

14.453: Problem Set #4

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1 Incomplete Markets and Asset Prices

This problem investigates the effects of market incompleteness on asset pricing following Constantinides and Duffie (1996). There is a continuum of individuals with identical CRRA preferences: $u(c) = c^{1-\gamma} / (1 - \gamma)$ and subjective discount factor is β .

- (a) Write down the Euler equation for individual i and asset j . This equation should relate the gross return R_{t+1}^j to the growth rate of consumption for individual i : c_{t+1}^i/c_t^i .
- (b) Now assume that the growth rate of individual consumption satisfies,

$$\frac{c_{t+1}^i}{c_t^i} = \frac{c_{t+1}}{c_t} \varepsilon_{t+1}^i, \quad (1)$$

where c_{t+1}/c_t is the growth rate of aggregate consumption. Here ε_{t+1}^i represents the idiosyncratic component of consumption growth. Conditional on σ_{t+1}^2 , c_{t+1}/c_t and R_{t+1}^j , assume ε_{t+1}^i is independent across individuals and $\log \varepsilon_{t+1}^i$ is distributed $N(-\frac{1}{2}\sigma_{t+1}^2, \sigma_{t+1}^2)$. The cross sectional variance parameter, σ_{t+1}^2 , is a random variable from the point of view of time t , it becomes known at time $t + 1$. Note that σ_{t+1}^2 is not assumed to be independent of c_{t+1}/c_t and R_{t+1}^j (i.e. the three variables may be correlated with each other).

Use the individual Euler equation from (a) to show that:

$$1 = \beta E_t \left[\Phi(\sigma_{t+1}^2) \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} R_{t+1}^j \right] \quad (2)$$

where

$$\Phi(\sigma_{t+1}^2) \equiv E \left[(\varepsilon_{t+1}^i)^{-\gamma} | \sigma_{t+1}^2 \right] = \exp \left\{ \frac{\gamma(1+\gamma)}{2} \sigma_{t+1}^2 \right\}.$$

- (c) Specialize the above by assuming that,

$$\sigma_{t+1}^2 = A - B \log \left(\frac{c_{t+1}}{c_t} \right).$$

Show that,

$$1 = \hat{\beta} E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\hat{\gamma}} R_{t+1}^j \right]$$

holds with,

$$\begin{aligned}\hat{\gamma} &= \gamma + \frac{1}{2} \gamma (1 + \gamma) B \\ \hat{\beta} &= \beta \exp \left\{ \frac{1}{2} \gamma (1 + \gamma) A \right\}\end{aligned}$$

- (d) How can these results help explain the equity premium puzzle?

2 Two-Sided Lack of Commitment: Stationary Allocations

Consider the environment of Alvarez and Jermann (2000), where the economy is populated by equal number of two types of agents with perfectly negatively correlated endowment, both with lack of commitment. Focus on the two-state case.

Based on the result that for the symmetric case with 2 agents and 2 shocks the optimal allocation eventually reached a "memory-less" (history independent) allocation, we now seek to characterize optimal stationary symmetric distributions.

- (a) Given (c^1, c^2) , show that

$$\begin{aligned}V^1(c^1, c^2) &= \frac{1}{1-\beta} \{ \omega u(c^1) + (1-\omega) u(c^2) \} \\ \text{where } \omega &= \frac{1-\beta p}{1+\beta-2p\beta} > \frac{1}{2}\end{aligned}$$

Moreover, show that ω is **decreasing in β and increasing in p** .

- (b) We call a stationary symmetric allocation *feasible* if it satisfies the resource and participation constraints:

$$c^1 + c^2 = e$$

$$V^1(c^1, c^2) \geq V^1(y^1, y^2) \quad (3)$$

$$V^2(c^1, c^2) \geq V^2(y^1, y^2) \quad (4)$$

Notice that autarky is always feasible.

Show that in any symmetric allocation (4) never binds. That is, show that whenever (3)

holds then (4) automatically holds with strict inequality.

(c) Show that full risk sharing is attainable if and only if:

$$u(e/2) \geq \omega u(y^1) + (1 - \omega) u(y^2) \quad (5)$$

(d) Here we use the comparative static results for ω found in part (a) and the result in part (c) to examine the parameters that affect the feasibility of risk sharing.

How do β and p affect the likelihood of full-risk sharing being feasible?

Show that for small enough spread between y^1 and y^2 (holding e constant) full risk sharing is not possible. Let utility take the form $u(c) = c^{1-\sigma} / (1 - \sigma)$ show that if σ is sufficiently close to 0 full risk is not feasible.

(e) If full risk sharing is not attainable we are interested in the best allocation that is feasible. Using your results from part (b) show that if (5) is not satisfied the best symmetric allocation satisfies

$$c^1 + c^2 = e$$

$$\omega [u(c^1) - u(y^1)] + (1 - \omega) [u(c^2) - u(y^2)] = 0$$

and $y^2 \leq c^2 \leq c^1 \leq y^1$ (i.e. satisfies the above two equations and has less variability than autarky).

(f) Here we compute numerically the optimal allocation for the case where the utility function is of the CRRA form: $u(c) = c^{1-\sigma} / (1 - \sigma)$.

Use the following parameters¹ $\beta = .65$, $p = 0.75$, $y^1 = 0.641$ and $y^2 = 0.359$. Plot the optimal c^1 and c^2 as functions of σ for the range $\sigma \in [1, 5]$ (i.e. use a grid over σ with enough points between 1 and 5)².

3 Investment with Fixed Costs

Consider the following problem of a firm under uncertainty in discrete time. At time t current profits net of user costs of capital are given by $\pi(k_t, z_t)$ where $z_t \in Z$ is the current shock to profits and k_t is the amount of capital used during period t . The function $\pi(k, z)$ is assumed to be strictly concave in k . Assume z_t follows a Markov process.

At the beginning of the period the current shock z_t is realized and the firm observes it.

¹These parameters imply a standard deviation for log-output of .29 and a first-order autocorrelation of .5, matching findings by Heaton and Lucas (1996) using the PSID.

²Hint: Make sure you first check for perfect risk sharing. If full risk sharing is available take that allocation. Otherwise compute the allocation that satisfies the requirements in part (e), which may imply autarky or some insurance (watch out: do not compute an allocation with more variability than autarky!).

The firm then decides whether or not to adjust its capital stock.

If the firm decides not to adjust its capital then its capital remains at the previous level k_{t-1} – we assume no depreciation. If instead the firm chooses to adjust its capital then it pays a fixed cost $c(z_t)$ and selects a new capital level k_t . The costs may depend on the shock z_t but *do not* depend on the new level of capital k_t chosen.

The firms objective is to maximize expected discounted profits net of any costs incurred for changing its capital stock. The firm discounts profits using a constant interest rate r .

a. Argue that k_{t-1}, z_t is the relevant state variable for a firm trying to decide whether or not to adjust capital. Set up the Bellman equation for the value of the firm $v(k_-, z)$ where k_- represents the previous period's capital stock and z represents the current shock.

b. Use your Bellman equation from (a) to argue that the optimal policy can be summarized by a region of inaction $K(z)$ and an optimal adjustment policy $k^*(z)$. For each $z \in Z$ the region of inaction $K(z)$ is a subset of \mathbb{R}_+ such that adjustment occurs in the current period if and only if $k_- \notin K(z)$. If adjustment occurs then the firm chooses $k^*(z)$ as its new capital stock. (note: you are not asked to actually solve for $K(z)$ or $k^*(z)$ just to show that the solution takes this form).

c. Is $k^*(z)$ equal to the unconstrained optimum level of capital, i.e. $k^*(z) = \arg \max_k \pi(k, z)$?

d. Can the region of inaction $K(z)$ always be summarized by “*sS* rules”, that is that for each state of the world the inaction region is a single closed interval so that there exists functions $\bar{k}(z)$ and $\underline{k}(z)$ such that

$$K(z) = \{k_- : \bar{k}(z) \geq k_- \geq \underline{k}(z)\}?$$

Why or why not?

e. Show that if the growth rate of z is i.i.d. (so that $z'/z = \varepsilon$ i.i.d.) and $\pi(k, z)$ and $c(z)$ are homogenous of degree one (constant returns to scale) so that $\pi(k, z) = s\pi(k/z, 1)$ and $c(z) = \bar{c}s$ then $v(k_-, z)$, $k^*(z)$, $K(z)$ are homogenous of degree one.

Define $\hat{k}_- = k_-/z$. What do these results imply about the region of inaction $\hat{K}(z)$ and the optimum adjustment $\hat{k}^*(z)$ for \hat{k}_- ?

f. Consider an industry with a continuum of ex-ante identical firms facing their own idiosyncratic z -shocks that are independent over time and across firms. Suppose these firms use *sS* rules for k_- as described in part (d). Briefly discuss how you would think of a steady state in such an economy – what kind of object is a possible equilibrium and what does it have to satisfy? What determines the steady state level of aggregate investment?