Macroeconomics II Part II, Lecture IV: The New-Keynesian Model: Basics

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Recap

- We've learned a lot about RBC: natural starting point for studying business cycles
- But! No role for monetary factors
- One may think *monetary factors* are key to many business cycle phenomena
- With this, we mean that fluctuations in nominal variables - money stock, nominal interest rates, inflation, etc. - cause fluctuations in real activity
- To get started to think about this, we will introduce the New-Keynesian Model
- The New-Keynesian Model = RBC with frictional price setting

Today's agenda

1. Evidence concering the effects of monetary policy
2. The vanilla NK model: Setup and equilibrium
3. The vanilla NK model: Determinacy

Evidence concering the effects of monetary policy

Monetary factors and real activity: how to approach the data?

- How to test the hypothesis that monetary factors matter for real activity?
- This is not easy, as monetary shocks are almost always coupled with other macroeconomic disturbances
- Four types of approaches:
  1. Historical analysis of unusual episodes, e.g.,
     - The Great Depression (e.g. Friedman & Schwartz, Book 1963)
     - Hyperinflations (e.g. Sargent-Velde, JPE 1995)
     - Sweden's experiment with using monetary policy for counteracting household debt (Coglianese-Olsson-Patterson, R&R AER 2024)
  2. Monetary regime shifts, e.g.,
     - Introduction of Volcker rule (Clarida-Galí-Gertler, QJE 2000)
  3. Monetary policy shocks
  4. Indirect inference based on other macroeconomic shocks
- Summary of evidence: i) monetary factors affect real activity, ii) a contraction in nominal demand causes a fall in real activity, iii) size of effect is context specific
- Approach 3 & 4 speak directly to the NK model; we will briefly discuss approach 3, following Ramey (HB Macro 2016)

Monetary policy shocks

- Current monetary policy regime (in most developed countries): use the interest rate on risk-free overnight bond-like instruments issued by the central bank (="policy rate") to "control" fluctuations in inflation and real activity

- Research question: What is the casual effect of a change in the policy rate on inflation and real activity?

- Data at hand: time series of the policy rate i_t and various macro aggregates

- Problem: Given our monetary regime, changes in monetary policy are super endogenous to changes in macro aggregates, e.g., inflation and real activity

- Operational question: How to isolate exogenous changes in the policy rate?
  - "Exogenous" = "Unexpected", given the various macro aggregates

Isolating exogenous shocks: an illustrative framework I

- With time series data, we can estimate a Vector AutoRegression (VAR)

$$X_t = \mathbf{B}X_{t-1} + \eta_t$$

  where, e.g., $X_t = [y_t, \pi_t, i_t]$ and η_t is vector of reduced-form residuals

- Assuming no omitted variable, the OLS estimate of $\hat{\mathbf{B}}$ allow us to estimate the impulse-response function to some *given* exogenous shock

- Temptation: interpret OLS residuals $\hat{\eta}$ as exogenous shocks

- But of course, this is not credible

- For exampple, an “unusually” high value of i_t could reflect both a shock and an endogenous reaction to a shock to π_t

Isolating exogenous shocks: an illustrative framework II

- Suppose the data-generating process (the "true" model) is

$$\mathbf{A}_0 X_t = \mathbf{A}_1 X_{t-1} + \epsilon_t$$

  where ϵ_t is vector of structural shocks

- $\mathbf{A}_0$ captures the contemporanous causal relationships between the endogenous variables

- The structural system can be written

$$X_t = \mathbf{A}_0^{-1} \mathbf{A}_1 X_{t-1} + \mathbf{A}_0^{-1} \epsilon_t$$

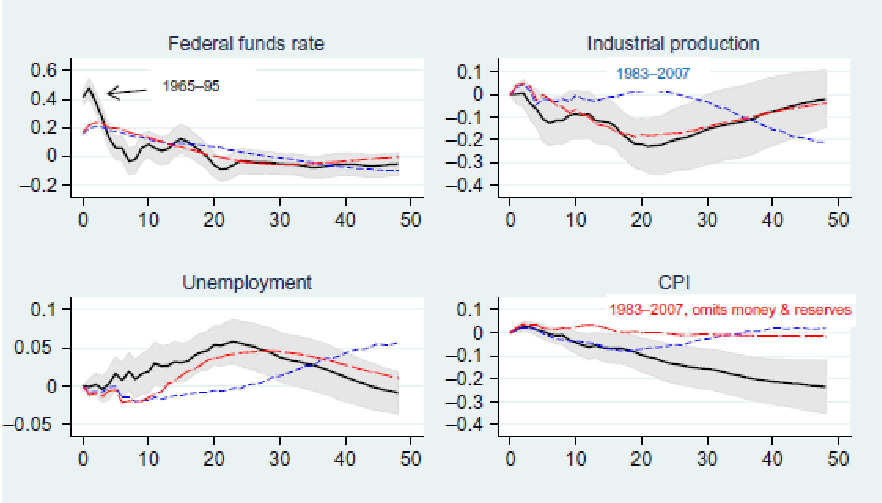
- Identification problem: How to extract the ϵ's from the η's?

- Solution: make assumptions on $\mathbf{A}_0$
  - Make assumptions = theory
  - Identification is always "structural"

Isolating exogenous shocks: common methods

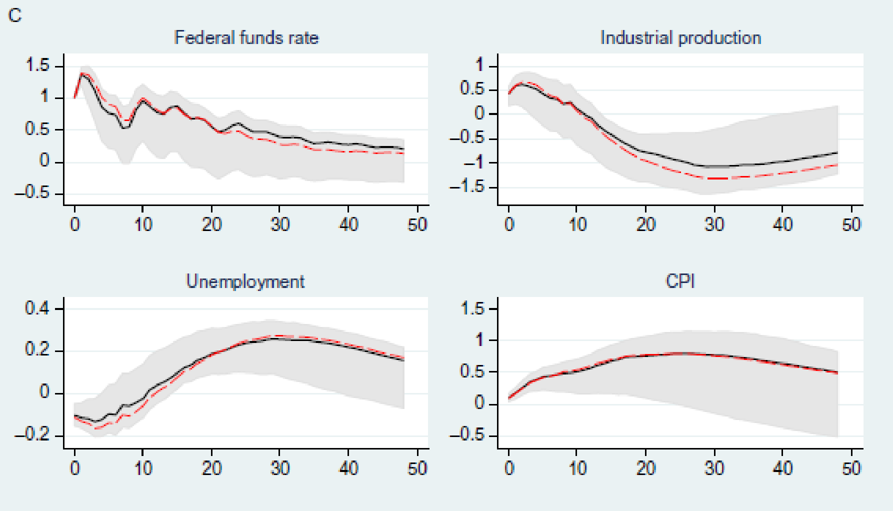
1. Cholesky decompostion/Recursive ordering (Sims, Ecmtra 1980; Bernanke-Blinder AER 1992)
    - Idea: Monetary policy can react to output/inflation within the same quarter, but output/inflation can only react to monetary policy with a lag
    - $\Rightarrow$ the contemporanous causal effect of MP on output/inflation is zero; justifies placing restrictions on A_0
2. Narrative shocks (Romer-Romer AER 2004)
    - Idea: Use the Fed's own forecast (the "Greenbook") to construct unexpected interest rate changes
    - The residual from regressing the policy rate on the forecast is the unexpected component, should not be correlated with other shocks
    - Can be used as instrument for OLS residuals in VAR system; justifies placing restrictions on A_0
3. High-frequency identification (Kuttner, JME 2001; Nakamura-Steinson QJE 2018)
    - Idea: discontinuous jumps in price of interest-rate forward contracts around policy announcement reflect unexpected policy change
    - Can be used as instrument for OLS residuals in VAR system; justifies placing restrictions on A_0

IRFs using recursive ordering



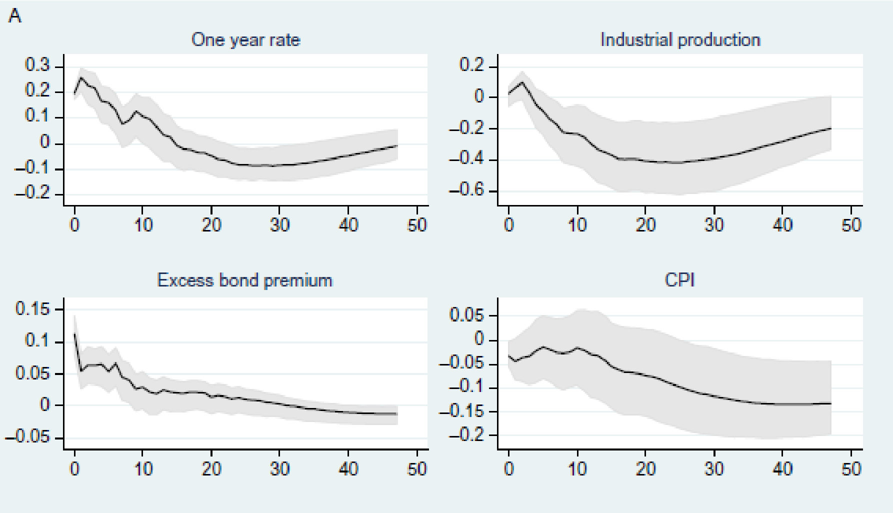
US monthly data, from Ramey (HB Macro 2016)

IRFs using narrative shocks



US monthly data, from Ramey (HB Macro 2016)

IRFs using HFI shocks



US monthly data, from Ramey (HB Macro 2016)

Monetary policy shocks: summary

- US data suggests a surprise increase in the policy rate leads to
  1. persistent hump-shaped decline in output
  2. small lagged decline in the price level (more uncertain)
- Very large literature exploring robustness, other countries, time periods etc.
  - Bauer-Czarnota-Klein (2025) constructs HFI shocks for Sweden, find similar effects
- Clearly inconsistent with models with frictionless price setting
- The NK model, in contrast, provides a starting point for interpreting this evidence

The New-Keynesian Model

NK Model: Overview

- Basic skeleton: RBC model with trade in nominal bonds
  - Older NK papers typically also an explicit role for money in facilitating transactions (e.g., CIA constraint, Money in the utility function)
  - We study the model in the cashless limit (which is arguably a very good approximation for transactions nowadays)
  - The price level (=the inverse of the price of money) still matters since bonds are denominated in units of money

- Two definining features:
  1. Monopolistic competition ⇒ firms are *price setters*, not price takers
  2. Frictions in price setting ⇒ some firms cannot freely set the price they would like

- To simplify, we assume no capital and first consider monetary policy shocks only (more shocks in next lecture)

The end product

- The log-linearized equilibrium of the vanilla NK model can be described by 3 equations

$$\begin{aligned} \text{DIS curve:} \qquad & \hat{y}_t = -(\hat{i}_t - E_t\pi_{t+1}) + E_t\hat{y}_{t+1} \\ \text{Phillips curve:} \qquad & \pi_t = \beta E_t\pi_{t+1} + \kappa\hat{y}_t \\ \text{Policy rule:} \qquad & \hat{i}_t = \phi\pi_t + \nu_t \end{aligned}$$

  where
  - y_t is output
  - π_t is inflation
  - i_t is the nominal interest rate
  - ν_t is the policy shock

How we will get there

- When presenting a model, a good practice is to
  1. State the assumptions
  2. Define a solution concept (equilibrium definition)
  3. Solve the model; when solving a model by log-linearization, the cook book says
     1. Start with making an equilibrium characterization
     2. Solve for the steady state
     3. Log-linearize the equilibrium characterization around the steady state

- We will not follow this recommended practice today, because the NK model is a bit more "messy" compared to RBC

- Instead, we will set up, characterize and log-linearize the agent's optimization problems in a sequential manner
  - By so doing, we will move back and forth between stating assumptions and solving the model

Household problem

- Program of the representative household

$$
\begin{aligned}
\max_{\{C_t, N_t, B_{t+1}\}} \quad & E_O \sum_{t=0}^{\infty} \beta^t \left[U(C_t) - V(N_t) \right] \\
\text{s.t} \quad & P_t C_t + Q_t B_{t+1} \leq W_t N_t + B_t + T_t \\
& C_t, N_t, B_{t+1} \geq 0
\end{aligned}
$$

with $U(C_t)$, $V(N_t)$ satisfying the usual regularity conditions

- Note:
  - Budget constraint denominated in units of money, P_t is the price level
  - Monopoly firm profits are returned to the household $\Rightarrow T_t \neq 0$
  - $W_t =$ nominal, not the real wage (conflicting with our notation for the RBC model)
  - The gross nominal return on bonds that pay in period $t+1$, $1/Q_t$ is known in period t
  - The gross real return, $R_t = 1/(Q_t \Pi_{t+1})$, is not known in period t
    - $\Pi_{t+1} = P_{t+1}/P_t$

Household problem optimality condtions

- Lagrangian to the household problem:

$$\mathbf{L} = E_0 \left(\sum_{t=0}^{\infty} \beta^t \left[U(C_t) - V(N_t) \right] - \sum_{t=0}^{\infty} \lambda_t \left[P_t C_t + Q_t B_{t+1} - W_t N_t - B_t - T_t \right] \right)$$

- First order conditions:

$$\begin{aligned} C_t : & \qquad \beta^t U'(C_t) - \lambda_t P_t = 0 \\ N_t : & \qquad -\beta^t V'(N_t) + \lambda_t W_t = 0 \\ B_{t+1} : & \qquad -\lambda_t Q_t + E_t \lambda_{t+1} = 0 \end{aligned}$$

- Put together:

$$\begin{aligned} \frac{W_t}{P_t} &= \frac{V'(N_t)}{U'(C_t)} \\ U'(C_t) &= E_t \left[\beta \frac{1}{Q_t \Pi_{t+1}} U'(C_{t+1}) \right] \end{aligned}$$

Log-linearizing household optimaility conditions

- Using our standard utility functions, $U(C) = \log C$ and $V(N) = \theta \frac{N^{1+\varphi}}{1+\varphi}$:

$$\begin{aligned} \frac{W_t}{P_t} &= C_t \theta N_t^{\varphi} \\ C_t^{-1} &= E_t \left[\beta \frac{1}{Q_t \Pi_{t+1}} C_{t+1}^{-1}) \right] \end{aligned}$$

- Taking logs:

$$\begin{aligned} w_t - p_t &= \log\theta + c_t + \varphi n_t \\ c_t &= -(i_t - E_t \pi_{t+1} - \xi) + E_t c_{t+1} \end{aligned}$$

where $i_t = \log\left(\frac{1}{Q_t}\right)$ is the net nominal interest rate and $\xi = -\log\beta$

- Substracting steady state:

$$\begin{aligned} \hat{w}_t - \hat{p}_t &= \hat{c}_t + \varphi \hat{n}_t \\ \hat{c}_t &= -(\hat{i}_t - E_t \pi_{t+1}) + E_t \hat{c}_{t+1} \end{aligned}$$

- Note: log utility implies that the elasticity of current consumption w.r.t. to the real interest rate is -1

Firms: 2 layers

- Final goods producers:
  - A representative firm operates in a competitive market
  - Take *differentiated* intermediate goods as input
  - Combine them using a production function that exhibits constant elasticity of substitution (CES)
  - $\Rightarrow$ CES demand function for intermediate goods
- Intermediate goods producers:
  - A continuum of firms operates under monopolistic competition
  - Take labor as input
  - produce differeniated intermediate goods
  - Set sale price to maximize discounted stream of profits, taking demand function as given
- Alternative and equivalent setup: only one production layer, but households have preferences over a CES bundle of goods

Final goods producers

- CES production technology

$$Y_t = \left(\int_0^1 Y_{it}^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$$

- Firms take prices as given to solve

$$\max_{\{Y_{it}\}} Y_t P_t - \int_0^1 Y_{it} P_{it} di$$

  where P_t is the price of the final good, and P_{it} are the prices of the intermediate goods

- Problem is static: no interdependence between profits in period t and $t+s$

Final goods producers optimality condition

- An interior solution requires the F.O.C. to hold:

$$P_t \frac{\partial Y_t}{\partial Y_{it}} - P_{it} = 0 \;\; \forall i \in [0,1]$$

with

$$\begin{aligned} \frac{\partial Y_t}{\partial Y_{it}} &= \frac{\epsilon}{\epsilon - 1}\left(\int_0^1 Y_{it}^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}-1} \times \frac{\epsilon-1}{\epsilon} Y_{it}^{\frac{\epsilon-1}{\epsilon}-1} \\ &= \left(\int_0^1 Y_{it}^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{1}{\epsilon-1}} \times Y_{it}^{\frac{-1}{\epsilon}} \\ &= Y_t^{\frac{1}{\epsilon}} Y_{it}^{\frac{-1}{\epsilon}} \end{aligned}$$

- So, F.O.C. can be written

$$Y_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\epsilon} Y_t \tag{1}$$

- $\Rightarrow Y_{it}$ has constant elasticity ϵ w.r.t P_{it}

- (1) is the demand function for intermediate goods Y_{it}

A price index

- The demand function implies that P_t can be interpreted as a price index
- Use the CES aggregator; manipulate and integrate:

$$
\begin{aligned}
& Y_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\epsilon} Y_t \\
\Leftrightarrow \quad & Y_{it}^{\frac{\epsilon-1}{\epsilon}} P_t^{\frac{\epsilon-1}{\epsilon}\times-\epsilon} = P_{it}^{\frac{\epsilon-1}{\epsilon}\times-\epsilon} Y_t^{\frac{\epsilon-1}{\epsilon}} \\
\Leftrightarrow \quad & \left(P_t^{1-\epsilon}\int_0^1 Y_{it}^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} = \left(Y_t^{\frac{\epsilon-1}{\epsilon}}\int_0^1 P_{it}^{1-\epsilon} di\right)^{\frac{\epsilon}{\epsilon-1}} \\
\Leftrightarrow \quad & P_t^{-\epsilon} Y_t = Y_t \left(\int_0^1 P_{it}^{1-\epsilon} di\right)^{\frac{\epsilon}{\epsilon-1}} \\
\Leftrightarrow \quad & P_t = \left(\int_0^1 P_{it}^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}
\end{aligned}
$$

- The price index $P_t = P_t(P_{it})$ is convex $\Rightarrow$ when the dispersion of P_{it} increases, the value of money, $1/P_t$, is lower

Intermediate goods producers

- A continuum of firms, indexed by $i \in [0, 1]$
- Each produce a different good Y_i
- No capital; no TFP shocks; CRS technology in labor:

$$Y_{it} = AN_{it}$$

- Monopolistic producers: set prices P_{it}, taking the demand curve (1) and aggregate variables as given
- We first consider their optimization problem in the case of flexible price setting, then add the pricing friction
- With flexible pricing, the optimization problem is static

Intermediate goods producers with flexible prices I

- Program:

$$\max_{P_{it}} \quad AN_{it}P_{it} - W_t N_{it}$$
$$\text{s.t.} \quad Y_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\epsilon} Y_t$$
$$Y_{it} = AN_{it}$$

- Equivalently:

$$\max_{P_{it}} \quad Y_{it}P_{it} - \Psi(Y_{it})$$
$$\text{s.t.} \quad Y_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\epsilon} Y_t$$

with nominal cost function $\Psi(Y_{it}) = W_t \frac{Y_{it}}{A}$

Intermediate goods producers with flexible prices II

- F.O.C:

$$\frac{\partial Y_{it}}{\partial P_{it}} P_{it} + Y_{it} - \psi_t \frac{\partial Y_{it}}{\partial P_{it}} = 0$$

where $\psi_t \equiv \frac{\partial \Psi(Y_{it})}{\partial Y_{it}} = \frac{W_t}{A}$ is nominal marginal cost

- Compute the derivative and plug in the demand constraint to find:

$$\begin{aligned} 0 &= Y_{it}(P_{it} - M\psi_t) \\ P_{it} &= M\psi_t \end{aligned}$$

where $M = \frac{\epsilon}{\epsilon - 1}$

- With lower elasticity ϵ, firms charge higher markups M

Intermediate goods producers with Calvo prices

- Now we understand the frictionless pricing problem
- Let's introduce the pricing friction, following Calvo (JME 1983)
- In each period, with probability θ, a firm i cannot change its price: $P_{it} = P_{it-1}$
- With probability $1 - \theta$, it can set whatever price it likes; denote its choice with P_t^*
- A price-resetting firm sets the price P_t^* to maximize expected discounted profits, during the time in which P_t^* is in place
- Since the firm is owned by the representative household, it discounts profits in period $t + s$ using the households' stochastic discount factor $M_{t,t+s} = \beta^s E_t \frac{U(C_{t+s})}{U(C_t)}$
- What is the firm program? How to interpret its optimality condition? (Do on Whiteboard)

Log-linearization

- Firm F.O.C. can be rewritten

$$0 = \sum_{s=0}^{\infty} \theta^s E_t \left[Q_{t,t+s} Y_{it+s|t} \left(P_t^* - M \times MC_{t+s} P_{t+s} \right) \right]$$

where $MC_{t+s} = \frac{\psi(Y_{it+s|t})}{P_{t+s}} = \frac{W_{t+s}}{AP_{t+s}}$ is real marginal cost

- In steady state:
  - $\bar{Q}_{t,t+s} = \beta^s$
  - $P^* = P = 1$ (last equality is just a normalization)
  - $P^* = M \times MC \times P$, and therefore $M \times MC = 1$

- Log-linearizing yields (Do this at home!!)

$$\begin{aligned} 0 &= \sum_{s=0}^{\infty} (\beta\theta)^s E_t \left(p_t^* - p_{t+s} - \widehat{mc}_{t+s} \right) \\ &= \sum_{s=0}^{\infty} (\beta\theta)^s E_t \left((p_t^* - p_{t-1}) - (p_{t+s} - p_{t-1}) - \widehat{mc}_{t+s} \right) \end{aligned}$$

or

$$p_t^* - p_{t-1} = (1 - \beta\theta) \sum_{s=0}^{\infty} (\beta\theta)^s E_t \left((p_{t+s} - p_{t-1}) + \widehat{mc}_{t+s} \right)$$

Optimality condition has a recursive structure

- Note that

$$
\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s E_t\left((p_{t+s}-p_{t-1}) + \widehat{mc}_{t+s}\right) \\
&= (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s E_t\left((p_{t+s}-p_t) + (p_t - p_{t-1}) + \widehat{mc}_{t+s}\right) \\
&= (1-\beta\theta)\widehat{mc}_t + \pi_t + \beta\theta(1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s E_t\left((p_{t+1+s}-p_t) + \widehat{mc}_{t+1+s}\right) \\
&= (1-\beta\theta)\widehat{mc}_t + \pi_t + \beta\theta(E_t p_{t+1}^* - p_t)
\end{aligned}
$$

Aggregation

- Calvo pricing implies, in general, a non-degenerate price distribution
- However, because resetting is random, the LOM for P_t can be expressed keeping track only of one moment, P_{t-1}:

$$\begin{aligned} P_t &= \left(\int_0^1 P_{it}^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}} \\ &= \left(\int_{i \in S(t)} P_{it-1}^{1-\epsilon} di + (1-\theta)(P_t^*)^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}} \\ &= \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}} \end{aligned}$$

  where $S(t) \in [0,1]$ is the set of non-resetters, and where we've used the Law of Large Numbers

- Hence

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon} \quad \Rightarrow \pi_t = (1-\theta)(p_t^* - p_{t-1})$$

Firm optimality + price law of motion $\Rightarrow$ Phillips curve

- Log-linear law of motion and firm optimality

$$\begin{aligned} \pi_t &= (1-\theta)(p_t^* - p_{t-1}) \\ p_t^* - p_{t-1} &= (1-\beta\theta)\widehat{mc}_t + \pi_t + \beta\theta(E_t p_{t+1}^* - p_t) \end{aligned}$$

- Together:

$$\pi_t = \beta E_t \pi_{t+1} + \lambda \widehat{mc}_t$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}$

- Phillips curve: firm set higher prices in response to 1) increases in real marginal cost and 2) increases in expected future prices

- Note! Linear production + constant TFP $\Rightarrow \widehat{mc}_t = \hat{w}_t - p_t$

Intermezzo: evidence on price-setting behavior

- Evidence of sticky prices is abundant
- But: is Calvo really a reasonable model of price-setting behavior?
- Alternative view: menu costs
  - Key difference 1: implies state-dependence, not time-dependence
  - Key difference 2: with menu costs, firm selection is endogenous, potentially implying much smaller real effects of monetary shocks (Golosov-Lucas, JPE 2007)
- Need micro data to tell them apart
  - Bils-Klenow (JPE 2004) exploit BLS price data
  - Nakamura-Steinsson (QJE 2008) exploit the CPI and PPI research database
  - Carlsson-Nordström Skans (AER 2013) exploit Swedish administrative data
    - Great advantage: with Swedish data, you can also estimate real costs of production
- My reading: truth seems to be somewhere in the middle

Final ingredient: a policy rule

- We assume the central bank sets the nominal interest rate according to a Taylor rule

$$\frac{1}{Q_t} = (1/\beta) * \Pi_t^{\phi} * \exp(\nu_t)$$

where

$$\nu_t = \rho_\nu \nu_{t-1} + \epsilon_t$$

- We interpret ϵ_t as a monetary policy shock

- This is a so called Taylor rule
  - Following Taylor (JME, 1993), rules of this sort (also with some weight on output) have proven to provide very good approximations of how monetary policy has been practiced in many countries since the 80's/90's

- Log linearizing:

$$\hat{i}_t = \phi \pi_t + \nu_t$$

where, again, $i_t = \log\left(\frac{1}{Q_t}\right)$

Market clearing

- Three markets: Goods, bonds, and labor
- Goods market:

$$C_t = Y_t \quad \Rightarrow \hat{c}_t = \hat{y}_t$$

- Bond market (no government debt):

$$B_t = 0 \quad \Rightarrow \hat{b}_t = 0$$

- Labor market

$$N_t = \int_0^1 N_{it} di \quad \Rightarrow ?$$

Log-linearizng labor market clearing

- Using the production function

$$N_t = \int_0^1 N_{it} di = \int_0^1 \frac{Y_{it}}{A} di$$

- Using the demand function

$$N_t = \int_0^1 \frac{Y_t}{A} \left(\frac{P_{it}}{P_t}\right)^{-\epsilon} di = \frac{Y_t}{A} D_t$$

where

$$D_t = \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\epsilon} di$$

  - D_t is a measure of price disperion, and therefore output dispersion
  - D_t drives a wedge in the aggregate production function; measures the efficiency cost of missallocation
  - One can prove that in a first order approximation, $\hat{d}_t = 0$ (See Galí, Ch. 3 appendix)

- Therefore, the log-linearized labor market clearing condition is simply

$$\hat{n}_t = \hat{y}_t$$

Summing up

- The log-linearized equilibrium is characterized by

$$
\begin{aligned}
\text{Intratemporal hh optimality:} \quad & \hat{\omega}_t = \hat{c}_t + \varphi \hat{n}_t \\
\text{Intertemporal hh optimality:} \quad & \hat{c}_t = -(\hat{i}_t - E_t \pi_{t+1}) + E_t \hat{c}_{t+1} \\
\text{Firm optimality:} \quad & \pi_t = \beta E_t \pi_{t+1} + \lambda \widehat{mc}_t \\
\text{Marginal cost:} \quad & \widehat{mc}_t = \hat{\omega}_t \\
\text{Goods clearing:} \quad & \hat{c}_t = \hat{y}_t \\
\text{Bonds clearing:} \quad & \hat{b}_t = 0 \\
\text{Labor clearing:} \quad & \hat{y}_t = \hat{n}_t \\
\text{Policy:} \quad & \hat{i}_t = \phi \pi_t + \nu_t
\end{aligned}
$$

  where $\hat{\omega}_t = \hat{w}_t - p_t$ is log deivations in the real wage

- 8 equations in 8 unknowns: $\{\hat{\omega}_t, \hat{c}_t, \hat{n}_t, \hat{i}_t, \pi_t, \widehat{mc}_t, \hat{b}_t\}$

- Also, the law of motion for exogenous shocks:

$$\nu_t = \rho_\nu \nu_{t-1} + \epsilon_t$$

Towards the 3-equation representation

- Consider the full system

$$
\begin{aligned}
\text{Intratemporal hh optimality:} \quad & \hat{\omega}_t = \hat{c}_t + \varphi \hat{n}_t \\
\text{Intertemporal hh optimality:} \quad & \hat{c}_t = -(\hat{i}_t - E_t \pi_{t+1}) + E_t \hat{c}_{t+1} \\
\text{Firm optimality:} \quad & \pi_t = \beta E_t \pi_{t+1} + \lambda \widehat{mc}_t \\
\text{Marginal cost:} \quad & \widehat{mc}_t = \hat{\omega}_t \\
\text{Goods clearing:} \quad & \hat{c}_t = \hat{y}_t \\
\text{Bonds clearing:} \quad & \hat{b}_t = 0 \\
\text{Labor clearing:} \quad & \hat{y}_t = \hat{n}_t \\
\text{Policy rule:} \quad & \hat{i}_t = \phi \pi_t + \nu_t
\end{aligned}
$$

- Firm optimality + intratemporal hh optimality + goods clearing + labor clearing produce the Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \hat{y}_t$$

where $\kappa = (1 + \varphi)\lambda$

Towards the 3-equation representation

- The log-linearized equilibrium is characterized by

Intratemporal hh optimality:	$\hat{\omega}_t = \hat{c}_t + \varphi \hat{n}_t$
Intertemporal hh optimality:	$\hat{c}_t = -(\hat{i}_t - E_t \pi_{t+1}) + E_t \hat{c}_{t+1}$
Firm optimality:	$\pi_t = \beta E_t \pi_{t+1} + \lambda \widehat{mc}_t$
Marginal cost:	$\widehat{mc}_t = \hat{\omega}_t$
Goods clearing:	$\hat{c}_t = \hat{y}_t$
Bonds clearing:	$\hat{b}_t = 0$
Labor clearing:	$\hat{y}_t = \hat{n}_t$
Policy rule:	$\hat{i}_t = \phi \pi_t + \nu_t$

- Intertemporal hh optimality + goods clearing produce the Dynamic IS curve

$$\hat{y}_t = -(\hat{i}_t - E_t \pi_{t+1}) + E_t \hat{y}_{t+1}$$

3-equation representation

- The log-linearized equilibrium can be characterized by

$$\begin{aligned} \text{DIS curve:} \quad & \hat{y}_t = -(\hat{i}_t - E_t\pi_{t+1}) + E_t\hat{y}_{t+1} \\ \text{Phillips curve:} \quad & \pi_t = \beta E_t\pi_{t+1} + \kappa\hat{y}_t \\ \text{Policy rule:} \quad & \hat{i}_t = \phi\pi_t + \nu_t \end{aligned}$$

- 3 equations in 3 unknowns: $\{\hat{y}_t, \hat{i}_t, \pi_t\}$!
- This is how the model is usually presented in the literature
- **Warning!:** Although very convenient, these equations mix multiple equilibrium relationships $\Rightarrow$ hard to extract a precise intuition about model mechanisms

Determinacy and Taylor rules

Determinacy

- Our system is

$$\begin{aligned}
\text{DIS curve:} \quad & \hat{y}_t = -(\hat{i}_t - E_t\pi_{t+1}) + E_t\hat{y}_{t+1} \\
\text{Phillips curve:} \quad & \pi_t = \beta E_t\pi_{t+1} + \kappa\hat{y}_t \\
\text{Policy rule:} \quad & \hat{i}_t = \phi\pi_t + \nu_t
\end{aligned}$$

- Before analyzing the response to some shock, we should ask: When is that response a unique solution?

- Recall Blanchard-Kahn condition: There exist a unique bounded solution to a autoregressive linear system of difference equations if and only if the system has the same number of eigenvalues inside the unit circle as the number of has forward-looking variables

- Insert monetary policy rule into DIS to eliminate one variable

$$\begin{aligned}
\text{DIS curve:} \quad & \hat{y}_t = -(\phi\pi_t - E_t\pi_{t+1}) + E_t\hat{y}_{t+1} - \nu_t \\
\text{Phillips curve:} \quad & \pi_t = \beta E_t\pi_{t+1} + \kappa\hat{y}_t
\end{aligned}$$

- How many forward-looking variables do we have here?

Some matrix algebra

- Our system

$$\begin{aligned} \text{DIS curve:} \qquad & \hat{y}_t = -(\phi\pi_t - E_t\pi_{t+1}) + E_t\hat{y}_{t+1} - \nu_t \\ \text{Phillips curve:} \qquad & \pi_t = \beta E_t\pi_{t+1} + \kappa\hat{y}_t \end{aligned}$$

- The system can be written as

$$\mathbf{A}_0\mathbf{x}_t = \mathbf{A}_1 E_t\mathbf{x}_{t+1} + \mathbf{B}_1\nu_t$$

where $\mathbf{x}_t = [\hat{y}_t, \pi_t]'$ and

$$\mathbf{A}_0 = \begin{bmatrix} 1 & \phi \\ -\kappa & 1 \end{bmatrix} \quad \mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 0 & \beta \end{bmatrix} \quad \mathbf{B}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- Rearrange to

$$\mathbf{x}_t = \mathbf{A}E_t\mathbf{x}_{t+1} + \mathbf{B}\nu_t$$

where $\mathbf{A} = \mathbf{A}_0^{-1}\mathbf{A}_1$ and $\mathbf{B} = \mathbf{A}_0^{-1}\mathbf{B}_1$

Some matrix algebra

- Do the algebra to find

$$\mathbf{A} = \Omega \begin{bmatrix} 1 & 1-\beta\phi \\ \kappa & \kappa+\beta \end{bmatrix} \quad \mathbf{B} = -\Omega \begin{bmatrix} 1 \\ \kappa \end{bmatrix}$$

  where $\Omega = \frac{1}{1+\kappa\phi}$

- Reminder: Eigenvalues of **A** given by solution to characteristic equation

$$\begin{aligned} & det(\mathbf{A} - \mathbb{I}\lambda) = 0 \\ \Leftrightarrow \quad & (\Omega - \lambda)(\Omega(\kappa+\beta) - \lambda) - \Omega^2(1-\beta\phi)\kappa = 0 \end{aligned}$$

- Quadratic equation in λ; both eigenvalues of **A** are inside the unit circle if and only if $\phi > 1$ (Bullard-Mitra JME 2002)

- Conversely, if $\phi \leq 1 \Rightarrow$ indeterminacy

- $\Rightarrow$ Unless monetary policy reacts sufficiently strong to inflation, we have multiple bounded equilibra!

How to think about this? Back to flexible prices...

- Following discussion draws heavily on Cochrane (JPE 2011)
- To learn what's going on, let's consider the flex-price model
- Under flexible price, firm optimality says:

$$
\begin{aligned}
P_{it} &= M\psi_t \\
\frac{P_{it}}{P_t} &= M\frac{\psi_t}{P_t} \quad \Rightarrow \widehat{mc}_t = 0
\end{aligned}
$$

- The equilibrium is thus described by

$$
\begin{aligned}
\text{Intratemporal hh optimality:} &\quad \hat{\omega}_t = \hat{c}_t + \varphi \hat{n}_t \\
\text{Intertemporal hh optimality:} &\quad \hat{c}_t = -(\hat{i}_t - E_t\pi_{t+1}) + E_t\hat{c}_{t+1} \\
\text{Firm optimality:} &\quad \widehat{mc}_t = 0 \\
\text{Marginal cost:} &\quad \widehat{mc}_t = \hat{\omega}_t \\
\text{Goods clearing:} &\quad \hat{c}_t = \hat{y}_t \\
\text{Bonds clearing:} &\quad \hat{b}_t = 0 \\
\text{Labor clearing:} &\quad \hat{y}_t = \hat{n}_t \\
\text{Policy rule:} &\quad \hat{i}_t = \phi\pi_t + \nu_t
\end{aligned}
$$

Equilibrium with flexible prices

- The equilibrium:

$$
\begin{aligned}
\text{Intratemporal hh optimality:} \quad & \hat{\omega}_t = \hat{c}_t + \varphi \hat{n}_t \\
\text{Intertemporal hh optimality:} \quad & \hat{c}_t = -(\hat{i}_t - E_t \pi_{t+1}) + E_t \hat{c}_{t+1} \\
\text{Firm optimality:} \quad & \widehat{mc}_t = 0 \\
\text{Marginal cost:} \quad & \widehat{mc}_t = \hat{\omega}_t \\
\text{Goods clearing:} \quad & \hat{c}_t = \hat{y}_t \\
\text{Bonds clearing:} \quad & \hat{b}_t = 0 \\
\text{Labor clearing:} \quad & \hat{y}_t = \hat{n}_t \\
\text{Policy rule:} \quad & \hat{i}_t = \phi \pi_t + \nu_t
\end{aligned}
$$

- Implying $\omega_t = \hat{c}_t = \hat{n}_t = \hat{y}_t = 0$ (monetary neutrality!)

- System reduces to

$$
\begin{aligned}
\text{DIS curve:} \quad & \hat{i}_t = E_t \pi_{t+1} \\
\text{Policy rule:} \quad & \hat{i}_t = \phi \pi_t + \nu_t
\end{aligned}
$$

or

$$E_t \pi_{t+1} = \phi \pi_t + \nu_t$$

Determinacy with $\phi > 1$

- Q: What bounded sequence of $\{\pi_t\}$ solves

$$E_t \pi_{t+1} = \phi \pi_t + \nu_t?$$

- Iterating forward, we have that

$$\pi_t = -\sum_{s=0}^{T} \frac{1}{\phi^{s+1}} \nu_{t+s} + \frac{1}{\phi^{T+1}} \mathbb{E}_t \pi_{t+T+1}$$

- Suppose $\phi > 1$, then there is the unique solution

$$\pi_t = -\sum_{s=0}^{\infty} \frac{1}{\phi^{s+1}} \nu_{t+s}$$

Indeterminacy with $\phi < 1$

- Now suppose $\phi < 1$, and consider our equilibrium condition

$$E_t \pi_{t+1} = \phi \pi_t + \nu_t$$

- Consider the sequence

$$\pi_0 = 3.2$$
$$\pi_{t+1} = \phi \pi_t + \nu_t \text{ if } t > 0$$

  This sequence is bounded and satisfies our condition, and is thus a bounded solution.

- More generally, any sequence that satisfies

$$\pi_{t+1} = \phi \pi_t + \nu_t + \delta_{it+1}$$

  with $E_t \delta_{it+1} = 0$ is a bounded solution

- The δ_{it+1} are sometimes referred to as sunspots

What's going on?

- $\phi > 1$ implies that if inflation happens to be "wrong", the policy rule will ensure that inflation grows without bound
  - Household optimality implies Fisher equation: $i_t = r + E_t \pi_{t+1}$
  - If inflation happens to be "too large" $\Rightarrow$ central bank raises interest rate $\Rightarrow$ inflation tomorrow rise
  - Put differently, the central bank "threats" with explosive dynamics
- In a model where inflation has no real effects, this is a completely ad hoc theory of inflation
  - Sure, we can restrict attention to bounded equilibria, but what is the economic argument to that?
- In a model where inflation has real effects, a transversality condition would eventually be broken, so it works logically
- Still very unrealistic - if we woke up and inflation was 20%, would you think the central bank would try to raise it to 200%?
- Very much in contrast with the "layman's" interpretation of Taylor rules, emphasizing stability, not determinacy:
  - Taylor's original interpretation: $\phi > 1$ implies real interest rate $r_t \approx i_t - \pi_t = (\phi - 1)\pi_t$ reacts to inflation $\Rightarrow$ stabilizes inflationary shocks

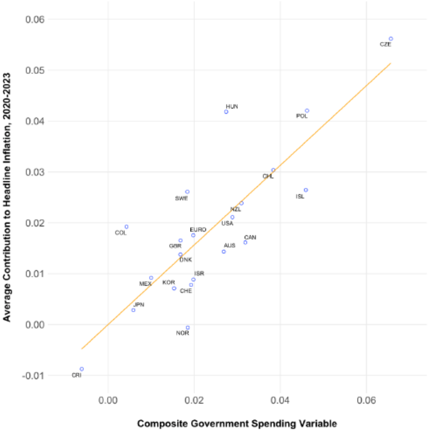
The Taylor rule theory of the Price level

- Taylor rule theory of the price level: determination by threats of explosive dynamics
- Leading theory in applied models, but foundations seem very shaky
- The possibility of explosive dynamics stems from the extremely forward-looking behavior in the model
  - If any variable jumps, agents update their expectations about the equilibrium path immediately
- $\Rightarrow$ much work has explored determinacy properties with alternative expectation formation processes, e.g., least-square learning
- See, e.g., Woodford (AnnualRE, 2013), Angeletos-Huo (AER 2021)

Alternative theories of the price level

- Back in the days: Quantity theory
  - See, e.g., Mitlon Friedman's AEA presidential address (AER 1968)
  - Central assertion: velocity v semi-exogenous in the formula
    $$YP = vM$$
  - Problem: v is highly endogenous to the interest rate (at $i = 0$, bonds are perfect subsitutes to money) and M is not controllable anymore
- Main competitor today: Fiscal Theory of the Price Level
  - See, e.g., Chris Sims' AEA presidential address (AER 2013)
  - Central assertion: B_t predetermined in government budget equation
    $$\frac{B_t}{P_t} = \sum_{s=0}^{\infty} Q_{t,t+s} S_{t+s}$$
    where S_{t+s} is the real government surplus stream
- Exciting research area!

FTPL explains cross-country inflation heterogeneity during COVID crisis?



Note: The sample is 2010-2023 for 21 economies (20 non-Euro zone and the Euro zone considered as an aggregate). The vertical axis has the average headline CPI inflation rate from 2020 to 2023, net of the estimated effects from the border dummy and the fixed effects for country and year. The horizontal axis has the composite government-spending variable: the ratio of general government primary spending to GDP (cumulation for 2020 and 2021 relative to that for 2019) divided by the ratio of gross public debt to GDP in 2019 and by the estimated duration of the public debt in 2019. The slope of the orange line equals the arithmetic average of the estimated coefficients on composite government spending from 2020 to 2023 (from Table A1).

- From Barro-Bianchi (2024)

Summing up

- Abundant evidence of monetary non-neutrality; In response to positive monetary policy shocks, real activity falls
- Basic NK model = RBC model + monopolistic competition and sticky prices + Taylor rule
- Taylor rule theory of inflation: inflation is bounded due to central bank making explosive threats
- Next class: Analysis of NK model's predictions in response to shocks