Aggregate Labor Market Dynamics
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Problem set 1
Due on February 18, 2016
Suggested solution

1 Endogenous separations in the DMP model

In the DMP model we discussed in class, we assumed that all separations are exogenous. In this problem set we will endogenize the job destruction decision by introducing a richer structure of productivity shocks. At some of the idiosyncratic productivities the match is profitable, but at some others it is not. A firm will now choose a reservation productivity at which it will terminate the match.

The setup is the same as the DMP model we studied in class, with one twist. Assume that a worker-firm match produces yx units of output where x is match-specific productivity and y is aggregate productivity. We call x *match-specific*, because workers and firms do not carry this productivity across matches, unmatched agents are all homogenous. We assume that shocks to x arrive at a Poisson rate λ, independent over time and across matches, and are drawn from distribution $G(x)$ with the support $[0,1]$. For simplicity, we assume that new jobs start with the highest productivity $x=1$. The rest of the setup is the same as we had in class. Firms and workers are risk-neutral and discount future at the rate ρ. Unemployed workers get flow benefit z while employed receive a wage $w(x)$. The wage is determined by Nash bargaining, where workers' bargaining power is γ. There is a free entry of firms, and the flow cost of posting a vacancy is c. Vacancies and unemployed workers match in a market where the total matches is described by a matching function $m(U,V)$. As before, use $p(\theta), q(\theta)$ to denote a probability that an unemployed worker meets a vacancy, and that a vacancy meets a worker, respectively.

Question 1.1 Let $W(x), U, J(x), V$ be value functions for an employed and unemployed worker, and filled and unfilled jobs. Argue that $J(x), W(x)$ are increasing in x.

Question 1.2 Job destruction will happen when the value of a job falls below that of a vacancy, $V=0$. Since $J(x)$ is increasing in x, the decision to separate is represented by a reservation productivity R below which jobs are destroyed. Taking R as given, formulate the Bellman equations, focusing on a steady-state analysis.

Answer

$$\begin{aligned}
\rho J(x) &= yx - w(x) + \lambda \int_R^1 \left(J(x') - J(x)\right) dG(x') + \lambda \int_0^R (0 - J(x))\, dG(x') \\
\rho V &= -c + q(\theta)\left[J(1) - V\right] \\
\rho W(x) &= w(x) + \lambda \int_R^1 \left(W(x') - W(x)\right) dG(x') + \lambda \int_0^R (U - W(x))\, dG(x') \\
\rho U &= z + p(\theta)\left[W(1) - U\right]
\end{aligned}$$

We can rewrite them as

$$\begin{aligned}
\rho J(x) &= yx - w(x) + \lambda \int_R^1 \left(J(x') - J(x)\right) dG(x') - \lambda J(x) \\
\rho V &= -c + q(\theta)\left[J(1) - V\right] \\
\rho W(x) &= w(x) + \lambda \int_R^1 W(x')\, dG(x') + \lambda G(R)\, U - \lambda W(x) \\
\rho U &= z + p(\theta)\left[W(1) - U\right]
\end{aligned}$$

■

Question 1.3 The wage is determined by Nash bargaining. We make the additional assumption that renegotiation takes place when a new idiosyncratic productivity x is drawn. Show that it leads to the same surplus splitting rule as in class:

$$\begin{aligned}
W(x) - U &= \gamma S(x) \\
J(x) - V &= (1-\gamma) S(x)
\end{aligned}$$

Answer The setup,

$$w^*(x) = \arg\max \left(W(w;x) - U\right)^{\gamma} \left(J(w;x) - V\right)^{1-\gamma}$$

Taking logs and then FOC, then

$$\gamma \frac{\partial W(w;x)/\partial w}{\left(W(w;x) - U\right)} = (1-\gamma) \frac{\partial J(w;x)/\partial w}{\left(J(w;x) - V\right)}.$$

Use that $\partial W(w;x)/\partial w = -\partial J(w;x)/\partial w$ and the definition of surplus, $S(x) = W(x) + J(x) - U - V$ to get the result. ■

Question 1.4 Find an expression for the wage $w(x)$ as a function of parameters of the model. To do it, combine value functions and surplus splitting rules. You should get an expression which is analogous to the one in class.

Answer Combining all value functions, we can find an expression a value function for the surplus

$$(\rho + \lambda) S(x) = yx - z + \lambda \int_R^1 S(x') \, dG(x') - p(\theta) \gamma S(1). \tag{1}$$

Now, subtracting $W(x)$ and U, we can find

$$(\rho + \lambda)(W(x) - U) = w(x) - z + \lambda \int_R^1 \left(W(x') - U\right) dG(x') - p(\theta)[W(1) - U]. \tag{2}$$

The surplus splitting rule implies $W(x) - U = \gamma S(x)$. Hence, multiply (1) and subtract (2) to get

$$\begin{aligned} 0 &= \gamma(yx - z) - (w(x) - z) \\ &\quad + p(\theta)[W(1) - U] - \gamma p(\theta) \gamma S(1) \end{aligned}$$

The integrals drop out due to surplus splitting rule. Using the surplus splitting again to manipulate the second line,

$$\gamma S(1) = W(1) - U = \gamma/(1-\gamma)[J(1) - V].$$

After substitution, we get

$$0 = \gamma(yx - z) - (w(x) - z) + p(\theta) \frac{\gamma}{1-\gamma} [J(1) - V].$$

Finally, substituting that $J(1) - V = c/q(\theta)$ we get the desired expression for the wage,

$$w(x) = \gamma yx + (1-\gamma) z + \gamma c\theta.$$

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Question 1.5 Write down a Bellman equation for the total surplus of the job, $S(x)$.

Answer We have

$$\begin{aligned}\rho J(x)+\rho W(x) &= yx+\lambda\int_R^1 \left(J\left(x'\right)+W\left(x'\right)\right)dG\left(x'\right)-\lambda\left(J(x)+W(x)\right)+\lambda G(R)U \\ \rho V+\rho U &= -c+q(\theta)\left[J(1)-V\right]+z+p(\theta)\left[W(1)-U\right] \\ \rho S(x) &= yx-z+\lambda\int_R^1 S\left(x'\right)dG\left(x'\right)-\lambda S(x)-p(\theta)\gamma S(1)\end{aligned}$$

Here the surplus depends on γ. ■

Question 1.6 Write down the law of motion for unemployment. Find the steady-state unemployment rate.

Answer It is

$$\dot{u}=\lambda G(R)(1-u)-p(\theta)u$$

In the steady state,

$$u^*=\frac{\lambda G(R)}{\lambda G(R)+p(\theta)}.$$

■

Question 1.7 Now we need to determine R. Argue that R is such that $S(R)=0$. Further, argue that all separations are mutually agreeable, meaning that a worker and a firm agree to terminate the match.

Answer The reservation productivity is such that a firm wants to separate, $J(R)-V=0$. From the surplus splitting rule it also follows that $S(R)=0$. Finally, using the fact that $W(R)-U=\gamma S(R)=0$, it also follows that the worker wants to separate at R, and hence the separations are mutually agreeable. ■

Question 1.8 Use the value function for $S(x)$ to find a value of $S(R)$. Use that $S(R)=0$ to find an equation for R.

Answer We have

$$\begin{aligned}\rho S(R) &= yR-z+\lambda\int_R^1 S\left(x'\right)dG\left(x'\right)-\lambda S(R)-p(\theta)\gamma S(1) \\ yR &= z-\lambda\int_R^1 S\left(x'\right)dG\left(x'\right)+p(\theta)\gamma S(1)\end{aligned}$$

■

Question 1.9 Use the expression for R and the value function for $S(x)$ to solve for $S(x)$.

You should get

$$S(x) = \frac{y(x-R)}{\rho+\lambda}. \tag{3}$$

Answer The value function

$$(\rho+\lambda) S(x) = yx - z + \lambda \int_R^1 S(x') \, dG(x') - p(\theta)\gamma S(1).$$

For $x = R$, we get

$$\begin{aligned}
(\rho+\lambda) S(R) &= 0 = yR - z + \lambda \int_R^1 S(x') \, dG(x') - p(\theta)\gamma S(1) \\
-yR &= -z + \lambda \int_R^1 S(x') \, dG(x') - p(\theta)\gamma S(1)
\end{aligned}$$

Substitute into the value function

$$\begin{aligned}
(\rho+\lambda) S(x) &= yx - yR \\
S(x) &= \frac{y(x-R)}{\rho+\lambda}
\end{aligned}$$

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Question 1.10 Combine the value for $S(x)$, the surplus splitting rule and the value function for $J(x)$ to find a new job creation curve,

$$\frac{c}{q(\theta)} = (1-\gamma)\, y \frac{1-R}{\rho+\lambda}.$$

Answer We know that $J(x) - V = (1-\gamma) S(x)$, and $J(x) = c/q(\theta)$, hence we get the above JC curve. ■

Question 1.11 Next, we will derive a job destruction curve which determines how firm choose the reservation productivity R. Substitute the expression for $S(x)$ from (3) into the integral term in the value function for $S(x)$, and evaluate it at $x = R$. You should get a condition which contains R,θ and then only parameters of the model.

Answer First notice that

$$\begin{aligned}
S(1) &= \frac{1}{1-\gamma} J(1) = \frac{1}{1-\gamma}\frac{c}{q(\theta)} = \frac{1}{1-\gamma}\frac{c\theta}{p(\theta)} \\
p(\theta) S(1) &= \frac{1}{1-\gamma} c\theta
\end{aligned}$$

Using (3) and the value function, together with the above expression,

$$
\begin{aligned}
0 &= yR - z + \lambda \int_R^1 \frac{y\left(x' - R\right)}{\rho + \lambda} dG\left(x'\right) - p\left(\theta\right)\gamma S\left(1\right) \\
0 &= yR - z + \frac{\lambda}{\rho + \lambda} y \int_R^1 \left(x' - R\right) dG\left(x'\right) - \theta c \frac{\gamma}{1 - \gamma}
\end{aligned}
$$

■

Question 1.12 The equilibrium with endogenous separations is determined by three equations in three unknowns (θ, R, u)

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\begin{aligned}
[JC] &: \frac{c}{q\left(\theta\right)} = \left(1 - \gamma\right) y \frac{1 - R}{\rho + \lambda} \\
[JD] &: 0 = yR - z + \frac{\lambda}{\rho + \lambda} y \int_R^1 \left(x' - R\right) dG\left(x'\right) - c\theta \frac{\gamma}{1 - \gamma} \\
[BC] &: u = \frac{\lambda G\left(R\right)}{\lambda G\left(R\right) + p\left(\theta\right)}
\end{aligned}
$$

Sketch two graphs. One with JC and JD curve in the (θ, R) space, with θ on the horizontal axis. Another one with BC and JC in the (u, v) space, with u on the horizontal axis. When drawing BC, assume that R is given.

Answer In the $\theta - R$ space, JC is decreasing, JD increasing. In the $u - v$ space, the BC is decreasing if we take R as a given constant, and JC is a ray from origin with an angle θ. ■

Question 1.13 Use the figures above to examine how the steady state values of $R, w\left(x\right), \theta, u, v$ change if y increases.

Answer After a positive productivity shock, JC moves up to the right, JD moves down to the right. The market tightness increases but R can go either way. If we assumed that vacancy posting costs c is proportional to aggregate productivity y, then JC would not contain y and R would decrease unambiguously. We will further assume that parameters of the model are such that R decreases when y increases. In the $u - v$ diagram, JC rotates upward and BC shifts down because R decreased. Hence, u declines but the effect on v is again ambiguous. ■

Question 1.14 Let's now think about out-of-steady-state dynamics in this model. Assume that unprofitable jobs can be shut down at any time without delay. Argue that R, θ and $w\left(x\right)$ are jump variables, and that they must be at their steady state values at all times.

Answer Notice that JC and JD equations do not depend on the "sticky" variable u. Hence, these variables immediately jump to their new steady state values after any unexpected parameter changes or for any initial unemployment rate u_0. The same argument holds for $w(x)$ - it depends on R and θ, but not on u. We get this result because the value of a newly created job is an increasing function of θ, ■

Question 1.15 Consider now dynamic adjustment to the new steady state after a *negative* aggregate productivity shock y. Argue that i) there will be a spike in the job destruction rate, ii) there will be a discrete jump in the unemployment rate after the shock.

Answer After the aggregate productivity decreases to y', the reservation productivity increases to $R' > R$. Thus, the mass of existing matches $\frac{G(R)-G(R')}{1-G(R)}(1-u)$ will be destroyed immediately after the shock. Workers become unemployed, hence the unemployment rate jumps. ■

Question 1.16 Does a model with endogenous job destruction rate imply a slower convergence to a new steady state after a negative aggregate productivity shock that the standard DMP? Suppose you calibrated the model with exogenous separations and the model with endogenous separations using the same moments, so that the steady state (before the unexpected shock) is the same.

Answer Think about the law of motion for unemployment. The only difference between this model and the standard DMP is in the separation rate, $\lambda G(R)$ instead of δ. If we calibrated these two models to the same steady state, we would have the same steady state separation rate, $\lambda G(R) = \delta$. After a negative productivity shock, the separation rate decreases to $\lambda G(R')$, and hence the adjustment to the new steady state will be slower here. ■

Question 1.17 This model implies an asymmetric response of the unemployment rate and job destruction to aggregate productivity shock – why? Think about how unemployment and job destruction respond after a positive aggregate productivity shock y and compare it to the response after a negative productivity shock.

Answer Consider a positive productivity shock $y' > y$. The reservation productivity decreases to $R' < R$. This does not have any immediate effect on existing matches because there are no matches with productivity below R'. There will be no jump in the unemployment rate, it will continuously adjust to the new level. This is different that the effect of a negative productivity shock where we observed spikes. ■

Question 1.18 What are the benefits of having endogenous separations in the model? Is this version of the model helping us explain some data better than the model with exogenous separations?

> **Answer** Job creation and job destruction co-exit; there is a BC component to the job destruction which we would not get with the exogenous separation rate. The asymmetric responses of the unemployment rate is also a good feature. ■

Question 1.19 Is the equilibrium efficient? Does Hosios condition continue to hold here? *HINT*: Write down the objective function for the social planner as

$$\max_{\theta_t, R_t} \int_0^\infty e^{-\rho t} \left(y\bar{X}_t + u_t z - c\theta_t u_t \right) dt$$

where $y\bar{X}_t$ is the total output of employed workers. Write down the law of motion for $\bar{X}_t$, which depends on θ and R. Treat $\bar{X}_t$ and u_t as state variables, formulate the current value Hamiltonian with μ_1 and μ_2 as co-states, and derive the optimality conditions. Then proceed as in class to derive the Hosios condition. Since we are interested in the steady state analysis, you can assume that the co-states are constant over time, $\dot{\mu}_1 = \dot{\mu}_2 = 0$.

> **Answer** Let's first write down the law of motion for the average productivity. Suppose that X_t is the

$$\begin{aligned} X_{t+\Delta t} &= X_t + p(\theta_t)(\Delta t) u_t \\ &+ \lambda \Delta t (1 - u_t) \int_R^1 \left(x' - \frac{\bar{X}_t}{1 - u_t} \right) dG(x') \\ &+ \lambda \Delta t (1 - u_t) \int_0^R \left(0 - \frac{\bar{X}_t}{1 - u_t} \right) dG(x') \end{aligned}$$

> The second term on the right of the first line shows new job creation when all workers start with match-specific productivity 1. The second line shows the fraction of workers who are hit by an idiosyncratic shock above R hence continue working. However, they lose their previous productivity– since workers are drawn randomly from the pool of existing matches, the expected productivity of this randomly drawn worker is just the average productivity, $\bar{X}_t / (1 - U_t)$. The third line considers the case when new productivity is below R and the match breaks down. Subtracting X_t and dividing by Δt yields

$$\dot{X}_t = p(\theta_t) u_t + \lambda (1 - u_t) \int_R^1 x' dG(x') - \lambda X_t$$

Then the social planner solves

$$\max_{\theta_t, R_t} \int_0^\infty e^{-\rho t} \left(y\bar{X}_t + u_t z - c\theta_t u_t \right) dt$$

$$\begin{aligned} s.t. \quad \dot{u}_t &= \lambda G(R)(1-u_t) - p(\theta_t) u_t \\ \dot{X}_t &= p(\theta_t) u_t + \lambda (1-u_t) \int_R^1 x' dG(x') - \lambda X_t \end{aligned}$$

The present value Hamiltonian is

$$\begin{aligned} H &= y\bar{X} + uz - c\theta u - \mu_1 \left[\lambda G(R)(1-u) - p(\theta) u \right] \\ &\quad + \mu_2 \left[p(\theta) u + \lambda (1-u) \int_R^1 x' dG(x') - \lambda \bar{X} \right] \end{aligned}$$

The optimality conditions are

$$\begin{aligned} \frac{\partial H}{\partial \mu_1} &= -\dot{u} & \qquad \frac{\partial H}{\partial \mu_2} &= \dot{X} \\ \frac{\partial H}{\partial u} &= \dot{\mu}_1 - \rho\mu_1 & \qquad \frac{\partial H}{\partial \bar{X}} &= -\dot{\mu}_2 + \rho\mu_2 \\ \frac{\partial H}{\partial \theta} &= 0 & \qquad \frac{\partial H}{\partial R} &= 0 \end{aligned}$$

where I defined costate μ_1 with the opposite sign, as in class. The FOCs are:

$$\dot{\mu}_1 - \rho\mu_1 = z - c\theta - \mu_1 \left[-\lambda G(R) - p(\theta) \right] + \mu_2 \left[p(\theta) - \lambda \int_R^1 x' dG(x') \right] \tag{4}$$

$$-\dot{\mu}_2 + \rho\mu_2 = y - \lambda\mu_2 \tag{5}$$

$$0 = -cu + \mu_1 p'(\theta) u + \mu_2 p'(\theta) u \tag{6}$$

$$0 = \mu_1 \lambda G'(R)(1-u) - \mu_2 \lambda (1-u) R dG(R) \tag{7}$$

Consider the steady state where $\dot{\mu}_1 = \dot{\mu}_2 = 0$. Then equations (5), (6) and (7) become

$$\mu_2 = \frac{y}{\rho + \lambda} \tag{8}$$

$$c = p'(\theta)(\mu_1 + \mu_2) \tag{9}$$

$$R = -\frac{\mu_1}{\mu_2} \tag{10}$$

Using results from class on the elasticity of the matching function $\varepsilon(\theta)$,

$$\begin{aligned} p'(\theta) &= q(\theta)(1 - \varepsilon(\theta)) \\ p(\theta) &= \theta q(\theta), \end{aligned}$$

and combining it with (8), (9), (10), we get a job creation condition:

$$\frac{c}{q(\theta)} = (1 - \varepsilon(\theta))\, y \frac{1-R}{\lambda+\rho}. \tag{11}$$

Finally, use (4), collect terms containing μ_1 and use the trick from class to write the term: $p(\theta) = (1-\varepsilon(\theta))\, p(\theta) + \varepsilon(\theta)\, p(\theta)$, use the fact that $p(\theta)(1-\varepsilon(\theta)) = \theta p'(\theta)$ so that you can use (9). You should getThus,

$$-\mu_1 (\rho + \lambda G(R) + \varepsilon(\theta)\, p(\theta)) = z + \varepsilon(\theta)\, p(\theta)\, \mu_2 - \mu_2 \lambda \int_R^1 x' dG(x').$$

Now eliminate μ_1 and μ_2 using that $\mu_1 = -R\mu_2$ and $\mu_2 = y/(\rho+\lambda)$ to arrive to a job destruction equation

$$yR - z + \frac{\lambda}{\rho+\lambda} y \int_R^1 (x' - R)\, dG(x') - \frac{\varepsilon(\theta)}{1-\varepsilon(\theta)} \theta c = 0.$$

Comparing JC and JD for the social planner and competitive equilibrium, we arrive again to the Hosios condition, $\gamma = \varepsilon(\theta)$. ■