1 Consumption Insurance

Throughout the chapter we will use the following notation to represent uncertainty. Let $s_t \in S_t$ be the current state of the economy and let $s^t = \{s_0, s_1, ..., s_t\}$ be the history up to time t, with $s^t \in S^t \equiv S_0 \times S_1 \times ... \times S_t$. Let $\pi(s^t)$ the probability of this history occurring. Let $y_t^i(s^t)$ be the individual i realization of endowment/income upon the realization of history s^t, with $\sum_{i \in I} y_t^i(s^t) = Y_t(s^t)$ denoting the aggregate endowment/income.

1.1 Two Benchmarks: Autarky and Complete Markets

If we consider the spectrum of all possible market arrangements, at the two extremes we find autarky and complete markets. Let's analyze these two cases first.

1.1.1 No Risk Sharing in Autarky

The simplest starting point to analyze consumption is an endowment economy where insurance markets to trade across states s_t at a given point in time t are completely absent, and there is no storage technology to transfer resources across periods (e.g., the consumption good is perishable). In this economy, an individual i who receives a random stream of income shocks $\left\{\{y_t^i(s^t)\}_{s^t \in S^t}\right\}_{t=0}^{\infty}$ has no other choice than consuming her income in every state

$$c_t^i\left(s^t\right) = y_t^i\left(s^t\right), \text{ for all } s^t, t \tag{1}$$

and equation (1) is also her budget constraint.

1.1.2 Full Risk Sharing in Complete Markets

AD budget constraint– The diametrically opposite benchmark is an endowment economy with a full set of insurance and financial markets. Here, every agent faces a time-zero Arrow-Debreu budget constraint of the form

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t\left(s^t\right)\left[c_t^i\left(s^t\right) - y_t^i\left(s^t\right)\right] = 0, \text{ for all } i \in I. \tag{2}$$

Hence, every possible transfer of income across states and time is possible, as long as the discounted consumption expenditures equal discounted income, for each individual.

SP solution– Using the First Welfare Theorem, we can characterize the complete markets competitive equilibrium allocations as the solution to the Pareto problem

$$\begin{aligned} & \max_{\left\{c_t^i(s^t)\right\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right) \sum_{i \in I} \alpha^i u\left(c_t^i\left(s^t\right)\right) \\ & s.t. \\ \sum_{i \in I} c_t^i\left(s^t\right) \quad = \quad & Y_t\left(s^t\right), \text{ for all } t, s^t \in S^t \end{aligned}$$

with solution

$$\beta^t \pi\left(s^t\right) \alpha^i u'\left(c_t^i\left(s^t\right)\right) = \theta_t\left(s^t\right),$$

which implies that for any pair of agents (i, j) the ratio of marginal utility is constant in every period and every state of the world, i.e.

$$\frac{u'\left(c_t^i\left(s^t\right)\right)}{u'\left(c_t^j\left(s^t\right)\right)} = \frac{\alpha^j}{\alpha^i}, \text{ for all } s^t, (i, j), \tag{3}$$

which is precisely the definition of *full risk-sharing (or full-insurance).*

For example, for the CRRA case, where $u\left(c_t^i\left(s_t\right)\right) = \frac{c_t^i(s_t)^{1-\sigma}}{1-\sigma}$, it is easy to derive that

$$\frac{c_t^i\left(s^t\right)}{c_t^j\left(s^t\right)} = \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}},$$

hence the ratio of consumption allocations is constant. Summing over $i = 1, ..., I$, this implies clearly that

$$c_t^j\left(s^t\right) = \left[\frac{\left(\alpha^j\right)^{\frac{1}{\sigma}}}{\sum_{i=1}^{I}\left(\alpha^i\right)^{\frac{1}{\sigma}}}\right] C_t\left(s^t\right), \tag{4}$$

so, individual consumption tracks perfectly aggregate consumption for every household.[1] Note that this result still holds in a production economy with capital accumulation. In complete markets there is a perfect separation between production of resources and distribution of output across households for consumption. Instead, in autarky, there is perfect correspondance between production and consumption.

[1] In general, full risk sharing does not mean constant consumption over time/across states. In the above model, individual consumption is constant over time and across states only if aggregate consumption is constant. Even if the aggregate endowment is constant, with flexible labor supply and non-separability between consumption and leisure in preferences, consumption may not be constant over time. The only statement that is always true in complete markets is that the ratio of marginal utility for any pair of agents is constant over time.

1.1.3 Empirical Implications

The full risk sharing hypothesis can be tested empirically. Under CRRA preferences, for example, equation (4) implies that the log-change in individual consumption should equal the log-change in aggregate consumption, for every individual, in every period. If we estimate from microdata the relationship

$$\Delta \log c_t^i = \beta_1 \Delta \log C_t + \beta_2 \Delta \log y_t^i + \varepsilon_t^i,$$

where y_t^i is current individual income, then the full risk-sharing hypothesis implies $(\beta_1 = 1, \beta_2 = 0)$. Contrast this prediction with the "autarky hypothesis" which implies $(\beta_1 = 0, \beta_2 = 1)$, i.e. consumption tracks perfectly current income. In general, they are both rejected, albeit the data seem to be much closer to full risk-sharing in many contexts.[2]

Result 2.0: A good model (empirically, at least) for consumption lies between autarky and full risk-sharing, i.e. it must be a model where agents have access to "partial" consumption insurance.

1.1.4 Partial Risk Sharing

It is useful to make a short detour here. How can we model partial consumption insurance? There are two approaches. The first one, which we can call the "endogenous incomplete markets" approach is to model explicitly the frictions that undermine full insurance (e.g., limited enforcement, or private information such as moral hazard and adverse selection). The second is the "exogenous incomplete markets" approach which suggests to model only the contracts/assets that we observe in the data (e.g., stock, bond, housing, etc.). The first approach is deeper than the second, but it yields usually a lot of state-contingent contracts in equilibrium that we do not see in the data, so the second approach is arguably more empirically useful.

Consider an example of the first approach. Suppose that the trades agreed upon at time $t = 0$ cannot be fully enforced. However, consumers who do not honor their contracts are excluded from financial markets forever and will leave in autarky thereon. Then, we can write "participation constraints" for agents, in every node s^t, of the form

$$\sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} \beta^{\tau-t} \pi(s^\tau) u\left(c_\tau^i(s^\tau)\right) \geq \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} \beta^{\tau-t} \pi(s^\tau) u\left(y_\tau^i(s^\tau)\right),$$

[2] See Mace (1991), Cochrane (1991).

where we have assumed that the default decision is made after consuming at t.These constraints state that agents always weakly prefers staying in the contract from $t+1$ onward than defaulting. Clearly, these constraints limit the amount of insurance that is available in equilibrium relative to an A-D equilibrium with full enforcement. At the end of the course, we will study economies with limited contract enforcement. For now, we focus on the "exogenous incomplete markets" approach.

1.2 Exogenous Restrictions on Trade: the Bond Economy

Sequential formulation of complete markets– It is useful to start from the sequential formulation version of the Arrow-Debreu constraint (2), i.e., a sequence of budget constraints, for every period t and history s^t, of the form

$$c_t^i\left(s^t\right) + \sum_{s_{t+1}\in S_{t+1}} q_t\left(s_{t+1}, s^t\right) a_{t+1}^i\left(s_{t+1}, s^t\right) = y_t^i\left(s^t\right) + a_t^i\left(s^t\right), \tag{5}$$

and a no Ponzi scheme condition that rules out excessively high debt at every node.[3] Here, $q\left(s_{t+1}, s^t\right)$ is the price at date t and state s^t of an Arrow security that pays one unit of consumption if state s_{t+1} occurs next period. Obviously, this sequential formulation of the complete markets model gives rise to the same conclusion, i.e., full risk sharing.[4]

Bond economy vs complete markets– The key difference between the bond economy and the complete markets model is in the set of securities that the household is allowed to trade. Under the bond economy, agents are restricted to trade only a non state-contingent asset. The budget constraint (5) is replaced by the more restrictive constraint, for every history s^t,

$$c_t^i\left(s^t\right) + q_t\left(s^t\right) a_{t+1}^i\left(s^t\right) = y_t^i\left(s^t\right) + a_t^i\left(s^{t-1}\right),$$

where $q_t\left(s^t\right)$ is the price at date t and state s^t of an asset that pays one unit of consumption next period, *independently* of the realization of the state s_{t+1}, i.e. it is a one-period bond.

[3] The notation $a_t^i\left(s_t, s^{t-1}\right)$ and the somewhat more concise formulation $a_t^i\left(s^t\right)$ are obviously equivalent, since $s^t = \left\{s_t, s^{t-1}\right\}$.

[4] It can be proved that the sequence of constraints in (5) and the no Ponzi-scheme condition

$$\lim_{t\to\infty} \sum_{s_{t+1}\in S_{t+1}} q_t\left(s_{t+1}, s^t\right) a_{t+1}^i\left(s_{t+1}, s^t\right) \geq 0$$

is equivalent to the Arrow-Debreu constraint (2). See chapter 8 in LS.

In other words, the agent is cut-off from every state-contingent insurance market and has only access to a simple financial instrument to transfer resources over time. The absence of insurance opportunities induces the consumer to hold a certain amount of the bond in order to smooth consumption.

We abstract from borrowing constraints for now, we only impose a No-Ponzi scheme condition stating that in the limit assets cannot be negative, i.e.

$$\lim_{t \to \infty} q_t \left(s^t\right) a^i_{t+1} \left(s^t\right) \geq 0,$$

Optimality implies that the weak inequality above holds with the $=$ sign.

To simplify the notation for the next sections, we assume away fluctuations in the aggregate endowment $Y_t\left(s^t\right)$, either deterministic or stochastic, hence

$$q_t \left(s^t\right) = q \equiv \frac{1}{1+r},$$

where r is the interest rate on a risk-free bond, and reformulate the budget constraint with lighter notation as

$$a_{t+1} = (1+r)\left(y_t + a_t - c_t\right), \tag{6}$$

i.e., we omit the explicit dependence on individual histories. We focus on the problem of a single individual, so we also omit the i subscript.

2 The Permanent Income Hypothesis (PIH)

The strict version of the PIH is a special case of the bond economy with two key assumptions: 1) households have quadratic utility

$$u\left(c\right) = b_1 c_t - \frac{1}{2} b_2 {c_t}^2,$$

and 2) the interest rate on the one-period bond equals the inverse of the discount rate, or $\beta\left(1+r\right) = 1$. Note that $u' > 0$ requires $c_t < b_1/b_2$ and $u'' < 0$ requires $b_2 > 0$.

Consumption as a random walk– From the consumption Euler equation:

$$b_1 - b_2 c_t = \beta\left(1+r\right) E_t\left(b_1 - b_2 c_{t+1}\right) \Rightarrow E_t c_{t+1} = c_t. \tag{7}$$

from which we recover the well known result that consumption is a martingale.[5]

[5] A martingale is a stochastic process (i.e., a sequence of random variables) $\{x_t\}$ which satisfies, at every t, $E_t x_{t+j} = x_t$ for any $j > 0$.

It is useful to note that from the law of iterated expectations and the martingale property:

$$E_t c_{t+2} = E_t \left[E_{t+1} c_{t+2} \right] = E_t c_{t+1} = c_t$$

and, more in general:

$$E_t c_{t+j} = c_t, \text{ for any } j \geq 0. \tag{8}$$

Iterating forward one period on the budget constraint (6), we obtain

$$c_t = y_t + a_t - \frac{1}{1+r} a_{t+1} = y_t + a_t - \frac{1}{1+r} \left[c_{t+1} - y_{t+1} + \frac{a_{t+2}}{1+r} \right]$$

and rearranging

$$c_t + \frac{1}{1+r} c_{t+1} = a_t + y_t + \frac{1}{1+r} y_{t+1} - \left(\frac{1}{1+r} \right)^2 a_{t+2}$$

If we keep iterating J times and use conditional expectations to deal with uncertain future realizations of income (and consumption), we arrive at

$$\sum_{j=0}^{J} \left(\frac{1}{1+r} \right)^j E_t c_{t+j} = a_t + \sum_{j=0}^{J} \left(\frac{1}{1+r} \right)^j E_t y_{t+j} + \left(\frac{1}{1+r} \right)^{J+1} E_t a_{t+J+1}$$

Taking the limit as $J \to \infty$ and using the No Ponzi scheme condition, we arrive at:

$$\begin{aligned} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j E_t c_{t+j} &= a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j E_t y_{t+j} \\ c_t &= \frac{r}{1+r} \left[a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j E_t y_{t+j} \right] = \frac{r}{1+r} \left(a_t + H_t \right) \end{aligned} \tag{9}$$

where the LHS of the second row uses property established in (8), and in the last equality we denoted human wealth, i.e. the expected discounted value of future earnings, with H_t. Recall that financial wealth is a_t. Define permanent income as the annuity value (i.e. $\frac{r}{1+r}$) of total (human and financial) wealth $W_t \equiv (a_t + H_t)$.[6] Therefore, we have the following result:

[6] The annuity value is defined as that portion that, when consumed every period, keeps the asset value constant. Note that (abstracting from y_t):

$$a_{t+1} = (1+r)(a_t - c_t) = (1+r)\left(a_t - \frac{r}{1+r} a_t \right) = a_t.$$

Result 2.1: If preferences are quadratic, and $\beta(1+r) = 1$, then consumption follows a martingale process and equals permanent income, i.e. the annuity value of human and financial wealth.

Certainty equivalence– Notice that, if one solves the non-stochastic version of the PIH problem stated earlier, from the FOC (7) one obtains $c_{t+1} = c_t$ and, by iterating forward on the budget constraint,

$$c_t = \frac{r}{1+r}\left[a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right].$$

Compared to equation (9), the above equation suggests that the consumption satisfies *certainty equivalence* in the sense that to obtain the solution of the stochastic problem, one can 1) solve the deterministic problem and 2) substitute conditional expectations of the forcing variables (y_{t+j}) in place of the variables themselves. Put differently, *the variance and higher moments of the income process do not matter for the determination of consumption.* This property descends directly from the linear-quadratic objective function.

Consumption dynamics– From (9), the change in consumption at time t equals

$$\Delta c_t = c_t - c_{t-1} = c_t - E_{t-1}c_t = \frac{r}{1+r}\left[W_t - E_{t-1}W_t\right],$$

where we have used the random walk property. Now, use the definition of total wealth W_t to define the innovation (i.e. the unexpected change) in permanent income, at time t as

$$\begin{aligned} W_t - E_{t-1}W_t &= a_t - E_{t-1}a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \left[E_t y_{t+j} - E_{t-1}\left(E_t y_{t+j}\right)\right], \qquad (10) \\ &= \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \left(E_t - E_{t-1}\right) y_{t+j}, \end{aligned}$$

where we have used the law of iterated expectations $E_{t-1}\left(E_t y_{t+j}\right) = E_{t-1}y_{t+j}$, and the fact that $a_t = E_{t-1}a_t$, since there is no uncertainty at time t about the evolution of wealth next period: just look at the budget constraint (6). Putting together (10) and the expression above for the change in consumption we arrive at

$$\Delta c_t = \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \left(E_t - E_{t-1}\right) y_{t+j}. \qquad (11)$$

This equation states another useful result:

Result 2.2: under the PIH, the change in consumption between $t-1$ and t *is proportional to the revision in expected earnings due to the new information ("news") accruing in that same time interval.*

2.1 Example with a Specific Income Shock

At this point, to make further progress, we need to make some assumptions on the statistical properties of the labor income process. We choose a specification that is very common in labor economics. We assume that labor income is the sum of two orthogonal components, a permanent component y_t^p which follows a martingale, and a transitory component u_t that is independently distributed over time:

$$\begin{aligned} y_t &= y_t^p + u_t, \\ y_t^p &= y_{t-1}^p + v_t. \end{aligned} \tag{12}$$

Note that v_t is the innovation to the permanent component, independently distributed over time. Assume that $E(v_t) = E(u_t) = 0$ and that the two shocks are orthogonal, $u_t \perp v_\tau$ for all pairs (t, τ).

Using $y_t^p = y_t - u_t$ from the first equation into the second equation, we obtain the representation

$$\begin{aligned} y_t - u_t &= y_{t-1} - u_{t-1} + v_t \\ y_t &= y_{t-1} + u_t - u_{t-1} + v_t \end{aligned} \tag{13}$$

With this structure of shocks in hand, we can express the change in consumption from (11) only as a function of the permanent innovation v_t and the transitory innovation u_t. Focusing on the RHS of (11) and ignoring for now the constant $r/(1+r)$, we have

$$\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (E_t - E_{t-1})\, y_{t+j} = (E_t - E_{t-1}) \left[y_t + \frac{1}{1+r} y_{t+1} + \left(\frac{1}{1+r}\right)^2 y_{t+2} + ... \right] \tag{14}$$

Using (13) into the first term of the RHS of equation (14)

$$\begin{aligned} (E_t - E_{t-1})\, y_t &= (E_t - E_{t-1}) \left[y_{t-1} + u_t - u_{t-1} + v_t \right] \\ &= u_t + v_t \end{aligned}$$

since all the terms in $t-1$ drop out because $E_t\left(x_{t-1}\right)=E_{t-1}\left(x_{t-1}\right)=x_{t-1}$. In other words, the unexpected change in income y_t (compared to the one-step-ahead forecast $E_{t-1}y_t$) is the sum of the permanent and the transitory innovations at time t.

Using (13) into the second term of the RHS of (14):

$$\begin{aligned}\left(E_t-E_{t-1}\right)y_{t+1} &= \left(E_t-E_{t-1}\right)\left[y_t+u_{t+1}-u_t+v_{t+1}\right] \\ &= u_t+v_t+\left(E_t-E_{t-1}\right)\left[u_{t+1}-u_t+v_{t+1}\right] \\ &= v_t,\end{aligned}$$

since all the terms indexed by $t+1$ drop out because $E_t\left(x_{t+1}\right)=E_{t-1}\left(x_{t+1}\right)=0$. At this point, it is easy to see that

$$\left(E_t-E_{t-1}\right)y_{t+j}=v_t \qquad \text{for any } j>1$$

In other words, the forecast revision between $t-1$ and t about income beyond time t equals the permanent innovation at time t.

Going back to expression (11), the innovation to permanent income is

$$\begin{aligned}\Delta c_t &= \frac{r}{1+r}\left[u_t+v_t+\frac{1}{1+r}v_t+\left(\frac{1}{1+r}\right)^2 v_t+...\right] \\ &= \frac{r}{1+r}\left[u_t+v_t\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j\right] \\ &= v_t+\frac{r}{1+r}u_t.\end{aligned}$$

Hence, households adjust their consumption responding to the annuitized change in income. This means that they will respond only weakly to purely transitory shocks (u_t), whereas they will respond one for one to permanent shocks (v_t). Indeed, the former shocks have only a small effect on permanent income, while the latter change permanent income one for one, by definition.

Note that under complete markets, we would have that $\Delta c_t=0$ since idiosyncratic shocks do not transmit into consumption. Thus, the bond economy is quite close to full insurance with respect to transitory shocks, but very far from it with respect to permanent shocks.

Identification of the shocks through panel data– Suppose that one has panel data on consumption and income for a cohort of households, $i=1,...,N$ followed over periods $t = 0,..,T$. Suppose that the variances of (u_t^i, v_t^i) change over time. Let var_t denote the cross-sectional variance (i.e., the variance across individuals of this cohort) at time t. If we are interested in identifying the variances of permanent and transitory shocks, we can simply use income data and the cross-sectional moment restrictions for $t = 1, ..., T$

$$\begin{aligned} var\left(\Delta y_t^i\right) &= var\left(v_t^i\right) + var\left(u_t^i\right) + var\left(u_{t-1}^i\right), \\ cov_t\left(\Delta y_{t-1}^i, \Delta y_t^i\right) &= -var_t\left(u_{t-1}^i\right). \end{aligned}$$

However, an altaernative is to use the restrictions imposed by the theory. Then, note that for $t = 1, ..., T$ we can write:

$$\begin{aligned} var\left(\Delta c_t^i\right) &= \left(\frac{r}{1+r}\right)^2 var\left(u_t^i\right) + var\left(v_t^i\right) \simeq var\left(v_t^i\right), \\ var\left(\Delta y_t^i\right) &= var\left(v_t^i\right) + var\left(u_t^i\right) + var\left(u_{t-1}^i\right), \\ cov\left(\Delta c_t^i, \Delta y_t^i\right) &= \left(\frac{r}{1+r}\right) var\left(u_t^i\right) + var\left(v_t^i\right) \end{aligned}$$

where the approximate equality in the first row holds for r "small". Therefore, it is easy to see that with data on consumption and income one can separately identify the variances of the underlying structural income shocks.

For example, if over a certain period of time we observe the variance of income rising, but the variance of consumption approximately flat, we should conclude that the rise in income uncertainty was mostly transitory. This is an important conclusion for policy, because it suggests that redistributing from the high-income to the low income agents may be largely a waste since agents can self-insure effectively transitory shocks on their own by borrowing and saving.

2.2 When are borrowing constraints binding?

So far, we have ignored the presence of borrowing constraints. We imposed a no-Ponzi scheme condition, but we never checked whether it's actually binding. Is it a good abstraction? The answer,as we show below, depends on the income process.

Wealth dynamics with borrowing constraints– First, note that, from the budget constraint (6)

$$a_{t+1} = (1+r)(y_t + a_t - c_t),$$

rearranging, we obtain an expression for the change in wealth

$$\Delta a_{t+1} = (1+r)\,y_t + r a_t - (1+r)\,c_t. \tag{15}$$

Substituting into the above equation the optimal consumption choice from (9) reproduced below

$$(1+r)\,c_t = r\left[a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] \tag{16}$$

we obtain

$$\begin{aligned}
\Delta a_{t+1} &= (1+r)\,y_t + r a_t - r\left[a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] = (1+r)\,y_t - r\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j} \\
&= (1+r)\,y_t - r y_t - r\left[\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] \\
&= y_t - \sum_{j=1}^{\infty}\left[\left(\frac{1}{1+r}\right)^{j-1} E_t y_{t+j} - \left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right]
\end{aligned}$$

where the last line uses the simple algebraic relationship

$$\frac{r}{(1+r)^j} = \frac{1}{(1+r)^{j-1}} - \frac{1}{(1+r)^j}.$$

Unfolding the expression in the sum listing first all the positive terms and then all the negative ones:

$$\begin{aligned}
\Delta a_{t+1} &= y_t - \left[E_t y_{t+1} + \left(\frac{1}{1+r}\right) E_t y_{t+2} + \ldots - \left(\frac{1}{1+r}\right) E_t y_{t+1} - \left(\frac{1}{1+r}\right)^2 E_t y_{t+2} - \ldots\right] \\
&= -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j-1} E_t \Delta y_{t+j}
\end{aligned}$$

Now, suppose that the income process follows a random walk, $y_t = y_{t-1} + \varepsilon_t$, with ε_t iid and $E(\varepsilon_t) = 0$. Then it is easy to see that $\Delta y_{t+j} = \varepsilon_{t+j}$ and $\Delta a_{t+1} = 0$. Therefore, the initial wealth endowment perpetuates itself (i.e., it is constant) so if the individual starts above the borrowing constraint, it will never be binding. The reason for this result is that wealth changes only if the individual is consuming just a part of its income (and saving the remaining part) in order to smooth consumption. With permanent shocks, all the income shock is consumed in every period and the individual also consumes the annuity value of wealth, which therefore remains constant.

However, if the income process is *iid*, we have that $\Delta y_{t+1} = \varepsilon_{t+1} - \varepsilon_t$, $\Delta y_{t+2} = \varepsilon_{t+2} - \varepsilon_{t+1}$, therefore

$$\begin{aligned}
\Delta a_{t+1} &= -\sum_{j=1}^{\infty} \left(\frac{1}{1+r}\right)^{j-1} E_t \Delta y_{t+j} = -\sum_{j=1}^{\infty} \left(\frac{1}{1+r}\right)^{j-1} E_t \left[\varepsilon_{t+j} - \varepsilon_{t+j-1}\right] \\
&= -E_t \left[\varepsilon_{t+1} - \varepsilon_t\right] - \frac{1}{1+r} E_t \left[\varepsilon_{t+2} - \varepsilon_{t+1}\right] - ... \\
&= \varepsilon_t
\end{aligned}$$

since all other terms are zero. This means that wealth follows a random walk and, as a result, any constraint on asset holdings will be binding with probability one sooner or later.

A simpler way to derive the same results is as follows. When income is a unit root ($y_t = y_{t-1} + \varepsilon_t$), from (16) we obtain

$$c_t = \frac{r}{1+r} a_t + y_t$$

since $E_t y_{t+j} = y_t$ which substituted into equation (15) yields $a_{t+1} = a_t$. When income is *iid* ($y_t = \varepsilon_t$) from (16) we have

$$c_t = \frac{r}{1+r} (a_t + y_t)$$

$E_t y_{t+j} = 0$ which substituted into equation (15) yields $\Delta a_{t+1} = \varepsilon_t$.

To conclude, whether ignoring borrowing constraint is troublesome or not depends on the specific income process. However, in general this result highlights the fact that borrowing constraints cannot be ignored.