Incomplete Markets

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Introduction

- So far: 'representative agent' framework where behavior of macro aggregates did not depend on individual heterogeneity
- Uncertainty was on the aggregate level
- Now: individual agents are hit by idiosyncratic shocks
- Shocks are not insurable because of incomplete markets

Introduction

- We will study the consequences of market incompleteness for individual consumption and savings behavior as well as for aggregate capital and equilibrium prices
- Will see that macro aggregates depend on individual heterogeneity
- Incomplete market models can be used to understand
  - origins of wealth inequality
  - wealth mobility
  - implications of financial market development

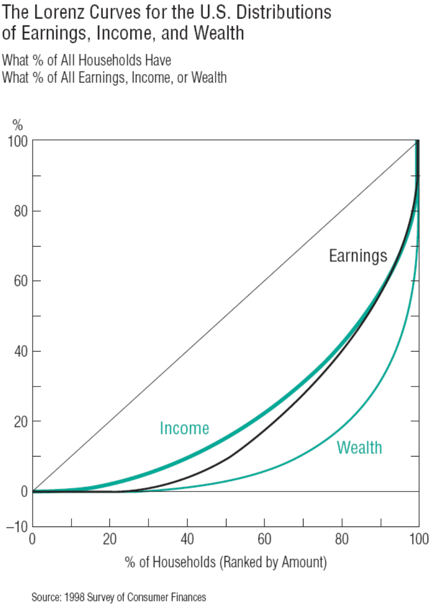


Figure: from Diaz-Giminez, Quadrini and Rios-Rull (1997)

Outline

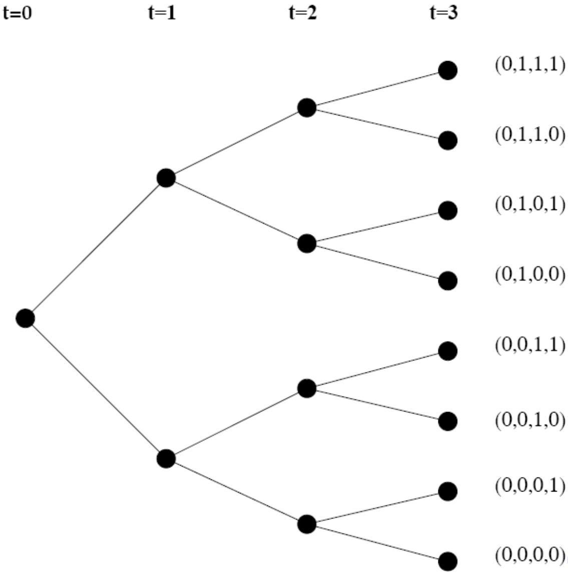
- Part I: Complete vs. incomplete markets
- Part II: Implications of market incompleteness for *individual* consumption and savings decisions
- Part III: Implications of market incompleteness for *aggregate* consumption and savings

Part I: Complete vs. Incomplete Markets

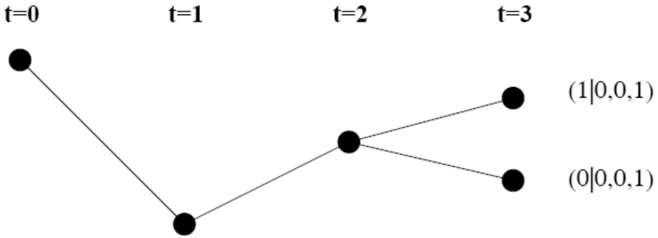
Uncertainty

- $s_t \in S_t$: current state of economy at time t
- $s^t = s_0, s_1, ..., s_t$: history up to time t
- $s^t \in S^t$, $S^t \equiv S_0 \times S_1 \times ...S_t$
- $\pi(s^t)$: probability that history s^t occurs
- $y_t^i(s^t)$: income of individual i upon realization of s^t
- $\sum_{i \in I} y_t^i(s^t) = Y_t(s^t)$

History: Example



Specific History: Example



Households Problem

- Consider the problem of an individual household $i \in I$ who wants to maximize expected lifetime utility:

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t))$$

with $u' > 0$, $u'' < 0$ and Inada conditions

Complete Markets: Arrow-Debreu

- First, consider an (endowment) economy with a full set of insurance and financial markets
- At time 0, individuals trade dated state-contingent claims (*Arrow securities*)
- Arrow-Debreu budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t)[c_t^i(s^t) - y_t^i(s^t)] = 0 \text{ for all } i \in I$$

- There is an asset for every possible contingency: complete markets
- Contracts on claims are perfectly enforceable

Households Problem

- Consider the problem of an individual household $i \in I$ who wants to maximize expected lifetime utility:

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t))$$

with $u' > 0$, $u'' < 0$ and Inada conditions

Complete Markets: Arrow-Debreu

- By the First Welfare Theorem, the competitive equilibrium can be characterized as the solution of the social planner problem:

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) \sum_{i \in I} \alpha^i u(c_t^i(s^t))$$

such that

$$\sum_{i \in I} c_t^i(s^t) = Y_t(s^t) \text{ for all } t, s^t \in S^t$$

CRRA preferences $\frac{c_t^i(s^t)^{1-\sigma}}{1-\sigma}$

- Risk sharing implies that

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}}, \text{ for all } t, s^t \in S^t$$

- Ratio of consumption between any two agents is constant across time and states
- Summing up over all $i \in I$:

$$\underbrace{\sum_{i \in I} c_t^i(s^t)}_{=C_t(s^t)(=Y_t(s^t))} = \sum_{i \in I} \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}} c_t^j(s^t)$$

- Individual consumption is a constant fraction of aggregate endowment:

$$c_t^j(s^t) = \frac{1}{(1/\alpha^j)^{\frac{1}{\sigma}} \sum_{i \in I} (\alpha^i)^{\frac{1}{\sigma}}} \cdot Y_t(s^t)$$

CRRA preferences

- If $Y_t(s^t) = \bar{Y}$, i.e. aggregate endowment is constant across states, then
- Then, $c_t^i(s^t) = \bar{c}^i$
- *Consumption is completely insured across time and states*

An example with heterogeneity

- Suppose preferences are $u(c) = \log c$
- Suppose there are two agents – A and B – whose endowments are deterministic and given by

$$
\begin{aligned}
y_t^A &= \begin{cases} 2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases} \\
y_t^B &= \begin{cases} 2 & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even} \end{cases}
\end{aligned}
$$

- Note: aggregate endowment is always $Y_t = y_t^A + y_t^B = 2$.

An example (cont.)

- Task: compute the competitive equilibrium
- From the welfare theorems, we can forumlate this as a planner problem with weights α^A and α^B (where $\alpha^A + \alpha^B = 1$)

$$
\begin{aligned}
& \max \sum_{t=1}^{\infty} \beta^t \left\{ \alpha^A u\left(c_t^A\right) + \alpha^B u\left(c_t^B\right) \right\} \\
& \text{subject to} \\
Y_t \quad = \quad & 2 = c_t^A + c_t^B
\end{aligned}
$$

An example (cont.)

- From the optimality condition above, we have

$$\frac{u'(c_t^A)}{u'(c_t^B)} = \frac{\alpha^B}{\alpha^A},$$

  which implies constant consumption $c_t^i = \bar{c}^i$.
- Need to solve for the planner weights α^A and α^B

An example (cont.)

- Recall: 2^{nd} welfare theorem says that any Pareto optimal allocation can be decentralized with transfers.
  - Need to focus on the allocation which does not have any transfers
- How compute net transfers?

An example (cont.)

- Define a transfer to e.g. agent A as

$$T_t^A = c_t^A - y_t^A$$

- Define net transfers as NPV of T_t^A:

$$NET^A = \sum_{t=1}^{\infty} p_t \cdot T_t^A,$$

where p_t is the Arrow-Debreu price of the period-t good

- But where does p_t come from?

An example (cont.)

- Need prices:
  - Use the information about consumption to back out the implied sequence $\{p_t\}_{t=1}^{\infty}$
  - Consider the Euler equation for any agent:

$$p_t \cdot u'\left(c_t^A\right) = p_{t+1} \cdot \beta u'\left(c_{t+1}^A\right)$$

  - Exploit that $c_t^A = c_{t+1}^A = \bar{c}^A$, so the Euler equation becomes

$$p_t = p_{t+1} \cdot \beta,$$

    which implies $p_{t+1}/p_t = 1/\beta$

An example (cont.)

- Normalize $p_1 = 1$ and obtain

$$NET^A = \sum_{t=1}^{\infty} \beta^{t-1} \cdot T_t^A,$$

- Now, insert the allocation for c_t^A and y_t^A and derive the value for α^A that ensures $NET^A = 0$

Empirical Relevance

- How well does the complete markets model describe the world?
- *"If the children of Noah had been able and willing to pool risks, Arrow-Debreu style, among themselves and their descendants, then the vast inequality we see today, within and across societies, would not exist."* (Lucas 1992, cited in Heathcote et al. 2009, p.5)

Empirical Relevance

- A more formal test is due to Mace (1991)
- Under perfect risk sharing and CRRA preferences, individual consumption should respond to aggregate income shocks, but not to individual income changes:

$$\Delta lnc_t^i = \alpha_1 \Delta \ln C_t + \alpha_2 \Delta \ln y_t^i + \epsilon_t^i$$

- Under the null hypothesis "complete markets", $\alpha_1 = 1$ and $\alpha_2 = 0$
- Under the null hypothesis "autarky", $\alpha_1 = 0$ and $\alpha_2 = 1$

Empirical Relevance

- Typically, both "complete markets" and "autarky" are rejected
- This suggest that there are markets on which households can buy some insurance against income fluctuations
- But because insurance is only partial, markets seem to be incomplete

Complete Markets: Sequential Trading

- Claims to one-period-ahead state-contingent consumption are traded in every period $t \geq 0$
- Budget constraint:

$$c_t^i(s^t) + \sum_{s_{t+1} \in S_{t+1}} q_t(s_{t+1}, s^t) a_{t+1}^i(s_{t+1}, s^t) = y_t^i(s^t) + a_t^i(s^t)$$

Complete Markets: Sequential Trading

Remark:

- With sequential trading, need to impose No-Ponzi scheme conditions:

$$\lim_{t\to\infty} \sum_{s^t \in S^t} q_t(s_t, s^{t-1}) a_t^i(s_t, s^{t-1}) \geq 0$$

- Ensures that the optimal consumption allocation is independent of the trading arrangement

Incomplete Markets

- Incomplete markets: agents are restricted to trade an uncontingent asset $a^i_{t+1}(s^t)$
- Budget constraint:

$$c^i_t(s^t) + q_t(s^t)a^i_{t+1}(s^t) = y^i_t(s^t) + a^i_t(s^{t-1})$$

Incomplete Markets

- $a^i_{t+1}(s^t)$ is a *zero-bond*, which ...
- ... pays 1 consumption unit in period $t+1$, *independently* of the realization of the state s_{t+1}
- ... is traded in period t with discount q_t

Incomplete Markets: History Dependence

- $a^i_{t+1}(s^t)$ depends on s^t, not on s^{t+1} as in the complete market case
- Consumption in $t+1$ depends on the realization of the state s_{t+1} and on the history s^t

Incomplete Markets: History Dependence

- We assume that households' i endowment in period t is time invariant functions of s_t: $y_t^i(s^t) = y^i(s_t)$
- s_t is governed by a Markov process, implying that endowments follow a Markov process as well
- In recursive version of households' problem, information about s^{t-1} and s_t is incorporated in y_t^i and a_t^i (state variables)
- No need to keep track of s^{t-1} and s_t
- For simplicity, we will ignore information structure even if we do not use recursive notation

Incomplete Markets: Constraints

- Use this to simplify budget constraint to

$$a_{t+1}^i = (1+r)(a_t^i + y_t^i - c_t^i)$$

- Assume, for simplicity, that $Y_t(s^t) = \bar{Y}$
- Assume economy is in steady state so prices are constant (will be verified later) $\Rightarrow q_t(s^t) = q$ and $q \equiv \frac{1}{1+r}$
- r is the interest rate of a risk-free bond
- The No-Ponzi Scheme condition reads as

$$\lim_{t\to\infty}\left[\left(\frac{1}{1+r}\right)^t a_t^i\right] \geq 0$$

Incomplete Markets: Self-Insurance

- Incomplete markets: there is no market for insurance against adverse shocks
- Agents have to 'self-insure' by managing a stock of a single asset to buffer their consumption
- In the next part, we will see that 'self-insurance' depends on (i) utility function, (ii) income process and (iii) existence of borrowing constraints

Part II: Implications of market incompleteness for individual consumption and savings decisions

Partial Equilibrium

- Focus on single-agent problems; drop household index "i"
- We study only the supply of assets: Partial Equilibrium
- Interest rate r is taken as given

Individual Household's Problem

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

such that

$$\begin{aligned} a_{t+1} &= (1+r)(a_t + y_t - c_t) \\ 0 &\leq E_0 \left[\lim_{t \to \infty} \left[\left(\frac{1}{1+r} \right)^t a_t \right] \right] \end{aligned}$$

Quadratic Utility

- 'Quadratic Utility'

$$u(c_t) = -\frac{1}{2}(c_t - \gamma)^2, \gamma > 0$$

- Marginal utility linear: $u'(c_t) = \gamma - c_t$
- If $\beta(1+r) = 1$, the Euler-Equation becomes

$$c_t = E_t c_{t+1}$$

- Consumption is a *martingale*

Definition: Martingale

- A (discrete-time) *martingale* is a (discrete-time) stochastic process X_1, X_2, X_3, ... that satisfies

$$\begin{aligned} E(|X_n|) &< \infty \\ E(X_{n+1}|X_1, X_2, \dots, X_n) &= X_n \end{aligned}$$

- A stochastic process is a *submartingale* if

$$E(X_{n+1}|X_1, X_2, \dots, X_n) \geq X_n$$

- A stochastic process is a *supermartingale* if

$$E(X_{n+1}|X_1, X_2, \dots, X_n) \leq X_n$$

Quadratic Utility

- 'Lifetime' budget constraint of an agent

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j} E_t c_{t+j} = a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j} E_t y_{t+j}$$

- Martingale property of consumption implies that

$$c_t = E_t c_{t+1} = E_t c_{t+2} \ldots = E_t c_{t+j}$$

Quadratic Utility: Permanent Income Hypothesis

- Re-writing budget constraint gives

$$c_t + \underbrace{\sum_{j=1}^{\infty} \left(\frac{1}{1+r}\right)^j E_t c_t}_{=\frac{c_t}{r}\text{'Annuity'}} = a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j E_t y_{t+j}$$

and

$$c_t = \underbrace{\frac{r}{1+r}\left[a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right]}_{\text{annuity value of total wealth: Permanent Income}}$$

- Consumption follows permanent income: 'Permanent Income Hypothesis' (Friedman 1954)

Quadratic Utility: Role of Income Uncertainty

- How does income uncertainty influence consumption?

$$\text{Income certain: } c_t = \frac{r}{1+r}\left[a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right]$$

$$\text{Income uncertain: } c_t = \frac{r}{1+r}\left[a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right]$$

- Given expectations about income, the agent behaves *as if income was certain*
- *Certainty Equivalence Principle* (Ljungvist/Sargent 2004): optimal consumption rule does not depend on variance and higher moments of the income process

Quadratic Utility: Role of Income Uncertainty

- Consumption is not insured against *unexpected* changes in income
- Notice difference to complete markets setting, where agents are fully insured

Permanent Income Hypothesis: Testable Implications

1. Consumption should not respond to *predictable* income changes
2. Consumption should respond one-to-one to *permanent* income shocks

Both implications are typically rejected - consumptions displays *excess sensitivity* w.r.t. predictable income changes and *excess smoothness* w.r.t. permanent income shocks

Permanent Income Hypothesis: Testable Implications

This suggests that there are important ingredients missing. We discuss the following two extensions:

- Borrowing constraints
- Prudence

Household's Problem with Quadratic Utility and Borrowing Constraints

- We will use the following setup to provide the answer:

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

such that

$$\begin{aligned} a_{t+1} &= (1+r)(a_t + y_t - c_t) \\ a_{t+1} &\geq -\underline{a} \end{aligned}$$

- Assume that $u(c_t) = -\frac{1}{2}(c_t - \gamma)^2, \gamma > 0$ and that $\underline{a}$ is such that No-Ponzi condition is fulfilled

Borrowing Constraints

- Let μ_t be the (current-value) Lagrange multiplier associated with the borrowing constraint $a_{t+1} \geq -\underline{a}$
- Let λ_t be the (current-value) Lagrange multiplier associated with the budget constraint $a_{t+1} - (1+r)(a_t + y_t - c_t) = 0$

Borrowing Constraints

Optimality conditions:

1. $\mu_t \geq 0$
2. Complementary slackness: $\mu_t(a_{t+1} + \underline{a}) = 0$
3. $u'(c_t) + \lambda_t(1 + r) = 0$
4. $\forall\ y_{t+1}$: $\pi(y_{t+1}|y_t)u'(c_{t+1}) + \lambda_{t+1}(1 + r) = 0\ \forall\ y_{t+1}$
5. $\beta^t \lambda_t + \beta^t \mu_t - \beta^{t+1} \sum_{y_{t+1}} (1 + r)\lambda_{t+1} = 0$

where $\pi(y_{t+1}|y_t)$: probability of y_{t+1}, given y_t (assume that node in the event tree is defined by a_t and y_t). Combining 3-5 yields

$$u'(c_t) = \beta(1 + r)E_t[u'(c_{t+1})] + \mu_t(1 + r)$$

where $E_t[u'(c_{t+1})] = \sum_{y_{t+1}} u'(c_{t+1})\pi(y_{t+1}|y_t)$

Borrowing Constraints

- Using $\beta(1+r) = 1$ and $u'(c_t) = \gamma - c_t$, we get

$$c_t = E_t c_{t+1} - \mu_t(1+r)$$

if $\mu_t > 0$ and

$$c_t^* = E_t c_{t+1}$$

otherwise

- If $\mu_t > 0$, $c_t < c_t^*$: binding borrowing constraints restrict consumption in period t

Borrowing Constraints

- Assume that $\mu_t = 0$
- Assume that constraint is expected to be binding in $t+1$: $E_t[\mu_{t+1}] > 0$
- Iterate Euler-Equation forward and use the law of iterated expectations:

$$\begin{aligned} c_t &= E_t c_{t+1} \Leftrightarrow \\ c_t &= E_t[E_{t+1}c_{t+2} - \mu_{t+1}(1+r)] \Leftrightarrow \\ c_t &= E_t[c_{t+2} - \mu_{t+1}(1+r)] \end{aligned}$$

- Borrowing constraint that is binding in expectations restricts consumption today

Borrowing Constraints

- (Potentially) binding borrowing constraints increase savings for 'self-insurance'
- Agents want to 'smooth out' impact of borrowing constraints on consumption

Quadratic Utility and Risk

- So far, we assumed quadratic utility $u(c_t) = -\frac{1}{2}(c_t - \gamma)^2$
- Hence, $u'(c_t) = \gamma - c_t$, $u''(c_t) = -1$ and $u'''(c_t) = 0$

Quadratic Utility and Risk

- Counter-intuitive implications: increasing absolute risk aversion in consumption
- Amount of consumption agents are willing to give up to avoid risk increases in consumption
- Arrow-Pratt measure of absolute risk aversion: $\alpha(c_t) \equiv \frac{-u''(c_t)}{u'(c_t)}$
- With quadratic utility: $\alpha(c_t) = \frac{1}{\gamma - c_t}$ and $\alpha_c(c_t) > 0$

Prudence

- The literature thus considers utility functions that show *decreasing* absolute risk aversion (DARA)
- Utility with DARA have $u'''(c_t) > 0$, i.e. marginal utility is convex
- This property is also called *prudence*
- It gives rise to an additional motive for self-insurance: *precautionary savings*

Precautionary Savings

- Euler-Equation (assuming $\beta(1+r)=1$):

$$u'(c_t) = E_t[u'(c_{t+1})]$$

- By convexity of the marginal utility ($u''' > 0$), Jensen's inequality implies that

$$E_t[u'(c_{t+1})] > u'(E_t[c_{t+1}])$$

- Because $u'' < 0$, we get $E_t[c_{t+1}] > c_t$
- If $u''' = 0$, we have $E_t[c_{t+1}] = c_t$
- Thus, savings are higher if $u''' > 0$

Precautionary Saving

- Uncertain income can cause savings for self-insurance even without binding borrowing constraints if $u''' > 0$
- If $u''' > 0$, consumption rule depends on higher moments of income process and certainty equivalence property does not apply anymore

Taking Stock

Incomplete markets give rise to two saving motives:[4]

- Consumption smoothing: driven by risk-aversion (i.e. $u'' < 0$). Saving responds to changes in expected (average income), as in PIH.
- Precautionary saving: driven by borrowing constraints and/or prudence (i.e. $u''' > 0$). Saving respond to change in income uncertainty.

[4] Strictly speaking, only the precautionary saving is due to incompleteness, since incompleteness implies that income uncertainty translates into consumption uncertainty, which is necessary to get precautionary saving. Consumption smoothing also operates when markets are complete and/or if income is nonstochastic but unevenly distributed over time. An additional (dis-)saving motive is (im-)patience, which depends on β and r.

Development of $\{c_t\}$ & $\{a_t\}$ over time

- With incomplete markets, consumption changes with time
- This is because of imperfect insurance
- How do $\{c_t\}$ and $\{a_t\}$ evolve as $t \to \infty$?
- Boundedness of state space is crucial for Part III, where we study a general equilibrium setup
- Recall from Macro I that boundedness was also useful to apply dynamic programming techniques

General Case

- We will use the following setup to provide the answer:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

such that

$$\begin{aligned} a_{t+1} &= (1+r)(a_t + y_t - c_t) \\ a_{t+1} &\geq -\underline{a} \end{aligned}$$

- Assume that $u'(c_t) > 0$, $u''(c_t) < 0$ and Inada conditions $\lim_{c_t \to 0} u'(c_t) = \infty$, $\lim_{c_t \to \infty} u'(c_t) = 0$

General Case

- First order necessary condition for optimality

$$u'(c_t) = \beta(1+r)E_t[u'(c_{t+1})] + \mu_t(1+r)$$

where μ_t is the Lagrange multiplier for borrowing constraint $a_{t+1} \geq -\underline{a}$

- Thus

$$u'(c_t) \geq \beta(1+r)E_t[u'(c_{t+1})]$$

General Case

- Rewrite FOC to
$$M_t \geq E_t M_{t+1}$$
where $M_t \equiv [\beta(1+r)]^t u'(c_t)$
- M_t is a *supermartingale*
- Because $u' > 0$ and $\lim_{c_t \to \infty} u'(c_t) = 0$, $M_t \geq 0$
- By the supermartingale convergence theorem, there exists a random variable $\bar{M}$ such that $\lim_{t \to \infty} M_t = \bar{M} < \infty$ almost surely (Ljungvist/Sargent (2004, p. 560)

General Case

To study the behavior of $\{c_t\}$ and $\{a_t\}$ with $t \to \infty$, we distinguish 3 cases

1. $\beta(1 + r) > 1$
2. $\beta(1 + r) = 1$
3. $\beta(1 + r) < 1$

Will $\{c_t\}$ and $\{a_t\}$ be bounded, i.e. $\{c_t\} \leq \bar{c} < \infty$ and $\{a_t\} \leq \bar{a} < \infty$, as $t \to \infty$?

$\beta(1+r) > 1$

$\{c_t\}$ and $\{a_t\}$ are unbounded and diverge as $t \to \infty$.

Proof:

- By supermartingale convergence: $\lim_{t\to\infty}[\beta(1+r)]^t u'(c_t) = \bar{M} < \infty$
- This implies $\lim_{t\to\infty} u'(c_t) = 0$ because $\lim_{t\to\infty}[\beta(1+r)]^t = \infty$
- $c_t \to \infty$ because $\lim_{c_t\to\infty} u'(c_t) = 0$
- It follows that $a_t \to \infty$ because $a_t \geq -\underline{a}$

$\beta(1+r)=1$

$\{c_t\}$ and $\{a_t\}$ are unbounded and diverge as $t \to \infty$.

Proof:

- With $\beta(1+r)=1$, Euler-Equation $u'(c_t) \geq E_t[u'(c_{t+1})]$
- Assume that $u'''(c_t) > 0$
- By Jensen's inequality, $u'(c_t) \geq E_t[u'(c_{t+1})] > u'(E_t[c_{t+1}])$
- It follows that $E_t[c_{t+1}] > c_t$ because of $u''(c_t) < 0$
- 'Ratchet effect': on average, consumption is increasing
- It follows that $a_t \to \infty$ because $a_t \geq -\underline{a}$

Assumption $u'''(c_t) > 0$ not needed if income is sufficiently stochastic (see LS, ch. 16.6)

$\beta(1+r) < 1$

If utility is of DARA class and income shocks are *iid*, $\{c_t\}$ and $\{a_t\}$ are bounded

Proof:

- See Schechtman/Escudero (1977)

Intuitively, DARA utility implies that agents are less worried about risk if they get richer. So, the richer they get, the less agents save for self-insurance. Thus, $\{a_t\}$ and $\{c_t\}$ converge.

Summary: Boundedness

Sufficient conditions for boundedness are:

- $\beta(1+r) < 1$
- DARA utility
- Income shocks are iid

For more general cases (e.g. income serially correlated across time), we can only hope that things work out fine

If T finite, $\{a_t\}$, $\{c_t\}$ will *always* be bounded

Part III: Implications of market incompleteness for aggregate consumption and savings decisions

General Equilibrium

- Before: single-agent problems, return r of asset was assumed to be exogenous
- Now: continuum of agents ('households')
- r is determined endgenously in general equilibrium

In the following, the model of Aiyagari (1994, QJE) is presented

General Equilibrium

What are the advantages of using a general equilibrium setup?

- Enforces scientific discipline: ratio between r,β is determined endogenously within the model - these parameters govern consumption/saving decisions
- Enables meaningful policy experiments: impact of price changes after a policy intervention can be traced out

General Equilibrium: Heterogeneity among Households

- Continuum of infinitely lived households with total measure 1
- Households are subject to uninsurable income shocks
- Households are *ex-ante* identical, i.e. before income shocks are realized
- Households differ *ex-post*, i.e. after income shocks have materialized, with respect to asset holdings, depending on the sequence of income shocks
- Incomplete markets models endogenously generate wealth inequality and social mobility

General Equilibrium: Ex-Post Heterogeneity

- Income shocks ϵ are idiosyncratic, i.e. *i.i.d.* across *households* (but not necessarily iid across *time*)
- Aggregate income is assumed to be constant
- Labor supply is inelastic: income shocks are interpreted as shocks to labor productivity
- Shocks follow Markov chain with transition matrix $\pi(\epsilon, \epsilon')$

Household's Problem

- Recursive formulation:

$$v(a, \epsilon; \lambda) = \max_{c,a'} \{u(c) + \beta \sum_{\epsilon' \in E} v(a', \epsilon'; \lambda')\pi(\epsilon, \epsilon')\}$$

s.t.

$$\begin{aligned} c + a' &= (1 + r(\lambda))a + w(\lambda)\epsilon \\ a' &\geq -\underline{a} \end{aligned}$$

where $r \equiv \tilde{r} - \delta$, ϵ are shocks to labor productivtiy and $\lambda(a, \epsilon)$ is the distribution of households across states

Budget Constraint

- Assets a are traded at price 1 today with return $1 + r$
- Assets contain claims to physical production capital
- Interest rate is determined by the marginal product of capital
- Timing assumption: (i) production takes place, (ii) income is distributed, (iii) households consume
- If r is stationary over time, this is equivalent to specification given above (zero-coupon bond)

Firms

- Competitive firms that produce a homogenous output good
- Firms act as price takers
- Production technology has constant returns to scale
- Hence, firm size is indeterminate: we consider a *single, representative firm*

Stationarity

- We want prices r and w to be time-invariant
- This requires the distribution of households across states to be invariant: the probability distribution will permanently reproduce itself
- $\lambda(a, \epsilon) = \lambda'(a, \epsilon) = \lambda^*(a, \epsilon)$

A **Stationary Competitive Recursive Equilibrium** is a value function $v(a, \epsilon)$, policy functions $a'(a, \epsilon)$ and $c(a, \epsilon)$, a probability distribution $\lambda^*(a, \epsilon)$, and positive real numbers (K, L, r, w) such that

1. The prices (w, r) satisfy
$$\begin{aligned} w &= F_L(K, L) \\ r &= F_K(K, L) - \delta \end{aligned}$$
where $F(K, N) = AK^\alpha L^{1-\alpha}$
2. Given r, w, the optimal policy functions $a'(a, \epsilon)$ and $c(a, \epsilon)$ solve the household's problem and v is the associated value function
3. The labor market clears: $L = \sum_{a,\epsilon} \epsilon \lambda^*(a, \epsilon)$
4. The capital market clears: $K = \sum_{a,\epsilon} a'(a, \epsilon)\lambda^*(a, \epsilon)$
5. The probability distribution $\lambda^*(a, \epsilon)$ is a stationary distribution associated with $a'(a, \epsilon)$ and $\pi(\epsilon, \epsilon')$; that is, it satisfies
$$\lambda^*(a', \epsilon') = \sum_{\epsilon} \sum_{a: a' = a'(a,\epsilon)} \lambda^*(a, \epsilon)\pi(\epsilon, \epsilon')$$

Existence and Uniqueness of Equilibrium

- General equilibrium requires market clearing in two markets: capital and labor (goods market clearing condition redundant by Walras' law)
- Two prices: r and w
- Because labor supply is exogenous, w is a function of r only
- General equilibrium is a function of r only

Determination of r: Demand for Capital

- $K(r)$: capital demand of the firm
- Implicitly defined such that marginal product equals interest rate:
$$r = F_K(K, L) - \delta$$
- From Inada conditions on F, $\lim_{r \to -\delta} K(r) = \infty$ and $\lim_{r \to \infty} K(r) = 0$
- $K(r)$ is continuous

Determination of r: Supply of Capital

- $A(r)$: capital supply of households:

$$A(r) = \sum_{a,\epsilon} a'(a, \epsilon; r)\lambda^*(a, \epsilon; r)$$

- We know that $\lim_{r \to \frac{1}{\beta} - 1} A(r) = \infty$ and $\lim_{r \to -\delta} A(r) < \infty$
- If $A(r)$ is continuous, an equilibrium with $r \in (-\delta, \frac{1}{\beta} - 1)$ exists

Is $A(r)$ Continuous?

- $a'(a, \epsilon; r)$ is continuous in r by (stochastic equivalent to) Corollary 6.1 in Acemoglu (2009).
- $\lambda^*(a, \epsilon; r)$ is continuous in r by Theorem 12.13 in Stokey/Lucas (1989)
- Necessary pre-requisite: Existence and uniqueness of invariant probability distribution $\lambda^*(a, \epsilon; r)$

Probability Measure

- We are interested under which condition a unique stationary probability distribution $\lambda^*(a', \epsilon')$ exists
- Let $Q : \lambda \to \lambda'$ be transition function of λ:

$$\lambda'(a', \epsilon') = \underbrace{\sum_{\epsilon} \sum_{a : a' = a'(a, \epsilon)} \pi(\epsilon, \epsilon')}_{\equiv Q} \lambda(a, \epsilon)$$

Probability Measure

Q needs to satisfy 3 conditions:

1. *Feller property*, for existence
2. *Mixing property*, for uniqueness
3. *Monotonicity*, for convergence

Then, by Theorem 12.12 in Stokey & Lucas (1989), there exists a unique invariant probability measure λ^* and Q ensures convergence, independently of the starting measure λ^0

Feller property, for existence

- Requires that Q maps continuous and bounded functions into the space of continuous and bounded functions
- Fulfilled because $a'(a, \epsilon)$ is continuous and bounded (if $\beta(1 + r) < 1$)
- Intuitively, the Feller property ensures that Q is indeed a transition function

Mixing property, for uniqueness

- Also called 'American Dream, American Nightmare' condition
- Requires that probability of moving away from worst state of the economy $(\underline{a}, \underline{\epsilon})$ in a finite number of periods is positive ("American Dream") and vice versa from the best state $(\bar{a}, \bar{\epsilon})$("American Nightmare")

Mixing property, for uniqueness

- Fulfilled since income shocks ϵ are transitory (income process is mean-reverting)
- They will increase (decrease) savings in response to high (low) shocks
- Start with $(\underline{a}, \underline{\epsilon})$ $((\bar{a}, \bar{\epsilon}))$
- Consider a long sequence of high (low) shocks: this event occurs with positive probability
- Household will move away from $(\underline{a}, \underline{\epsilon})$ $((\bar{a}, \bar{\epsilon}))$

Mixing property, for uniqueness

- Counterexample, where mixing property does not hold:

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Households stay in the state they started in
- *Any* measure is invariant

Monotonicity, for convergence

- If $\lambda(a, \epsilon)$ is high, then $\lambda'(a, \epsilon)$ will be high too
- Requires 'positive autocorrelation' of Q
- Fulfilled if $a'(a, \epsilon)$ increasing and Markov chain is monotone (e.g. features positive autocorrelation in the income process)

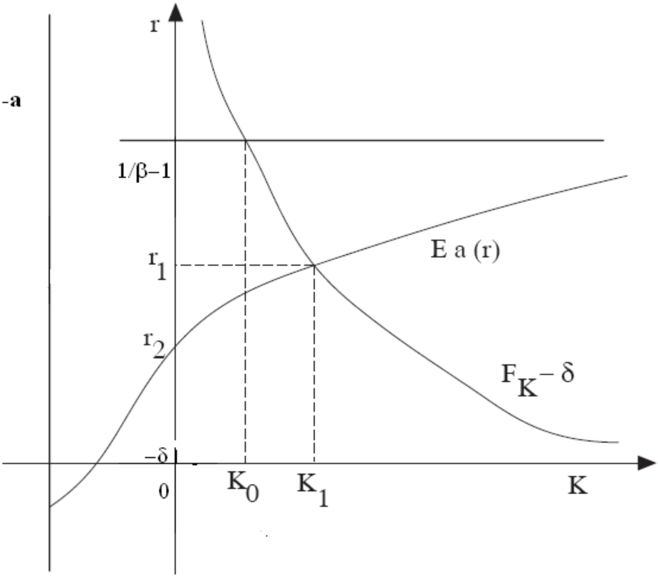
Monotonicity, for convergence

- Counterexample, where monotoniticty does not hold:

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Households keep switching back and forth between the two states
- No convergence possible

A Possible Equilibrium



Aggregate Precautionary Savings

- Compare aggregate capital stock in the economy with earnings uncertainty (K_1) to economy without earnings uncertainty/complete markets (K_0)[5]
- K_0 corresponds to the stationary capital stock of the representative agent economy, where $\beta(1+r) = 1$
- Uninsurable idiosyncratic shocks boosts aggregate savings and lowers r such that $\beta(1+r) < 1$

[5] It is convention to call $K_1 - K_0$ aggregate precautionary savings. This is because all other saving motives - except precautionary saving - are also at work when there is no uncertainty (see footnote 4). See footnote 15 in LS ch. 17 for an economy that allows for a decomposition of all motives in general equilibrium.

Uniqueness?

- We do not know whether there is a unique r^* that clears the capital market
- This is due to the unknown relative strength of substitution/income effect: sign of $A_r(r)$ is unclear
- Moreover, the effects of changes in r on the asset distribution are unknown

Applications

Models with incomplete markets are used to understand

- Emergence of wealth inequality
- Quantitative importance of precautionary savings for aggregate capital stock
- Asset pricing
- Welfare implications of earnings uncertainty

See Heathcote, Storesletten and Violante (2009, Annual Review of Economics) and Guvenen (2011, NBER WP 17622)