Recursive Competitive Equilibrium

References: L&S (3rd edition) chapter 12 and 18.15

1 The Neoclassical Growth Model (again!)

- Recall the model:
  - Preferences
$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$
  - Technology
$$\begin{aligned} Y_t &= F(K_t, L_t) \\ K_{t+1} &= (1-\delta) K_t + I_t \\ C_t + I_t &= Y_t \\ K_0 & \text{ given} \end{aligned}$$
- Competitive Equilibrium:
  - Allocation $\{Y_t, C_t, K_t, L_t\}_{t=0}^{\infty}$
  - Prices $\{p_t, w_t, r_t^K\}_{t=0}^{\infty}$

  such that
  - Households and firms maximize
  - Markets clear
- What we did so far with this model

1. Characterized equilibrium directly:
   - Solve household sequence problem

- Show that firm's problem is static
- Find equilibrium prices

2. Appealed to the First Welfare Theorem

- Competitive Equilibrium coincides with social planner problem
- Social planner problem can be represented recursively

1.1 Why it's not straightforward to represent the household problem recursively

- If household has k_t today and k_{t+1} tomorrow, then this period's utility is can be found as follows:
  - Sequential form of the budget constraint:

$$k_{t+1} = \left(1 - \delta + r_t^K\right) k_t + w_t - c_t$$
$$\Rightarrow c_t = \left(1 - \delta + r_t^K\right) k_t + w_t - k_{t+1}$$

  - Replace in utility:

$$u\left(\left(1 - \delta + r_t^K\right) k_t + w_t - k_{t+1}\right)$$

- If the prices r_t^K and w_t were constant, we could write

$$V(k) = \max_{k' \in [0, (1-\delta+r^K)k+w]} u\left(\left(1 - \delta + r^K\right) k + w - k'\right) + \beta V(k')$$

  and solve to household's problem recursively

- But the prices r_t^K and w_t can be time-varying!

1.2 An aggregate state variable

- Idea: state variable summarizes everything you need to know about the economy
- We need to find X such that prices and X' are a function of X only
- For the (deterministic) NGM, the aggregate capital stock K is a sufficient state variable.
- Prices gross interest rate $R(K)$, capital rental rate $r^K(K)$ and wage $w(K)$ (endogenous)
- A "law of motion" for K such that $K' = G(K)$

- Household and firm take these as given
- State variables for the household: K and wealth a
- Household problem:

$$V(a,K) = \max_{a' \in [0, R(K)a + w(K)]} u\left(R(K)a + w(K) - a'\right) + \beta V\left(a', G(K)\right) \tag{1}$$

- Firm problem:

$$\max_{k,L} f(k,L) - w(K)k - r^K(K)L \tag{2}$$

- No arbitrage between capital and borrowing/lending:

$$R(K) = 1 - \delta + r^K(K) \tag{3}$$

1.3 Definition of Recursive Competitive Equilibrium

Definition 1. A recursive competitive equilibrium is:

- a set of prices $\left\{R(K), r^K(K), w(K)\right\}$
- a law of motion $K' = G(K)$
- a value function $V(a,K)$
- a policy function for households $a' = g(a,K)$
- factor demands for the firm $k^D(K)$ and $L^D(K)$

such that

- The value function V and policy function g solve (1), taking $\left\{R(K), r^K(K), w(K)\right\}$ and $G(K)$ as given
- $k(K) = K$ and $L(K) = 1$ solve (2), taking $\left\{r^K(K), w(K)\right\}$ as given
- the no-arbitrage condition (3) is satisfied
- The law of motion G is consistent with individual choice:

$$g(K,K) = G(K) \tag{4}$$

- Interpretation of each condition

- Rational expectations built into the definition
- "Small k, big K" formulation

1.4 Re-derivation of stuff we already knew about the NGM

- $$V(a,K) = \max_{a' \in [0, R(K)a + w(K)]} u\left(R(K)a + w(K) - a'\right) + \beta V\left(a', G(K)\right) \tag{5}$$

- Household FOC:

$$-u'\left(R(K)a + w(K) - g(a,K)\right) + \beta \frac{\partial V}{\partial a}\left(g(a,K), G(K)\right) = 0$$

- Envelope condition:

$$\frac{\partial V}{\partial a}(a,K) = R(K)\, u'\left(R(K)a + w(K) - g(a,K)\right)$$
$$\frac{\partial V}{\partial a}\left(g(a,K), G(K)\right) = R(G(K))\, u'\left(R(G(K))\, g(a,K) + w(G(K)) - g\left(g(a,K), G(K)\right)\right)$$

- Imply Euler equation:

$$-u'\left(R(K)a + w(K) - g(a,K)\right) + \beta\left[R(G(K))\, u'\left(R(G(K))\, g(a,K) + w(G(K)) - g\left(g(a,K), G(K)\right)\right.\right.$$

- Using the consistency condition (4), constant returns and (3):

$$-u'\left(f(K,L) + 1 - \delta - G(K)\right) + \beta\left[\left(1 - \delta + f'\left(G(K), L\right)\right) u'\left(f\left(G(K), L\right) + 1 - \delta - G(G(K))\right)\right] = 0$$

which, as we know, is the same as what the planner would choose

1.5 Computation

- One approach that (usually) works:

  1. Guess $G(K)$
  2. Find expressions for $r^K(K)$ and $w(k)$ from the firm problem and $R(K)$ from (3)
  3. Solve the household's recursive problem in the standard way and find the policy function $g(K)$
  4. Update the guess of $G(K)$ using (4)

- You can think of this whole operation as a mapping that starts with some $G : X \to X$ and returns a different G
- There is no guarantee that this mapping will be a contraction

1.6 When is this especially useful?

- Problems where:
  - The sequence problem is difficult
  - The assumption of the FWT don't hold, so a planner's solution might not coincide with the competitive equilibrium
- Example: an economy with externalities
- Production function is

$$f(k, L, K)$$

e.g.

$$k^\alpha L^{1-\alpha} K^\gamma$$

- Interpretation: learning by doing
- Social planner would solve

$$V(K) = \max_{K'} u\left(f(K, L, K) + (1-\delta) K - K'\right) + \beta V(K')$$

- Equilibrium conditions are just like in the NGM
- If you tried to solve the planner's problem you would have the Euler equation

$$u'(c_t) = \beta\left[1 - \delta + (\alpha + \gamma) K^{\alpha+\gamma-1} L^{1-\alpha}\right] u'(c_{t+1})$$

whereas in equilibrium you have

$$u'(c_t) = \beta\left[1 - \delta + \alpha K^{\alpha+\gamma-1} L^{1-\alpha}\right] u'(c_{t+1})$$

2 Extension to Stochastic Problems

- The concept of Recursive Competitive Equilibrium extend readily to the stochastic environement

- Consider the NGM with stochastic productivity (also known as the Real Business Cycle model)
- The production function is

$$\theta(s) f(k, L)$$

  where s follows a Markov process.
- Assume there are complete markets for one-period-ahead Arrow securities
- Aggregate state variable is $X = (K, s)$
- The "law of motion" of X is a Markov process, defined by

$$\Pi(X'|X)$$

  - Π is endogenous because K' is endogenous
- We need to specify prices:
  - $w(X)$: wage
  - $r^K(X)$: rental rate of capital
  - $q(X, X')$: price of one-period-ahead Arrow securities
- Household solves

$$V(a, X) = \max_{a'(X')} u\left(w(K) + a - \sum_{X'} q(X, X') a'(X')\right) + \beta \sum_{X'} \Pi(X'|X) V(a'(X'), X') \tag{6}$$

- For this stochastic example, the definition of Recursive Competitive Equilibrium is modified as follows
  - Note that here the timing assumption is slightly different and a enters the budget constraint instead of Ra
  - Π takes the place of G as the (now stochastic) law of motion
  - But Π is derived from the exogenous Markov process for s plus the law of motion for K, which we can write as $K' = G(X)$, so the consitency condition (4) becomes

$$g(K, X) = G(X)$$

– Prices of Arrow securities appear instead of the single interest rate R, so the no-arbitrage condition becomes

$$1 = \sum_{X'} q\left(X, X'\right)\left[r^K\left(X'\right) + 1 - \delta\right]$$

3 Aggregate Shocks and Idiosyncratic Risk Together

3.1 The Problem

- Suppose we want to analyze how an economy with uninsured idiosyncratic risk responds to aggregate shocks
- First Welfare Theorem doesn't hold because we have incomplete markets
- Sequence problem for each household is complicated because of all the individual + aggregate histories we need to keep track of
- Ideal:
  - State variable: $\{\Gamma, \theta\}$
  - Γ is the joint distribution of asset holdings A_i and effective-labour-endowmentshocks s_i. Γ_A and Γ_s are the marginal distributions.
  - θ is the aggregate productivity shock (or any other kind of aggregate shock)
  - The capital stock is

$$K\left(\Gamma\right) = \int A d\Gamma_A\left(A\right)$$

  - Factor prices are:[1]

$$\begin{aligned} w\left(\Gamma, \theta\right) &= F_L\left(K(\Gamma), \theta, L\right) \\ r^K\left(\Gamma, \theta\right) &= F_K\left(K(\Gamma), \theta, L\right) \\ R\left(\Gamma, \theta\right) &= r^K\left(\Gamma, \theta\right) + 1 - \delta \end{aligned}$$

  - so the household problem is

$$V\left(A, s, \Gamma, \theta\right) = \max_{c, A'} u(c) + \beta \mathbb{E}\left[V\left(A', s', H\left(\Gamma, \theta\right), \theta'\right) | s, \Gamma, \theta\right] \tag{7}$$
$$s.t. \quad A' \leq l(s) w\left(\Gamma, \theta\right) - c + R\left(\Gamma, \theta\right) A$$
$$A' \geq -b$$

[1] Implicit in this formulation is th assumption that all savings are in the form of capital. Because there is aggregate risk, the return on savings will not be risk-free. There is still no insurance against idiosyncratic risk.

where H is the law of motion of Γ, and equilibrium requires that H (which the household takes as given) be consistent with household decisions.

- Why does the household need to know $\{\Gamma, \theta\}$ and the transition rule H? In order to forecast future factor prices.
  - This is just like in the NGM (or any other competitive model). The only reason to care about the aggregate is to forecast prices.
- Computational problem: infinite-dimensional state variable

3.2 The Krusell and Smith [1998] approximate solution

- Assume that agents believe that current and future prices depend only on the first I moments of the distribution Γ (rather than the full distribution)
- E.g. if $I = 2$, agents believe future prices depend only on
  - today's mean asset holidngs $\int Ad\Gamma_A(A)$
  - today's mean labour-endowment shocks $\int l(s)\, d\Gamma_s(s)$, which maybe we can assume is a constant
  - the variance of asset holdings
  - the variance of labour-endowment shocks, which maybe we can also assume is a constant
  - the covariance of asset holdings and labour-endowment shocks
  - (and of course future realizations of aggregate shocks)
  - (but not on the skewness of distributions, etc.)
- Note that for curent prices, this is exactly true even for $m = 1$ because only the total capital stock matters. The approximation is with regards to how higher moments of the **current** wealth\productivity distribution affect **future** capital stocks.
- The reason why higher moments matter is that the policy function $A'(A, s)$ is nonlinear in A!
- Let m be the vector of I moments. Agents believe there is a law of motion $m' = H_I(m, \theta)$
  - Note that agents will continually be a little bit surprised that m' turns out not to be exactly equal to $H_I(m, \theta)$
- Given a conjectured law of motion H_I, there will be optimal decision rules that yield **actual** laws of motion for the **true** Γ

  - Note: there will be no **actual** law of motion for m because m' will in general depend on the higher moments and not just on m

- Over a simulated path for many individuals using the optimal decision rule under H_I, one can find the best approximation $\hat{H}_I$ to a law of motion for m (for instance, by linear regression of m' on m and θ) within some class of possible transition functions (for instance, linear functions)

- An approximate equilibrium is given by:

  - A conjectured law of motion for m called H_I
  - Decision rules for the agents

  such that

  - The decision rules for the agents are optimal given H_I
  - $H_I = \hat{H}_I$ (i.e. the conjectured law of motion is equal to the best possible prediction of m' given m)

- Computationally, the steps are:

1. Guess H_I
2. Solve consumer problem
3. Simulate the behaviour of many consumers over time
4. For each period compute m
5. Estimate the best-fit $\hat{H}_I$, for instance by regression of m' on m and θ
6. Iterate until $\hat{H}_I = H_I$

- Then you need to check whether the fit is good (for instance, by checking the R^2 of predicting m' on the basis of m), to see if the agents will make large mistakes by relying on H_I

3.3 Approximate aggregation

- In calibrated example, Krusell and Smith [1998] find that the $R^2 = 0.999998$ with $I = 1$
- Forecasting K'on the basis of K alone works extremely well!
- Despite the fact that the policy function is nonlinear, it is not far from linear for most agents

  - This comes from calibration with not-very-persistent unemployment risk and low-ish risk aversion

- Even if (in other calibrations) there were many agents with nonlinear policy functions, these would be the poor ones and most of the wealth is held by the wealthy - for those guys, the policy function is almost linear

- (But in plain Aiyagari [1994] we don't really match the welath distribution. Doesn't that matter?)

  - One way to match the wealth distribution is to add discount-rate shocks, so that the patient become wealthy. Approximate aggregation still holds

References

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Per Krusell and Anthony A Smith. Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896, 1998.