - \hat{k}_{t+1}) = \beta U'(f(\hat{k}_{t+1}) - \hat{k}_{t+2}) f'(\hat{k}_{t+1})$$

   Note that for every $\hat{k}_T \in (0, \bar{k})$ there exists a unique such sequence, and this sequence is strictly increasing in $\hat{k}_T$

3. If $\hat{k}_0 = k_0$ then we are done and have found a solution. If $\hat{k}_0 > k_0$ go to step 1 and lower the guess for $\hat{k}_T$. If instead $\hat{k}_0 < k_0$ go to step 1 and increase the guess for $\hat{k}_T$.

The Infinite Horizon Case

Now let us turn to the infinite horizon problem and let's see how far we can get with the Euler equation approach. Remember that the problem was to solve

$$\begin{aligned} w(\bar{k}_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\ 0 &\leq k_{t+1} \leq f(k_t) \\ k_0 &= \bar{k}_0 > 0 \text{ given} \end{aligned}$$

Since the period utility function is strictly concave and the constraint set is convex, the first order conditions constitute necessary conditions for an optimal sequence $\{k^*_{t+1}\}_{t=0}^{\infty}$ (a proof of this is a formalization of the variational argument I spelled out when discussing the intuition for the Euler equation). As a reminder, the Euler equations were

$$\beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = U'(f(k_t) - k_{t+1}) \qquad \text{for all } t = 0, \ldots, t, \ldots \tag{3.8}$$

Again this is a second order difference equation, but now we only have an initial condition for k_0, but no terminal condition since there is no terminal time period.

In a lot of applications, the transversality condition substitutes for the missing terminal condition. Let us first state and then interpret the TVC[7]

$$\lim_{t\to\infty} \underbrace{\beta^t U'(f(k_t)-k_{t+1})f'(k_t)}_{\substack{\text{value in discounted} \\ \text{utility terms of one} \\ \text{more unit of capital}}} \underbrace{k_t}_{\substack{\text{Total} \\ \text{Capital} \\ \text{Stock}}} = 0$$

The transversality condition states that the value of the capital stock k_t, when measured in terms of discounted utility, goes to zero as time goes to infinity. Note that this condition does not require that the capital stock itself converges to zero in the limit, only that the (shadow) value of the capital stock has to converge to zero.

The transversality condition is a tricky beast, and you may spend some

[7] Often one can find an alternative statement of the TVC in the literature:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0$$

where λ_t is the Lagrange multiplier on the constraint

$$c_t + k_{t+1} = f(k_t)$$

in the social planner problem in which consumption is not yet substituted out in the objective function. From the first order condition we have

$$\begin{aligned} \beta^t U'(c_t) &= \lambda_t \\ \beta^t U'(f(k_t)-k_{t+1}) &= \lambda \text{t} \end{aligned}$$

Hence the TVC becomes

$$\lim_{t\to\infty} \beta^t U'(f(k_t)-k_{t+1})k_{t+1} = 0$$

This condition is equvalent to the condition given in the main text, as shown by the following argument (which uses the Euler equation)

$$\begin{aligned} 0 &= \lim_{t\to\infty} \beta^t U'(f(k_t)-k_{t+1})k_{t+1} \\ &= \lim_{t\to\infty} \beta^{t-1} U'(f(k_{t-1})-k_t)k_t \\ &= \lim_{t\to\infty} \beta^{t-1} \beta U'(f(k_t)-k_{t+1})f'(k_t)k_t \\ &= \lim_{t\to\infty} \beta^t U'(f(k_t)-k_{t+1})f'(k_t)k_t \end{aligned}$$

which is the TVC in the main text.

more time on it as the semester progresses. For now we just state the following theorem (see Stokey and Lucas, p. 98):

Theorem 12 *Let U, β and F (and hence f) satisfy assumptions 1. and 2. Then an allocation $\{k_{t+1}\}_{t=0}^{\infty}$ that satisfies the Euler equations and the transversality condition solves the sequential social planners problem, for a given k_0.*

This theorem states that under certain assumptions the Euler equations and the transversality condition are jointly sufficient for a solution to the social planners problem in sequential formulation. Stokey et al., p. 98-99 prove this theorem. Note that in their proof they do not use the boundedness assumption on U, and thus the result applies to unbounded utility functions as well (such as CRRA utility), as long as U satisfies the other assumptions.

Also note that we have said nothing about the necessity of the TVC. We have (loosely) argued that the Euler equations are necessary conditions, but is the TVC necessary, i.e. does every solution to the sequential planning problem have to satisfy the TVC? This turns out to be a hard problem, and there is not a very general result for this. However, for the log-case (with $f's$ satisfying our assumptions), Ekelund and Scheinkman (1985) show that the TVC is in fact a necessary condition. Refer to their paper and to the related results by Peleg and Ryder (1972) and Weitzman (1973) for further details. From now on we assert that the TVC is necessary and sufficient for optimization under the assumptions we made on f, U, but you should remember that these assertions remain to be proved.

But now we take these theoretical results for granted and proceed with our example of $U(c) = \ln(c)$, $f(k) = k^{\alpha}$. For these particular functional forms, the TVC becomes

$$\begin{aligned} & \lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t \\ = & \lim_{t\to\infty} \frac{\alpha\beta^t k_t^{\alpha}}{k_t^{\alpha} - k_{t+1}} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - \frac{k_{t+1}}{k_t^{\alpha}}} \\ = & \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t} \end{aligned}$$

We also repeat the first order difference equation derived from the Euler equations

$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$$

We can't solve the Euler equations form $\{z_t\}_{t=0}^{\infty}$ backwards, but we can solve it forwards, conditional on guessing an initial value for z_0. We show that only one guess for z_0 yields a sequence that does not violate the TVC or the nonnegativity constraint on capital or consumption.

1. $z_0 < \alpha\beta$. From Figure **??** we see that in finite time $z_t < 0$, violating the nonnegativity constraint on capital

2. $z_0 > \alpha\beta$. Then from Figure 3 we see that $\lim_{t\to\infty} z_t = 1$. (Note that, in fact, every $z_0 > 1$ violate the nonnegativity of consumption and hence is not admissible as a starting value). We will argue that all these paths violate the TVC.

3. $z_0 = \alpha\beta$. Then $z_t = \alpha\beta$ for all $t > 0$. For this path (which obviously satisfies the Euler equations) we have that

$$\lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - \alpha\beta} = 0$$

   and hence this sequence satisfies the TVC. From the sufficiency of the Euler equation jointly with the TVC we conclude that the sequence $\{z_t\}_{t=0}^{\infty}$ given by $z_t = \alpha\beta$ is an optimal solution for the sequential social planners problem. Translating into capital sequences yields as optimal policy $k_{t+1} = \alpha\beta k_t^{\alpha}$, with k_0 given. But this is exactly the constant saving rate policy that we derived as optimal in the recursive problem.

Now we pick up the unfinished business from point 2. Note that we asserted above (citing Ekelund and Scheinkman) that for our particular example the TVC is also a necessary condition, i.e. any sequence $\{k_{t+1}\}_{t=0}^{\infty}$ that does not satisfy the TVC can't be an optimal solution.

Since all sequences $\{z_t\}_{t=0}^{\infty}$ from case 2. above converge to 1, in the TVC both the nominator and the denominator go to zero. Let us linearly approximate z_{t+1} around the steady state $z = 1$. This gives

$$\begin{aligned}
z_{t+1} &= 1 + \alpha\beta - \frac{\alpha\beta}{z_t} := g(z_t) \\
z_{t+1} &\approx g(1) + (z_t - 1)g'(z_t)|_{z_t=1} \\
&= 1 + (z_t - 1)\left(\frac{\alpha\beta}{z_t^2}\right)|_{z_t=1} \\
&= 1 + \alpha\beta(z_t - 1) \\
(1 - z_{t+1}) &\approx \alpha\beta(1 - z_t) \\
&\approx (\alpha\beta)^{t-k+1}(1 - z_k) \qquad \text{for all } k
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{t\to\infty} \frac{\alpha\beta^{t+1}}{1 - z_{t+1}} &\approx \lim_{t\to\infty} \frac{\alpha\beta^{t+1}}{(\alpha\beta)^{t-k+1}(1 - z_k)} \\
&= \lim_{t\to\infty} \frac{\beta^k}{\alpha^{t-k}(1 - z_k)} = \infty
\end{aligned}$$

as long as $0 < \alpha < 1$. Hence non of the sequences contemplated in 2. can be an optimal solution, and our solution found in 3. is indeed the unique optimal solution to the infinite-dimensional social planner problem. Therefore in this specific case the Euler equation approach, augmented by the TVC works. But as with the guess-and-verify method this is very unique to the specific example at hand. Therefore for the general case we can't rely on pencil and paper, but have to resort to computational techniques.

To make sure that these techniques give the desired answer, we have to study the general properties of the functional equation associated with the sequential social planner problem and the relation of its solution to the solution of the sequential problem. We will do this in chapters 4 and 5. Before this we will show that, by solving the social planners problem we have, in effect, solved for a (the) competitive equilibrium in this economy. But first we will analyze the properties of the solution to the social planner problem a bit further.

3.2.5 Steady States and the Modified Golden Rule

A steady state is defined as a social optimum or competitive equilibrium in which allocations are constant over time, $c_t = c^*$ and $k_{t+1} = k^*$. In general, we can only expect for a steady state to arise for the right initial condition, that is, we need $k_0 = k^*$. Even if $k_0 \neq k^*$, the allocation may over time converge to (c^*, k^*); in that case we call (c^*, k^*) an (asymptotically) stable steady steady state. We can use our previous results to sharply characterize steady states.

The Euler equations for the social planner problem read as

$$\begin{aligned} \beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) &= U'(f(k_t) - k_{t+1}) \text{ or} \\ \beta U'(c_{t+1}) f'(k_{t+1}) &= U'(c_t). \end{aligned}$$

In a steady state $c_t = c_{t+1} = c^*$ and thus

$$\begin{aligned} \beta f'(k^*) &= 1 \\ f'(k^*) &= 1 + \rho \end{aligned}$$

where ρ is the time discount rate. Recalling the definition of $f'(k) = F_k(k, 1) + 1 - \delta$ we obtain the so-called modified golden rule

$$F_k(k^*, 1) - \delta = \rho$$

that is, the social planner sets the marginal product of capital, net of depreciation, equal to the time discount rate. As we will see below, the net real interest rate in a competitive equilibrium equals $F_k(k, 1) - \delta$, so the modified golden rule can be restated as equating the real interest rate and the time discount rate. Note that we derived exactly the same result in our simple pure exchange economy in chapter 2.

For our example above with log-utility, Cobb-Douglas production and $\delta = 1$ we find that

$$\begin{aligned} \alpha (k^*)^{\alpha - 1} &= \rho + 1 = \frac{1}{\beta} \\ k^* &= (\alpha\beta)^{\frac{1}{1-\alpha}}. \end{aligned}$$

One can also find the steady state level of capital by exploiting the optimal policy function from the recursive solution of the problem, $k' = \alpha\beta k^\alpha$. Setting $k' = k$ and solving we find again $k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$. Also note that from any initial capital stock $k_0 > 0$ the optimal sequence chosen by the social planner $\{k^*_{t+1}\}$ converges to $k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$. This is no accident: the unique steady state of the neoclassical growth model is globally asymptotically stable in general. We will show this in the continuous time version of the model in chapter 8.

Note that the name modified golden rule comes from the following consideration: the resource constraint reads as

$$c_t = f(k_t) - k_{t+1}$$

and in the steady state

$$c = f(k) - k.$$

The capital stock that maximizes consumption per capita, called the (original) golden rule k^g, therefore satisfies

$$\begin{aligned} f'(k^g) &= 1 \text{ or} \\ F_k(k^g, 1) - \delta &= 0 \end{aligned}$$

Thus the social planner finds it optimal to set capital $k^* < k^g$ in the long run because he respects the impatience of the representative household.

3.2.6 A Remark About Balanced Growth

So far we have abstracted both from population growth as well as technological progress. As a consequence there is no long-run growth in aggregate and in per-capita variables as the economy converges to its long-run steady state. Thus the neoclassical growth model does not generate long-run growth.

Fortunately this shortcoming is easily fixed. So now assume that the population is growing at rate n, so that at time t the size of the population is $N_t = (1+n)^t$. Furthermore assume that there is labor-augmenting technological progress, so that output at date t is produced according to the production function

$$F(K_t, N_t(1+g)^t)$$

where K_t is the total capital stock in the economy. In the model with population growth there is some choice as to what the objective function of the social planner (and the household in the competitive equilibrium) ought to be. Either per capita lifetime utility

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{3.9}$$

or lifetime utility of the entire dynasty

$$\sum_{t=0}^{\infty} N_t \beta^t U(c_t) \tag{3.10}$$

is being maximized. We will go with the first formulation, but also give results for the second (nothing substantial changes, just some adjustments in the algebra are required). The resource constraint reads as

$$(1+n)^t c_t + K_{t+1} = F(K_t, (1+n)^t (1+g)^t) + (1-\delta)K_t.$$

Now we define growth-adjusted per capita variables as

$$\begin{aligned} \tilde{c}_t &= \frac{c_t}{(1+g)^t} \\ \tilde{k}_t &= \frac{k_t}{(1+g)^t} = \frac{K_t}{(1+n)^t(1+g)^t} \end{aligned}$$

and divide the resource constraint by $(1+n)^t(1+g)^t$ to obtain

$$\tilde{c}_t + (1+n)(1+g)\tilde{k}_{t+1} = F(\tilde{k}_t, 1) + (1-\delta)\tilde{k}_t. \tag{3.11}$$

In order to be able to analyze this economy and to obtain a balanced growth path we now assume that the period utility function is of CRRA form $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. We can then rewrite the objective function (3.9) as

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} &= \sum_{t=0}^{\infty} \beta^t \frac{(\tilde{c}_t(1+g)^t)^{1-\sigma}}{1-\sigma} \\ &= \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} \end{aligned}$$

where $\tilde{\beta} = \beta(1+g)^{1-\sigma}$. Note that had we assumed (3.10) as our objective, only our definition of $\tilde{\beta}$ would change[8]; it would now read as $\tilde{\beta} = \beta(1+n)(1+g)^{1-\sigma}$.

Given these adjustments we can rewrite the growth-deflated social planner problem as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\left(f(\tilde{k}_t) - (1+g)(1+n)\tilde{k}_{t+1}\right)^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} 0 &\leq (1+g)(1+n)\tilde{k}_{t+1} \leq f(\tilde{k}_t) \\ \tilde{k}_0 &= k_0 \text{ given} \end{aligned}$$

[8] In either case one must now assume $\tilde{\beta} < 1$ which entails a joint assumption on the parameters β, σ, g of the model.

and all the analysis from above goes through completely unchanged. In particular, all the recursive techniques apply and the Euler equation techniques remain the same

A balanced growth path is a socially optimal allocation (or a competitive equilibrium) where all variables grow at a constant rate (this rate may vary across variables). Given our deflation above a balanced growth path in the variables $\{c_t, k_{t+1}\}$ corresponds to constant steady state for $\{\tilde{c}_t, \tilde{k}_{t+1}\}$. The Euler equations associated with the social planner problem above is

$$(1+n)(1+g)\left(\tilde{c}_t\right)^{-\sigma} = \tilde{\beta}\left(\tilde{c}_{t+1}\right)^{-\sigma}\left[F_k(\tilde{k}_{t+1}, 1) + (1-\delta)\right]. \tag{3.12}$$

Evaluated at the steady state this reads as

$$(1+n)(1+g) = \tilde{\beta}\left[F_k(\tilde{k}^*, 1) + (1-\delta)\right]$$

Defining $\tilde{\beta} = \frac{1}{1+\tilde{\rho}}$ we have

$$(1+n)(1+g)(1+\tilde{\rho}) = F_k(\tilde{k}^*, 1) + (1-\delta)$$

or, as long as the terms $ng, n\tilde{\rho}, g\tilde{\rho}$ are sufficiently small, the new modified golden rule reads as

$$F_k(\tilde{k}^*, 1) - \delta = n + g + \tilde{\rho}.$$

Note that the original golden rule in this growing economy is defined as maximizing, growth-deflated per capita consumption

$$\tilde{c} = f(\tilde{k}) - (1+g)(1+n)\tilde{k}$$

and thus (approximately)

$$F_k(\tilde{k}^g, 1) - \delta = n + g.$$

Once the optimal growth-deflated variables $\{\tilde{c}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}$ have been determined, the true variables of interest can trivially be computed as

$$\begin{aligned} c_t &= (1+g)^t \tilde{c}_t \text{ and } k_{t+1} = (1+g)^t \tilde{k}_{t+1} \\ C_t &= (1+n)^t (1+g)^t \tilde{c}_t \text{ and } K_{t+1} = (1+n)^t (1+g)^t \tilde{k}_{t+1}. \end{aligned}$$

Overall we conclude that the model with population growth and technological progress is no harder to analyze than the benchmark model. All we have to do is to redefine the time discount factor, deflate all per-capita variables by technological progress, all aggregate variables in addition by population growth, and pre-multiply effective capital tomorrow by $(1+n)(1+g)$.

3.3 Competitive Equilibrium Growth

Suppose we have solved the social planners problem for a Pareto efficient allocation $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$. What we are genuinely interested in are allocations and prices that arise when firms and consumers interact in markets. In this section we will discuss the connection between Pareto optimal allocations and allocations arising in a competitive equilibrium. For the discussion of Pareto optimal allocations it did not matter who owns what in the economy, since the planner was allowed to freely redistribute endowments across agents. For a competitive equilibrium the question of ownership is crucial. We make the following assumption on the ownership structure of the economy: we assume that consumers own all factors of production (i.e. they own the capital stock at all times) and rent it out to the firms. We also assume that households own the firms, i.e. are claimants of the firms' profits.

Now we have to specify the equilibrium concept and the market structure. We assume that the final goods market and the factor markets (for labor and capital services) are perfectly competitive, which means that households as well as firms take prices are given and beyond their control. We assume that there is a single market at time zero in which goods for all future periods are traded. After this market closes, in all future periods the agents in the economy just carry out the trades they agreed upon in period 0. We assume that all contracts are perfectly enforceable. This market is often called Arrow-Debreu market structure and the corresponding competitive equilibrium an Arrow-Debreu equilibrium.

For each period there are three goods that are traded:

1. The final output good, y_t that can be used for consumption c_t or investment i_t purposes of the household. Let p_t denote the price of the period t final output good, quoted in period 0. We let the period 0 output good be the numeraire and thus normalize $p_0 = 1$.

2. Labor services n_t. Let w_t be the price of one unit of labor services delivered in period t, quoted in period 0, in terms of the period t consumption good. Hence w_t is the real wage; it tells how many units of the *period* t consumption goods one can buy for the receipts for one unit of labor. The wage in terms of the numeraire, the period 0 output good is $p_t w_t$.

3. Capital services k_t. Let r_t be the rental price of one unit of capital

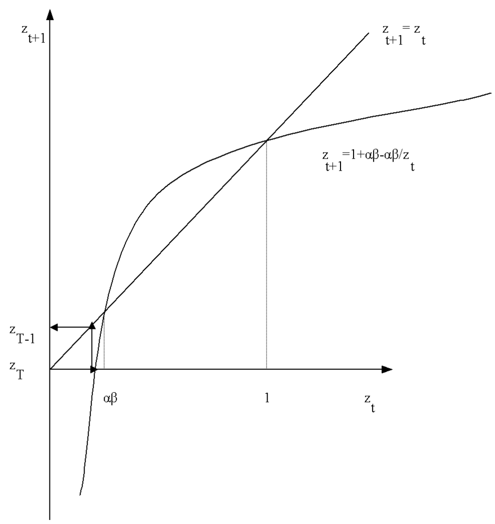

Figure 3.4: Flows of Goods and Payments in Neoclassical Growth Model

services delivered in period t, quoted in period 0, in terms of the period t consumption good. Note that r_t is the real rental rate of capital; the rental rate in terms of the numeraire good is $p_t r_t$.

Figure 3.3 summarizes the flows of goods and payments in the economy (note that, since all trade takes place in period 0, no payments are made after period 0).

3.3.1 Definition of Competitive Equilibrium

Now we will define a competitive equilibrium for this economy. Let us first look at firms. Without loss of generality assume that there is a single, representative firm that behaves competitively.[9]

The representative firm's problem is, given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$,

[9] As we will show below this is an innocuous assumption as long as the technology features constant returns to scale.

to solve:

$$\begin{aligned}
\pi &= \max_{\{y_t,k_t,n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t n_t) \qquad (3.13)\\
s.t. \quad y_t &= F(k_t, n_t) \text{ for all } t \geq 0\\
y_t, k_t, n_t &\geq 0
\end{aligned}$$

Hence firms chose an infinite sequence of inputs $\{k_t, n_t\}$ to maximize total profits π. Since in each period all inputs are rented (the firm does not make the capital accumulation decision), there is nothing dynamic about the firm's problem and it will separate into an infinite number of static maximization problems.

Households instead face a fully dynamic problem in this economy. They own the capital stock and hence have to decide how much labor and capital services to supply, how much to consume and how much capital to accumulate. Taking prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ as given the representative consumer solves

$$\max_{\{c_t,i_t,x_{t+1},k_t,n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \qquad (3.14)$$

$$\begin{aligned}
s.t. \quad \sum_{t=0}^{\infty} p_t(c_t + i_t) &\leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t n_t) + \pi\\
x_{t+1} &= (1-\delta)x_t + i_t \qquad \text{all } t \geq 0\\
0 &\leq n_t \leq 1, 0 \leq k_t \leq x_t \qquad \text{all } t \geq 0\\
c_t, x_{t+1} &\geq 0 \qquad \text{all } t \geq 0\\
& x_0 \text{ given}
\end{aligned}$$

A few remarks are in order. First, there is only one, time zero budget constraint, the so-called Arrow-Debreu budget constraint, as markets are only open in period 0. Secondly we carefully distinguish between the capital stock x_t and capital services that households supply to the firm. Capital services are traded and hence have a price attached to them, the capital stock x_t remains in the possession of the household, is never traded and hence does not have a price attached to it.[10] We have implicitly assumed two things about

[10]This is not quite correct since we do not require investment i_t to be positive. If household choose $c_t < -i_t < 0$, households transform part of the capital stock back into final output goods and sell it back to the firm at price p_t.

technology: a) as previously stated the capital stock depreciates no matter whether it is rented out to the firm and used in production or not and b) there is a technology for households that transforms one unit of the capital stock at time t into one unit of capital services at time t. The constraint $k_t \leq x_t$ then states that households cannot provide more capital services than the capital stock at their disposal produces. Also note that we only require the capital stock to be nonnegative, but not investment. We are now ready to define a competitive equilibrium for this economy.

Definition 13 *A Competitive Equilibrium (Arrow-Debreu Equilibrium) consists of prices* $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ *and allocations for the firm* $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ *and the household* $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ *such that*

1. *Given prices* $\{p_t, w_t, r_t\}_{t=0}^{\infty}$*, the allocation of the representative firm* $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ *solves* (3.13)

2. *Given prices* $\{p_t, w_t, r_t\}_{t=0}^{\infty}$*, the allocation of the representative household* $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ *solves* (3.14)

3. *Markets clear*

$$
\begin{aligned}
y_t &= c_t + i_t \text{ (Goods Market)} \\
n_t^d &= n_t^s \text{ (Labor Market)} \\
k_t^d &= k_t^s \text{ (Capital Services Market)}
\end{aligned}
$$

3.3.2 Characterization of the Competitive Equilibrium and the Welfare Theorems

Firms

Let us start with a partial characterization of the competitive equilibrium. First of all we simplify notation and denote by $k_t = k_t^d = k_t^s$ the equilibrium demand and supply of capital services. Similarly $n_t = n_t^d = n_t^s$. It is straightforward to show that in any equilibrium $p_t > 0$ for all t, since the utility function is strictly increasing in consumption (and therefore consumption demand would be infinite at a zero price). But then, since the production function exhibits positive marginal products, $r_t, w_t > 0$ in any competitive equilibrium because otherwise factor demands would become unbounded.

Now let us analyze the problem of the representative firm. As stated earlier, the firms does not face a dynamic decision problem as the variables chosen at period t, (y_t, k_t, n_t) do not affect the constraints nor returns (profits) at later periods. The static profit maximization problem for the representative firm is given by

$$\pi_t = \max_{k_t, n_t \geq 0} p_t \left(F(k_t, n_t) - r_t k_t - w_t n_t\right)$$

Since the firm take prices as given, the usual "factor price equals marginal product" conditions arise

$$\begin{aligned} r_t &= F_k(k_t, n_t) \\ w_t &= F_n(k_t, n_t) \end{aligned} \tag{3.15}$$

Substituting marginal products for factor prices in the expression for profits implies that in equilibrium the profits the firms earns in period t are equal to

$$\pi_t = p_t \left(F(k_t, n_t) - F_k(k_t, n_t)k_t - F_n(k_t, n_t)n_t\right) = 0$$

The fact that profits are equal to zero is a consequence of perfect competition (and the associated marginal product pricing conditions (3.15)) *and* the assumption that the production function F exhibits constant returns to scale (that is, it is homogeneous of degree 1):

$$F(\lambda k, \lambda n) = \lambda F(k, n) \text{ for all } \lambda > 0$$

Euler's theorem[11] states that for any function that is homogeneous of degree 1 payments to production factors exhaust output, or formally:

$$F(k_t, n_t) = F_k(k_t, n_t)k_t + F_n(k_t, n_t)n_t$$

[11]Euler's theorem states that for any function that is homogeneous of degree k and differentiable at $x \in \mathbf{R}^L$ we have

$$kf(x) = \sum_{i=1}^{L} x_i \frac{\partial f(x)}{\partial x_i}$$

Proof. Since f is homogeneous of degree k we have for all $\lambda > 0$

$$f(\lambda x) = \lambda^k f(x)$$

Differentiating both sides with respect to λ yields

$$\sum_{i=1}^{L} x_i \frac{\partial f(\lambda x)}{\partial x_i} = k\lambda^{k-1} f(x)$$

Therefore total profits of the representative firm π_t are equal to zero in equilibrium in every period, and thus overall profits $\pi = 0$ in equilibrium as well. This result of course also implies that the owner of the firm, the representative household, will not receive any profits in equilibrium either.

We now return to the point that with a CRTS production technology the assumption of having a single representative (competitively behaving) firm is innocuous. In fact, with CRTS the number of firms is indeterminate in a competitive equilibrium; it could be one firm, two firms each operating at half the scale of the one firm or 10 million firms. To see this, first note that constant returns to scale imply that the marginal products of labor and capital are homogeneous of degree 0: for all $\lambda > 0$ we have[12]

$$\begin{aligned} F_k(\lambda k_t, \lambda n_t) &= F_k(k_t, n_t) \\ F_k(\lambda k_t, \lambda n_t) &= F_k(k_t, n_t). \end{aligned}$$

Taking $\lambda = \frac{1}{n}$ we obtain

$$F_k(k/n, 1) = F_k(k, n).$$

Therefore equation

$$r_t = F_k(k_t, n_t) = F_k(k_t/n_t, 1) \tag{3.16}$$

implies that all firms that we might assume to exist (the single representative firm or the 10 million firms) in equilibrium would operate with exactly the same capital-labor ratio determined by (3.16). Only that ratio is pinned down

---

Setting $\lambda = 1$ yields

$$\sum_{i=1}^{L} x_i \frac{\partial f(x)}{\partial x_i} = kf(x)$$

■

[12]For any $\lambda > 0$, since F has CRTS, we have:

$$F(\lambda k, \lambda n) = \lambda F(k, n)$$

Differentiate this expression with respect to one of the inputs, say k, to obtain

$$\begin{aligned} \lambda F_k(\lambda k, \lambda n) &= \lambda F_k(k, n) \\ F_k(\lambda k, \lambda n) &= F_k(k, n) \end{aligned}$$

and thus the marginal product of capital is homogeneous of degree 0 in its argument (of course the same can be derived for the marginal product of labor).

by the marginal product pricing conditions[13], but not the scale of operation of each firm. So whether total output is produced by one representative (still competitively behaving) firm with output

$$F(k_t, n_t) = n_t F(k_t/n_t, 1)$$

or n_t firms, each with one worker and output $F(k_t/n_t, 1)$ is both indeterminate and irrelevant for the equilibrium, and without loss of generality we can restrict attention to a single representative firm.

Households

Let's now turn to the representative household. Given that output and factor prices have to be positive in equilibrium it is clear that the utility maximizing choices of the household entail

$$\begin{aligned} n_t &= 1, k_t = x_t \\ i_t &= k_{t+1} - (1-\delta)k_t \end{aligned}$$

From the equilibrium condition in the goods market we also obtain

$$F(k_t, 1) = c_t + k_{t+1} - (1-\delta)k_t$$

and thus

$$f(k_t) = c_t + k_{t+1}.$$

Since utility is strictly increasing in consumption, we also can conclude that the Arrow-Debreu budget constraint holds with equality in equilibrium. Using these results we can rewrite the household problem as

$$\begin{aligned} & & \max_{\{c_t, k_{t=1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\ s.t. \quad \sum_{t=0}^{\infty} p_t (c_t + k_{t+1} - (1-\delta)k_t) &=& \sum_{t=0}^{\infty} p_t (r_t k_t + w_t) \\ c_t, k_{t+1} &\geq& 0 \qquad \text{all } t \geq 0 \\ & & k_0 \text{ given} \end{aligned}$$

[13]Note that the other condition

$$w_t = F_n(k_t, n_t) = F_n(k_t/n_t, 1)$$

does not help here (but it does imply that r_t and w_t are inversely related in any competitive equilibrium, since F_k is strictly decreasing in k_t/n_t and F_n is strictly increasing in it.

Again the first order conditions are necessary for a solution to the household optimization problem. Attaching μ to the Arrow-Debreu budget constraint and ignoring the nonnegativity constraints on consumption and capital stock we get as first order conditions[14] with respect to c_t, c_{t+1} and k_{t+1}

$$\begin{aligned}
\beta^t U'(c_t) &= \mu p_t \\
\beta^{t+1} U'(c_{t+1}) &= \mu p_{t+1} \\
\mu p_t &= \mu(1-\delta+r_{t+1})p_{t+1}
\end{aligned}$$

Combining yields the Euler equation

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r_{t+1}-\delta}$$

and thus

$$\frac{(1-\delta+r_{t+1})\,\beta U'(c_{t+1})}{U'(c_t)} = 1$$

Note that the net real interest rate in this economy is given by $r_{t+1}-\delta$. When a household saves one unit of consumption for tomorrow, she can rent it out tomorrow of a rental rate r_{t+1}, but a fraction δ of the one unit depreciates, so the net return on her saving is $r_{t+1}-\delta$. In these lecture notes we sometimes let r_{t+1} denote the net real interest rate, sometimes the real rental rate of capital; the context will always make clear which of the two concepts r_{t+1} stands for.

Now we use the marginal pricing condition and the fact that we defined $f(k_t) = F(k_t,1) + (1-\delta)k_t$

$$r_t = F_k(k_t,1) = f'(k_t) - (1-\delta)$$

and the market clearing condition from the goods market

$$c_t = f(k_t) - k_{t+1}$$

[14]That the nonnegativity constraints on consumption do not bind follows directly from the Inada conditions. The nonnegativity constraints on capital could potentially bind if we look at the household problem in isolation. However, since from the production function $k_t = 0$ implies $F(0,1) = 0$ and $F_k(0,1) = \infty$. Thus in equilibrium r_t would be bid up to the point where $k_t > 0$ is optimal for the household. Anticipating this we take the shortcut and ignore the corners with respect to capital holdings. But you should be aware of the fact that we did something here that was not very clean, we used equilibrium logic before carrying out the maximization problem of the household. This is fine here, but may lead to a lot of problems when used in other circumstances.

in the Euler equation to obtain

$$\frac{f'(k_{t+1})\beta U'(f(k_{t+1}) - k_{t+2})}{U'(f(k_t) - k_{t+1})} = 1 \tag{3.17}$$

which is exactly the same Euler equation as in the social planners problem.

Also recall that for the social planner problem, in addition we needed to make sure that the value of the capital stock the social planner chose converged to zero in the limit: one version of the transversality condition we stated there was

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0$$

where λ_t was the Lagrange multiplier (social shadow cost) on the resource constraint. The Euler equation and TVC were jointly sufficient for a maximizing sequence of capital stocks. The same is true here: in addition to the Euler equation we need to make sure that in the limit the value of the capital stock carried forward by the household converges to zero[15]:

$$\lim_{t\to\infty} p_t k_{t+1} = 0$$

But using the first order condition yields

$$\begin{aligned}
\lim_{t\to\infty} p_t k_{t+1} &= \frac{1}{\mu}\lim_{t\to\infty} \beta^t U'(c_t)k_{t+1} \\
&= \frac{1}{\mu}\lim_{t\to\infty} \beta^{t-1} U'(c_{t-1})k_t \\
&= \frac{1}{\mu}\lim_{t\to\infty} \beta^{t-1}\beta U'(c_t)(1-\delta+r_t)k_t \\
&= \frac{1}{\mu}\lim_{t\to\infty} \beta^t U'(f(k_t)-k_{t+1})f'(k_t)k_t
\end{aligned}$$

where the Lagrange multiplier μ on the Arrow-Debreu budget constraint is positive since the budget constraint is strictly binding. Note that this is

[15]We implicitly assert here that for the assumptions we made on U, f the Euler conditions with the TVC are jointly sufficient and they are *both* necessary.

Note that Stokey et al. in Chapter 2.3, when they discuss the relation between the planning problem and the competitive equilibrium allocation use the finite horizon case, because for this case, under the assumptions made the Euler equations are both necessary and sufficient for both the planning problem and the household optimization problem, so they don't have to worry about the TVC.

exactly the same TVC as for the social planners problem (as stated in the main text).

Hence an allocation of capital $\{k_{t+1}\}_{t=0}^{\infty}$ satisfies the necessary and sufficient conditions for being a Pareto optimal allocations if and only if it satisfies the necessary and sufficient conditions for being part of a competitive equilibrium (always subject to the caveat about the necessity of the TVC in both problems).

This last statement is our version of the fundamental theorems of welfare economics for the particular economy that we consider. The first welfare theorem states that a competitive equilibrium allocation is Pareto efficient (under very general assumptions). The second welfare theorem states that any Pareto efficient allocation can be decentralized as a competitive equilibrium with transfers (under much more restrictive assumptions), i.e. there exist prices and redistributions of initial endowments such that the prices, together with the Pareto efficient allocation is a competitive equilibrium for the economy with redistributed endowments.

In particular, when dealing with an economy with a representative agent (i.e. when restricting attention to type-identical allocations), whenever the second welfare theorem applies we can solve for Pareto efficient allocations by solving a social planners problem and be sure that all Pareto efficient allocations are competitive equilibrium allocations (since there is nobody to redistribute endowments to/from). If, in addition, the first welfare theorem applies we can be sure that we found *all* competitive equilibrium allocations.

Also note an important fact. The first welfare theorem is usually easy to prove, whereas the second welfare theorem is substantially harder, in particular in infinite-dimensional spaces. Suppose we have proved the first welfare theorem and we have established that there exists a unique Pareto efficient allocation (this in general requires representative agent economies and restrictions to type-identical allocations, but in these environments boils down to showing that the social planners problem has a unique solution). Then we have established that, if there is a competitive equilibrium, its allocation has to equal the Pareto efficient allocation. Of course we still need to prove existence of a competitive equilibrium, but this is not surprising given the intimate link between the second welfare theorem and the existence proof.

Back to our economy at hand. Once we have determined the equilibrium sequence of capital stocks $\{k_{t+1}\}_{t=0}^{\infty}$ we can construct the rest of the competitive equilibrium. In particular equilibrium allocations are given by

$$\begin{aligned} c_t &= f(k_t) - k_{t+1} \\ y_t &= F(k_t, 1) \\ i_t &= y_t - c_t \\ n_t &= 1 \end{aligned}$$

for all $t \geq 0$. Finally we can construct factor equilibrium prices as

$$\begin{aligned} r_t &= F_k(k_t, 1) \\ w_t &= F_n(k_t, 1) \end{aligned}$$

Finally, the prices of the final output good can be found as follows. We have already normalized $p_0 = 1$. From the Euler equations for the household in then follows that

$$\begin{aligned} p_{t+1} &= \frac{\beta U'(c_{t+1})}{U'(c_t)} p_t \\ \frac{p_{t+1}}{p_t} &= \frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{1}{1 + r_{t+1} - \delta} \\ p_{t+1} &= \frac{\beta^{t+1} U'(c_{t+1})}{U'(c_0)} = \prod_{\tau=0}^{t} \frac{1}{1 + r_{\tau+1} - \delta} \end{aligned}$$

and we have constructed a complete competitive equilibrium, conditional on having found $\{k_{t+1}\}_{t=0}^{\infty}$.

To summarize, section 3.2 discussed how to solve for the optimal allocations of the social planner problem using recursive techniques (analytically for an example, numerically for the general case). Chapters 4 and 5 will give the theoretical-mathematical background for this discussion. In section 3.3 we then discussed how to decentralize this allocation as a competitive (Arrow-Debreu) equilibrium by demonstrating that the optimal allocation, together with appropriately chosen prices, satisfy all household and firm optimality and all equilibrium conditions.

3.3.3 Sequential Markets Equilibrium

We now briefly state the definition of a sequential markets equilibrium for this economy. This definition is useful in its own right (given that the equivalence between an Arrow Debreu equilibrium and a sequential markets equilibrium continues to apply), but also prepares the definition of a recursive competitive equilibrium in the next subsection.

In a sequential markets equilibrium households (who own the capital stock) take sequences of wages and interest rates as given and in every period chooses consumption and capital to be brought into tomorrow. In every period the consumption/investment good is the numeraire an its price normalized to 1. Thus the representative household solves

$$\begin{aligned} & \max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \qquad (3.18) \\ & \text{s.t.} \\ c_t + k_{t+1} - (1-\delta)k_t &= w_t + r_t k_t \\ c_t, k_{t+1} &\geq 0 \\ & k_0 \text{ given} \end{aligned}$$

Firms solve a sequence of static problems (since households, not firms own the capital stock). Taking wages and rental rates of capital as given the firm's problem is given as

$$\max_{k_t,n_t \geq 0} F(k_t, n_t) - w_t n_t - r_t k_t. \tag{3.19}$$

Thus we can define a sequential markets equilibrium as

Definition 14 *A sequential markets equilibrium is a sequence of prices $\{w_t, r_t\}_{t=0}^{\infty}$, allocations for the representative household $\{c_t, k_{t+1}^s\}_{t=0}^{\infty}$ and allocations for the representative firm $\{n_t^d, k_t^d\}_{t=0}^{\infty}$ such that*

1. *Given k_0 and $\{w_t, r_t\}_{t=0}^{\infty}$, allocations for the representative household $\{c_t, k_{t+1}^s\}_{t=0}^{\infty}$ solve the household maximization problem* (3.18)

2. *For each $t \geq 0$, given (w_t, r_t) the firm allocation (n_t^d, k_t^d) solves the firms' maximization problem* (3.19).

3. *Markets clear: for all $t \geq 0$*

$$\begin{aligned} n_t^d &= 1 \\ k_t^d &= k_t^s \\ F(k_t^d, n_t^d) &= c_t + k_{t+1}^s - (1-\delta)k_t^s \end{aligned}$$

Note that the notation implicitly uses that $k_0^s = k_0$. The characterization of equilibrium allocations and prices is identical to that of the Arrow-Debreu equilibrium.[16] In particular, once we have solved for a Pareto-optimal allocation, it can straightforwardly be decentralized as a SM equilibrium

3.3.4 Recursive Competitive Equilibrium

We have argued that in general the social planner problem needs to be solved recursively. In models where the equilibrium is not Pareto-efficient and it is not straightforward to solve in sequential or Arrow-Debreu formulation one often proceeds as follows. First, one makes the dynamic decision problems (here only the household problem is dynamic) recursive and then defines and computes a *Recursive Competitive Equilibrium.* While this is not strictly necessary for the neoclassical growth model (since we can obtain the equilibrium from the social planner problem) we now want to show how to define a recursive competitive equilibrium in this economy.

A useful starting point is typically the sequential formulation of the problem. The first question is what are the appropriate state variables for the household, that is, what is the minimal information the household requires to solve its dynamic decision problem from today on. Certainly the households own capital stock at the beginning of the period, k. In addition, the household needs to know w and r, which in turn are determined by the marginal products of the aggregate production function, evaluated at the *aggregate* capital stock K and labor supply $N = 1$. While it may seem redundant to distinguish k and K (they are surely intimately related in equilibrium) it is absolutely crucial to do so in order to avoid mistakes when solving the household recursive problem. Thus the state variables of the household are given by (k, K) and the control variables are today's consumption c and the capital stock being brought into tomorrow, k'.

The Bellman equation characterizing the household problem is then given

[16] Note that we have taken care of the need for a no Ponzi condition by requiring that $k_{t+1} \geq 0$.

by

$$\begin{aligned}
v(k,K) &= \max_{c,k'\geq 0}\{U(c)+\beta v(k',K')\} \quad (3.20)\\
&\text{s.t.}\\
c+k' &= w(K)+(1+r(K)-\delta)k\\
K' &= H(K)
\end{aligned}$$

The last equation is called the aggregate law of motion: the (as of yet unknown) function H describes how the aggregate capital stock evolves between today and tomorrow, which the household needs to know, given that K' enters the value function tomorrow. It now is clear why we need to distinguish k and K. Without that distinction the household would perceive that by choosing k' it would affect future prices $w(k')$ and $r(k')$. While this is true in equilibrium, by our competitive behavior assumption it is exactly this influence the household does *not* take into account when making decisions. To clarify this the (k,K) notation is necessary. The solution of the household problem is given by a value function v and two policy functions $c=C(k,K)$ and $k'=G(k,K)$.

On the firm side we could certainly formulate the maximization problem and define optimal policy functions, but since there is nothing dynamic about the firm problem we will go ahead and use the firm's first order conditions, evaluated at the aggregate capital stock, to define the wage and return functions

$$w(K) = F_l(K,1) \quad (3.21)$$
$$r(K) = F_k(K,1). \quad (3.22)$$

We then have the following definition

Definition 15 *A recursive competitive equilibrium is a value function* $v:\mathbf{R}_+^2\to\mathbf{R}$ *and policy functions* $C,G:\mathbf{R}_+^2\to\mathbf{R}_+$ *for the representative household, pricing functions* $w,r:\mathbf{R}_+\to\mathbf{R}_+$ *and an aggregate law of motion* $H:\mathbf{R}_+\to\mathbf{R}_+$ *such that*

1. *Given the functions* w,r *and* H, *the value function* v *solves the Bellman equation* (3.20) *and* C,G *are the associated policy functions.*

2. *The pricing functions satisfy* (3.21)-(3.22).

3. *Consistency*
$$H(K) = G(K,K)$$

4. *For all* $K \in \mathbf{R}_+$
$$C(K,K) + G(K,K) = F(K,1) + (1-\delta)K$$

The one condition that is not straightforward is the consistency condition 3. It simply states that the law of motion for the aggregate capital stock is consistent with the household capital accumulation decision as the households' individual asset holdings coincide with the aggregate capital stock.

As with the social planner problem we hope to prove that the recursive formulation of the household problem is equivalent to the sequential formulation, so that by solving the former (which is computationally feasible) we also have solved the latter (which is the problem we are interested in, but can in general not solve). But as before it requires a proof that asserting this equivalence is in fact justified. This equivalence result between the sequential and the recursive problem is (again, as before) nothing else but the principle of optimality which we discuss in generality in chapter 5.1.

3.4 Mapping the Model to Data: Calibration

So far we have studied the theoretical properties of the neoclassical growth model and described how to solve for equilibria and socially efficient allocations numerically, for given values of the parameters. In the final section of this chapter we discuss a simple method to select (estimate) these parameters in practice, so that the model can be used for a quantitative analysis of the real world and for counterfactual analysis.

The method we describe is called calibration. The idea of this method is to first choose a set of empirical facts that the model should match. The parameters of the model are then selected so that the equilibrium allocations and prices implied by the model matches these facts. Evidently the fact that the model fits these empirical observations cannot be treated as success of the model. To argue that the model is useful requires the empirical evaluation of the model predictions along dimensions the model was it *not* calibrated to.

Before choosing parameters we specify functional forms of the period utility function and production function. We select a CRRA utility function to enable the model to possess a balanced growth path and a Cobb-Douglas production function for reasons clarified below.

$$U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$$
$$F(K,N) = K^{\alpha}\left[(1+g)^t N\right]^{1-\alpha}$$

The model is then fully specified by the technology parameters (α, δ, g), the demographics parameter n, and the preference parameters (β, σ).

Since the neoclassical growth model is meant to explain long-run growth we choose as empirical targets long run averages of particular variables in U.S. data and choose the six parameters such that the long run equilibrium of the model (the balanced growth path, BGP) matches these facts. In order to make this procedure operational we first have to take a stance on how long a time period lasts in the model. We choose the length of the period as one year.

The first set of parameters in the model can be chosen directly from the data. In the model the long run population growth rate is (by assumption) n. For U.S. the long run annual average population growth rate for the last century is about 1.1%. Thus we choose $n = 0.011$. Similarly, in the BGP the growth rate of output (GDP) per capita is given by g. In the data, per capita GDP has grown at an average annual rate of about 1.8%. Consequently we select $g = 0.018$.

For the remaining parameters we use equilibrium relationships to inform their choice. In equilibrium the wage equals the marginal product of labor,

$$w_t = (1-\alpha)K_t^{\alpha}N_t^{-\alpha}\left[(1+g)^t\right]^{1-\alpha}$$

Thus the labor share of income is given by

$$\frac{w_t N_t}{Y_t} = 1-\alpha.$$

Note that this fact holds not only in the BGP, but in every period.[17] In U.S. the labor share of income has averaged about 2/3, and thus we choose $\alpha = 1/3$.

[17]Note that the only production function with CRTS that also has constant factor shares (independent of the level of inputs) is the Cobb-Douglas production function which explains both our choice as well as its frequent use. Also note that this production function has a constant elasticity of substitution between capital and labor inputs, and that this elasticity of substitution equals 1.

In order to calibrate the depreciation rate δ we use the relationship between gross investment and the capital stock:

$$\begin{aligned}
I_t &= K_{t+1} - (1-\delta)K_t \\
&= (1+n)^{t+1}(1+g)^{t+1}\tilde{k}_{t+1} - (1-\delta)(1+n)^t(1+g)^t\tilde{k}_t \\
&= \left[(1+n)(1+g) - (1-\delta)\right](1+n)^t(1+g)^t\tilde{k} \\
&= \left[(1+n)(1+g) - (1-\delta)\right]K_t
\end{aligned}$$

Thus the investment-capital ratio in the BGP is given by

$$\frac{I_t}{K_t} = \frac{I_t/Y_t}{K_t/Y_t} = \left[(1+n)(1+g) - (1-\delta)\right] \approx n + g + \delta$$

In the data the share of investment in GDP averages about $I/Y \approx 0.2$ and the capital-output ratio averages $K/Y \approx 3$. Using the selections $g = 1.8\%$ and $n = 1.1\%$ from above then yields $\delta \approx 4\%$.

Finally, to pin down the preference parameters we turn to the remaining key equilibrium condition, the Euler equation for the representative household (see (3.12)). With CRRA utility and growth it reads as

$$(1+n)(1+g)\left(\tilde{c}_t\right)^{-\sigma} = (1+r_{t+1}-\delta)\tilde{\beta}\left(\tilde{c}_{t+1}\right)^{-\sigma} \tag{3.23}$$

In the BGP

$$\begin{aligned}
(1+n)(1+g) &= (1+r-\delta)\beta(1+g)^{1-\sigma} \\
\beta(1+g)^{-\sigma} &= \frac{1+n}{1+r-\delta}
\end{aligned} \tag{3.24}$$

Now we note that in the competitive equilibrium the rental rate on capital is given by

$$r_t = \alpha K_t^{\alpha-1}\left[(1+g)^t N_t\right]^{1-\alpha} = \alpha\frac{Y_t}{K_t}$$

We have already chosen $\alpha = 0.33$ and targeted a capital-output ratio of $K/Y = 3$. The rental rate is then given by $r = 0.33/3 = 0.11$ and real interest rate by $r - \delta = 7\%$.

Given $n = 1.1\%$ and $g = 1.8\%$ and $r - \delta = 7\%$ equation (3.24) provides a relationship between the preference parameters β and σ:

$$\beta(1.018)^{-\sigma} = 0.944.$$

First we note that in the absence of growth, $g = 0$, this relationship uniquely pins down β, but contains no information about σ. If $g > 0$, the parameters β and σ are only jointly determined. In models without risk the typical approach to deal with this problem is to choose σ based on information about the IES $\frac{1}{\sigma}$ outside the model.[18]

One can estimate an equation of the form (3.23), preferably after having taken logs, with aggregate consumption data. Doing so Hall (1982) finds $\frac{1}{\sigma} = 0.1$. One could do the same using micro household data, which Attanasio (and a large subsequent literature) has done with various coauthors in several important papers (1993, 1995) and find a range $\frac{1}{\sigma} \in [0.3, 0.8]$, and possibly higher values for particular groups. Finally, Lucas argues that cross-country differences in g are large, relative to cross-country differences in $r - \delta$ (and n). Thus, conditional on all countries sharing same preference parameters relation, condition (3.24) suggests a value for the IES of $\frac{1}{\sigma} \geq 1$. If we take $\sigma = 1$ (log-case), then $\beta = 0.961$ (i.e. $\rho = 3.9\%$). We summarize our preferred calibration of the model in the following table.

Calibration: Summary		
Param.	Value	Target
g	1.8%	g in Data
n	1.1%	n in Data
α	0.33	$\frac{wN}{Y}$
δ	4%	$\frac{I/Y}{K/Y}$
σ	1	Outside Evid.
β	0.961	K/Y

The calibration approach to select parameter values is frequently used in business cycle analysis. Once we have augmented the model with sources for fluctuations (e.g. technology shocks) as we will do in chapter 6.4, the parameters of the model are chosen such that the model replicates long run growth observations, as just discussed. It is then evaluated based on its ability to generate business cycle fluctuations of plausible size and length, as well as the appropriate co-movement of the economic variables of interest (e.g. productivity, output, investment, consumption and hours worked).

[18] In models with risk σ is not only a measure of the IES, but also of risk aversion and return (prices) of risky assets might provide additional information that helps to pin down σ with equilibrium relationships of the model.

Chapter 4

Mathematical Preliminaries

We want to study functional equations of the form

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

where F is the period return function (such as the utility function) and Γ is the constraint set. Note that for the neoclassical growth model $x = k, y = k'$ and $F(k, k') = U(f(k) - k')$ and $\Gamma(k) = \{k' \in \mathbf{R} : 0 \leq k \leq f(k)\}$

In order to so we define the following operator T

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

This operator T takes the function v as input and spits out a new function Tv. In this sense T is like a regular function, but it takes as inputs not scalars $z \in \mathbf{R}$ or vectors $\mathbf{z} \in \mathbf{R}^n$, but functions v from some subset of possible functions. A solution to the functional equation is then a fixed point of this operator, i.e. a function v^* such that

$$v^* = Tv^*$$

We want to find out under what conditions the operator T has a fixed point (existence), under what conditions it is unique and under what conditions we can start from an arbitrary function v and converge, by applying the operator T repeatedly, to v^*. More precisely, by defining the sequence of functions $\{v_n\}_{n=0}^{\infty}$ recursively by $v_0 = v$ and $v_{n+1} = Tv_n$ we want to ask under what conditions $\lim_{n \to \infty} v_n = v^*$.

In order to make these questions (and the answers to them) precise we have to define the domain and range of the operator T and we have to define what we mean by lim . This requires the discussion of complete metric spaces. In the next subsection I will first define what a metric space is and then what makes a metric space complete.

Then I will state and prove the contraction mapping theorem. This theorem states that an operator T, defined on a metric space, has a unique fixed point if this operator T is a contraction (I will obviously first define what a contraction is). Furthermore it assures that from any starting guess v repeated applications of the operator T will lead to its unique fixed point.

Finally I will prove a theorem, Blackwell's theorem, that provides sufficient condition for an operator to be a contraction. We will use this theorem to prove that for the neoclassical growth model the operator T is a contraction and hence the functional equation of our interest has a unique solution.

4.1 Complete Metric Spaces

Definition 16 *A metric space is a set S and a function $d : S \times S \to \mathbf{R}$ such that for all $x, y, z \in S$*

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ *if and only if* $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

The function d is called a metric and is used to measure the distance between two elements in S. The third property is usually referred to as symmetry, and the fourth property as triangle inequality (because of its geometric interpretation in $\mathbf{R}$ Examples of metric spaces (S, d) include[1]

---

[1] A function $f : X \to \mathbf{R}$ is said to be bounded if there exists a constant $K > 0$ such that $|f(x)| < K$ for all $x \in X$.

Let S be any subset of $\mathbf{R}$. A number $u \in \mathbf{R}$ is said to be an upper bound for the set S if $s \leq u$ for all $s \in S$. The supremum of S, $\sup(S)$ is the smallest upper bound of S.

Every set in $\mathbf{R}$ that has an upper bound has a supremum (imposed by the completeness axiom). For sets that are unbounded above some people say that the supremum does not exist, others write $\sup(S) = \infty$. We will follow the second convention.

Example 17 *$S = \mathbf{R}$ with metric $d(x, y) = |x - y|$*

Example 18 *$S = \mathbf{R}$ with metric* $d(x, y) = \begin{cases} 1 \text{ if } & x \neq y \\ 0 \text{ otherwise} \end{cases}$

Example 19 *$S = l_\infty = \{x = \{x\}_{t=0}^{\infty} \, | x_t \in \mathbf{R}$, all $t \geq 0$ and $\sup_t |x_t| < \infty\}$ with metric $d(x, y) = \sup_t |x_t - y_t|$*

Example 20 *Let $X \subseteq \mathbf{R}^l$ and $S = C(X)$ be the set of all continuous and bounded functions $f : X \to \mathbf{R}$. Define the metric $d : C(X) \times C(X) \to \mathbf{R}$ as $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$. Then (S, d) is a metric space*

A few remarks: the space l_∞ (with corresponding norm) will be important when we discuss the welfare theorems as naturally consumption allocations for models with infinitely lived consumers are infinite sequences. Why we want to require these sequences to be bounded will become clearer later.

The space $C(X)$ with norm d as defined above will be used immediately as we will define the domain of our operator T to be $C(X)$, i.e. T uses as inputs continuous and bounded functions.

Let us prove that some of the examples are indeed metric spaces. For the first example the result is trivial.

Claim 21 *$S = \mathbf{R}$ with metric* $d(x, y) = \begin{cases} 1 \text{ if } & x \neq y \\ 0 \text{ otherwise} \end{cases}$ *is a metric space*

Proof. We have to show that the function d satisfies all three properties in the definition. The first three properties are obvious. For the forth property: if $x = z$, the result follows immediately. So suppose $x \neq z$. Then $d(x, z) = 1$. But then either $y \neq x$ or $y \neq z$ (or both), so that $d(x, y) + d(y, z) \geq 1$ ■

Claim 22 *l_∞ together with the* sup*-metric is a metric space*

Also note that $\sup(S) = \max(S)$, whenever the latter exists. What the sup buys us is that it always exists even when the max does not. A simle example

$$S = \left\{ -\frac{1}{n} : n \in \mathbf{N} \right\}$$

For this example $\sup(S) = 0$ whereas $\max(S)$ does not exist.

Proof. Take arbitrary $x, y, z \in l_\infty$. From the basic triangle inequality on $\mathbf{R}$ we have that $|x_t - y_t| \leq |x_t| + |y_t|$. Hence, since $\sup_t |x_t| < \infty$ and $\sup_t |y_t| < \infty$, we have that $\sup_t |x_t - y_t| < \infty$. Property 1 is obvious. If $x = y$ (i.e. $x_t = y_t$ for all t), then $|x_t - y_t| = 0$ for all t, hence $\sup_t |x_t - y_t| = 0$. Suppose $x \neq y$. Then there exists T such that $x_T \neq y_T$, hence $|x_T - y_T| > 0$, hence $\sup_t |x_t - y_t| > 0$

Property 3 is obvious since $|x_t - y_t| = |y_t - x_t|$, all t. Finally for property 4. we note that for all t

$$|x_t - z_t| \leq |x_t - y_t| + |y_t - z_t|$$

Since this is true for all t, we can apply the sup to both sides to obtain the result (note that the sup on both sides is finite). ■

Claim 23 *$C(X)$ together with the* sup-*norm is a metric space*

Proof. Take arbitrary $f, g \in C(X)$. $f = g$ means that $f(x) = g(x)$ for all $x \in X$. Since f, g are bounded, $\sup_{x \in X} |f(x)| < \infty$ and $\sup_{x \in X} |f(x)| < \infty$, so $\sup_{x \in X} |f(x) - g(x)| < \infty$. Property 1. through 3. are obvious and for property 4. we use the same argument as before, including the fact that $f, g \in C(X)$ implies that $\sup_{x \in X} |f(x) - g(x)| < \infty$. ■

4.2 Convergence of Sequences

The next definition will make precise the meaning of statements of the form $\lim_{n \to \infty} v_n = v^*$. For an arbitrary metric space (S, d) we have the following definition.

Definition 24 *A sequence $\{x_n\}_{n=0}^\infty$ with $x_n \in S$ for all n is said to converge to $x \in S$, if for every $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbf{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N_\varepsilon$. In this case we write* $\lim_{n \to \infty} x_n = x$.

This definition basically says that a sequence $\{x_n\}_{n=0}^\infty$ converges to a point if we, for every distance $\varepsilon > 0$ we can find an index N_ε so that the sequence of x_n is not more than ε away from x after the N_ε element of the sequence. Also note that, in order to verify that a sequence converges, it is usually necessary to know the x to which it converges in order to apply the definition directly.

Example 25 *Take $S = \mathbf{R}$ with $d(x, y) = |x - y|$. Define $\{x_n\}_{n=0}^{\infty}$ by $x_n = \frac{1}{n}$. Then $\lim_{n\to\infty} x_n = 0$. This is straightforward to prove, using the definition. Take any $\varepsilon > 0$. Then $d(x_n, 0) = \frac{1}{n}$. By taking $N_\varepsilon = \frac{2}{\varepsilon}$ we have that for $n \geq N_\varepsilon$, $d(x_n, 0) = \frac{1}{n} \leq \frac{1}{N_\varepsilon} = \frac{\varepsilon}{2} < \varepsilon$ (if $N_\varepsilon = \frac{2}{\varepsilon}$ is not an integer, take the next biggest integer).*

For easy examples of sequences it is no problem to guess the limit. Note that the limit of a sequence, if it exists, is always unique (you should prove this for yourself). For not so easy examples this may not work. There is an alternative criterion of convergence, due to Cauchy.[2]

Definition 26 *A sequence $\{x_n\}_{n=0}^{\infty}$ with $x_n \in S$ for all n is said to be a Cauchy sequence if for each $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$*

Hence a sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence if for every distance $\varepsilon > 0$ we can find an index N_ε so that the elements of the sequence do not differ by more than by ε.

Example 27 *Take $S = \mathbf{R}$ with $d(x, y) = |x - y|$. Define $\{x_n\}_{n=0}^{\infty}$ by $x_n = \frac{1}{n}$. This sequence is a Cauchy sequence. Again this is straightforward to prove. Fix $\varepsilon > 0$ and take any $n, m \in \mathbf{N}$. Without loss of generality assume that $m > n$. Then $d(x_n, x_m) = \frac{1}{n} - \frac{1}{m} < \frac{1}{n}$. Pick $N_\varepsilon = \frac{2}{\varepsilon}$ and we have that for $n, m \geq N_\varepsilon$, $d(x_n, 0) < \frac{1}{n} \leq \frac{1}{N_\varepsilon} = \frac{\varepsilon}{2} < \varepsilon$. Hence the sequence is a Cauchy sequence.*

So it turns out that the sequence in the last example both converges and is a Cauchy sequence. This is not an accident. In fact, one can prove the following

Theorem 28 *Suppose that (S, d) is a metric space and that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to $x \in S$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.*

Proof. Since $\{x_n\}_{n=0}^{\infty}$ converges to x, there exists $M_{\frac{\varepsilon}{2}}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq M_{\frac{\varepsilon}{2}}$. Therefore if $n, m \geq N_\varepsilon$ we have that $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ (by the definition of convergence and the triangle inequality). But then for any $\varepsilon > 0$, pick $N_\varepsilon = M_{\frac{\varepsilon}{2}}$ and it follows that for all $n, m \geq N_\varepsilon$ we have $d(x_n, x_m) < \varepsilon$ ■

[2] Augustin-Louis Cauchy (1789-1857) was the founder of modern analysis. He wrote about 800 (!) mathematical papers during his scientific life.

Example 29 *Take* $S = \mathbf{R}$ *with* $d(x,y) = \begin{cases} 1 \text{ if } & x \neq y \\ 0 \text{ otherwise} \end{cases}$. *Define* $\{x_n\}_{n=0}^{\infty}$ *by* $x_n = \frac{1}{n}$. *Obviously* $d(x_n, x_m) = 1$ *for all* $n, m \in \mathbf{N}$. *Therefore the sequence is not a Cauchy sequence. It then follows from the preceding theorem (by taking the contrapositive) that the sequence cannot converge. This example shows that, whenever discussing a metric space, it is absolutely crucial to specify the metric.*

This theorem tells us that every convergent sequence is a Cauchy sequence. The reverse does not always hold, but it is such an important property that when it holds, it is given a particular name.

Definition 30 *A metric space* (S, d) *is complete if every Cauchy sequence* $\{x_n\}_{n=0}^{\infty}$ *with* $x_n \in S$ *for all* n *converges to some* $x \in S$.

Note that the definition requires that the limit x has to lie within S. We are interested in complete metric spaces since the Contraction Mapping Theorem deals with operators $T : S \to S$, where (S, d) is required to be a complete metric space. Also note that there are important examples of complete metric spaces, but other examples where a metric space is not complete (and for which the Contraction Mapping Theorem does not apply).

Example 31 *Let* S *be the set of all continuous, strictly decreasing functions on* $[1, 2]$ *and let the metric on* S *be defined as* $d(f, g) = \sup_{x \in [1,2]} |f(x) - g(x)|$. *I claim that* (S, d) *is not a complete metric space. This can be proved by an example of a sequence of functions* $\{f_n\}_{n=0}^{\infty}$ *that is a Cauchy sequence, but does not converge within* S. *Define* $f_n : [0, 1] \to \mathbf{R}$ *by* $f_n(x) = \frac{1}{nx}$. *Obviously all* f_n *are continuous and strictly decreasing on* $[1, 2]$,

hence $f_n \in S$ *for all* n. *Let us first prove that this sequence is a Cauchy sequence. Fix* $\varepsilon > 0$ *and take* $N_\varepsilon = \frac{2}{\varepsilon}$. *Suppose that* $m, n \geq N_\varepsilon$ *and without* *loss of generality assume that $m > n$. Then*

$$\begin{aligned}
d(f_n, f_m) &= \sup_{x\in[1,2]} \left|\frac{1}{nx} - \frac{1}{mx}\right| \\
&= \sup_{x\in[1,2]} \frac{1}{nx} - \frac{1}{mx} \\
&= \sup_{x\in[1,2]} \frac{m-n}{mnx} \\
&= \frac{m-n}{mn} = \frac{1-\frac{n}{m}}{n} \\
&\leq \frac{1}{n} \leq \frac{1}{N_\varepsilon} = \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

Hence the sequence is a Cauchy sequence. But since for all $x \in [1,2]$, $\lim_{n\to\infty} f_n(x) = 0$, the sequence converges to the function f, defined as $f(x) = 0$, for all $x \in [1,2]$. But obviously, since f is not strictly decreasing, $f \notin S$. Hence (S,d) is not a complete metric space. Note that if we choose S to be the set of all continuous and decreasing (or increasing) functions on $\mathbf{R}$, then S, together with the sup*-norm, is a complete metric space.*

Example 32 *Let $S = \mathbf{R}^L$ and $d(x,y) = \sqrt[L]{\sum_{l=1}^{L} |x_l - y_l|^L}$. (S,d) is a complete metric space. This is easily proved by proving the following three lemmata (which is left to the reader).*

1. *Every Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbf{R}^L$ is bounded*
2. *Every bounded sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbf{R}^L$ has a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ converging to some $x \in \mathbf{R}^L$ (Bolzano-Weierstrass Theorem)*
3. *For every Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathbf{R}^L$, if a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ converges to $x \in \mathbf{R}^L$, then the entire sequence $\{x_n\}_{n=0}^{\infty}$ converges to $x \in \mathbf{R}^L$.*

Example 33 *This last example is very important for the applications we are interested in. Let $X \subseteq R^L$ and $C(X)$ be the set of all bounded continuous functions $f : X \to \mathbf{R}$ with d being the* sup*-norm. Then $(C(X),d)$ is a complete metric space.*

Proof. (This follows SLP, pp. 48) We already proved that $(C(X), d)$ is a metric space. Now we want to prove that this space is complete. Let $\{f_n\}_{n=0}^{\infty}$ be an arbitrary sequence of functions in $C(X)$ which is Cauchy. We need to establish the existence of a function $f \in C(X)$ such that for all $\varepsilon > 0$ there exists N_ε satisfying $\sup_{x\in X} |f_n(x) - f(x)| < \varepsilon$ for all $n \geq N_\varepsilon$.

We will proceed in three steps: a) find a candidate for f, b) establish that the sequence $\{f_n\}_{n=0}^{\infty}$ converges to f in the sup-norm and c) show that $f \in C(X)$.

1. Since $\{f_n\}_{n=0}^{\infty}$ is Cauchy, for each $\varepsilon > 0$ there exists M_ε such that $\sup_{x\in X} |f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geq M_\varepsilon$. Now fix a particular $x \in X$. Then $\{f_n(x)\}_{n=0}^{\infty}$ is just a sequence of numbers. Now

   $$|f_n(x) - f_m(x)| \leq \sup_{y\in X} |f_n(y) - f_m(y)| < \varepsilon$$

   Hence the sequence of numbers $\{f_n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbf{R}$. Since $\mathbf{R}$ is a complete metric space, $\{f_n(x)\}_{n=0}^{\infty}$ converges to some number, call it $f(x)$. By repeating this argument for all $x \in X$ we derive our candidate function f; it is the pointwise limit of the sequence of functions $\{f_n\}_{n=0}^{\infty}$.

2. Now we want to show that $\{f_n\}_{n=0}^{\infty}$ converges to f as constructed above. Hence we want to argue that $d(f_n, f)$ goes to zero as n goes to infinity. Fix $\varepsilon > 0$. Since $\{f_n\}_{n=0}^{\infty}$ is Cauchy, it follows that there exists N_ε such that $d(f_n, f_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$. Now fix $x \in X$. For any $m \geq n \geq N_\varepsilon$ we have (remember that the norm is the sup-norm)

   $$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq d(f_n, f_m) + |f_m(x) - f(x)| \\ &\leq \frac{\varepsilon}{2} + |f_m(x) - f(x)| \end{aligned}$$

   But since $\{f_n\}_{n=0}^{\infty}$ converges to f pointwise, we have that $|f_m(x) - f(x)| < \frac{\varepsilon}{2}$ for all $m \geq N_\varepsilon(x)$, where $N_\varepsilon(x)$ is a number that may (and in general does) depend on x. But then, since $x \in X$ was arbitrary, $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N_\varepsilon$ (the key is that this N_ε does *not* depend on x). Therefore $\sup_{x\in X} |f_n(x) - f(x)| = d(f_n, f) \leq \varepsilon$ and hence the sequence $\{f_n\}_{n=0}^{\infty}$ converges to f.

3. Finally we want to show that $f \in C(X)$, i.e. that f is bounded and continuous. Since $\{f_n\}_{n=0}^{\infty}$ lies in $C(X)$, all f_n are bounded, i.e. there is a sequence of numbers $\{K_n\}_{n=0}^{\infty}$ such that $\sup_{x\in X}|f_n(x)| \leq K_n$. But by the triangle inequality, for arbitrary n

$$\begin{aligned}
\sup_{x\in X}|f(x)| &= \sup_{x\in X}|f(x)-f_n(x)+f_n(x)| \\
&\leq \sup_{x\in X}|f(x)-f_n(x)| + \sup_{x\in X}|f_n(x)| \\
&\leq \sup_{x\in X}|f(x)-f_n(x)| + K_n
\end{aligned}$$

But since $\{f_n\}_{n=0}^{\infty}$ converges to f, there exists N_ε such that $\sup_{x\in X}|f(x)-f_n(x)| < \varepsilon$ for all $n \geq N_\varepsilon$. Fix an ε and take $K = K_{N_\varepsilon} + 2\varepsilon$. It is obvious that $\sup_{x\in X}|f(x)| \leq K$. Hence f is bounded. Finally we prove continuity of f. Let us choose the metric on $\mathbf{R}^L$ to be $||x-y|| = \sqrt[L]{\sum_{l=1}^{L}|x_l-y_l|^L}$. We need to show that for every $\varepsilon > 0$ and every $x \in X$ there exists a $\delta(\varepsilon, x) > 0$ such that if $||x-y|| < \delta(\varepsilon, x)$ then $|f(x)-f(y)| < \varepsilon$, for all $x, y \in X$. Fix ε and x. Pick a k large enough so that $d(f_k, f) < \frac{\varepsilon}{3}$ (which is possible as $\{f_n\}_{n=0}^{\infty}$ converges to f). Choose $\delta(\varepsilon, x) > 0$ such that $||x-y|| < \delta(\varepsilon, x)$ implies $|f_k(x) - f_k(y)| < \frac{\varepsilon}{3}$. Since all $f_n \in C(X)$, f_k is continuous and hence such a $\delta(\varepsilon, x) > 0$ exists. Now

$$\begin{aligned}
|f(x)-f(y)| &\leq |f(x)-f_k(x)| + |f_k(x)-f_k(y)| + |f_k(y)-f(y)| \\
&\leq d(f, f_k) + |f_k(x)-f_k(y)| + d(f_k, f) \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

■

4.3 The Contraction Mapping Theorem

Now we are ready to state the theorem that will give us the existence and uniqueness of a fixed point of the operator T, i.e. existence and uniqueness of a function v^* satisfying $v^* = Tv^*$. Let (S, d) be a metric space. Just to clarify, an operator T (or a mapping) is just a function that maps elements of S into some other space. The operator that we are interested in maps functions into functions, but the results in this section apply to any metric space. We start with a definition of what a contraction mapping is.

Definition 34 *Let (S,d) be a metric space and $T : S \to S$ be a function mapping S into itself. The function T is a contraction mapping if there exists a number $\beta \in (0,1)$ satisfying*

$$d(Tx,Ty) \leq \beta d(x,y) \text{ for all } x,y \in S$$

The number β is called the modulus of the contraction mapping

A geometric example of a contraction mapping for $S = [0,1], d(x,y) = |x-y|$ is contained in SLP, p. 50. Note that a function that is a contraction mapping is automatically a continuous function, as the next lemma shows

Lemma 35 *Let (S,d) be a metric space and $T : S \to S$ be a function mapping S into itself. If T is a contraction mapping, then T is continuous.*

Proof. Remember from the definition of continuity we have to show that for all $s_0 \in S$ and all $\varepsilon > 0$ there exists a $\delta(\varepsilon, s_0)$ such that whenever $s \in S, d(s,s_0) < \delta(\varepsilon, s_0)$, then $d(Ts,Ts_0) < \varepsilon$. Fix an arbitrary $s_0 \in S$ and $\varepsilon > 0$ and pick $\delta(\varepsilon, s_0) = \varepsilon$. Then

$$d(Ts,Ts_0) \leq \beta d(s,s_0) \leq \beta\delta(\varepsilon,s_0) = \beta\varepsilon < \varepsilon$$

■

We now can state and prove the contraction mapping theorem. Let by $v_n = T^n v_0 \in S$ denote the element in S that is obtained by applying the operator T n-times to v_0, i.e. the n-th element in the sequence starting with an arbitrary v_0 and defined recursively by $v_n = Tv_{n-1} = T(Tv_{n-2}) = \cdots = T^n v_0$. Then we have

Theorem 36 *Let (S,d) be a complete metric space and suppose that $T : S \to S$ is a contraction mapping with modulus β. Then a) the operator T has exactly one fixed point $v^* \in S$ and b) for any $v_0 \in S$, and any $n \in \mathbf{N}$ we have*

$$d(T^n v_0, v^*) \leq \beta^n d(v_0, v^*)$$

A few remarks before the proof. Part a) of the theorem tells us that there is a $v^* \in S$ satisfying $v^* = Tv^*$ and that there is only one such $v^* \in S$. Part b) asserts that from any starting guess v_0, the sequence $\{v_n\}_{n=0}^{\infty}$ as defined recursively above converges to v^* at a geometric rate of β. This last part is important for computational purposes as it makes sure that we, by repeatedly applying T to any (as crazy as can be) initial guess $v_0 \in S$, will eventually converge to the unique fixed point and it gives us a lower bound on the speed of convergence. But now to the proof.

Proof. First we prove part a) Start with an arbitrary v_0. As our candidate for a fixed point we take $v^* = \lim_{n\to\infty} v_n$. We first have to establish that the sequence $\{v_n\}_{n=0}^{\infty}$ in fact converges to a function v^*. We then have to show that this v^* satisfies $v^* = Tv^*$ and we then have to show that there is no other $\hat{v}$ that also satisfies $\hat{v} = T\hat{v}$

Since by assumption T is a contraction

$$\begin{aligned} d(v_{n+1}, v_n) &= d(Tv_n, Tv_{n-1}) \leq \beta d(v_n, v_{n-1}) \\ &= \beta d(Tv_{n-1}, Tv_{n-2}) \leq \beta^2 d(v_{n-1}, v_{n-2}) \\ &= \cdots = \beta^n d(v_1, v_0) \end{aligned}$$

where we used the way the sequence $\{v_n\}_{n=0}^{\infty}$ was constructed, i.e. the fact that $v_{n+1} = Tv_n$. For any $m > n$ it then follows from the triangle inequality that

$$\begin{aligned} d(v_m, v_n) &\leq d(v_m, v_{m-1}) + d(v_{m-1}, v_n) \\ &\leq d(v_m, v_{m-1}) + d(v_{m-1}, v_{m-2}) + \cdots + d(v_{n+1}, v_n) \\ &\leq \beta^m d(v_1, v_0) + \beta^{m-1} d(v_1, v_0) + \cdots \beta^n d(v_1, v_0) \\ &= \beta^n \left(\beta^{m-n-1} + \cdots + \beta + 1\right) d(v_1, v_0) \\ &\leq \frac{\beta^n}{1-\beta} d(v_1, v_0) \end{aligned}$$

By making n large we can make $d(v_m, v_n)$ as small as we want. Hence the sequence $\{v_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since (S, d) is a complete metric space, the sequence converges in S and therefore $v^* = \lim_{n\to\infty} v_n$ is well-defined.

Now we establish that v^* is a fixed point of T, i.e. we need to show that $Tv^* = v^*$. But

$$Tv^* = T\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} T(v_n) = \lim_{n\to\infty} v_{n+1} = v^*$$

Note that the fact that $T\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} T(v_n)$ follows from the continuity of T.[3]

[3]Almost by definition. Since T is continuous for every $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such

Now we want to prove that the fixed point of T is unique. Suppose there exists another $\hat{v} \in S$ such that $\hat{v} = T\hat{v}$ and $\hat{v} \neq v^*$. Then there exists $c > 0$ such that $d(\hat{v}, v^*) = a$. But

$$0 < a = d(\hat{v}, v^*) = d(T\hat{v}, Tv^*) \leq \beta d(\hat{v}, v^*) = \beta a$$

a contradiction. Here the second equality follows from the fact that we assumed that both $\hat{v}, v^*$ are fixed points of T and the inequality follows from the fact that T is a contraction.

We prove part b) by induction. For $n = 0$ (using the convention that $T^0 v = v$) the claim automatically holds. Now suppose that

$$d(T^k v_0, v^*) \leq \beta^k d(v_0, v^*)$$

We want to prove that

$$d(T^{k+1} v_0, v^*) \leq \beta^{k+1} d(v_0, v^*)$$

But

$$d(T^{k+1} v_0, v^*) = d(T\left(T^k v_0\right), Tv^*) \leq \beta d(T^k v_0, v^*) \leq \beta^{k+1} d(v_0, v^*)$$

where the first inequality follows from the fact that T is a contraction and the second follows from the induction hypothesis. ■

The following corollary, which I will state without proof, will be very useful in establishing properties (such as continuity, monotonicity, concavity) of the unique fixed point v^* and the associated policy correspondence.

Corollary 37 *Let (S, d) be a complete metric space, and let $T : S \to S$ be a contraction mapping with fixed point $v \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$, then $v \in S'$. If in addition $T(S') \subseteq S'' \subseteq S'$, then $v \in S''$.*

The Contraction Theorem is is extremely useful in order to establish that our functional equation of interest has a unique fixed point. It is, however, not very operational as long as we don't know how to determine whether a given operator is a contraction mapping. There is some good news, however.

that $d(v_n - v^*) < \delta(\varepsilon)$ implies $d(T(v_n) - T(v^*)) < \varepsilon$. Hence the sequence $\{T(v_n)\}_{n=0}^{\infty}$ converges and $\lim_{n\to\infty} T(v_n)$ is well-defined. We showed that $\lim_{n\to\infty} v_n = v^*$. Hence both $\lim_{n\to\infty} T(v_n)$ and $\lim_{n\to\infty} v_n$ are well-defined. Then obviously $\lim_{n\to\infty} T(v_n) = T(v^*) = T(\lim_{n\to\infty} v_n)$.

Blackwell, in 1965 provided sufficient conditions for an operator to be a contraction mapping. It turns out that these conditions can be easily checked in a lot of applications. Since they are only sufficient however, failure of these conditions does not imply that the operator is not a contraction. In these cases we just have to look somewhere else. Here is Blackwell's theorem.

Theorem 38 *Let $X \subseteq \mathbf{R}^L$ and $B(X)$ be the space of bounded functions $f : X \to \mathbf{R}$ with the d being the* sup*-norm. Let $T : B(X) \to B(X)$ be an operator satisfying*

1. *Monotonicity: If $f, g \in B(X)$ are such that $f(x) \leq g(x)$ for all $x \in X$, then $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.*

2. *Discounting: Let the function $f + a$, for $f \in B(X)$ and $a \in \mathbf{R}_+$ be defined by $(f + a)(x) = f(x) + a$ (i.e. for all x the number a is added to $f(x)$). There exists $\beta \in (0, 1)$ such that for all $f \in B(X), a \geq 0$ and all $x \in X$*
$$[T(f + a)](x) \leq [Tf](x) + \beta a$$

   If these two conditions are satisfied, then the operator T is a contraction with modulus β.

Proof. In terms of notation, if $f, g \in B(X)$ are such that $f(x) \leq g(x)$ for all $x \in X$, then we write $f \leq g$. We want to show that if the operator T satisfies conditions 1. and 2. then there exists $\beta \in (0, 1)$ such that for all $f, g \in B(X)$ we have that $d(Tf, Tg) \leq \beta d(f, g)$.

Fix $x \in X$. Then $f(x) - g(x) \leq \sup_{y \in X} |f(y) - g(y)|$. But this is true for all $x \in X$. So using our notation we have that $f \leq g + d(f, g)$ (which means that for any value of $x \in X$, adding the constant $d(f, g)$ to $g(x)$ gives something bigger than $f(x)$.

But from $f \leq g + d(f, g)$ it follows by monotonicity that

$$\begin{aligned} Tf &\leq T[g + d(f, g)] \\ &\leq Tg + \beta d(f, g) \end{aligned}$$

where the last inequality comes from discounting. Hence we have

$$Tf - Tg \leq \beta d(f, g)$$

Switching the roles of f and g around we get

$$-(Tf - Tg) \leq \beta d(g, f) = \beta d(f, g)$$

(by symmetry of the metric). Combining yields

$$\begin{aligned} (Tf)(x) - (Tg)(x) &\leq \beta d(f, g) \text{ for all } x \in X \\ (Tg)(x) - (Tf)(x) &\leq \beta d(f, g) \text{ for all } x \in X \end{aligned}$$

Therefore

$$\sup_{x \in X} |(Tf)(x) - (Tg)(x)| = d(Tf, Tg) \leq \beta d(f, g)$$

and T is a contraction mapping with modulus β. ■

Note that do not require the functions in $B(X)$ to be continuous. It is straightforward to prove that $(B(X), d)$ is a complete metric space once we proved that $(B(X), d)$ is a complete metric space. Also note that we could restrict ourselves to continuous and bounded functions and Blackwell's theorem obviously applies. Note however that Blackwells theorem requires the metric space to be a space of functions, so we lose generality as compared to the Contraction mapping theorem (which is valid for any complete metric space). But for our purposes it is key that, once Blackwell's conditions are verified we can invoke the CMT to argue that our functional equation of interest has a unique solution that can be obtained by repeated iterations on the operator T.

We can state an alternative version of Blackwell's theorem

Theorem 39 *Let $X \subseteq \mathbf{R}^L$ and $B(X)$ be the space of bounded functions $f : X \to \mathbf{R}$ with the d being the* sup-*norm. Let $T : B(X) \to B(X)$ be an operator satisfying*

1. *Monotonicity: If $f, g \in B(X)$ are such that $f(x) \leq g(x)$ for all $x \in X$, then $(Tf)(x) \geq (Tg)(x)$ for all $x \in X$.*
2. *Discounting: Let the function $f + a$, for $f \in B(X)$ and $a \in \mathbf{R}_+$ be defined by $(f + a)(x) = f(x) + a$ (i.e. for all x the number a is added to $f(x)$). There exists $\beta \in (0, 1)$ such that for all $f \in B(X), a \geq 0$ and all $x \in X$*

$$[T(f - a)](x) \leq [Tf](x) + \beta a$$

If these two conditions are satisfied, then the operator T is a contraction with modulus β.

The proof is identical to the first theorem and hence omitted.

As an application of the mathematical structure we developed let us look back at the neoclassical growth model. The operator T corresponding to our functional equation was

$$Tv(k) = \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta v(k')\}$$

Define as our metric space $(B[0,\infty), d)$ the space of bounded functions on $[0,\infty)$ with d being the sup-norm. We want to argue that this operator has a unique fixed point and we want to apply Blackwell's theorem and the CMT. So let us verify that all the hypotheses for Blackwell's theorem are satisfied.

1. First we have to verify that the operator T maps $B[0,\infty)$ into itself (this is very often forgotten). So if we take v to be bounded, since we assumed that U is bounded, then Tv is bounded. Note that you may be in big trouble here if U is not bounded.[4]

2. How about monotonicity. It is obvious that this is satisfied. Suppose $v \leq w$. Let by $g_v(k)$ denote an optimal policy (need not be unique) corresponding to v. Then for all $k \in (0,\infty)$

$$\begin{aligned} Tv(k) &= U(f(k) - g_v(k)) + \beta v(g_v(k)) \\ &\leq U(f(k) - g_v(k)) + \beta w(g_v(k)) \\ &\leq \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta w(k')\} \\ &= Tw(k) \end{aligned}$$

Even by applying the policy $g_v(k)$ (which need not be optimal for the situation in which the value function is w) gives higher $Tw(k)$ than $Tv(k)$. Choosing the policy for w optimally does only improve the value $(Tv)(k)$.

[4]Somewhat surprisingly, in many applications the problem is that u is not bounded below; unboundedness from above is sometimes easy to deal with.

We made the assumption that $f \in C^2$ $f' > 0, f'' < 0, \lim_{k \searrow 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 1 - \delta$. Hence there exists a unique $\hat{k}$ such that $f(\hat{k}) = \hat{k}$. Hence for all $k_t > \hat{k}$ we have $k_{t+1} \leq f(k_t) < k_t$. Therefore we can effectively restrict ourselves to capital stocks in the set $[0, \max(k_0, \hat{k})]$. Hence, even if u is not bounded above we have that for all feasible paths policies $u(f(k) - k') \leq u(f(\max(k_0, \hat{k})) < \infty$, and hence by sticking a function v into the operator that is bounded above, we get a Tv that is bounded above. Lack of boundedness from below is a much harder problem in general.

3. Discounting. This also straightforward

$$\begin{aligned} T(v+a)(k) &= \max_{0\leq k'\leq f(k)} \{U(f(k)-k') + \beta(v(k')+a)\} \\ &= \max_{0\leq k'\leq f(k)} \{U(f(k)-k') + \beta v(k')\} + \beta a \\ &= Tv(k) + \beta a \end{aligned}$$

Hence the neoclassical growth model with bounded utility satisfies the Sufficient conditions for a contraction and there is a unique fixed point to the functional equation that can be computed from any starting guess v_0 be repeated application of the T-operator.

One can also prove some theoretical properties of the Howard improvement algorithm using the Contraction Mapping Theorem and Blackwell's conditions. Even though we could state the results in much generality, we will confine our discussion to the neoclassical growth model. Remember that the Howard improvement algorithm iterates on feasible policies [TBC]

4.4 The Theorem of the Maximum

An important theorem is the theorem of the maximum. It will help us to establish that, if we stick a continuous function f into our operator T, the resulting function Tf will also be continuous and the optimal policy function will be continuous in an appropriate sense.

We are interested in problems of the form

$$h(x) = \max_{y\in\Gamma(x)} \{f(x,y)\}$$

The function h gives the value of the maximization problem, conditional on the state x. We define

$$G(x) = \{y \in \Gamma(x) : f(x,y) = h(x)\}$$

Hence G is the set of all choices y that attain the maximum of f, given the state x, i.e. $G(x)$ is the set of argmax'es. Note that $G(x)$ need not be single-valued.

In the example that we study the function f will consist of the sum of the current return function r and the continuation value v and the constraint set describes the resource constraint. The theorem of the maximum is also widely used in microeconomics. There, most frequently x consists of prices and income, f is the (static) utility function, the function h is the indirect utility function, Γ is the budget set and G is the set of consumption bundles that maximize utility at $x = (p, m)$.

Before stating the theorem we need a few definitions. Let X, Y be arbitrary sets (in what follows we will be mostly concerned with the situations in which X and Y are subsets of Euclidean spaces. A correspondence $\Gamma : X \Rightarrow Y$ maps each element $x \in X$ into a subset $\Gamma(x)$ of Y. Hence the image of the point x under Γ may consist of more than one point (in contrast to a function, in which the image of x always consists of a singleton).

Definition 40 *A compact-valued correspondence $\Gamma : X \Rightarrow Y$ is upper-hemicontinuous at a point x if $\Gamma(x) \neq \varnothing$ and if for all sequences $\{x_n\}$ in X converging to some $x \in X$ and all sequences $\{y_n\}$ in Y such that $y_n \in \Gamma(x_n)$ for all n, there exists a convergent subsequence of $\{y_n\}$ that converges to some $y \in \Gamma(x)$. A correspondence is upper-hemicontinuous if it is upper-hemicontinuous at all $x \in X$.*

A few remarks: by talking about convergence we have implicitly assumed that X and Y (together with corresponding metrics) are metric spaces. Also, a correspondence is compact-valued, if for all $x \in X, \Gamma(x)$ is a compact set. Also this definition requires Γ to be compact-valued. With this additional requirement the definition of upper hemicontinuity actually corresponds to the definition of a correspondence having a closed graph. See, e.g. Mas-Colell et al. p. 949-950 for details.

Definition 41 *A correspondence $\Gamma : X \Rightarrow Y$ is lower-hemicontinuous at a point x if $\Gamma(x) \neq \varnothing$ and if for every $y \in \Gamma(x)$ and every sequence $\{x_n\}$ in X converging to $x \in X$ there exists $N \geq 1$ and a sequence $\{y_n\}$ in Y converging to y such that $y_n \in \Gamma(x_n)$ for all $n \geq N$. A correspondence is lower-hemicontinuous if it is lower-hemicontinuous at all $x \in X$.*

Definition 42 *A correspondence $\Gamma : X \Rightarrow Y$ is continuous if it is both upper-hemicontinuous and lower-hemicontinuous.*

Note that a single-valued correspondence (i.e. a function) that is upper-hemicontinuous is continuous. Now we can state the theorem of the maximum.

Theorem 43 *Let $X \subseteq \mathbf{R}^L$ and $Y \subseteq \mathbf{R}^M$, let $f : X \times Y \to \mathbf{R}$ be a continuous function, and let $\Gamma : X \rightrightarrows Y$ be a compact-valued and continuous correspondence. Then $h : X \to \mathbf{R}$ is continuous and $G : X \to Y$ is nonempty, compact-valued and upper-hemicontinuous.*

The proof is somewhat tedious and omitted here (you probably have done it in micro anyway).

Chapter 5

Dynamic Programming

5.1 The Principle of Optimality

In the last section we showed that under certain conditions, the functional equation (FE)

$$v(x) = \sup_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\}$$

has a unique solution which is approached from any initial guess v_0 at geometric speed. What we were really interested in, however, was a problem of sequential form (SP)

$$\begin{aligned} w(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ s.t.\ x_{t+1} &\in \Gamma(x_t) \\ x_0 &\in X \text{ given} \end{aligned}$$

Note that I replaced max with sup since we have not made any assumptions so far that would guarantee that the maximum in either the functional equation or the sequential problem exists. In this section we want to find out under what conditions the functions v and w are equal and under what conditions optimal sequential policies $\{x_{t+1}\}_{t=0}^{\infty}$ are equivalent to optimal policies $y = g(x)$ from the recursive problem, i.e. under what conditions the principle of optimality holds. It turns out that these conditions are very mild.

In this section I will try to state the main results and make clear what they mean; I will not prove the results. The interested reader is invited to consult Stokey and Lucas or Bertsekas. Unfortunately, to make our results precise additional notation is needed. Let X be the set of possible values that the state x can take. X may be a subset of a Euclidean space, a set of functions or something else; we need not be more specific at this point. The correspondence $\Gamma : X \Rightarrow X$ describes the feasible set of next period's states y, given that today's state is x. The graph of Γ, A is defined as

$$A = \{(x, y) \in X \times X : y \in \Gamma(x)\}$$

The period return function $F : A \rightarrow \mathbf{R}$ maps the set of all feasible combinations of today's and tomorrow's state into the reals. So the fundamentals of our analysis are (X, F, β, Γ). For the neoclassical growth model F and β describe preferences and X, Γ describe the technology.

We call any sequence of states $\{x_t\}_{t=0}^{\infty}$ a plan. For a given initial condition x_0, the set of feasible plans $\Pi(x_0)$ from x_0 is defined as

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} : x_{t+1} \in \Gamma(x_t)\}$$

Hence $\Pi(x_0)$ is the set of sequences that, for a given initial condition, satisfy all the feasibility constraints of the economy. We will denote by $\bar{x}$ a generic element of $\Pi(x_0)$. The two assumptions that we need for the principle of optimality are basically that for any initial condition x_0 the social planner (or whoever solves the problem) has at least one feasible plan and that the total return (the total utility, say) from all feasible plans can be evaluated. That's it. More precisely we have

Assumption 1: $\Gamma(x)$ is nonempty for all $x \in X$

Assumption 2: For all initial conditions x_0 and all feasible plans $\bar{x} \in \Pi(x_0)$

$$\lim_{n\rightarrow\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

exists (although it may be $+\infty$ or $-\infty$).

Assumption 1 does not require much discussion: we don't want to deal with an optimization problem in which the decision maker (at least for some initial conditions) can't do anything. Assumption 2 is more subtle. There are various ways to verify that assumption 2 is satisfied, i.e. various sets of sufficient conditions for assumption 2 to hold. Assumption 2 holds if

1. F is bounded and $\beta \in (0, 1)$. Note that boundedness of F is not enough. Suppose $\beta = 1$ and $F(x_t, x_{t+1}) = \begin{cases} 1 \text{ if } t \text{ even} \\ -1 \text{ if } t \text{ odd} \end{cases}$ Obviously F

is bounded, but since $\sum_{t=0}^{n}\beta^t F(x_t,x_{t+1}) = \begin{cases} 1 \text{ if } n \text{ even} \\ 0 \text{ if } n \text{ odd} \end{cases}$, the limit in assumption 2 does not exist. If $\beta \in (0,1)$ then $\sum_{t=0}^{n}\beta^t F(x_t,x_{t+1}) = \begin{cases} 1-\beta^{\frac{n}{2}}+\beta^n \text{ if } n \text{ even} \\ 1-\beta^{\frac{n}{2}} \text{ if } n \text{ odd} \end{cases}$ and therefore $\lim_{n\to\infty}\sum_{t=0}^{n}\beta^t F(x_t,x_{t+1})$ exists and equals 1. In general the joint assumption that F is bounded and $\beta \in (0,1)$ implies that the sequence $y_n = \sum_{t=0}^{n}\beta^t F(x_t,x_{t+1})$ is Cauchy and hence converges. In this case $\lim y_n = y$ is obviously finite.

2. Define $F^+(x,y) = \max\{0, F(x,y)\}$ and $F^-(x,y) = \max\{0,-F(x,y)\}$. Then assumption 2 is satisfied if for all $x_0 \in X$, all $\bar{x} \in \Pi(x_0)$, either

$$\begin{aligned} \lim_{n\to\infty}\sum_{t=0}^{n}\beta^t F^+(x_t,x_{t+1}) &< +\infty \text{ or} \\ \lim_{n\to\infty}\sum_{t=0}^{n}\beta^t F^-(x_t,x_{t+1}) &< +\infty \end{aligned}$$

or both. For example, if $\beta \in (0,1)$ and F is bounded above, then the first condition is satisfied, if $\beta \in (0,1)$ and F is bounded below then the second condition is satisfied.

3. Assumption 2 is satisfied if for every $x_0 \in X$ and every $\bar{x} \in \Pi(x_0)$ there are numbers (possibly dependent on $x_0, \bar{x}$) $\theta \in (0, \frac{1}{\beta})$ and $0 < c < +\infty$ such that for all t

$$F(x_t,x_{t+1}) \leq c\theta^t$$

Hence F need not be bounded in any direction for assumption 2 to be satisfied. As long as the returns from the sequences do not grow too fast (at rate higher than $\frac{1}{\beta}$) we are still fine .

I would conclude that assumption 2 is rather weak (I can't think of any interesting economic example where assumption1 is violated, but let me know if you come up with one). A final piece of notation and we are ready to state some theorems.

Define the sequence of functions $u_n : \Pi(x_0) \to \mathbf{R}$ by

$$u_n(\bar{x}) = \sum_{t=0}^{n}\beta^t F(x_t,x_{t+1})$$

For each feasible plan u_n gives the total discounted return (utility) up until period n. If assumption 2 is satisfied, then the function $u : \Pi(x_0) \to \bar{\mathbf{R}}$

$$u(\bar{x}) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

is also well-defined, since under assumption 2 the limit exists. The range of u is $\bar{\mathbf{R}}$, the extended real line, i.e. $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ since we allowed the limit to be plus or minus infinity. From the definition of u it follows that under assumption 2

$$w(x_0) = \sup_{\bar{x}\in\Pi(x_0)} u(\bar{x})$$

Note that by construction, whenever w exists, it is unique (since the supremum of a set is always unique). Also note that the way I have defined w above only makes sense under assumption 1. and 2., otherwise w is not well-defined.

We have the following theorem, stating the principle of optimality.

Theorem 44 *Suppose (X, Γ, F, β) satisfy assumptions 1. and 2. Then*

1. *the function w satisfies the functional equation (FE)*
2. *if for all $x_0 \in X$ and all $\bar{x} \in \Pi(x_0)$ a solution v to the functional equation (FE) satisfies*

$$\lim_{n\to\infty} \beta^n v(x_n) = 0 \tag{5.1}$$

then $v = w$

I will skip the proof, but try to provide some intuition. The first result states that the supremum function from the sequential problem (which is well-defined under assumption 1. and 2.) solves the functional equation. This result, although nice, is not particularly useful for us. We are interested in solving the sequential problem and in the last section we made progress in solving the functional equation (not the other way around).

But result 2. is really key. It states a condition under which a solution to the functional equation (which we know how to compute) is a solution to the sequential problem (the solution of which we desire). Note that the functional equation (FE) may (or may not) have several solution. We haven't made enough assumptions to use the CMT to argue uniqueness. However, only one of these potential several solutions can satisfy (5.1) since if it does, the theorem tells us that it has to equal the supremum function w (which is necessarily unique). The condition (5.1) is somewhat hard to interpret (and SLP don't even try), but think about the following. We saw in the first lecture that for infinite-dimensional optimization problems like the one in (SP) a transversality condition was often necessary and (even more often) sufficient (jointly with the Euler equation). The transversality condition rules out as suboptimal plans that postpone too much utility into the distant future. There is no equivalent condition for the recursive formulation (as this formulation is basically a two period formulation, today vs. everything from tomorrow onwards). Condition (5.1) basically requires that the continuation utility from date n onwards, discounted to period 0, should vanish in the time limit. In other words, this puts an upper limit on the growth rate of continuation utility, which seems to substitute for the TVC. It is not clear to me how to make this intuition more rigorous, though.

A simple, but quite famous example, shows that the condition (5.1) has some bite. Consider the following consumption problem of an infinitely lived household. The household has initial wealth $x_0 \in X = \mathbf{R}$. He can borrow or lend at a gross interest rate $1 + r = \frac{1}{\beta} > 1$. So the price of a bond that pays off one unit of consumption is $q = \beta$. There are no borrowing constraints, so the sequential budget constraint is

$$c_t + \beta x_{t+1} \leq x_t$$

and the nonnegativity constraint on consumption, $c_t \geq 0$. The household values discounted consumption, so that her maximization problem is

$$\begin{aligned} w(x_0) &= \sup_{\{(c_t, x_{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t c_t \\ s.t.\ 0 &\leq c_t \leq x_t - \beta x_{t+1} \\ & x_0 \text{ given} \end{aligned}$$

Since there are no borrowing constraint, the consumer can assure herself infinite utility by just borrowing an infinite amount in period 0 and then rolling over the debt by even borrowing more in the future. Such a strategy is called a Ponzi-scheme -see the hand-out. Hence the supremum function equals $w(x_0) = +\infty$ for all $x_0 \in X$. Now consider the recursive formulation (we denote by x current period wealth x_t, by y next period's wealth and substitute out for consumption $c_t = x_t - \beta x_{t+1}$ (which is OK given monotonicity of preferences)

$$v(x) = \sup_{y \leq \frac{x}{\beta}} \{x - \beta y + \beta v(y)\}$$

Obviously the function $w(x) = +\infty$ satisfies this functional equation (just plug in w on the right side, since for all x it is optimal to let y tend to $-\infty$ and hence $v(x) = +\infty$. This should be the case from the first part of the previous theorem. But the function $\check{v}(x) = x$ satisfies the functional equation, too. Using it on the right hand side gives, for an arbitrary $x \in X$

$$\sup_{y \leq \frac{x}{\beta}} \{x - \beta y + \beta y\} = \sup_{y \leq \frac{x}{\beta}} x = x = \check{v}(x)$$

Note, however that the second part of the preceding theorem does not apply for $\check{v}$ since the sequence $\{x_n\}$ defined by $x_n = \frac{x_0}{\beta^n}$ is a feasible plan from $x_0 > 0$ and

$$\lim_{n \to \infty} \beta^n v(x_n) = \lim_{n \to \infty} \beta^n x_n = x_0 > 0$$

Note however that the second part of the theorem gives only a sufficient condition for a solution v to the functional equation being equal to the supremum function from (SP), but not a necessary condition. Also w itself does not satisfy the condition, but is evidently equal to the supremum function. So whenever we can use the CMT (or something equivalent) we have to be aware of the fact that there may be several solutions to the functional equation, but at most one the several is the function that we look for.

Now we want to establish a similar equivalence between the sequential problem and the recursive problem with respect to the optimal policies/plans. The first observation. Solving the functional equation gives us optimal policies $y = g(x)$ (note that g need not be a function, but could be a correspondence). Such an optimal policy induces a feasible plan $\{\hat{x}_{t+1}\}_{t=0}^{\infty}$ in the following fashion: $x_0 = \hat{x}_0$ is an initial condition, $\hat{x}_1 \in g(\hat{x}_0)$ and recursively $\hat{x}_{t+1} = g(\hat{x}_t)$. The basic question is how a plan constructed from a solution to the functional equation relates to a plan that solves the sequential problem. We have the following theorem.

Theorem 45 *Suppose (X, Γ, F, β) satisfy assumptions 1. and 2.*

1. *Let $\bar{x} \in \Pi(x_0)$ be a feasible plan that attains the supremum in the sequential problem. Then for all $t \geq 0$*

$$w(\bar{x}_t) = F(\bar{x}_t, \bar{x}_{t+1}) + \beta w(\bar{x}_{t+1})$$

2. *Let $\hat{x} \in \Pi(x_0)$ be a feasible plan satisfying, for all $t \geq 0$*

$$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$$

and additionally[1]

$$\limsup_{t\to\infty} \beta^t w(\hat{x}_t) \leq 0 \tag{5.2}$$

Then $\hat{x}$ attains the supremum in (SP) for the initial condition x_0.

What does this result say? The first part says that any optimal plan in the sequence problem, together with the supremum function w as value function satisfies the functional equation for all t. Loosely it says that any optimal plan from the sequential problem is an optimal policy for the recursive problem (once the value function is the right one).

Again the second part is more important. It says that, for the "right" fixed point of the functional equation w the corresponding policy g generates a plan $\hat{x}$ that solves the sequential problem if it satisfies the additional limit condition. Again we can give this condition a loose interpretation as standing in for a transversality condition. Note that for any plan $\{\hat{x}_t\}$ generated from a policy g associated with a value function v that satisfies (5.1) condition (5.2) is automatically satisfied. From (5.1) we have

$$\lim_{t\to\infty} \beta^t v(x_t) = 0$$

for any feasible $\{x_t\} \in \Pi(x_0)$, all x_0. Also from Theorem 32 $v = w$. So for any plan $\{\hat{x}_t\}$ generated from a policy g associated with $v = w$ we have

$$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$$

and since $\lim_{t\to\infty} \beta^t v(\hat{x}_t)$ exists and equals to 0 (since v satisfies (5.1)), we have

$$\limsup_{t\to\infty} \beta^t v(\hat{x}_t) = 0$$

[1]The limit superior of a bounded sequence $\{x_n\}$ is the infimum of the set V of real numbers v such that only a finite number of elements of the sequence strictly exceed v. Hence it is the largest cluster point of the sequence $\{x_n\}$.

and hence (5.2) is satisfied. But Theorem 33.2 is obviously not redundant as there may be situations in which Theorem 32.2 does not apply but 33.2 does. Let us look at the following example, a simple modification of the saving problem from before. Now however we impose a borrowing constraint of zero.

$$\begin{aligned} w(x_0) &= \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (x_t - \beta x_{t+1}) \\ s.t.\ 0 &\leq x_{t+1} \leq \frac{x_t}{\beta} \\ & x_0 \text{ given} \end{aligned}$$

Writing out the objective function yields

$$\begin{aligned} w_0(x_0) &= (x_0 - \beta x_1) + (x_1 - \beta x_2) + \ldots \\ &= x_0 \end{aligned}$$

Now consider the associated functional equation

$$v(x) = \max_{0 \leq x' \leq \frac{x}{\beta}} \{x - \beta x' + v(x')\}$$

Obviously one solution of this functional equation is $v(x) = x$ and by Theorem 32.1 is rightly follows that w satisfies the functional equation. However, for v condition (5.1) fails, as the feasible plan defined by $x_t = \frac{x_0}{\beta^t}$ shows. Hence Theorem 32.2 does not apply and we can't conclude that $v = w$ (although we have verified it directly, there may be other examples for which this is not so straightforward). Still we can apply Theorem 33.2 to conclude that certain plans are optimal plans. Let $\{\hat{x}_t\}$ be defined by $\hat{x}_0 = x_0, \hat{x}_t = 0$ all $t > 0$. Then

$$\limsup_{t\to\infty} \beta^t w(\hat{x}_t) = 0$$

and we can conclude by Theorem 33.2 that this plan is optimal for the sequential problem. There are tons of other plans for which we can apply the same logic to shop that they are optimal, too (which shows that we obviously can't make any claim about uniqueness). To show that condition (5.2) has some bite consider the plan defined by $\hat{x}_t = \frac{x_0}{\beta^t}$. Obviously this is a feasible plan satisfying

$$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$$

but since for all $x_0 > 0$

$$\limsup_{t \to \infty} \beta^t w(\hat{x}_t) = x_0 > 0$$

Theorem 33.2 does not apply and we can't conclude that $\{\hat{x}_t\}$ is optimal (as in fact this plan is not optimal).

So basically we have a prescription what to do once we solved our functional equation: pick the right fixed point (if there are more than one, check the limit condition to find the right one, if possible) and then construct a plan from the policy corresponding to this fixed point. Check the limit condition to make sure that the plan so constructed is indeed optimal for the sequential problem. Done.

Note, however, that so far we don't know anything about the number (unless the CMT applies) and the shape of fixed point to the functional equation. This is not quite surprising given that we have put almost no structure onto our economy. By making further assumptions one obtains sharper characterizations of the fixed point(s) of the functional equation and thus, in the light of the preceding theorems, about the solution of the sequential problem.

5.2 Dynamic Programming with Bounded Returns

Again we look at a functional equation of the form

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

We will now assume that $F : X \times X$ is bounded and $\beta \in (0, 1)$. We will make the following two assumptions throughout this section

Assumption 3: X is a convex subset of $\mathbf{R}^L$ and the correspondence $\Gamma : X \Rightarrow X$ is nonempty, compact-valued and continuous.

Assumption 4: The function $F : A \to \mathbf{R}$ is continuous and bounded, and $\beta \in (0, 1)$

We immediately get that assumptions 1. and 2. are satisfied and hence the theorems of the previous section apply. Define the policy correspondence connected to any solution to the functional equation as

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

and the operator T on $C(X)$

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

Here $C(X)$ is the space of bounded continuous functions on X and we use the sup-metric as metric. Then we have the following

Theorem 46 *Under Assumptions 3. and 4. the operator T maps $C(X)$ into itself. T has a unique fixed point v and for all $v_0 \in C(X)$*

$$d(T^n v_0, v) \leq \beta^n d(v_0, v)$$

The policy correspondence G belonging to v is compact-valued and upper-hemicontinuous

Now we add further assumptions on the structure of the return function F, with the result that we can characterize the unique fixed point of T better.

Assumption 5: For fixed y, $F(., y)$ is strictly increasing in each of its L components.

Assumption 6: Γ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$.

The result we get out of these assumptions is strict monotonicity of the value function.

Theorem 47 *Under Assumptions 3. to 6. the unique fixed point v of T is strictly increasing.*

We have a similar result in spirit if we make assumptions about the curvature of the return function and the convexity of the constraint set.

Assumption 7: F is strictly concave, i.e. for all $(x, y), (x', y') \in A$ and $\theta \in (0, 1)$

$$F[\theta(x, y) + (1 - \theta)(x', y')] \geq \theta F(x, y) + (1 - \theta)F(x', y')$$

and the inequality is strict if $x \neq x'$

Assumption 8: Γ is convex in the sense that for $\theta \in [0, 1]$ and $x, x' \in X$, the fact $y \in \Gamma(x), y' \in \Gamma(x')$

$$\theta y + (1 - \theta)y' \in \Gamma(\theta x + (1 - \theta)x')$$

Again we find that the properties assumed about F extend to the value function.

Theorem 48 *Under Assumptions 3.-4. and 7.-8. the unique fixed point of v is strictly concave and the optimal policy is a single-valued continuous function, call it g.*

Finally we state a result about the differentiability of the value function, the famous envelope theorem (some people call it the Benveniste-Scheinkman theorem).

Assumption 9: F is continuously differentiable on the interior of A.

Theorem 49 *Under assumptions 3.-4. and 7.-9. if $x_0 \in int(X)$ and $g(x_0) \in int(\Gamma(x_0))$, then the unique fixed point of T, v is continuously differentiable at x_0 with*

$$\frac{\partial v(x_0)}{\partial x^i} = \frac{\partial F(x_0, g(x_0))}{\partial x^i}$$

where $\frac{\partial v(x_0)}{\partial x^i}$ denotes the derivative of v with respect to its i-th component, evaluated at x_0.

This theorem gives us an easy way to derive Euler equations from the recursive formulation of the neoclassical growth model. Remember the functional equation

$$v(k) = \max_{0 \le k' \le f(k)} U(f(k) - k') + \beta v(k')$$

Taking first order conditions with respect to k' (and ignoring corner solutions) we get

$$U'(f(k) - k') = \beta v'(k')$$

Denote by $k' = g(k)$ the optimal policy. The problem is that we don't know v'. But now we can use Benveniste-Scheinkman to obtain

$$v'(k) = U'(f(k) - g(k))f'(k)$$

Using this in the first order condition we obtain

$$\begin{aligned} U'(f(k) - g(k)) &= \beta v'(k') = \beta U'(f(k') - g(k'))f'(k') \\ &= \beta f'(g(k))U'(f(g(k)) - g(g(k))) \end{aligned}$$

Denoting $k = k_t$, $g(k) = k_{t+1}$ and $g(g(k)) = k_{t+2}$ we obtain our usual Euler equation

$$U'(f(k_t) - k_{t+1}) = \beta f'(k_{t+1})U'(f(k_{t+1}) - k_{t+2})$$

Chapter 6

Models with Risk

In this section we will introduce a basic model with risk and complete financial markets, in order to establish some notation and extend our discussion of efficient economies to this important case. We will also derive two substantive results, namely that risk will be perfectly shared across households (in a sense to be made precise), and that for the pricing of assets the distribution of endowments (incomes) across households is irrelevant. Then, as a first application, we will look at the stochastic neoclassical growth model, which forms the basis for a particular theory of business cycles, the so called "Real Business Cycle" (RBC) theory. In this section we will be a bit loose with our treatment of risk, in that we will not explicitly discuss probability spaces that form the formal basis of our representation of risk.

6.1 Basic Representation of Risk

The basic novelty of models with risk is the formal representation of this risk and the ensuing description of the information structure that agents have. We start with the notion of an event $s_t \in S$. The set $S = \{\eta_1, , \ldots, \eta_N\}$ of possible *events* that can happen in period t is assumed to be finite and the same for all periods t. If there is no room for confusion we use the notation $s_t = 1$ instead of $s_t = \eta_1$ and so forth. For example S may consist of all weather conditions than can happen in the economy, with $s_t = 1$ indicating sunshine in period t, $s_t = 2$ indicating cloudy skies, $s_t = 3$ indicating rain and so forth.[1] As another example, consider the economy from Section 2, but now with random endowments. In each period one of the two agents has endowment 0 and the other has endowment 2, but who has what is random, with $s_t = 1$ indicating that agent 1 has high endowment and $s_t = 2$ indicating that agent 2 has high endowment at period t. The set of possible events for this example is given by $S = \{1, 2\}$

An event history $s^t = (s_0, s_1, \ldots s_t)$ is a vector of length $t+1$ summarizing the realizations of all events up to period t. Formally (and with some abuse of notation), with $S^t = S \times S \times \ldots \times S$ denoting the $t+1$-fold product of S, event history $s^t \in S^t$ lies in the set of all possible event histories of length t.

By $\pi_t(s^t)$ let denote the probability of a particular event history. We assume that $\pi_t(s^t) > 0$ for all $s^t \in S^t$, for all t. For our example economy, if $s^2 = (1, 1, 2)$ then $\pi_t(s^2)$ is the probability that agent 1 has high endowment in period $t = 0$ *and* $t = 1$ *and* agent 2 has high endowment in period 2. Figure 6.1 summarizes the concepts introduced so far, for the case in which $S = \{1, 2\}$ is the set of possible events that can happen in every period. Note that the sets S^t of possible event histories of length t become fairly big very rapidly even when the set of events itself is small, which poses computational problems when dealing with models with risk.

All commodities of our economy, instead of being indexed by time t as before, now also have to be indexed by event histories s^t. In particular, an allocation for the example economy of Section 2, but now with risk, is given by

$$(c^1, c^2) = \{c_t^1(s^t), c_t^2(s^t)\}_{t=0, s^t \in S^t}^{\infty}$$

with the interpretation that $c_t^i(s^t)$ is consumption of agent i in period t if event history s^t has occurred. Note that consumption in period t of agents are allowed to (and in general will) vary with the history of events that have occurred in the past.

Now we are ready to specify to remaining elements of the economy. With respect to endowments, these also take the general form

$$(e^1, e^2) = \{e_t^1(s^t), e_t^2(s^t)\}_{t=0, s^t \in S^t}^{\infty}$$

[1] Technically speaking s_t is a random variable with respect to some underlying probability space $(\Omega, \mathcal{A}, P)$, where Ω is some set of basis events with generic element ω, $\mathcal{A}$ is a sigma algebra on Ω and P is a probability measure.

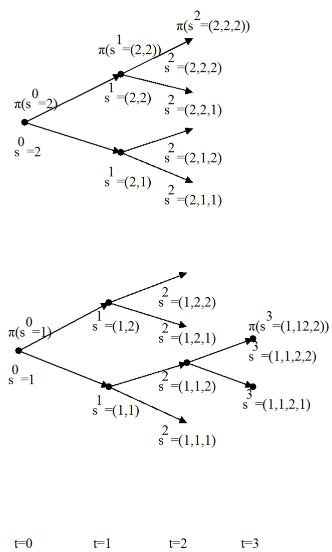

Figure 6.1: Event Tree in Models with Risk

and for the particular example

$$e_t^1(s^t) = \begin{cases} 2 & \text{if } s_t = 1 \\ 0 & \text{if } s_t = 2 \end{cases}$$
$$e_t^2(s^t) = \begin{cases} 0 & \text{if } s_t = 1 \\ 2 & \text{if } s_t = 2 \end{cases}$$

i.e. for the particular example endowments in period t only depend on the realization of the event s_t, not on the entire history. Nothing, however, would prevent us from specifying more general endowment patterns.

Now we specify preferences. We assume that households maximize *expected* lifetime utility where E_0 is the expectation operator at period 0, prior to any realization of risk (in particular the risk with respect to s_0). Given our notation just established, assuming that preferences admit a von-Neumann Morgenstern utility function representation we represent households' preferences by

$$u(c^i) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) U(c_t^i(s^t))$$

This completes our description of this simple stochastic endowment economy.

6.2 Definitions of Equilibrium

Again there are two possible market structures that we can work with. The Arrow-Debreu market structure turns out to be easier than the sequential markets market structure, so we will start with it. Again there is an equivalence theorem that relates the equilibrium of the two markets structures, once we allow the asset market structure for the sequential markets market structure to be rich enough.

6.2.1 Arrow-Debreu Market Structure

As usual with Arrow-Debreu, trade takes place at period 0, *before* any risk has been realized (in particular, before s_0 has been realized). As with allocations, Arrow-Debreu prices have to be indexed by event histories in addition to time, so let $p_t(s^t)$ denote the price of one unit of consumption, quoted at period 0, delivered at period t if (and only if) event history s^t has been realized. Given this notation, the definition of an AD-equilibrium is identical to the case without risk, with the exception that, since goods and prices are not only indexed by time, but also by histories, we have to sum over both time and histories in the individual households' budget constraint.

Definition 50 *A (competitive) Arrow-Debreu equilibrium are prices* $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ *and allocations* $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,2}$ *such that*

1. *Given* $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$, *for* $i=1,2$, $\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ *solves*

$$\max_{\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t\in S^t} \beta^t \pi_t(s^t) U(c_t^i(s^t)) \tag{6.1}$$

s.t.

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} \hat{p}_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t\in S^t} \hat{p}_t(s^t) e_t^i(s^t) \tag{6.2}$$

$$c_t^i(s^t) \geq 0 \text{ for all } t, \text{ all } s^t \in S^t \tag{6.3}$$

2.

$$\hat{c}_t^1(s^t) + \hat{c}_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t) \text{ for all } t, \text{ all } s^t \in S^t \tag{6.4}$$

Note that there is again only one budget constraint, and that the market clearing condition has to hold date by date, event history by event history. Also note that, when computing equilibria, one can normalize the price of only one commodity to 1, and consumption at the same date, but for different event histories are different commodities. That means that if we normalize $p_0(s_0 = 1) = 1$ we can't also normalize $p_0(s_0 = 2) = 1$. Finally, there are no probabilities in the budget constraint. Equilibrium prices will reflect the probabilities of different event histories, but there is no scope for these probabilities in the Arrow-Debreu budget constraint directly.

It is relatively straightforward to characterize equilibrium prices. Taking first order conditions with respect to $c_t^i(s^t)$ and $c_0^i(s_0)$ yields

$$\begin{aligned} \beta^t \pi_t(s^t) U'(c_t^i(s^t)) &= \mu p_t(s^t) \\ \pi_0(s_0) U'(c_0^i(s_0)) &= \mu p_0(s_0) \end{aligned}$$

and combining yields

$$\frac{p_t(s^t)}{p_0(s_0)} = \beta^t \frac{\pi_t(s^t)}{\pi_0(s_0)} \frac{U'(c_t^i(s^t))}{U'(c_0^i(s_0))} \tag{6.5}$$

for all t, s^t and all agents i. This immediately implies that

$$\frac{U'(c_t^1(s^t))}{U'(c_0^1(s_0))} = \frac{U'(c_t^2(s^t))}{U'(c_0^2(s_0))}$$

or

$$\frac{U'(c_t^2(s^t))}{U'(c_t^1(s^t))} = \frac{U'(c_0^2(s_0))}{U'(c_0^1(s_0))} \text{ for all } s^t.$$

for all s^t. That is, the ratio of marginal utilities between the two agents is constant over time and across states of the world. In addition, if households have CRRA period utility, the above equation implies that

$$\left(\frac{c_t^2(s^t)}{c_t^1(s^t)}\right)^{-\sigma} = \left(\frac{c_0^2(s^0)}{c_0^1(s^0)}\right)^{-\sigma}$$

that is, the ratio of consumption between the two agents is constant over time. Denoting the aggregate endowment by

$$e_t(s^t) = \sum_i e_t^i(s^t)$$

the resource constraint then implies that for both agents

$$c_t^i(s^t) = \theta^i e_t(s^t) \tag{6.6}$$

where θ^i is the constant share of aggregate endowment household i consumes. Using this result in equation (6.5) we find, after normalizing $p_0(s_0) = 1$ for the particular s_0 we have chosen, that

$$\begin{aligned} p_t(s^t) &= \beta^t \frac{\pi_t(s^t)}{\pi_0(s_0)} \left(\frac{c_t^i(s^t)}{c_0^i(s_0)}\right)^{-\sigma} \\ &= \beta^t \frac{\pi_t(s^t)}{\pi_0(s_0)} \left(\frac{e_t(s^t)}{e_0(s_0)}\right)^{-\sigma}. \end{aligned} \tag{6.7}$$

Thus the price of consumption at node s^t is declining with t because of discounting, it is the higher the more likely node s^t is realized and is declining in the availability of consumption at this node (as measured by the aggregate endowment $e_t(s^t)$).[2]

[2]We have to be a bit careful with prices at initial nodes $\hat{s}_0 \neq s_0$ (because we can only normalize one price to one). These prices are given by

$$\frac{p_0(\hat{s}_0)}{p_0(s_0)} = \frac{\pi_0(\hat{s}_0)}{\pi_0(s_0)} \left(\frac{e_0(\hat{s}_0)}{e_0(s_0)}\right)^{-\sigma}.$$

Equations (6.6) and (6.7) have important implications. Turning to equation (6.6), it implies that endowment risk is perfectly shared. The only endowment risk that affects consumption of each household i is aggregate risk, that is, fluctuations in the aggregate endowment $e_t(s^t)$. These shocks are born by all households equally, in that consumption of all households falls by the same fraction as the aggregate endowment plummets. In contrast, shocks to individual endowments $e_t^i(s^t)$ that do not affect the aggregate endowment (because household i is small in the aggregate, or because endowments of households i and j are perfectly negatively correlated) in turn do not impact individual consumption since they are perfectly diversified across households. In this sense, the economy exhibits perfect risk sharing (of individual risks).

Equation (6.7) shows that Arrow Debreu equilibrium prices (and thus all other asset prices, as discussed below) only depend the stochastic process for the *aggregate* endowment $e_t(s^t)$, but not on how these endowments are distributed across households. This implies that if we consider an economy with a representative household whose endowment process equals the aggregate endowment process $\{e_t(s^t)\}$ and who has CRRA risk aversion utility with the same coefficient σ as all the households in our economy, then the Arrow Debreu prices (and thus all other asset prices) in the representative agent economy are identical to the ones we have determined in our economy in (6.7). That is, for asset pricing purposes we might as well study the representative agent economy (whose equilibrium allocations are of course trivial to solve in an endowment economy since the representative agent just eats her endowment in every period).

6.2.2 Pareto Efficiency

The definition of Pareto efficiency is identical to that of the certainty case; the first welfare theorem goes through without any changes (in particular, the proof is identical, apart from changes in notation). We state both for completeness

Definition 51 *An allocation* $\{(c_t^1(s^t), c_t^2(s^t))\}_{t=0,s^t\in S^t}^{\infty}$ *is feasible if*

1. $$c_t^i(s^t) \geq 0 \text{ for all } t, \text{ all } s^t \in S^t, \text{ for } i = 1, 2$$

2. $$c_t^1(s^t) + c_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t) \text{ for all } t, \text{ all } s^t \in S^t$$

Definition 52 *An allocation* $\{(c_t^1(s^t), c_t^2(s^t))\}_{t=0,s^t\in S^t}^{\infty}$ *is Pareto efficient if it is feasible and if there is no other feasible allocation* $\{(\tilde{c}_t^1(s^t), \tilde{c}_t^2(s^t))\}_{t=0,s^t\in S^t}^{\infty}$ *such that*

$$\begin{aligned} u(\tilde{c}^i) &\geq u(c^i) \text{ for both } i = 1,2 \\ u(\tilde{c}^i) &> u(c^i) \text{ for at least one } i = 1,2 \end{aligned}$$

Proposition 53 *Let* $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,2}$ *be a competitive equilibrium allocation. Then* $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,2}$ *is Pareto efficient.*

Note that we could have obtained the above characterization of equilibrium allocations and prices from following the Negishi approach, that is, by solving a social planner problem and using the transfer functions to compute the appropriate welfare weights.

6.2.3 Sequential Markets Market Structure

Now let trade take place sequentially in each period (more precisely, in each period, event-history pair). Without risk we allowed trade in consumption and in one-period IOU's. For the equivalence between Arrow-Debreu and sequential markets with risk, this is not enough. We introduce one period contingent IOU's, financial contracts bought in period t that pay out one unit of the consumption good in $t+1$ only for a particular realization of $s_{t+1} = j$ tomorrow.[3] So let $q_t(s^t, s_{t+1} = j)$ denote the price at period t of a contract that pays out one unit of consumption in period $t+1$ if (and only if) tomorrow's event is $s_{t+1} = j$. These contracts are often called Arrow securities, contingent claims or one-period insurance contracts. Let $a_{t+1}^i(s^t, s_{t+1})$ denote the quantities of these Arrow securities bought (or sold) at period t by agent i.

The period t, event history s^t budget constraint of agent i is given by

$$c_t^i(s^t) + \sum_{s_{t+1}\in S} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \leq e_t^i(s^t) + a_t^i(s^t)$$

[3] A full set of one-period Arrow securities is sufficient to make markets "sequentially complete", in the sense that any (nonnegative) consumption allocation is attainable with an appropriate sequence of Arrow security holdings $\{a_{t+1}(s^t, s_{t+1})\}$ satisfying all sequential markets budget constraints.

Note that agents purchase Arrow securities $\{a_{t+1}^i(s^t, s_{t+1})\}_{s_{t+1}\in S}$ for all contingencies $s_{t+1} \in S$ that can happen tomorrow, but that, once s_{t+1} is realized, only the $a_{t+1}^i(s^{t+1})$ corresponding to the particular realization of s_{t+1} becomes the asset position that he starts the current period with. We assume that $a_0^i(s_0) = 0$ for all $s_0 \in S$.

We then have the following

Definition 54 *A SM equilibrium is allocations* $\left\{\left(\hat{c}_t^i(s^t), \left\{\hat{a}_{t+1}^i(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\right)_{i=1,2}\right\}_{t=0,s^t\in S^t}^{\infty}$, *and prices for Arrow securities* $\{\hat{q}_t(s^t, s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}\in S}^{\infty}$ *such that*

1. *Given* $\{\hat{q}_t(s^t, s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}\in S}^{\infty}$, *for all* i, $\{\hat{c}_t^i(s^t), \left\{\hat{a}_{t+1}^i(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\}_{t=0,s^t\in S^t}^{\infty}$ *solves*

$$
\begin{aligned}
& \max_{\{c_t^i(s^t),\left\{a_{t+1}^i(s^t,s_{t+1})\right\}_{s_{t+1}\in S}\}_{t=0,s^t\in S^t}^{\infty}} u(c^i) \\
& s.t. \\
c_t^i(s^t) + \sum_{s_{t+1}\in S} \hat{q}_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) &\leq e_t^i(s^t) + a_t^i(s^t) \text{ for all } t, s^t \in S^t \\
c_t^i(s^t) &\geq 0 \text{ for all } t, s^t \in S^t \\
a_{t+1}^i(s^t, s_{t+1}) &\geq -\bar{A}^i \text{ for all } t, s^t \in S^t, s_{t+1} \in S
\end{aligned}
$$

2. *For all* $t \geq 0$

$$
\begin{aligned}
\sum_{i=1}^{2} \hat{c}_t^i(s^t) &= \sum_{i=1}^{2} e_t^i(s^t) \text{ for all } t, s^t \in S^t \\
\sum_{i=1}^{2} \hat{a}_{t+1}^i(s^t, s_{t+1}) &= 0 \text{ for all } t, s^t \in S^t \text{ and all } s_{t+1} \in S
\end{aligned}
$$

Note that we have a market clearing condition in the asset market *for each* Arrow security being traded for period $t+1$.

6.2.4 Equivalence between Market Structures

As before we can establish the equivalence, in terms of equilibrium outcomes, between the Arrow-Debreu and the sequential markets structure. Without repeating the details (which are identical to the discussion in chapter 2, mutatis mutandis), the key to the argument is the map between Arrow-Debreu prices and prices for Arrow securities, given by

$$q_t(s^t, s_{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} \tag{6.8}$$

$$p_t(s^t) = p_0(s_0) * q_0(s_0, s_1) * \ldots q_{t-1}(s^{t-1}, s_t). \tag{6.9}$$

6.2.5 Asset Pricing

With Arrow Debreu prices (and sequential market prices from (6.8)) in hand we can now price any additional asset in this economy. Consider an arbitrary asset j, defined by the dividends $d^j = \{d_t^j(s^t)\}$ it pays in each node s^t. The dividend $d_t^j(s^t)$ is simply a claim to $d_t^j(s^t)$ units of the consumption good at node s^t of the event tree. Thus the time zero (cum dividend) price of such an asset is given by

$$P_0^j(d) = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) d_t^j(s^t),$$

that is, it is the value of all consumption goods the asset delivers at all future dates and states. The ex-dividend price of such an asset at node s^t, expressed in terms of period t consumption good is given by

$$P_t^j(d; s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) d_\tau^j(s^\tau)}{p_t(s^t)}$$

that is, the value of all future dividends, translated into the node s^t consumption good.

Most of asset pricing work with asset returns rather than asset prices. So let us define the one-period gross realized real return of an asset j between s^t and s^{t+1} as

$$R_{t+1}^j(s^{t+1}) = \frac{P_{t+1}^j(d; s^{t+1}) + d_{t+1}^j(s^{t+1})}{P_t^j(d; s^t)}$$

Let us consider a few examples that make these definitions clear.

Example 55 *Consider an Arrow security from the Sequential Markets equilibrium above that is purchased in s^t and pays off one unit of consumption* *in state $\hat{s}_{t+1}$ and nothing in all other states s_{t+1} (and nothing beyond period $t+1$). Then its price at s^t is given by*

$$P_t^A(d;s^t) = \frac{p_{t+1}(\hat{s}^{t+1})}{p_t(s^t)} = q_t(s^t, \hat{s}_{t+1})$$

and the associated gross realized return between s^t and $\hat{s}^{t+1} = (s^t, \hat{s}_{t+1})$ is

$$\begin{aligned} R_{t+1}^A(\hat{s}^{t+1}) &= \frac{0+1}{p_{t+1}(\hat{s}^{t+1})/p_t(s^t)} \\ &= \frac{p_t(s^t)}{p_{t+1}(\hat{s}^{t+1})} = \frac{1}{q_t(s^t, \hat{s}_{t+1})} \end{aligned}$$

and $R_{t+1}^A(s^{t+1}) = 0$ for all $s_{t+1} \neq s_{t+1}$.

Example 56 *Now consider a one period risk-free bond, that is, an asset that is purchased at s^t and pays one unit of consumption at all events s_{t+1} tomorrow. Its price at s^t is given by*

$$P_t^B(d;s^t) = \frac{\sum_{s^{t+1}|s^t} p_{t+1}(s^{t+1})}{p_t(s^t)} = \sum_{s_{t+1}} q_t(s^t, s_{t+1})$$

and its realized return is given by

$$\begin{aligned} R_{t+1}^B(s^{t+1}) &= \frac{1}{P_t^B(d;s^t)} \\ &= \frac{1}{\sum_{s_{t+1}} q_t(s^t, s_{t+1})} = R_{t+1}^B(s^t) \end{aligned}$$

which from the perspective of s^t is nonstochastic (since it does not depend on s_{t+1}). Hence the name risk-free bond.

Example 57 *A stock that pays as dividend the aggregate endowment in each period (a so-called Lucas tree) has a price per share (if the total number of shares outstanding is one) of:*

$$P_t^S(d;s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} p_\tau(s^\tau) e_\tau(s^\tau)}{p_t(s^t)}$$

Example 58 *An option to buy one share of the Lucas tree at time T (at all nodes) for a price K has a price $P_t^{call}(s^t)$ at node s^t given by*

$$P_t^{call}(s^t) = \sum_{s^T|s^t} \frac{p_T(s^T)}{p_t(s^t)} \max\left\{P_T^S(d;s^T) - K, 0\right\}$$

Such an option is called a call option. A put option is the option to sell the same asset, and its price given by

$$P_t^{put}(s^t) = \sum_{s^T|s^t} \frac{p_T(s^T)}{p_t(s^t)} \max\left\{K - P_T^S(d;s^T), 0\right\}.$$

The price K is called the strike price (and easily could be made dependent on s^T, too).

6.3 Markov Processes

So far we haven't specified the exact stochastic structure of risk. In particular, in no sense have we assumed that the random variables s_t and s_τ, $\tau > t$ are independent over time or time-dependent in a simple way. Our theory is completely general along this dimension; to make it implementable (analytically or numerically), however, one typically has to assume a particular structure of the risk.

In particular, for the computation of equilibria or socially efficient allocations using recursive techniques it is useful to assume that the s_t's follow a discrete time (time is discrete), discrete state (the number of values s_t can take is finite) time homogeneous Markov chain. Let by

$$\pi(j|i) = \text{prob}(s_{t+1} = j|s_t = i)$$

denote the conditional probability that the state in $t+1$ equals $j \in S$ if the state in period t equals $s_t = i \in S$. Time homogeneity means that π is not indexed by time. Given that $s_{t+1} \in S$ and $s_t \in S$ and S is a finite set, $\pi(.|.)$

can be represented by an $N \times N$-matrix of the form

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \vdots & \cdots & \pi_{1N} \\ \pi_{21} & & & \vdots & & \vdots \\ \vdots & & & \vdots & & \vdots \\ \pi_{i1} & \cdots & \cdots & \pi_{ij} & \cdots & \pi_{iN} \\ \vdots & & & \vdots & & \vdots \\ \pi_{N1} & \cdots & \cdots & \vdots & \cdots & \pi_{NN} \end{pmatrix}$$

with generic element $\pi_{ij} = \pi(j|i) =$prob$(s_{t+1} = j|s_t = i)$. Hence the i-th row gives the probabilities of going from state i today to all the possible states tomorrow, and the j-th column gives the probability of landing in state j tomorrow conditional of being in an arbitrary state i today. Since $\pi_{ij} \geq 0$ and $\sum_j \pi_{ij} = 1$ for all i (for all states today, one has to go somewhere tomorrow), the matrix π is a so-called stochastic matrix.

Suppose the probability distribution over states today is given by the N-dimensional column vector $P_t = (p_t^1, \ldots, p_t^N)^T$ and risk is described by a Markov chain of the from above. Note that $\sum_i p_t^i = 1$. Then the probability of being in state j tomorrow is given by

$$p_{t+1}^j = \sum_i \pi_{ij} p_t^i$$

i.e. by the sum of the conditional probabilities of going to state j from state i, weighted by the probabilities of starting out in state i today. More compactly we can write

$$P_{t+1} = \pi^T P_t.$$

We have the following

Definition 59 *A distribution* $\Pi \in \mathbf{R}_+^N$ *that satisfies*

$$\Pi = \pi^T \Pi$$

is called a stationary distribution associatted with the Markov chain π*.*

A stationary distribution has the property that if one starts out today with a distribution over states Π then tomorrow one ends up with the *same* distribution over states Π. From the theory of stochastic matrices we have the following result, stated here without proof

Theorem 60 *Associated with every Markov transition matrix π is at least one stationary distribution Π. It is the eigenvector (normalized to length 1) associated with the eigenvalue $\lambda = 1$ of π^T.*

Thus note that every stochastic matrix π has (at least) one eigenvalue equal to 1. If there is only one such eigenvalue, then there is a unique stationary distribution, if there are multiple eigenvalues of length 1, then there a multiple stationary distributions (in fact a continuum of them).

Also note that the Markov assumption restricts the conditional probability distribution of s_{t+1} to depend only on the realization of s_t, but not on realizations of s_{t-1}, s_{t-2} and so forth. This obviously is a severe restriction on the possible randomness that we allow, but it also means that the nature of risk for period $t+1$ is completely described by the realization of s_t, which is crucial when formulating these economies recursively. We have to start the Markov process out at period 0, so let by $\Pi(s_0)$ denote the probability that the state in period 0 is s_0. Given our Markov assumption the probability of a particular event history can then be written as

$$\pi_{t+1}(s^{t+1}) = \pi(s_{t+1}|s_t) * \pi(s_t|s_{t-1}) \ldots * \pi(s_1|s_0) * \Pi(s_0)$$

Example 61 *Suppose that $N = 2$. Let the transition matrix be symmetric, that is*

$$\pi = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

for some $p \in (0,1)$. Then the unique invariant distribution is $\Pi(s) = 0.5$ for both s.

Example 62 *Let*

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then any distribution over the two states is an invariant distribution.

6.4 Stochastic Neoclassical Growth Model

In this section we will briefly consider a stochastic extension to the deterministic neoclassical growth model. The stochastic neoclassical growth model is the workhorse for half of modern business cycle theory; everybody doing real business cycle theory uses it. It also forms an important ingredient of many New Keynesian business cycle models. I therefore think that it is useful to expose you to this model, even though you may decide not to do business cycle research theory in the future.

The economy is populated by a large number of identical households. For convenience we normalize the number of households to 1. In each period three goods are traded, labor services n_t, capital services k_t and the final output good y_t, which can be used for consumption c_t or investment i_t.

1. Technology:
$$y_t = e^{z_t} F(k_t, n_t)$$
where z_t is a technology shock. F is assumed to have the usual properties, i.e. has constant returns to scale, positive but declining marginal products and it satisfies the INADA conditions. We assume that the technology shock has unconditional mean 0 and follows a N-state Markov chain. Let $Z = \{z_1, z_2, \ldots z_N\}$ be the state space of the Markov chain, i.e. the set of values that z_t can take on. Let $\pi = (\pi_{ij})$ denote the Markov transition matrix and Π the stationary distribution of the chain (ignore the fact that in some of our applications Π will not be unique). Let $\pi(z'|z) = prob(z_{t+1} = z'|z_t = z)$. In most of the applications we will take $N = 2$. The evolution of the capital stock is given by
$$k_{t+1} = (1-\delta)k_t + i_t$$
and the composition of output is given by
$$y_t = c_t + i_t$$
Note that the set Z takes the role of S in our general formulation of risk, and z^t corresponds to s^t.

2. Preferences:
$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ with } \beta \in (0,1)$$
The period utility function is assumed to have the usual properties.

3. Endowment: each household has an initial endowment of capital, k_0 and one unit of time in each period. These endowments are *not* stochastic.

4. Information: The variable z_t, the only source of risk in this model, is publicly observable. We assume that in period 0 z_0 has not been realized, but is drawn from the stationary distribution Π. All agents are perfectly informed that the technology shock follows the Markov chain π with initial distribution Π.

A lot of the things that we did for the case without risk go through almost unchanged for the stochastic model. Specifically, we could prove the first welfare theorem and characterize competitive equilibrium allocations using the social planner problem. The only key difference is that now commodities have to be indexed not only by time, but also by histories of productivity shocks z^t, since goods delivered at different nodes of the event tree are different commodities, even though they have the same physical characteristics.

6.4.1 Social Planner Problem in Recursive Formulation

For the recursive formulation of the social planners problem, note that the current state of the economy now not only includes the capital stock k that the planner brings into the current period, but also the current state of the technology z. This is due to the fact that current production depends on the current technology shock, but also due to the fact that the probability distribution of tomorrow's shocks $\pi(z'|z)$ depends on the current shock, due to the Markov structure of the shocks. Also note that even if the social planner chooses capital stock k' for tomorrow today, lifetime utility from tomorrow onwards is uncertain, due to the risk of z'. These considerations, plus the usual observation that $n_t = 1$ is optimal, give rise to the following Bellman equation

$$v(k, z) = \max_{0 \leq k' \leq e^z F(k,1) + (1-\delta)k} \left\{ U(e^z F(k, 1) + (1 - \delta)k - k') + \beta \sum_{z'} \pi(z'|z) v(k', z') \right\}.$$

However, the model discussed so far is not quite yet a satisfactory business cycle model since it does not permit fluctuations in labor input of the sort that characterize business cycles in the real world. For this we require households to value leisure, so that the period utility function becomes

$$U(c_t, l_t) = U(c_t, 1 - n_t)$$

and the recursive formulation of the planner problem reads as

$$v(k,z) = \max_{\substack{0 \le k' \le e^z F(k,1)+(1-\delta)k \\ 0 \le n \le 1}} \left\{ U(e^z F(k,n) + (1-\delta)k - k', 1-n) + \beta \sum_{z'} \pi(z'|z) v(k',z') \right\}$$

This version of the model is also often called the *Real Business Cycle* model, since it ascribes the origins of fluctuations in aggregate economic activity to real shocks, those to total factor productivity e^z.

The first order conditions for the maximization problem (assuming differentiability of the value function) read as

$$e^z F_n(k,n) = \frac{U_2(c,1-n)}{U_1(c,1-n)} \tag{6.10}$$

and

$$U_1(c,1-n) = \beta \sum_{z'} \pi(z'|z) v'(k',z') \tag{6.11}$$

where U_1 is the marginal utility of consumption, U_2 is the marginal utility of leisure and v' is the first derivative of the value function with respect to its first argument. The envelope condition reads as

$$v'(k,z) = \left(e^z F_k(k,n) + 1 - \delta\right) U_1(c,1-n). \tag{6.12}$$

Using this in equation (6.11) we obtain

$$U_1(c,1-n) = \beta \sum_{z'} \pi(z'|z) \left(e^{z'} F_k(k',n') + 1 - \delta\right) U_1(c',1-n'). \tag{6.13}$$

Thus the key optimality conditions of the stochastic neoclassical growth model with endogenous labor supply, often referred to as the real business cycle model, are (6.10) and (6.13). Equation (6.10), the intratemporal optimality condition, states that at the optimum the marginal rate of substitution between leisure and consumption is equated to the marginal product of labor (the wage, in the decentralized equilibrium). Equation (6.13) is the standard intertemporal Euler equation, now equating the marginal utility of consumption today to the expected marginal utility of consumption tomorrow, adjusted by the time discount factor β and the stochastic rate of return on capital, $e^{z'} F_k(k',n') + 1 - \delta$, which in turn equals the gross real interest rate in the competitive equilibrium.

6.4.2 Recursive Competitive Equilibrium

The definition of a recursive competitive equilibrium proceeds in exactly the same way as for the deterministic neoclassical growth model. We then have

Definition 63 *A recursive competitive equilibrium is a value function* $v : \mathbf{R}_+^3 \to \mathbf{R}$ *and policy functions* $c, n, g : \mathbf{R}_+^3 \to \mathbf{R}_+$ *for the representative household, a labor demand function for the representative firm* $N : \mathbf{R}_+^2 \to \mathbf{R}_+$*, pricing functions* $w, r : \mathbf{R}_+^2 \to \mathbf{R}_+$ *and an aggregate law of motion* $H : \mathbf{R}_+^2 \to \mathbf{R}_+$ *such that*

1. *Given the functions* w, r *and* H*, the value function* v *solves the Bellman equation*

$$\begin{aligned} v(k,z,K) &= \max_{c,k',n\geq 0} \left\{ U(c,n) + \beta \sum_{z'\in Z} \pi(z'|z) v(k',z',K') \right\} \\ & \quad s.t. \\ c + k' &= w(z,K)n + (1 + r(z,K) - \delta)k \\ K' &= H(z,K) \end{aligned}$$

   and c, n, g *are the associated policy functions.*

2. *The labor demand and pricing functions satisfy*

$$\begin{aligned} w(z,K) &= e^z F_n(K, N(z,K)) \\ r(z,K) &= e^z F_k(K, N(z,K)). \end{aligned}$$

3. *Consistency*

$$H(z,K) = g(K,z,K)$$

4. *For all* $K \in \mathbf{R}_+$

$$\begin{aligned} c(K,z,K) + g(K,z,K) &= e^z F(K, N(z,K)) + (1-\delta)K \\ N(z,K) &= n(K,z,K) \end{aligned}$$

Note that by using the first order conditions and the envelope condition of the household problem we can arrive at exactly the same intratemporal and intertemporal optimality conditions as from the recursive social planner problem, once we substitute out prices with marginal productivities from the firm's optimality conditions.

Chapter 7

The Two Welfare Theorems

In this section we will present the two fundamental theorems of welfare economics for economies in which the commodity space is a general (real) vector space, which is not necessarily finite dimensional. Since in macroeconomics we often deal with agents or economies that live forever, usually a finite dimensional commodity space is not sufficient for our analysis. The significance of the welfare theorems, apart from providing a normative justification for studying competitive equilibria is that planning problems characterizing Pareto optima are usually easier to solve that equilibrium problems, the ultimate goal of our theorizing.

Our discussion will follow Stokey et al. (1989), which in turn draws heavily on results developed by Debreu (1954).

7.1 What is an Economy?

We first discuss how what an economy is in Arrow-Debreu language. An economy $E = ((X_i, u_i)_{i \in I}, (Y_j)_{j \in J})$ consists of the following elements

1. A list of commodities, represented by the commodity space S. We require S to be a normed (real) vector space with norm $\|.\|$.[1]

[1] For completeness we state the following definitions

Definition 64 *A real vector space is a set S (whose elements are called vectors) on which are defined two operations*

- *Addition $+ : S \times S \to S$. For any $x, y \in S$, $x + y \in S$.*

2. A finite set of people $i \in I$. Abusing notation I will by I denote both the set of people and the number of people in the economy.

3. Consumption sets $X_i \subseteq S$ for all $i \in I$. We will incorporate the restrictions that households endowments place on the x_i in the description of the consumption sets X_i.

4. Preferences representable by utility functions $u_i : S \to \mathbf{R}$.

5. A finite set of firms $j \in J$. The same remark about notation as above applies.

- *Scalar Multiplication* $\cdot : \mathbf{R} \times S \to S$. *For any* $\alpha \in \mathbf{R}$ *and any* $x \in S$, $\alpha x \in S$ *that satisfy the following algebraic properties: for all* $x, y \in S$ *and all* $\alpha, \beta \in \mathbf{R}$

  (a) $x + y = y + x$

  (b) $(x + y) + z = x + (y + z)$

  (c) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

  (d) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$

  (e) $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$

  (f) *There exists a null element* $\theta \in S$ *such that*

$$\begin{aligned} x + \theta &= x \\ 0 \cdot x &= \theta \end{aligned}$$

  (g) $1 \cdot x = x$

Definition 65 *A normed vector space is a vector space is a vector space* S *together with a norm* $\|.\| : S \to \mathbf{R}$ *such that for all* $x, y \in S$ *and* $\alpha \in \mathbf{R}$

(a) $\|x\| \geq 0$, *with equality if and only if* $x = \theta$

(b) $\|\alpha \cdot x\| = |\alpha| \, \|x\|$

(c) $\|x + y\| \leq \|x\| + \|y\|$

Note that in the first definition the adjective real refers to the fact that scalar multiplication is done with respect to a real number. Also note the intimate relation between a norm and a metric defined above. A norm of a vector space S, $\|.\| : S \to \mathbf{R}$ induces a metric $d : S \times S \to \mathbf{R}$ by

$$d(x, y) = \|x - y\|$$

6. Technology sets $Y_j \subseteq S$ for all $j \in J$. Let by

$$Y = \sum_{j \in J} Y_j = \left\{ y \in S : \exists (y_j)_{j \in J} \text{ such that } y = \sum_{j \in J} y_j \text{ and } y_j \in Y_j \text{ for all } j \in J \right\}$$

denote the aggregate production set.

A private ownership economy $\tilde{E} = ((X_i, u_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J})$ consists of all the elements of an economy and a specification of ownership of the firms $\theta_{ij} \geq 0$ with $\sum_{i \in I} \theta_{ij} = 1$ for all $j \in J$. The entity θ_{ij} is interpreted as the share of ownership of household ι to firm j, i.e. the fraction of total profits of firm j that household i is entitled to.

With our formalization of the economy we can also make precise what we mean by an externality. An economy is said to exhibit an externality if household i's consumption set X_i or firm j's production set Y_j is affected by the choice of household k's consumption bundle x_k or firm m's production plan y_m. Unless otherwise stated we assume that we deal with an economy without externalities.

Definition 66 *An allocation is a tuple* $[(x_i)_{i \in I}, (y_j)_{j \in J}] \in S^{I \times J}$.

In the economy people supply factors of production and demand final output goods. We follow Debreu and use the convention that negative components of the x_i's denote factor inputs and positive components denote final goods. Similarly negative components of the y_j's denote factor inputs of firms and positive components denote final output of firms.

Definition 67 *An allocation* $[(x_i)_{i \in I}, (y_j)_{j \in J}] \in S^{I \times J}$ *is feasible if*

1. $x_i \in X_i$ *for all* $i \in I$

2. $y_j \in Y_j$ *for all* $j \in J$

3. *(Resource Balance)*

$$\sum_{i \in I} x_i = \sum_{j \in J} y_j$$

Note that we require resource balance to hold with equality, ruling out free disposal. If we want to allow free disposal we will specify this directly as part of the description of technology.

Definition 68 *An allocation* $[(x_i)_{i\in I},(y_j)_{j\in J}]$ *is Pareto optimal if*

1. *it is feasible*
2. *there does not exist another feasible allocation* $[(x_i^*)_{i\in I},(y_j^*)_{j\in J}]$ *such that*

$$\begin{aligned} u_i(x_i^*) &\geq u_i(x_i) \text{ for all } i\in I \\ u_i(x_i^*) &> u_i(x_i) \text{ for at least one } i\in I \end{aligned}$$

Note that if $I=J=1$ then[2] for an allocation $[x,y]$ resource balance requires $x=y$, the allocation is feasible if $x\in X\cap Y$, and the allocation is Pareto optimal if

$$x\in \arg\max_{z\in X\cap Y} u(z)$$

Also note that the definition of feasibility and Pareto optimality are identical for economies E and private ownership economies $\tilde{E}$. The difference comes in the definition of competitive equilibrium and there in particular in the formulation of the resource constraint. The discussion of competitive equilibrium requires a discussion of prices at which allocations are evaluated. Since we deal with possibly infinite dimensional commodity spaces, prices in general cannot be represented by a finite dimensional vector. To discuss prices for our general environment we need a more general notion of a price system. This is necessary in order to state and prove the welfare theorems for infinitely lived economies that we are interested in.

7.2 Dual Spaces

A price system attaches to every bundle of the commodity space S a real number that indicates how much this bundle costs. If the commodity space is a finite (say $k-$) dimensional Euclidean space, then the natural thing to

[2] The assumption that $J=1$ is not at all restrictive if we restrict our attention to constant returns to scale technologies. Then, in any competitive equilibrium profits are zero and the number of firms is indeterminate in equilibrium; without loss of generality we then can restrict attention to a single representative firm. If we furthermore restrict attention to identical people and type identical allocations, then de facto $I=1$. Under which assumptions the restriction to type identical allocations is justified will be discussed below.

do is to represent a price system by a k-dimensional vector $p = (p_1, \ldots p_k)$, where p_l is the price of the l-th component of a commodity vector. The price of an entire point of the commodity space is then $\phi(s) = \sum_{l=1}^{k} s_l p_l$. Note that every $p \in \mathbf{R}^k$ represents a function that maps $S = \mathbf{R}^k$ into $\mathbf{R}$. Obviously, since for a given p and all $s, s' \in S$ and all $\alpha, \beta \in \mathbf{R}$

$$\phi(\alpha s + \beta s') = \sum_{l=1}^{k} p_l(\alpha s_l + \beta s_l') = \alpha \sum_{l=1}^{k} p_l s_l + \beta \sum_{l=1}^{k} p_l s_l' = \alpha\phi(s) + \beta\phi(s')$$

the mapping associated with p is linear. We will take as a price system for an arbitrary commodity space S a continuous linear functional defined on S. The next definition makes the notion of a continuous linear functional precise.

Definition 69 *A linear functional ϕ on a normed vector space S (with associated norm $\|\|_S$) is a function $\phi : S \to \mathbf{R}$ that maps S into the reals and satisfies*

$$\phi(\alpha s + \beta s') = \alpha\phi(s) + \beta\phi(s') \text{ for all } s, s' \in S, \text{ all } \alpha, \beta \in \mathbf{R}$$

The functional ϕ is continuous if $\|s_n - s\|_S \to 0$ implies $|\phi(s_n) - \phi(s)| \to 0$ for all $\{s_n\}_{n=0}^{\infty} \in S, s \in S$. The functional ϕ is bounded if there exists a constant $M \in \mathbf{R}$ such that $|\phi(s)| \leq M \|s\|_S$ for all $s \in S$. For a bounded linear functional ϕ we define its norm by

$$\|\phi\|_d = \sup_{\|s\|_S \leq 1} |\phi(s)|$$

Fortunately it is rather easy to verify whether a linear functional is continuous and bounded. Stokey et al. state and prove a theorem that states that a linear functional is continuous if it is continuous at a particular point $s \in S$ and that it is bounded if (and only if) it is continuous. Hence a linear functional is bounded and continuous if it is continuous at a single point.

For any normed vector space S the space

$$S^* = \{\phi : \phi \text{ is a continuous linear functional on } S\}$$

is called the (algebraic) dual (or conjugate) space of S. With addition and scalar multiplication defined in the standard way S^* is a vector space, and with the norm $\|\|_d$ defined above S^* is a normed vector space as well. Note (you should prove this[3]) that even if S is not a complete space, S^* is a complete space and hence a Banach space (a complete normed vector space). Let us consider several examples that will be of interest for our economic applications.

Example 70 *For each $p \in [1, \infty)$ define the space l_p by*

$$l_p = \{x = \{x_t\}_{t=0}^{\infty} : x_t \in \mathbf{R}, \text{ for all } t; \ \|x\|_p = \left(\sum_{t=0}^{\infty} |x_t|^p\right)^{\frac{1}{p}} < \infty\}$$

with corresponding norm $\|x\|_p$. For $p = \infty$, the space l_∞ is defined correspondingly, with norm $\|x\|_\infty = \sup_t |x_t|$. For any $p \in [1, \infty)$ define the conjugate index q by

$$\frac{1}{p} + \frac{1}{q} = 1$$

For $p = 1$ we define $q = \infty$. We have the important result that for any $p \in [1, \infty)$ the dual of l_p is l_q. This result can be proved by using the following theorem (which in turn is proved by Luenberger (1969), p. 107.)

Theorem 71 *Every continuous linear functional ϕ on l_p, $p \in [1, \infty)$, is representable uniquely in the form*

$$\phi(x) = \sum_{t=0}^{\infty} x_t y_t \tag{7.1}$$

where $y = \{y_t\} \in l_q$. Furthermore, every element of l_q defines an element of the dual of l_p, l_p^* in this way, and we have

$$\|\phi\|_d = \|y\|_q = \begin{cases} \left(\sum_{t=0}^{\infty} |y_t|^q\right)^{\frac{1}{q}} & \text{if } 1 < p < \infty \\ \sup_t |y_t| & \text{if } p = 1 \end{cases}$$

Let's first understand what the theorem gives us. Take any space l_p (note that the theorem does NOT make any statements about l_∞). Then the theorem states that its dual is l_q. The first part of the theorem states that $l_q \subseteq l_p^*$. Take any element $\phi \in l_p^*$. Then there exists $y \in l_q$ such that ϕ

[3] *After* you are done with this, check Kolmogorov and Fomin (1970), p. 187 (Theorem 1) for their proof.

is representable by y. In this sense $\phi \in l_q$. The second part states that any $y \in l_q$ defines a functional ϕ on l_p by (7.1). Given its definition, ϕ is obviously continuous and hence bounded. Finally the theorem assures that the norm of the functional ϕ associated with y is indeed the norm associated with l_q. Hence $l_p^* \subseteq l_q$.

As a result of the theorem, whenever we deal with l_p, $p \in [1, \infty)$ as commodity space we can restrict attention to price systems that can be represented by a vector $p = (p_0, p_1, \ldots p_t, \ldots)$ and hence have a straightforward economic interpretation: p_t is the price of the good at period t and the cost of a consumption bundle x is just the sum of the cost of all its components.

For reasons that will become clearer later the most interesting commodity space for infinitely lived economies, however, is l_∞. And for this commodity space the previous theorem does not make any statements. It would suggest that the dual of l_∞ is l_1, but this is not quite correct, as the next result shows.

Proposition 72 *The dual of l_∞ contains l_1. There are $\phi \in l_\infty^*$ that are not representable by an element $y \in l_1$*

Proof. For the first part for any $y \in l_1$ define $\phi : l_\infty \to \mathbf{R}$ by

$$\phi(x) = \sum_{t=0}^{\infty} x_t y_t$$

We need to show that ϕ is linear and continuous. Linearity is obvious. For continuity we need to show that for any sequence $\{x^n\} \in l_\infty$ and $x \in l_\infty$, $\|x^n - x\| = \sup_t |x_t^n - x_t| \to 0$ implies $|\phi(x^n) - \phi(x)| \to 0$. Since $y \in l_1$ there exists M such that $\sum_{t=0}^{\infty} |y_t| < M$. Since $\sup_t |x_t^n - x_t| \to 0$, for all $\delta > 0$ there exists $N(\delta)$ such that fro all $n > N(\delta)$ we have $\sup_t |x_t^n - x_t| < \delta$. But then for any $\varepsilon > 0$, taking $\delta(\varepsilon) = \frac{\varepsilon}{2M}$ and $N(\varepsilon) = N(\delta(\varepsilon))$, for all $n > N(\varepsilon)$

$$\begin{aligned}
|\phi(x^n) - \phi(x)| &= \left| \sum_{t=0}^{\infty} x_t^n y_t - \sum_{t=0}^{\infty} x_t y_t \right| \\
&\leq \sum_{t=0}^{\infty} |y_t(x_t^n - x_t)| \\
&\leq \sum_{t=0}^{\infty} |y_t| \cdot |x_t^n - x_t| \\
&\leq M\delta(e) = \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

The second part we prove via a counter example after we have proved the second welfare theorem. ■

The second part of the proposition is somewhat discouraging in that it asserts that, when dealing with l_∞ as commodity space we may require a price system that does not have a natural economic interpretation. It is true that there is a subspace of l_∞ for which l_1 is its dual. Define the space c_0 (with associated sup-norm) as

$$c_0 = \{x \in l_\infty : \lim_{t\to\infty} x_t = 0\}$$

We can prove that l_1 is the dual of c_0. Since $c_0 \subseteq l_\infty$ and $l_1 \subseteq l_\infty^*$, obviously $l_1 \subseteq c_0^*$. It remains to show that any $\phi \in c_0^*$ can be represented by a $y \in l_1$. [TO BE COMPLETED]

7.3 Definition of Competitive Equilibrium

Corresponding to our two notions of an economy and a private ownership economy we have two definitions of competitive equilibrium that differ in their specification of the individual budget constraints.

Definition 73 *A competitive equilibrium is an allocation* $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ *and a continuous linear functional* $\phi : S \to \mathbf{R}$ *such that*

1. *for all* $i \in I$, x_i^0 *solves* $\max u_i(x)$ *subject to* $x \in X_i$ *and* $\phi(x) \leq \phi(x_i^0)$
2. *for all* $j \in J$, y_j^0 *solves* $\max \phi(y)$ *subject to* $y \in Y_j$
3. $\sum_{i\in I} x_i^0 = \sum_{j\in J} y_j^0$

In this definition we have obviously ignored ownership of firms. If, however, all Y_j are convex cones, the technologies exhibit constant returns to scale, profits are zero in equilibrium and this definition of equilibrium is equivalent to the definition of equilibrium for a private ownership economy (under appropriate assumptions on preferences such as local nonsatiation). Note that condition 1. is equivalent to requiring that for all $i \in I$, $x \in X_i$ and $\phi(x) \leq \phi(x_i^0)$ implies $u_i(x) \leq u_i(x_i^0)$ which states that all bundles that are cheaper than x_i^0 must not yield higher utility. Again note that we made no reference to the value of an individuals' endowment or firm ownership.

Definition 74 *A competitive equilibrium for a private ownership economy is an allocation* $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ *and a continuous linear functional* $\phi : S \to \mathbf{R}$ *such that*

1. *for all* $i \in I$, x_i^0 *solves* $\max u_i(x)$ *subject to* $x \in X_i$ *and* $\phi(x) \leq \sum_{j\in J} \theta_{ij}\phi(y_j^0)$
2. *for all* $j \in J$, y_j^0 *solves* $\max \phi(y)$ *subject to* $y \in Y_j$
3. $\sum_{i\in I} x_i^0 = \sum_{j\in J} y_j^0$

We can interpret $\sum_{j\in J} \theta_{ij}\phi(y_j^0)$ as the value of the ownership that household i holds to all the firms of the economy.

7.4 The Neoclassical Growth Model in Arrow-Debreu Language

Let us look at the neoclassical growth model presented in Section 2. We will adopt the notation so that it fits into our general discussion. Remember that in the economy the representative household owned the capital stock and the representative firm, supplied capital and labor services and bought final output from the firm. A helpful exercise would be to repeat this exercise under the assumption that the firm owns the capital stock. The household had unit endowment of time and initial endowment of $\bar{k}_0$ of the capital stock. To make our exercise more interesting we assume that the household values consumption and leisure according to instantaneous utility function $U(c, l)$, where c is consumption and l is leisure. The technology is described by $y = F(k, n)$ where F exhibits constant returns to scale. For further details refer to Section 2. Let us represent this economy in Arrow-Debreu language.

- $I = J = 1, \theta_{ij} = 1$
- Commodity Space S: since three goods are traded in each period (final output, labor and capital services), time is discrete and extends to infinity, a natural choice is $S = l_\infty^3 = l_\infty \times l_\infty \times l_\infty$. That is, S consists of all three-dimensional infinite sequences that are bounded in the sup-norm, or

$$S = \{s = (s^1, s^2, s^3) = \{(s_t^1, s_t^2, s_t^3)\}_{t=0}^\infty : s_t^i \in \mathbf{R}, \sup_t \max_i \left|s_t^i\right| < \infty\}$$

Obviously S, together with the sup-norm, is a (real) normed vector space. We use the convention that the first component of s denotes the output good (and hence is required to be positive), whereas the second and third components denote labor and capital services, respectively. Again following the convention these inputs are required to be negative.

- Consumption Set X :

  $X = \{\{x_t^1, x_t^2, x_t^3\} \in S : x_0^3 \geq -\bar{k}_0, -1 \leq x_t^2 \leq 0, x_t^3 \leq 0, x_t^1 \geq 0, x_t^1 - (1-\delta)x_t^3 + x_{t+1}^3 \geq 0$ fo

  We do not distinguish between capital and capital services here; this can be done by adding extra notation and is an optional homework. The constraints indicate that the household cannot provide more capital in the first period than the initial endowment, can't provide more than one unit of labor in each period, holds nonnegative capital stock and is required to have nonnegative consumption. Evidently $X \subseteq S$.

- Utility function $u : X \rightarrow \mathbf{R}$ is defined by

  $$u(x) = \sum_{t=0}^{\infty} \beta^t U(x_t^1 - (1-\delta)x_t^3 + x_{t+1}^3, 1 + x_t^2)$$

  Again remember the convention than labor and capital (as inputs) are negative.

- Aggregate Production Set Y :

  $Y = \{\{y_t^1, y_t^2, y_t^3\} \in S : y_t^1 \geq 0, y_t^2 \leq 0, y_t^3 \leq 0, y_t^1 = F(-y_t^3, -y_t^2) \text{ for all } t\}$

  Note that the aggregate production set reflects the technological constraints in the economy. It does not contain any constraints that have to do with limited supply of factors, in particular $-1 \leq y_t^2$ is not imposed.

- An allocation is $[x, y]$ with $x, y \in S$. A feasible allocation is an allocation such that $x \in X, y \in Y$ and $x = y$. An allocation is Pareto optimal is it is feasible and if there is no other feasible allocation $[x^*, y^*]$ such that $u(x^*) > u(x)$.

- A price system ϕ is a continuous linear functional $\phi : S \rightarrow \mathbf{R}$. If ϕ has inner product representation, we represent it by $p = (p^1, p^2, p^3) = \{(p_t^1, p_t^2, p_t^3)\}_{t=0}^{\infty}$.

- A competitive equilibrium for this private ownership economy is an allocation $[x^*, y^*]$ and a continuous linear functional such that

  1. y^* maximizes $\phi(y)$ subject to $y \in Y$
  2. x^* maximizes $u(x)$ subject to $x \in X$ and $\phi(x) \leq \phi(y^*)$
  3. $x^* = y^*$

Note that with constant returns to scale $\phi(y^*) = 0$. With inner product representation of the price system the budget constraint hence becomes

$$\phi(x) = p \cdot x = \sum_{t=0}^{\infty} \sum_{i=1}^{3} p_t^i x_t^i \leq 0$$

Remembering our sign convention for inputs and mapping $p_t^1 = p_t$, $p_t^2 = p_t w_t$, $p_t^3 = p_t r_t$ we obtain the same budget constraint as in Section 2.

7.5 A Pure Exchange Economy in Arrow-Debreu Language

Suppose there are I individuals that live forever. There is one nonstorable consumption good in each period. Individuals order consumption allocations according to

$$u_i(c_i) = \sum_{t=0}^{\infty} \beta_i^t U(c_t^i)$$

They have deterministic endowment streams $e^i = \{e_t^i\}_{t=0}^{\infty}$. Trade takes place at period 0. The standard definition of a competitive (Arrow-Debreu) equilibrium would go like this:

Definition 75 *A competitive equilibrium are prices* $\{p_t\}_{t=0}^{\infty}$ *and allocations* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i \in I}$ *such that*

1. *Given* $\{p_t\}_{t=0}^{\infty}$*, for all* $i \in I$*,* $\{\hat{c}_t^i\}_{t=0}^{\infty}$ *solves* $\max_{c^i \geq 0} u_i(c_i)$ *subject to*

$$\sum_{t=0}^{\infty} p_t(c_t^i - e_t^i) \leq 0$$

2.

$$\sum_{i\in I} c_t^i = \sum_{i\in I} e_t^i \textit{ for all } t$$

We briefly want to demonstrate that we can easily write this economy in our formal language. What goes on is that the household sells his endowment of the consumption good to the market and buys consumption goods from the market. So even though there is a single good in each period we find it useful to have two commodities in each period. We also introduce an artificial technology that transforms one unit of the endowment in period t into one unit of the consumption good at period t. There is a single representative firm that operates this technology and each consumer owns share θ_i of the firm, with $\sum_{i\in I}\theta_i = 1$. We then have the following representation of this economy

- $S = l_\infty^2$. We use the convention that the first good is the consumption good to be consumed, the second good is the endowment to be sold as input by consumers. Again we use the convention that final output is positive, inputs are negative.
- $X_i = \{x \in S : x_t^1 \geq 0, -e_t^i \leq x_t^2 \leq 0\}$
- $u_i : X_i \to \mathbf{R}$ defined by

$$u_i(x) = \sum_{t=0}^{\infty} \beta_i^t U(x_t^1)$$

- Aggregate production set

$$Y = \{y \in S : y_t^1 \geq 0, y_t^2 \leq 0, y_t^1 = -y_t^2\}$$

- Allocations, feasible allocations and Pareto efficient allocations are defined as before.
- A price system ϕ is a continuous linear functional $\phi : S \to \mathbf{R}$. If ϕ has inner product representation, we represent it by $p = (p^1, p^2) = \{(p_t^1, p_t^2)\}_{t=0}^{\infty}$.
- A competitive equilibrium $[(x^{i*})_{i\in I}, y, \phi]$ for this private ownership economy defined as before.

- Note that with constant returns to scale in equilibrium we have $\phi(y^*) = 0$. With inner product representation of the price system in equilibrium also $p_t^1 = p_t^2 = p_t$. The budget constraint hence becomes

$$\phi(x) = p \cdot x = \sum_{t=0}^{\infty} \sum_{i=1}^{2} p_t^i x_t^i \leq 0$$

Obviously (as long as $p_t > 0$ for all t) the consumer will choose $x_t^{i2} = -e_t^i$, i.e. sell all his endowment. The budget constraint then takes the familiar form

$$\sum_{t=0}^{\infty} p_t(c_t^i - e_t^i) \leq 0$$

The purpose of this exercise was to demonstrate that, although in the remaining part of the course we will describe the economy and define an equilibrium in the first way, whenever we desire to prove the welfare theorems we can represent any pure exchange economy easily in our formal language and use the machinery developed in this section (if applicable).

7.6 The First Welfare Theorem

The first welfare theorem states that every competitive equilibrium allocation is Pareto optimal. The only assumption that is required is that people's preferences be locally nonsatiated. The proof of the theorem is unchanged from the one you should be familiar with from micro last quarter

Theorem 76 *Suppose that for all i, all $x \in X_i$ there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X_i converging to x with $u(x_n) > u(x)$ for all n (local nonsatiation). If an allocation $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ and a continuous linear functional ϕ constitute a competitive equilibrium, then the allocation $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ is Pareto optimal.*

Proof. The proof is by contradiction. Suppose $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$, ϕ is a competitive equilibrium.

Step 1: We show that for all i, all $x \in X_i$, $u(x) \geq u(x_i^0)$ implies $\phi(x) \geq \phi(x_i^0)$. Suppose not, i.e. suppose there exists i and $x \in X_i$ with $u(x) \geq u(x_i^0)$ and $\phi(x) < \phi(x_i^0)$. Let $\{x_n\}$ in X_i be a sequence converging to x with $u(x_n) >$

$u(x)$ for all n. Such a sequence exists by our local nonsatiation assumption. By continuity of ϕ there exists an n such that $u(x_n) > u(x) \geq u(x_i^0)$ and $\phi(x_n) < \phi(x_i^0)$, violating the fact that x_i^0 is part of a competitive equilibrium.

Step 2: For all i, all $x \in X_i$, $u(x) > u(x_i^0)$ implies $\phi(x) > \phi(x_i^0)$. This follows directly from the fact that x_i^0 is part of a competitive equilibrium.

Step 3: Now suppose $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ is not Pareto optimal. Then there exists another feasible allocation $[(x_i^*)_{i\in I}, (y_j^*)_{j\in J}]$ such that $u(x_i^*) \geq u(x_i^0)$ for all i and with strict inequality for some i. Since $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ is a competitive equilibrium allocation, by step 1 and 2 we have

$$\phi(x_i^*) \geq \phi(x_i^0)$$

for all i, with strict inequality for some i. Summing up over all individuals yields

$$\sum_{i\in I} \phi(x_i^*) > \sum_{i\in I} \phi(x_i^0) < \infty$$

The last inequality comes from the fact that the set of people I is finite and that for all i, $\phi(x_i^0)$ is finite (otherwise the consumer maximization problem has no solution). By linearity of ϕ we have

$$\phi\left(\sum_{i\in I} x_i^*\right) = \sum_{i\in I} \phi(x_i^*) > \sum_{i\in I} \phi(x_i^0) = \phi\left(\sum_{i\in I} x_i^0\right)$$

Since both allocations are feasible we have that

$$\begin{aligned} \sum_{i\in I} x_i^0 &= \sum_{j\in J} y_j^0 \\ \sum_{i\in I} x_i^* &= \sum_{j\in J} y_j^* \end{aligned}$$

and hence

$$\phi\left(\sum_{j\in J} y_j^*\right) > \phi\left(\sum_{j\in J} y_j^0\right)$$

Again by linearity of ϕ

$$\sum_{j\in J} \phi(y_j^*) > \sum_{j\in J} \phi(y_j^0)$$

Figure 7.1: Separating Hyperplanes in the Second Welfare Theorem

and hence for at least one $j \in J$, $\phi(y_j^*) > \phi(y_j^0)$. But $y_j^* \in Y_j$ and we obtain a contradiction to the hypothesis that $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ is a competitive equilibrium allocation. ■

Several remarks are in order. It is crucial for the proof that the set of individuals is finite, as will be seen in our discussion of overlapping generations economies. Also our equilibrium definition seems odd as it makes no reference to endowments or ownership in the budget constraint. For the preceding theorem, however, this is not a shortcoming. Since we start with a competitive equilibrium we know the value of each individual's consumption allocation. By local nonsatiation each consumer exhausts her budget and hence we implicitly know each individual's income (the value of endowments and firm ownership, if specified in a private ownership economy).

7.7 The Second Welfare Theorem

The second welfare theorem provides a converse to the first welfare theorem. Under suitable assumptions it states that for any Pareto-optimal allocation there exists a price system such that the allocation together with the price system form a competitive equilibrium. It may at first be surprising that the second welfare theorem requires much more stringent assumptions than the first welfare theorem. Remember, however, that in the first welfare theorem we start with a competitive equilibrium whereas in the proof of the second welfare we have to carry out an existence proof. Comparing the assumptions of the second welfare theorem with those of existence theorems makes clear the intimate relation between them.

As in micro we will use a separating hyperplane theorem to establish the existence of a price system that decentralizes a given allocation $[x, y]$. The price system is nothing else than a hyperplane that separates the aggregate production set from the set of consumption allocations that are jointly preferred by all consumers. Figure 6 illustrates this general principle.

In lieu of Figure 6 it is not surprising that several convexity assumptions have to be made to prove the second welfare theorem. We will come back to this when we discuss each specific assumption. First we state the separating hyperplane that we will use for our proof. Obviously we can't use the standard theorems commonly used in micro[4] since our commodity space in a general real vector space (possibly infinite dimensional).

We will apply the geometric form of the Hahn-Banach theorem. For this we need the following definition

Definition 77 *Let S be a normed real vector space with norm $\|\|_S$. Define by*

$$b(x,\varepsilon) = \{s \in S : \|x - s\|_S < \varepsilon\}$$

the open ball of radius ε around x. The interior of a set $A \subseteq S$, $\mathring{A}$ is defined to be

$$\mathring{A} = \{x \in A : \exists \varepsilon > 0 \text{ with } b(x,\varepsilon) \subseteq A\}$$

Hence the interior of a set A consists of all the points in A for which we can find a open ball (no matter how small) around the point that lies entirely in A. We then have the following

Theorem 78 *(Geometric Form of the Hahn-Banach Theorem): Let $A, Y \subset S$ be convex sets and assume that*

$$\begin{aligned} \text{either } Y \text{ has an interior point and } A \cap \mathring{Y} &= \emptyset \\ \text{or } S \text{ is finite dimensional and } A \cap Y &= \emptyset \end{aligned}$$

Then there exists a continuous linear functional ϕ, not identically zero on S, and a constant c such that

$$\phi(y) \le c \le \phi(x) \text{ for all } x \in A \text{ and all } y \in Y$$

For the proof of the Hahn-Banach theorem in its several forms see Luenberger (1969), p. 111 and p. 133. For the case that S is finite dimensional this theorem is rather intuitive in light of Figure 6. But since we are interested in commodity spaces with infinite dimensions (typically $S = l_p$, for $p \in [1, \infty]$), we usually have to prove that the aggregate production set Y has an interior point in order to apply the Hahn-Banach theorem. We will two things now: a) prove by example that the requirement of an interior point is an assumption that cannot be dispensed with if S is not finite dimensional b) show that this assumption de facto rules out using $S = l_p$, for $p \in [1, \infty)$, as commodity space when one wants to apply the second welfare theorem.

For the first part consider the following

[4] See MasColell et al., p. 948. This theorem is usually attributed to Minkowski.

Example 79 *Consider as commodity space*

$$S = \{\{x_t\}_{t=0}^{\infty} : x_t \in \mathbf{R} \text{ for all } t,\ \|x\|_S = \sum_{t=0}^{\infty} \beta^t |x_t| < \infty\}$$

for some $\beta \in (0,1)$. Let $A = \{\theta\}$ and

$$Y = \{x \in S : |x_t| \leq 1 \text{ for all } t\}$$

Obviously $A, B \subset S$ are convex sets. In some sense $\theta = (0, 0, \ldots, 0, \ldots)$ lies in the middle of Y, but it does not lie in the interior of Y. Suppose it did, then there exists $\varepsilon > 0$ such that for all $x \in S$ such that

$$\|x - \theta\|_S = \sum_{t=0}^{\infty} \beta^t |x_t| < \varepsilon$$

we have $x \in Y$. But for any $\varepsilon > 0$, define $t(\varepsilon) = \frac{\ln(\frac{\varepsilon}{2})}{\ln(\beta)} + 1$. Then $x = (0, 0, \ldots, x_{t(\varepsilon)} = 2, 0, \ldots) \notin Y$ satisfies $\sum_{t=0}^{\infty} \beta^t |x_t| = 2\beta^{t(\varepsilon)} < \varepsilon$. Since this is true for all $\varepsilon > 0$, this shows that θ is not in the interior of Y, or $A \cap \overset{\circ}{Y} = \emptyset$. A very similar argument shows that no $s \in S$ is in the interior of Y, i.e. $\overset{\circ}{Y} = \emptyset$. Hence the only hypothesis for the Hahn-Banach theorem that fails is that Y has an interior point. We now show that the conclusion of the theorem fails. Suppose, to the contrary, that there exists a continuous linear functional ϕ on S with $\phi(s) \neq 0$ for some $\bar{s} \in S$ and

$$\phi(y) \leq c \leq \phi(\theta) \text{ for all } y \in Y$$

Obviously $\phi(\theta) = \phi(0 \cdot \bar{s}) = 0$ by linearity of ϕ. Hence it follows that for all $y \in Y$, $\phi(y) \leq 0$. Now suppose there exists $\bar{y} \in Y$ such that $\phi(\bar{y}) < 0$. But since $-\bar{y} \in Y$, by linearity $\phi(-\bar{y}) = -\phi(\bar{y}) > 0$ a contradiction. Hence $\phi(y) = 0$ for all $y \in Y$. From this it follows that $\phi(s) = 0$ for all $s \in S$ (why?), contradicting the conclusion of the theorem.

As we will see in the proof of the second welfare theorem, to apply the Hahn-Banach theorem we have to assure that the aggregate production set has nonempty interior. The aggregate production set in many application will be (a subset) of the positive orthant of the commodity space. The problem with taking l_p, $p \in [1, \infty)$ as the commodity space is that, as the next proposition shows, the positive orthant

$$l_p^+ = \{x \in l_p : x_t \geq 0 \text{ for all } t\}$$

has empty interior. The good thing about l_∞ is that is has a nonempty interior. This justifies why we usually use it (or its k-fold product space) as commodity space.

Proposition 80 *The positive orthant of l_p, $p \in [0, \infty)$ has an empty interior. The positive orthant of l_∞ has nonempty interior.*

Proof. For the first part suppose there exists $x \in l_p^+$ and $\varepsilon > 0$ such that $b(x, \varepsilon) \subseteq l_p^+$. Since $x \in l_p$, $x_t \to 0$, i.e. $x_t < \frac{\varepsilon}{2}$ for all $t \geq T(\varepsilon)$. Take any $\tau > T(\varepsilon)$ and define z as

$$z_t = \begin{cases} x_t \text{ if } t \neq \tau \\ x_t - \frac{\varepsilon}{2} \text{ if } t = \tau \end{cases}$$

Evidently $z_\tau < 0$ and hence $z \notin l_p^+$. But since

$$\|x - z\|_p = \left(\sum_{t=0}^{\infty} |x_t - z_t|^p \right)^{\frac{1}{p}} = |x_\tau - z_\tau| = \frac{\varepsilon}{2} < \varepsilon$$

we have $z \in b(x, \varepsilon)$, a contradiction. Hence the interior of l_p^+ is empty, the Hahn-Banach theorem doesn't apply and we can't use it to prove the second welfare theorem.

For the second part it suffices to construct an interior point of l_∞^+. Take $x = (1, 1, \ldots, 1, \ldots)$ and $\varepsilon = \frac{1}{2}$. We want to show that $b(x, \varepsilon) \subseteq l_\infty^+$. Take any $z \in b(x, \varepsilon)$. Clearly $z_t \geq \frac{1}{2} \geq 0$. Furthermore

$$\sup_t |z_t| \leq 1\frac{1}{2} < \infty$$

Hence $z \in l_\infty^+$. ■

Now let us proceed with the statement and the proof of the second welfare theorem. We need the following assumptions

1. For each $i \in I$, X_i is convex.

2. For each $i \in I$, if $x, x' \in X_i$ and $u_i(x) > u_i(x')$, then for all $\lambda \in (0, 1)$

$$u_i(\lambda x + (1 - \lambda)x') > u_i(x')$$

3. For each $i \in I$, u_i is continuous.

4. The aggregate production set Y is convex

5. Either Y has an interior point or S is finite-dimensional.

Note that the second assumption is sometimes referred to as strict quasi-concavity[5] of the utility functions. It implies that the upper contour sets

$$A_x^i = \{z \in X_i : u_i(z) \geq u_i(x)\}$$

are convex, for all i, all $x \in X_i$. Without the convexity assumption 1. assumption 2 would not be well-defined as without convex X_i, $\lambda x + (1-\lambda)x' \notin X_i$ is possible, in which case $u_i(\lambda x + (1-\lambda)x')$ is not well-defined. I mention this since otherwise 1. is not needed for the following theorem. Also note that it is assumption 5 that has no counterpart to the theorem in finite dimensions. It only is required to use the appropriate separating hyperplane theorem in the proof. With these assumptions we can state the second welfare theorem

Theorem 81 *Let $[(x_i^0), (y_j^0)]$ be a Pareto optimal allocation and assume that for some $h \in I$ there is a $\hat{x}_h \in X_h$ with $u_h(\hat{x}_h) > u_h(x_h^0)$. Then there exists a continuous linear functional $\phi : S \rightarrow \mathbf{R}$, not identically zero on S, such that*

1. *for all $j \in J$, $y_j^0 \in \arg\max_{y \in Y_j} \phi(y)$*

2. *for all $i \in I$ and all $x \in X_i$, $u_i(x) \geq u_i(x_i^0)$ implies $\phi(x) \geq \phi(x_i^0)$*

Several comments are in order. The theorem states that (under the assumptions of the theorem) any Pareto optimal allocation can be supported by a price system as a quasi-equilibrium. By definition of Pareto optimality the allocation is feasible and hence satisfies resource balance. The theorem also guarantees profit maximization of firms. For consumers, however, it only guarantees that x_i^0 minimizes the cost of attaining utility $u_i(x_i^0)$, but not utility maximization among the bundles that cost no more than $\phi(x_i^0)$, as would be required by a competitive equilibrium. You also may be used to a version of this theorem that shows that a Pareto optimal allocation can be made into an equilibrium with transfers. Since here we haven't defined ownership and in the equilibrium definition make no reference to the value of endowments or firm ownership (i.e. do NOT require the budget constraint

[5] To me it seems that quasi-concavity is enough for the theorem to hold as quasi-concavity is equivalent to convex upper contour sets which all one needs in the proof.

to hold), we can abstract from transfers, too. The proof of the theorem is similar to the one for finite dimensional commodity spaces.

Proof. Let $[(x_i^0),(y_j^0)]$ be a Pareto optimal allocation and $A^i_{x_i^0}$ be the upper contour sets (as defined above) with respect to x_i^0, for all $i \in I$. Also let $\mathring{A}^i_{x_i^0}$ to be the interior of $A^i_{x_i^0}$, i.e.

$$\mathring{A}^i_{x_i^0} = \{z \in X_i : u_i(z) > u_i(x_i^0)\}$$

By assumption 2. the $A^i_{x_i^0}$ are convex and hence $\mathring{A}^i_{x_i^0}$ is convex. Furthermore $x_i^0 \in A^i_{x_i^0}$, so the $A^i_{x_i^0}$ are nonempty. By one of the hypotheses of the theorem there is some $h \in I$ there is a $\hat{x}_h \in X_h$ with $u_h(\hat{x}_h) > u_h(x_h^0)$. For that h, $\mathring{A}^h_{x_h^0}$ is nonempty. Define

$$A = \mathring{A}^h_{x_h^0} + \sum_{i \neq h} A^i_{x_i^0}$$

A is the set of all aggregate consumption bundles that can be split in such a way as to give every agent at least as much utility and agent h strictly more utility than the Pareto optimal allocation $[(x_i^0),(y_j^0)]$. As A is the sum of nonempty convex sets, so is A. Obviously $A \subset S$. By assumption Y is convex. Since $[(x_i^0),(y_j^0)]$ is a Pareto optimal allocation $A \cap Y = \emptyset$. Otherwise there is an aggregate consumption bundle $x^* \in A \cap Y$ that can be produced (as $x^* \in Y$) and Pareto dominates x^0 (as $x^* \in A$), contradicting Pareto optimality of $[(x_i^0),(y_j^0)]$. With assumption 5. we have all the assumptions we need to apply the Hahn-Banach theorem. Hence there exists a continuous linear functional ϕ on S, not identically zero, and a number c such that

$$\phi(y) \leq c \leq \phi(x) \text{ for all } x \in A, \text{ all } y \in Y$$

It remains to be shown that $[(x_i^0),(y_j^0)]$ together with ϕ satisfy conclusions 1 and 2, i.e. constitute a quasi-equilibrium.

First note that the closure of A is $\bar{A} = \sum_{i\in I} A^i_{x_i^0}$ since by continuity of u_h (assumption 3.) the closure of $\mathring{A}^h_{x_h^0}$ is $A^h_{x_h^0}$. Therefore, since ϕ is continuous, $c \leq \phi(x)$ for all $x \in \bar{A} = \sum_{i\in I} A^i_{x_i^0}$.

Second, note that, since $[(x_i^0),(y_j^0)]$ is Pareto optimal, it is feasible and hence $y^0 \in Y$

$$x^0 = \sum_{i\in I} x_i^0 = \sum_{j\in J} y_j^0 = y^0$$

Obviously $x^0 \in \bar{A}$. Therefore $\phi(x^0) = \phi(y^0) \leq c \leq \phi(x^0)$ which implies $\phi(x^0) = \phi(y^0) = c$.

To show conclusion 1 fix $j \in J$ and suppose there exists $\tilde{y}_j \in Y_j$ such that $\phi(\tilde{y}_j) > \phi(y_j^0)$. For $k \neq j$ define $\tilde{y}_k = y_k^0$. Obviously $\tilde{y} = \sum_j \tilde{y}_j \in Y$ and $\phi(\tilde{y}) > \phi(y^0) = c$, a contradiction to the fact that $\phi(y) \leq c$ for all $y \in Y$. Therefore y_j^0 maximizes $\phi(z)$ subject to $z \in Y_j$, for all $j \in J$.

To show conclusion 2 fix $i \in I$ and suppose there exists $\tilde{x}_i \in X_i$ with $u_i(\tilde{x}_i) \geq u_i(x_i^0)$ and $\phi(\tilde{x}_i) < \phi(x_i^0)$. For $l \neq i$ define $\tilde{x}_l = x_l^0$. Obviously $\tilde{x} = \sum_i \tilde{x}_i \in \bar{A}$ and $\phi(\tilde{x}) < \phi(x^0) = c,$ a contradiction to the fact that $\phi(x) \geq c$ for all $x \in \bar{A}$. Therefore x_i^0 minimizes $\phi(z)$ subject to $u_i(z) \geq u_i(x_i^0), z \in X_i$.
■

We now want to provide a condition that assures that the quasi-equilibrium in the previous theorem is in fact a competitive equilibrium, i.e. is not only cost minimizing for the households, but also utility maximizing. This is done in the following

Remark 82 *Let the hypotheses of the second welfare theorem be satisfied and let ϕ be a continuous linear functional that together with $[(x_i^0), (y_j^0)]$ satisfies the conclusions of the second welfare theorem. Also suppose that for all $i \in I$ there exists $x_i' \in X_i$ such that*

$$\phi(x_i') < \phi(x_i^0)$$

Then $[(x_i^0), (y_j^0), \phi]$ constitutes a competitive equilibrium

Note that, in order to verify the additional condition -the existence of a cheaper point in the consumption set for each $i \in I$- we need a candidate price system ϕ that already passed the test of the second welfare theorem. It is not, as the assumptions for the second welfare theorem, an assumptions on the fundamentals of the economy alone.

Proof. We need to prove that for all $i \in I$, all $x \in X_i$, $\phi(x) \leq \phi(x_i^0)$ implies $u_i(x) \leq u_i(x_i^0)$. Pick an arbitrary $i \in I$, $x \in X_i$ satisfying $\phi(x) \leq \phi(x_i^0)$. Define

$$x_\lambda = \lambda x_i' + (1 - \lambda)x \text{ for all } \lambda \in (0, 1)$$

Since by assumption $\phi(x_i') < \phi(x_i^0)$ and $\phi(x) \leq \phi(x_i^0)$ we have by linearity of ϕ

$$\phi(x_\lambda) = \lambda\phi(x_i') + (1 - \lambda)\,\phi(x) < \phi(x_i^0) \text{ for all } \lambda \in (0, 1)$$

Figure 7.2: Example for Problems with Second Welfare Theorem

Since x_0^i by assumption is part of a quasi-equilibrium and (by convexity of X_i we have $x_\lambda \in X_i$), $u_i(x_\lambda) \geq u_i(x_i^0)$ implies $\phi(x_\lambda) \geq \phi(x_i^0)$, or by contraposition $\phi(x_\lambda) < \phi(x_i^0)$ implies $u_i(x_\lambda) < u_i(x_i^0)$ for all $\lambda \in (0,1)$. But then by continuity of u_i we have $u_i(x) = \lim_{\lambda\to 0} u_i(x_\lambda) \leq u_i(x_i^0)$ as desired. ■

As shown by an example in Stokey et al. the assumption on the existence of a cheaper point cannot be dispensed with when wanting to make sure that a quasi-equilibrium is in fact a competitive equilibrium. In Figure 7 we draw the Edgeworth box of a pure exchange economy. Consumer B's consumption set is the entire positive orthant, whereas consumer A's consumption set is the are above the line marked by $-p$, as indicated by the broken lines. Both consumption sets are convex, the upper contour sets are convex and close as for standard utility functions satisfying assumptions 2. and 3. Point E clearly represents a Pareto optimal allocation (since at E consumer B's utility is globally maximized subject to the allocation being feasible). Furthermore E represents a quasi-equilibrium, since at prices p both consumers minimize costs subject to attaining at least as much utility as with allocation E. However, at prices p (obviously the only candidate for supporting E as competitive equilibrium since tangent to consumer B's indifference curve through E) agent A obtains higher utility at allocation E' with the same cost as with E, hence $[E,p]$ is not a competitive equilibrium. The remark fails because at candidate prices p there is no consumption allocation for A that is feasible (in X_A) and cheaper. This demonstrates that the cheaper-point assumption cannot be dispensed with in the remark. This concludes the discussion of the second welfare theorem.

The last thing we want to do in this section is to demonstrate that our choice of l_∞ as commodity space is not without problems either. We argued earlier that l_p, $p \in [1,\infty)$ is not an attractive alternative. Now we use the second welfare theorem to show that for certain economies the price system needed (whose existence is guaranteed by the theorem) need not lie in l_1, i.e. does not have a representation as a vector $p = (p_0, p_1, \ldots, p_t, \ldots)$. This is bad in the sense that then the price system we get from the theorem does not have a natural economic interpretation. After presenting such a pathological example we will briefly discuss possible remedies.

Example 83 *Let $S = l_\infty$. There is a single consumer and a single firm. The aggregate production set is given by*

$$Y = \{y \in S : 0 \leq y_t \leq 1 + \frac{1}{t},\ \textit{for all } t\}$$

The consumption set is given by

$$X = \{x \in S : x_t \geq 0 \textit{ for all } t\}$$

The utility function $u : X \to \mathbf{R}$ is

$$u(x) = \inf_t x_t$$

[TO BE COMPLETED]

7.8 Type Identical Allocations

[TO BE COMPLETED]

Chapter 8

The Overlapping Generations Model

In this section we will discuss the second major workhorse model of modern macroeconomics, the Overlapping Generations (OLG) model, due to Allais (1947), Samuelson (1958) and Diamond (1965). The structure of this section will be as follows: we will first present a basic pure exchange version of the OLG model, show how to analyze it and contrast its properties with those of a pure exchange economy with infinitely lived agents. The basic differences are that in the OLG model

- competitive equilibria may be Pareto suboptimal
- (outside) money may have positive value
- there may exist a continuum of equilibria

We will demonstrate these properties in detail via examples. We will then discuss the Ricardian Equivalence hypothesis (the notion that, given a stream of government spending the financing method of the government -taxes or budget deficits- does not influence macroeconomic aggregates) for both the infinitely lived agent model as well as the OLG model. Finally we will introduce production into the OLG model to discuss the notion of dynamic inefficiency. The first part of this section will be based on Kehoe (1989), Geanakoplos (1989), the second section on Barro (1974) and the third section on Diamond (1965). Other good sources of information include Blanchard and Fischer (1989), chapter 3, Sargent and Ljungquist, chapter 8 and Azariadis, chapter 11 and 12.

8.1 A Simple Pure Exchange Overlapping Generations Model

Let's start by repeating the infinitely lived agent model to which we will compare the OLG model. Suppose there are I individuals that live forever. There is one nonstorable consumption good in each period. Individuals order consumption allocations according to

$$u_i(c_i) = \sum_{t=0}^{\infty} \beta_i^t U(c_t^i)$$

Agents have deterministic endowment streams $e^i = \{e_t^i\}_{t=0}^{\infty}$. Trade takes place at period 0. The standard definition of an Arrow-Debreu equilibrium goes like this:

Definition 84 *A competitive equilibrium are prices* $\{p_t\}_{t=0}^{\infty}$ *and allocations* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i\in I}$ *such that*

1. *Given* $\{p_t\}_{t=0}^{\infty}$, *for all* $i \in I$, $\{\hat{c}_t^i\}_{t=0}^{\infty}$ *solves* $\max_{c^i \geq 0} u_i(c_i)$ *subject to*

$$\sum_{t=0}^{\infty} p_t(c_t^i - e_t^i) \leq 0$$

2. $$\sum_{i\in I} \hat{c}_t^i = \sum_{i\in I} e_t^i \text{ for all } t$$

What are the main shortcomings of this model that have lead to the development of the OLG model? The first criticism is that individuals apparently do not live forever, so that a model with finitely lived agents is needed. We will see later that we can give the infinitely lived agent model an interpretation in which individuals lived only for a finite number of periods, but, by having an altruistic bequest motive, act so as to maximize the utility of the entire dynasty, which in effect makes the planning horizon of the agent infinite. So infinite lives in itself are not as unsatisfactory as it may seem. But if people live forever, they don't undergo a life cycle with low-income youth, high income middle ages and retirement where labor income drops to zero. In the infinitely lived agent model every period is like the next (which makes it so useful since this stationarity renders dynamic programming techniques easily applicable). So in order to analyze issues like social security, the effect of taxes on retirement decisions, the distributive effects of taxes vs. government deficits, the effects of life-cycle saving on capital accumulation one needs a model in which agents experience a life cycle and in which people of different ages live at the same time in the economy. This is why the OLG model is a very useful tool for applied policy analysis. Because of its interesting (some say, pathological) theoretical properties, it is also an area of intense study among economic theorists.

8.1.1 Basic Setup of the Model

Let us describe the model formally now. Time is discrete, $t = 1, 2, 3, \ldots$ and the economy (but not its people) lives forever. In each period there is a single, nonstorable consumption good. In each time period a new generation (of measure 1) is born, which we index by its date of birth. People live for two periods and then die. By (e_t^t, e_{t+1}^t) we denote generation t's endowment of the consumption good in the first and second period of their live and by (c_t^t, c_{t+1}^t) we denote the consumption allocation of generation t. Hence in time t there are two generations alive, one old generation $t-1$ that has endowment e_t^{t-1} and consumption c_t^{t-1} and one young generation t that has endowment e_t^t and consumption c_t^t. In addition, in period 1 there is an initial old generation 0 that has endowment e_1^0 and consumes c_1^0. In some of our applications we will endow the initial generation with an amount of outside money[1] m. We will NOT assume $m \geq 0$. If $m \geq 0$, then m can be interpreted straightforwardly as fiat money, if $m < 0$ one should envision the initial old people having borrowed from some institution (which is, however, outside the model) and m is the amount to be repaid.

In the next Table 1 we demonstrate the demographic structure of the economy. Note that there are both an infinite number of periods as well as well as an infinite number of agents in this economy. This "double infinity" has been cited to be the major source of the theoretical peculiarities of the OLG model (prominently by Karl Shell).

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[1]Money that is, on net, an asset of the private economy, is "outside money". This includes fiat currency issued by the government. In contrast, inside money (such as bank deposits) is both an asset as well as a liability of the private sector (in the case of deposits an asset of the deposit holder, a liability to the bank).

Table 1

		Time			
G		1	2	... t	$t+1$
e	0	(c_1^0, e_1^0)			
n	1	(c_1^1, e_1^1)	(c_2^1, e_2^1)		
e	$\vdots$			$\ddots$	
r	$t-1$			(c_t^{t-1}, e_t^{t-1})	
a	t			(c_t^t, e_t^t)	(c_{t+1}^t, e_{t+1}^t)
t.	$t+1$				$(c_{t+1}^{t+1}, e_{t+1}^{t+1})$

Preferences of individuals are assumed to be representable by an additively separable utility function of the form

$$u_t(c) = U(c_t^t) + \beta U(c_{t+1}^t)$$

and the preferences of the initial old generation is representable by

$$u_0(c) = U(c_1^0)$$

We shall assume that U is strictly increasing, strictly concave and twice continuously differentiable. This completes the description of the economy. Note that we can easily represent this economy in our formal Arrow-Debreu language from Chapter 7 since it is a standard pure exchange economy with infinite number of agents and the peculiar preference and endowment structure $e_s^t = 0$ for all $s \neq t, t+1$ and $u_t(c)$ only depending on c_t^t, c_{t+1}^t. You should complete the formal representation as a useful homework exercise.

The following definitions are straightforward

Definition 85 *An allocation is a sequence $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^\infty$. An allocation is feasible if $c_t^{t-1}, c_t^t \geq 0$ for all $t \geq 1$ and*

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1$$

An allocation $c_1^0, \{(c_t^t, c_{t+1}^t)\}_{t=1}^\infty$ is Pareto optimal if it is feasible and if there is no other feasible allocation $\hat{c}_0^1, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^\infty$ such that

$$\begin{aligned} u_t(\hat{c}_t^t, \hat{c}_{t+1}^t) &\geq u_t(c_t^t, c_{t+1}^t) \text{ for all } t \geq 1 \\ u_0(\hat{c}_1^0) &\geq u_0(c_1^0) \end{aligned}$$

with strict inequality for at least one $t \geq 0$.

We now define an equilibrium for this economy in two different ways, depending on the market structure. Let p_t be the price of one unit of the consumption good at period t. In the presence of money (i.e. $m \neq 0$) we will take money to be the numeraire. This is important since we can only normalize the price of one commoditiy to 1, so with money no further normalizations are admissible. Of course, without money we are free to normalize the price of one other commodity. Keep this in mind for later. We now have the following

Definition 86 *Given m, an Arrow-Debreu equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ and prices $\{p_t\}_{t=1}^{\infty}$ such that*

1. *Given $\{p_t\}_{t=1}^{\infty}$, for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ solves*

$$\max_{(c_t^t, c_{t+1}^t) \geq 0} u_t(c_t^t, c_{t+1}^t) \tag{8.1}$$

$$s.t. \;\; p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t \tag{8.2}$$

2. *Given p_1, $\hat{c}_1^0$ solves*

$$\max_{c_1^0} u_0(c_1^0)$$

$$s.t. \;\; p_1 c_1^0 \leq p_1 e_1^0 + m \tag{8.3}$$

3. *For all $t \geq 1$ (Resource Balance or goods market clearing)*

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \textit{ for all } t \geq 1$$

As usual within the Arrow-Debreu framework, trading takes place in a hypothetical centralized market place at period 0 (even though the generations are not born yet).[2] There is an alternative definition of equilibrium that assumes sequential trading. Let r_{t+1} be the interest rate from period t to period $t+1$ and s_t^t be the savings of generation t from period t to period $t+1$. We will look at a slightly different form of assets in this section.

[2] When naming this definition after Arrow-Debreu I make reference to the market *structure* that is envisioned under this definition of equilibrium. Others, including Geanakoplos, refer to a particular *model* when talking about Arrow-Debreu, the standard general equilibrium model encountered in micro with finite number of simultaneously living agents. I hope this does not cause any confusion.

Previously we dealt with one-period IOU's that had price q_t in period t and paid out one unit of the consumption good in $t+1$ (so-called zero bonds). Now we consider assets that cost one unit of consumption in period t and deliver $1+r_{t+1}$ units tomorrow. Equilibria with these two different assets are obviously equivalent to each other, but the latter specification is easier to interpret if the asset at hand is fiat money.

We define a Sequential Markets (SM) equilibrium as follows:

Definition 87 *Given m, a sequential markets equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$ and interest rates $\{r_t\}_{t=1}^{\infty}$ such that*

1. *Given $\{r_t\}_{t=1}^{\infty}$ for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves*

$$\max_{(c_t^t, c_{t+1}^t) \geq 0, s_t^t} u_t(c_t^t, c_{t+1}^t)$$

$$s.t.\ c_t^t + s_t^t \leq e_t^t \tag{8.4}$$

$$c_{t+1}^t \leq e_{t+1}^t + (1+r_{t+1})s_t^t \tag{8.5}$$

2. *Given r_1, $\hat{c}_1^0$ solves*

$$\max_{c_1^0} u_0(c_1^0)$$

$$s.t.\ c_1^0 \leq e_1^0 + (1+r_1)m$$

3. *For all $t \geq 1$ (Resource Balance or goods market clearing)*

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1 \tag{8.6}$$

In this interpretation trade takes place sequentially in spot markets for consumption goods that open in each period. In addition there is an asset market through which individuals do their saving. Remember that when we wrote down the sequential formulation of equilibrium for an infinitely lived consumer model we had to add a shortsale constraint on borrowing (i.e. $s_t \geq -A$) in order to prevent Ponzi schemes, the continuous rolling over of higher and higher debt. This is not necessary in the OLG model as people live for a finite (two) number of periods (and we, as usual, assume perfect enforceability of contracts)

Given that the period utility function U is strictly increasing, the budget constraints (8.4) and (8.5) hold with equality. Take budget constraint (8.5) for generation t and (8.4) for generation $t+1$ and sum them up to obtain

$$c_{t+1}^{t} + c_{t+1}^{t+1} + s_{t+1}^{t+1} = e_{t+1}^{t} + e_{t+1}^{t+1} + (1 + r_{t+1})s_t^t$$

Now use equation (8.6) to obtain

$$s_{t+1}^{t+1} = (1 + r_{t+1})s_t^t$$

Doing the same manipulations for generation 0 and 1 gives

$$s_1^1 = (1 + r_1)m$$

and hence, using repeated substitution one obtains

$$s_t^t = \Pi_{\tau=1}^{t}(1 + r_\tau)m \tag{8.7}$$

This is the market clearing condition for the asset market: the amount of saving (in terms of the period t consumption good) has to equal the value of the outside supply of assets, $\Pi_{\tau=1}^{t}(1 + r_\tau)m$. Strictly speaking one should include condition (8.7) in the definition of equilibrium. By Walras' law however, either the asset market or the good market equilibrium condition is redundant.

There is an obvious sense in which equilibria for the Arrow-Debreu economy (with trading at period 0) are equivalent to equilibria for the sequential markets economy. For $r_{t+1} > -1$ combine (8.4) and (8.5) into

$$c_t^t + \frac{c_{t+1}^t}{1 + r_{t+1}} = e_t^t + \frac{e_{t+1}^t}{1 + r_{t+1}}$$

Divide (8.2) by $p_t > 0$ to obtain

$$c_t^t + \frac{p_{t+1}}{p_t}c_{t+1}^t = e_t^t + \frac{p_{t+1}}{p_t}e_{t+1}^t$$

Furthermore divide (8.3) by $p_1 > 0$ to obtain

$$c_1^0 \leq e_1^0 + \frac{m}{p_1}$$

We then can straightforwardly prove the following proposition

Proposition 88 *Let allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^\infty$ and prices $\{p_t\}_{t=1}^\infty$ constitute an Arrow-Debreu equilibrium with $p_t > 0$ for all $t \geq 1$. Then there exists a corresponding sequential market equilibrium with allocations $\tilde{c}_1^0, \{(\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{s}_t^t)\}_{t=1}^\infty$ and interest rates $\{r_t\}_{t=1}^\infty$with*

$$\begin{aligned}
\tilde{c}_t^{t-1} &= \hat{c}_t^{t-1} \text{ for all } t \geq 1 \\
\tilde{c}_t^t &= \hat{c}_t^t \text{ for all } t \geq 1
\end{aligned}$$

Furthermore, let allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^\infty$ and interest rates $\{r_t\}_{t=1}^\infty$ constitute a sequential market equilibrium with $r_t > -1$ for all $t \geq 0$. Then there exists a corresponding Arrow-Debreu equilibrium with allocations $\tilde{c}_1^0, \{(\tilde{c}_t^t, \tilde{c}_{t+1}^t)\}_{t=1}^\infty$ and prices $\{p_t\}_{t=1}^\infty$ such that

$$\begin{aligned}
\tilde{c}_t^{t-1} &= \hat{c}_t^{t-1} \text{ for all } t \geq 1 \\
\tilde{c}_t^t &= \hat{c}_t^t \text{ for all } t \geq 1
\end{aligned}$$

Proof. The proof is similar to the infinite horizon counterpart. Given equilibrium Arrow-Debreu prices $\{p_t\}_{t=1}^\infty$ define interest rates as

$$\begin{aligned}
1 + r_{t+1} &= \frac{p_t}{p_{t+1}} \\
1 + r_1 &= \frac{1}{p_1}
\end{aligned}$$

and savings

$$\tilde{s}_t^t = e_t^t - \hat{c}_t^t$$

It is straightforward to verify that the allocations and prices so constructed constitute a sequential markets equilibrium.

Given equilibrium sequential markets interest rates $\{r_t\}_{t=1}^\infty$ define Arrow-Debreu prices by

$$\begin{aligned}
p_1 &= \frac{1}{1 + r_1} \\
p_{t+1} &= \frac{p_t}{1 + r_{t+1}}
\end{aligned}$$

Again it is straightforward to verify that the prices and allocations so constructed form an Arrow-Debreu equilibrium. ■

Note that the requirement on interest rates is weaker for the OLG version of this proposition than for the infinite horizon counterpart. This is due to the particular specification of the no-Ponzi condition used. A less stringent condition still ruling out Ponzi schemes would lead to a weaker condtion in the proposition for the infinite horizon economy also.

Also note that with this equivalence we have that

$$\Pi_{\tau=1}^{t}(1+r_\tau)m = \frac{m}{p_t}$$

so that the asset market clearing condition for the sequential markets economy can be written as

$$p_t s_t^t = m$$

i.e. the demand for assets (saving) equals the outside supply of assets, m. Note that the demanders of the assets are the currently young whereas the suppliers are the currently old people. From the equivalence we can also see that the return on the asset (to be interpreted as money) equals

$$\begin{aligned}
1+r_{t+1} &= \frac{p_t}{p_{t+1}} = \frac{1}{1+\pi_{t+1}} \\
(1+r_{t+1})(1+\pi_{t+1}) &= 1 \\
r_{t+1} &\approx -\pi_{t+1}
\end{aligned}$$

where π_{t+1} is the inflation rate from period t to $t+1$. As it should be, the real return on money equals the negative of the inflation rate.

8.1.2 Analysis of the Model Using Offer Curves

Unless otherwise noted in this subsection we will focus on Arrow-Debreu equilibria. Gale (1973) developed a nice way of analyzing the equilibria of a two-period OLG economy graphically, using offer curves. First let us assume that the economy is stationary in that $e_t^t = w_1$ and $e_{t+1}^t = w_2$, i.e. the endowments are time invariant. For given $p_t, p_{t+1} > 0$ let by $c_t^t(p_t, p_{t+1})$ and $c_{t+1}^t(p_t, p_{t+1})$ denote the solution to maximizing (8.1) subject to (8.2) for all $t \geq 1$. Given our assumptions this solution is unique. Let the excess demand functions y and z be defined by

$$\begin{aligned}
y(p_t, p_{t+1}) &= c_t^t(p_t, p_{t+1}) - e_t^t \\
&= c_t^t(p_t, p_{t+1}) - w_1 \\
z(p_t, p_{t+1}) &= c_{t+1}^t(p_t, p_{t+1}) - w_2
\end{aligned}$$

These two functions summarize, for given prices, all implications that consumer optimization has for equilibrium allocations. Note that from the Arrow-Debreu budget constraint it is obvious that y and z only depend on the ratio $\frac{p_{t+1}}{p_t}$, but not on p_t and p_{t+1} separately (this is nothing else than saying that the excess demand functions are homogeneous of degree zero in prices, as they should be). Varying $\frac{p_{t+1}}{p_t}$ between 0 and ∞ (not inclusive) one obtains a locus of optimal excess demands in (y, z) space, the so called offer curve. Let us denote this curve as

$$(y, f(y)) \tag{8.8}$$

where it is understood that f can be a correspondence, i.e. multi-valued. A point on the offer curve is an optimal excess demand function for *some* $\frac{p_{t+1}}{p_t} \in (0, \infty)$. Also note that since $c_t^t(p_t, p_{t+1}) \geq 0$ and $c_{t+1}^t(p_t, p_{t+1}) \geq 0$ the offer curve obviously satisfies $y(p_t, p_{t+1}) \geq -w_1$ and $z(p_t, p_{t+1}) \geq -w_2$. Furthermore, since the optimal choices obviously satisfy the budget constraint, i.e.

$$\begin{aligned} p_t y(p_t, p_{t+1}) + p_{t+1} z(p_t, p_{t+1}) &= 0 \\ \frac{z(p_t, p_{t+1})}{y(p_t, p_{t+1})} &= -\frac{p_t}{p_{t+1}} \end{aligned} \tag{8.9}$$

Equation (8.9) is an equation in the two unknowns (p_t, p_{t+1}) for a given $t \geq 1$. Obviously $(y, z) = (0, 0)$ is on the offer curve, as for appropriate prices (which we will determine later) no trade is the optimal trading strategy. Equation (8.9) is very useful in that for a given point on the offer curve $(y(p_t, p_{t+1}), z(p_t, p_{t+1}))$ in y-z space with $y(p_t, p_{t+1}) \neq 0$ we can immediately read off the price ratio at which these are the optimal demands. Draw a straight line through the point (y, z) and the origin; the slope of that line equals $-\frac{p_t}{p_{t+1}}$. One should also note that if $y(p_t, p_{t+1})$ is negative, then $z(p_t, p_{t+1})$ is positive and vice versa. Let's look at an example

Example 89 *Let $w_1 = \varepsilon$, $w_2 = 1 - \varepsilon$, with $\varepsilon > 0$. Also let $U(c) = \ln(c)$ and $\beta = 1$. Then the first order conditions imply*

$$p_t c_t^t = p_{t+1} c_{t+1}^t \tag{8.10}$$

Figure 8.1: Offer Curves in OLG Models

and the optimal consumption choices are

$$c_t^t(p_t, p_{t+1}) = \frac{1}{2}\left(\varepsilon + \frac{p_{t+1}}{p_t}(1-\varepsilon)\right) \tag{8.11}$$

$$c_{t+1}^t(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_t}{p_{t+1}}\varepsilon + (1-\varepsilon)\right) \tag{8.12}$$

the excess demands are given by

$$y(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_{t+1}}{p_t}(1-\varepsilon) - \varepsilon\right) \tag{8.13}$$

$$z(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_t}{p_{t+1}}\varepsilon - (1-\varepsilon)\right) \tag{8.14}$$

Note that as $\frac{p_{t+1}}{p_t} \in (0, \infty)$ varies, y varies between $-\frac{\varepsilon}{2}$ and ∞ and z varies between $-\frac{(1-\varepsilon)}{2}$ and ∞. Solving z as a function of y by eliminating $\frac{p_{t+1}}{p_t}$ yields

$$z = \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2} \text{ for } y \in (-\frac{\varepsilon}{2}, \infty) \tag{8.15}$$

This is the offer curve $(y, z) = (y, f(y))$. We draw the offer curve in Figure 8

The discussion of the offer curve takes care of the first part of the equilibrium definition, namely optimality. It is straightforward to express goods market clearing in terms of excess demand functions as

$$y(p_t, p_{t+1}) + z(p_{t-1}, p_t) = 0 \tag{8.16}$$

Also note that for the initial old generation the excess demand function is given by

$$z_0(p_1, m) = \frac{m}{p_1}$$

so that the goods market equilibrium condition for the first period reads as

$$y(p_1, p_2) + z_0(p_1, m) = 0 \tag{8.17}$$

Graphically in (y, z)-space equations (8.16) and (8.17) are straight lines through the origin with slope -1. All points on this line are resource feasible. We therefore have the following procedure to find equilibria for this economy for a given initial endowment of money m of the initial old generation, using the offer curve (8.8) and the resource feasibility constraints (8.16) and (8.17).

1. Pick an initial price p_1 (note that this is NOT a normalization as in the infinitely lived agent model since the value of p_1 determines the real value of money $\frac{m}{p_1}$ the initial old generation is endowed with; we have already normailzed the price of money). Hence we know $z_0(p_1, m)$. From (8.17) this determines $y(p_1, p_2)$.

2. From the offer curve (8.8) we determine $z(p_1, p_2) \in f(y(p_1, p_2))$. Note that if f is a correspondence then there are multiple choices for z.

3. Once we know $z(p_1, p_2)$, from (8.16) we can find $y(p_2, p_3)$ and so forth. In this way we determine the entire equilibrium consumption allocation

$$\begin{aligned}
c_1^0 &= z_0(p_1, m) + w_2 \\
c_t^t &= y(p_t, p_{t+1}) + w_1 \\
c_{t+1}^t &= z(p_t, p_{t+1}) + w_2
\end{aligned}$$

4. Equilibrium prices can then be found, given p_1 from equation (8.9). Any initial p_1 that induces, in such a way, sequences $c_1^0, \{(c_t^t, c_{t+1}^t), p_t\}_{t=1}^{\infty}$ such that the consumption sequence satisfies $c_t^{t-1}, c_t^t \geq 0$ is an equilibrium for given money stock. This already indicates the possibility of a lot of equilibria for this model, a fact that we will demonstrate below.

This algorithm can be demonstrated graphically using the offer curve diagram. We add the line representing goods market clearing, equation (8.16). In the (y, z)-plane this is a straight line through the origin with slope -1. This line intersects the offer curve at least once, namely at the origin. Unless we have the degenerate situation that the offer curve has slope -1 at the origin, there is (at least) one other intersection of the offer curve with the goods clearing line. These intersection will have special significance as they will represent stationary equilibria. As we will see, there is a load of other equilibria as well. We will first describe the graphical procedure in general and then look at some examples. See Figure 9.

Figure 8.2: Using Offer Curves in OLG Models

Given any m (for concreteness let $m > 0$) pick $p_1 > 0$. This determines $z_0 = \frac{m}{p_1} > 0$. Find this quantity on the z-axis, representing the excess demand of the initial old generation. From this point on the z-axis go horizontally to the goods market line, from there down to the y-axis. The point on the y-axis represents the excess demand function of generation 1 when young. From this point $y_1 = y(p_1, p_2)$ go vertically to the offer curve, then horizontally to the z-axis. The resulting point $z_1 = z(p_1, p_2)$ is the excess demand of generation 1 when old. Then back horizontally to the goods market clearing condition and down yields $y_2 = y(p_2, p_3)$, the excess demand for the second generation and so on. This way the entire equilibrium consumption allocation can be constructed. Equilibrium prices are easily found from equilibrium allocations with (8.9), given p_1. In such a way we construct an entire equilibrium graphically.

Let's now look at some example.

Example 90 *Reconsider the example with isoelastic utility above. We found the offer curve to be*

$$z = \frac{\varepsilon(1-\varepsilon)}{4y+2\varepsilon} - \frac{1-\varepsilon}{2} \text{ for } y \in (-\frac{\varepsilon}{2}, \infty)$$

The goods market equilibrium condition is

$$y + z = 0$$

Now let's construct an equilibrium for the case $m = 0$, for zero supply of outside money. Following the procedure outlined above we first find the excess demand function for the initial old generation $z_0(m, p_1) = 0$ for all $p_1 > 0$. Then from goods market $y(p_1, p_2) = -z_0(m, p_1) = 0$. From the offer curve

$$\begin{aligned} z(p_1, p_2) &= \frac{\varepsilon(1-\varepsilon)}{4y(p_1,p_2)+2\varepsilon} - \frac{1-\varepsilon}{2} \\ &= \frac{\varepsilon(1-\varepsilon)}{2\varepsilon} - \frac{1-\varepsilon}{2} \\ &= 0 \end{aligned}$$

and continuing we find $z(p_t, p_{t+1}) = y(p_t, p_{t+1}) = 0$ for all $t \geq 1$. This implies that the equilibrium allocation is $c_t^{t-1} = 1-\varepsilon, c_t^t = \varepsilon$. In this equilibrium every *consumer eats his endowment in each period and no trade between generations takes place. We call this equilibrium the autarkic equilibrium. Obviously we can't determine equilibrium prices from equation* (8.9)*. However, the first order conditions imply that*

$$\frac{p_{t+1}}{p_t} = \frac{c_t^t}{c_{t+1}^t} = \frac{\varepsilon}{1-\varepsilon}$$

For $m = 0$ *we can, without loss of generality, normalize the price of the first period consumption good* $p_1 = 1$*. Note again that only for* $m = 0$ *this normalization is innocuous, since it does not change the real value of the stock of outside money that the initial old generation is endowed with. With this normalization the sequence* $\{p_t\}_{t=1}^{\infty}$ *defined as*

$$p_t = \left(\frac{\varepsilon}{1-\varepsilon}\right)^{t-1}$$

together with the autarkic allocation form an (Arrow-Debreu)-equilibrium. Obviously any other price sequence $\{\bar{p}_t\}$ *with* $\bar{p}_t = \alpha p_t$ *for any* $\alpha > 1$*, is also an equilibrium price sequence supporting the autarkic allocation as equilibrium. This is not, however, what we mean by the possibility of a continuum of equilibria in OLG-model, but rather the usual feature of standard competitive equilibria that the equilibrium prices are only determined up to one normalization. In fact, for this example with* $m = 0$*, the autarkic equilibrium is the unique equilibrium for this economy.*[3] *This is easily seen. Since the initial old generation has no money, only its endowments* $1 - \varepsilon$*, there is no way for them to consume more than their endowments. Obviously they can always assure to consume at least their endowments by not trading, and that is what they do for any* $p_1 > 0$ *(obviously* $p_1 \leq 0$ *is not possible in equilibrium). But then from the resource constraint it follows that the first young generation must consume their endowments when young. Since they haven't saved anything, the best they can do when old is to consume their endowment again. But then the next young generation is forced to consume their endowments and so forth. Trade breaks down completely. For this allocation to be an equilibrium prices must be such that at these prices all generations*

[3] The fact that the autarkic is the only equilibrium is specific to pure exchange OLG-models with agents living for only two periods. Therefore Samuelson (1958) considered three-period lived agents for most of his analysis.

actually find it optimal not to trade, which yields the prices below.[4]

Note that in the picture the second intersection of the offer curve with the resource constraint (the first is at the origin) occurs in the forth orthant. This need not be the case. If the slope of the offer curve at the origin is less than one, we obtain the picture above, if the slope is bigger than one, then the second intersection occurs in the second orthant. Let us distinguish between these two cases more carefully. In general, the price ratio supporting the autarkic equilibrium satisfies

$$\frac{p_t}{p_{t+1}} = \frac{U'(e_t^t)}{\beta U'(e_{t+1}^t)} = \frac{U'(w_1)}{\beta U'(w_2)}$$

and this ratio represents the slope of the offer curve at the origin. With this in mind define the autarkic interest rate (remember our equivalence result from above) as

$$1 + \bar{r} = \frac{U'(w_1)}{\beta u'(w_2)}$$

Gale (1973) has invented the following terminology: when $\bar{r} < 0$ he calls this the Samueson case, whereas when $\bar{r} \geq 0$ he calls this the classical case.[5] As

[4]If you look at Sargent and Ljungquist (1999), Chapter 8, you will see that they claim to construct several equilibria for exactly this example. Note, however, that their equilibrium definition has as feasibility constraint

$$c_t^{t-1} + c_t^t \leq e_t^{t-1} + e_t^t$$

and all the equilibria apart from the autarkic one constructed above have the feature that for $t = 1$

$$c_1^0 + c_1^1 < e_1^0 + e_1^1$$

which violate feasibility in the way we have defined it. Personally I find the free disposal assumption not satisfactory; it makes, however, their life easier in some of the examples to follow, whereas in my discussion I need more handwaving. You'll see.

[5]More generally, the Samuelson case is defined by the condition that savings of the young generation be positive at an interest rate equal to the population growth rate n. So far we have assumed $n = 0$, so the Samuelson case requires saving to be positive at zero interest rate. We stated the condition as $\bar{r} < 0$. But if the interest rate at which the young don't save (the autarkic allocation) is smaller than zero, then at the higher interest rate of zero they will save a positive amount, so that we can define the Samuelson case as in the text, provided that savings are strictly increasing in the interest rate. This in turn requires the assumption that first and second period consumption are strict gross substitutes, so that the offer curve is not backward-bending. In the homework you will encounter an example in which this assumption is not satisfied.

it will turn out and will be demonstrated below autarkic equilibria are not Pareto optimal in the Samuelson case whereas they are in the classical case.

8.1.3 Inefficient Equilibria

The preceding example can also serve to demonstrate our first major feature of OLG economies that sets it apart from the standard infinitely lived consumer model with finite number of agents: competitive equilibria may be not be Pareto optimal. For economies like the one defined at the beginning of the section the two welfare theorems were proved and hence equilibria are Pareto optimal. Now let's see that the equilibrium constructed above for the OLG model may not be.

Note that in the economy above the aggregate endowment equals to 1 in each period. Also note that then the value of the aggregate endowment at the equilibrium prices, given by $\sum_{t=1}^{\infty} p_t$. Obviously, if $\varepsilon < 0.5$, then this sum converges and the value of the aggregate endowment is finite, whereas if $\varepsilon \geq 0.5$, then the value of the aggregate endowment is infinite. Whether the value of the aggregate endowment is infinite has profound implications for the welfare properties of the competitive equilibrium. In particular, using a similar argument as in the standard proof of the first welfare theorem you can show (and will do so in the homework) that if $\sum_{t=1}^{\infty} p_t < \infty$, then the competitive equilibrium allocation for this economy (and in general for any pure exchange OLG economy) is Pareto-efficient. If, however, the value of the aggregate endowment is infinite (at the equilibrium prices), then the competitive equilibrium MAY not be Pareto optimal. In our current example it turns out that if $\varepsilon > 0.5$, then the autarkic equilibrium is not Pareto efficient, whereas if $\varepsilon = 0.5$ it is. Since interest rates are defined as

$$r_{t+1} = \frac{p_t}{p_{t+1}} - 1$$

$\varepsilon \dot{<} 0.5$ implies $r_{t+1} = \frac{1-\varepsilon}{\varepsilon} - 1 = \frac{1}{\varepsilon} - 2$. Hence $\varepsilon < 0.5$ implies $r_{t+1} > 0$ (the classical case) and $\varepsilon \geq 0.5$ implies $r_{t+1} < 0$. (the Samuelson case). Inefficiency is therefore associated with low (negative interest rates). In fact, Balasko and Shell (1980) show that the autarkic equilibrium is Pareto optimal if and only if

$$\sum_{t=1}^{\infty} \prod_{\tau=1}^{t} (1 + r_{\tau+1}) = +\infty$$

where $\{r_{t+1}\}$ is the sequence of autarkic equilibrium interest rates.[6] Obviously the above equation is satisfied if and only if $\varepsilon \leq 0.5$.

Let us briefly demonstrate the first claim (a more careful discussion is left for the homework). To show that for $\varepsilon > 0.5$ the autarkic allocation (which is the unique equilibrium allocation) is not Pareto optimal it is sufficient to find another feasible allocation that Pareto-dominates it. Let's do this graphically in Figure 10. The autarkic allocation is represented by the origin (excess demand functions equal zero). Consider an alternative allocation represented by the intersection of the offer curve and the resource constraint. We want to argue that this point Pareto dominates the autarkic allocation. First consider an arbitrary generation $t \geq 1$. Note that the indifference curve through the origin must lie to the outside of the offer curve (they are equal at the origin, but everywhere else the indifference curve lies below). Why: the autarkic point can be chosen at all price ratios. Thus a point on the offer

[6] Rather than a formal proof (which is quite involved), let's develop some intuition for why low interest rates are associated with inefficiency. Take the autarkic allocation and try to construct a Pareto improvement. In particular, give additional $\delta_0 > 0$ units of consumption to the initial old generation. This obviously improves this generation's life. From resource feasibilty this requires taking away δ_0 from generation 1 in their first period of life. To make them not worse of they have to recieve δ_1 in additional consumption in their second period of life, with δ_1 satisfying

$$\delta_0 U'(e_1^1) = \delta_1 \beta U'(e_2^1)$$

or

$$\begin{aligned} \delta_1 &= \delta_0 \frac{U'(e_1^1)}{\beta U'(e_2^1)} \\ &= \delta_0 (1 + r_2) > 0 \end{aligned}$$

and in general

$$\delta_t = \delta_0 \prod_{\tau=1}^{t} (1 + r_{\tau+1})$$

are the required transfers in the second period of generation t's life to compensate for the reduction of first period consumption. Obviously such a scheme does not work if the economy ends at fine time T since the last generation (that lives only through youth) is worse off. But as our economy extends forever, such an intergenerational transfer scheme is feasible provided that the δ_t don't grow too fast, i.e. if interest rates are sufficiently small. But if such a transfer scheme is feasible, then we found a Pareto improvement over the original autarkic allocation, and hence the autarkic equilibrium allocation is not Pareto efficient.

Figure 8.3: Pareto Optimality in OLG Models

curve was chosen when the autarkic allocation was affordable, and therefore must represent a higher utility. This demonstrates that the alternative point marked in the figure (which is both on the offer curve as well as the resource constraint, the line with slope -1) is at least as good as the autarkic allocation for all generations $t \geq 1$. What about the initial old generation? In the autarkic allocation it has $c_1^0 = 1 - \varepsilon$, or $z_0 = 0$. In the new allocation it has $z_0 > 0$ as shown in the figure, so the initial old generation is strictly better off in this new allocation. Hence the alternative allocation Pareto-dominates the autarkic equilibrium allocation, which shows that this allocation is not Pareto-optimal. In the homework you are asked to make this argument rigorous by actually computing the alternative allocation and then arguing that it Pareto-dominates the autarkic equilibrium.

What in our graphical argument hinges on the assumption that $\varepsilon > 0.5$. Remember that for $\varepsilon \leq 0.5$ we have said that the autarkic allocation is actually Pareto optimal. It turns out that for $\varepsilon < 0.5$, the intersection of the resource constraint and the offer curve lies in the fourth orthant instead of in the second as in Figure 10. It is still the case that every generation $t \geq 1$ at least weakly prefers the alternative to the autarkic allocation. Now, however, this alternative allocation has $z_0 < 0$, which makes the initial old generation worse off than in the autarkic allocation, so that the argument does not work. Finally, for $\varepsilon = 0.5$ we have the degenerate situation that the slope of the offer curve at the origin is -1, so that the offer curve is tangent to the resource line and there is no second intersection. Again the argument does not work and we can't argue that the autarkic allocation is not Pareto optimal. It is an interesting optional exercise to show that for $\varepsilon = 0.5$ the autarkic allocation is Pareto optimal.

Now we want to demonstrate the second and third feature of OLG models that set it apart from standard Arrow-Debreu economies, namely the possibility of a continuum of equilibria and the fact that outside money may have positive value. We will see that, given the way we have defined our equilibria, these two issues are intimately linked. So now let us suppose that $m \neq 0$. In our discussion we will assume that $m > 0$, the situation for $m < 0$ is symmetric. We first want to argue that for $m > 0$ the economy has a continuum of equilibria, not of the trivial sort that only prices differ by a constant, but that allocations differ across equilibria. Let us first look at equilibria that are stationary in the following sense:

Definition 91 *An equilibrium is stationary if* $c_t^{t-1} = c^o$, $c_t^t = c^y$ *and* $\frac{p_{t+1}}{p_t} = a$, *where* a *is a constant.*

Given that we made the assumption that each generation has the same endowment structure a stationary equilibrium necessarily has to satisfy $y(p_t, p_{t+1}) = y$, $z_0(m, p_1) = z(p_t, p_{t+1}) = z$ for all $t \geq 1$. From our offer curve diagram the only candidates are the autarkic equilibrium (the origin) and any other allocations represented by intersections of the offer curve and the resource line. We will discuss the possibility of an autarkic equilibrium with money later. With respect to other stationary equilibria, they all have to have prices $\frac{p_{t+1}}{p_t} = 1$, with p_1 such that $(\frac{m}{p_1}, -\frac{m}{p_1})$ is on the offer curve. For our previous example, for any $m \neq 0$ we find the stationary equilibrium by solving for the intersection of offer curve and resource line

$$\begin{aligned} y + z &= 0 \\ z &= \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2} \end{aligned}$$

This yields a second order polynomial in y

$$-y = \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2}$$

whose one solution is $y = 0$ (the autarkic allocation) and the other solution is $y = \frac{1}{2} - \varepsilon$, so that $z = -\frac{1}{2} + \varepsilon$. Hence the corresponding consumption allocation has

$$c_t^{t-1} = c_t^t = \frac{1}{2} \text{ for all } t \geq 1$$

In order for this to be an equilibrium we need

$$\frac{1}{2} = c_1^0 = (1-\varepsilon) + \frac{m}{p_1}$$

hence $p_1 = \frac{m}{\varepsilon - 0.5} > 0$. Therefore a stationary equilibrium (apart from autarky) only exists for $m > 0$ and $\varepsilon > 0.5$ or $m < 0$ and $\varepsilon < 0.5$. Also note that the choice of p_1 is not a matter of normalization: any multiple of p_1 will not yield a stationary equilibrium. The equilibrium prices supporting the stationary allocation have $p_t = p_1$ for all $t \geq 1$. Finally note that this equilibrium, since it features $\frac{p_{t+1}}{p_t} = 1$, has an inflation rate of $\pi_{t+1} = -r_{t+1} = 0$. It is exactly this equilibrium allocation that we used to prove that, for $\varepsilon > 0.5$, the autarkic equilibrium is not Pareto-efficient.

How about the autarkic allocation? Obviously it is stationary as $c_t^{t-1} = 1-\varepsilon$ and $c_t^t = \varepsilon$ for all $t \geq 1$. But can it be made into an equilibrium if $m \neq 0$. If we look at the sequential markets equilibrium definition there is no problem: the budget constraint of the initial old generation reads

$$c_1^0 = 1 - \varepsilon + (1 + r_1)m$$

So we need $r_1 = -1$. For all other generations the same arguments as without money apply and the interest sequence satisfying $r_1 = -1$, $r_{t+1} = \frac{1-\varepsilon}{\varepsilon} - 1$ for all $t \geq 1$, together with the autarkic allocation forms a sequential market equilibrium. In this equilibrium the stock of outside money, m, is not valued: the initial old don't get any goods in exchange for it and future generations are not willing to ever exchange goods for money, which results in the autarkic, no-trade situation. To make autarky an Arrow-Debreu equilibrium is a bit more problematic. Again from the budget constraint of the initial old we find

$$c_1^0 = 1 - \varepsilon + \frac{m}{p_1}$$

which, for autarky to be an equilibrium requires $p_1 = \infty$, i.e. the price level is so high in the first period that the stock of money de facto has no value. Since for all other periods we need $\frac{p_{t+1}}{p_t} = \frac{\varepsilon}{1-\varepsilon}$ to support the autarkic allocation, we have the obscure requirement that we need price *levels* to be infinite with well-defined finite price *ratios.* This is unsatisfactory, but there is no way around it unless we a) change the equilibrium definition (see Sargent and Ljungquist) or b) let the economy extend from the infinite past to the infinite future (instead of starting with an initial old generation, see Geanakoplos) or c) treat money somewhat as a residual, as something almost endogenous (see Kehoe) or d) make some consumption good rather than money the numeraire (with nonmonetary equilibria corresponding to situations in which money has a price of zero in terms of real consumption goods). For now we will accept autarky as an equilibrium even with money and we will treat it as identical to the autarkic equilibrium without money (because indeed in the sequential markets formulation only r_1 changes and in the Arrow Debreu formulation only p_1 changes, although in an unsatisfactory fashion).

8.1.4 Positive Valuation of Outside Money

In our construction of the nonautarkic stationary equilibrium we have already demonstrated our second main result of OLG models: outside money may have positive value. In that equilibrium the initial old had endowment $1-\varepsilon$ but consumed $c_1^0 = \frac{1}{2}$. If $\varepsilon > \frac{1}{2}$, then the stock of outside money, m, is valued in equilibrium in that the old guys can exchange m pieces of intrinsically worthless paper for $\frac{m}{p_1} > 0$ units of period 1 consumption goods.[7] The currently young generation accepts to transfer some of their endowment to the old people for pieces of paper because they expect (correctly so, in equilibrium) to exchange these pieces of paper against consumption goods when they are old, and hence to achieve an intertemporal allocation of consumption goods that dominates the autarkic allocation. Without the outside asset, again, this economy can do nothing else but remain in the possibly dismal state of autarky (imagine $\varepsilon = 1$ and log-utility). This is why the social contrivance of money is so useful in this economy. As we will see later, other institutions (for example a pay-as-you-go social security system) may achieve the same as money.

Before we demonstrate that, apart from stationary equilibria (two in the example, usually at least only a finite number) there may be a continuum of other, nonstationary equilibria we take a little digression to show for the general infinitely lived agent endowment economies set out at the beginning of this section money cannot have positive value in equilibrium.

Proposition 92 *In pure exchange economies with a finite number of infinitely lived agents there cannot be an equilibrium in which outside money is valued.*

Proof. Suppose, to the contrary, that there is an equilibrium $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}, \{\hat{p}_t\}_{t=1}^{\infty}$ for initial endowments of outside money $(m^i)_{i\in I}$ such that $\sum_{i\in I} m^i \neq 0$. Given the assumption of local nonsatiation each consumer in equilibrium satisfies the Arrow-Debreu budget constraint with equality

$$\sum_{t=1}^{\infty} \hat{p}_t \hat{c}_t^i = \sum_{t=1} \hat{p}_t e_t^i + m^i < \infty$$

[7] In finance lingo money in this equilibrium is a "bubble". The fundamental value of an assets is the value of its dividends, evaluated at the equilibrium Arrow-Debreu prices. An asset is (or has) a bubble if its price does not equal its fundamental value. Obviuosly, since money doesn't pay dividends, its fundamental value is zero and the fact that it is valued positively in equilibrium makes it a bubble.

Summing over all individuals $i \in I$ yields

$$\sum_{t=1}^{\infty} \hat{p}_t \sum_{i \in I} \left(\hat{c}_t^i - e_t^i \right) = \sum_{i \in I} m^i$$

But resource feasibility requires $\sum_{i \in I} \left(\hat{c}_t^i - e_t^i \right) = 0$ for all $t \geq 1$ and hence

$$\sum_{i \in I} m^i = 0$$

a contradiction. This shows that there cannot exist an equilibrium in this type of economy in which outside money is valued in equilibrium. Note that this result applies to a much wider class of standard Arrow-Debreu economies than just the pure exchange economies considered in this section. ■

Hence we have established the second major difference between the standard Arrow-Debreu general equilibrium model and the OLG model.

Continuum of Equilibria

We will now go ahead and demonstrate the third major difference, the possibility of a whole continuum of equilibria in OLG models. We will restrict ourselves to the specific example. Again suppose $m > 0$ and $\varepsilon > 0.5$.[8] For any p_1 such that $\frac{m}{p_1} < \varepsilon - \frac{1}{2} > 0$ we can construct an equilibrium using our geometric method before. From the picture it is clear that all these equilibria have the feature that the equilibrium allocations over time converge to the autarkic allocation, with $z_0 > z_1 > z_2 > \ldots z_t > 0$ and $\lim_{t \to \infty} z_t = 0$ and $0 > y_t > \ldots y_2 > y_1$ with $\lim_{t \to \infty} y_t = 0$. We also see from the figure that, since the offer curve lies below the -45^0-line for the part we are concerned with that $\frac{p_1}{p_2} < 1$ and $\frac{p_t}{p_{t+1}} < \frac{p_{t-1}}{p_t} < \ldots < \frac{p_1}{p_2} < 1$, implying that prices are increasing with $\lim_{t \to \infty} p_t = \infty$. Hence all the nonstationary equilibria feature inflation, although the inflation rate is bounded above by $\pi_\infty = -r_\infty = 1 - \frac{1-\varepsilon}{\varepsilon} = 2 - \frac{1}{\varepsilon} > 0$. The real value of money, however, declines to zero in the limit.[9] Note that, although all nonstationary equilibria so constructed in the limit converge to the same allocation (autarky), they differ

[8] You should verify that if $\varepsilon \leq 0.5$, then $\bar{r} \geq 0$ and the only equilibrium with $m > 0$ is the autarkic equilibrium in which money has no value. All other possible equilibrium paths eventually violate nonnegativity of consumption.

[9] But only in the limit. It is crucial that the real value of money is not zero at finite t, since with perfect foresight as in this model generation t would anticipate the fact that money would lose all its value, would not accept it from generation $t-1$ and all monetary equilibria would unravel, with only the autarkic euqilibrium surviving.

in the sense that at any finite t, the consumption allocations and price ratios (and levels) differ across equilibria. Hence there is an entire continuum of equilibria, indexed by $p_1 \in (\frac{m}{\varepsilon-0.5}, \infty)$. These equilibria are arbitrarily close to each other. This is again in stark contrast to standard Arrow-Debreu economies where, generically, the set of equilibria is finite and all equilibria are locally unique.[10] For details consult Debreu (1970) and the references therein.

Note that, if we are in the *Samuelson case* $\bar{r} < 0$, then (and only then) all these equilibria are Pareto-ranked.[11] Let the equilibria be indexed by p_1. One can show, by similar arguments that demonstrated that the autarkic equilibrium is not Pareto optimal, that these equilibria are Pareto-ranked: let $p_1, \hat{p}_1 \in (\frac{m}{\varepsilon-0.5}, \infty)$ with $p_1 > \hat{p}_1$, then the equilibrium corresponding to $\hat{p}_1$ Pareto-dominates the equilibrium indexed by p_1. By the same token, the *only* Pareto optimal equilibrium allocation is the nonautarkic stationary monetary equilibrium.

8.1.5 Productive Outside Assets

We have seen that with a positive supply of an outside asset with no intrinsic value, $m > 0$, then in the Samuelson case (for which the slope of the offer curve is smaller than one at the autarkic allocation) we have a continuum of equilibria. Now suppose that, instead of being endowed with intrinsically useless pieces of paper the initial old are endowed with a Lucas tree that yields dividends $d > 0$ in terms of the consumption good in each period. In a lot of ways this economy seems a lot like the previous one with money. So it should have the same number and types of equilibria!? The definition of equilibrium (we will focus on Arrow-Debreu equilibria) remains the same, apart from the resource constraint which now reads

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t + d$$

[10] Generically in this context means, for almost all endowments, i.e. the set of possible values for the endowments for which this statement does not hold is of measure zero. Local uniquenes means that in for every equilibrium price vector there exists ε such that any ε-neighborhood of the price vector does not contain another equilibrium price vector (apart from the trivial ones involving a different normalization).

[11] Again we require the assumption that consumption in the first and the second period are strict gross substitutes, ruling out backward-bending offer curves.

Figure 8.4: Productive Outside Assets in the OLG Model

and the budget constraint of the initial old generation which now reads

$$p_1 c_1^0 \leq p_1 e_1^0 + d \sum_{t=1}^{\infty} p_t$$

Let's analyze this economy using our standard techniques. The offer curve remains completely unchanged, but the resource line shifts to the right, now goes through the points $(y, z) = (d, 0)$ and $(y, z) = (0, d)$. Let's look at Figure 11.

It appears that, as in the case with money $m > 0$ there are two stationary and a continuum of nonstationary equilibria. The point (y_1, z_0) on the offer curve indeed represents a stationary equilibrium. Note that the constant equilibrium price ratio satisfies $\frac{p_t}{p_{t+1}} = \alpha > 1$ (just draw a ray through the origin and the point and compare with the slope of the resource constraint which is -1). Hence we have, after normalization of $p_1 = 1$, $p_t = \left(\frac{1}{\alpha}\right)^{t-1}$ and therefore the value of the Lucas tree in the first period equals

$$d \sum_{t=1}^{\infty} \left(\frac{1}{\alpha}\right)^{t-1} < \infty$$

How about the other intersection of the resource line with the offer curve, (y_1', z_0')? Note that in this hypothetical stationary equilibrium $\frac{p_t}{p_{t+1}} = \gamma < 1$, so that $p_t = \left(\frac{1}{\gamma}\right)^{t-1} p_1$. Hence the period 0 value of the Lucas tree is infinite and the consumption of the initial old exceed the resources available in the economy in period 1. This obviously cannot be an equilibrium. Similarly all equilibrium paths starting at some point z_0'' converge to this stationary point, so for all hypothetical nonstationary equilibria we have $\frac{p_t}{p_{t+1}} < 1$ for t large enough and again the value of the Lucas tree remains unbounded, and these paths cannot be equilibrium paths either. We conclude that in this economy there exists a unique equilibrium, which, by the way, is Pareto optimal.

This example demonstrates that it is not the existence of a long-lived outside asset that is responsible for the existence of a continuum of equilibria. What is the difference? In all monetary equilibria apart from the stationary nonautarkic equilibrium (which exists for the Lucas tree economy, too) the price level goes to infinity, as in the hypothetical Lucas tree equilibria that

Figure 8.5: Endogenous Cycles in OLG Models

turned out not to be equilibria. What is crucial is that money is intrinsically useless and does not generate real stuff so that it is possible in equilibrium that prices explode, but the real value of the dividends remains bounded. Also note that we were to introduce a Lucas tree with negative dividends (the initial old generation is an eternal slave, say, of the government and has to come up with d in every period to be used for government consumption), then the existence of the whole continuum of equilibria is restored.[12]

8.1.6 Endogenous Cycles

Not only is there a possibility of a continuum of equilibria in the basic OLG-model, but these equilibria need not take the monotonic form described above. Instead, equilibria with cycles are possible. In Figure 12 we have drawn an offer curve that is backward bending. In the homework you will see an example of preferences that yields such a backward bending offer curve, for a rather normal utility function.

Let $m > 0$ and let p_1 be such that $z_0 = \frac{m}{p_1}$. Using our geometric approach we find $y_1 = y(p_1, p_2)$ from the resource line, $z_1 = z(p_1, p_2)$ from the offer curve (ignore for the moment the fact that there are several z_1 will do; this merely indicates that the multiplicity of equilibria is of even higher order than previously demonstrated). From the resource line we find $y_2 = y(p_2, p_3)$ and from the offer curve $z_2 = z(p_2, p_3) = z_0$. After period $t = 2$ the economy repeats the cycle from the first two periods. The equilibrium allocation is of the form

$$
\begin{aligned}
c_t^{t-1} &= \begin{cases} c^{ol} = z_0 - w_2 \text{ for } t \text{ odd} \\ c^{oh} = z_1 - w_2 \text{ for } t \text{ even} \end{cases} \\
c_t^t &= \begin{cases} c^{yl} = y_1 - w_1 \text{ for } t \text{ odd} \\ c^{yh} = y_2 - w_1 \text{ for } t \text{ even} \end{cases}
\end{aligned}
$$

[12] Also note that the fact that in the unique equilibrium $\lim_{t\to\infty} p_t = 0$ has to be true (otherwise the Lucas tree cannot have finite value) implies that this equilibrium cannot be made into a monetary equilibrium, since $\lim_{t\to\infty} \frac{m}{p_t} = \infty$ and the real value of money would eventually exceed the aggregate endowment of the economy for any $m > 0$.

with $c^{ol} < c^{oh}$, $c^{yl} < c^{yh}$. Prices satisfy

$$\begin{aligned}
\frac{p_t}{p_{t+1}} &= \begin{cases} \alpha^h \text{ for } t \text{ odd} \\ \alpha^l \text{ for } t \text{ even} \end{cases} \\
\pi_{t+1} &= -r_{t+1} = \begin{cases} \pi^l < 0 \text{ for } t \text{ odd} \\ \pi^h > 0 \text{ for } t \text{ even} \end{cases}
\end{aligned}$$

Consumption of generations fluctuates in a two period cycle, with odd generations eating little when young and a lot when old and even generations having the reverse pattern. Equilibrium returns on money (inflation rates) fluctuate, too, with returns from odd to even periods being high (low inflation) and returns being low (high inflation) from even to odd periods. Note that these cycles are purely endogenous in the sense that the environment is completely stationary: nothing distinguishes odd and even periods in terms of endowments, preferences of people alive or the number of people. It is not surprising that some economists have taken this feature of OLG models to be the basis of a theory of endogenous business cycles (see, for example, Grandmont (1985)). Also note that it is not particularly difficult to construct cycles of length bigger than 2 periods.

8.1.7 Social Security and Population Growth

The pure exchange OLG model renders itself nicely to a discussion of a pay-as-you-go social security system. It also prepares us for the more complicated discussion of the same issue once we have introduced capital accumulation. Consider the simple model without money (i.e. $m = 0$). Also now assume that the population is growing at constant rate n, so that for each old person in a given period there are $(1 + n)$ young people around. Definitions of equilibria remain unchanged, apart from resource feasibility that now reads

$$c_t^{t-1} + (1+n)c_t^t = e_t^{t-1} + (1+n)e_t^t$$

or, in terms of excess demands

$$z(p_{t-1}, p_t) + (1+n)y(p_t, p_{t+1}) = 0$$

This economy can be analyzed in exactly the same way as before with noticing that in our offer curve diagram the slope of the resource line is not -1 anymore, but $-(1 + n)$. We know from above that, without any government intervention, the unique equilibrium in this case is the autarkic equilibrium. We now want to analyze under what conditions the introduction of a pay-as-you-go social security system in period 1 (or any other date) is welfare-improving. We again assume stationary endowments $e_t^t = w_1$ and $e_{t+1}^t = w_2$ for all t. The social security system is modeled as follows: the young pay social security taxes of $\tau \in [0, w_1)$ and receive social security benefits b when old. We assume that the social security system balances its budget in each period, so that benefits are given by

$$b = \tau(1+n)$$

Obviously the new unique competitive equilibrium is again autarkic with endowments $(w_1 - \tau, w_2 + \tau(1+n))$ and equilibrium interest rates satisfy

$$1 + r_{t+1} = 1 + r = \frac{U'(w_1 - \tau)}{\beta U'(w_2 + \tau(1+n))}$$

Obviously for any $\tau > 0$, the initial old generation receives a windfall transfer of $\tau(1+n) > 0$ and hence unambiguously benefits from the introduction. For all other generations, define the equilibrium lifetime utility, as a function of the social security system, as

$$V(\tau) = U(w_1 - \tau) + \beta U(w_2 + \tau(1+n))$$

The introduction of a small social security system is welfare improving if and only if $V'(\tau)$, evaluated at $\tau = 0$, is positive. But

$$\begin{aligned} V'(\tau) &= -U'(w_1 - \tau) + \beta U'(w_2 + \tau(1+n))(1+n) \\ V'(0) &= -U'(w_1) + \beta U'(w_2)(1+n) \end{aligned}$$

Hence $V'(0) > 0$ if and only if

$$n > \frac{U'(w_1)}{\beta U'(w_2)} - 1 = \bar{r}$$

where $\bar{r}$ is the autarkic interest rate. Hence the introduction of a (marginal) pay-as-you-go social security system is welfare improving if and only if the population growth rate exceeds the equilibrium (autarkic) interest rate, or, to use our previous terminology, if we are in the Samuelson case where autarky is not a Pareto optimal allocation. Note that social security has the same function as money in our economy: it is a social institution that transfers resources between generations (backward in time) that do not trade among each other in equilibrium. In enhancing intergenerational exchange not provided by the market it may generate allocations that are Pareto superior to the autarkic allocation, in the case in which individuals private marginal rate of substitution $1+\bar{r}$ (at the autarkic allocation) falls short of the social intertemporal rate of transformation $1+n$.

If $n > \bar{r}$ we can solve for optimal sizes of the social security system analytically in special cases. Remember that for the case with positive money supply $m > 0$ but no social security system the unique Pareto optimal allocation was the nonautarkic stationary allocation. Using similar arguments we can show that the sizes of the social security system for which the resulting equilibrium allocation is Pareto optimal is such that the resulting autarkic equilibrium interest rate is at least equal to the population growth rate, or

$$1+n \leq \frac{U'(w_1-\tau)}{\beta U'(w_2+\tau(1+n))}$$

For the case in which the period utility function is of logarithmic form this yields

$$\begin{aligned} 1+n &\leq \frac{w_2+\tau(1+n)}{\beta(w_1-\tau)} \\ \tau &\geq \frac{\beta}{1+\beta}w_1 - \frac{w_2}{(1+\beta)(1+n)} = \tau^*(w_1,w_2,n,\beta) \end{aligned}$$

Note that τ^* is the unique size of the social security system that maximizes the lifetime utility of the representative generation. For any smaller size we could marginally increase the size and make the representative generation better off and increase the windfall transfers to the initial old. Note, however, that any $\tau > \tau^*$ satisfying $\tau \leq w_1$ generates a Pareto optimal allocation, too: the representative generation would be better off with a smaller system, but the initial old generation would be worse off. This again demonstrates the weak requirements that Pareto optimality puts on an allocation. Also note that the "optimal" size of social security is an increasing function of first period income w_1, the population growth rate n and the time discount factor β, and a decreasing function of the second period income w_2.

So far we have assumed that the government sustains the social security system by forcing people to participate.[13] Now we briefly describe how such a system may come about if policy is determined endogenously. We make the following assumptions. The initial old people can decide upon the size of the social security system $\tau_0 = \tau^{**} \geq 0$. In each period $t \geq 1$ there is a majority vote as to whether the current system is to be kept or abolished. If the majority of the population in period t favors the abolishment of the system, then $\tau_t = 0$ and no payroll taxes or social security benefits are paid. If the vote is in favor of the system, then the young pay taxes τ^{**} and the old receive $(1+n)\tau^{**}$. We assume that $n > 0$, so the current young generation determines current policy. Since current voting behavior depends on expectations about voting behavior of future generations we have to specify how expectations about the voting behavior of future generations is determined. We assume the following expectations mechanism (see Cooley and Soares (1999) for a more detailed discussion of justifications as well as shortcomings for this specification of forming expectations):

$$\tau_{t+1}^e = \begin{cases} \tau^{**} \text{ if } \tau_t = \tau^{**} \\ 0 \text{ otherwise} \end{cases} \tag{8.18}$$

that is, if young individuals at period t voted down the original social security system then they expect that a newly proposed social security system will be voted down tomorrow. Expectations are rational if $\tau_t^e = \tau_t$ for all t. Let $\tau = \{\tau_t\}_{t=0}^{\infty}$ be an arbitrary sequence of policies that is feasible (i.e. satisfies $\tau_t \in [0, w_1)$)

Definition 93 *A rational expectations politico-economic equilibrium, given our expectations mechanism is an allocation rule $\hat{c}_1^0(\tau), \{(\hat{c}_t^t(\tau), \hat{c}_{t+1}^t(\tau))\}$, price rule $\{\hat{p}_t(\tau)\}$ and policies $\{\hat{\tau}_t\}$ such that*[14]

1. *for all $t \geq 1$, for all feasible τ, and given $\{\hat{p}_t(\tau)\}$,*

$$\begin{aligned} (\hat{c}_t^t, \hat{c}_{t+1}^t) &\in \arg \max_{(c_t^t, c_{t+1}^t) \geq 0} V(\tau_t, \tau_{t+1}) = U(c_t^t) + \beta U(c_{t+1}^t) \\ s.t.\ p_t c_t^t + p_{t+1} c_{t+1}^t &\leq p_t (w_1 - \tau_t) + p_{t+1} (w_2 + (1+n)\tau_{t+1}) \end{aligned}$$

[13] This section is not based on any reference, but rather my own thoughts. Please be aware of this and read with caution.

[14] The dependence of allocations and prices on τ is implicit from now on.

2. *for all feasible* τ, *and given* $\{\hat{p}_t(\tau)\}$,

$$\begin{aligned} \hat{c}_1^0 &\in \arg\max_{c_1^0 \geq 0} V(\tau_0, \tau_1) = U(c_1^0) \\ s.t.\ p_1 c_1^0 &\leq p_1(w_2 + (1+n)\tau_1) \end{aligned}$$

3.

$$c_t^{t-1} + (1+n)c_t^t = w_2 + (1+n)w_1$$

4. *For all* $t \geq 1$

$$\hat{\tau}_t \in \arg\max_{\theta \in \{0, \tau^{**}\}} V(\theta, \tau_{t+1}^e)$$

where τ_{t+1}^e *is determined according to* (8.18)

5.

$$\hat{\tau}_0 \in \arg\max_{\theta \in [0, w_1)} V(\theta, \hat{\tau}_1)$$

6. *For all* $t \geq 1$

$$\tau_t^e = \hat{\tau}_t$$

Conditions 1-3 are the standard economic equilibrium conditions for any arbitrary sequence of social security taxes. Condition 4 says that all agents of generation $t \geq 1$ vote rationally and sincerely, given the expectations mechanism specified. Condition 5 says that the initial old generation implements the best possible social security system (for themselves). Note the constraint that the initial generation faces in its maximization: if it picks θ too high, the first regular generation (see condition 4) may find it in its interest to vote the system down. Finally the last condition requires rational expectations with respect to the formation of policy expectations.

Political equilibria are in general very hard to solve unless one makes the economic equilibrium problem easy, assumes simple voting rules and simplifies as much as possible the expectations formation process. I tried to do all of the above for our discussion. So let find an (the!) political economic equilibrium. First notice that for any policy the equilibrium allocation will be autarky since there is no outside asset. Hence we have as equilibrium allocations and prices for a given policy τ

$$\begin{aligned}
c_t^{t-1} &= w_2 + (1+n)\tau_t \\
c_t^t &= w_1 - \tau_t \\
p_1 &= 1 \\
\frac{p_t}{p_{t+1}} &= \frac{U'(w_1 - \tau_t)}{\beta U'(w_2 + (1+n)\tau_t)}
\end{aligned}$$

Therefore the only equilibrium element to determine are the optimal policies. Given our expectations mechanism for any choice of $\tau_0 = \tau^{**}$, when would generation t vote the system τ^{**} down when young? If it does, given the expectation mechanism, it would not receive benefits when old (a newly installed system would be voted down right away, according to the generations' expectation). Hence

$$V(0, \tau_{t+1}^e) = V(0,0) = U(w_1) + \beta U(w_2)$$

Voting to keep the system in place yields

$$V(\tau^{**}, \tau_{t+1}^e) = V(\tau^{**}, \tau^{**}) = U(w_1 - \tau^{**}) + \beta U(w_2 + (1+n)\tau^{**})$$

and a vote in favor requires

$$V(\tau^{**}, \tau^{**}) \geq V(0,0) \tag{8.19}$$

But this is true for all generations, including the first regular generation. Given the assumption that we are in the Samuelson case with $n > \bar{r}$ there exists a $\tau^{**} > 0$ such that the above inequality holds. Hence the initial old generation can introduce a positive social security system with $\tau_0 = \tau^{**} > 0$ that is not voted down by the next generation (and hence by no generation) and creates positive transfers for itself. Obviously, then, the optimal choice is to maximize $\tau_0 = \tau^{**}$ subject to (8.19), and the equilibrium sequence of policies satisfies $\hat{\tau}_t = \tau^{**}$ where $\tau^{**} > 0$ satisfies

$$U(w_1 - \tau^{**}) + \beta U(w_2 + (1+n)\tau^{**}) = U(w_1) + \beta U(w_2)$$

Note that since the offer curve lies everywhere above the indifference curve through the no-social security endowment point (w_1, w_2), we know that the indifference curve thorugh that point intersects the resource line to the northwest of the intersection of resource line and offer curve (in the Samuelson case). But this implies that $\tau^{**} > \tau^{*}$ (which was defined as the level of social security that maximizes lifetime utility of a typical generation). Consequently the politico-equilibrium social security tax rate is bigger than the one maximizing welfare for the typical generation: by having the right to set up the system first the initial old can steer the economy to an equilibrium that is better for them (and worse for all future generations) than the one implied by tax rate τ^{*}.

8.2 The Ricardian Equivalence Hypothesis

How should the government finance a given stream of government expenditures, say, for a war? There are two principal ways to levy revenues for a government, namely to tax current generations or to issue government debt in the form of government bonds the interest and principal of which has to be paid later.[15] The question then arise what the macroeconomic consequences of using these different instruments are, and which instrument is to be preferred from a normative point of view. The Ricardian Equivalence Hypothesis claims that it makes no difference, that a switch from one instrument to the other does not change real allocations and prices in the economy. Therefore this hypothesis, is also called Modigliani-Miller theorem of public finance.[16] It's origin dates back to the classical economist David Ricardo (1772-1823). He wrote about how to finance a war with annual expenditures of £20 millions and asked whether it makes a difference to finance the £20 millions via current taxes or to issue government bonds with infinite maturity (so-called consols) and finance the annual interest payments of £1 million in all future years by future taxes (at an assumed interest rate of 5%). His conclusion was (in "Funding System") that

> in the point of the economy, there is no real difference in either of the modes; for twenty millions in one payment [or] one million per annum for ever ... are precisely of the same value

Here Ricardo formulates and explains the equivalence hypothesis, but immediately makes clear that he is sceptical about its empirical validity

[15] I will restrict myself to a discussion of real economic models, in which fiat money is absent. Hence the government cannot levy revenue via seignorage.

[16] When we discuss a theoretical model, Ricardian equivalence will take the form of a theorem that either holds or does not hold, depending on the assumptions we make. When discussing whether Ricardian equivalence holds empirically, I will call it a hypothesis.

> ...but the people who pay the taxes never so estimate them, and therefore do not manage their affairs accordingly. We are too apt to think, that the war is burdensome only in proportion to what we are at the moment called to pay for it in taxes, without reflecting on the probable duration of such taxes. It would be difficult to convince a man possessed of £20,000, or any other sum, that a perpetual payment of £50 per annum was equally burdensome with a single tax of £1,000.

Ricardo doubts that agents are as rational as they should, according to "in the point of the economy", or that they rationally believe not to live forever and hence do not have to bear part of the burden of the debt. Since Ricardo didn't believe in the empirical validity of the theorem, he has a strong opinion about which financing instrument ought to be used to finance the war

> war-taxes, then, are more economical; for when they are paid, an effort is made to save to the amount of the whole expenditure of the war; in the other case, an effort is only made to save to the amount of the interest of such expenditure.

Ricardo thought of government debt as one of the prime tortures of mankind. Not surprisingly he strongly advocates the use of current taxes. We will, after having discussed the Ricardian equivalence hypothesis, briefly look at the long-run effects of government debt on economic growth, in order to evaluate whether the phobia of Ricardo (and almost all other classical economists) about government debt is in fact justified from a theoretical point of view. Now let's turn to a model-based discussion of Ricardian equivalence.

8.2.1 Infinite Lifetime Horizon and Borrowing Constraints

The Ricardian Equivalence hypothesis is, in fact, a theorem that holds in a fairly wide class of models. It is most easily demonstrated within the Arrow-Debreu market structure of infinite horizon models. Consider the simple infinite horizon pure exchange model discussed at the beginning of the section. Now introduce a government that has to finance a given exogenous stream of government expenditures (in real terms) denoted by $\{G_t\}_{t=1}^{\infty}$. These government expenditures do not yield any utility to the agents (this assumption is not at all restrictive for the results to come). Let p_t denote the Arrow-Debreu price at date 0 of one unit of the consumption good delivered at period t. The government has initial outstanding real debt[17] of B_1 that is held by the public. Let b_1^i denote the initial endowment of government bonds of agent i. Obviously we have the restriction

$$\sum_{i \in I} b_1^i = B_1$$

Note that b_1^i is agent i's entitlement to period 1 consumption that the government owes to the agent. In order to finance the government expenditures the government levies lump-sum taxes: let τ_t^i denote the taxes that agent i pays in period t, denoted in terms of the period t consumption good. We define an Arrow-Debreu equilibrium with government as follows

Definition 94 *Given a sequence of government spending $\{G_t\}_{t=1}^{\infty}$ and initial government debt B_1 and $(b_1^i)_{i \in I}$ an Arrow-Debreu equilibrium are allocations $\{(\hat{c}_t^i)_{i \in I}\}_{t=1}^{\infty}$, prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i \in I}\}_{t=1}^{\infty}$ such that*

1. *Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i \in I}\}_{t=1}^{\infty}$ for all $i \in I$, $\{\hat{c}_t^i\}_{t=1}^{\infty}$ solves*

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t^i) \tag{8.20}$$

$$\text{s.t.} \sum_{t=1}^{\infty} \hat{p}_t (c_t + \tau_t^i) \leq \sum_{t=1}^{\infty} \hat{p}_t e_t^i + \hat{p}_1 b_1^i$$

2. *Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ the tax policy satisfies*

$$\sum_{t=1}^{\infty} \hat{p}_t G_t + \hat{p}_1 B_1 = \sum_{t=1}^{\infty} \sum_{i \in I} \hat{p}_t \tau_t^i$$

3. *For all $t \geq 1$*

$$\sum_{i \in I} \hat{c}_t^i + G_t = \sum_{i \in I} e_t^i$$

[17] I.e. the government owes real consumption goods to its citizens.

In an Arrow-Debreu definition of equilibrium the government, as the agent, faces a single intertemporal budget constraint which states that the total value of tax receipts is sufficient to finance the value of all government purchases plus the initial government debt. From the definition it is clear that, with respect to government tax policies, the only thing that matters is the total value of taxes $\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i$ that the individual has to pay, but not the timing of taxes. It is then straightforward to prove the Ricardian Equivalence theorem for this economy.

Theorem 95 *Take as given a sequence of government spending* $\{G_t\}_{t=1}^{\infty}$ *and initial government debt* $B_1, (b_1^i)_{i\in I}$. *Suppose that allocations* $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, *prices* $\{\hat{p}_t\}_{t=1}^{\infty}$ *and taxes* $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ *form an Arrow-Debreu equilibrium. Let* $\{(\hat{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$ *be an arbitrary alternative tax system satisfying*

$$\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i = \sum_{t=1}^{\infty} \hat{p}_t \hat{\tau}_t^i \text{ for all } i \in I$$

Then $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, $\{\hat{p}_t\}_{t=1}^{\infty}$ *and* $\{(\hat{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$ *form an Arrow-Debreu equilibrium.*

There are two important elements of this theorem to mention. First, the sequence of government expenditures is taken as fixed and exogenously given. Second, the condition in the theorem rules out redistribution among individuals. It also requires that the new tax system has the same cost to each individual *at the old equilibrium prices* (but not necessarily at alternative prices).

Proof. This is obvious. The budget constraint of individuals does not change, hence the optimal consumption choice at the old equilibrium prices does not change. Obviously resource feasibility is satisfied. The government budget constraint is satisfied due to the assumption made in the theorem. ∎

A shortcoming of the Arrow-Debreu equilibrium definition and the preceding theorem is that it does not make explicit the substitution between current taxes and government deficits that may occur for two equivalent tax systems $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ and $\{(\hat{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$. Therefore we will now reformulate this economy sequentially. This will also allow us to see that one of the main assumptions of the theorem, the absence of borrowing constraints is crucial for the validity of the theorem.

As usual with sequential markets we now assume that markets for the consumption good and one-period loans open every period. We restrict ourselves to government bonds and loans with one year maturity, which, in this environment is without loss of generality (note that there is no risk) and will not distinguish between borrowing and lending between two agents an agent an the government. Let r_{t+1} denote the interest rate on one period loans from period t to period $t+1$. Given the tax system and initial bond holdings each agent i now faces a sequence of budget constraints of the form

$$c_t^i + \frac{b_{t+1}^i}{1+r_{t+1}} \leq e_t^i - \tau_t^i + b_t^i \tag{8.21}$$

with b_1^i given. In order to rule out Ponzi schemes we have to impose a no Ponzi scheme condition of the form $b_t^i \geq -a_t^i(r, e^i, \tau)$ on the consumer, which, in general may depend on the sequence of interest rates as well as the endowment stream of the individual and the tax system. We will be more specific about the exact from of the constraint later. In fact, we will see that the exact specification of the borrowing constraint is crucial for the validity of Ricardian equivalence.

The government faces a similar sequence of budget constraints of the form

$$G_t + B_t = \sum_{i \in I} \tau_t^i + \frac{B_{t+1}}{1+r_{t+1}} \tag{8.22}$$

with B_1 given. We also impose a condition on the government that rules out government policies that run a Ponzi scheme, or $B_t \geq -A_t(r, G, \tau)$. The definition of a sequential markets equilibrium is standard

Definition 96 *Given a sequence of government spending $\{G_t\}_{t=1}^{\infty}$ and initial government debt $B_1, (b_1^i)_{i\in I}$ a Sequential Markets equilibrium is allocations $\{\left(\hat{c}_t^i, \hat{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, interest rates $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$ and government policies $\{(\tau_t^i)_{i\in I}, B_{t+1}\}_{t=1}^{\infty}$ such that*

1. *Given interest rates $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ for all $i \in I$, $\{\hat{c}_t^i, \hat{b}_{t+1}^i\}_{t=1}^{\infty}$ maximizes (8.20) subject to (8.21) and $b_{t+1}^i \geq -a_t^i(\hat{r}, e^i, \tau)$ for all $t \geq 1$.*

2. *Given interest rates $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$, the government policy satisfies (8.22) and $B_{t+1} \geq -A_t(\hat{r}, G)$ for all $t \geq 1$*

3. For all $t \geq 1$

$$\begin{aligned}\sum_{i\in I}\hat{c}_t^i + G_t &= \sum_{i\in I} e_t^i \\ \sum_{i\in I}\hat{b}_{t+1}^i &= B_{t+1}\end{aligned}$$

We will particularly concerned with two forms of borrowing constraints. The first is the so called natural borrowing or debt limit: it is that amount that, at given sequence of interest rates, the consumer can maximally repay, by setting consumption to zero in each period. It is given by

$$an_t^i(\hat{r}, e, \tau) = \sum_{\tau=1}^{\infty} \frac{e_{t+\tau}^i - \tau_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})}$$

where we define $\prod_{j=t+1}^{t}(1+\hat{r}_{j+1}) = 1$. Similarly we set the borrowing limit of the government at its natural limit

$$An_t(\hat{r}, \tau) = \sum_{\tau=1}^{\infty} \frac{\sum_{i\in I}\tau_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})}$$

The other form is to prevent borrowing altogether, setting $a0_t^i(\hat{r}, e) = 0$ for all i, t. Note that since there is positive supply of government bonds, such restriction does not rule out saving of individuals in equilibrium. We can make full use of the Ricardian equivalence theorem for Arrow-Debreu economies one we have proved the following equivalence result

Proposition 97 *Fix a sequence of government spending* $\{G_t\}_{t=1}^{\infty}$ *and initial government debt* $B_1, (b_1^i)_{i\in I}$. *Let allocations* $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, *prices* $\{\hat{p}_t\}_{t=1}^{\infty}$ *and taxes* $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ *form an Arrow-Debreu equilibrium. Then there exists a corresponding sequential markets equilibrium with the natural debt limits* $\{\left(\tilde{c}_t^i, \tilde{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, $\{\tilde{r}_t\}_{t=1}^{\infty}$, $\{(\tilde{\tau}_t^i)_{i\in I}, \tilde{B}_{t+1}\}_{t=1}^{\infty}$ *such that*

$$\begin{aligned}\hat{c}_t^i &= \tilde{c}_t^i \\ \tau_t^i &= \tilde{\tau}_t^i \text{ for all } i, \text{ all } t\end{aligned}$$

Reversely, let allocations $\{\left(\hat{c}_t^i, \hat{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, *interest rates* $\{\hat{r}_t\}_{t=1}^{\infty}$ *and government policies* $\{(\tau_t^i)_{i\in I}, B_{t+1}\}_{t=1}^{\infty}$ *form a sequential markets equilibrium with* *natural debt limits. Suppose that it satisfies*

$$\begin{aligned}
\hat{r}_{t+1} &> -1, \text{ for all } t \geq 1 \\
\sum_{t=1}^{\infty} \frac{e_t^i - \tau_t^i}{\prod_{j=1}^{t-1}(1+\hat{r}_{j+1})} &< \infty \text{ for all } i \in I \\
\sum_{\tau=1}^{\infty} \frac{\sum_{i\in I} \tau_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau}(1+\hat{r}_{j+1})} &< \infty
\end{aligned}$$

Then there exists a corresponding Arrow-Debreu equilibrium $\{(\tilde{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, $\{\tilde{p}_t\}_{t=1}^{\infty}$, $\{(\tilde{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$ *such that*

$$\begin{aligned}
\hat{c}_t^i &= \tilde{c}_t^i \\
\tau_t^i &= \tilde{\tau}_t^i \text{ for all } i, \text{ all } t
\end{aligned}$$

Proof. The key to the proof is to show the equivalence of the budget sets for the Arrow-Debreu and the sequential markets structure. Normalize $\hat{p}_1 = 1$ and relate equilibrium prices and interest rates by

$$1 + \hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}} \tag{8.23}$$

Now look at the sequence of budget constraints and assume that they hold with equality (which they do in equilibrium, due to the nonsatiation assumption)

$$c_1^i + \frac{b_2^i}{1+\hat{r}_2} = e_1^i - \tau_1^i + b_1^i \tag{8.24}$$

$$c_2^i + \frac{b_3^i}{1+\hat{r}_3} = e_2^i - \tau_2^i + b_2^i \tag{8.25}$$

$$\vdots$$

$$c_t^i + \frac{b_{t+1}^i}{1+\hat{r}_{t+1}} = e_t^i - \tau_t^i + b_t^i \tag{8.26}$$

Substituting for b_2^i from (8.25) in (8.24) one gets

$$c_1^i + \tau_1^i - e_1^i + \frac{c_2^i + \tau_2^i - e_2^i}{1+\hat{r}_2} + \frac{b_3^i}{(1+\hat{r}_2)(1+\hat{r}_3)} = b_1^i$$

and in general

$$\sum_{t=1}^{T} \frac{c_t - e_t}{\prod_{j=1}^{t-1}(1+\hat{r}_{j+1})} + \frac{b_{T+1}^i}{\prod_{j=1}^{T}(1+\hat{r}_{j+1})} = b_1^i$$

Taking limits on both sides gives, using (8.23)

$$\sum_{t=1}^{\infty} \hat{p}_t(c_t^i + \tau_t^i - e_t^i) + \lim_{T\to\infty} \frac{b_{T+1}^i}{\prod_{j=1}^{T}(1+\hat{r}_{j+1})} = b_1^i$$

Hence we obtain the Arrow-Debreu budget constraint if and only if

$$\lim_{T\to\infty} \frac{b_{T+1}^i}{\prod_{j=1}^{T}(1+\hat{r}_{j+1})} = \lim_{T\to\infty} \hat{p}_{T+1} b_{T+1}^i \geq 0$$

But from the natural debt constraint

$$\begin{aligned} \hat{p}_{T+1} b_{T+1}^i &\geq -\hat{p}_{T+1} \sum_{\tau=1}^{\infty} \frac{e_{t+\tau}^i - \tau_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})} = -\sum_{\tau=T+1}^{\infty} \hat{p}_t(e_\tau^i - \tau_\tau^i) \\ &= -\sum_{\tau=1}^{\infty} \hat{p}_t(e_\tau^i - \tau_\tau^i) + \sum_{\tau=1}^{T} \hat{p}_t(e_\tau^i - \tau_\tau^i) \end{aligned}$$

Taking limits with respect to both sides and using that by assumption $\sum_{t=1}^{\infty} \frac{e_t^i - \tau_t^i}{\prod_{j=1}^{t-1}(1+\hat{r}_{j+1})} = \sum_{t=1}^{\infty} \hat{p}_t(e_\tau^i - \tau_\tau^i) < \infty$ we have

$$\lim_{T\to\infty} \hat{p}_{T+1} b_{T+1}^i \geq 0$$

So at equilibrium prices, with natural debt limits and the restrictions posed in the proposition a consumption allocation satisfies the Arrow-Debreu budget constraint (at equilibrium prices) if and only if it satisfies the sequence of budget constraints in sequential markets. A similar argument can be carried out for the budget constraint(s) of the government. The remainder of the proof is then straightforward and left to the reader. Note that, given an Arrow-Debreu equilibrium consumption allocation, the corresponding bond holdings for the sequential markets formulation are

$$b_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{c}_{t+\tau}^i + \tau_{t+\tau}^i - e_{t+\tau}^i}{\prod_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})}$$

■

As a straightforward corollary of the last two results we obtain the Ricardian equivalence theorem for sequential markets with natural debt limits (under the weak requirements of the last proposition).[18] Let us look at a few examples

Example 98 *(Financing a war) Let the economy be populated by $I = 1000$ identical people, with $U(c) = \ln(c)$, $\beta = 0.5$*

$$e_t^i = 1$$

and $G_1 = 500$ (the war), $G_t = 0$ for all $t > 1$. Let $b_1 = B_1 = 0$. Consider two tax policies. The first is a balanced budget requirement, i.e. $\tau_1 = 0.5$, $\tau_t = 0$ for all $t > 1$. The second is a tax policy that tries to smooth out the cost of the war, i.e. sets $\tau_t = \tau = \frac{1}{3}$ for all $t \geq 1$. Let us look at the equilibrium for the first tax policy. Obviously the equilibrium consumption allocation (we restrict ourselves to type-identical allocations) has

$$\hat{c}_t^i = \begin{cases} 0.5 \text{ for } t = 1 \\ 1 \text{ for } t \geq 1 \end{cases}$$

and the Arrow-Debreu equilibrium price sequence satisfies (after normalization of $p_1 = 1$) $p_2 = 0.25$ and $p_t = 0.25 * 0.5^{t-2}$ for all $t > 2$. The level of government debt and the bond holdings of individuals in the sequential markets economy satisfy

$$B_t = b_t = 0 \text{ for all } t$$

Interest rates are easily computed as $r_2 = 3$, $r_t = 1$ for $t > 2$. The budget constraint of the government and the agents are obviously satisfied. Now consider the second tax policy. Given resource constraint the previous equilibrium allocation and price sequences are the only candidate for an equilibrium

[18] An equivalence result with even less restrictive assumptions can be proved under the specification of a bounded shortsale constraint

$$\inf_t b_t^i < \infty$$

instead of the natural debt limit. See Huang and Werner (1998) for details.

under the new policy. Let's check whether they satisfy the budget constraints of government and individuals. For the government

$$\begin{aligned}
\sum_{t=1}^{\infty} \hat{p}_t G_t + \hat{p}_1 B_1 &= \sum_{t=1}^{\infty} \sum_{i \in I} \hat{p}_t \tau_t^i \\
500 &= \frac{1}{3} \sum_{t=1}^{\infty} 1000 \hat{p}_t \\
&= \frac{1000}{3} (1 + 0.25 + \sum_{t=3}^{\infty} 0.25 * 0.5^{t-2}) \\
&= 500
\end{aligned}$$

and for the individual

$$\begin{aligned}
\sum_{t=1}^{\infty} \hat{p}_t (c_t + \tau_t^i) &\leq \sum_{t=1}^{\infty} \hat{p}_t e_t^i + \hat{p}_1 b_1^i \\
\frac{5}{6} + \frac{4}{3} \sum_{t=2}^{\infty} \hat{p}_t &\leq \sum_{t=1}^{\infty} \hat{p}_t \\
\frac{1}{3} \sum_{t=2}^{\infty} \hat{p}_t &= \frac{1}{6} \leq \frac{1}{6}
\end{aligned}$$

Finally, for this tax policy the sequence of government debt and private bond holdings are

$$B_t = \frac{2000}{3}, b_2 = \frac{2}{3} \text{ for all } t \geq 2$$

i.e. the government runs a deficit to finance the war and, in later periods, uses taxes to pay interest on the accumulated debt. It never, in fact, retires the debt. As proved in the theorem both tax policies are equivalent as the equilibrium allocation and prices remain the same after a switch from tax to deficit finance of the war.

The Ricardian equivalence theorem rests on several important assumptions. The first is that there are perfect capital markets. If consumers face binding borrowing constraints (e.g. for the specification requiring $b_{t+1}^i \geq 0$), or if, with risk, not a full set of contingent claims is available, then Ricardian equivalence may fail. Secondly one has to require that all taxes are lump-sum. Non-lump sum taxes may distort relative prices (e.g. labor income taxes distort the relative price of leisure) and hence a change in the timing of taxes may have real effects. All taxes on endowments, whatever form they take, are lump-sum, not, however consumption taxes. Finally a change from one to another tax system is assumed to not redistribute wealth among agents. This was a maintained assumption of the theorem, which required that the total tax bill that each agent faces was left unchanged by a change in the tax system. In a world with finitely lived overlapping generations this would mean that a change in the tax system is not supposed to redistribute the tax burden among different generations.

Now let's briefly look at the effect of borrowing constraints. Suppose we restrict agents from borrowing, i.e. impose $b_{t+1}^i \geq 0$, for all i, all t. For the government we still impose the old restriction on debt, $B_t \geq -An_t(\hat{r}, \tau)$. We can still prove a limited Ricardian result

Proposition 99 *Let* $\{G_t\}_{t=1}^{\infty}$ *and* $B_1, (b_1^i)_{i\in I}$ *be given and let allocations* $\{\left(\hat{c}_t^i, \hat{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, *interest rates* $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$ *and government policies* $\{(\tau_t^i)_{i\in I}, B_{t+1}\}_{t=1}^{\infty}$ *be a Sequential Markets equilibrium with no-borrowing constraints for which* $\hat{b}_{t+1}^i > 0$ *for all* i, t. *Let* $\{(\tilde{\tau}_t^i)_{i\in I}, \tilde{B}_{t+1}\}_{t=1}^{\infty}$ *be an alternative government policy such that*

$$\tilde{b}_{t+1}^i = \sum_{\tau=t+1}^{\infty} \frac{\hat{c}_\tau^i + \tilde{\tau}_\tau^i - e_\tau^i}{\prod_{j=t+2}^{\tau}(1+\hat{r}_{j+1})} \geq 0 \tag{8.27}$$

$$G_t + \tilde{B}_t = \sum_{i\in I} \tilde{\tau}_t^i + \frac{\tilde{B}_{t+1}}{1+\hat{r}_{t+1}} \text{ for all } t \tag{8.28}$$

$$\tilde{B}_{t+1} \geq -An_t(\hat{r}, \tau) \tag{8.29}$$

$$\sum_{\tau=1}^{\infty} \frac{\tilde{\tau}_\tau^i}{\prod_{j=1}^{\tau-1}(1+\hat{r}_j)} = \sum_{\tau=1}^{\infty} \frac{\tau_\tau^i}{\prod_{j=1}^{\tau-1}(1+\hat{r}_{j+1})} \tag{8.30}$$

Then $\{\left(\hat{c}_t^i, \tilde{b}_{t+1}^i\right)_{i\in I}\}_{t=1}^{\infty}$, $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$ *and* $\{(\tilde{\tau}_t^i)_{i\in I}, \tilde{B}_{t+1}\}_{t=1}^{\infty}$ *is also a sequential markets equilibrium with no-borrowing constraint.*

The conditions that we need for this theorem are that the change in the tax system is not redistributive (condition (8.30)), that the new government policies satisfy the government budget constraint and debt limit (conditions (8.28) and (8.29)) and that the new bond holdings of each individual that are required to satisfy the budget constraints of the individual at old consumption allocations do not violate the no-borrowing constraint (condition (8.27)).

Proof. This proposition to straightforward to prove so we will sketch it here only. Budget constraints of the government and resource feasibility are obviously satisfied under the new policy. How about consumer optimization? Given the equilibrium prices and under the imposed conditions both policies induce the same budget set of individuals. Now suppose there is an i and allocation $\{\bar{c}_t^i\} \neq \{\hat{c}_t^i\}$ that dominates $\{\hat{c}_t^i\}$. Since $\{\bar{c}_t^i\}$ was affordable with the old policy, it must be the case that the associated bond holdings under the old policy, $\{\bar{b}_{t+1}^i\}$ violated one of the no-borrowing constraints. But then, by continuity of the price functional and the utility function there is an allocation $\{\check{c}_t^i\}$ with associated bond holdings $\{\check{b}_{t+1}^i\}$ that is affordable under the old policy and satisfies the no-borrowing constraint (take a convex combination of the $\{\hat{c}_t^i, \hat{b}_{t+1}^i\}$ and the $\{\bar{c}_t^i, \bar{b}_{t+1}^i\}$, with sufficient weight on the $\{\hat{c}_t^i, \hat{b}_{t+1}^i\}$ so as to satisfy the no-borrowing constraints). Note that for this to work it is crucial that the no-borrowing constraints are not binding under the old policy for $\{\hat{c}_t^i, \hat{b}_{t+1}^i\}$. You should fill in the mathematical details ■

Let us analyze an example in which, because of the borrowing constraints, Ricardian equivalence fails.

Example 100 *Consider an economy with 2 agents, $U^i = \ln(c)$, $\beta_i = 0.5$, $b_1^i = B_1 = 0$. Also $G_t = 0$ for all t and endowments are*

$$
\begin{aligned}
e_t^1 &= \begin{cases} 2 \text{ if } t \text{ odd} \\ 1 \text{ if } t \text{ even} \end{cases} \\
e_t^2 &= \begin{cases} 1 \text{ if } t \text{ odd} \\ 2 \text{ if } t \text{ even} \end{cases}
\end{aligned}
$$

As first tax system consider

$$
\begin{aligned}
\tau_t^1 &= \begin{cases} 0.5 \text{ if } t \text{ odd} \\ -0.5 \text{ if } t \text{ even} \end{cases} \\
e_t^2 &= \begin{cases} -0.5 \text{ if } t \text{ odd} \\ 0.5 \text{ if } t \text{ even} \end{cases}
\end{aligned}
$$

Obviously this tax system balances the budget. The equilibrium allocation with no-borrowing constraints evidently is the autarkic (after-tax) allocation

$c_t^i = 1.5$, *for all* i, t. *From the first order conditions we obtain, taking account the nonnegativity constraint on* b_{t+1}^i *(here* $\lambda_t \geq 0$ *is the Lagrange multiplier on the budget constraint in period* t *and* μ_{t+1} *is the Lagrange multiplier on the nonnegativity constraint for* b_{t+1}^i*)*

$$\begin{aligned} \beta^{t-1} U'(c_t^i) &= \lambda_t \\ \beta^t U(c_{t+1}^i) &= \lambda_{t+1} \\ \frac{\lambda_t}{1+r_{t+1}} &= \lambda_{t+1} + \mu_{t+1} \end{aligned}$$

Combining yields

$$\frac{U'(c_t^i)}{\beta U'(c_{t+1}^i)} = \frac{\lambda_t}{\lambda_{t+1}} = 1 + r_{t+1} + \frac{(1+r_t)\mu_{t+1}}{\lambda_{t+1}}$$

Hence

$$\begin{aligned} \frac{U'(c_t^i)}{\beta U'(c_{t+1}^i)} &\geq 1 + r_{t+1} \\ &= 1 + r_{t+1} \text{ if } b_{t+1}^i > 0 \end{aligned}$$

The equilibrium interest rates are given as $r_{t+1} \leq 1$, *i.e. are indeterminate. Both agents are allowed to save, and at* $r_{t+1} > 1$ *they would do so (which of course can't happen in equilibrium as there is zero net supply of assets). For any* $r_{t+1} \leq 1$ *the agents would like to borrow, but are prevented from doing so by the no-borrowing constraint, so any of these interest rates is fine as equilibrium interest rates. For concreteness let's take* $r_{t+1} = 1$ *for all* t.[19] *Then the total bill of taxes for the first consumer is* $\frac{1}{3}$ *and* $-\frac{1}{3}$ *for the second agent. Now lets consider a second tax system that has* $\tau_1^1 = \frac{1}{3}$, $\tau_1^2 = -\frac{1}{3}$ *and* $\tau_t^i = 0$ *for all* $i, t \geq 2$. *Obviously now the equilibrium allocation changes to* $c_t^1 = \frac{5}{3}$, $c_1^2 = \frac{4}{3}$ *and* $c_t^i = e_t^i$ *for all* $i, t \geq 2$. *Obviously the new tax system satisfies the government budget constraint and does not redistribute among agents. However, equilibrium allocations change. Furthermore, equilibrium interest rate change to* $r_2 = \frac{3}{2.5}$ *and* $r_t = 0$ *for all* $t \geq 3$. *Ricardian equivalence fails.*[20]

[19] These are the interest rates that would arise under natural debt limits, too.

[20] In general it is very hard to solve for equilibria with no-borrowing constraints analytically, even in partial equilibrium with fixed exogenous interest rates, even more so in gneral equilibrium. So if the above example seems cooked up, it is, since it is about the only example I know how to solve without going to the computer. We will see this more explicitly once we talk about Deaton's (1991) EC piece.

8.2.2 Finite Horizon and Operative Bequest Motives

It should be clear from the above discussion that one only obtains a very limited Ricardian equivalence theorem for OLG economies. Any change in the timing of taxes that redistributes among generations is in general not neutral in the Ricardian sense. If we insist on representative agents within one generation and purely selfish, two-period lived individuals, then in fact any change in the timing of taxes can't be neutral unless it is targeted towards a particular generation, i.e. the tax change is such that it decreases taxes for the currently young only and increases them for the old next period. Hence, with sufficient generality we can say that Ricardian equivalence does not hold for OLG economies with purely selfish individuals.

Rather than to demonstrate this obvious point with another example we now briefly review Barro's (1974) argument that under certain conditions finitely lived agents will behave as if they had infinite lifetime. As a consequence, Ricardian equivalence is re-established. Barro's (1974) article "Are Government Bonds Net Wealth?" asks exactly the Ricardian question, namely does an increase in government debt, financed by future taxes to pay the interest on the debt increase the net wealth of the private sector? If yes, then current consumption would increase, aggregate saving (private plus public) would decrease, leading to an increase in interest rate and less capital accumulation. Depending on the perspective, countercyclical fiscal policy[21] is effective against the business cycle (the Keynesian perspective) or harmful for long term growth (the classical perspective). If, however, the value of government bonds if completely offset by the value of future higher taxes for each individual, then government bonds are not net wealth of the private sector, and changes in fiscal policy are neutral.

Barro identified two main sources for why future taxes are not exactly offsetting current tax cuts (increasing government deficits): a) finite lives of agents that lead to intergenerational redistribution caused by a change in the timing of taxes b) imperfect private capital markets. Barro's paper focuses on the first source of nonneutrality.

Barro's key result is the following: in OLG-models finiteness of lives does not invalidate Ricardian equivalence as long as current generations are connected to future generations by a chain of operational intergenerational, altruistically motivated transfers. These may be transfers from old to young

[21]By fiscal policy in this section we mean the financing decision of the government for a given *exogenous* path of government expenditures.

via bequests or from young to old via social security programs. Let us look at his formal model.[22]

Consider the standard pure exchange OLG model with two-period lived agents. There is no population growth, so that each member of the old generation (whose size we normalize to 1) has exactly one child. Agents have endowment $e_t^t = w$ when young and no endowment when old. There is a government that, for simplicity, has 0 government expenditures but initial outstanding government debt B. This debt is denominated in terms of the period 1 (or any other period) consumption good. The initial old generation is endowed with these B units of government bonds. We assume that these government bonds are zero coupon bonds with maturity of one period. Further we assume that the government keeps its outstanding government debt constant and we assume a constant one-period real interest rate r on these bonds.[23] In order to finance the interest payments on government debt the government taxes the currently young people. The government budget constraint gives

$$\frac{B}{1+r} + \tau = B$$

The right hand side is the old debt that the government has to retire in the current period. On the left hand side we have the revenue from issuing new debt, $\frac{B}{1+r}$ (remember that we assume zero coupon bonds, so $\frac{1}{1+r}$ is the price of one government bond today that pays 1 unit of the consumption good tomorrow) and the tax revenue. With the assumption of constant government debt we find

$$\tau = \frac{rB}{1+r}$$

and we assume $\frac{rB}{1+r} < w$.

Now let's turn to the budget constraints of the individuals. Let by a_t^t denote the savings of currently young people for the second period of their lives and by a_{t+1}^t denote the savings of the currently old people for the next generation, i.e. the old people's bequests. We require bequests to be non-negative, i.e. $a_{t+1}^t \geq 0$. In our previous OLG models obviously $a_{t+1}^t = 0$ was the only optimal choice since individuals were completely selfish. We

[22] I will present a simplified, pure exchange version of his model to more clearly isolate his main point.

[23] This assumption is justified since the resulting equilibrium allocation (there is no money!) is the autarkic allocation and hence the interest rate always equals the autarkic interest rate.

will see below how to induce positive bequests when discussing individuals' preferences. The budget constraints of a representative generation are then given by

$$\begin{aligned} c_t^t + \frac{a_t^t}{1+r} &= w - \tau \\ c_{t+1}^t + \frac{a_{t+1}^t}{1+r} &= a_t^t + a_t^{t-1} \end{aligned}$$

The budget constraint of the young are standard; one may just remember that assets here are zero coupon bonds: spending $\frac{a_t^t}{1+r}$ on bonds in the current period yields a_t^t units of consumption goods tomorrow. We do not require a_t^t to be positive. When old the individuals have two sources of funds: their own savings from the previous period and the bequests a_t^{t-1} from the previous generation. They use it to buy own consumption and bequests for the next generation. The total expenditure for bequests of a currently old individual is $\frac{a_{t+1}^t}{1+r}$ and it delivers funds to her child next period (that has then become old) of a_{t+1}^t. We can consolidate the two budget constraints to obtain

$$c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{(1+r)^2} = w + \frac{a_t^{t-1}}{1+r} - \tau$$

Since the total lifetime resources available to generation t are given by $e_t = w + \frac{a_t^{t-1}}{1+r} - \tau$, the lifetime utility that this generation can attain is determined by e. The budget constraint of the initial old generation is given by

$$c_1^0 + \frac{a_1^0}{1+r} = B$$

With the formulation of preferences comes the crucial twist of Barro. He assumes that individuals are altruistic and care about the well-being of their descendant.[24] Altruistic here means that the parents genuinely care about the utility of their children and leave bequests for that reason; it is not that the parents leave bequests in order to induce actions of the children that yield utility to the parents.[25] Preferences of generation t are represented by

$$u_t(c_t^t, c_{t+1}^t, a_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t) + \alpha V_{t+1}(e_{t+1})$$

[24] Note that we only assume that the agent cares only about her immediate descendant, but (possibly) not at all about grandchildren.

[25] This strategic bequest motive does not necessarily help to reestablish Ricardian equivalence, as Bernheim, Shleifer and Summers (1985) show.

where $V_{t+1}(e_{t+1})$ is the maximal utility generation $t+1$ can attain with lifetime resources $e_{t+1} = w + \frac{a_{t+1}^t}{1+r} - \tau$, which are evidently a function of bequests $a_{t+1}^t$from generation t.[26] We make no assumption about the size of α as compared to β, but assume $\alpha \in (0,1)$. The initial old generation has preferences represented by

$$u_0(c_1^0, a_1^0) = \beta U(c_1^0) + \alpha V_1(e_1)$$

The equilibrium conditions for the goods and the asset market are, respectively

$$\begin{aligned} c_t^{t-1} + c_t^t &= w \text{ for all } t \geq 1 \\ a_t^{t-1} + a_t^t &= B \text{ for all } t \geq 1 \end{aligned}$$

Now let us look at the optimization problem of the initial old generation

$$\begin{aligned} V_0(B) &= \max_{c_1^0, a_1^0 \geq 0} \left\{ \beta U(c_1^0) + \alpha V_1(e_1) \right\} \\ \text{s.t. } c_1^0 + \frac{a_1^0}{1+r} &= B \\ e_1 &= w + \frac{a_1^0}{1+r} - \tau \end{aligned}$$

Note that the two constraints can be consolidated to

$$c_1^0 + e_1 = w + B - \tau \tag{8.31}$$

This yields optimal decision rules $c_1^0(B)$ and $a_1^0(B)$ (or $e_1(B)$). Now assume that the bequest motive is operative, i.e. $a_1^0(B) > 0$ and consider the Ricardian experiment of government: increase initial government debt marginally by ΔB and repay this additional debt by levying higher taxes on the first young generation. Clearly, in the OLG model without bequest motives such a change in fiscal policy is not neutral: it increases resources available to the initial old and reduces resources available to the first regular generation. This

[26] To formulate the problem recurively we need separability of the utility function with respect to time and utility of children. The argument goes through without this, but then it can't be clarified using recursive methods. See Barro's original paper for a more general discussion. Also note that he, in all likelihood, was not aware of the full power of recursive techniques in 1974. Lucas (1972) seminal paper was probably the first to make full use of recursive techiques in (macro) economics.

will change consumption of both generations and interest rate. What happens in the Barro economy? In order to repay the ΔB, from the government budget constraint taxes for the young have to increase by

$$\Delta\tau = \Delta B$$

since by assumption government debt from the second period onwards remains unchanged. How does this affect the optimal consumption and bequest choice of the initial old generation? It is clear from (8.31) that the optimal choices for c_1^0 and e_1 do not change as long as the bequest motive was operative before.[27] The initial old generation receives additional transfers of bonds of magnitude ΔB from the government and increases its bequests a_1^0 by $(1+r)\Delta B$ so that lifetime resources available to their descendants (and hence their allocation) is left unchanged. Altruistically motivated bequest motives just undo the change in fiscal policy. Ricardian equivalence is restored.

This last result was just an example. Now let's show that Ricardian equivalence holds in general with operational altruistic bequests. In doing so we will de facto establish between Barro's OLG economy and an economy with infinitely lived consumers and borrowing constraints. Again consider the problem of the initial old generation (and remember that, for a given tax rate and wage there is a one-to-one mapping between e_{t+1} and a_{t+1}^t

$$\begin{aligned} V_0(B) &= \max_{\substack{c_1^0, a_1^0 \geq 0 \\ c_1^0 + \frac{a_1^0}{1+r} = B}} \left\{\beta U(c_1^0) + \alpha V_1(a_1^0)\right\} \\ &= \max_{\substack{c_1^0, a_1^0 \geq 0 \\ c_1^0 + \frac{a_1^0}{1+r} = B}} \left\{ \beta U(c_1^0) + \alpha \max_{\substack{c_1^1, c_2^1, a_2^1 \geq 0, a_1^1 \\ c_1^1 + \frac{a_1^1}{1+r} = w - \tau \\ c_2^1 + \frac{a_2^1}{1+r} = a_1^1 + a_1^0}} \left\{U(c_1^1) + \beta U(c_2^1) + \alpha V_2(a_2^1)\right\} \right\} \end{aligned}$$

[27]If the bequest motive was not operative, i.e. if the constraint $a_1^0 \geq 0$ was binding, then by increasing B may result in an increase in c_1^0 and a decrease in e_1.

But this maximization problem can be rewritten as

$$\max_{c_1^0, a_1^0, c_1^1, c_2^1, a_2^1 \geq 0, a_1^1} \left\{ \beta U(c_1^0) + \alpha U(c_1^1) + \alpha\beta U(c_2^1) + \alpha^2 V_2(a_2^1) \right\}$$

$$\begin{aligned} \text{s.t. } c_1^0 + \frac{a_1^0}{1+r} &= B \\ c_1^1 + \frac{a_1^1}{1+r} &= w - \tau \\ c_2^1 + \frac{a_2^1}{1+r} &= a_1^1 + a_1^0 \end{aligned}$$

or, repeating this procedure infinitely many times (which is a valid procedure only for $\alpha < 1$), we obtain as implied maximization problem of the initial old generation

$$\max_{\{(c_t^{t-1}, c_t^t, a_t^{t-1})\}_{t=1}^{\infty} \geq 0} \left\{ \beta U(c_1^0) + \sum_{t=1}^{\infty} \alpha^t \left(U(c_t^t) + \beta U(c_{t+1}^t) \right) \right\}$$

$$\begin{aligned} s.t.\ c_1^0 + \frac{a_1^0}{1+r} &= B \\ c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{(1+r)^2} &= w - \tau + \frac{a_t^{t-1}}{1+r} \end{aligned}$$

i.e. the problem is equivalent to that of an infinitely lived consumer that faces a no-borrowing constraint. This infinitely lived consumer is peculiar in the sense that her periods are subdivided into two subperiods, she eats twice a period, c_t^t in the first subperiod and c_{t+1}^t in the second subperiod, and the relative price of the consumption goods in the two subperiods is given by $(1 + r)$. Apart from these reinterpretations this is a standard infinitely lived consumer with no-borrowing constraints imposed on her. Consequently one obtains a Ricardian equivalence proposition similar to proposition 99, where the requirement of "operative bequest motives" is the equivalent to condition (8.27). More generally, this argument shows that an OLG economy with two period-lived agents and operative bequest motives is formally equivalent to an infinitely lived agent model.

Example 101 *Suppose we carry out the Ricardian experiment and increase initial government debt by ΔB. Suppose the debt is never retired, but the required interest payments are financed by permanently higher taxes. The tax* *increase that is needed is (see above)*

$$\Delta\tau = \frac{r\Delta B}{1+r}$$

Suppose that for the initial debt level $\{(\hat{c}_t^{t-1}, \hat{c}_t^t, \hat{a}_t^{t-1})\}_{t=1}^{\infty}$ *together with* $\hat{r}$ *is an equilibrium such that* $\hat{a}_t^{t-1} > 0$ *for all* t. *It is then straightforward to verify that* $\{(\hat{c}_t^{t-1}, \hat{c}_t^t, \tilde{a}_t^{t-1})\}_{t=1}^{\infty}$ *together with* $\hat{r}$ *is an equilibrium for the new debt level, where*

$$\tilde{a}_t^{t-1} = \hat{a}_t^{t-1} + (1+\hat{r})\Delta B \text{ for all } t$$

i.e. in each period savings increase by the increased level of debt, plus the provision for the higher required tax payments. Obviously one can construct much more complicated tax experiments that are neutral in the Ricardian sense, provided that for the original tax system the non-borrowing constraints never bind (i.e. that bequest motives are always operative). Also note that Barro discussed his result in the context of a production economy, an issue to which we turn next.

8.3 Overlapping Generations Models with Production

So far we have ignored production in our discussion of OLG-models. It may be the case that some of the pathodologies of the OLG-model appear only in pure exchange versions of the model. Since actual economies feature capital accumulation and production, these pathodologies then are nothing to worry about. However, we will find out that, for example, the possibility of inefficient competitive equilibria extends to OLG models with production. The issues of whether money may have positive value and whether there exists a continuum of equilibria are not easy for production economies and will not be discussed in these notes.

8.3.1 Basic Setup of the Model

As much as possible I will synchronize the discussion here with the discrete time neoclassical growth model in Chapter 2 and the pure exchange OLG model in previous subsections. The economy consists of individuals and firms. Individuals live for two periods By N_t^t denote the number of young people in period t, by N_t^{t-1} denote the number of old people at period t. Normalize the size of the initial old generation to 1, i.e. $N_0^0 = 1$. We assume that people do not die early, so $N_t^t = N_{t+1}^t$. Furthermore assume that the population grows at constant rate n, so that $N_t^t = (1+n)^t N_0^0 = (1+n)^t$. The total population at period t is therefore given by $N_t^{t-1} + N_t^t = (1+n)^t(1 + \frac{1}{1+n})$.

The representative member of generation t has preferences over consumption streams given by

$$u(c_t^t, c_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t)$$

where U is strictly increasing, strictly concave, twice continuously differentiable and satisfies the Inada conditions. All individuals are assumed to be purely selfish and have no bequest motives whatsoever. The initial old generation has preferences

$$u(c_1^0) = U(c_1^0)$$

Each individual of generation $t \geq 1$ has as endowments one unit of time to work when young and no endowment when old. Hence the labor force in period t is of size N_t^t with maximal labor supply of $1 * N_t^t$. Each member of the initial old generation is endowed with capital stock $(1+n)\bar{k}_1 > 0$.

Firms has access to a constant returns to scale technology that produces output Y_t using labor input L_t and capital input K_t rented from households i.e. $Y_t = F(K_t, L_t)$. Since firms face constant returns to scale, profits are zero in equilibrium and we do not have to specify ownership of firms. Also without loss of generality we can assume that there is a single, representative firm, that, as usual, behaves competitively in that it takes as given the rental prices of factor inputs (r_t, w_t) and the price for its output. Defining the capital-labor ratio $k_t = \frac{K_t}{L_t}$ we have by constant returns to scale

$$y_t = \frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$$

We assume that f is twice continuously differentiable, strictly concave and satisfies the Inada conditions.

8.3.2 Competitive Equilibrium

The timing of events for a given generation t is as follows

1. At the beginning of period t production takes place with labor of generation t and capital saved by the now old generation $t-1$ from the previous period. The young generation earns a wage w_t

2. At the end of period t the young generation decides how much of the wage income to consume, c_t^t, and how much to save for tomorrow, s_t^t. The saving occurs in form of physical capital, which is the only asset in this economy

3. At the beginning of period $t+1$ production takes place with labor of generation $t+1$ and the saved capital of the now old generation t. The return on savings equals $r_{t+1} - \delta$, where again r_{t+1} is the rental rate of capital and δ is the rate of depreciation, so that $r_{t+1} - \delta$ is the real interest rate from period t to $t+1$.

4. At the end of period $t+1$ generation t consumes its savings plus interest rate, i.e. $c_{t+1}^t = (1 + r_{t+1} - \delta)s_t^t$ and then dies.

We now can define a sequential markets equilibrium for this economy

Definition 102 *Given $\bar{k}_1$, a sequential markets equilibrium is allocations for households $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$, allocations for the firm $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^{\infty}$ and prices $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^{\infty}$ such that*

1. *For all $t \geq 1$, given $(\hat{w}_t, \hat{r}_{t+1})$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves*

$$\max_{c_t^t, c_{t+1}^t \geq 0, s_t^t} U(c_t^t) + \beta U(c_{t+1}^t)$$

$$\begin{aligned} s.t.\ c_t^t + s_t^t &\leq \hat{w}_t \\ c_{t+1}^t &\leq (1 + \hat{r}_{t+1} - \delta)s_t^t \end{aligned}$$

2. *Given $\bar{k}_1$ and $\hat{r}_1$, $\hat{c}_1^0$ solves*

$$\max_{c_1^0 \geq 0} U(c_1^0)$$

$$s.t.\ c_1^0 \leq (1 + \hat{r}_1 - \delta)(1+n)\bar{k}_1$$

3. *For all $t \geq 1$, given $(\hat{r}_t, \hat{w}_t)$, $(\hat{K}_t, \hat{L}_t)$ solves*

$$\max_{K_t, L_t \geq 0} F(K_t, L_t) - \hat{r}_t K_t - \hat{w}_t L_t$$

4. For all $t \geq 1$

(a) *(Goods Market)*

$$N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$$

(b) *(Asset Market)*

$$N_t^t \hat{s}_t^t = \hat{K}_{t+1}$$

(c) *(Labor Market)*

$$N_t^t = \hat{L}_t$$

The first two points in the equilibrium definition are completely standard, apart from the change in the timing convention for the interest rate. For firm maximization we used the fact that, given that the firm is renting inputs in each period, the firms intertemporal maximization problem separates into a sequence of static profit maximization problems. The goods market equilibrium condition is standard: total consumption plus gross investment equals output. The labor market equilibrium condition is obvious. The asset or capital market equilibrium condition requires a bit more thought: it states that total saving of the currently young generation makes up the capital stock for tomorrow, since physical capital is the only asset in this economy. Alternatively think of it as equating the total supply of capital in form the saving done by the now young, tomorrow old generation and the total demand for capital by firms next period.[28] It will be useful to single out particular equilibria and attach a certain name to them.

Definition 103 *A steady state (or stationary equilibrium) is* $(\bar{k}, \bar{s}, \bar{c}_1, \bar{c}_2, \bar{r}, \bar{w})$ *such that the sequences* $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$, $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^{\infty}$ *and* $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^{\infty}$,

[28] To define an Arrow-Debreu equilibrium is quite standard here. Let p_t the price of the consumption good at period t, $r_t p_t$ the nominal rental price of capital and $w_t p_t$ the nominal wage. Then the household and the firms problems are in the neoclassical growth model, in the household problem taking into account that agents only live for two periods.

defined by

$$\begin{aligned}
\hat{c}_t^t &= \bar{c}_1 \\
\hat{c}_t^{t-1} &= \bar{c}_2 \\
\hat{s}_t^t &= \bar{s} \\
\hat{r}_t &= \bar{r} \\
\hat{w}_t &= \bar{w} \\
\hat{K}_t &= \bar{k} * N_t^t \\
\hat{L}_t &= N_t^t
\end{aligned}$$

are an equilibrium, for given initial condition $\bar{k}_1 = \bar{k}$.

In other words, a steady state is an equilibrium for which the allocation (per capita) is constant over time, given that the initial condition for the initial capital stock is exactly right. Alternatively it is allocations and prices that satisfy all the equilibrium conditions apart from possibly obeying the initial condition.

We can use the goods and asset market equilibrium to derive an equation that equates saving to investment. By definition gross investment equals $\hat{K}_{t+1} - (1-\delta)\hat{K}_t$, whereas savings equals that part of income that is not consumed, or

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t) - \left(N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1}\right)$$

But what is total saving equal to? The currently young save $N_t^t \hat{s}_t^t$, the currently old dissave $\hat{s}_{t-1}^{t-1} N_{t-1}^{t-1} = (1-\delta)\hat{K}_t$ (they sell whatever capital stock they have left).[29] Hence setting investment equal to saving yields

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = N_t^t \hat{s}_t^t - (1-\delta)\hat{K}_t$$

or our asset market equilibrium condition

$$N_t^t \hat{s}_t^t = \hat{K}_{t+1}$$

[29] By definition the saving of the old is their total income minus their total consumption. Their income consists of returns on their assets and hence their total saving is

$$\begin{aligned}
& \left[(r_t s_{t-1}^{t-1} - c_t^{t-1}\right] N_{t-1}^{t-1} \\
= \quad & -(1-\delta) s_{t-1}^{t-1} N_{t-1}^{t-1} = -(1-\delta) K_t
\end{aligned}$$

Now let us start to characterize the equilibrium It will turn out that we can describe the equilibrium completely by a first order difference equation in the capital-labor ratio k_t. Unfortunately it will have a rather nasty form in general, so that we can characterize analytic properties of the competitive equilibrium only very partially. Also note that, as we will see later, the welfare theorems do not apply so that there is no social planner problem that will make our lives easier, as was the case in the infinitely lived consumer model (which I dubbed the discrete-time neoclassical growth model in Section 3).

From now on we will omit the hats above the variables indicating equilibrium elements as it is understood that the following analysis applies to equilibrium sequences. From the optimization condition for capital for the firm we obtain

$$r_t = F_K(K_t, L_t) = F_K\left(\frac{K_t}{L_t}, 1\right) = f'(k_t)$$

because partial derivatives of functions that are homogeneous of degree 1 are homogeneous of degree zero. Since we have zero profits in equilibrium we find that

$$w_t L_t = F(K_t, L_t) - r_t K_t$$

and dividing by L_t we obtain

$$w_t = f(k_t) - f'(k_t)k_t$$

i.e. factor prices are completely determined by the capital-labor ratio. Investigating the households problem we see that its solution is completely characterized by a saving function (note that given our assumptions on preferences the optimal choice for savings exists and is unique)

$$\begin{aligned} s_t^t &= s\left(w_t, r_{t+1}\right) \\ &= s\left(f(k_t) - f'(k_t)k_t, f'(k_{t+1})\right) \end{aligned}$$

so optimal savings are a function of this and next period's capital stock. Obviously, once we know s_t^t we know c_t^t and c_{t+1}^t from the household's budget constraint. From Walras law one of the market clearing conditions is redundant. Equilibrium in the labor market is straightforward as

$$L_t = N_t^t = (1+n)^t$$

So let's drop the goods market equilibrium condition.[30] Then the only condition left to exploit is the asset market equilibrium condition

$$\begin{aligned} s_t^t N_t^t &= K_{t+1} \\ s_t^t &= \frac{K_{t+1}}{N_t^t} = \frac{N_{t+1}^{t+1}}{N_t^t}\frac{K_{t+1}}{N_{t+1}^{t+1}} \\ &= (1+n)\frac{K_{t+1}}{L_{t+1}} \\ &= (1+n)k_{t+1} \end{aligned}$$

Substituting in the savings function yields our first order difference equation

$$k_{t+1} = \frac{s\left(f(k_t) - f'(k_t)k_t, f'(k_{t+1})\right)}{1+n} \tag{8.32}$$

where the exact form of the saving function obviously depends on the functional form of the utility function U. As starting value for the capital-labor ratio we have $\frac{K_1}{L_1} = \frac{(1+n)\bar{k}_1}{N_1^1} = \bar{k}_1$. So in principle we could put equation (8.32) on a computer and solve for the entire sequence of $\{k_{t+1}\}_{t=1}^{\infty}$ and hence for the entire equilibrium. Note, however, that equation (8.32) gives k_{t+1} only as an implicit function of k_t as k_{t+1} appears on the right hand side of the equation as well. So let us make an attempt to obtain analytical properties of this equation. Before, let's solve an example.

Example 104 *Let $U(c) = \ln(c)$, $n = 0, \beta = 1$ and $f(k) = k^{\alpha}$, with $\alpha \in (0,1)$. The choice of log-utility is particularly convenient as the income and substitution effects of an interest change cancel each other out; saving is independent of r_{t+1}. As we will see later it is crucial whether the income or substitution effect for an interest change dominates in the saving decision, i.e. whether*

$$s_{r_{t+1}}(w_t, r_{t+1}) \lesseqgtr 0$$

[30] In the homework you are asked to do the analysis with dropping the asset market instead of the goods market equilibrium condition. Keep the present analysis in mind when doing this question.

But let's proceed. The saving function for the example is given by

$$
\begin{aligned}
s(w_t, r_{t+1}) &= \frac{1}{2} w_t \\
&= \frac{1}{2}\left(k_t^{\alpha} - \alpha k_t^{\alpha}\right) \\
&= \frac{1-\alpha}{2} k_t^{\alpha}
\end{aligned}
$$

so that the difference equation characterizing the dynamic equilibrium is given by

$$k_{t+1} = \frac{1-\alpha}{2} k_t^{\alpha}$$

There are two steady states for this differential equation, $k_0 = 0$ and $k^* = \left(\frac{1-\alpha}{2}\right)^{\frac{1}{1-\alpha}}$. The first obviously is not an equilibrium as interest rates are infinite and no solution to the consumer problem exists. From now on we will ignore this steady state, not only for the example, but in general. Hence there is a unique steady state equilibrium associated with k^*. From any initial condition $\bar{k}_1 > 0$, there is a unique dynamic equilibrium $\{k_{t+1}\}_{t=1}^{\infty}$ converging to k^* described by the first order difference equation above.

Unfortunately things are not always that easy. Let us return to the general first order difference equation (8.32) and discuss properties of the saving function. Let, us for simplicity, assume that the saving function s is differentiable in both arguments (w_t, r_{t+1}).[31] Since the saving function satisfies the first order condition

$$U'(w_t - s(w_t, r_{t+1})) = \beta U'((1 + r_{t+1} - \delta)s(w_t, r_{t+1})) * (1 + r_{t+1} - \delta)$$

we use the Implicit Function Theorem (which is applicable in this case) to obtain

$$
\begin{aligned}
s_{w_t}(w_t, r_{t+1}) &= \frac{U''(w_t - s(w_t, r_{t+1}))}{U''(w_t - s(w_t, r_{t+1})) + \beta U''((1 + r_{t+1} - \delta)s(w_t, r_{t+1}))(1 + r_{t+1} - \delta)^2} \in (0, 1) \\
s_{r_{t+1}}(w_t, r_{t+1}) &= \frac{-\beta U'((1 + r_{t+1} - \delta)s(.,.)) - \beta U''((1 + r_{t+1} - \delta)s(.,.))(1 + r_{t+1} - \delta)s(.,.)}{U''(w_t - s(.,.)) + \beta U''((1 + r_{t+1} - \delta)s(.,.))(1 + r_{t+1} - \delta)^2} \gtrless
\end{aligned}
$$

[31] One has to invoke the implicit function theorem (and check its conditions) on the first order condition to insure differentiability of the savings function. See Mas-Colell et al. p. 940-942 for details.

Figure 8.6: Capital Dynamics in the OLG Model with Production

Given our assumptions optimal saving increases in first period income w_t, but it may increase or decrease in the interest rate. You may verify from the above formula that indeed for the log-case $s_{r_{t+1}}(w_t, r_{t+1}) = 0$. A lot of theoretical work focused on the case in which the saving function increases with the interest rate, which is equivalent to saying that the substitution effect dominates the income effect (and equivalent to assuming that consumption in the two periods are strict gross substitutes).

Equation (8.32) traces out a graph in (k_t, k_{t+1}) space whose shape we want to characterize. Differentiating both sides of (8.32) with respect to k_t we obtain[32]

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_{w_t}(w_t, r_{t+1})f''(k_t)k_t + s_{r_{t+1}}(w_t, r_{t+1})f'(k_{t+1})\frac{dk_{t+1}}{dk_t}}{1+n}$$

or rewriting

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_{w_t}(w_t, r_{t+1})f''(k_t)k_t}{1+n-s_{r_{t+1}}(w_t, r_{t+1})f''(k_{t+1})}$$

Given our assumptions on f the nominator of the above expression is strictly positive for all $k_t > 0$. If we assume that $s_{r_{t+1}} \geq 0$, then the (k_t, k_{t+1})-locus is upward sloping. If we allow $s_{r_{t+1}} < 0$, then it may be downward sloping.

Figure 13 shows possible shapes of the (k_t, k_{t+1})-locus under the assumption that $s_{r_{t+1}} \geq 0$. We see that even this assumption does not place a lot of restrictions on the dynamic behavior of our economy. Without further assumptions it may be the case that, as in case A there is no steady state with positive capital-labor ratio. Starting from any initial capital-per worker level the economy converges to a situation with no production over time. It may be that, as in case C, there is a unique positive steady state k_C^* and this steady state is globally stable (for state space excluding 0). Or it is possible that there are multiple steady states which alternate in being locally stable (as k_B^*) and unstable (as k_B^{**}) as in case B. Just about any dynamic behavior is possible and in order to deduce further qualitative properties we must either specify special functional forms or make assumptions about endogenous variables, something that one should avoid, if possible.

[32]Again we appeal to the Implicit function theorem that guarantees that k_{t+1} is a differentiable function of k_t with derivative given below.

We will proceed however, doing exactly this. For now let's *assume* that there exists a unique positive steady state. Under what conditions is this steady state locally stable? As suggested by Figure 13 stability requires that the saving locus intersects the 45^0-line from above, provided the locus is upward sloping. A necessary and sufficient condition for local stability at the assumed unique steady state k^* is that

$$\left|\frac{-s_{w_t}(w(k^*), r(k^*))f''(k^*)k^*}{1+n-s_{r_{t+1}}(w(k^*), r(k^*))f''(k^*)}\right| < 1$$

If $s_{r_{t+1}} < 0$ it may be possible that the slope of the saving locus is negative. Under the condition above the steady state is still locally stable, but it exhibits oscillatory dynamics. If we require that the unique steady state is locally stable and that the dynamic equilibrium is characterized by monotonic adjustment to the unique steady state we need as necessary and sufficient condition

$$0 < \frac{-s_{w_t}(w(k^*), r(k^*))f''(k^*)k^*}{1+n-s_{r_{t+1}}(w(k^*), r(k^*))f''(k^*)} < 1$$

The procedure to make sufficient assumptions that guarantee the existence of a well-behaved dynamic equilibrium and then use exactly these assumption to deduce qualitative comparative statics results (how does the steady state change as we change δ, n or the like) is called Samuelson's correspondence principle, as often exactly the assumptions that guarantee monotonic local stability are sufficient to draw qualitative comparative statics conclusions. Diamond (1965) uses Samuelson's correspondence principle extensively and we will do so, too, assuming from now on that above inequalities hold.

8.3.3 Optimality of Allocations

Before turning to Diamond's (1965) analysis of the effect of public debt let us discuss the dynamic optimality properties of competitive equilibria. Consider first steady state equilibria. Let c_1^*, c_2^* be the steady state consumption levels when young and old, respectively, and k^* be the steady state capital labor ratio. Consider the goods market clearing (or resource constraint)

$$N_t^t \hat{c}_t^t + N_t^{t-1}\hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$$

Divide by $N_t^t = \hat{L}_t$ to obtain

$$\hat{c}_t^t + \frac{\hat{c}_t^{t-1}}{1+n} + (1+n)\hat{k}_{t+1} - (1-\delta)\hat{k}_t = f(k_t) \tag{8.33}$$

and use the steady state allocations to obtain

$$c_1^* + \frac{c_2^*}{1+n} + (1+n)k^* - (1-\delta)k^* = f(k^*)$$

Define $c^* = c_1^* + \frac{c_2^*}{1+n}$ to be total (per worker) consumption in the steady state. We have that

$$c^* = f(k^*) - (n+\delta)k^*$$

Now suppose that the steady state equilibrium satisfies

$$f'(k^*) - \delta < n \tag{8.34}$$

something that may or may not hold, depending on functional forms and parameter values. We claim that this steady state is not Pareto optimal. The intuition is as follows. Suppose that (8.34) holds. Then it is possible to decrease the capital stock per worker marginally, and the effect on per capita consumption is

$$\frac{dc^*}{dk^*} = f'(k^*) - (n+\delta) < 0$$

so that a marginal decrease of the capital stock leads to higher available overall consumption. The capital stock is inefficiently high; it is so high that its marginal productivity $f'(k^*)$ is outweighed by the cost of replacing depreciated capital, δk^* and provide newborns with the steady state level of capital per worker, nk^*. In this situation we can again pull the Gamov trick to construct a Pareto superior allocation. Suppose the economy is in the steady state at some arbitrary date t and suppose that the steady state satisfies (8.34). Now consider the alternative allocation: at date t reduce the capital stock per worker to be saved to the next period, k_{t+1}, by a marginal $\Delta k^* < 0$ to $k^{**} = k^* + \Delta k^*$ and keep it at k^{**} forever. From (8.33) we obtain

$$c_t = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}$$

The effect on per capita consumption from period t onwards is

$$\begin{aligned}
\Delta c_t &= -(1+n)\Delta k^* > 0 \\
\Delta c_{t+\tau} &= f'(k^*)\Delta k^* + [1-\delta-(1+n)]\Delta k^* \\
&= [f'(k^*) - (\delta+n)]\,\Delta k^* > 0
\end{aligned}$$

In this way we can increase total per capita consumption in every period. Now we just divide the additional consumption between the two generations alive in a given period in such a way that make both generations better off, which is straightforward to do, given that we have extra consumption goods to distribute in every period. Note again that for the Gamov trick to work it is crucial to have an infinite hotel, i.e. that time extends to the infinite future. If there is a last generation, it surely will dislike losing some of its final period capital (which we assume is eatable as we are in a one sector economy where the good is a consumption as well as investment good). A construction of a Pareto superior allocation wouldn't be possible. The previous discussion can be summarized in the following proposition

Proposition 105 *Suppose a competitive equilibrium converges to a steady state satisfying* (8.34). *Then the equilibrium allocation is not Pareto efficient, or, as often called, the equilibrium is dynamically inefficient.*

When comparing this result to the pure exchange model we see the direct parallel: an allocation is inefficient if the interest rate (in the steady state) is smaller than the population growth rate, i.e. if we are in the Samuelson case. In fact, we repeat a much stronger result by Balasko and Shell that we quoted earlier, but that also applies to production economies. A feasible allocation is an allocation $c_1^0, \{c_t^t, c_{t+1}^t, k_{t+1}\}_{t=1}^{\infty}$ that satisfies all negativity constraints and the resource constraint (8.33). Obviously from the allocation we can reconstruct s_t^t and K_t. Let $r_t = f'(k_t)$ denote the marginal products of capital per worker. Maintain all assumptions made on U and f and let n_t be the population growth rate from period $t-1$ to t. We have the following result

Theorem 106 *Cass (1972)*[33]*, Balasko and Shell (1980). A feasible allocation is Pareto optimal if and only if*

$$\sum_{t=1}^{\infty} \prod_{\tau=1}^{t} \frac{(1 + r_{\tau+1} - \delta)}{(1 + n_{\tau+1})} = +\infty$$

As an obvious corollary, alluded to before we have that a steady state equilibrium is Pareto optimal (or dynamically efficient) if and only if $f'(k^*) - \delta \geq n$.

That dynamic inefficiency is not purely an academic matter is demonstrated by the following example

[33] The first reference of this theorem is in fact Cass (1972), Theorem 3.

Example 107 *Consider the previous example with log utility, but now with population growth n and time discounting β. It is straightforward to compute the steady state unique steady state as*

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

so that

$$r^* = \frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)}$$

and the economy is dynamically inefficient if and only if

$$\frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)} - \delta < n$$

Let's pick some reasonable numbers. We have a 2-period OLG model, so let us interpret one period as 30 years. α corresponds to the capital share of income, so $\alpha = .3$ is a commonly used value in macroeconomics. The current yearly population growth rate in the US is about 1%, so lets pick $(1+n) = (1+0.01)^{30}$. Suppose that capital depreciates at around 6% per year, so choose $(1-\delta) = 0.94^{30}$. This yields $n = 0.35$ and $\delta = 0.843$. Then for a yearly subjective discount factor $\beta_y \geq 0.998$, the economy is dynamically inefficient. Dynamic inefficiency therefore is definitely more than just a theoretical curiousum. If the economy features technological progress of rate g, then the condition for dynamic inefficiency becomes (approximately) $f'(k^*) < n+\delta+g$. If we assume a yearly rate of technological progress of 2%, then with the same parameter values for $\beta_y \geq 0.971$ we obtain dynamic inefficiency. Note that there is a more immediate way to check for dynamic inefficiency in an actual economy: since in the model $f'(k^*) - \delta$ is the real interest rate and $g + n$ is the growth rate of real GDP, one may just check whether the real interest rate is smaller than the growth rate in long-run averages.

If the competitive equilibrium of the economy features dynamic inefficiency its citizens save more than is socially optimal. Hence government programs that reduce national saving are called for. We already have discussed such a government program, namely an unfunded, or pay-as-you-go social security system. Let's briefly see how such a program can reduce the capital stock of an economy and hence leads to a Pareto-superior allocation, provided that the initial allocation without the system was dynamically inefficient.

Suppose the government introduces a social security system that taxes people the amount τ when young and pays benefits of $b = (1+n)\tau$ when old. For simplicity we assume balanced budget for the social security system as well as lump-sum taxation. The budget constraints of the representative individual change to

$$\begin{aligned} c_t^t + s_t^t &= w_t - \tau \\ c_{t+1}^t &= (1 + r_{t+1} - \delta)s_t^t + (1+n)\tau \end{aligned}$$

We will repeat our previous analysis and first check how individual savings react to a change in the size of the social security system. The first order condition for consumer maximization is

$$U'(w_t - \tau - s_t^t) = \beta U'((1 + r_{t+1} - \delta)s_t^t + (1+n)\tau) * (1 + r_{t+1} - \delta)$$

which implicitly defines the optimal saving function $s_t^t = s(w_t, r_{t+1}, \tau)$. Again invoking the implicit function theorem we find that

$$\begin{aligned} & -U''(w_t - \tau - s(.,.,.)) \left(1 + \frac{ds}{d\tau}\right) \\ = \;\; & \beta U''((1 + r_{t+1} - \delta)s(.,.,.) + (1+n)\tau) * (1 + r_{t+1} - \delta) * \left((1 + r_{t+1} - \delta)\frac{ds}{d\tau} + 1 + n\right) \end{aligned}$$

or

$$\frac{ds}{d\tau} = s_\tau = -\frac{U''() + (1+n)\beta U''(.)(1 + r_{t+1} - \delta)}{U''(.) + \beta U''(.)(1 + r_{t+1} - \delta)^2} < 0$$

Therefore the bigger the pay-as-you-go social security system, the smaller is the private savings of individuals, holding factor prices constant. This however, is only the partial equilibrium effect of social security. Now let's use the asset market equilibrium condition

$$\begin{aligned} k_{t+1} &= \frac{s(w_t, r_{t+1}, \tau)}{1+n} \\ &= \frac{s\left(f(k_t) - f'(k_t)k_t, f'(k_{t+1}, \tau\right)}{1+n} \end{aligned}$$

Now let us investigate how the equilibrium (k_t, k_{t+1})-locus changes as τ changes. For *fixed* k_t, how does $k_{t+1}(k_t)$ changes as τ changes. Again using the implicit function theorem yields

$$\frac{dk_{t+1}}{d\tau} = \frac{s_{r_{t+1}} f''(k_{t+1}) \frac{dk_{t+1}}{d\tau} + s_\tau}{1+n}$$

Figure 8.7: The Dynamics of the Neoclassical Growth Model

and hence

$$\frac{dk_{t+1}}{d\tau} = \frac{s_\tau}{1+n-s_{r_{t+1}}f''(k_{t+1})}$$

The nominator is negative as shown above; the denominator is positive by our assumption of monotonic local stability (this is our first application of Samuelson's correspondence principle). Hence $\frac{dk_{t+1}}{d\tau} < 0$, the locus (always under the maintained monotonic stability assumption) tilts downwards, as shown in Figure 14.

We can conduct the following thought experiment. Suppose the economy converged to its old steady state k^* and suddenly, at period T, the government *unanticipatedly* announces the introduction of a (marginal) pay-as-you go system. The saving locus shifts down, the new steady state capital labor ratio declines and the economy, over time, converges to its new steady state. Note that over time the interest rate increases and the wage rate declines. Is the introduction of a marginal pay-as-you-go social security system welfare improving? It depends on whether the old steady state capital-labor ratio was inefficiently high, i.e. it depends on whether $f'(k^*) - \delta < n$ or not. Our conclusions about the desirability of social security remain unchanged from the pure exchange model.

8.3.4 The Long-Run Effects of Government Debt

Diamond (1965) discusses the effects of government debt on long run capital accumulation. He distinguishes between government debt that is held by foreigners, so-called external debt, and government debt that is held by domestic citizens, so-called internal debt. Note that the second case is identical to Barro's analysis if we abstract from capital accumulation and allow altruistic bequest motives. In fact, in Diamond's environment with production, but altruistic and operative bequests a similar Ricardian equivalence result as before applies. In this sense Barro's neutrality result provides the benchmark for Diamond's analysis of the internal debt case, and we will see how the absence of operative bequests leads to real consequences of different levels of internal debt.

External Debt

Suppose the government has initial outstanding debt, denoted in real terms, of B_1. Denote by $b_t = \frac{B_t}{L_t} = \frac{B_t}{N_t^t}$ the debt-labor ratio. All government bonds have maturity of one period, and the government issues new bonds[34] so as to keep the debt-labor ratio constant at $b_t = b$ over time. Bonds that are issued in period $t-1$, B_t, are required to pay the same gross interest as domestic capital, namely $1+r_t-\delta$, in period t when they become due. The government taxes the current young generation in order to finance the required interest payments on the debt. Taxes are lump sum and are denoted by τ. The budget constraint of the government is then

$$B_t(1 + r_t - \delta) = B_{t+1} + N_t^t \tau$$

or, dividing by N_t^t, we get, under the assumption of a constant debt-labor ratio,

$$\tau = (r_t - \delta - n)b$$

For the previous discussion of the model nothing but the budget constraint of young individuals changes, namely to

$$\begin{aligned} c_t^t + s_t^t &= w_t - \tau \\ &= w_t - (r_t - \delta - n)b \end{aligned}$$

In particular the asset market equilibrium condition does not change as the outstanding debt is held exclusively by foreigners, by assumption. As before we obtain a saving function $s(w_t - (r_t - \delta - n)b, r_{t+1})$ as solution to the households optimization problem, and the asset market equilibrium condition reads as before

$$k_{t+1} = \frac{s(w_t - (r_t - \delta - n)b, r_{t+1})}{1+n}$$

Our objective is to determine how a change in the external debt-labor ratio changes the steady state capital stock and the interest rate. This can be answered by examining $s()$. Again we will apply Samuelson's correspondence principle. Assuming monotonic local stability of the unique steady state is equivalent to assuming

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_{w_t}(.,.)f''(k_t)(k_t + b)}{1 + n - s_{r_{t+1}}(.,.)f''(k_{t+1})} \in (0,1) \tag{8.35}$$

[34] As Diamond (1965) let us specify these bonds as interest-bearing bonds (in contrast to zero-coupon bonds). A bond bought in period t pays (interst plus principal) $1+r_{t+1}-\delta$ in period $t+1$.

In order to determine how the saving locus in (k_t, k_{t+1}) space shifts we apply the Implicit Function Theorem to the asset equilibrium condition to find

$$\frac{dk_{t+1}}{db} = \frac{-s_{w_t}(.,.)\,(f'(k_t) - \delta - n)}{1 + n - s_{r_{t+1}}(.,.)f''(k_{t+1})}$$

so the sign of $\frac{dk_{t+1}}{db}$ equals the negative of the sign of $f'(k_t) - \delta - n$ under the maintained assumption of monotonic local stability. Suppose we are at a steady state k^* corresponding to external debt to labor ratio b^*. Now the government marginally increases the debt-labor ratio. If the old steady state was not dynamically inefficient, i.e. $f'(k^*) \geq \delta + n$, then the saving locus shifts down and the new steady state capital stock is lower than the old one. Diamond goes on to show that in this case such an increase in government debt leads to a reduction in the utility level of a generation that lives in the new rather than the old steady state. Note however that, because of transition generations this does not necessarily mean that marginally increasing external debt leads to a Pareto-inferior allocation. For the case in which the old equilibrium is dynamically inefficient an increase in government debt shifts the saving locus upward and hence increases the steady state capital stock per worker. Again Diamond shows that now the effects on steady state utility are indeterminate.

Internal Debt

Now we assume that government debt is held exclusively by own citizens. The tax payments required to finance the interest payments on the outstanding debt take the same form as before. Let's assume that the government issues new government debt so as to keep the debt-labor ratio $\frac{B_t}{L_t}$ constant over time at $\tilde{b}$. Hence the required tax payments are given by

$$\tau = (r_t - \delta - n)\tilde{b}$$

Again denote the new saving function derived from consumer optimization by $s(w_t - (r_t - \delta - n)\tilde{b}, r_{t+1})$. Now, however, the equilibrium asset market condition changes as the savings of the young not only have to absorb the supply of the physical capital stock, but also the supply of government bonds newly issued. Hence the equilibrium condition becomes

$$N_t^t s(w_t - (r_t - \delta - n)\tilde{b}, r_{t+1}) = K_{t+1} + B_{t+1}$$

or, dividing by $N_t^t = L_t$, we obtain

$$k_{t+1} = \frac{s(w_t - (r_t - \delta - n)\tilde{b}, r_{t+1})}{1+n} - \tilde{b}$$

To determine the shift in the saving locus in (k_t, k_{t+1}) we again implicitly differentiate to obtain

$$\frac{dk_{t+1}}{d\tilde{b}} = \frac{-s_{w_t}(.,.)(r_t - \delta - n) + s_{r_{t+1}} f''(k_{t+1}) \frac{dk_{t+1}}{d\tilde{b}}}{1+n} - 1$$

and hence

$$\frac{dk_{t+1}}{d\tilde{b}} = \frac{-s_{w_t}(.,.)(f'(k_t) - \delta - n) - (1+n)}{(1+n) - s_{r_{t+1}} f''(k_{t+1})}.$$

Now we again assume that

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_{w_t}(.,.)f''(k_t)(k_t + \tilde{b})}{1+n - s_{r_{t+1}}(.,.)f''(k_{t+1})} = \phi(k_t, k_{t+1}, \tilde{b}) \in (0,1)$$

to assure monotonic stability of the steady state. Then

$$\begin{aligned} \frac{dk_{t+1}}{d\tilde{b}} &= \frac{[-s_{w_t}(.,.)(f'(k_t) - \delta - n) - (1+n)]}{(1+n) - s_{r_{t+1}} f''(k_{t+1})} \\ &= \left[\frac{\phi(k_t, k_{t+1}, \tilde{b})}{-f''(k_t) s_{w_t}(.,.)(k_t + \tilde{b})} \right] \cdot [s_{w_t}(.,.)(\delta + n - f'(k_t)) - (1+n)] \end{aligned}$$

The term in the first brackets is positive since $\phi \in (0,1)$, $s_w \in (0,1)$ and $-f''(k_t)(k_t + \tilde{b}) > 0$. The term in the second brackets is negative since

$$s_{w_t}(.,.)(\delta + n - f'(k_t)) \leq s_{w_t}(.,.)(\delta + n) < 1 + n.$$

Thus

$$\frac{dk_{t+1}}{d\tilde{b}} < 0.$$

The $k_{t+1}(k_t)$ curve unambiguously shifts down with an increase in internal debt $\tilde{b}$, leading to a decline in the steady state capital stock per worker. Diamond, again only comparing steady state utilities, shows that if the initial steady state was dynamically efficient, then an increase in internal debt leads to a reduction in steady state welfare, whereas if the initial steady state was dynamically inefficient, then an increase in internal government debt leads to a increase in steady state welfare. Here the intuition is again clear: if the economy has accumulated too much capital, then increasing the supply of alternative assets leads to a interest-driven "crowding out" of demand for physical capital, which is a good thing given that the economy possesses too much capital. In the efficient case the reverse logic applies. In comparison with the external debt case we obtain clearer welfare conclusions for the dynamically inefficient case. For external debt an increase in debt is not necessarily good even in the dynamically inefficient case because it requires higher tax payments, which, in contrast to internal debt, leave the country and therefore reduce the available resources to be consumed (or invested). This negative effect balances against the positive effect of reducing the inefficiently high capital stock, so that the overall effects are indeterminate. In comparison to Barro (1974) we see that without operative bequests the level of outstanding government bonds influences real equilibrium allocations: Ricardian equivalence breaks down.

Chapter 9

Continuous Time Growth Theory

> I do not see how one can look at figures like these without seeing them as representing possibilities. Is there some action a government could take that would lead the Indian economy to grow like Indonesia's or Egypt's? If so, what exactly? If not, what is it about the nature of India that makes it so? The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else. [Lucas 1988, p. 5]

So much for motivation. We are doing growth in continuous time since I think you should know how to deal with continuous time models as a significant fraction of the economic literature employs continuous time, partly because in certain instances the mathematics becomes easier. In continuous time, variables are functions of time and one can use calculus to analyze how they change over time.

9.1 Stylized Growth and Development Facts

> Data! Data! Data! I can't make bricks without clay. [Sherlock Holmes]

In this part we will briefly review the main stylized facts characterizing economic growth of the now industrialized countries and the main facts

Figure 9.1: Real GDP per Capita

characterizing the level and change of economic development of not yet industrialized countries.

9.1.1 Kaldor's Growth Facts

The British economist Nicholas Kaldor pointed out the following stylized growth facts (empirical regularities of the growth process) for the US and for most other industrialized countries.

1. Output (real GDP) per worker $y = \frac{Y}{L}$ and capital per worker $k = \frac{K}{L}$ grow over time at relatively constant and positive rate. See Figure 9.1.1.

2. They grow at similar rates, so that the ratio between capital and output, $\frac{K}{Y}$ is relatively constant over time

3. The real return to capital r (and the real interest rate $r-\delta$) is relatively constant over time.

4. The capital and labor shares are roughly constant over time. The capital share α is the fraction of GDP that is devoted to interest payments on capital, $\alpha = \frac{rK}{Y}$. The labor share $1-\alpha$ is the fraction of GDP that is devoted to the payments to labor inputs; i.e. to wages and salaries and other compensations: $1-\alpha = \frac{wL}{Y}$. Here w is the real wage.

These stylized facts motivated the development of the neoclassical growth model, the Solow growth model, to be discussed below. The Solow model has spectacular success in explaining the stylized growth facts by Kaldor.

9.1.2 Development Facts from the Summers-Heston Data Set

In addition to the growth facts we will be concerned with how income (per worker) levels and growth rates vary across countries in different stages of their development process. The true test of the Solow model is to what extent it can explain differences in income levels and growth rates across countries,

Figure 9.2: Relative Incomes around the World

the so called development facts. As we will see in our discussion of Mankiw, Romer and Weil (1992) the verdict is mixed.

Now we summarize the most important facts from the Summers and Heston's panel data set. This data set follows about 100 countries for 30 years and has data on income (production) levels and growth rates as well as population and labor force data. In what follows we focus on the variable income per worker. This is due to two considerations: a) our theory (the Solow model) will make predictions about exactly this variable b) although other variables are also important determinants for the standard of living in a country, income per worker (or income per capita) may be the most important variable (for the economist anyway) and other determinants of well-being tend to be highly positively correlated with income per worker.

Before looking at the data we have to think about an important measurement issue. Income is measured as GDP, and GDP of a particular country is measured in the currency of that particular country. In order to compare income between countries we have to convert these income measures into a common unit. One option would be exchange rates. These, however, tend to be rather volatile and reactive to events on world financial markets. Economists which study growth and development tend to use PPP-based exchange rates, where PPP stands for Purchasing Power Parity. All income numbers used by Summers and Heston (and used in these notes) are converted to $US via PPP-based exchange rates.

Here are the most important facts from the Summers and Heston data set:

1. Enormous variation of per capita income across countries: the poorest countries have about 5% of per capita GDP of US per capita GDP. This fact makes a statement about dispersion in income *levels.* When we look at Figure **??**, we see that out of the 104 countries in the data set, 37 in 1990 and 38 in 1960 had per worker incomes of less than 10% of the US level.

   The richest countries in 1990, in terms of per worker income, are Luxembourg, the US, Canada and Switzerland with over $30,000, the poorest countries, without exceptions, are in Africa. Mali, Uganda, Chad, Central African Republic, Burundi, Burkina Faso all have income per

Figure 9.3: Growth Rates around the World

worker of less than \$1000. Not only are most countries extremely poor compared to the US, but most of the world's *population* is poor relative to the US.

2. Enormous variation in *growth rates* of per worker income. This fact makes a statement about *changes* of levels in per capita income. Figure 2 shows the distribution of average yearly growth rates from 1960 to 1990.

   The majority of countries grew at average rates of between 1% and 3% (these are growth rates for *real GDP per worker*). Note that some countries posted average growth rates in excess of 6% (Singapore, Hong Kong, Japan, Taiwan, South Korea) whereas other countries actually shrunk, i.e. had negative growth rates (Venezuela, Nicaragua, Guyana, Zambia, Benin, Ghana, Mauretania, Madagascar, Mozambique, Malawi, Uganda, Mali). We will sometimes call the first group growth miracles, the second group growth disasters. Note that not only did the disasters' relative position worsen, but that these countries experienced *absolute* declines in living standards. The US, in terms of its growth experience in the last 30 years, was in the middle of the pack with a growth rate of real per worker GDP of 1.4% between 1960 and 1990.

3. Growth rates determine economic fate of a country over longer periods of time. How long does it take for a country to double its per capita GDP if it grows at average rate of $g\%$ per year? A good rule of thumb: $70/g$ years (this rule of thumb is due to Nobel Price winner Robert E. Lucas (1988)).[1] Growth rates are not constant over time for a given country. This can easily be demonstrated for the US. GDP per worker

[1] Let y_T denote GDP per capita in period T and y_0 denote period 0 GDP per capita in a particular country. Suppose the growth rate of GDP per capita is constant at g, i.e. $100 * g\%$. Then

$$y_T = y_0 e^{gT}$$

Suppose we want to double GDP per capita in T years. Then

$$2 = \frac{y_T}{y_0} = e^{gT}$$

Figure 9.4: Income per Capita in the Long Run

in 1990 was \$36,810. If GDP would always have grown at 1.4%, then for the US GDP per worker would have been about \$9,000 in 1900, \$2,300 in 1800, \$570 in 1700, \$140 in 1600, \$35 in 1500 and so forth. Economic historians (and common sense) tells us that nobody can survive on \$35 per year (estimates are that about \$300 are necessary as minimum income level for survival). This indicates that the US (or any other country) cannot have experienced sustained positive growth for the last millennium or so. In fact, prior to the era of modern economic growth, which started in England in the late 1800th century, per worker income levels have been almost constant at subsistence levels. This can be seen from Figure 3, which compiles data from various historical sources.

The start of modern economic growth is sometimes referred to as the Industrial Revolution. It is the single most significant economic event in history and has, like no other event, changed the economic circumstances in which we live. Hence modern economic growth is a quite recent phenomenon, and so far has occurred only in Western Europe and its offsprings (US, Canada, Australia and New Zealand) as well as recently in East Asia.

4. Countries change their *relative* position in the (international) income distribution. Growth disasters fall, growth miracles rise, in the relative cross-country income distribution. A classical example of a growth disaster is Argentina. At the turn of the century Argentina had a per-worker income that was comparable to that in the US. In 1990 the per-worker income of Argentina was only on a level of one third of the US, due to a healthy growth experience of the US and a disastrous growth performance of Argentina. Countries that dramatically moved up in the relative income distribution include Italy, Spain, Hong Kong,

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or

$$\begin{aligned} \ln(2) &= gT \\ T^* &= \frac{\ln(2)}{g} = \frac{100 * \ln(2)}{g(\text{in } \%)} \end{aligned}$$

Since $100 * \ln(2) \approx 70$, the rule of thumb follows.

Japan, Taiwan and South Korea, countries that moved down are New Zealand, Venezuela, Iran, Nicaragua, Peru and Trinidad&Tobago.

In the next section we have two tasks: to construct a model, the Solow model, that a) can successfully explain the stylized growth facts b) investigate to which extent the Solow model can explain the development facts.

9.2 The Solow Model and its Empirical Evaluation

The basic assumptions of the Solow model are that there is a single good produced in our economy and that there is no international trade, i.e. the economy is closed to international goods and factor flows. Also there is no government. It is also assumed that all factors of production (labor, capital) are fully employed in the production process. We assume that the labor force, $L(t)$ grows at constant rate $n > 0$, so that, by normalizing $L(0) = 1$ we have that

$$L(t) = e^{nt}L(0) = e^{nt}$$

The model consists of two basic equations, the neoclassical aggregate production function and a capital accumulation equation.

1. Neoclassical aggregate production function

   $$Y(t) = F(K(t), A(t)L(t))$$

   We assume that F has constant returns to scale, is strictly concave and strictly increasing, twice continuously differentiable, $F(0, .) = F(., 0) = 0$ and satisfies the Inada conditions. Here $Y(t)$ is total output, $K(t)$ is the capital stock at time t and $A(t)$ is the level of technology at time t. We normalize $A(0) = 1$, so that a worker in period t provides the same labor input as $A(t)$ workers in period 0. We call $A(t)L(t)$ labor input in labor efficiency units (rather than in raw number of bodies) or effective labor at date t. We assume that

   $$A(t) = e^{gt}$$

   i.e. the level of technology increases at continuous rate $g > 0$. We interpret this as technological progress: due to the invention of new technologies or "ideas" workers get more productive over time. This exogenous technological progress, which is not explained within the model is the key driving force of economic growth in the Solow model. One of the main criticisms of the Solow model is that it does not provide an endogenous explanation for why technological progress, the driving force of growth, arises. Romer (1990) and Jones (1995) pick up exactly this point. We model technological progress as making labor more effective in the production process. This form of technological progress is called labor augmenting or Harrod-neutral technological progress.[2] In order to analyze the model we seek a representation in variables that remain stationary over time, so that we can talk about steady states and dynamics around the steady state. Obviously, since the number of workers as well as technology grows exponentially, total output and capital (even per capita or per worker) will tend to grow. However, expressing all variables of the model in per effective labor units there is hope to arrive at a representation of the model in which the endogenous variables are stationary. Hence we divide both sides of the production function by the effective labor input $A(t)L(t)$ to obtain (using the constant returns to scale assumption)[3]

$$\xi(t) = \frac{Y(t)}{A(t)L(t)} = \frac{F(K(t), A(t)L(t))}{A(t)L(t)} = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) = f(\kappa(t)) \tag{9.1}$$

where $\xi(t) = \frac{Y(t)}{A(t)L(t)}$ is output per effective labor input and $\kappa(t) = \frac{K(t)}{A(t)L(t)}$ is the capital stock perfect labor input. From the assumptions made on F it follows that f is strictly increasing, strictly concave, twice continuously differentiable, $f(0) = 0$ and satisfies the Inada condition. Equation (9.1) summarizes our assumptions about the production technology of the economy.

[2]Alternative specifications of the production functions are $F(AK, L)$ in which case technological progress is called capital augmenting or Solow neutral technological progress, and $AF(K, L)$ in which case it is called Hicks neutral technological progress. For the way we will define a balanced growth path below it is only Harrod-neutral technological progress (at least for general production functions) that guarantees the existence of a balanced growth path in the Solow model.

[3]In terms of notation I will use uppercase variables for aggregate variables, lowercase for per-worker variables and the corresponding greek letter for variables per effective labor units. Since there is no greek y I use ξ for per capita output

2. Capital accumulation equation and resource constraint

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{9.2}$$

$$\dot{K}(t) + \delta K(t) = Y(t) - C(t) \tag{9.3}$$

The change of the capital stock in period t, $\dot{K}(t)$ is given by gross investment in period t, $sY(t)$ minus the depreciation of the old capital stock $\delta K(t)$. We assume $\delta \geq 0$. Since we have a closed economy model gross investment is equal to national saving (which is equal to saving of the private sector, since there is no government). Here s is the fraction of total output (income) in period t that is saved, i.e. not consumed. The important assumption implicit in equation (9.2) is that households save a constant fraction s of output (income), regardless of the level of income. This is a strong assumption about the *behavior* of households that is not endogenously derived from within a model of utility-maximizing agents (and the Cass-Koopmans-Ramsey model relaxes exactly this assumption). Remember that the discrete time counterpart of this equation was

$$\begin{aligned} K_{t+1} - K_t &= sY_t - \delta K_t \\ K_{t+1} - (1-\delta)K_t &= Y_t - C_t \end{aligned}$$

Now we can divide both sides of equation (9.2) by $A(t)L(t)$ to obtain

$$\frac{\dot{K}(t)}{A(t)L(t)} = s\xi(t) - \delta\kappa(t) \tag{9.4}$$

Expanding the left hand side of equation (9.4) gives

$$\frac{\dot{K}(t)}{A(t)L(t)} = \frac{\dot{K}(t)}{K(t)}\frac{K(t)}{A(t)L(t)} = \frac{\dot{K}(t)}{K(t)}\kappa(t) \tag{9.5}$$

But

$$\frac{\dot{\kappa}(t)}{\kappa(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} - \frac{\dot{A}(t)}{A(t)} = \frac{\dot{K}(t)}{K(t)} - n - g$$

Hence

$$\frac{\dot{K}(t)}{K(t)} = \frac{\dot{\kappa}(t)}{\kappa(t)} + n + g \tag{9.6}$$

Combining equations (9.5) and (9.6) with (9.4) yields

$$\begin{aligned}
\frac{\dot{K}(t)}{A(t)L(t)} &= \frac{\dot{K}(t)}{K(t)}\kappa(t) = \left(\frac{\dot{\kappa}(t)}{\kappa(t)} + n + g\right)\kappa(t) && (9.7)\\
\dot{\kappa}(t) + \kappa(t)(n+g) &= s\xi(t) - \delta\kappa(t) && (9.8)\\
\dot{\kappa}(t) &= s\xi(t) - (n+g+\delta)\kappa(t) && (9.9)
\end{aligned}$$

This is the capital accumulation equation in per-effective worker terms. Combining this equation with the production function gives the fundamental differential equation of the Solow model

$$\dot{\kappa}(t) = sf(\kappa(t)) - (n+g+\delta)\kappa(t) \tag{9.10}$$

Technically speaking this is a first order nonlinear ordinary differential equation, and it completely characterizes the evolution of the economy for any initial condition $\kappa(0) = K(0)$. Once we have solved the differential equation for the capital per effective labor path $\kappa(t)_{t\in[0,\infty)}$ the rest of the endogenous variables are simply given by

$$\begin{aligned}
k(t) &= \kappa(t)A(t) = e^{gt}\kappa(t)\\
K(t) &= e^{(n+g)t}\kappa(t)\\
y(t) &= e^{gt}f(\kappa(t))\\
Y(t) &= e^{(n+g)t}f(\kappa(t))\\
C(t) &= (1-s)e^{(n+g)t}f(\kappa(t))\\
c(t) &= (1-s)e^{gt}f(\kappa(t))
\end{aligned}$$

9.2.1 The Model and its Implications

Analyzing the qualitative properties of the model amounts to analyzing the differential equation (9.10). Unfortunately this differential equation is nonlinear, so there is no general method to explicitly solve for the function $\kappa(t)$. We can, however, analyze the differential equation graphically. Before doing this, however, let us look at a (I think the only) particular example for which we actually can solve the equation analytically

Example 108 *Let $f(\kappa) = \kappa^{\alpha}$ (i.e. $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$). The fundamental differential equation becomes*

$$\dot{\kappa}(t) = s\kappa(t)^{\alpha} - (n+g+\delta)\kappa(t) \tag{9.11}$$

with $\kappa(0) > 0$ *given. A steady state of this equation is given by* $\kappa(t) = \kappa^*$ *for which* $\dot{\kappa}(t) = 0$ *for all* t. *There are two steady states, the trivial one at* $\kappa = 0$ *(which we will ignore from now on, as it is only reached if* $\kappa(0) = 0$*) and the unique positive steady state* $\kappa^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$. *Now let's solve the differential equation. This equation is, in fact, a special case of the so-called Bernoulli equation. Let's do the following substitution of variables. Define* $v(t) = \kappa(t)^{1-\alpha}$. *Then*

$$\dot{v}(t) = (1-\alpha)\kappa(t)^{-\alpha} * \dot{\kappa}(t) = \frac{(1-\alpha)\dot{\kappa}(t)}{\kappa(t)^{\alpha}}$$

Dividing both sides of (9.11) *by* $\frac{\kappa(t)^{\alpha}}{1-\alpha}$ *yields*

$$\frac{(1-\alpha)\dot{\kappa}(t)}{\kappa(t)^{\alpha}} = (1-\alpha)s - (1-\alpha)(n+g+\delta)\kappa(t)^{1-\alpha}$$

and now making the substitution of variables

$$\dot{v}(t) = (1-\alpha)s - (1-\alpha)(n+g+\delta)v(t)$$

which is a linear ordinary first order (nonhomogeneous) differential equation, which we know how to solve.[4] *The general solution to the homogeneous equation takes the form*

$$v_g(t) = Ce^{-(1-\alpha)(n+g+\delta)t}$$

where C *is an arbitrary constant. A particular solution to the nonhomogeneous equation is*

$$v_p(t) = \frac{s}{n+g+\delta} = v^* = (\kappa^*)^{1-\alpha}$$

Hence all solutions to the differential equation take the form

$$\begin{aligned} v(t) &= v_g(t) + v_p(t) \\ &= v^* + Ce^{-(1-\alpha)(n+g+\delta)t} \end{aligned}$$

[4] An excellent reference for economists is Gandolfo, G. "Economic Dynamics: Methods and Models". There are thousands of math books on differential equations, e.g. Boyce, W. and DiPrima, R. "Elementary Differential Equations and Boundary Value Problems"

Now we use the initial condition $v(0) = \kappa(0)^{1-\alpha}$ *to determine the constant* C

$$\begin{aligned} v(0) &= v^* + C \\ C &= v(0) - v^* \end{aligned}$$

Hence the solution to the initial value problem is

$$v(t) = v^* + (v(0) - v^*)\, e^{-(1-\alpha)(n+g+\delta)t}$$

and substituting back κ *for* v *we obtain*

$$\kappa(t)^{1-\alpha} = (\kappa^*)^{1-\alpha} + \left(\kappa(0)^{1-\alpha} - (\kappa^*)^{1-\alpha}\right) e^{-(1-\alpha)(n+g+\delta)t}$$

and hence

$$\kappa(t) = \left[\frac{s}{n+g+\delta} + \left(\kappa(0)^{1-\alpha} - \frac{s}{n+g+\delta}\right) e^{-(1-\alpha)(n+g+\delta)t}\right]^{\frac{1}{1-\alpha}}$$

Note that $\lim_{t\to\infty} \kappa(t) = \left[\frac{s}{n+g+\delta}\right]^{\frac{1}{1-\alpha}} = \kappa^*$ *regardless of the value of* $\kappa(0) > 0$. *In other words the unique steady state capital per labor efficiency unit is locally (globally if one restricts attention to strictly positive capital stocks) asymptotically stable*

For a general production function one can't solve the differential equation explicitly and has to resort to graphical analysis. In Figure **??** we plot the two functions $(n+\delta+g)\kappa(t)$ and $sf(\kappa(t))$ against $\kappa(t)$. Given the properties of f it is clear that both curves intersect twice, once at the origin and once at a unique positive κ^* and $(n+\delta+g)\kappa(t) < sf(\kappa(t))$ for all $\kappa(t) < \kappa^*$ and $(n+\delta+g)\kappa(t) > sf(\kappa(t))$ for all $\kappa(t) > \kappa^*$. The steady state solves

$$\frac{sf(\kappa^*)}{k^*} = n + \delta + g$$

Since the change in κ is given by the difference of the two curves, for $\kappa(t) < \kappa^*$ κ increases, for $\kappa(t) > \kappa^*$ it decreases over time and for $\kappa(t) = \kappa^*$ it remains constant. Hence, as for the example above, also in the general case there exists a unique positive steady state level of the capital-labor-efficiency ratio that is locally asymptotically stable. Hence in the long run κ settles down at κ^* for any initial condition $\kappa(0) > 0$.

Figure 9.5: Capital in the Solow Model

Once the economy has settled down at κ^*, output, consumption and capital per worker grow at constant rates g and total output, capital and consumption grow at constant rates $g+n$. A situation in which the endogenous variables of the model grow at constant (not necessarily the same) rates is called a Balanced Growth Path (henceforth BGP). A steady state is a balanced growth path with growth rate of 0.

9.2.2 Empirical Evaluation of the Model

Kaldor's Growth Facts

Can the Solow model reproduce the stylized growth facts? The prediction of the model is that in the long run output per worker and capital per worker both grow at positive and constant rate g, the growth rate of technology. Therefore the capital-labor ratio k is constant, as observed by Kaldor. The other two stylized facts have to do with factor prices. Suppose that output is produced by a single competitive firm that faces a rental rate of capital $r(t)$ and wage rate $w(t)$ for one unit of *raw* labor (i.e. not labor in efficiency units). The firm rents both input at each instant in time and solves

$$\max_{K(t),L(t)\geq 0} F(K(t),A(t)L(t)) - r(t)K(t) - w(t)L(t)$$

Profit maximization requires

$$\begin{aligned} r(t) &= F_K(K(t),A(t)L(t)) \\ w(t) &= A(t)F_L(K(t),A(t)L(t)) \end{aligned}$$

Given that F is homogenous of degree 1, F_K and F_L are homogeneous of degree zero, i.e.

$$\begin{aligned} r(t) &= F_K\left(\frac{K(t)}{A(t)L(t)},1\right) \\ w(t) &= A(t)F_L\left(\frac{K(t)}{A(t)L(t)},1\right) \end{aligned}$$

In a balanced grow path $\frac{K(t)}{A(t)L(t)} = \kappa(t) = \kappa^*$ is constant, so the real rental rate of capital is constant and hence the real interest rate is constant. The wage rate increases at the rate of technological progress, g. Finally we can compute capital and labor shares. The capital share is given as

$$\alpha = \frac{r(t)K(t)}{Y(t)}$$

which is constant in a balanced growth path since the rental rate of capital is constant and $Y(t)$ and $K(t)$ grow at the same rate $g+n$. Hence the unique balanced growth path of the Solow model, to which the economy converges from any initial condition, reproduces all four stylized facts reported by Kaldor. In this dimension the Solow model is a big success and Solow won the Nobel price for it in 1989.

The Summers-Heston Development Facts

How can we explain the large difference in per capita income levels across countries? Assume first that all countries have access to the same production technology, face the same population growth rate and have the same saving rate. Then the Solow model predicts that all countries over time converge to the same balanced growth path represented by κ^*. All countries' per capita income converges to the path $y(t) = A(t)\kappa^*$, equal for all countries under the assumption of the same technology, i.e. same $A(t)$ process. Hence, so the prediction of the model, eventually per worker income (GDP) is equalized internationally. The fact that we observe large differences in per worker incomes across countries in the data must then be due to different initial conditions for the capital stock, so that countries differ with respect to their relative distance to the common BGP. Poorer countries are just further away from the BGP because they started with lesser capital stock, but will eventually catch up. This implies that poorer countries temporarily should grow faster than richer countries, according to the model. To see this, note that the growth rate of output per worker $\gamma_y(t)$ is given by

$$\begin{aligned} \gamma_y(t) &= \frac{\dot{y}(t)}{y(t)} = g + \frac{f'(\kappa(t))\dot{\kappa}(t)}{f(\kappa(t))} \\ &= g + \frac{f'(\kappa(t))}{f(\kappa(t))}\left(sf(\kappa(t)) - (n+\delta+g)\kappa(t)\right) \end{aligned}$$

Since $\frac{f'(\kappa(t))}{f(\kappa(t))}$ is positive and decreasing in $\kappa(t)$ and $(sf(\kappa(t)) - (n+\delta+g)\kappa(t))$ is decreasing in $\kappa(t)$ for two countries with $\kappa^1(t) < \kappa^2(t) < \kappa^*$ we have

Figure 9.6: Long Run Convergence of Income Levels

Figure 9.7: Long-Run Convergence of Income Levels

$\gamma_y^1(t) > \gamma_y^2(t) > 0$, i.e. countries that a further away from the balanced growth path grow more rapidly. The hypothesis that all countries' per worker income eventually converges to the same balanced growth path, or the somewhat weaker hypothesis that initially poorer countries grow faster than initially richer countries is called *absolute convergence*. If one imposes the assumptions of equality of technology and savings rates across countries, then the Solow model predicts absolute convergence. This implication of the model has been tested empirically by several authors. The data one needs is a measure of "initially poor vs. rich" and data on growth rates from "initially" until now. As measure of "initially poor vs. rich" the income per worker (in $US) of different countries at some initial year has been used.

In Figure 9.2.2 we use data for a long time horizon for 16 now industrialized countries. Clearly the level of GDP per capita in 1885 is negatively correlated with the growth rate of GDP per capita over the last 100 years across countries. So this figure lends support to the convergence hypothesis. We get the same qualitative picture when we use more recent data for 22 industrialized countries: the level of GDP per worker in 1960 is negatively correlated with the growth rate between 1960 and 1990 across this group of countries, as Figure 9.2.2 shows. This result, however, may be due to the way we selected countries: the very fact that these countries are industrialized countries means that they must have caught up with the leading country (otherwise they wouldn't be called industrialized countries now). This important point was raised by Bradford deLong (1988)

Let us take deLongs point seriously and look at the correlation between initial income levels and subsequent growth rates for the whole cross-sectional sample of Summers-Heston. Figure 9.2.2 doesn't seem to support the convergence hypothesis: for the whole sample initial levels of GDP per worker are pretty much uncorrelated with consequent growth rates. In particular, it doesn't seem to be the case that most of the very poor countries, in particular in Africa, are catching up with the rest of the world, at least not until 1990 (or until 2002 for that matter).

So does Figure 9.2.2 constitute the big failure of the Solow model? After

Figure 9.8: Failure of Long Run Convergence of Income Levels

all, for the big sample of countries it didn't seem to be the case that poor countries grow faster than rich countries. But isn't that what the Solow model predicts? Not exactly: the Solow model predicts that countries that are further away from *their* balanced growth path grow faster than countries that are closer to their balanced growth path (always assuming that the rate of technological progress is the same across countries). This hypothesis is called *conditional* convergence. The "conditional" means that we have to condition on characteristics of countries that may make them have different steady states $\kappa^*(s, n, \delta)$ (they still should grow at the *same rate* eventually, after having converged to their steady states) to determine which countries should grow faster than others. So the fact that poor African countries grow slowly even though they are poor may be, according to the conditional convergence hypothesis, due to the fact that they have a low balanced growth path and are already close to it, whereas some richer countries grow fast since they have a high balanced growth path and are still far from reaching it.

To test the conditional convergence hypothesis economists basically do the following: they compute the steady state output per worker[5] that a country should possess in a given initial period, say 1960, given n, s, δ measured in this country's data. Then they measure the actual GDP per worker in this period and build the difference. This difference indicates how far away this particular country is away from its balanced growth path. This variable, the difference between hypothetical steady state and actual GDP per worker is then plotted against the growth rate of GDP per worker from the initial period to the current period. If the hypothesis of conditional convergence were true, these two variables should be negatively correlated across countries: countries that are further away from their from their balanced growth path should grow faster. Jones' (1998) Figure 3.8 provides such a plot. In contrast to Figure 9.2.2 he finds that, once one conditions on country-specific steady states, poor (relative to their steady) tend to grow faster than rich countries. So again, the Solow model is quite successful qualitatively.

Now we want to go one step further and ask whether the Solow model can predict the *magnitude* of cross-country income differences once we allow

[5]Which is proportional to the balanced growth path for output per worker (just multiply it by the constant $A(1960)$).

parameters that determine the steady state to vary across countries. Such a quantitative exercise was carried out in the influential paper by Mankiw, Romer and Weil (1992). The authors "want to take Robert Solow seriously", i.e. investigate whether the quantitative predictions of his model are in line with the data. More specifically they ask whether the model can explain the enormous cross-country variation of income per worker. For example in 1985 per worker income of the US was 31 times as high as in Ethiopia.

There is an obvious way in which the Solow model can account for this number. Suppose we constrain ourselves to balanced growth paths (i.e. ignore the convergence discussion that relies on the assumptions that countries have not yet reached their BGP's). Then, by denoting $y^{US}(t)$ as per worker income in the US and $y^{ETH}(t)$ as per worker income in Ethiopia in time t we find that along BGP's, with assumed Cobb-Douglas production function

$$\frac{y^{US}(t)}{y^{ETH}(t)} = \frac{A^{US}(t)}{A^{ETH}(t)} * \frac{\left(\frac{s^{US}}{n^{US}+g^{US}+\delta^{US}}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{s^{ETH}}{n^{ETH}+g^{ETH}+\delta^{ETH}}\right)^{\frac{\alpha}{1-\alpha}}} \tag{9.12}$$

One easy way to get the income differential is to assume large enough differences in levels of technology $\frac{A^{US}(t)}{A^{ETH}(t)}$. One fraction of the literature has gone this route; the hard part is to justify the large differences in levels of technology when technology transfer is relatively easy between a lot of countries.[6] The other fraction, instead of attributing the large income differences to differences in A attributes the difference to variation in savings (investment) and population growth rates. Mankiw et al. take this view. They assume that there is in fact no difference across countries in the production technologies used, so that $A^{US}(t) = A^{ETH}(t) = A(0)e^{gt}$, $g^{ETH} = g^{US}$ and $\delta^{ETH} = \delta^{US}$. Assuming balanced growth paths and Cobb-Douglas production we can write

$$y^i(t) = A(0)e^{gt}\left(\frac{s^i}{n^i+\delta+g}\right)^{\frac{\alpha}{1-\alpha}}$$

where i indexes a country. Taking natural logs on both sides we get

$$\ln(y^i(t)) = \ln(A(0)) + gt + \frac{\alpha}{1-\alpha}\ln(s^i) - \frac{\alpha}{1-\alpha}\ln(n^i+\delta+g)$$

Given this linear relationship derived from the theoretical model it very tempting to run this as a regression on cross-country data. For this, how-

[6] See, e.g. Parente and Prescott (1994, 1999).

ever, we need a stochastic error term which is nowhere to be detected in the model. Mankiw et al. use the following assumption

$$\ln(A(0)) = a + \varepsilon^i \tag{9.13}$$

where a is a constant (common across countries) and ε^i is a country specific random shock to the (initial) level of technology that may, according to the authors, represent not only variations in production technologies used, but also climate, institutions, endowments with natural resources and the like. Using this and assuming that the time period for the cross sectional data on which the regression is run is $t = 0$ (if $t = T$ that only changes the constant[7]) we obtain the following linear regression

$$\begin{aligned}
\ln(y^i) &= a + \frac{\alpha}{1-\alpha}\ln(s^i) - \frac{\alpha}{1-\alpha}\ln(n^i + \delta + g) + \varepsilon^i \\
\ln(y^i) &= a + b_1 \ln(s^i) + b_2 \ln(n^i + \delta + g) + \varepsilon^i \qquad (9.14)
\end{aligned}$$

Note that the variation in ε^i across countries, according to the underlying model, are attributed to variations in technology. Hence the regression results will tell us how much of the variation in cross-country per-worker income is due to variations in investment and population growth rates, and how much is due to *random* differences in the level of technology. This is, if we take (9.13) literally, how the regression results have to be interpreted. If we want to estimate (9.14) by OLS, the identifying assumption is that the ε^i are uncorrelated with the other variables on the right hand side, in particular the investment and population growth rate. Given the interpretation the authors offered for ε^i I invite you all to contemplate whether this is a good assumption or not. Note that the regression equation also implies restrictions on the parameters to be estimated: if the specification is correct, then one expects the estimated $\hat{b}_1 = -\hat{b}_2$. One may also impose this constraint a priori on the parameter values and do constrained OLS. Apparently the results don't change much from the unrestricted estimation. Also, given that the production function is Cobb-Douglas, α has the interpretation as capital share, which is observable in the data and is thought to be around .25-0.5 for most countries, one would expect $\hat{b}_1 \in [\frac{1}{3}, 1]$ a priori. This is an important test for whether the specification of the regression is correct.

[7] Note that we do not use the time series dimension of the data, only the cross-sectional, i.e. cross-country dimension.

With respect to data, y^i is taken to be real GDP divided by working age population in 1985, n^i is the average growth rate of the working-age population[8] from $1960-85$ and s is the average share of real investment[9] from real GDP between $1960-85$. Finally they assume that $g+\delta=0.05$ for all countries.

Table 2 reports their results for the unrestricted OLS-estimated regression on a sample of 98 countries (see their data appendix for the countries in the sample)

Table 2

$\hat{a}$	$\hat{b}_1$	$\hat{b}_2$	$\bar{R}^2$
5.48 (1.59)	1.42 (0.14)	-1.48 (0.12)	0.59

The basic results supporting the Solow model are that the $\hat{b}_i$ have the right sign, are highly statistically significant and are of similar size. Most importantly, a major fraction of the cross-country variation in per-worker incomes, namely about 60% is accounted for by the variations in the explanatory variables, namely investment rates and population growth rates. The rest, given the assumptions about where the stochastic error term comes from, is attributed to random variations in the level of technology employed in particular countries.

That seems like a fairly big success of the Solow model. However, the size of the estimates $\hat{b}_i$ indicates that the implied required capital shares on average have to lie around $\frac{2}{3}$ rather than $\frac{1}{3}$ usually observed in the data. This is both problematic for the success of the model and points to a direction of improvement of the model.

Let's first understand where the high coefficients come from. Assume that $n^{US}=n^{ETH}=n$ (variation in population growth rates is too small to make a significant difference) and rewrite (9.12) as (using the assumption of same technology, the differences are assumed to be of stochastic nature)

$$\frac{y^{US}(t)}{y^{ETH}(t)}=\left(\frac{s^{US}}{s^{ETH}}\right)^{\frac{\alpha}{1-\alpha}}$$

[8]This implicitly assumes a constant labor force participation rate from $1960-85$.

[9]Private as well as government (gross) investment.

To generate a spread of incomes of 31, for $\alpha = \frac{1}{3}$ one needs a ratio of investment rates of 961 which is obviously absurdly high. But for $\alpha = \frac{2}{3}$ one only requires a ratio of 5.5. In the data, the measured ratio is about 3.9 for the US versus Ethiopia. This comes pretty close (population growth differentials would almost do the rest). Obviously this is a back of the envelope calculation involving only two countries, but it demonstrates the core of the problem: there is substantial variation in investment and population growth rates across countries, but if the importance of capital in the production process is as low as the commonly believed $\alpha = \frac{1}{3}$, then these variations are nowhere nearly high enough to generate the large income differentials that we observe in the data. Hence the regression forces the estimated α up to two thirds to make the variations in s^i (and n^i) matter sufficiently much.

So if we can't change the data to give us a higher capital share and can't force the model to deliver the cross-country spread in incomes given reasonable capital shares, how can we rescue the model? Mankiw, Romer and Weil do a combination of both. Suppose you reinterpret the capital stock as broadly containing not only the physical capital stock, but also the stock of human capital and you interpret part of labor income as return to not just raw physical labor, but as returns to human capital such as education, then possibly a capital share of two thirds is reasonable. In order to do this reinterpretation on the data, one better first augments the model to incorporate human capital as well.

So now let the aggregate production function be given by

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} \left(A(t)L(t)\right)^{1-\alpha-\beta}$$

where $H(t)$ is the stock of human capital. We assume $\alpha + \beta < 1$, since if $\alpha + \beta = 1$, there are constant returns to scale in accumulable factors alone, which prevents the existence of a balanced growth path (the model basically becomes an AK-model to be discussed below. We will specify below how to measure human capital (or better: investment into human capital) in the data. The capital accumulation equations are now given by

$$\begin{aligned} \dot{K}(t) &= s_k Y(t) - \delta K(t) \\ \dot{H}(t) &= s_h Y(t) - \delta H(t) \end{aligned}$$

Expressing all equations in per-effective labor units yields (where $\eta(t) =$

$\frac{H(t)}{A(t)L(t)}$

$$\begin{aligned}
\xi(t) &= \kappa(t)^{a}\eta(t)^{\beta} \\
\dot{\kappa}(t) &= s_k\xi(t) - (n+\delta+g)\kappa(t) \\
\dot{\eta}(t) &= s_h\xi(t) - (n+\delta+g)\eta(t)
\end{aligned}$$

Obviously a unique positive steady state exists which can be computed as before

$$\begin{aligned}
\kappa^* &= \left(\frac{s_k^{1-\beta}s_h^{\beta}}{n+\delta+g}\right)^{\frac{1}{1-\alpha-\beta}} \\
\eta^* &= \left(\frac{s_k^{\alpha}s_h^{1-\alpha}}{n+\delta+g}\right)^{\frac{1}{1-\alpha-\beta}} \\
\xi^* &= (\kappa^*)^{\alpha}(\eta^*)^{\beta}
\end{aligned}$$

and the associated balanced growth path has

$$\begin{aligned}
y(t) &= A(0)e^{gt}\xi^* \\
&= A(0)e^{gt}\left(\frac{s_k^{1-\beta}s_h^{\beta}}{n+\delta+g}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{s_k^{\alpha}s_h^{1-\alpha}}{n+\delta+g}\right)^{\frac{\beta}{1-\alpha-\beta}}
\end{aligned}$$

Taking logs yields

$$\ln(y(t)) = \ln(A(0)) + gt + b_1\ln(s_k) + b_2\ln(s_h) + b_3\ln(n+\delta+g)$$

where $b_1 = \frac{\alpha}{1-\alpha-\beta}$, $b_2 = \frac{\beta}{1-\alpha-\beta}$ and $b_3 = -\frac{\alpha+\beta}{1-\alpha-\beta}$. Making the same assumptions about how to bring a stochastic component into the completely deterministic model yields the regression equation

$$\ln(y^i) = a + b_1\ln(s_k^i) + b_2\ln(s_h^i) + b_3\ln(n^i+\delta+g) + \varepsilon^i$$

The main problem in estimating this regression (apart from the validity of the orthogonality assumption of errors and instruments) is to construct reasonable data for the savings rate of human capital. Ideally we would measure all the resources flowing into investment that increases the stock of human capital, including investment into education, health and so forth. For now let's limit attention to investment into education. Mankiw et al.'s measure of the investment rate of education is the fraction of the total working age population that goes to secondary school, as found in data collected by the UNESCO, i.e.

$$s_h = \frac{S}{L}$$

where S is the number of people in the labor force that go to school (and forgo wages as unskilled workers) and L is the total labor force. Why may this be a good proxy for the investment share of output into education? Investment expenditures for education include new buildings of the universities, salaries of teachers, and most significantly, the forgone wages of the students in school. Let's assume that forgone wages are the only input for human capital investment (if the other inputs are proportional to this measure, the argument goes through unchanged). Let the people in school forgo wages w_L as unskilled workers. Total forgone earnings are then $w_L S$ and the investment share of output into human capital is $\frac{w_L S}{Y}$. But the wage of an unskilled worker is given (under perfect competition) by its marginal product

$$w_L = (1-\alpha-\beta)K(t)^\alpha H(t)^\beta A(t)^{1-\alpha-\beta} L(t)^{-\alpha-\beta}$$

so that

$$\frac{w_L S}{Y} = \frac{w_L L S}{Y L} = (1-\alpha-\beta)\frac{S}{L} = (1-\alpha-\beta)s_h$$

so that the measure that the authors employ is proportional to a "theoretically more ideal" measure of the human capital savings rate. Noting that $\ln((1-\alpha-\beta)s_h) = \ln(1-\alpha-\beta) + \ln(s_h)$ one immediately see that the proportionality factor will only affect the estimate of the constant, but not the estimates of the b_i.

The results of estimating the augmented regression by OLS are given in Table 3

Table 3

$\hat{a}$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\bar{R}^2$
6.89 (1.17)	0.69 (0.13)	0.66 (0.07)	-1.73 (0.41)	0.78

The results are quite remarkable. First of all, almost 80% of the variation of cross-country income differences is explained by differences in savings rates in physical and human capital This is a huge number for cross-sectional regressions. Second, all parameter estimates are highly significant and have the right sign. In addition we (i.e. Mankiw, Romer and Weil) seem to have found a remedy for the excessively high implied estimates for α. Now the estimates for b_i imply almost precisely $\alpha = \beta = \frac{1}{3}$ and the one overidentifying restriction on the $b_i^{\prime}s$ can't be rejected at standard confidence levels (although $\hat{b}_3$ is a bit high). The final verdict is that with respect to explaining cross-country income differences an augmented version of the Solow model does remarkably well. This is, as usual subject to the standard quarrels that there may be big problems with data quality and that their method is not applicable for non-Cobb-Douglas technology. On a more fundamental level the Solow model has methodological problems and Mankiw et al.'s analysis leaves several questions wide open:

1. The assumption of a constant saving rate is a strong behavioral assumption that is not derived from any underlying utility maximization problem of rational agents. Our next topic, the discussion of the Cass-Koopmans-Ramsey model will remedy exactly this shortcoming

2. The driving force of economic growth, technological progress, is model-exogenous; it is assumed, rather than endogenously derived. We will pick this up in our discussion of endogenous growth models.

3. The cross-country variation of per-worker income is attributed to variations in investment rates, which are taken to be exogenous. What is then needed is a theory of why investment rates differ across countries. I can provide you with interesting references that deal with this problem, but we will not talk about this in detail in class.

But now let's turn to the first of these points, the introduction of endogenous determination of household's saving rates.

9.3 The Ramsey-Cass-Koopmans Model

In this section we discuss the first logical extension of the Solow model. Instead of assuming that households save at a fixed, exogenously given rate s we will analyze a model in which agents actually make economic decisions; in particular they make the decision how much of their income to consume in the current period and how much to save for later. This model was first analyzed by the British mathematician and economist Frank Ramsey. He died in 1930 at age 29 from tuberculosis, not before he wrote two of the most influential economics papers ever to be written. We will discuss a second pathbreaking idea of his in our section on optimal fiscal policy. Ramsey's ideas were taken up independently by David Cass and Tjelling Koopmans in 1965 and have now become the second major workhorse model in modern macroeconomics, besides the OLG model discussed previously. In fact, in Section 3 of these notes we discussed the discrete-time version of this model and named it the neoclassical growth model. Now we will in fact incorporate economic growth into the model, which is somewhat more elegant to do in continuous time, although there is nothing conceptually difficult about introducing growth into the discrete-time version- a useful exercise.

9.3.1 Mathematical Preliminaries: Pontryagin's Maximum Principle

Intriligator, Chapter 14

9.3.2 Setup of the Model

Our basic assumptions made in the previous section are carried over. There is a representative, infinitely lived family (dynasty) in our economy that grows at population growth rate $n > 0$ over time, so that, by normalizing the size of the population at time 0 to 1 we have that $L(t) = e^{nt}$ is the size of the family (or population) at date t. We will treat this dynasty as a single economic agent. There is no risk in this economy and all agents are assumed to have perfect foresight.

Production takes place with a constant returns to scale production function

$$Y(t) = F(K(t), A(t)L(t))$$

where the level of technology grows at constant rate $g > 0$, so that, normalizing $A(0) = 1$ we find that the level of technology at date t is given by $A(t) = e^{gt}$. The aggregate capital stock evolves according to

$$\dot{K}(t) = F(K(t), A(t)L(t)) - \delta K(t) - C(t) \tag{9.15}$$

i.e. the net change in the capital stock is given by that fraction of output that is not consumed by households, $C(t)$ or by depreciation $\delta K(t)$. Alternatively, this equation can be written as

$$\dot{K}(t) + \delta K(t) = F(K(t), A(t)L(t)) - C(t)$$

which simply says that aggregate gross investment $\dot{K}(t) + \delta K(t)$ equals aggregate saving $F(K(t), A(t)L(t)) - C(t)$ (note that the economy is closed and there is no government). As before this equation can be expressed in labor-intensive form: define $c(t) = \frac{C(t)}{L(t)}$ as consumption per capita (or worker) and $\zeta(t) = \frac{C(t)}{A(t)L(t)}$ as consumption per labor efficiency unit (the Greek symbol is called a "zeta"). Then we can rewrite (9.15) as, using the same manipulations as before

$$\dot{\kappa}(t) = f(\kappa(t)) - \zeta(t) - (n + \delta + g)\kappa(t) \tag{9.16}$$

Again f is assumed to have all the properties as in the previous section. We assume that the initial endowment of capital is given by $K(0) = \kappa(0) = \kappa_0 > 0$

So far we just discussed the technology side of the economy. Now we want to describe the preferences of the representative family. We assume that the family values streams of per-capita consumption $c(t)_{t\in[0,\infty)}$ by

$$u(c) = \int_0^\infty e^{-\rho t} U(c(t))dt$$

where $\rho > 0$ is a time discount factor. Note that this implicitly discounts utility of agents that are born at later periods. Ramsey found this to be unethical and hence assumed $\rho = 0$. Here $U(c)$ is the instantaneous utility or felicity function.[10] In most of our discussion we will assume that the period utility function is of constant relative risk aversion (CRRA) form, i.e.

$$U(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} \text{ if } \sigma \neq 1 \\ \ln(c) \text{ if } \sigma = 1 \end{cases}$$

[10] An alternative, so-called Benthamite (after British philosopher Jeremy Bentham) felicity function would read as $L(t)U(c(t))$. Since $L(t) = e^{nt}$ we immediately see

$$\begin{aligned} & e^{-\rho t} L(t) U(c(t)) \\ = \; & e^{-(\rho - n)t} U(c(t)) \end{aligned}$$

and hence we would have the same problem with adjusted time discount factor, and we would need to make the additional assumption that $\rho > n$.

Under our assumption of CRRA[11] we can rewrite

$$\begin{aligned} e^{-\rho t}U(c(t)) &= e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma} \\ &= e^{-\rho t}\frac{(\zeta(t)e^{gt})^{1-\sigma}}{1-\sigma} \\ &= e^{-(\rho-g(1-\sigma))t}\frac{\zeta(t)^{1-\sigma}}{1-\sigma} \end{aligned}$$

and we assume $\rho > g(1-\sigma)$. Define $\hat{\rho} = \rho - g(1-\sigma)$. We therefore can rewrite the utility function of the dynasty as

$$u(\zeta) = \int_0^\infty e^{-\hat{\rho}t}\frac{\zeta(t)^{1-\sigma}}{1-\sigma}dt \tag{9.17}$$

$$= \int_0^\infty e^{-\hat{\rho}t}U(\zeta(t))dt \tag{9.18}$$

where $\sigma = 1$ is understood to be the log-case. As before note that, once we know the variables $\kappa(t)$ and $\zeta(t)$ we can immediately determine per capita consumption $c(t) = \zeta(t)e^{gt}$ and the per capita capital stock $k(t) = \kappa(t)e^{gt}$ and output $y(t) = e^{gt}f(\kappa(t))$. Aggregate consumption, output and capital stock can be deduced similarly.

This completes the description of the environment. We will now, in turn, describe Pareto optimal and competitive equilibrium allocations and argue (heuristically) that they coincide.

9.3.3 Social Planners Problem

The first question is how a social planner would allocate consumption and saving over time. Note that in this economy there is a single agent, so the problem of the social planner is reduced from the OLG model to only intertemporal (and not also intergenerational) allocation of consumption. An allocation is a pair of functions $\kappa(t) : [0,\infty) \to \mathbf{R}$ and $\zeta(t) : [0,\infty) \to \mathbf{R}$.

[11] Some of the subsequent analysis could be carried out with more general assumptions on the period utility functions. However for the existence of a balanced growth path one has to assume CRRA, so I don't see much of a point in higher degree of generality that in some point of the argument has to be dispensed with anyway.

For an extensive discussion of the properties of the CRRA utility function see the appendix to Chapter 2 and HW1.

Definition 109 *An allocation* (κ, ζ) *is feasible if it satisfies* $\kappa(0) = \kappa_0$, $\kappa(t), \zeta(t) \geq 0$ *and* (9.16) *for all* $t \in [0, \infty)$.

Definition 110 *An allocation* (κ^*, ζ^*) *is Pareto optimal if it is feasible and if there is no other feasible allocation* $(\hat{\kappa}, \hat{\zeta})$ *such that* $u(\hat{\zeta}) > u(\zeta^*)$.

It is obvious that (κ^*, ζ^*) is Pareto optimal, if and only if it solves the social planner problem

$$\max_{(\kappa,\zeta)\geq 0} \int_0^\infty e^{-\hat{\rho}t} U(\zeta(t))dt \tag{9.19}$$

$$\begin{aligned} \text{s.t. } \dot{\kappa}(t) &= f(\kappa(t)) - \zeta(t) - (n + \delta + g)\kappa(t) \\ \kappa(0) &= \kappa_0 \end{aligned}$$

This problem can be solved using Pontryagin's maximum principle. The state variable in this problem is $\kappa(t)$ and the control variable is $\zeta(t)$. Let by $\lambda(t)$ denote the co-state variable corresponding to $\kappa(t)$. Forming the present value Hamiltonian and ignoring nonnegativity constraints[12] yields

$$\mathcal{H}(t, \kappa, \zeta, \lambda) = e^{-\hat{\rho}t} U(\zeta(t)) + \lambda(t) \left[f(\kappa(t)) - \zeta(t) - (n + \delta + g)\kappa(t) \right]$$

Sufficient conditions for an optimal solution to the planners problem (9.19) are[13]

$$\begin{aligned} \frac{\partial \mathcal{H}(t, \kappa, \zeta, \lambda)}{\partial \zeta(t)} &= 0 \\ \dot{\lambda}(t) &= -\frac{\partial \mathcal{H}(t, \kappa, \zeta, \lambda)}{\partial \kappa(t)} \\ \lim_{t\to\infty} \lambda(t)\kappa(t) &= 0 \end{aligned}$$

The last condition is the so-called transversality condition (TVC). This yields

$$e^{-\hat{\rho}t} U'(\zeta(t)) = \lambda(t) \tag{9.20}$$

$$\dot{\lambda}(t) = -\left(f'(\kappa(t)) - (n + \delta + g) \right) \lambda(t) \tag{9.21}$$

$$\lim_{t\to\infty} \lambda(t)\kappa(t) = 0 \tag{9.22}$$

[12] Given the functional form assumptions this is unproblematic.

[13] I use present value Hamiltonians. You should do the same derivation using current value Hamiltonians, as, e.g. in Intriligator, Chapter 16.

plus the constraint

$$\dot{\kappa}(t) = f(\kappa(t)) - \zeta(t) - (n+\delta+g)\kappa(t) \tag{9.23}$$

Now we eliminate the co-state variable from this system. Differentiating (9.20) with respect to time yields

$$\dot{\lambda}(t) = e^{-\hat{\rho}t}U''(\zeta(t))\dot{\zeta}(t) - \hat{\rho}e^{-\hat{\rho}t}U'(\zeta(t))$$

or, using (9.20)

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{\dot{\zeta}(t)U''(\zeta(t))}{U'(\zeta(t))} - \hat{\rho} \tag{9.24}$$

Combining (9.24) with (9.21) yields

$$\frac{\dot{\zeta}(t)U''(\zeta(t))}{U'(\zeta(t))} = -\left(f'(\kappa(t)) - (n+\delta+g+\hat{\rho})\right) \tag{9.25}$$

or multiplying both sides by $\zeta(t)$ yields

$$\dot{\zeta}(t)\frac{\zeta(t)U''(\zeta(t))}{U'(\zeta(t))} = -\left(f'(\kappa(t)) - (n+\delta+g+\hat{\rho})\right)\zeta(t)$$

Using our functional form assumption on the utility function $U(\zeta) = \frac{\zeta^{1-\sigma}}{1-\sigma}$ we obtain for the coefficient of relative risk aversion $-\frac{\zeta(t)U''(\zeta(t))}{U'(\zeta(t))} = \sigma$ and hence

$$\dot{\zeta}(t) = \frac{1}{\sigma}\left(f'(\kappa(t)) - (n+\delta+g+\hat{\rho})\right)\zeta(t)$$

Note that for the isoelastic case ($\sigma = 1$) we have that $\hat{\rho} = \rho$ and hence the equation becomes

$$\dot{\zeta}(t) = \left(f'(\kappa(t)) - (n+\delta+g+\rho)\right)\zeta(t)$$

The transversality condition can be written as

$$\lim_{t\to\infty} \lambda(t)\kappa(t) = \lim_{t\to\infty} e^{-\hat{\rho}t}U'(\zeta(t))\kappa(t) = 0$$

Hence any allocation (κ, ζ) that satisfies the system of nonlinear ordinary differential equations

$$\begin{aligned} \dot{\zeta}(t) &= \frac{1}{\sigma}\left(f'(\kappa(t)) - (n+\delta+g+\hat{\rho})\right)\zeta(t) && (9.26) \\ \dot{\kappa}(t) &= f(\kappa(t)) - \zeta(t) - (n+\delta+g)\kappa(t) && (9.27) \end{aligned}$$

with the initial condition $\kappa(0) = \kappa_0$ and terminal condition (TVC)

$$\lim_{t\to\infty} e^{-\hat{\rho}t} U'(\zeta(t))\kappa(t) = 0$$

is a Pareto optimal allocation. We now want to analyze the dynamic system (9.26) − (9.27) in more detail.

Steady State Analysis

Before analyzing the full dynamics of the system we look at the steady state of the optimal allocation. A steady state satisfies $\dot{\zeta}(t) = \dot{\kappa}(t) = 0$. Hence from equation (9.26) we have[14], denoting steady state capital and consumption per efficiency units by ζ^* and κ^*

$$f'(\kappa^*) = (n + \delta + g + \hat{\rho}) \tag{9.28}$$

The unique capital stock κ^* satisfying this equation is called the modified golden rule capital stock.

The "modified" comes from the following consideration. Suppose there is no technological progress, then the modified golden rule capital stock $\kappa^* = k^*$ satisfies

$$f'(k^*) = (n + \delta + \rho) \tag{9.29}$$

The golden rule capital stock is that capital stock per worker k^g that maximizes per-capita consumption. The steady state capital accumulation condition (without technological progress) is (see (9.27))

$$c = f(k) - (n + \delta)k$$

Hence the original golden rule capital stock satisfies[15]

$$f'(k^g) = n + \delta$$

and hence $k^* < k^g$. The social planner optimally chooses a capital stock per worker k^* below the one that would maximize consumption per capita. So

[14] There is the trivial steady state $\kappa^* = \zeta^* = 0$. We will ignore this steady state from now on, as it only is optimal when $\kappa(0) = \kappa_0$.

[15] Note that the golden rule capital stock had special significance in OLG economies. In particular, any steady state equilibrium with capital stock above the golden rule was shown to be dynamically inefficient.

even though the planner could increase every person's steady state consumption by increasing the capital stock, taking into account the impatience of individuals the planner finds it optimal not to do so.

Equation (9.28) or (9.29) indicate that the exogenous parameters governing individual time preference, population and technology growth determine the interest rate and the marginal product of capital. The production technology then determines the unique steady state capital stock and the unique steady state consumption from (9.27) as

$$\zeta^* = f(\kappa^*) - (n + \delta + g)\kappa^*$$

The Phase Diagram

It is in general impossible to solve the two-dimensional system of differential equations analytically, even for the simple example for which we obtain an analytical solution in the Solow model. A powerful tool when analyzing the dynamics of continuous time economies turn out to be so-called phase diagrams. Again, the dynamic system to be analyzed is

$$\begin{aligned}\dot{\zeta}(t) &= \frac{1}{\sigma}\left(f'(\kappa(t)) - (n + \delta + g + \hat{\rho})\right)\zeta(t)\\ \dot{\kappa}(t) &= f(\kappa(t)) - \zeta(t) - (n + \delta + g)\kappa(t)\end{aligned}$$

with initial condition $\kappa(0) = \kappa_0$ and terminal transversality condition $\lim_{t\to\infty} e^{-\hat{\rho}t}U'(\zeta(t))\kappa(t) = 0$. We will analyze the dynamics of this system in (κ, ζ) space. For any given value of the pair $(\kappa, \zeta) \geq 0$ the dynamic system above indicates the change of the variables $\kappa(t)$ and $\zeta(t)$ over time. Let us start with the first equation.

The locus of values for (κ, ζ) for which $\dot{\zeta}(t) = 0$ is called an isocline; it is the collection of all points (κ, ζ) for which $\dot{\zeta}(t) = 0$. Apart from the trivial steady state we have $\dot{\zeta}(t) = 0$ if and only if $\kappa(t)$ satisfies $f'(\kappa(t)) - (n + \delta + g + \hat{\rho}) = 0$, or $\kappa(t) = \kappa^*$. Hence in the (κ, ζ) plane the isocline is a vertical line at $\kappa(t) = \kappa^*$. Whenever $\kappa(t) > \kappa^*$ (and $\zeta(t) > 0$), then $\dot{\zeta}(t) < 0$, i.e. $\zeta(t)$ declines. We indicate this in Figure 9.3.3 with vertical arrows downwards at all points (κ, ζ) for which $\kappa < \kappa^*$. Reversely, whenever $\kappa < \kappa^*$ we have that $\dot{\zeta}(t) > 0$, i.e. $\zeta(t)$ increases. We indicate this with vertical arrows upwards at all points (κ, ζ) at which $\kappa < \kappa^*$. Similarly we determine the isocline corresponding to the equation $\dot{\kappa}(t) = f(\kappa(t)) - \zeta(t) - (n + \delta + g)\kappa(t)$. Setting $\dot{\kappa}(t) = 0$ we obtain all points in (κ, ζ)-plane for which $\dot{\kappa}(t) = 0$, or $\zeta(t) = f(\kappa(t)) - (n + \delta + g)\kappa(t)$. Obviously for $\kappa(t) = 0$ we have $\zeta(t) = 0$. The

Figure 9.9: The Dynamics of the Neoclassical Growth Model

Figure 9.10: The Dynamics of the Neoclassical Growth Model

curve is strictly concave in $\kappa(t)$ (as f is strictly concave), has its maximum at $\kappa^g > \kappa^*$ solving $f'(\kappa^g) = (n + \delta + g)$ and again intersects the horizontal axis for $\kappa(t) > \kappa^g$ solving $f(\kappa(t)) = (n + \delta + g)\kappa(t)$. Hence the isocline corresponding to $\dot{\kappa}(t) = 0$ is hump-shaped with peak at κ^g.

For all (κ, ζ) combinations above the isocline we have $\zeta(t) > f(\kappa(t)) - (n + \delta + g)\kappa(t)$, hence $\dot{\kappa}(t) < 0$ and hence $\kappa(t)$ is decreasing. This is indicated by horizontal arrows pointing to the left in Figure 9.3.3. Correspondingly, for all (κ, ζ) combinations below the isocline we have $\zeta(t) < f(\kappa(t)) - (n + \delta + g)\kappa(t)$ and hence $\dot{\kappa}(t) > 0$; i.e. $\kappa(t)$ is increasing, which is indicated by arrows pointing to the right.

Note that we have one initial condition for the dynamic system, $\kappa(0) = \kappa_0$. The arrows indicate the direction of the dynamics, starting from $\kappa(0)$. However, one initial condition is generally not enough to pin down the behavior of the dynamic system over time, i.e. there may be several time paths of $(\kappa(t), \zeta(t))$ that are an optimal solution to the social planners problem. The question is, basically, how the social planner should choose $\zeta(0)$. Once this choice is made the dynamic system as described by the phase diagram uniquely determines the optimal path of capital and consumption. Possible such paths are traced out in Figure 9.3.3.

We now want to argue two things: a) for a given $\kappa(0) > 0$ any choice $\zeta(0)$ of the planner leading to a path not converging to the steady state (κ^*, ζ^*) cannot be an optimal solution and b) there is a unique stable path leading to the steady state. The second property is called-saddle-path stability of the steady state and the unique stable path is often called a saddle path (or a one-dimensional stable manifold).

Let us start with the first point. There are three possibilities for any path starting with arbitrary $\kappa(0) > 0$; they either go to the unique steady state, they lead to the point E (as trajectories starting from points A or C), or they go to points with $\kappa = 0$ such as trajectories starting at B or D. Obviously trajectories like A and C that don't converge to E violate the nonnegativity of consumption $\zeta(t) = 0$ in finite amount of time. But a trajectory converging asymptotically to E violates the transversality condition

$$\lim_{t\to\infty} e^{-\hat{\rho}t}U'(\zeta(t))\kappa(t) = 0$$

As the trajectory converges to E, $\kappa(t)$ converges to a $\bar{\kappa} > \kappa^g > \kappa^* > 0$ and from (9.25) we have, since $\frac{dU'(\zeta(t))}{dt} = \dot{\zeta}(t)U''(\zeta(t))$

$$\frac{\frac{dU'(\zeta(t))}{dt}}{U'(\zeta(t))} = -f'(\kappa(t)) + (n + \delta + g + \hat{\rho}) > \hat{\rho} > 0$$

i.e. the growth rate of marginal utility of consumption is bigger than $\hat{\rho}$ as the trajectory approaches A. Given that κ approaches $\bar{\kappa}$ it is clear that the transversality condition is violated for all those trajectories.

Now consider trajectories like B or D. If, in finite amount of time, the trajectory hits the ζ-axis, then $\kappa(t) = \zeta(t) = 0$ from that time onwards, which, given the Inada conditions imposed on the utility function can't be optimal. It may, however, be possible that these trajectories asymptotically go to $(\kappa, \zeta) = (0, \infty)$. That this can't happen can be shown as follows. From (9.27) we have

$$\dot{\kappa}(t) = f(\kappa(t)) - \zeta(t) - (n + \delta + g)\kappa(t)$$

which is negative for all $\kappa(t) < \kappa^*$. Differentiating both sides with respect to time yields

$$\frac{d\dot{\kappa}(t)}{dt} = \frac{d^2\kappa(t)}{dt^2} = \left(f'(\kappa(t)) - (n + \delta + g)\right)\dot{\kappa}(t) - \dot{\zeta}(t) < 0$$

since along a possible asymptotic path $\dot{\zeta}(t) > 0$. So not only does $\kappa(t)$ decline, but it declines at increasing pace. Asymptotic convergence to the ζ-axis, however, would require $\kappa(t)$ to decline at a decreasing pace. Hence all paths like B or D have to reach $\kappa(t) = 0$ at finite time and therefore can't be optimal. These arguments show that only trajectories that lead to the unique positive steady state (κ^*, ζ^*) can be optimal solutions to the planner problem

In order to prove the second claim that there is a *unique* such path for each possible initial condition $\kappa(0)$ we have to analyze the dynamics around the steady state.

Dynamics around the Steady State

We can't solve the system of differential equations explicitly even for simple examples. But from the theory of linear approximations we know that in a neighborhood of the steady state the dynamic behavior of the nonlinear system is characterized by the behavior of the linearized system around the steady state. Remember that the first order Taylor expansion of a function $f : \mathbf{R}^n \to \mathbf{R}$ around a point $x^* \in \mathbf{R}^n$ is given by

$$f(x) = f(x^*) + \nabla f(x^*) \cdot (x - x^*)$$

where $\nabla f(x^*) \in \mathbf{R}^n$ is the gradient (vector of partial derivatives) of f at x^*. In our case we have $x^* = (\kappa^*, \zeta^*)$, and two functions f, g defined as

$$\begin{aligned}
\dot{\zeta}(t) &= f(\kappa(t), \zeta(t)) = \frac{1}{\sigma}\left(f'(\kappa(t)) - (n + \delta + g + \hat{\rho})\right)\zeta(t) \\
\dot{\kappa}(t) &= g(\kappa(t), \zeta(t)) = f(\kappa(t)) - \zeta(t) - (n + \delta + g)\kappa(t)
\end{aligned}$$

Obviously we have $f(\kappa^*, \zeta^*) = g(\kappa^*, \zeta^*) = 0$ since (κ^*, ζ^*) is a steady state. Hence the linear approximation around the steady state takes the form

$$\begin{aligned}
\begin{pmatrix} \dot{\zeta}(t) \\ \dot{\kappa}(t) \end{pmatrix} &\approx \left.\begin{pmatrix} \frac{1}{\sigma}\left(f'(\kappa(t)) - (n + \delta + g + \hat{\rho})\right) & \frac{1}{\sigma} f''(\kappa(t))\zeta(t) \\ -1 & f'(\kappa(t)) - (n + \delta + g) \end{pmatrix}\right|_{(\zeta(t),\kappa(t)) = (\zeta^*,\kappa^*)} \cdot \left(\begin{matrix} \zeta \\ \kappa \end{matrix} \right. \\
&= \begin{pmatrix} 0 & \frac{1}{\sigma} f''(\kappa^*)\zeta^* \\ -1 & \hat{\rho} \end{pmatrix} \cdot \begin{pmatrix} \zeta(t) - \zeta^* \\ \kappa(t) - \kappa^* \end{pmatrix}
\end{aligned}$$

This two-dimensional linear difference equation can now be solved analytically. It is easiest to obtain the qualitative properties of this system by reducing it two a single second order differential equation. Differentiate the second equation with respect to time to obtain

$$\ddot{\kappa}(t) = -\dot{\zeta}(t) + \hat{\rho}\dot{\kappa}(t)$$

Defining $\beta = -\frac{1}{\sigma} f''(\kappa^*)\zeta^* > 0$ and substituting in from (9.30) for $\dot{\zeta}(t)$ yields

$$\begin{aligned}
\ddot{\kappa}(t) &= \beta\left(\kappa(t) - \kappa^*\right) + \hat{\rho}\dot{\kappa}(t) \\
\ddot{\kappa}(t) - \hat{\rho}\dot{\kappa}(t) - \beta\kappa(t) &= -\beta\kappa^* \qquad (9.31)
\end{aligned}$$

We know how to solve this second order differential equation; we just have to find the general solution to the homogeneous equation and a particular solution to the nonhomogeneous equation, i.e.

$$\kappa(t) = \kappa_g(t) + \kappa_p(t)$$

It is straightforward to verify that a particular solution to the nonhomogeneous equation is given by $\kappa_p(t) = \kappa^*$. With respect to the general solution to the homogeneous equation we know that its general form is given by

$$\kappa_g(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where C_1, C_2 are two constants and λ_1, λ_2 are the two roots of the characteristic equation

$$\begin{aligned} \lambda^2 - \hat{\rho}\lambda - \beta &= 0 \\ \lambda_{1,2} &= \frac{\hat{\rho}}{2} \pm \sqrt{\beta + \frac{\hat{\rho}^2}{4}} \end{aligned}$$

We see that the two roots are real, distinct and one is bigger than zero and one is less than zero. Let λ_1 be the smaller and λ_2 be the bigger root. The fact that one of the roots is bigger, one is smaller than one implies that *locally* around the steady state the dynamic system is saddle-path stable, i.e. there is a unique stable manifold (path) leading to the steady state. For any value other than $C_2 = 0$ we will have $\lim_{t\to\infty} \kappa(t) = \infty$ (or $-\infty$) which violates feasibility. Hence we have that

$$\kappa(t) = \kappa^* + C_1 e^{\lambda_1 t}$$

(remember that $\lambda_1 < 0$). Finally C_1 is determined by the initial condition $\kappa(0) = \kappa_0$ since

$$\begin{aligned} \kappa(0) &= \kappa^* + C_1 \\ C_1 &= \kappa(0) - \kappa^* \end{aligned}$$

and hence the solution for κ is

$$\kappa(t) = \kappa^* + (\kappa(0) - \kappa^*)\, e^{\lambda_1 t}$$

and the corresponding solution for ζ can be found from

$$\begin{aligned} \dot{\kappa}(t) &= -\zeta(t) + \zeta^* + \hat{\rho}\,(\kappa(t) - \kappa^*) \\ \zeta(t) &= \zeta^* + \hat{\rho}\,(\kappa(t) - \kappa^*) - \dot{\kappa}(t) \end{aligned}$$

by simply using the solution for $\kappa(t)$. Hence for any given $\kappa(0)$ there is a unique optimal path $(\kappa(t), \zeta(t))$ which converges to the steady state (κ^*, ζ^*).

Note that the speed of convergence to the steady state is determined by $|\lambda_1| = \left|\frac{\hat{\rho}}{2} - \sqrt{-\frac{1}{\sigma}f''(\kappa^*)\zeta^* + \frac{\hat{\rho}^2}{4}}\right|$ which is increasing in $-\frac{1}{\sigma}$ and decreasing in $\hat{\rho}$. The higher the intertemporal elasticity of substitution, the more are individuals willing to forgo early consumption for later consumption an the more rapid does capital accumulation towards the steady state occur. The higher the effective time discount rate $\hat{\rho}$, the more impatient are households and the stronger they prefer current over future consumption, inducing a lower rate of capital accumulation.

So far what have we showed? That only paths converging to the unique steady state can be optimal solutions and that locally, around the steady state this path is unique, and therefore was referred to as saddle path. This also means that any potentially optimal path must hit the saddle path in finite time.

Hence there is a unique solution to the social planners problem that is graphically given as follows. The initial condition κ_0 determines the starting point of the optimal path $\kappa(0)$. The planner then optimally chooses $\zeta(0)$ such as to jump on the saddle path. From then on the optimal sequences $(\kappa(t), \zeta(t))_{t\in[0,\infty)}$ are just given by the segment of the saddle path from $\kappa(0)$ to the steady state. Convergence to the steady state is asymptotic, monotonic (the path does not jump over the steady state) and exponential. This indicates that eventually, once the steady state is reached, per capita variables grow at constant rates g and aggregate variables grow at constant rates $g + n$:

$$\begin{aligned}
c(t) &= e^{gt}\zeta^* \\
k(t) &= e^{gt}\kappa^* \\
y(t) &= e^{gt}f(\kappa^*) \\
C(t) &= e^{(n+g)t}\zeta^* \\
K(t) &= e^{(n+g)t}\kappa^* \\
Y(t) &= e^{(n+g)t}f(\kappa^*)
\end{aligned}$$

Hence the long-run behavior of this model is identical to that of the Solow model; it predicts that the economy converges to a balanced growth path at which all per capita variables grow at rate g and all aggregate variables grow at rate $g + n$. In this sense we can understand the Cass-Koopmans-Ramsey model as a micro foundation of the Solow model, with predictions that are quite similar.

9.3.4 Decentralization

In this subsection we want to demonstrate that the solution to the social planners problem does correspond to the (unique) competitive equilibrium allocation and we want to find prices supporting the Pareto optimal allocation as a competitive equilibrium.

In the decentralized economy there is a single representative firm that rents capital and labor services to produce output. As usual, whenever the firm does not own the capital stock its intertemporal profit maximization problem is equivalent to a continuum of static maximization problems

$$\max_{K(t),L(t)\geq 0} F(K(t), A(t)+(t)) - r(t)K(t) - w(t)L(t) \tag{9.32}$$

taking $w(t)$ and $r(t)$, the real wage rate and rental rate of capital, respectively, as given.

The representative household (dynasty) maximizes the family's utility by choosing per capita consumption and per capita asset holding at each instant in time. Remember that preferences were given as

$$u(c) = \int_0^\infty e^{-\rho t} U(c(t))dt \tag{9.33}$$

The only asset in this economy is physical capital[16] on which the return is $r(t) - \delta$. As before we could introduce notation for the real interest rate $i(t) = r(t) - \delta$ but we will take a shortcut and use $r(t) - \delta$ in the period household budget constraint. This budget constraint (in per capita terms, with the consumption good being the numeraire) is given by

$$c(t) + \dot{a}(t) + na(t) = w(t) + (r(t) - \delta)\, a(t) \tag{9.34}$$

where $a(t) = \frac{A(t)}{L(t)}$ are per capita asset holdings, with $a(0) = \kappa_0$ given. Note that the term $na(t)$ enters because of population growth: in order to, say, keep the per-capita assets constant, the household has to spend $na(t)$ units to account for its growing size.[17] As with discrete time we have to impose a condition on the household that rules out Ponzi schemes. At the same time

[16] Introducing a second asset, say government bonds, is straightforward and you should do it as an exercise.

[17] The household's budget constraint in aggregate (not per capita) terms is

$$C(t) + \dot{A}(t) = L(t)w(t) + (r(t) - \delta)\, A(t)$$

we do not prevent the household from temporarily borrowing (for the households a is perceived as an arbitrary asset, not necessarily physical capital). A standard condition that is widely used is to require that the household debt holdings in the limit do no grow at a faster rate than the interest rate, or alternatively put, that the time zero value of household debt has to be nonnegative in the limit.

$$\lim_{t\to\infty} a(t)e^{-\int_0^t (r(\tau)-\delta-n)d\tau} \geq 0 \tag{9.35}$$

Note that with a path of interest rates $r(t)-\delta$, the value of one unit of the consumption good at time t in units of the period consumption good is given by $e^{-\int_0^t (r(\tau)-\delta)d\tau}$. We immediately have the following definition of equilibrium

Definition 111 *A sequential markets equilibrium are allocations for the household* $(c(t),a(t))_{t\in[0,\infty)}$, *allocations for the firm* $(K(t),L(t))_{t\in[0,\infty)}$ *and prices* $(r(t),w(t))_{t\in[0,\infty)}$ *such that*

1. *Given prices* $(r(t),w(t))_{t\in[0,\infty)}$ *and* κ_0, *the allocation* $(c(t),a(t))_{t\in[0,\infty)}$ *maximizes* (9.33) *subject to* (9.34), *for all* t, *and* (9.35) *and* $c(t)\geq 0$.

2. *Given prices* $(r(t),w(t))_{t\in[0,\infty)}$, *the allocation* $(K(t),L(t))_{t\in[0,\infty)}$ *solves* (9.32)

3.

$$\begin{aligned} L(t) &= e^{nt} \\ L(t)a(t) &= K(t) \\ L(t)c(t)+\dot{K}(t)+\delta K(t) &= F(K(t),L(t)) \end{aligned}$$

This definition is completely standard; the three market clearing conditions are for the labor market, the capital market and the goods market, respectively. Note that we can, as for the discrete time case, define an Arrow-Debreu equilibrium and show equivalence between Arrow-Debreu equilibria

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Dividing by $L(t)$ yields

$$c(t)+\frac{\dot{A}(t)}{L(t)} = w(t)+(r(t)-\delta)\,a(t)$$

and expanding $\frac{\dot{A}(t)}{L(t)}$ gives the result in the main text.

and sequential market equilibria under the imposition of the no Ponzi condition (9.35). A heuristic argument will do here. Rewrite (9.34) as

$$c(t) = w(t) + (r(t) - \delta)\, a(t) - \dot{a}(t) - na(t)$$

then multiply both sides by $e^{-\int_0^t (r(\tau)-n-\delta)d\tau}$ and integrate from $t = 0$ to $t = T$ to get

$$\begin{aligned} \int_0^T c(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt &= \int_0^T w(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt \\ &\quad - \int_0^T \left[\dot{a}(t) - (r(t) - n - \delta)\, a(t)\right] e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt \end{aligned} \tag{9.36}$$

But if we define

$$F(t) = a(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau}$$

then

$$\begin{aligned} F'(t) &= \dot{a}(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt - a(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} \left[r(t) - \delta - n\right] \\ &= \left[\dot{a}(t) - (r(t) - n - \delta)\, a(t)\right] e^{-\int_0^t (r(\tau)-n-\delta)d\tau} \end{aligned}$$

so that (9.36) becomes

$$\begin{aligned} \int_0^T c(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt &= \int_0^T w(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt + F(0) - F(T) \\ &= \int_0^T w(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt + a(0) - a(T)e^{-\int_0^T (r(\tau)-n-\delta)d\tau} \end{aligned}$$

Now taking limits with respect to T and using (9.35) yields

$$\int_0^\infty c(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt = \int_0^\infty w(t)e^{-\int_0^t (r(\tau)-n-\delta)d\tau} dt + a(0)$$

or defining Arrow-Debreu prices as $p(t) = e^{-\int_0^t (r(\tau)-\delta)d\tau}$ we have

$$\int_0^\infty p(t)C(t)dt = \int_0^\infty p(t)L(t)w(t)dt + a(0)L(0)$$

where $C(t) = L(t)c(t)$ and we used the fact that $L(0) = 1$. But this is a standard Arrow-Debreu budget constraint. Hence by imposing the correct no Ponzi condition we have shown that the collection of sequential budget constraints is equivalent to the Arrow Debreu budget constraint with appropriate prices

$$p(t) = e^{-\int_0^t (r(\tau)-\delta)d\tau}$$

The rest of the proof that the set of Arrow-Debreu equilibrium allocations equals the set of sequential markets equilibrium allocations is obvious.[18]

We now want to characterize the equilibrium; in particular we want to show that the resulting dynamic system is identical to that arising for the social planner problem, suggesting that the welfare theorems hold for this economy. From the firm's problem we obtain

$$\begin{aligned} r(t) &= F_K(K(t), A(t)L(t)) = F_K\left(\frac{K(t)}{A(t)L(t)}, 1\right) \\ &= f'(\kappa(t)) \end{aligned} \tag{9.37}$$

and by zero profits in equilibrium

$$\begin{aligned} w(t)L(t) &= F(K(t), A(t)L(t)) - r(t)K(t) \\ \omega(t) &= \frac{w(t)}{A(t)} = f(\kappa(t)) - f'(\kappa(t))\kappa(t) \\ w(t) &= A(t)\left(f(\kappa(t)) - f'(\kappa(t))\kappa(t)\right) \end{aligned} \tag{9.38}$$

From the goods market equilibrium condition we find as before (by dividing by $A(t)L(t)$)

$$\begin{aligned} L(t)c(t) + \dot{K}(t) + \delta K(t) &= F(K(t), L(t)) \\ \dot{\kappa}(t) &= f(\kappa(t)) - (n + \delta + n)\kappa(t) - \zeta(t) \end{aligned} \tag{9.39}$$

Now we analyze the household's decision problem. First we rewrite the utility function and the household's budget constraint in intensive form. Making the assumption that the period utility is of CRRA form we again obtain (9.17). With respect to the individual budget constraint we obtain (again by

[18]Note that no equilbrium can exist for prices satisfying

$$\lim_{t\to\infty} p(t)L(t) = \lim_{t\to\infty} e^{-\int_0^t (r(\tau)-\delta-n)d\tau} > 0$$

because otherwise labor income of the family is unbounded.

dividing by $A(t)$)

$$\begin{aligned} c(t) + \dot{a}(t) + na(t) &= w(t) + (r(t) - \delta)\, a(t) \\ \dot{\alpha}(t) &= \omega(t) + (r(t) - (\delta + n + g))\, \alpha(t) - \zeta(t) \end{aligned}$$

where $\alpha(t) = \frac{a(t)}{A(t)}$. The individual state variable is the per-capita asset holdings in intensive form $\alpha(t)$ and the individual control variable is $\zeta(t)$. Forming the Hamiltonian yields

$$\mathcal{H}(t, \alpha, \zeta, \lambda) = e^{-\hat{\rho}t} U(\zeta(t)) + \lambda(t) \left[\omega(t) + (r(t) - (\delta + n + g))\, \alpha(t) - \zeta(t)\right]$$

The first order condition yields

$$e^{-\hat{\rho}t} U'(\zeta(t)) = \lambda(t) \tag{9.40}$$

and the time derivative of the Lagrange multiplier is given by

$$\dot{\lambda}(t) = -\left[r(t) - (\delta + n + g)\right] \lambda(t) \tag{9.41}$$

The transversality condition is given by

$$\lim_{t \to \infty} \lambda(t)\alpha(t) = 0$$

Now we proceed as in the social planners case. We first differentiate (9.40) with respect to time to obtain

$$e^{-\hat{\rho}t} U''(\zeta(t))\dot{\zeta}(t) - \hat{\rho} e^{-\hat{\rho}t} U'(\zeta(t)) = \dot{\lambda}(t)$$

and use this and (9.40) to substitute out for the costate variable in (9.41) to obtain

$$\begin{aligned} \frac{\dot{\lambda}(t)}{\lambda(t)} &= -\left[r(t) - (\delta + n + g)\right] \\ &= -\hat{\rho} + \frac{U''(\zeta(t))\dot{\zeta}(t)}{U'(\zeta(t))} \\ &= -\hat{\rho} - \sigma \frac{\dot{\zeta}(t)}{\zeta(t)} \end{aligned}$$

or

$$\dot{\zeta}(t) = \frac{1}{\sigma} \left[r(t) - (\delta + n + g + \hat{\rho})\right] \zeta(t)$$

Note that this condition has an intuitive interpretation: if the interest rate is higher than the effective subjective time discount factor, the individual values consumption tomorrow relatively higher than the market and hence $\dot{\zeta}(t) > 0$, i.e. consumption is increasing over time.

Finally we use the profit maximization conditions of the firm to substitute $r(t) = f'(\kappa(t))$ to obtain

$$\dot{\zeta}(t) = \frac{1}{\sigma}\left[f'(\kappa(t)) - (\delta + n + g + \hat{\rho})\right]\zeta(t)$$

Combining this with the resource constraint (9.39) gives us the same dynamic system as for the social planners problem, with the same initial condition $\kappa(0) = \kappa_0$. And given that the capital market clearing condition reads $L(t)a(t) = K(t)$ or $\alpha(t) = \kappa(t)$ the transversality condition is identical to that of the social planners problem. Obviously the competitive equilibrium allocation coincides with the (unique) Pareto optimal allocation; in particular it also possesses the saddle path property. Competitive equilibrium prices are simply given by

$$\begin{aligned} r(t) &= f'(\kappa(t)) \\ w(t) &= A(t)\left(f(\kappa(t)) - f'(\kappa(t))\kappa(t)\right) \end{aligned}$$

Note in particular that real wages are growing at the rate of technological progress along the balanced growth path. This argument shows that in contrast to the OLG economies considered before here the welfare theorems apply. In fact, this section should be quite familiar to you; it is nothing else but a repetition of Chapter 3 in continuous time, executed to make you familiar with continuous time optimization techniques. In terms of economics, the current model provides a micro foundation of the basic Solow model. It removes the problem of a constant, exogenous saving rate. However the engine of growth is, as in the Solow model, exogenously given technological progress. The next step in our analysis is to develop models that do not *assume* economic growth, but rather derive it as an equilibrium phenomenon. These models are therefore called *endogenous* growth models (as opposed to exogenous growth models).

9.4 Endogenous Growth Models

The second main problem of the Solow model, which is shared with the Cass-Koopmans model of growth is that growth is exogenous: without exogenous technological progress there is no sustained growth in per capita income and consumption. In this sense growth in these models is more assumed rather than derived endogenously as an equilibrium phenomenon. The key assumption driving the result, that, absent technological progress the economy will converge to a no-growth steady state is the assumption of diminishing marginal product to the production factor that is accumulated, namely capital. As economies grow they accumulate more and more capital, which, with decreasing marginal products, yields lower and lower returns. Absent technological progress this force drives the economy to the steady state. Hence the key to derive sustained growth without assuming it being created by exogenous technological progress is to pose production technologies in which marginal products to accumulable factors are not driven down as the economy accumulates these factors.

We will start our discussion of these models with a stylized version of the so called AK-model, then turn to models with externalities as in Romer (1986) and Lucas (1988) and finally look at Romer's (1990) model of endogenous technological progress.

9.4.1 The Basic AK-Model

Even though the basic AK-model may seem unrealistic it is a good first step to analyze the basic properties of most one-sector competitive endogenous growth models. The basic structure of the economy is very similar to the Cass-Koopmans model. Assume that there is *no* technological progress. The representative household again grows in size at population growth rate $n > 0$ and its preferences are given by

$$U(c) = \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

Its budget constraint is again given by

$$c(t) + \dot{a}(t) + na(t) = w(t) + (r(t) - \delta)\, a(t)$$

with initial condition $a(0) = k_0$. We impose the same condition to rule out Ponzi schemes as before

$$\lim_{t\to\infty} a(t) e^{-\int_0^t (r(\tau)-\delta-n)d\tau} \geq 0$$

The main difference to the previous model comes from the specification of technology. We assume that output is produced by a constant returns to scale technology only using capital

$$Y(t) = AK(t)$$

The aggregate resource constraint is, as before, given by

$$\dot{K}(t) + \delta K(t) + C(t) = Y(t)$$

This completes the description of the model. The definition of equilibrium is completely standard and hence omitted. Also note that this economy does not feature externalities, tax distortions or the like that would invalidate the welfare theorems. So we could, in principle, solve a social planners problem to obtain equilibrium allocations and then find supporting prices. Given that for this economy the competitive equilibrium itself is straightforward to characterize we will take a shot at it directly.

Let's first consider the household problem. Forming the Hamiltonian and carrying out the same manipulations as for the Cass-Koopmans model yields as Euler equation (note that there is no technological progress here)

$$\begin{aligned}
\dot{c}(t) &= \frac{1}{\sigma}\left[r(t) - (n + \delta + \rho)\right] c(t) \\
\gamma_c(t) &= \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[r(t) - (n + \delta + \rho)\right]
\end{aligned}$$

The transversality condition is given as

$$\lim_{t\to\infty} \lambda(t)a(t) = \lim_{t\to\infty} e^{-\rho t}c(t)^{-\sigma}a(t) \tag{9.42}$$

The representative firm's problem is as before

$$\max_{K(t),L(t)\geq 0} AK(t) - r(t)K(t) - w(t)L(t)$$

and yields as marginal cost pricing conditions

$$\begin{aligned}
r(t) &= A \\
w(t) &= 0
\end{aligned}$$

Hence the marginal product of capital and therefore the real interest rate are constant across time, independent of the level of capital accumulated in the economy. Plugging into the consumption Euler equation yields

$$\gamma_c(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[A - (n + \delta + \rho)\right]$$

i.e. the consumption growth rate is constant (always, not only along a balanced growth path) and equal to $A-(n+\delta+\rho)$. Integrating both sides with respect to time, say, until time t yields

$$c(t) = c(0)e^{\frac{1}{\sigma}[A-(n+\delta+\rho)]t} \tag{9.43}$$

where $c(0)$ is an endogenous variable that yet needs to be determined. We now make the following assumptions on parameters

$$\begin{aligned} [A - (n + \delta + \rho)] &> 0 &(9.44)\\ \frac{1-\sigma}{\sigma}\left[A - (n+\delta) - \frac{\rho}{1-\sigma}\right] &= \phi < 0 &(9.45) \end{aligned}$$

The first assumption, requiring that the interest rate exceeds the population growth rate plus the time discount rate, will guarantee positive growth of per capita consumption. It basically requires that the production technology is productive enough to generate sustained growth. The second assumption assures that utility from a consumption stream satisfying (9.43) remains bounded since

$$\begin{aligned} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt &= \int_0^\infty e^{-\rho t}\frac{c(0)^{1-\sigma}e^{\frac{1-\sigma}{\sigma}[A-(n+\delta+\rho)]t}}{1-\sigma}dt\\ &= \frac{c(0)^{1-\sigma}}{1-\sigma}\int_0^\infty e^{\left[\frac{1-\sigma}{\sigma}\left[A-(n+\delta)-\frac{\rho}{1-\sigma}\right]\right]t}dt\\ &< \infty \text{ if and only if } \frac{1-\sigma}{\sigma}\left[A-(n+\delta)-\frac{\rho}{1-\sigma}\right] < 0 \end{aligned}$$

From the aggregate resource constraint we have

$$\begin{aligned} \dot{K}(t) + \delta K(t) + C(t) &= AK(t)\\ c(t) + \dot{k}(t) &= Ak(t) - (n+\delta)k(t) &(9.46) \end{aligned}$$

Dividing both sides by $k(t)$ yields

$$\gamma_k(t) = \frac{\dot{k}(t)}{k(t)} = A - (n+\delta) - \frac{c(t)}{k(t)}$$

In a balanced growth path $\gamma_k(t)$ is constant over time, and hence $k(t)$ is proportional to $c(t)$, which implies that along a balanced growth path

$$\gamma_k(t) = \gamma_c(t) = A - (n+\delta+\rho)$$

i.e. not only do consumption and capital grow at constant rates (this is by definition of a balanced growth path), but they grow at the *same* rate $A-(n+\delta+\rho)$. We already saw that consumption always grows at a constant rate in this model. We will now argue that capital does, too, right away from $t=0$. In other words, we will show that transition to the (unique) balanced growth path is immediate.

Plugging in for $c(t)$ in equation (9.46) yields

$$\dot{k}(t) = -c(0)e^{\frac{1}{\sigma}[A-(n+\delta+\rho)]t} + Ak(t) - (n+\delta)k(t)$$

which is a first order nonhomogeneous differential equation. The general solution to the homogeneous equation is

$$k_g(t) = C_1 e^{(A-n-\delta)t}$$

A particular solution to the nonhomogeneous equation is (verify this by plugging into the differential equation)

$$k_p(t) = \frac{-c(0)e^{\frac{1}{\sigma}[A-(n+\delta+\rho)]t}}{\phi}$$

Hence the general solution to the differential equation is given by

$$k(t) = C_1 e^{(A-n-\delta)t} - \frac{c(0)}{\phi}e^{\frac{1}{\sigma}[A-(n+\delta+\rho)]t}$$

where $\phi = \frac{1-\sigma}{\sigma}\left[A-(n+\delta)-\frac{\rho}{1-\sigma}\right] < 0$. Now we use that in equilibrium $a(t) = k(t)$. From the transversality condition we have that, using (9.43)

$$\begin{aligned}
\lim_{t\to\infty} e^{-\rho t}c(t)^{-\sigma}k(t) &= \lim_{t\to\infty} e^{-\rho t}c(0)^{-\sigma}e^{-[A-(n+\delta+\rho)]t}\left[C_1 e^{(A-n-\delta)t} - \frac{c(0)}{\phi}e^{\frac{1}{\sigma}[A-(n+\delta+\rho)]t}\right] \\
&= c(0)^{-\sigma}\left[C_1 \lim_{t\to\infty} e^{[-\rho-A+n+\delta+\rho+A-n-\delta]t} - \frac{c(0)}{\phi}\lim_{t\to\infty} e^{\left[-\rho-A+n+\delta+\rho+\frac{1}{\sigma}[A-(n+\delta\right.}\right. \\
&= c(0)^{-\sigma}\left[C_1 - \frac{c(0)}{\phi}\lim_{t\to\infty} e^{\frac{1-\sigma}{\sigma}\left[A-(n+\delta)-\frac{\rho}{1-\sigma}\right]}\right] = 0 \text{ if and only if } C_1 = 0
\end{aligned}$$

because of the assumed inequality in (9.45). Hence

$$k(t) = -\frac{c(0)}{\phi} e^{\frac{1}{\sigma}[A-(n+\delta+\rho)]t} = -\frac{c(t)}{\phi}$$

i.e. the capital stock is proportional to consumption. Since we already found that consumption *always* grows at a constant rate $\gamma_c = A - (n + \delta + \rho)$, so does $k(t)$. The initial condition $k(0) = k_0$ determines the level of capital, consumption $c(0) = -\phi k(0)$ and output $y(0) = Ak(0)$ that the economy starts from; subsequently all variables grow at constant rate $\gamma_c = \gamma_k = \gamma_y$. Note that in this model the transition to a balanced growth path from any initial condition $k(0)$ is immediate.

In this simple model we can explicitly compute the saving rate for any point in time. It is given by

$$s(t) = \frac{Y(t) - C(t)}{Y(t)} = \frac{Ak(t) - c(t)}{Ak(t)} = 1 + \frac{\phi}{A} = s \in (0, 1)$$

i.e. the saving rate is constant over time (as in the original Solow model and in contrast to the Cass-Koopmans model where the saving rate is only constant along a balanced growth path).

In the Solow and Cass-Koopmans model the growth rate of the economy was given by $\gamma_c = \gamma_k = \gamma_y = g$, the growth rate of technological progress. In particular, savings rates, population growth rates, depreciation and the subjective time discount rate affect per capita income *levels*, but not growth rates. In contrast, in the basic AK-model the growth rate of the economy is affected positively by the parameter governing the productivity of capital, A and negatively by parameters reducing the willingness to save, namely the effective depreciation rate $\delta + n$ and the degree of impatience ρ. Any policy affecting these parameters in the Solow or Cass-Koopmans model have only level, but no growth rate effects, but have growth rate effects in the AK-model. Hence the former models are sometimes referred to as "income level models" whereas the others are referred to as "growth rate models".

With respect to their empirical predictions, the AK-model does not predict convergence. Suppose all countries share the same characteristics in terms of technology and preferences, and only differ in terms of their initial capital stock. The Solow and Cass-Koopmans model then predict absolute convergence in income levels and higher growth rates in poorer countries, whereas the AK-model predicts no convergence whatsoever. In fact, since all countries share the same growth rate and all economies are on the balanced growth path immediately, initial differences in per capita capital and hence per capita income and consumption persist forever and completely. The absence of decreasing marginal products of capital prevents richer countries to slow down in their growth process as compared to poor countries. If countries differ with respect to their characteristics, the Solow and Cass-Koopmans model predict conditional convergence. The AK-model predicts that different countries grow at different rates. Hence it may be possible that the gap between rich and poor countries widen or that poor countries take over rich countries. Hence one important test of these two competing theories of growth is an empirical exercise to determine whether we in fact see absolute and/or conditional convergence. Note that we discuss the predictions of the basic AK-model with respect to convergence at length here because the following, more sophisticated models will share the qualitative features of the simple model.

9.4.2 Models with Externalities

The main assumption generating sustained growth in the last chapter was the presence of constant returns to scale with respect to production factors that are, in contrast to raw labor, accumulable. Otherwise eventually decreasing marginal products set in and bring the growth process to a halt. One obvious unsatisfactory element of the previous model was that labor was not needed for production and that therefore the capital share equals one. Even if one interprets capital broadly as including physical capital, this assumption may be rather unrealistic. We, i.e. the growth theorist faces the following dilemma: on the one hand we want constant returns to scale to accumulable factors, on the other hand we want labor to claim a share of income, on the third hand we can't deal with increasing returns to scale on the firm level as this destroys existence of competitive equilibrium. (At least) two ways out of this problem have been proposed: a) there may be increasing returns to scale on the firm level, but the firm does not perceive it this way because part of its inputs come from positive externalities beyond the control of the firm b) a departure from perfect competition towards monopolistic competition. We will discuss the main contributions in both of these proposed resolutions.

Romer (1986)

We consider a simplified version of Romer's (1986) model. This model is very similar in spirit and qualitative results to the one in the previous section. However, the production technology is modified in the following form. Firms are indexed by $i \in [0,1]$, i.e. there is a continuum of firms of measure 1 that behave competitively. Each firm produces output according to the production function

$$y_i(t) = F(k_i(t), l_i(t)K(t))$$

where $k_i(t)$ and $l_i(t)$ are labor and capital input of firm i, respectively, and $K(t) = \int k_i(t)di$ is the average capital stock in the economy at time t. We assume that firm i, when choosing capital input $k_i(t)$, does not take into account the effect of $k_i(t)$ on $K(t)$.[19] We make the usual assumption on F: constant returns to scale with respect to the two inputs $k_i(t)$ and $l_i(t)K(t)$, positive but decreasing marginal products (we will denote by F_1 the partial derivative with respect to the first, by F_2 the partial derivative with respect to the second argument), and Inada conditions.

Note that F exhibits increasing returns to scale with respect to all three factors of production

$$\begin{aligned} F(\lambda k_i(t), \lambda l_i(t)\lambda K(t)) &= F(\lambda k_i(t), \lambda^2 l_i(t)K(t)) > \lambda F(k_i(t), l_i(t)K(t)) \text{ for all } \lambda > 1 \\ F(\lambda k_i(t), \lambda [l_i(t)K(t)]) &= \lambda F(k_i(t), l_i(t)K(t)) \end{aligned}$$

but since the firm does not realize its impact on $K(t)$, a competitive equilibrium will exist in this economy. It will, however, in general not be Pareto optimal. This is due to the externality in the production technology of the firm: a higher aggregate capital stock makes individual firm's workers more productive, but firms do not internalize this effect of the capital input decision on the aggregate capital stock. As we will see, this will lead to less investment and a lower capital stock than socially optimal.

The household sector is described as before, with standard preferences and initial capital endowments $k(0) > 0$. For simplicity we abstract from population growth (you should work out the model with population growth).

[19] Since we assume that there is a continuum of firms this assumption is completely rigourous as

$$\int_0^1 k_i(t)di = \int_0^1 \tilde{k}_i(t)di$$

as long as $k_i(t) = \tilde{k}_i(t)$ for all but countably many agents.

However we assume that the representative household in the economy has a size of L identical people (we will only look at type identical allocations). We do this in order to discuss "scale effects", i.e. the dependence of income levels and growth rates on the size of the economy.

Since this economy is not quite as standard as before we define a competitive equilibrium

Definition 112 *A competitive equilibrium are allocations* $(\hat{c}(t), \hat{a}(t))_{t\in[0,\infty)}$ *for the representative household, allocations* $(\hat{k}_i(t), \hat{l}_i(t))_{t\in[0,\infty),i\in[0,1]}$ *for firms, an aggregate capital stock* $\hat{K}(t)_{t\in[0,\infty)}$ *and prices* $(\hat{r}(t), \hat{w}(t))_{t\in[0,\infty)}$ *such that*

1. *Given* $(\hat{r}(t), \hat{w}(t))_{t\in[0,\infty)}$ $(\hat{c}(t), \hat{a}(t))_{t\in[0,\infty)}$ *solve*

$$\begin{aligned} & \max_{(c(t),a(t))_{t\in[0,\infty)}} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt \\ s.t.\ c(t)+\dot{a}(t) &= \hat{w}(t)+(\hat{r}(t)-\delta)\,a(t) \text{ with } a(0)=k(0) \text{ given} \\ c(t) &\geq 0 \\ \lim_{t\to\infty} a(t)e^{-\int_0^t(\hat{r}(\tau)-\delta)d\tau} &\geq 0 \end{aligned}$$

2. *Given* $\hat{r}(t), \hat{w}(t)$ *and* $\hat{K}(t)$ *for all* t *and all* i, $\hat{k}_i(t), \hat{l}_i(t)$ *solve*

$$\max_{k_i(t),l_i(t)\geq 0} F(k_i(t), l_i(t)\hat{K}(t)) - \hat{r}(t)k_i(t) - \hat{w}(t)l_i(t)$$

3. *For all* t

$$\begin{aligned} L\hat{c}(t)+\dot{\hat{K}}(t)+\hat{K}(t)\delta(t) &= \int_0^1 F(\hat{k}_i(t), \hat{l}_i(t)\hat{K}(t))di \\ \int_0^1 \hat{l}_i(t)di &= L \\ \int_0^1 \hat{k}_i(t)di &= L\hat{a}(t) \end{aligned}$$

4. *For all* t

$$\int_0^1 \hat{k}_i(t)di = \hat{K}(t)$$

The first element of the equilibrium definition is completely standard. In the firm's maximization problem the important feature is that the *equilibrium* average capital stock is taken as given by individual firms. The market clearing conditions for goods, labor and capital are straightforward. Finally the last condition imposes rational expectations: what individual firms perceive to be the average capital stock in equilibrium is the average capital stock, given the firms' behavior, i.e. equilibrium capital demand.

Given that all L households are identical it is straightforward to define a Pareto optimal allocation and it is easy to see that it must solve the following social planners problem[20]

$$\begin{aligned} & \max_{(c(t),K(t))_{t\in[0,\infty)}\geq 0} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt \\ \text{s.t. } Lc(t)+\dot{K}(t)+\delta K(t) &= F(K(t),K(t)L) \text{ with } K(0)=Lk(0) \text{ given} \end{aligned}$$

[20] The social planner has the power to dictate how much each firm produces and how much inputs to allocate to that firm. Since production has no intertemporal links it is obvious that the planners maximization problem can solved in two steps: first the planner decides on aggregate variables $c(t)$ and $K(t)$ and then she decides how to allocate aggregate inputs L and $K(t)$ between firms. The second stage of this problem is therefore

$$\max_{l_i(t),k_i(t)\geq 0} \int_0^1 F\left(k_i(t), l_i(t)\left[\int_0^1 k_j(t)dj\right]\right)di$$

$$\begin{aligned} \text{s.t.} \int_0^1 k_i(t) &= K(t) \\ \int_0^1 l_i(t) &= L(t) \end{aligned}$$

i.e. given the aggregate amount of capital chosen the planner decides how to best allocate it. Let μ and λ denote the Lagrange multipliers on the two constraints.

First order conditions with respect to $l_i(t)$ imply that

$$F_2\left(k_i(t), l_i(t)K(t)\right)K(t) = \lambda$$

or, since F_2 is homogeneous of degree zero

$$F_2\left(\frac{k_i(t)}{l_i(t)}, K(t)\right)K(t) = \lambda$$

which indicates that the planner allocates inputs so that each firm has the same capital labor ratio. Denote this common ratio by

$$\begin{aligned} \phi &= \frac{k_i(t)}{l_i(t)} \text{ for all } i\in[0,1] \\ &= \frac{K(t)}{L} \end{aligned}$$

But then total output becomes

$$\begin{aligned} \int_0^1 F\left(k_i(t), l_i(t)\left[\int_0^1 k_j(t)dj\right]\right)di &= \int_0^1 k_i(t)F(1,\frac{K(t)}{\phi})di \\ &= F(1,\frac{K(t)}{\phi})K(t) \\ & F(K(t),K(t)L) \end{aligned}$$

How much production the planner allocates to each firm hence does not matter; the only important thing is that she equalizes capital-labor ratios across firms. Once she does, the production possibilies for any given choice of $K(t)$ are given by $F(K(t),K(t)L)$.

Note that the social planner, in contrast to the competitively behaving firms, internalizes the effect of the average (aggregate) capital stock on labor productivity. Let us start with this social planners problem. Forming the Hamiltonian and manipulation the optimality conditions yields as socially optimal growth rate for consumption

$$\gamma_c^{SP}(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[F_1(K(t), K(t)L) + F_2(K(t), K(t)L)L - (\delta + \rho)\right]$$

Note that, since F is homogeneous of degree one, the partial derivatives are homogeneous of degree zero and hence

$$\begin{aligned} F_1(K(t), K(t)L) + F_2(K(t), K(t)L)L &= F_1(1, \frac{K(t)L}{K(t)}) + F_2(1, \frac{K(t)L}{K(t)})L \\ &= F_1(1, L) + F_2(1, L)L \end{aligned}$$

and hence the growth rate of consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[F_1(1, L) + F_2(1, L)L - (\delta + \rho)\right]$$

is constant over time. By dividing the aggregate resource constraint by $K(t)$ we find that

$$L\frac{c(t)}{K(t)} + \frac{\dot{K}(t)}{K(t)} + \delta = F(1, L)$$

and hence along a balanced growth path $\gamma_K^{SP} = \gamma_k^{SP} = \gamma_c^{SP}$. As before the transition to the balanced growth path is immediate, which can be shown invoking the transversality condition as before.

Now let's turn to the competitive equilibrium. From the household problem we immediately obtain as Euler equation

$$\gamma_c^{CE}(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[r(t) - (\delta + \rho)\right]$$

The firm's profit maximization condition implies

$$r(t) = F_1(k_i(t), l_i(t)K(t))$$

But since all firms are identical and hence choose the same allocations[21] we have that

$$\begin{aligned} k_i(t) &= k(t) = \int_0^1 k(t)di = K(t) \\ l_i(t) &= L \end{aligned}$$

and hence

$$r(t) = F_1(K(t), K(t)L) = F_1(1, L)$$

Hence the growth rate of per capita consumption in the competitive equilibrium is given by

$$\gamma_c^{CE}(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[F_1(1, L) - (\delta + \rho)\right]$$

and is constant over time, not only in the steady state. Doing the same manipulation with resource constraint we see that along a balanced growth path the growth rate of capital has to equal the growth rate of consumption, i.e. $\gamma_K^{CE} = \gamma_k^{CE} = \gamma_c^{CE}$. Again, in order to obtain sustained endogenous growth we have to assume that the technology is sufficiently productive, or

$$F_1(1, L) - (\delta + \rho) > 0$$

Using arguments similar to the ones above we can show that in this economy transition to the balanced growth path is immediate, i.e. there are no transition dynamics.

Comparing the growth rates of the competitive equilibrium with the socially optimal growth rates we see that, since $F_2(1, L)L > 0$ the competitive economy grows inefficiently slow, i.e. $\gamma_c^{CE} < \gamma_c^{SP}$. This is due to the fact that competitive firms do not internalize the productivity-enhancing effect of higher average capital and hence under-employ capital, compared to the social optimum. Put otherwise, the private returns to investment (saving) are too low, giving rise to underinvestment and slow capital accumulation. Compared to the competitive equilibrium the planner chooses lower period zero consumption and higher investment, which generates a higher growth rate. Obviously welfare is higher in the socially optimal allocation than under

[21] This is without loss of generality. As long as firms choose the same capital-labor ratio (which they have to in equilibrium), the scale of operation of any particular firm is irrelevant.

the competitive equilibrium allocation (since the planner can always choose the competitive equilibrium allocation, but does not find it optimal in general to do so). In fact, under special functional form assumptions on F we could derive both competitive and socially optimal allocations directly and compare welfare, showing that the lower initial consumption level that the social planner dictates is more than offset by the subsequently higher consumption growth.

An obvious next question is what type of policies would be able to remove the inefficiency of the competitive equilibrium? The answer is obvious once we realize the source of the inefficiency. Firms do not take into account the externality of a higher aggregate capital stock, because at the equilibrium interest rate it is optimal to choose exactly as much capital input as they do in a competitive equilibrium. The private return to capital (i.e. the private marginal product of capital in equilibrium equals $F_1(1, L)$ whereas the social return equals $F_1(1, L) + F_2(1, L)L$. One way for the firms to internalize the social returns in their private decisions is to pay them a subsidy of $F_2(1, L)L$ for each unit of capital hired. The firm would then face an effective rental rate of capital of

$$r(t) - F_2(1, L)L$$

per unit of capital hired and would hire more capital. Since all factor payments go to private households, total capital income from a given firm is given by $[r(t) + F_2(1, L)L]\, k_i(t)$, i.e. given by the (now lower) return on capital plus the subsidy. The higher return on capital will induce the household to consume less and save more, providing the necessary funds for higher capital accumulation. These subsidies have to be financed, however. In order to reproduce the social optimum as a competitive equilibrium with subsidies it is important not to introduce further distortions of private decisions. A lump sum tax on the representative household in each period will do the trick, not however a consumption tax (at least not in general) or a tax that taxes factor income at different rates.

The empirical predictions of the Romer model with respect to the convergence discussion are similar to the predictions of the basic AK-model and hence not further discussed. An interesting property of the Romer model and a whole class of models following this model is the presence of scale effects. Realizing that $F_1(1, L) = F_1(\frac{1}{L}, 1)$ and $F_1(1, L) + F_2(1, L)L = F(1, L)$ (by

Euler's theorem) we find that

$$\begin{aligned}
\frac{\partial \gamma_c^{CE}}{\partial L} &= -\frac{1}{\sigma L^2} F_{11}(1, L) > 0 \\
\frac{\partial \gamma_c^{SP}}{\partial L} &= \frac{F_2(1, L)}{\sigma} > 0
\end{aligned}$$

i.e. that the growth rate of a country should grow with its size (more precisely, with the size of its labor force). This result is basically due to the fact that the higher the number of workers, the more workers benefit form the externality of the aggregate (average) capital stock. Note that this scale effect would vanish if, instead of the aggregate capital stock K the aggregate capital stock per worker $\frac{K}{L}$ would generate the externality. The prediction of the model that countries with a bigger labor force are predicted to grow faster has led some people to dismiss this type of endogenous models as empirically relevant. Others have tried, with some, but not big success, to find evidence for a scale effect in the data. The question seems unsettled for now, but I am sceptical whether this prediction of the model(s) can be identified in the data.

Lucas (1988)

Whereas Romer (1986) stresses the externalities generated by a high economy-wide capital stock, Lucas (1988) focuses on the effect of externalities generated by human capital. You will write a good thesis because you are around a bunch of smart colleagues with high average human capital from which you can learn. In other respects Lucas' model is very similar in spirit to Romer (1986), unfortunately much harder to analyze. Hence we will only sketch the main elements here.

The economy is populated by a continuum of identical, infinitely lived households that are indexed by $i \in [0, 1]$. They value consumption according to standard CRRA utility. There is a single consumption good in each period. Individuals are endowed with $h_i(0) = h_0$ units of human capital and $k_i(0) = k_0$ units of physical capital. In each period the households make the following decisions

- what fraction of their time to spend in the production of the consumption good, $1 - s_i(t)$ and what fraction to spend on the accumulation of new human capital, $s_i(t)$. A household that spends $1 - s_i(t)$ units of time in the production of the consumption good and has a level of human capital of $h_i(t)$ supplies $(1 - s_i(t))h_i(t)$ units of effective labor, and hence total labor income is given by $(1 - s_i(t))h_i(t)w(t)$

- how much of the current labor income to consume and how much to save for tomorrow

The budget constraint of the household is then given as

$$c_i(t) + \dot{a}_i(t) = (r(t) - \delta)a_i(t) + (1 - s_i(t))h_i(t)w(t)$$

Human capital is assumed to accumulate according to the accumulation equation

$$\dot{h}_i(t) = \theta h_i(t)s_i(t) - \delta h_i(t)$$

where $\theta > 0$ is a productivity parameter for the human capital production function. Note that this formulation implies that the time cost needed to acquire an extra 1% of human capital is constant, independent of the level of human capital already acquired. Also note that for human capital to the engine of sustained endogenous economic growth it is absolutely crucial that there are no decreasing marginal products of h in the production of human capital; if there were then eventually the growth in human capital would cease and the growth in the economy would stall.

A household then maximizes utility by choosing consumption $c_i(t)$, time allocation $s_i(t)$ and asset levels $a_i(t)$ as well as human capital levels $h_i(t)$, subject to the budget constraint, the human capital accumulation equation, a standard no-Ponzi scheme condition and nonnegativity constraints on consumption as well as human capital, and the constraint $s_i(t) \in [0, 1]$. There is a single representative firm that hires labor $L(t)$ and capital $K(t)$ for rental rates $r(t)$ and $w(t)$ and produces output according to the technology

$$Y(t) = AK(t)^{\alpha}L(t)^{1-\alpha}H(t)^{\beta}$$

where $\alpha \in (0, 1), \beta > 0$. Note that the firm faces a production externality in that the average level of human capital in the economy, $H(t) = \int_0^1 h_i(t)di$ enters the production function positively. The firm acts competitive and treats the average (or aggregate) level of human capital as exogenously given. Hence the firm's problem is completely standard. Note, however, that because of the externality in production (which is beyond the control of the firm and not internalized by individual households, although higher average human capital means higher wages) this economy again will feature inefficiency of competitive equilibrium allocations; in particular it is to be expected that the competitive equilibrium features underinvestment in human capital.

The market clearing conditions for the goods market, labor market and capital market are

$$\begin{aligned}\int_0^1 c_i(t)di + \dot{K}(t) + \delta K(t) &= AK(t)^\alpha L(t)^{1-\alpha} H(t)^\beta \\ \int_0^1 (1 - s_i(t)h_i(t))\, di &= L(t) \\ \int_0^1 a_i(t)di &= K(t)\end{aligned}$$

Rational expectations require that the average level of human capital that is expected by firms and households coincides with the level that households in fact choose, i.e.

$$\int_0^1 h_i(t)di = H(t)$$

The definition of equilibrium is then straightforward as is the definition of a Pareto optimal allocation (if, since all agents are ex ante identical, we confine ourselves to type-identical allocations, i.e. all individuals have the same welfare weights in the objective function of the social planner). The social planners problem that solves for Pareto optimal allocations is given as

$$\max_{(c(t),s(t),H(t),K(t))_{t\in[0,\infty)}\geq 0} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt$$

$$\begin{aligned}\text{s.t. } c(t) + \dot{K}(t) + \delta K(t) &= AK(t)^\alpha((1 - s(t)H(t))^{1-\alpha} H(t)^\beta \text{ with } K(0) = k_0 \text{ given} \\ \dot{H}(t) &= \theta H(t)s(t) - \delta H(t) \text{ with } H(0) = h_0 \text{ given} \\ s(t) &\in [0, 1]\end{aligned}$$

This model is already so complex that we can't do much more than simply determine growth rates of the competitive equilibrium and a Pareto optimum, compare them and discuss potential policies that may remove the inefficiency of the competitive equilibrium. In this economy a balanced growth path is an allocation (competitive equilibrium or social planners) such that consumption, physical and human capital and output grow at constant rates (which need not equal each other) and the time spent in human capital accumulation is constant over time.

Let's start with the social planner's problem. In this model we have two state variables, namely $K(t)$ and $H(t)$, and two control variables, namely $s(t)$ and $c(t)$. Obviously we need two co-state variables and the whole dynamical system becomes more messy. Let $\lambda(t)$ be the co-state variable for $K(t)$ and $\mu(t)$ the co-state variable for $H(t)$. The Hamiltonian is $\dot{\mu}$

$$\begin{aligned} & \mathcal{H}(c(t), s(t), K(t), H(t), \lambda(t), \mu(t), t) \\ = \; & e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda(t)\left[AK(t)^{\alpha}\left((1-s(t)H(t)\right)^{1-\alpha}H(t)^{\beta} - \delta K(t) - c(t)\right] \\ & +\mu(t)\left[\theta H(t)s(t) - \delta H(t)\right] \end{aligned}$$

The first order conditions are

$$e^{-\rho t}c(t)^{-\sigma} = \lambda(t) \tag{9.47}$$

$$\mu(t)\theta H(t) = \lambda(t)(1-\alpha)\left[\frac{AK(t)^{\alpha}\left((1-s(t)H(t)\right)^{1-\alpha}H(t)^{\beta}}{(1-s(t))}\right] \tag{9.48}$$

The co-state equations are

$$\dot{\lambda}(t) = -\lambda(t)\alpha\left[\frac{AK(t)^{\alpha}\left((1-s(t)H(t)\right)^{1-\alpha}H(t)^{\beta}}{K(t)} - \delta\right] \tag{9.49}$$

$$\dot{\mu}(t) = -\lambda(t)(1-\alpha+\beta)\left[\frac{AK(t)^{\alpha}\left((1-s(t)H(t)\right)^{1-\alpha}H(t)^{\beta}}{H(t)}\right] - \mu(t)\left[\theta s(t) - \delta\right] \tag{9.50}$$

Define $Y(t) = AK(t)^{\alpha}\left((1-s(t)H(t)\right)^{1-\alpha}H(t)^{\beta}$. Along a balanced growth path we have

$$\begin{aligned}
\frac{\dot{Y}(t)}{Y(t)} &= \gamma_Y(t) = \gamma_Y \\
\frac{\dot{c}(t)}{c(t)} &= \gamma_c(t) = \gamma_c \\
\frac{\dot{K}(t)}{K(t)} &= \gamma_K(t) = \gamma_K \\
\frac{\dot{H}(t)}{H(t)} &= \gamma_H(t) = \gamma_H \\
\frac{\dot{\lambda}(t)}{\lambda(t)} &= \gamma_\lambda(t) = \gamma_\lambda \\
\frac{\dot{\mu}(t)}{\mu(t)} &= \gamma_\mu(t) = \gamma_\mu \\
s(t) &= s
\end{aligned}$$

Let's focus on BGP's. From the definition of $Y(t)$ we have (by log-differentiating)

$$\gamma_Y = a\gamma_K + (1-\alpha+\beta)\gamma_H \tag{9.51}$$

From the human capital accumulation equation we have

$$\gamma_H = \theta s - \delta \tag{9.52}$$

From the Euler equation we have

$$\gamma_c = \frac{1}{\sigma}\left[\alpha\frac{Y(t)}{K(t)} - (\delta+\rho)\right] \tag{9.53}$$

and hence

$$\gamma_Y = \gamma_K \tag{9.54}$$

From the resource constraint it then follows that

$$\gamma_c = \gamma_Y = \gamma_K \tag{9.55}$$

and therefore

$$\gamma_K = \frac{1-\alpha+\beta}{1-\alpha}\gamma_H \tag{9.56}$$

From the first order conditions we have

$$\gamma_\lambda = -\rho - \sigma\gamma_c \tag{9.57}$$
$$\gamma_\mu = \gamma_\lambda + \gamma_Y - \gamma_H \tag{9.58}$$

Divide (9.47) by $\mu(t)$ and isolate $\frac{\lambda(t)}{\mu(t)}$ to obtain

$$\frac{\lambda(t)}{\mu(t)} = \frac{\theta H(t)(1-s(t))}{(1-\alpha)Y(t)}$$

Do the same with (9.50) to obtain

$$\frac{\lambda(t)}{\mu(t)} = -\left(\gamma_\mu + \gamma_H\right)\frac{H(t)}{(1-\alpha+\beta)Y(t)}$$

Equating the last two equations yields

$$-\frac{(\gamma_\mu + \gamma_H)}{(1-\alpha+\beta)} = \frac{\theta(1-s)}{(1-\alpha)}$$

Using (9.58) and (9.55) and (9.52) and (9.56) we finally arrive at

$$\gamma_c = \frac{1}{\sigma}\left[\frac{(\theta-\delta)(1-\alpha+\beta)}{1-\alpha} - \rho\right]$$

The other growth rates and the time spent with the accumulation of human capital can then be easily deduced form the above equations. Be aware of the algebra.

In general, due to the externality the competitive equilibrium will not be Pareto optimal; in particular, agents may underinvest into human capital. From the firms problem we obtain the standard conditions (from now on we leave out the i index for households

$$\begin{aligned} r(t) &= \alpha\frac{Y(t)}{K(t)} \\ w(t) &= (1-\alpha)\frac{Y(t)}{L(t)} = (1-\alpha)\frac{Y(t)}{(1-s(t))h(t)} \end{aligned}$$

Form the Lagrangian for the representative household with state variables $a(t), h(t)$ and control variables $s(t), c(t)$

$$\begin{aligned} H &= e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda(t)\left[\left(r(t)-\delta\right)a(t) + (1-s(t))h(t)w(t) - c(t)\right] \\ &\quad +\mu(t)\left[\theta h(t)s(t) - \delta h(t)\right] \end{aligned}$$

The first order conditions are

$$\begin{aligned} e^{-\rho t}c(t)^{-\sigma} &= \lambda(t) & (9.59)\\ \lambda(t)h(t)w(t) &= \mu(t)\theta h(t) & (9.60) \end{aligned}$$

and the derivatives of the co-state variables are given by

$$\begin{aligned} \dot{\lambda}(t) &= -\lambda(t)(r(t)-\delta) & (9.61)\\ \dot{\mu}(t) &= -\lambda(t)(1-s(t))w(t) - \mu(t)(\theta s(t)-\delta) & (9.62) \end{aligned}$$

Imposing balanced growth path conditions gives

$$\begin{aligned} \gamma_c &= \frac{1}{\sigma}(-\gamma_\lambda - \rho)\\ \gamma_\lambda &= \gamma_\mu - \gamma_w = \gamma_\mu - \gamma_Y + \gamma_h\\ \gamma_c &= \gamma_Y = \gamma_K\\ \gamma_h &= \frac{1-\alpha}{1-\alpha+\beta}\gamma_Y \end{aligned}$$

Hence

$$\gamma_\lambda = \gamma_\mu - \left(\frac{\beta}{1-\alpha+\beta}\right)\gamma_c$$

Using (9.60) and (9.62) we find

$$\gamma_\mu = \delta - \theta$$

and hence

$$\begin{aligned} \gamma_c &= \frac{1}{\sigma}(\theta - \left(\frac{\beta}{1-\alpha+\beta}\right)\gamma_c - (\rho+\rho))\\ \gamma_c^{CE} &= \frac{1}{\sigma + \frac{\beta}{1-\alpha+\beta}}(\theta - (\rho+\rho)) \end{aligned}$$

Compare this to the growth rate a social planner would choose

$$\gamma_c^{SP} = \frac{1}{\sigma}\left[\frac{(\theta-\delta)(1-\alpha+\beta)}{1-\alpha} - \rho\right]$$

We note that if $\beta = 0$ (no externality), then both growth rates are identical ((as they should since then the welfare theorems apply). If, however

$\beta > 0$ and the externality from human capital is present, then if both growth rates are positive, tedious algebra can show that $\gamma_c^{CE} < \gamma_c^{SP}$. The competitive economy grows slower than optimal since the private returns to human capital accumulation are lower than the social returns (agents don't take the externality into account) and hence accumulate to little human capital, lowering the growth rate of human capital.

9.4.3 Models of Technological Progress Based on Monopolistic Competition: Variant of Romer (1990)

In this section we will present a model in which technological progress, and hence economic growth, is the result of a conscious effort of profit maximizing agents to invent new ideas and sell them to other producers, in order to recover their costs for invention.[22] We envision a world in which competitive software firms hire factor inputs to produce new software, which is then sold to intermediate goods producers who use it in the production of a new intermediate good, which in turn is needed for the production of a final good which is sold to consumers. In this sense the Romer model (and its followers, in particular Jones (1995)) are sometimes referred to as endogenous growth models, whereas the previous growth models are sometimes called only semi-endogenous growth models.

Setup of the Model

Production in the economy is composed of three sectors. There is a final goods producing sector in which all firms behave perfectly competitive. These firms have the following production technology

$$Y(t) = L(t)^{1-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}}$$

where $Y(t)$ is output, $L(t)$ is labor input of the final goods sector and $x_i(t)$ is the input of intermediate good i in the production of final goods. $\frac{1}{\mu}$ is elasticity of substitution between two inputs (i.e. measures the slope of

[22] I changed and simplified the model a bit, in order to obtain analytic solutions and make results coparable to previous sections. The model is basically a continuous time version of the model described in Jones and Manuelli (1998), section 6.

isoquants), with $\mu = 0$ being the special case in which intermediate inputs are perfect substitutes. For $\mu \to \infty$ we approach the Leontieff technology. Evidently this is a constant returns to scale technology, and hence, without loss of generality we can normalize the number of final goods producers to 1.

At time t there is a continuum of differentiated intermediate goods indexed by $i \in [0, A(t)]$, where $A(t)$ will evolve endogenously as described below. Let $A_0 > 0$ be the initial level of technology. Technological progress in this model takes the form of an increase in the variety of intermediate goods. For $0 < \mu < 1$ this will expand the production possibility frontier (see below). We will assume this restriction on μ to hold.

Each differentiated product is produced by a single, monopolistically competitive firm. This firm has bought the patent for producing good i and is the only firm that is entitled to produce good i. The fact, however, that the intermediate goods are substitutes in production limits the market power of this firm. Each intermediate goods firm has the following constant returns to scale production function to produce the intermediate good

$$x_i(t) = al_i(t)$$

where $l_i(t)$ is the labor input of intermediate goods producer i at date t and $a > 0$ is a technology parameter, common across firms, that measures labor productivity in the intermediate goods sector. We assume that the intermediate goods producers act competitively in the labor market

Finally there is a sector producing new "ideas", patents to new intermediate products. The technology for this sector is described by

$$\dot{A}(t) = bX(t)$$

Note that this technology faces constant returns to scale in the production of new ideas in that $X(t)$ is the only input in the production of new ideas. The parameter b measures the productivity of the production of new ideas: if the ideas producers buy $X(t)$ units of the final good for their production of new ideas, they generate $bX(t)$ new ideas.

Planner's Problem

Before we go ahead and more fully describe the equilibrium concept for this economy we first want to solve for Pareto-optimal allocations. As usual we specify consumer preferences as

$$u(c) = \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

The social planner then solves[23]

$$\begin{aligned}
& \max_{c(t), l_i(t), x_i(t), A(t), L(t), X(t) \geq 0} \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \\
\text{s.t. } c(t) + X(t) &= L(t)^{1-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}} \\
L(t) + \int_0^{A(t)} l_i(t) di &= 1 \\
x_i(t) &= a l_i(t) \text{ for all } i \in [0, A(t)] \\
\dot{A}(t) &= bX(t)
\end{aligned}$$

This problem can be simplified substantially. Since $\mu \in (0,1)$ it is obvious that $x_i(t) = x_j(t) = x(t)$ for all $i, j \in [0, A(t)]$ and $l_i(t) = l_j(t) = l(t)$ for all $i, j \in [0, A(t)]$.[24]. Also use the fact that $L(t) = 1 - A(t)l(t)$ to obtain the

[23]Note that there is no physical capital in this model. Romer (1990) assumes that intermediate goods producers produce a durable intermediate good that they then rent out every period. This makes the intermediate goods capital goods, which slightly complicates the analysis of the model. See the original article for further details.

[24]Suppose there are only two intermediate goods and one wants to

$$\begin{aligned}
& \max_{l_1(t), l_2(t) \geq 0} \left(\sum_{i=1}^{2} a l_i(t)^{1-\mu} \right)^{\frac{\alpha}{1-\mu}} \\
\text{s.t. } l_1(t) + l_2(t) &= L
\end{aligned}$$

For $\mu \in (0,1)$ the isoquant

$$\left(\sum_{i=1}^{2} a l_i(t)^{1-\mu} \right)^{\frac{\alpha}{1-\mu}} = C > 0$$

is strictly convex, with slope strictly bigger than one in absolute value. Given the above constraint, the maximum is interior and the first order conditions imply $l_1(t) = l_2(t)$ immediately. The same logic applies to the integral, where, strictly speaking, we have to add an "almost everywhere" (since sets of Lebesgue measure zero leave the integral unchanged). Note that for $\mu \leq 0$ the above argument doesn't work as we have corner solutions.

constraint set

$$\begin{aligned}
c(t)+X(t) &= L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}} \\
&= L(t)^{1-\alpha}\left((al(t))^{1-\mu}\int_0^{A(t)} di\right)^{\frac{\alpha}{1-\mu}} \\
&= L(t)^{1-\alpha}\left(A(t)\left(a\frac{1-L(t)}{A(t)}\right)^{1-\mu}\right)^{\frac{\alpha}{1-\mu}} \\
&= a^{\alpha}L(t)^{1-\alpha}\left(1-L(t)\right)^{\alpha}A(t)^{\frac{\alpha\mu}{1-\mu}} \\
\dot{A}(t) &= bX(t)
\end{aligned}$$

Finally we note that the optimal allocation of labor solves the static problem of

$$\max_{L(t)\in[0,1]} L(t)^{1-\alpha}\left(1-L(t)\right)^{\alpha}$$

with solution $L(t)=1-\alpha$. So finally we can write the social planners problem as

$$\begin{aligned}
u(c) &= \int_0^{\infty} e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt \\
\text{s.t. } c(t)+\frac{\dot{A}(t)}{b} &= CA(t)^{\eta} \qquad (9.63)
\end{aligned}$$

where $C=a^{a}(1-\alpha)^{1-\alpha}\alpha^{\alpha}$ and $\eta=\frac{\alpha\mu}{1-\mu}>0$ and with $A(0)=A_0$ given. Note that if $0<\mu<1$, this model boils down to the standard Cass-Koopmans model, whereas if $\eta=1$ we obtain the basic AK-model. Finally, if $\eta>1$ the model will exhibit accelerating growth. Forming the Hamiltonian and manipulating the first order conditions yields

$$\gamma_c(t)=\frac{1}{\sigma}\left[b\eta CA(t)^{\eta-1}-\rho\right]$$

Hence along a balanced growth path $A(t)^{\eta-1}$ has to remain constant over time. From the ideas accumulation equation we find

$$\frac{\dot{A}(t)}{A(t)}=\frac{bX(t)}{A(t)}$$

which implies that along a balanced growth path X and A grow at the same rate. Dividing () by $A(t)$ yields

$$\frac{c(t)}{A(t)} + \frac{\dot{A}(t)}{bA(t)} = CA(t)^{\eta-1}$$

which implies that c grows at the same rate as A and X.

We see that for $\eta < 1$ the economy behaves like the neoclassical growth model: from $A(0) = A_0$ the level of technology converges to the steady state A^* satisfying

$$\begin{aligned} \frac{b\eta C}{(A^*)^{1-\eta}} &= \rho \\ X^* &= 0 \\ c^* &= C\,(A^*)^{\eta} \end{aligned}$$

Without exogenous technological progress sustained economic growth in per capita income and consumption is infeasible; the economy is saddle path stable as the Cass-Koopmans model.

If $\eta = 1$, then the balanced growth path growth rate is

$$\gamma_c(t) = \frac{1}{\sigma}\left[b\eta C - \rho\right] > 0$$

provided that the technology producing new ideas, manifested in the parameter b, is productive enough to sustain positive growth. Now the model behaves as the AK-model, with constant positive growth possible and immediate convergence to the balanced growth path. Note that a condition equivalent to (9.45) is needed to ensure convergence of the utility generated by the consumption stream. Finally, for $\eta > 1$ (and $A_0 > 1$) we can show that the growth rate of consumption (and income) increases over time. Remember again that $\eta = \frac{\alpha\mu}{1-\mu}$, which, a priori, does not indicate the size of η. What empirical predictions the model has therefore crucially depends on the magnitudes of the capital share α and the intratemporal elasticity of substitution between inputs, μ.

Decentralization

We have in mind the following market structure. There is a single representative final goods producing firm that faces the constant returns to scale production technology as discussed above. The firm sells final output at time t for price $p(t)$ and hires labor $L(t)$ for a (nominal) wage $w(t)$. It also buys intermediate goods of all varieties for prices $p_i(t)$ per unit. The final goods firm acts competitively in all markets. The final goods producer makes zero profits in equilibrium (remember CRTS). The representative producer of new ideas in each period buys final goods $X(t)$ as inputs for price $p(t)$ and sells a new idea to a new intermediate goods producer for price $\kappa(t)$. The idea producer behaves competitively and makes zero profits in equilibrium (remember CRTS). There is free entry in the intermediate goods producing sector. Each new intermediate goods producer has to pay the fixed cost $\kappa(t)$ for the idea and will earn subsequent profits $\pi(\tau)$, $\tau \geq t$ since he is a monopolistic competition, by hiring labor $l_i(t)$ for wage $w(t)$ and selling output $x_i(t)$ for price $p_i(t)$. Each intermediate producer takes as given the entire demand schedule of the final producer $x_i^d(\overrightarrow{p}(t))$, where $\overrightarrow{p} = (p, w, (p_i)_{i\in[0,A(t)]}$. We denote by $\overrightarrow{p}_{-1}$ all prices but the price of intermediate good i. Free entry drives net profits to zero, i.e. equates $\kappa(t)$ and the (appropriately discounted) stream of future profits. Now let's define a market equilibrium (note that we can't call it a competitive equilibrium anymore because the intermediate goods producers are monopolistic competitors).

Definition 113 *A market equilibrium is prices $(\hat{p}(t), \hat{\kappa}(t), \hat{p}_i(t)_{i\in[0,A(t)}, \hat{w}(t))_{t\in[0,\infty)}$, allocations for the household $\hat{c}(t)_{t\in[0,\infty)}$, demands for the final goods producer $(\hat{L}(\overrightarrow{p}(t)), \hat{x}_i^d(\overrightarrow{p}(t))_{i\in[0,A(t)]})_{t\in[0,\infty)}$, allocations for the intermediate goods producers $((\hat{x}_i^s(t), \hat{l}_i(t))_{i\in[0,A(t)})_{t=[0,\infty)}$ and allocations for the idea producer $(\hat{A}(t), \hat{X}(t))_{t=[0,\infty)}$ such that*

1. *Given $\hat{\kappa}(0), (\hat{p}(t), \hat{w}(t))_{t\in[0,\infty)}$, $\hat{c}(t)_{t\in[0,\infty)}$ solves*

$$\begin{aligned} & \max_{c(t)\geq 0} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt \\ s.t. \int_0^\infty p(t)c(t)dt &= \int_0^\infty w(t)dt + \hat{\kappa}(0)A_0 \end{aligned}$$

2. *For each i, t, given $\overrightarrow{\hat{p}}_{-i}(t), \hat{w}(t)$, and $\hat{x}_i^d(\overrightarrow{p}(t)$, $(\hat{x}_i^s(t), \hat{l}_i(t), \hat{p}_i(t))$ solves*

$$\begin{aligned} \hat{\pi}_i(t) &= \max_{x_i(t), l_i(t), p_i(t)\geq 0} p_i(t)x_i^d(\overrightarrow{\hat{p}}(t)) - w(t)l_i(t) \\ s.t.\ x_i(t) &= x_i^d(\overrightarrow{\hat{p}}(t)) \\ x_i(t) &= al_i(t) \end{aligned}$$

3. *For each t, and each $\overrightarrow{p} \geq 0$, $(\hat{L}(\overrightarrow{p}(t)), \hat{x}_i^d(\overrightarrow{p}(t))$ solves*

$$\max_{L(t),x_i(t)\geq 0} \hat{p}(t)L(t)^{1-\alpha}\left(\int_0^{\hat{A}(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}} - \hat{w}(t)L(t) - \int_0^{\hat{A}(t)} \hat{p}_i(t)x_i(t)di$$

4. *Given $(\hat{p}(t), \hat{c}(t))_{t\in[0,\infty}$, $(\hat{A}(t), \hat{X}(t))_{t=[0,\infty)}$ solves*

$$\begin{aligned} \max \int_0^\infty c(t)\dot{A}(t) &- \int_0^\infty p(t)X(t)dt \\ s.t.\ \dot{A}(t) &= bX(t) \text{ with } A(0) = A_0 \text{ given} \end{aligned}$$

5. *For all t*

$$\begin{aligned} \hat{L}(\overrightarrow{\hat{p}}(t))^{1-\alpha}\left(\int_0^{\hat{A}(t)} \hat{x}_i^d(\overrightarrow{\hat{p}}(t))^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}} &= \hat{X}(t) + \hat{c}(t) \\ \hat{x}_i^s(t) &= \hat{x}_i^d(\overrightarrow{\hat{p}}(t)) \text{ for all } i \in [0, \hat{A}(t)] \\ \hat{L}(t) + \int_0^{\hat{A}(t)} \hat{l}_i(t)di &= 1 \end{aligned}$$

6. *For all t, all $i \in \hat{A}(t)$*

$$\hat{\kappa}(t) = \int_t^\infty \hat{\pi}_i(\tau)d\tau$$

Several remarks are in order. First, note that in this model there is no physical capital. Hence the household only receives income from labor and from selling initial ideas (of course we could make the idea producers own the initial ideas and transfer the profits from selling them to the household). The key equilibrium condition involves the intermediate goods producers. They, by assumption, are monopolistic competitors and hence can set prices, taking as given the entire demand schedule of the final goods producer. Since the intermediate goods are substitutes in production, the demand for intermediate good i depends on all intermediate goods prices. Note that the intermediate goods producer can only set quantity or price, the other is dictated by the demand of the final goods producer. The required labor input follows from the production technology. Since we require the entire demand schedule for the intermediate goods producers we require the final goods producer to solve its maximization problem for all conceivable (positive) prices. The profit maximization requirement for the ideas producer is standard (remember that he behave perfectly competitive by assumption). The equilibrium conditions for final goods, intermediate goods and labor market are straightforward. The final condition is the zero profit condition for new entrants into intermediate goods production, stating that the price of the pattern must equal to future profits.

It is in general very hard to solve for an equilibrium explicitly in these type of models. However, parts of the equilibrium can be characterized quite sharply; in particular optimal pricing policies of the intermediate goods producers. Since the differentiated product model is widely used, not only in growth, but also in monetary economics and particularly in trade, we want to analyze it more carefully.

Let's start with the final goods producer. First order conditions with respect to $L(t)$ and $x_i(t)$ entail[25]

$$w(t) = (1-\alpha)p(t)L(t)^{-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}} = \frac{(1-\alpha)p(t)Y(t)}{L(t)} \tag{9.64}$$

$$p_i(t) = \alpha p(t)L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}-1} x_i(t)^{-\mu} \tag{9.65}$$

or

$$\begin{aligned} x_i(t)^{\mu}p_i(t) &= \alpha p(t)L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}-1} \quad \text{for all } i\in[0,A(t)] \\ &= \frac{\alpha p(t)Y(t)}{\int_0^{A(t)} x_i(t)^{1-\mu}di} \end{aligned}$$

Hence the demand for input $x_i(t)$ is given by

$$x_i(t) = \left(\frac{p(t)}{p_i(t)}\right)^{\frac{1}{\mu}}\left(\frac{\alpha Y(t)}{\int_0^{A(t)} x_i(t)^{1-\mu}di}\right)^{\frac{1}{\mu}} \tag{9.66}$$

$$= \left(\frac{p(t)}{p_i(t)}\right)^{\frac{1}{\mu}} \alpha Y(t)^{\frac{\mu+\alpha-1}{\alpha\mu}} L(t)^{\frac{(1-\mu)(1-\alpha)}{\alpha\mu}} \tag{9.67}$$

[25]Strictly speaking we should worry about corners. However, by assumption $\mu\in(0,1)$ will assure that for equilibrium *prices* corners don't occur

As it should be, demand for intermediate input i is decreasing in its relative price $\frac{p(t)}{p_i(t)}$. Now we proceed to the profit maximization problem of the typical intermediate goods firm. Taking as given the demand schedule derived above, the firm solves (using the fact that $x_i(t) = al_i(t)$

$$\begin{aligned} & \max_{p_i(t)} p_i(t)x_i(t) - \frac{w(t)x_i(t)}{a} \\ = \ & x_i(t)\left(p_i(t) - \frac{w(t)}{a}\right) \end{aligned}$$

The first order condition reads (note that $p_i(t)$ enters $x_i(t)$ as shown in (9.67)

$$x_i(t) - \frac{1}{\mu p_i(t)} x_i(t)\left(p_i(t) - \frac{w(t)}{a}\right) = 0$$

and hence

$$\begin{aligned} 1 &= \frac{1}{\mu} - \frac{w(t)}{\mu a p_i(t)} \\ p_i(t) &= \frac{w(t)}{a(1-\mu)} \end{aligned} \tag{9.68}$$

A perfectly competitive firm would have price $p_i(t)$ equal marginal cost $\frac{w(t)}{a}$. The pricing rule of the monopolistic competitor is very simple, he charges a constant markup $\frac{1}{1-\mu} > 1$ over marginal cost. Note that the markup is the lower the lower μ. For the special case in which the intermediate goods are perfect substitutes in production, $\mu = 0$ and there is no markup over marginal cost. Perfect substitutability of inputs forces the monopolistic competitor to behave as under perfect competition. On the other hand, the closer μ gets to 1 (in which case the inputs are complements), the higher the markup the firms can charge. Note that this pricing policy is valid not only in a balanced growth path. indicating that

Another important implication is that all firms charge the same price, and therefore have the same scale of production. So let $x(t)$ denote this common output of firms and $\tilde{p}(t) = \frac{w(t)}{a(1-\mu)}$ the common price of intermediate producers. Profits of every monopolistic competitor are given by

$$\begin{aligned} \pi(t) &= \tilde{p}(t)x(t) - \frac{w(t)x(t)}{a} \\ &= \mu x(t)\tilde{p}(t) \\ &= \frac{\mu\alpha p(t)Y(t)}{A(t)} \end{aligned} \tag{9.69}$$

We see that in the case of perfect substitutes profits are zero, whereas profits increase with declining degree of substitutability between intermediate goods.[26]

Using the above results in equations (9.64) and (9.66) yields

$$\begin{aligned} w(t)L(t) &= (1-\alpha)p(t)Y(t) && (9.70)\\ A(t)x(t)\tilde{p}(t) &= \alpha p(t)Y(t) && (9.71) \end{aligned}$$

We see that for the final goods producer factor payments to labor, $w(t)L(t)$ and to intermediate goods, $A(t)x(t)\tilde{p}(t)$, exhaust the value of production $p(t)Y(t)$ so that profits are zero as they should be for a perfectly competitive firm with constant returns to scale. From the labor market equilibrium condition we find

$$L(t) = 1 - \frac{A(t)x(t)}{a} \tag{9.72}$$

and output is given from the production function as

$$Y(t) = L(t)^{1-\alpha}x(t)^{\alpha}A(t)^{\frac{\alpha}{1-\mu}} \tag{9.73}$$

and is used for consumption and investment into new ideas

$$Y(t) = c(t) + X(t) \tag{9.74}$$

We assumed that the ideas producer is perfectly competitive. Then it follows immediately, given the technology

$$\begin{aligned} \dot{A}(t) &= bX(t) \\ A(t) &= A(0) + \int_0^t X(\tau)d\tau && (9.75) \end{aligned}$$

that

$$\kappa(t) = \frac{p(t)}{b} \tag{9.76}$$

The zero profit-free entry condition then reads (using (9.69))

$$\frac{p(t)}{\kappa} = \mu\alpha \int_t^{\infty} \frac{p(\tau)Y(\tau)}{A(\tau)}d\tau \tag{9.77}$$

[26] This is not a precise argument. One has to consider the general equilibrium effects of changes in μ on $p(t), Y(t), A(t)$ which is, in fact, quite tricky.

Finally, let us look at the household maximization problem. Note that, in the absence of physical capital or any other long-lived asset household problem does not have any state variable. Hence the household problem is a standard maximization problem, subject to a single budget constraint. Let λ be the Lagrange multiplier associated with this constraint. The first order condition reads

$$e^{-\rho t}c(t)^{-\sigma} = \lambda p(t)$$

Differentiating this condition with respect to time yields

$$-\sigma e^{-\rho t}c(t)^{-\sigma-1}\dot{c}(t) - \rho e^{-\rho t}c(t)^{-\sigma} = \lambda \dot{p}(t)$$

and hence

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left(-\frac{\dot{p}(t)}{p(t)} - \rho\right) \tag{9.78}$$

i.e. the growth rate of consumption equals the rate of deflation minus the time discount rate. In summary, the entire market equilibrium is characterized by the 10 equations (9.68) and (9.70) to (9.78) in the 10 variables $x(t), c(t), X(t), Y(t), L(t), A(t), \kappa(t), p(t), w(t), \tilde{p}(t)$, with initial condition $A(0) = A_0$. Since it is, in principle, extremely hard to solve this entire system we restrict ourselves to a few more interesting results.

First we want to solve for the fraction of labor devoted to the production of final goods, $L(t)$. Remember that the social planner allocated a fraction $1-\alpha$ of all labor to this sector. From (9.72) we have that $L(t) = 1 - \frac{A(t)x(t)}{a}$. Dividing (9.71) by (9.70) yields

$$\begin{aligned}
\frac{\alpha}{1-\alpha} &= \frac{A(t)x(t)\tilde{p}(t)}{w(t)L(t)} = \frac{A(t)x(t)}{aL(t)(1-\mu)} \\
\frac{A(t)x(t)}{a} &= \frac{\alpha(1-\mu)L(t)}{1-\alpha}
\end{aligned}$$

and hence

$$\begin{aligned}
L(t) &= 1 - \frac{A(t)x(t)}{a} = 1 - \frac{\alpha(1-\mu)L(t)}{1-\alpha} \\
L(t) &= \frac{1-\alpha}{1-\alpha\mu} > 1-\alpha
\end{aligned}$$

Hence in the market equilibrium more workers work in the final goods sector and less in the intermediate goods sector than socially optimal. The intuition for this is simple: since the intermediate goods sector is monopolistically competitive, prices are higher than optimal (than social shadow prices) and output is lower than optimal; differently put, final goods producers substitute away from expensive intermediate goods into labor. Obviously labor input in the intermediate goods sector is lower than in the social optimum and hence

$$A^{ME}(t)x^{ME}(t) < A^{SP}(t)x^{SP}(t)$$

Again these relationships hold always, not just in the balanced growth path.

Now let's focus on a balanced growth path where all variables grow at constant, possibly different rate. Obviously, since $L(t) = \frac{1-\alpha}{1-\alpha\mu}$ we have that $g_L = 0$. From the labor market equilibrium $g_A = -g_x$. From constant markup pricing we have $g_w = g_{\tilde{p}}$. From (9.75) we have $g_A = g_X$ and from the resource constraint (9.74) we have $g_A = g_X = g_c = g_Y$. Then from (9.70) and (9.71) we have that

$$\begin{aligned} g_w &= g_Y + g_P \\ g_{\tilde{p}} &= g_Y + g_P \end{aligned}$$

From the production function we find that

$$\begin{aligned} g_Y &= \alpha g_x + \frac{\alpha}{1-\mu} g_A \\ &= \frac{\alpha\mu}{1-\mu} g_A \end{aligned}$$

Hence a balanced growth path exists if and only if $g_Y = 0$ or $\eta = \frac{\alpha\mu}{1-\mu} = 1$. The first case corresponds to the standard Solow or Cass-Koopmans model: if $\eta < 1$ the model behaves as the neoclassical growth model with asymptotic convergence to the no-growth steady state (unless there is exogenous technological progress). The case $\eta = 1$ delivers (as in the social planners problem) a balanced growth path with sustained positive growth, whereas $\eta > 1$ yields explosive growth (for the appropriate initial conditions).

Let's assume $\eta = 1$ for the moment. Then $g_Y = g_A$ and hence $\frac{Y(t)}{A(t)} = \frac{Y(0)}{A_0}$ =constant. The no entry-zero profit condition in the BGP can be written as, since $p(\tau) = p(t)e^{g_p(\tau - t)}$ for all $\tau \geq t$

$$\begin{aligned}
\frac{p(t)}{b} &= \mu\alpha\frac{Y(0)}{A_0}\int_t^\infty p(t)^{g_p(\tau-t)}d\tau \\
1 &= -b\mu\alpha\frac{Y(0)}{A_0 g_p} \\
g_p &= \frac{\dot{p}(t)}{p(t)} = -\left(b\mu\alpha\frac{Y(0)}{A_0}\right) < 0
\end{aligned}$$

Finally, from the consumption Euler equation

$$g_c = \frac{1}{\sigma}\left(b\mu\alpha\frac{Y(0)}{A_0} - \rho\right)$$

But now note that

$$\begin{aligned}
Y(0) &= L(0)^{1-\alpha}x(0)^\alpha A(0)^{\frac{\alpha}{1-\mu}} \\
&= L(0)^{1-\alpha}\left(x(0)A(0)\right)^\alpha A(0)^{\frac{\alpha\mu}{1-\mu}} \\
&= L(0)^{1-\alpha}\left(x(0)A(0)\right)^\alpha A(0)
\end{aligned}$$

under the assumption that $\eta = 1$. Hence, using (9.72)

$$\begin{aligned}
\frac{Y(0)}{A_0} &= L(0)^{1-\alpha}\left(x(0)A(0)\right)^\alpha \\
&= L(0)^{1-\alpha}\left(a(1 - L(0)\right)^\alpha \\
&= L(0)a^\alpha \\
&= \frac{1-\alpha}{1-\alpha\mu}a^a
\end{aligned}$$

Therefore finally

$$g_c = g_Y = g_A = \frac{1}{\sigma}\left(ba^a\mu\alpha\frac{1-\alpha}{1-\alpha\mu} - \rho\right)$$

is the competitive equilibrium growth rate in the balanced growth path under the assumption that $\eta = 1$. Comparing this to the growth rate that the social planner would choose

$$\gamma_c(t) = \frac{1}{\sigma}\left[ba^a(1-\alpha)^{1-\alpha}\alpha^\alpha - \rho\right]$$

We see that for $\mu\alpha \leq 1$ the social planner would choose a higher balanced growth path growth rate than the market equilibrium BGP growth rate. The market power of the intermediate goods producers leads to lower production of intermediate goods and hence less resources for consumption and new inventions, which drive growth in this model.[27]

This completes our discussion of endogenous growth theory. The Romer-type model discussed last can, appropriately interpreted, nest the standard Solow-Cass-Koopmans type neoclassical growth models as well as the early AK-type growth models. In addition it achieves to make the growth rate of the economy truly endogenous: the economy grows because inventors of new ideas consciously expend resources to develop new ideas and sell them to intermediate producers that use them in the production of a new product.

[27]Note however that there is an effect of market power in the opposite direction. Since in the market equilibrium the intermediate goods producers make profits due to their (competitive) monopoly position, and the ideas inventors can extract these profits by selling new designs, due to the free entry condition, they have too big an incentive to invent new intermediate goods, relative to the social optimum. For big μ and big α this may, in fact, lead to an inefficiently *high* growth rate in the market equilibrium.

Chapter 10

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