INTERNAL AND EXTERNAL ADJUSTMENT COSTS

IN THE THEORY OF FIXED INVESTMENT

(lecture notes for Investment Theory

for First Year Graduate Macroeconomics)

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Introduction

In these notes I present three simple models of investment. I start with the neoclassical model of Jorgenson (1963). I show that this model has some undesirable features such as extremely large investment rates when there are small changes in economic conditions. I then introduce internal adjustment costs and show the dynamic responses of the economy to unanticipated and anticipated changes in economic conditions. The prediction of this model is that the shadow price (marginal q) is a sufficient statistic for investment. I then show the conditions under which this important shadow price is equivalent to some observable variable. Following Hayashi (1982), I show that if the adjustment cost technology displays constant returns to scale, the capital goods are homogeneous and the stock market is efficient, then the unobservable marginal q is equal to the observable average or Tobin's q.

I next present a model of investment with external adjustment costs. I argue that this alternative approach is more relevant for certain types of investment such as residential investment. I also show the dynamic responses to unanticipated and anticipated changes in the economic environment.

Finally, I argue that the models of internal and external adjustment costs are very similar and the apparent differences are due to assumptions other than the fact that the costs are internal rather than external to the firm.

(1) The Neoclassical Model of Investment.

In the neoclassical model described by Jorgenson (1963 and 1967) and others, firms are assumed to produce output using two inputs, labor (L_t) and capital (K_t), and sell at price p_t. Labor services are hired in a

spot market at a wage w_t and the capital stock is the sum of previous gross investments minus depreciations. It is assumed that firms can buy and sell (invest and disinvest) [1] unlimited amounts of capital at a constant price p_{It} .[2] In order to simplify the initial analysis, it will be convenient to start by assuming that the slope of

the transformation technology is one. This is equivalent to saying that capital and output are the same good (so their relative price is one).

We will also assume that the degree of capital utilization is not a choice variable for firms (so all capital available is used at all points in time). Technology is such that capital wears out at a constant rate, O (capital goods that "break" with the passage of time). Before machines break, however, they are all equally productive. In other words, "modern" machines are not more productive than older ones.[3]

The term A_t can be thought of as some productivity parameter that can be interpreted as the level of

technology. It can also be interpreted as the share of income that firms keep after taxes (that is, define $A_t = 1 - \tau_t$, where O is the tax rate on output). The functional form $F()$ will be assumed to be

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[1] We assume that investment is not irreversible so, at any point in time, firms can decrease their capital stock or disinvest.

[2] In other words, firms can transform their output into capital inputs by using a linear transformation technology with slope p_t/p_{It}.

[3] In a more general model we could have firms choosing how intensively they want to use the available units of capital (which would imply faster depreciation rates for capital used more intensively). We could also have the productivity of capital goods differ depending on the time when they were purchased (or "vintage" to which they belong). This kind of model would be called "vintage capital models".

neoclassical. That is, $F()$ is assumed to exhibit constant returns to scale (where scale is defined as K and L),

to be concave, twice differentiable, and to satisfy the Inada conditions:

(1.0) $\quad Y_t = A_t F(K_t, L_t)$

$\quad$ (A) $\quad F(\beta K, \beta L) = \beta F(K, L)$ $\quad$ (constant returns to scale)

$\quad$ (B) $\quad F K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0, \left(F_{KK} F_{LL} - (F_{KL})^2 \right) > 0$ $\quad$ (positive but

diminishing marginal products of capital and labor and overall concavity)

$\quad$ (C) $\quad \lim_{K \to 0} F_K(K, L) = \infty$ $\quad$ and $\quad \lim_{K \to \infty} F_K(K, L) = 0$ $\quad$ (Inada Conditions)

$\quad$ It is interesting to note that if these three properties apply, then capital and labor are essential in the

sense that no output can be produced without some positive amount of each of the two inputs.[4] That

is: $F(0, L) = 0$ $\quad$ and $\quad F(K, 0) = 0$.

$\quad$ Firms choose, labor, L_t, investment, I_t, and capital, K_t , so as to maximize the present value of all future

cash flows taking the constant interest rate as given.

(1.1) $\quad V = \int_0^\infty e^{-rt} \left[P_t A_t F(K_t, L_t) - w_t L_t - P_t I_t \right] dt$

subject to the accumulation constraint

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[4] See Barro and Sala-i-Martin (1995) for a derivation of this result.

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(1.2) $\dot{K}_t = I_t$

where we simplify the analysis by assuming that the depreciation rate is zero (we can think, alternatively, that

F() is output net of depreciation. We assume that the firm inherits some positive amount of capital at time

zero:

(1.3) $K_0 > 0$ given.

We also impose a terminal condition known as a non-ponzi condition that requires that, in the limit of the

planning horizon, the present value of assets cannot be negative $\lim_{t \to \infty} e^{-rt} K_t \geq 0$.

In order to solve the problem above we can form the following Hamiltonian

(1.4) $H() = e^{-rt}\left(p_t A_t F(K_t, L_t) - w_t L_t - p_{It} I_t \right) + \lambda_t I_t$

where Q_t is the shadow price of investment at time t. It represents the contribution of a unit of capital at time t

to the value of the firm as measured at time zero (the contribution of the constraint to the objective function).

Because it represents the (shadow) value of a time-t machine as measured at time zero, this is called the

present value shadow price. This cumbersome shadow price can be easily transformed into a "current value

shadow price" (the value of a time-t machine as measured at that same moment in time) by simply multiplying

by the current value factor, e^{rt} . That is, we define the current value shadow price as $q_t = e^{rt} \lambda_t$. For

practical purposes, it is important to remember that the first order conditions have to be derived first using

λ_t . That is, the first order conditions to this problem are $H_L = 0$, $H_I = 0$,

$-\dot{\lambda} = H_K$ and $\lim_{t \to \infty} \lambda_t K_t = 0$. Where H_x represents the partial derivative of H with respect to x and a

dot on top of a variable represents the partial derivative of that variable with respect to time. Once the FOC

have been derived using λ_t they can be rewritten using q_t . If we want the FOC to be expressed in

terms of the current value shadow price, we will have to take into account the relation betwen the two. In our

particular case the FOC are

(1.6) FOC with respect to *L*: $A F_L = \dfrac{w}{p}$

(1.7) w.r.t. *I*: $-e^{-rt} p_{It} + \lambda_t = 0 \quad \to \quad -p_t + q_t = 0$

(1.8) w.r.t. *K*: $-\dot{\lambda} = e^{-rt} (pAF_K) \quad \to \quad -\dot{q} e^{-rt} + rq e^{-rt} = e^{-rt} (pAF_K)$

(1.9) $\lim_{t \to \infty} \lambda_t K_t = 0 \quad \to \quad \lim_{t \to \infty} e^{-rt} q_t K_t = 0$

where time subscripts have been omitted when no ambiguities arise. Condition (1.6) just says that the firm will

hire labor up to the point where its marginal product equals the real wage. It corresponds to a labor demand

equation relating labor as a negative function of the wage rate. Condition (1.7) says that the current (shadow)

value of a unit of investment is worth exactly its cost, p_{It} . Notice that this seems a sensible investment rule:

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if the marginal revenue from an extra unit of capital that costs one is larger than one, firms will buy it (so they

will invest a positive amount). They will keep investing until the market cost of an extra unit of capital, P_{It} ,

exactly equals the marginal revenue, q_t . By appropriate choice of numeraire we can set $P_{It} = 1$ for all

periods. Notice that this, together with (1.7) implies that $\dot{q}$ is zero all the time.

Condition (1.8) can be rewritten as

(1.8)' $A\, F_K = r$

which is the condition we always get in static models of the firm. In words it says that firms invest (purchase

capital) up to the point where the marginal product of capital equals the return on alternative assets, the real

interest rate. Thus, at the optimum, they are indifferent between purchasing one more unit of capital that yields

the marginal product and purchasing a bond that yields r.

The final optimality condition, called Transversality Condition (TVC), can be interpreted in terms of the

complementary slackness condition in the Kuhn Tucker theorem (see the mathematical appendix of Barro

and Sala-i-Martin (1995) for a derivation of this result). To get an economic intuition, it will be useful to

consider a finite horizon version of the problem. Suppose for a second that the firm has a planning horizon of

T periods (after period , $T < \infty$, the firm drops dead). The TVC in this case would

be $e^{-rT}\, q_T\, K_T = 0$. That is, the present value of the amount of capital left in the "last period" times its

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shadow price must be zero. This means that if, for any reason, the quantity that firms choose to leave after they "die" at time T is positive, it must be the case that its value (the present value of its marginal contribution to profits) is zero. And vice versa, if the value is positive, then it must be the case that firms choose not to leave any amount of capital. The TVC in (1.9) can be thought as the infinite horizon version of the condition I just described, with "final period" being now infinity.[5]

Notice that the neoclassical model just described says that firms will keep hiring capital and labor until their marginal products are always equal to the relative prices . Of course this implies that discrete changes in w or r will trigger discrete changes in L and K. The constraint (1.2), however, says that investment is the derivative of K with respect to time. If the capital stock "jumps" instantaneously, this means that the derivative at that instant is infinity. It follows that any discrete change in r will trigger INFINITELY positive or negative investment rates (investment is the slope of the path of capital which is vertical whenever there is a jump). Whenever interest rates do not change, the investment rate will be zero. In other words, we will only observe either zero or plus or minus infinity investment rates. Hence, this particular model does not predict that investment is a negative function of the interest rate, which is what economists like to use (in fact, the investment function is not a well defined function).[6]

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[5] This intuition is correct for the particular case we deal with in this lecture notes. In general, the intuition behind the TVC may be a bit more complicated. See the mathematical appendix of Barro and Sala-i-Martin (1995).

[6] One might be tempted to blame this cumbersome result to the fact that we are dealing with a continuous-time model. This is only partly true. If time were discrete, investment would not be infinity. But it would still be the case that firms want to do all their investment in the first period after the change in interest rates or level of technology. That is, if the period of analysis is a week or a month or a quarter, all investment would take place in the first week, month or quarter and nothing would happen in the following period. Hence, the volatility of investment would still be predicted to be extremely large.

A second type of problem with the neoclassical model arises when we consider heterogeneous firms with different production functions. If all firms face the same interest rate but different marginal product of capital (MPK), all investment in the economy will take place in the firm with the highest MPK. A similar type of situation arises when we consider the world economy where all countries face the same "world real interest rate" but different countries have different levels of capital (so the poorest country has the highest MPK). If capital is free to move across borders, the neoclassical model of investment predicts that ALL the investment in the world will take place in the poorest country. This is clearly a counterfactual implication of the model.

A third potential source of unrealistic behavior is that current investment is independent of future marginal products of capital. Recall that the equalization of marginal product to interest rate yields the **desired** level of capital and that investment is then equal to the difference between the existing and the desired capital stocks. Hence, investment is a function of both the existing capital stock and the real interest rate, but is independent of future marginal products of capital. If firms know that the marginal product will increase at some point t_1 in the future, their best strategy is not to do anything until that moment arrives at which point they will discretely increase the amount of capital to the new desired level. In other words, because firms can discretely get the desired capital level at every moment in time, it does not pay them to plan for the future since future changes in business conditions will be absorbed by future discrete changes in capital stocks. Economists tend to think that future changes in business conditions have effects on today investment decisions. To get rid of this result we need a theory that makes firms willing to smooth investment over time. One way of introducing such a willingness to smooth investment is to make it costly to invest or disinvest large amounts of capital at once. This is the idea behind the concept of **adjustment costs**.

One final note, in this section we assumed that the interest rate was exogenously given to the firm. This

seems a reasonable assumption if we want to think about the behavior of individual firms. Yet, as macroeconomists, we are more concerned about aggregate investment. Aggregate investment both depends upon and affects the interest rate of the economy. That is, in the aggregate the interest rate is endogenous. We can endogenize the real interest rate by embodying the individual neoclassical firms just described in a general equilibrium model where there are also consumers and the interest rate is determined by the equalization between the desired investment by firms and desired savings by households. This will give raise to the Neoclassical model of economic growth of Ramsey (1928), Cass (1965) and Koopmans (1965). A thorough description of such a model can be found in Barro and Sala-i-Martin (1995).

(2) Internal Adjustment Costs: The q-Theory of Investment.

(i) Setup of the problem and First Order Conditions

Following Eisner and Strotz (1963), Gould (1968), Lucas (1967), Treadway (1969), Uzawa (1969), Abel and Blanchard (1983) and Hayashi (1982), we will now imagine that firms behave exactly as just described, except that there are some installation or adjustment costs. By that we mean that , like in the neoclassical model, one unit output can be transformed into one unit of capital. This capital (which we will call "uninstalled capital") is not useful until it is installed. Unlike the neoclassical model, firms have to pay some installation or adjustment costs in order to install or uninstall capital. These adjustment costs are foregone resources within the firm: for example computers can be purchased at price p but they cannot be used until they have been properly installed. The installation process requires that some of the workers stop working in the production line for some of the time. Hence, by installing the new computer the firm foregoes some resources, which we call internal adjustment costs.

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Let us denote the cost of installing ONE unit of capital by O($\mathbf{\cdot}$). This is an increasing function of desired investment (the more firms wants to invest at once, the more people they need to put to install machines and the more output they forego) and a negative function of the existing capital stock (firms that have a lot of physical capital are firms that have invested a lot in the past and, therefore, are firms that have a lot of experience in installing; this experience allows them to install capital at a lower unit cost). Notice that without this latter assumption, it would be optimal for firms to divide themselves into little units of infinitesimal size and invest a little bit in each unit. Following Abel and Blanchard (1983) we will further assume that the unit adjustment cost is a function of I/K with $\varphi'(I/K) > 0$ and $2\varphi' + (I/K)\varphi'' \geq 0$. We will also imagine that zero investment bears no installation cost, $\varphi(0) = 0$.

It is interesting to note that our formulation implies that it is costly to both invest and disinvest (invest negative amounts). The unit cost of investing some negative amount is negative so the actual cost (negative cost times negative amount of units) is positive. Finally, we are implicitly assuming that replacing depreciated capital does not involve adjustment costs (recall that we defined *AF()* and *I* as output and investment net of depreciation). One could argue that Gross Investment is the relevant unit here since getting rid of old, depreciated machines is also costly. See Eisner and Strotz (1963) and Lucas (1967) for a discussion on this point. One example of unit and total adjustment cost functions is depicted in Figures 1 and 2 respectively.

As in Section 1, firms are assumed to maximize the present value of all future cash flows

(2.1) $\quad V = \int_0^\infty e^{-rt} \left[P_t A_t F(K_t, L_t) - w_t L_t - I_t(1 + \varphi(I_t/K_t))\right] dt$

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subject to the constraints

(2.2) $\dot{K}_t = I_t$

(2.3) $K_0 > 0$ given and subject to the non-ponzi condition $\lim_{t \to \infty} e^{-rt} K_t \geq 0$.

where, again, A_t is some productivity parameter that can be interpreted as the level of technology or as an

after-tax factor. The total cost of investment is sum of the purchasing cost, I, plus the installation

cost, $I\varphi(I/K)$. Again, we normalize the price of output (and therefore investment, because they are the

same good) to one. As we did in the last section we can form the present value Hamiltonian by defining a

current value shadow price $q_t = e^{rt}\lambda_t$ where λ_t is the present value of the marginal contribution of

capital to profits (present value shadow price).

(2.4) $H() = e^{-rt}\left[A_t F(K_t, L_t) - w_t L_t - I_t\left(1 + \varphi\left(\frac{I_t}{K_t}\right)\right)\right] + \lambda_t I_t$

The first order conditions are the following:

(2.6) FOC with respect to L: $AF_L = w$

(2.7) w.r.t I:

$$e^{-rt}\left(-(1 + \varphi(I_t/K_t) + (I_t/K_t)\varphi'(I_t/K_t)\right) = \lambda_t \quad \rightarrow$$

$$(1 + \varphi(I_t/K_t) + (I_t/K_t)\varphi'(I_t/K_t)) = q_t$$

(2.8) w.r.t. K:
$$-\dot{\lambda}_t = e^{-rt}(AF_K + (I_t/K_t)^2 \varphi'(I_t/K_t)) \quad \rightarrow$$

$$-(\dot{q}_t e^{-rt} - rq_t e^{-rt}) = e^{-rt}(AF_K + (I_t/K_t)^2 \varphi'(I_t/K_t))$$

(2.9) TVC: $\quad \lim_{t \to \infty} \lambda_t K_t = 0 \quad \rightarrow \quad \lim_{t \to \infty} e^{-rt} q_t K_t = 0$

Condition (2.6) is equivalent to (1.6) and says that optimizing firms will hire workers up to the point

where the marginal product of labor equals to marginal cost, the real wage. Condition (2.7) is a bit more

interesting than (1.7). It says that q_t (the current shadow value of investment) is a function of the

investment ratio *I/K*. Hence, q_t can be written as

(2.7)' $\quad q_t = g(I_t/K_t)$

Since $2\varphi' + (I/K)\varphi'' \geq 0$ holds, we have that *g'>0*. Notice that *g(0)=1* which means that when

there is no investment, there are no adjustment costs and therefore the (shadow) value of one more unit of

investment is exactly its market price, 1 (we are back to the neoclassical model of the previous section).

Since *g(▾)* is a monotonic function, it can be inverted so as to get *I/K* as function of *q*.

(2.7)'' $\quad \dfrac{I_t}{K_t} = h(q_t)$

with *h(1)=0* and $h'() > 0$. This is a very important result. First, it means that the only thing that firms need

to observe in order to make investment decisions is q_t, the shadow price of investment. In other words, q_t is a

"sufficient statistic" for investment in the sense that it embodies all useful information about the environment.

Further, when q is larger than one, firms will invest positive (but finite) amounts. When q is less than one the

investment rate will negative (but finite). The intuition is simple: the market cost of purchasing a machine is 1.

If firms know that they can buy capital at price 1 and get an increase in (the present value of) revenue larger

than 1 ($q>1$), they will buy. The installation costs will prevent any firm from investing "too much at once" since

these extra costs increase with investment. In some sense, when firms want to have more capital, the value of

installed capital goods is larger than the value of uninstalled capital since uninstalled capital has to pay some

installation or adjustment cost before it can become productive. Essentially, installed and uninstalled capital

are two different goods whose "relative (shadow) price" is q_t.

In a sense, equation (2.7)" can be interpreted as a demand for investment that relates investment to its

(shadow) price. The negative relation is due to the desire to substitute across goods. This demand function is

depicted in Figure 3. In this model the supply of investment goods is infinitely elastic at q_t. Thus, given q,

actual investment is determined by (2.7)".

The question, then, is what determines q? The answer is given by the third FOC. (2.8), which can be

rewritten as

$$(2.8)' \quad r = \frac{\dot{q} + [AF_k + (I/K)^2 \varphi/(I/K)]}{q}$$

This first order condition says that, at the optimum, the firm will be indifferent between investing and

purchasing bonds so that the rates of return to the two should be the same. The left hand side of equation

(2.8)' is the return to bonds (the real interest rate). The right hand side represents the return to investment. It

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is the sum of the marginal product of capital (AF_K), the change in the (shadow) value of capital or capital gains, ($\dot{q}$) and the gains of reduced adjustment costs due to a larger scale $([I/K]^2 \varphi'[I/K])$. The

differential equation (2.8)' can be solved forward between zero and infinity and use the TVC (2.9) to get rid of the limiting term. The resulting expression for the shadow price at time zero, q_0, is

$$(2.10) \quad q_0 = \int_0^\infty e^{-rt}(AF_K + (I/K)^2 \varphi/(I/K))\, dt$$

That is, the current value or price of installed capital is the present value of all future marginal contributions of capital to profits, and again, marginal contributions of capital to profits are the sum of the direct marginal product of capital plus the reduction in adjustment costs associated with a larger firm size. Contrary to what we found in the neoclassical model, anticipated future movements in the marginal product of capital (due to taxation, exogenous discoveries of new technologies, etc) affect the current valuation of newly installed capital - q - and therefore, current investment. The reason is that, because of the existence of adjustment costs, it does not pay for firms to invest large amounts prior to the increase of the marginal product of capital. It pays to start investing long before that, so they start today. Notice also that the interest rate also affects investment negatively...but it does so through its effect on q.

(ii) Capital, Investment and shadow price dynamics.

The next step is to study the dynamic behavior of the firm. Substitute (2.7)" into (2.8)' and get

$$(2.11) \quad \dot{q} = rq - [AF_K + h(q)^2 \varphi'(h(q))]$$

Equation (2.2) can be rewritten using (2.7)" to get

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(2.2)' $\dot{K} = K h(q)$

Equations (2.11) and (2.2)' form a system of two differential equations with two unknowns, q and K.

We also have an initial condition given by (2.3) and a terminal condition given by (2.9). The steady state

involves a constant value of K and q implied by the equations $\dot{k} = 0$ and $\dot{q} = 0$:

(2.12) $I^*/K^* = 0$

(2.14) $q^* = 1$

(2.15) $AF_K(K^*) = r$

That is, in the steady state, the shadow value of an extra unit of installed capital equals its market cost,

1. Therefore, the desired investment rate is zero. Notice that this is what we got in the neoclassical case with

no adjustment costs. This is reasonable given that when investment is close to zero, the adjustment costs

become unimportant which is what we assumed for neoclassical firms. In order to analyze the dynamics we

can either construct a phase diagram or get an algebraic solution of the problem by linearizing around the

steady state. We will first do the graphical analysis.

(iii) Graphical Analysis.

The phase diagram can be constructed by picturing the two steady state loci. Notice that there are two

"jumping" or control variables, q and I, but equation (2.7)'' says that there is a one to one relationship

between the two so we can exclude one of them from the analysis. We will use q. From (2.2) the $\dot{K} = 0$

schedule implies that $I/K=0$ which, by virtue of (2.7)' can be graphed as a horizontal line at $q=1$. We do that

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in Figure 4. For values of q larger than 1, $\dot{K} > 0$ so the arrows point east. This reflects the fact that if the

marginal revenue from an extra unit of capital is larger than the marginal cost (equal to one) firms will proceed

with the investment which, of course increases the capital stock. The opposite is true for values of q smaller

than 1. Equation (2.11) says that the $\dot{q} = 0$ schedule is

(2.15) $\quad 0 = rq - [AF_K + h(q)^2 \varphi'(h(q))]$

This implicitly defines q as a function of K. No explicit functional forms can be found in general, but the slope

of this implicit function can be found with the implicit function theorem

(2.16) $\quad \dfrac{dq}{dK} = - \dfrac{-AF_K}{r - 2h(q)\varphi'(h(q)) - h(q)^2 \varphi''(h(q))h'(q)}$

The numerator is positive because the production function is concave. Since $h>0$, $h'>0$ and

$2\varphi' + h\varphi'' > 0$, the denominator has an ambiguous sign. Around steady state, however, we know that

$h(q^*)=0$ so the denominator is equal to $r>0$. Thus the slope of the $\dot{q} = 0$ schedule is negative around

steady state as depicted in Figure 4 (for higher values of q, this schedule is actually upward sloping). To the

left of the schedule (larger q), $\dot{q}$ is negative so arrows point south and the opposite is true to its right of the

schedule. The steady state exhibits saddle path stability with only one stable arm converging to the steady

state point.

As we can see in Figure 4 there are (infinitely) many paths that satisfy the first order conditions (2.11) and (2.2)'. We know, however that the only perfect foresight path that satisfies all the first order must also satisfy the transversality condition. This requires that the sequence of capital and shadow prices (and associated investment rates since, remember, investment is a monotonic function of the shadow price q) be such that the economy ends up in the steady state. Hence, the unique optimal behavior of the firm is described by a movement along the stable arm (which is the only path that can take the economy to the steady state).

Consider the case where the initial capital stock inherited by the firm K_0 is less than the steady state K^*. If the shadow price is larger than the q_0 that lies in the stable arm (such as point (b) in figure 4), the economy will find itself in a place where both q and K grow at increasing rates. This path will lead to the violation of the TVC. If, on the contrary the shadow price at time zero is less than q_0 (such as point (a) in figure 4) it will start falling and, eventually, so will capital. The path will eventually hit either the vertical or horizontal axes in finite time. If K becomes zero in finite time, q will jump discretely (because $\lim_{t \to \infty} F_K(0) = \infty$) which will violate

condition (2.8). If we hit the horizontal axes in finite time, q becomes negative in finite time. Rational individuals cannot possibly expect such a sequence of capital losses (reductions in the value of installed capital) followed by a negative value of installed capital given that the marginal product of capital is positive for all positive capital stocks.

Notice that the path converging to the steady state satisfies all the first order conditions, including transversality: at the steady state, the capital stock is constant at $K^*>0$ whereas $q=1$. Hence, the limit of the

shadow price times the capital stock times the discounting factor must be zero, $\lim_{t \to \infty} e^{-rt} q_t K_t = 0$, so the

transversality condition is satisfied. Thus, the only possible perfect foresight path is the stable arm.

(iv) Permanent Productivity Shocks or tax Cuts.

We are now ready to analyze the behavior of the firm when it faces exogenous changes in the

parameters of the model. Suppose, first, that there is an exogenous, unexpected, and permanent increase in

the productivity parameter A, due to a technological innovation or to a reduction in the product tax rates

which increase the after tax cash flow for the firm.[7]

Since the shock is a surprise to agents, the first order conditions (2.6) and (2.9) may not be satisfied at

the initial moment. Notice, however, that since individuals know that there will be no more surprises in the

future, the FOC will be satisfied at all future points in time. In particular, after time zero the shadow price will

move according to (2.8)' which implies smooth movements in q. Hence, individuals will not expect the

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[7] The existence of "unexpected" shocks in a model of perfect foresight seems a bit strange.
After all, if individuals have perfect foresight, why didn't they foresee this shock also?. The alternative is,
however, to model the whole economy in a stochastic setting where individuals assign a given known
probability to any productivity shock. The analysis in this stochastic setup would be to allow for an
innovation or shock to occur at time zero and to shut down all future innovations to see how the firm
reacts at all points in time. This exercise, which is sometimes called "impulse response" is analogous to
the "once in a lifetime shock" analyzed in these notes, and the results, therefore, would be very much
the same. Hence, instead of doing this more complete and complicated alternative we will assume that
firms assigned a zero probability of a shock at every point in time in the past. For some reason one of
such shocks occurs today, but individuals think that the chance of that happening again is zero.

shadow price to jump in the future,[8] although q may jump at moment zero,[9] when they learn about the existence of this surprise shock.

A permanent increase in A implies a shift of the $\dot{q} = 0$ schedule to the right and does not change the $\dot{K} = 0$ one. The new dynamics (which in Figure 5 are denoted with solid arrows) apply immediately after the shift. Figure 5 tells us therefore that q will immediately jump upwards, this will trigger positive investment and a higher long-run capital stock.

What is the intuition behind this result?. An increase in A implies a higher (after tax) marginal product of capital. Hence, immediately after firms find out about the increase in A, they want to hire more capital and increase production. The new desired capital level is, in fact, K^{**} the new steady state capital stock. But (contrary to what happened in the neoclassical model with no installation costs) they do not jump immediately to that desired level because that would entail infinite investment rates and, consequently, infinite costs of adjustment. Hence, they smooth investment over time. This, of course, means that the value of installed machines is higher than uninstalled ones since they can already be used with the new superior technology. Hence, the shadow price of installed machines jumps immediately. As firms keep investing positive amounts, diminishing returns to capital take over so each additional unit of capital is less desirable because of a

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[8] In order for q to jump in the future, the right hand side of the $\dot{q}$ equation (2.11) would have to be infinity and, for positive values of K and q, it is never infinity.

[9] The FOC start applying at time zero so any jump in q between a second before zero and zero (that is, any initial jump in q) would not violate the first order condition (2.11).

diminishing marginal product. This implies that the shadow value of installed relative to uninstalled capital falls

as K grows. The process stops when firms achieve the new desired capital stock, K^*. The time paths of I, K

and q are depicted in Figure 6. Note that both q and I jump immediately, although they do not go to infitity,

and they remain high but decreasing all the way to their steady state values. This, of course, implies an

increasing but concave path for the capital stock, which asymptotically reaches its new steady state level K^{**}.

(v) Anticipated Productivity Shock or Tax Cut

Let us consider what happens when we ANNOUNCE at t=0 that A will increase permanently at some

point in the future, $t_1>0$. In other words, the firm gets the information at time zero that the level of technology

will improve in the future. The firm knows that between times 0 and $t_1 > 0$, the available technology will still

be the old one.

As shown in Figure 2, at time $t_1 > 0$, the $\dot{q} = 0$ locus will shift to the right. Up to then, however,

the economy will still be governed by the "old arrows". The first order conditions say that the optimal

behavior of the firm must involve (1) no future jumps in q and (2) finish in the new steady state capital stock,

K**, which is the only point that does not violate the transversailty condition. In order prevent future jumps in

q - which would violate condition (2.11) - it must be the case that we land exactly on the new stable arm at

exactly t_1 . This is the case because at t_1 the new dynamics (denoted with solid arrows in Figure 7) will

apply. If at that point we are not on the stable arm, the economy will blow up for the same reasons we

mentioned in section (iii).

Before time t_1, the system will still be governed by the old dynamics. The question is, then, starting

from the old steady state, what is the path for q and K?. Let us find the answer by eliminating all the paths that

are not optimal. Suppose first that the initial q does not jump at all. The old dynamics (denoted by dotted

arrows in Figure 7) tell us that both q and K stay in the old steady state until t_1. But then, in order to be

exactly on the new stable arm at t_1, q will have to jump at some point in the future. This is not optimum since it

would violate condition (2.11). Hence, q must jump today.

Suppose now that it jumps all the way to the new stable arm. The old, dotted, dynamics say that both q

and K will start to increase after the initial jump, so we will start getting away from the new steady state. It will

therefore be impossible for us to land on the stable arm at exactly t_1 so this possibility is not optimal either.

It follows that q will jump, but not all the way to the new stable arm. The exact size of the jump will be such

that after a period of increasing q and K, we land on the new stable arm at exactly t_1. After that moment,

the new dynamics (solid arrows) take over so we follow the new stable arm all the way to the new steady

state. The time paths for K, I and q are depicted in Figure 8. Both q and I jump immediately and keep

increasing until time t_1. At that point they start falling until they reach their old steady state values. The

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capital stock starts increasing from time zero (due to positive investment rates). It does so in a convex manner

between times 0 and t_1 (because investment is increasing over time) and in a concave way after t_1

(because of falling, although positive investment rates). There is a "kink" in the paths of q and I at time t_1 .

We can see in equation (2.11) that $\dot{q}$ is a function of A, K and q. Since at time t_1, neither K nor q jump and

A increases discretely, it must be the case that there is a discrete decrease in $\dot{q}$. This translates into a 'kink'

in q. The same is true for I since it is a monotonic function of q.

The important point of this exercise is that the anticipation of future changes in the marginal product

have an effect on today's investment. Recall that this was not true in the neoclassical model where firms were

able to discretely adjust to any desired capital level without any transition or adjustment cost. The intuition is

the following. Firms learn at time 0 that the marginal product of capital will be higher in the future. The stock

of capital they want to have at exactly t_1 increases. Due to the existence of adjustment costs, it will be very

costly to increase the capital level right before the increase in A takes place. Hence, they will start

accumulating capital today even though the marginal product is still low. In a way, internal adjustment costs

lead firms to smooth investment over time.

(vi) Temporary Productivity Shocks or Tax Cuts.

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We consider now a temporary productivity shock. That is, today (t=0) we learn that, taking place

immediately, the level of technology increases to a higher level, and we also learn today that it will go back to

the original level at exactly t_1 . Real world examples of temporary productivity shocks would include a

drought, an increase in the price of oil or a temporary reduction in tax rates. In terms of our model, this means

that the firm knows that the $\dot{K} = 0$ schedule does not move whereas the $\dot{q} = 0$ one temporarily shifts

to the right. At moment t_1 , the $\dot{q} = 0$ schedule will go back to the initial position and the dynamics

governing the system will be the old ones again.

In order to solve the model, we need, again, to take into account that (1) q cannot jump after the initial

moment (a jump would violate condition (2.11)) and (2) that, whatever we do in the short term, we need to

move q and K in such a way that we end up in the steady state (otherwise, the TVC will be violated). These

two conditions imply that, at exactly t_1 we must be ON the OLD stable arm (because we must end up in

the OLD steady state since, in the long run, the level of technology will be again the one we had till today). In

the meantime, however, the dynamics are governed by the new technology so the question is: what are the

movements in q, I and K (if any)?. As usual, we will proceed by eliminating non-optimal paths.

The new improved technology makes installed capital more productive. The question is whether its

shadow value (and the level of investment) increases or not. Suppose first that q does not jump at all at time

zero. Since the new dynamics (denoted by solid arrows in Figure 9) start applying immediately, both q and K

will start falling. Note that there is no way we will be on the old stable arm at t_1 so this is not a feasible

possibility.

If we jump exactly to the new stable arm, we follow the new dynamics until period t_1. We will then

move along the new stable arm, so we will not be able to land on the old stable arm at exactly time t_1. The

same is true if we jump to a q higher than the one corresponding to the new stable arm since the new

dynamics takes us towards the northeast.

Hence, it must be the case that we jump less than before. After the initial moment we follow the new

dynamics until period t_1. Thus, since q is larger than 1, we invest positive amounts but both q and I are

falling over time. This will keep happening until we reach point, $\hat{t} < t_1$, where q takes value 1. At this

point investment goes from positive to negative so K starts to fall in a continuous fashion. Both q and K will

keep falling until, at exactly the moment t_1 we smoothly land on the stable arm and the old dynamics

(represented by the dotted arrows in figure 9) take over. From that point on, both q and K increase till they

reach their old steady state values. The paths for q, I and K just described are depicted in Figure 10.

Intuitively, why did all this happen?. Firms want to take advantage of a temporarily high marginal

product of capital by temporarily having a lot of capital. In the absence of adjustment costs they would want

to discretely increase the amount of capital at time zero and discretely reduce the stock at t_1 . But the

existence of adjustment costs prevents them from doing so, yet they still can take partial advantage of the new

situation. To that effect, at the moment they learn about the shock they start investing a lot. They come as

close to the maximum amount of capital as possible taking into account that, after t_1 they will want to go

back to the initial level. Hence, at some point, the present value of all future marginal products (q) falls below

one which indicates firms that it is time to decumulate. They keep disinvesting until they reach the old steady

state capital stock.

(3) Marginal q versus Average q.

(i) Hayashi's Theorem.

We have thus far developed quite a nice theory that says that the only variable that matters for

investment decisions is q. As we mentioned earlier, however, q is not really a price that can be observed in a

market, but a shadow price or Lagrange multiplier. Although shadow prices have clear economic

interpretations, they cannot be directly observed by econometricians. Thus, unless we can relate q to some

observable variable, the theory developed in the previous section has little empirical content.

Fortunately, Hayashi showed that if (1) the production function and the total adjustment cost functions

are homogeneous of degree one (that is constant returns to scale), (2) the capital goods are all homogenous

and identical, and (3) the stock market is efficient, then the shadow price q is equivalent to the ratio of the

market value of a firm divided by the replacement cost of capital. This ratio is often called Tobin's **average q**

as opposed to the **marginal q.** The term marginal q refers to the shadow price. Notice that our two functions

satisfy the homogeneity-of-degree-one requirements of the Hayashi theorem since

(3.1) $F(\beta K, \beta L) = \beta F(K, L)$

(3,2) $\beta I \varphi (\beta I / \beta K) = \beta (I \varphi (I/K))$

Thus, we know that in our particular case the shadow price will be equal to the stock market value of a

unit of capital. Let us show that, in fact, in our model Marginal and **average q** are identical. Let again q be

the shadow price of capital and K be the capital stock. Let us take the derivative of the product of K and q.

(3.3) $\dfrac{d(q_t K_t)}{dt} = \dot{q}_t K_t + q_t \dot{K}_t$

Plug (2.8)' and (2.2) in (3.3) to get

(3.4) $\dfrac{d(q_t K_t)}{dt} = K \left[rq - AF_K - \left(\dfrac{I}{K} \right)^2 \varphi'(I/K) \right] + I \left[1 + \varphi (I/K) + \dfrac{I}{K} \varphi'(I/K) \right] =$

$$= -KAF_K + rKq + I(1 + \varphi(I/K))$$

where use of (2.7) has been made. By Euler's theorem we know that, if F() is homogeneous of degree one

(that is, if it exhibits constant returns to scale), then $K(AF_K) + L(AF_L) = AF(K, L)$ (this is where

the homogeneity of degree one assumption plays an important role). We can use this equality together with

condition (2.6) to get

$$(3.5) \quad \frac{d(q_t K_t)}{dt} = rKq - [AF(K,L) - wL - I(1 + \varphi(I/K))]$$

Which is a first order ordinary differential equation in the variable Kq. Solve this equation forward to get

$$(3.6) \quad q_0 K_0 = \int_0^\infty e^{-rt}[AF(K,L) - wL - I(1 + \varphi(I/K))]\, dt$$

Notice that the right hand side of (3.6) is the present value of all future dividends. We know that **if the stock market is efficient**, this present value will be exactly equal to the stock market price of a firm. Let us denote such stock market value by V_0 so (3.6) becomes

$$(3.7) \quad q_0 = \frac{V_0}{K_0}$$

In other words, the shadow price of capital (marginal q) is equal to the ratio of the (stock) market value of a firm to the replacement cost of its capital (Tobin's average q). Hence, if we want to test the theory we just need to collect data on market values of firms, replacement costs and construct a measure of average q. The theory predicts that this measure is a sufficient statistic for a firm's investment.

(ii) Should Managers Always React to Stock Market Signals?

Does this mean that managers should just observe the stock market value of an additional machine and make their investment decisions by comparing that price to one?. There are a number of reasons why this may not be entirely true. Here we present two of them.

(ii,a) Heterogeneous Capital Goods.

We have proceeded under the assumption that capital goods were homogeneous. As mentioned earlier,

however, capital goods in the real world are not. Most innovations and technological improvements are embodied in the machinery so capital goods of different vintages have different technological attributes. If this is the case, technological disturbances like the ones studied above may shift the average and the marginal values of q in opposite directions. Consider technological process in the world of computer microprocessors. When Pentium III was introduced in the market it made the machines with Pentium II partly obsolete. The stock market value of old (installed) Pentium II machines dropped dramatically and, therefore so did the AVERAGE q of firms that used them. The shadow price of new machines (that is, MARGINAL q), on the other hand, increased since firms wanted to have Pentium III computers installed instead of the old Pentium II. Thus MARGINAL q was larger than one and firms proceeded to invest, as predicted by the theory. This is an example in which AVERAGE and MARGINAL q moved in opposite directions and econometricians that assumed that average and marginal q are the same found a negative correlation between q (based on stock market) and investment.

Some people argue that this effect was particularly important during the seventies when sharp oil increases made the oil-intensive capital goods obsolete. The market value of most firms -average q- fell sharply. The marginal profit derived from installing new oil-saving capital goods -marginal q-, in the other hand, probably increased. Thus, the average and marginal values of q moved in opposite directions. Again empirical studies that rely on the observable average q failed to capture such episodes since measured q and investment move in opposite directions. This is one of the problems that makes the empirical implementation of the theory quite difficult.

(ii,b) The stock market may not be efficient.

A second reason why managers may not want to react to the stock market is that markets may not be efficient. Recall that in the proof of the Hayashi theorem we said that the stock market value of a firm -V_0- was equal to the present value of all its future dividends. This is true if the stock market functions efficiently. Some economists, however, argue that stock market prices may depart from fundamentals for a variety of reasons. One of them is the existence of so called bubbles, self fulfilling increases in stock prices totally unrelated to changes in fundamentals. The question is: if managers know that the present value of all future marginal revenue derived from an additional machine is less than the stock market price, should they still purchase the machine, sell stock and cash the difference?. This seems like a reasonable strategy since it generates unbelievable amounts of free revenue for the firm. Hence firms should react to high prices of stock by investing even if they know that these high prices do not correspond to fundamentals.

In the presence of bubbles, however, one could argue that instead of buying a machine and selling stock it would be better to sell stock and buy treasury bills: treasury bills yield a safe return which is probably higher than the machine which we know is not going to give so much revenue (no matter what the stock market says). Under these conditions, investors should not react to the stock market whenever they know that it is over valued.

Yet one could counter argue that if stock brokers are overvaluing the firm because they are less informed than managers, when they observe the managers buying treasury bills and selling stock they will understand that the managers know that the firm is overvalued, which in turn may burst the bubble. Thus, in order to keep stock brokers fooled, managers should invest whenever stock prices are high, even if it is due to non-fundamental reasons.

And we could go on and on. The point is that it is not theoretically clear whether firms should always

react to the stock market valuation of their firm regardless of whether markets are valuing fundamentals correctly or not. Whether they actually do or not remains an unanswered empirical question (see Barro (1989) and Blanchard-Rhee-Summers (1990)).

(4) Linearization around Steady State.

Let us now solve the model analytically by linearizing around the steady state. Let us recall that the optimal behavior of the firm can be described by the system of equations

(4.1) $\dot{K}_t = h(q_t) K_t$

(4.2) $\dot{q} = rq - [AF_K + h(q)^2 \varphi'(h(q))]$

and the two initial and terminal conditions are

(4.3) $K_0 > 0$ given

(4.4) $\lim_{t \to \infty} e^{-rt} q_t K_t = 0$

The steady state values of q and K can be found by setting $\dot{K}$ and $\dot{q}$ equal to zero

(4.5) $q^* = 1$

(4.6) $K^* = F_K^{-1}\left(\frac{r}{A}\right)$

We can apply a first-order Taylor expansion series around the steady state

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(4.7) $\dot{K} = h(1)[K_t - K^*] + K^* h'(1)[q_t - 1]$

(4.8) $\dot{q} = -AF_{KK}^*[K_t - K^*] + (r - 2h(1)\phi'(h(1)) - h(1)^2\phi''(h(1))h'(1))[q_t - 1]$

Notice that $h(1) = 0$ so the two equations simplify and can be expressed in matrix notation as:

$$\begin{bmatrix} \dot{K} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & K^* h'(1) \\ -A_{KK}^* & r \end{bmatrix} \begin{bmatrix} K - K^* \\ q - 1 \end{bmatrix}$$

The determinant of the associated matrix is $AF_{KK}K^* h'(1)$ which is negative. Hence, there are

two real roots of opposite signs and, therefore, the system exhibits saddle path stability. The eigenvalues can

be found by solving the following equation[10]

$$(4.9) \quad \det \begin{bmatrix} -\lambda & K^* h'(1) \\ -A_{KK}^* & r - \lambda \end{bmatrix} = 0$$

which is a quadratic equation in λ. The solution to this quadratic equation is

$$(4.10) \quad \lambda_t = \frac{r \pm \sqrt{r^2 - AF_{KK}^* h'(1)k^*}}{2}$$

Which corresponds to two values of λ, one positive and one negative. Let $-\lambda_1$ be the negative

eigenvalue and λ_2 be the positive one (notice that we define $-\lambda_1$ to be negative so that $\lambda_1 > 0$ is

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[10]See the mathematical appendix of Barro and Sala-i-Martin (1995).

positive). The solution for K_t and q_t is given by the two time equations

(4.11) $K_t - K^* = \psi_{11} e^{-\lambda_1 t} + \psi_{12} e^{\lambda_2 t}$

(4.12) $q_t - 1 = \psi_{21} e^{-\lambda_1 t} + \psi_{22} e^{\lambda_2 t}$

where $\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}$ are constants to be determined. These constants are related to each other

through the two eigenvectors associated with $-O_1$ and O_2 respectively. The eigenvector associated with $-O_1$

requires

(4.14) $\lambda_1 \psi_{11} + K^* h'(1) \psi_{21} = 0$

which implies that the relation between ψ_{11} and ψ_{21} is $\psi_{21} = \dfrac{-\lambda_1 \psi_{11}}{K^* h'(1)}$. The relation between

ψ_{12} and ψ_{22} can also be found but it will not be necessary for the following reason. Notice that λ_2

is positive and larger than r. This means that, unless ψ_{12} or ψ_{22} are equal to zero (and notice that they

will be proportional to each other because they are the two components of the same eigenvector so if one is

zero so will be the other and if one is nonzero so will be the other), both q and K will grow at rates larger

than r. Of course this would violate the TVC in (4.4). Thus we must set both ψ_{12} and ψ_{22} to zero. We

can finally make use of the fact that we know K_0 to evaluate equation (4.11) at time zero

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$(4.11)'$ $K_0 - K^* = \psi_{11}\, e^{-\lambda_1 0}$

which, of course, means that the constant O_{11} is equal to $K_0\text{-}K^*$. Equation (4.14) says that

$$\psi_{21} = -\frac{(K_0 - K^*)\lambda_1}{K^* h'(1)}$$ which identifies the initial value of q (at the stable arm)

(4.15) $q_0 = 1 - \dfrac{\lambda_1 (K_0 - K^*)}{K^* h'(1)}$

Equation (4.15) describes the stable arm in Figure 4 around the steady state. It says that whenever K_0

is smaller than the steady state, capital stock, K^*, q_0 will be larger than 1 and vice versa. In other words, the

stable arm is above the steady state for small capital and below the steady state for larger capital stocks so it

is downward-sloping. Recall that this is the same result we got in the graphical analysis.

(5) External Adjustment Costs.

In section (1) we saw that the neoclassical firms with no installation costs wanted to have the "desired

capital stock" at all points in time, even if that entailed infinite investment rates. In section 2 we saw that if

every time firms have to invest they must forego some resources (*internal adjustment costs*) the problem of

infinite investment rates was solved. We will now see another way to solve the problem. Following Clower

(1954), Witte (1963), and Foley and Sidrauski (1970), we will think about an economy where there are two

firms. The first one is the neoclassical firm of section (1), which produces some manufactured output Y using

capital and labor according to technology (1.0). This firm faces no internal adjustment costs. It purchases

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capital from some producers of capital goods at price P_I. The key assumption here is that the technology that produces these capital goods exhibits Deceasing Returns to Scale (DRS) so the marginal cost function is increasing in the amount of investment goods demanded. Thus, the price of investment is an increasing function of the quantity invested. Notice that this assumption will prevent producers of manufactured goods from demanding infinite amounts, since infinite investment entails infinite marginal cost, and infinite prices. It is in this sense that we call this the **external adjustment cost** approach to investment: it is not the firms that demand investment who forego resources whenever they install new machines. Rather, they face an increasing price in the amount of investment they demand (so she still faces adjustment costs but they are external to the firm). We analyze the two firms separately

(i) Producers of Capital Goods.

Producers of investment goods face a decreasing returns to scale (DRS) technology. Minimization of costs subject to this DRS production function yields a cost function of the form

(5.1) $C_t = C(I_t)$

where $C(0)=0$, $C'(I)>0$ for $I>0$ and $C'(0)=0$, $C''(I)>0$ for $I \blacktriangledown 0$ and $\lim_{I \to \infty} C'(I) = \infty$. In other words,

marginal cost is increasing and the marginal cost of producing an infinite flow of capital goods is infinity. One key assumption is that this cost function depends on investment, I, but not on the capital stock, K. We could follow the analysis of the previous section and assume that firms also learn from past experiences (learning by doing) and that the cost is a negative function of the existing capital stock. We could further assume that costs are a function of I/K, rather than I. We will proceed, however, under the assumption that the cost function,

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C() is a function of *I*, and we will get back to this point later on.

Firms will be assumed to behave competitively in that they maximize profits taking the price of

investment goods, P_I, as given. Their program is, therefore

$$(5.2) \quad \max \quad V(0) = \int_0^\infty e^{-rt} \left[P_{It} I_t - C(I_t) \right] dt$$

Notice that there are no dynamic constraints to this firm, so the first order conditions require the equalization

of price to marginal cost, that is,

$$(5.3) \quad P_I = C'(I)$$

at all points in time. Notice that this optimizing condition is the static one. The reason behind this similarity is

the absence of dynamic constraints which means that the dynamic problem is nothing but a sequence of static

problems so maximizing profits over the whole horizon is identical to maximizing them period by period.

Because *C'* is a monotonic function of *I* and *C''>0*, then equation (5.3) can be inverted to yield an

optimal investment supply as a function of the price of investment

$$(5.4) \quad I = h(P_I)$$

where *h'()>0*. This can be thought as a supply function of investment goods. We will combine this supply

function with the infinite elastic demand function we derived in section 1 to yield the equilibrium rate of

investment for the economy.

(ii) Producers of Manufactured Goods.

The behavior of producers of manufactured goods is similar to the one already described in section (1).

They buy labor at *w* and capital goods at P_I, which they take as given. With these inputs they produce some

output which they sell at P_Y. Since capital and output are different goods we will not make the assumption

that $AF(K,L)$ is output net of depreciation $(F()=G()-\delta K$ where $G()$ is gross output). Firms are assumed to

maximize the present value of all future net cash flows

$$(5.5) \quad \max V(0) = \int_0^{\infty} e^{-rt} \left[P_{Y_t} A_t F(K_t, L_t) - w_t L_t - P_{I_t} I_t \right] dt$$

subject to the constraints

$$(5.6) \quad \dot{K}_t = I_t - \delta K_t$$

(5.7) $K_0 > 0$ given.

where P_Y is the price of output and P_I is the price of investment goods. Notice that net investment - $\dot{K}$ - is

equal to gross investment - I - minus depreciation - δK. This implies that the part investment used to

replace depreciated capital pays the same price - P_I - as investment designed to increase the stock of capital.

To solve the optimization program we can form the Hamiltonian

$$(5.8) \quad H() = e^{-rt} \left(P_{Y_t} A_t F(K_t, L_t) - w_t L_t - P_{I_t} I_t \right) + \lambda_t (I_t - \delta K_t)$$

where λ is the present value shadow price of investment (marginal contribution of investment at time t to

profits at time zero). We can define a current value shadow price $\mu_t = \lambda_t e^{rt}$. The first order conditions

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to this problem are

(5.9) $AF_L = \dfrac{w}{P_Y}$

(5.10) $- e^{-rt} P_I + \lambda = 0 \quad \rightarrow \quad P_I = \mu$

(5.11) $- \dot{\lambda}_t = e^{-rt} P_Y AF_K + \lambda_t(-\delta) \quad \rightarrow \quad -\left(\dot{\mu}\, e^{-rt} - r\mu e^{-rt} \right) = e^{-rt} (P_Y AF_K - \mu\delta)$

(5.12) $\lim\limits_{t \to \infty} \lambda_t K_t = 0 \quad \rightarrow \quad \lim\limits_{t \to \infty} e^{-rt} \mu_t K_t = 0$

Equation (5.9) corresponds to (1.6) and says that firms will hire workers until the marginal product of

labor equals the real wage. Condition (5.10) is similar to (1.7) and it says that the shadow price of investment

is equal to the price of investment. Firms will be at the optimum whenever they are indifferent between

investing and not investing an additional unit of capital. This will happen when the contribution of an additional

machine to revenue, O, is equal to its market cost, P_I. Equation (5.11) expresses the dynamic behavior of the

shadow price of investment. Equation (5.12) is the usual transversality condition. Using (5.10), equation

(5.11) can be rewritten as

(5.14) $\dot{P}_I = (r + \delta) P_I - P_Y AF_K$

which is the first dynamic equation we need to characterize the solution to the model. We can make use of the

supply equation (5.4) and the constraint (5.6) to get the second dynamic equation

(5.15) $\dot{K} = h(P_I) - \delta K$

Equations (5.14) and (5.15) form a system of two first order differential equations that, together with

the initial condition (5.7) and the terminal condition (5.12) characterize the solution of the model.

(iii) A Housing Market Interpretation.

The housing market seems one where this external adjustment costs are particularly important. It seems plausible that the marginal cost of constructing additional buildings is increasing in the number of buildings being constructed simultaneously. Hence, the model above is potentially a good description of the behavior of residential investment. Let us reinterpret K as the stock of housing so I_t is residential investment or housing construction at an instant in time and P_{It} is the price of a new home at that particular time. Let us imagine that the construction companies sell the new residential units to some real estate agents (which corresponds to what we called "producers of manufactured goods) at price P_{It}. These real estate agents combine labor (maybe for maintenance) and the total stock of housing to generate housing services, which they rent at rate P_Y. Hence we can think of $P_YAF(K,L)$ as the total amount of "rents" or income received by the real estate agents. A particular functional form for $AF()$ is $AF(K,L)=K$ which means that rental income is proportional to the value of the existing stock of houses. In our model P_Y is exogenous, but a simple extension would include consumers whose utility function depends on housing services. Maximization of utility subject to budget constraints will yield demand for housing services which, together with the supply that comes from the present model, determines the equilibrium rents of the economy. Equation (5.14) can be reinterpreted if we rewrite it as follows

$$(5.14)' \quad r = \frac{\dot{P}_I + P_Y A F_K - \delta P_I}{P_I}$$

The rate of return to purchasing a house is equal to the marginal contribution of the new house to rents plus the capital gains (increase in price) minus depreciation. Notice that what matters is P_YAF_K, the contribution of the EXTRA house to rents -or the rents associated with this extra unit of housing- rather than

$P_Y AF(K,L)$, the total amount of rents received by the real estate developers. In our example where

$AF(K,L)=K$, we have that $P_Y AF_K$ is equal to the rental rate P_Y. Equation (5.14)' says that this return to

investing in a house must equal the real interest rate (which is the real return of purchasing a safe bond). In

other words, real estate agents will purchase houses from construction firms up to the point where they are

indifferent between one more house and a bond with real return i. Equation (5.14) can be integrated forward

to get

$$(5.14)'' \quad P_{I0} = \int_0^{\overline{\infty}} e^{-(r+\delta)t} P_{Yt} AF_K(t)\, dt$$

that is, the price of a house must equal the present value of all future rents associated with it $(P_Y AF_K)$. Finally,

condition (5.15) just says that the net supply of housing is equal to the gross supply (which is an increasing

function of the price, given the increasing marginal cost assumption) minus depreciation. Equations (5.14) and

(5.15) closely correspond to the arbitrage condition and residential investment supply in, for instance, the

Poterba (1984) model of housing.

(iv) Graphical Analysis.

We can now analyze the dynamics of the system. As usual, we will display the $\dot{K} = 0$

and $\dot{P}_I = 0$ schedules in a diagram, and analyze the dynamics around steady state. Condition (5.14) says

that the $\dot{P_I} = 0$ schedule is a negative relation between P_I and K, given that F_{KK} is negative. That is

(5.17) $\quad P_I = \dfrac{P_Y A F_K}{r+\delta}$

This relation is depicted in Figure 11. Notice that for P_I's higher than those on the schedule, $\dot{P_I}$ is

positive so the arrows point north. The opposite is true for points below the schedule. Equation (5.15) implies

a that the $\dot{K} = 0$ schedule is a positive relation between P_I and K.

(5.19) $\quad \dfrac{dP_I}{dK} = - \dfrac{-\delta}{h'(P_I)} > 0$

where $h'()>0$ because the marginal cost in the supply of investment goods is increasing. The $\dot{K} = 0$

schedule is also depicted in Figure 12. Notice that points above it imply $\dot{K} > 0$ (so arrows point east) and

points below it imply $\dot{K} < 0$ (so arrows point west). It follows that the system displays saddle path

stability. The TVC ensures that we will lie on the stable arm. The initial condition, K_0 tells us where in the

stable arm we start. Suppose as an example that the initial K_0 is smaller than the steady state K^*. The phase

diagram says that the initial price will be larger than the steady state one. The reason is that since the initial

stock of capital (or houses) is smaller than the steady state, producers of manufactured goods (or real estate

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agents) demand a large amount of investment. The marginal cost of producing such a large amount is large

and that pushes the price P_I above steady state. As the capital stock is getting closer to the steady state level,

the desired investment gets close to zero and the market price reaches its steady state level.

(vi) Effects of Abolishing a Law of Rent Control (or decrease in tax rates or positive

productivity shocks).

We can now use this model to analyze the dynamic behavior of the economy when we confront it with

different type of shocks. Let us start by thinking about the effects of abolishing a law that has been reducing

rents below the market rate. In our model, this corresponds to an exogenous and permanent increase in P_Y.

Shocks that produce the same effect would be permanent productivity shocks in the manufacturing sector or

permanent decreases in taxes paid by manufacturers (increase in A). Notice that this implies an upward shift

in the $\dot{P_I} = 0$ schedule and no shift in the $\dot{K} = 0$ schedule. This is depicted in Figure 12. Notice that

there is an immediate jump in housing prices. This leads to positive residential investment and, consequently,

to a steady increase in the stock of houses until a steady state with larger stock is reached. An interesting

feature of this model (to which we will come back later) is that the steady state housing price is higher than the

initial one. The reason is that, in order to maintain a constant stock of houses or capital, firms have to

purchase the replacement for depreciates at price P_I. As we saw, one of the implications of abolishing the

rent control law is that the steady state capital stock is higher and, consequently, total steady-state

depreciation also increases. Hence, the steady state gross investment (equal to total depreciation since net

investment is zero) is higher and so is the marginal cost of producing it. This is why the steady-state price of capital goods is larger. The time paths for K, P_I and I are depicted in Figure 14.

This model can be used to analyze a number of shocks and policies: temporary and anticipated shocks to the productivity of the manufacturing sector, changes in residential property taxes or tax rates on rental income. We are not going to do these here because the analysis is analogous to that of the previous section.

The model can also be extended to analyze the effects of exogenous increases in population growth on the price of houses over time (see Mankiw and Weil (1989) for such an exercise).

(6) The Relation between External and Internal Adjustment Costs.

We have now seen that the introduction of either internal or external adjustment costs generates theories of investment that appear to be more sensible than the neoclassical theory without adjustment costs. A natural question to ask is whether the two theories are really different. After all, adjustment costs are adjustment costs. Internal and external adjustment costs are conceptually two different animals: by definition, internal costs are technological costs that have to be paid before we can use the newly purchased machinery. External costs are pecuniary cost that the users pay to the producers of capital goods.

Yet costs that are external to the firm are not external to the economy. Thus, from a macroeconomic perspective there should be no distinction between internal and external adjustment costs. We can see this point by reformulating the whole model of external costs as follows. Suppose that the producer of capital goods is not a separate firm but, instead, a division within the manufacturer. Whenever the manufacturer wants more capital goods, he goes to the "investment division" and asks for them. This division produces them with its Decreasing Returns to Scale Technology and gives them to the manufacturing division. Recall

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that in section (1) we said that in the neoclassical model, manufacturers could decide to give up some output and transform it into capital with a transformation technology that exhibited CRS (in fact we assumed capital and manufactured goods were the same goods so the transformation technology had a unit slope). We are now changing the transformation technology and assuming that it is DRS (see Figure 15). In other words, if we want an additional unit of capital we need to give up a lot or a little of manufactured output depending on whether we are already transforming a lot or a little. This is the essence of the external adjustment costs. But notice that this is not very different from saying that there is an "investment division" that transforms output into "uninstalled capital" with a CRS technology and there is also an "installation division" that transforms uninstalled capital into installed capital with a DRS technology (see Figure 15). This, of course, corresponds to the internal adjustment cost model.

When we consider all divisions together, there should be no difference between having an investment division with a DRS technology or an investment division with a CRS technology plus an installation division with DRS technology. Thus at the macroeconomic level, the two models should be the same. If we compare Figures 5 and 12, however, we could be tempted to argue that this is not true. After all, recall that the $\dot{K} ? 0$ schedule in the internal cost model was a horizontal line at q=1 while the same schedule for the external cost model was an upward sloping line. This had the implication that permanent increases in productivity increased the steady state capital stock in both models BUT IT INCREASED THE STEADY STATE PRICE IN THE EXTERNAL COST MODEL ONLY. This is true. The question is whether this is the result of modeling internal as opposed to external adjustment costs or the result of some other assumption. Our claim is that it is due to the particular assumptions on the functional forms of the adjustment cost.

First, for the case of external adjustment costs, we assumed functional forms that yield marginal costs as functions of the absolute LEVEL of GROSS investment. Under these circumstances, steady states with high capital stocks, and therefore high levels of replacement investment, are steady states with high marginal costs. This is why the $\dot{K} = 0$ schedule was upward sloping. Of course we could have a flat $\dot{K} = 0$ schedule

by assuming that replacement investment does not have to pay adjustment costs. This can be seen by substituting (5.3) by $c'(I - \delta K) = P_I$. With such a marginal cost function, the capital accumulation

condition (5.4) becomes

(5.4)' $\dot{K} = I - \delta K = h(P_I)$

which, of course, implies a flat $\dot{K} = 0$ schedule at $P_I = h^{-1}(0)$ which is a constant.

There is another, perhaps more important difference between the internal and external adjustment cost models we have shown in these notes and that is the functional forms assumed. In the case of internal adjustment costs we assumed that unit costs were a function of I/K. We argued that the reason why adjustment costs depended negatively on the stock of capital was that firms "learned" to reduce adjustment costs from past investment experiences, which add up to the existing stock of capital. On the other hand, we assumed that the investment technology in the external adjustment cost case was such that the marginal cost was a function of I (not of I/K) so we could write $P_I = c'(I)$. One could also think that these firms are subject to "learning by doing" and that their marginal costs are reduced if they have invested a lot in the past. In other

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words, the marginal cost would depend positively on I and negatively on K:

$$(5.3)' \quad P_I = c'(I/K)$$

the accumulation equation (5.4) would become

$$(5.4)'' \quad \dot{K} = I - \delta K = K[h(P_I) - \delta]$$

where *h( )* is the inverse of the marginal cost function, *c'( )*. Equation (5.4)' also yields a flat $\dot{K} = 0$

schedule, at $P_I = h^{-1}(\delta) = c'(\delta)$, just as in the internal adjustment cost model.

Summarizing, from a macroeconomic perspective the two models of investment are the same because costs that are external to the firm are not external to the economy. The seemingly different behavior of steady state prices in our two models is entirely due to the (seemingly arbitrary) different assumptions about depreciation and the functional form of the adjustment technology. In particular, the key assumptions are whether replacing depreciated capital involves the adjustment costs or not and whether adjustment costs are a function of the level of investment or the rate of investment per unit of capital. The two models will be essentially different only if there are compelling reasons why the marginal internal adjustment cost should be fundamentally different from the marginal external adjustment cost.

(8) Conclusions

We first presented a version of the Neoclassical model of investment and argued that it had some undesirable features such as infinite investment rates at points where the desired capital stock changed.

We then expanded the neoclassical model to include internal adjustment costs and saw that the infinite investment rates feature disappeared. We showed that, under constant returns to scale, the model was directly testable since **marginal q** was equal to **average q**. Some problems arise, however, when capital goods are heterogeneous or when financial markets are not fully efficient.

We then allowed for external adjustment costs. This application was particularly interesting for explaining residential investment. We saw that the model presented could be seen as a micro story behind some housing market models such as Poterba (1984).

Finally, we argued that from a macroeconomic perspective, the two models (internal and external cost models) were equivalent if the assumptions on depreciation and the functional form of the adjustment technology are the same. In particular, the key questions are whether replacing depreciated capital bears adjustment costs or not and whether the marginal adjustment cost function involves absolute levels of investment or rates of investment per unit of capital.

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