1 Life-cycle economies

In this section, we introduce an explicit life-cycle dimension into the incomplete markets model. To close the model, we need to design an economy with generations that overlap so that the population structure is constant. We will use this model to study the evolution of inequality over the life cycle. Here are some basic facts we should keep in mind.

1. Wage inequality rises almost linearly. The variance of log wages rises by 0.30 point from age 25 to age 55.

2. Consumption inequality also rises over the life-cycle, but the increase is smaller, around 1/3 of the rise in wage inequality.

3. Hours inequality is fairly flat over the life cycle.

Before getting to the incomplete markets model, we will try to explain these life-cycle inequality patterns as a complete markets allocation.

1.1 Complete markets

We follow Storesletten, Telmer and Yaron, (2001). Consider the following economic environment. Time is discrete and indexed by $t = 0, 1, ...$ and the economy is stationary.

Demographics: The economy is populated by $J + 1$ overlapping cohorts of individuals. Each cohort is born with measure one and it is indexed by their birth date τ. Individuals indexed by i are born at age $j = 0$ and die for sure after reaching age J. They survive up to age j with probability φ^j. By definition, age j is equal to $t - \tau$, i.e., current date minus birth date.

Preferences: Intratemporal utility, as a function of consumption and hours, of an agent i is given by:

$$u(c, h) = \frac{c^{1-\gamma}}{1-\gamma} - A\frac{h^{1+\sigma}}{1+\sigma}.$$

Note that $1/\gamma$ is the intertemporal elasticity of substitution and $1/\sigma$ is the intertemporal labor supply elasticity, or Frisch labor supply elasticity.

Endowments: Individual productivity has a deterministic cohort- and age-specific component $\kappa_{j\tau}$ and a stochastic, idiosyncratic component $\varepsilon_{i,j,\tau}$ which could be, also, cohort specific. For example, the wage-age profile of some cohorts can start higher and be steeper than others. Or, the variance of the shocks $\varepsilon_{i,j,\tau}$ could be larger for some cohorts than for others. These two components (κ, ε) are multiplicative.

Technology: The production technology is linear in aggregate efficiency units of labor, $Y = zN$, where N are the aggregate efficiency-weighted hours worked, i.e., the sum over the population of $n_{i,j,\tau} = \kappa_{j\tau}\varepsilon_{i,j,\tau}h_{i,j,\tau}$.

We wish to use a planner's problem to characterize the complete markets allocations. Consider a planner with social welfare function putting positive weights only on individuals alive today only (i.e., the planner puts zero weight on all future cohorts). Its social welfare function is:

$$\max_{\{c_{ij\tau},h_{ij\tau}\}} \sum_{\tau=t-J}^{t} \sum_{j=t-\tau}^{J} \beta^{j-(t-\tau)}\varphi^{j} \int \lambda_{i\tau} \left[\frac{c_{ij\tau}^{1-\gamma}}{1-\gamma} - A\frac{h_{ij\tau}^{1+\sigma}}{1+\sigma} \right] d\mu_{\tau}$$

where $\lambda_{i\tau}$ is the weight put on individual i of cohort τ and μ_{τ} is the cross-sectional distribution (i.e., it integrates to one) of agents in cohort τ.

The aggregate resource constraint at every date t obeys the equation:

$$0 = \sum_{\tau=t-J}^{t} \varphi^{t-\tau} \int \left(z\kappa_{t-\tau,\tau}\varepsilon_{i,t-\tau,\tau}h_{i,t-\tau,\tau} - c_{i,t-\tau,\tau}\right) d\mu_{\tau}.$$

Denote by θ_t the Lagrange multiplier on the resource constraint at date t.

At date t, the planner's FOC with respect to consumption of individual i belonging to birth cohort τ yields:

$$\varphi^{t-\tau}\lambda_{i\tau}c_{i,t-\tau,\tau}^{-\gamma} = \varphi^{t-\tau}\theta_t \tag{1}$$

which establishes that the ratio of marginal utility of consumption between any two agents (i,τ) and (i',τ') is constant over time and equal to the ratio of planner weights, which is the definition of complete markets.

Taking logs of (1), using the notation $j = t - \tau$ and rearranging, we arrive at:

$$\begin{aligned} \log \lambda_{i\tau} - \gamma \log c_{i,j,\tau} &= \log \theta_t \\ \log c_{i,j,\tau} &= \frac{1}{\gamma} \left[\log \lambda_{i\tau} - \log \theta_t \right]. \end{aligned}$$

The cross-sectional variance of log consumption across agents i conditional on age $j = t - \tau$, given by

$$var_j \left(\log c_{ij\tau} \right) = \frac{1}{\gamma^2} var \left(\log \lambda_{i\tau} \right)$$

is independent of age (albeit it could depend on cohort). Hence, we have reached three results:

1. Consumption inequality within a cohort can be positive only if the planner's weights differ across agents belonging to the same cohort. Remember, from the Negishi approach, that weights differ if initial endowments of wealth differ, for example.

2. Consumption inequality can vary across cohorts, if the distribution of planner weights differ across cohorts.

3. Consumption dispersion cannot rise over the life cycle under complete markets with separable preferences. Indeed, the discrepancy between the growth in consumption dispersion and the growth of wage dispersion can be interpreted as a metric of risk sharing in the economy. The larger this gap, the lower risk sharing.

From the intratemporal FOC for an agent i of cohort τ at date t is:

$$\lambda_{i\tau} A h^{\sigma}_{i,t-\tau,\tau} = \theta_t z \kappa_{t-\tau,\tau} \varepsilon_{i,t-\tau,\tau}. \tag{2}$$

Rearranging (2), taking logs, and using $j = t - \tau$ we reach:

$$\log h_{i,j,\tau} = \frac{1}{\sigma} \log \left(\frac{z\theta_t}{A} \right) - \frac{1}{\sigma} \log \left(\lambda_{i\tau} \right) + \frac{1}{\sigma} \log \varepsilon_{i,j,\tau} + \frac{1}{\sigma} \log \kappa_{j\tau}.$$

The cross-sectional variance of log hours across agents i for cohort τ conditional on age j is:

$$var_j \left(\log h_{ij\tau} \right) = \frac{1}{\sigma^2} var \left(\log \lambda_{i\tau} \right) + \frac{1}{\sigma^2} var_j \left(\log \varepsilon_{ij\tau} \right)$$

which shows that in complete markets hours inequality grows over the life cycle if the unconditional variance of the shock grows over the life cycle. Empirically, as discussed earlier, the variance of hourly wages increases linearly over the life-cycle which implies that $\varepsilon_{ij\tau}$ has a permanent component, e.g.,

$$\log \varepsilon_{i,j,\tau} = \log \varepsilon_{i,j-1,\tau} + v_{i,j,\tau}$$

which implies

$$var_j \left(\ln \varepsilon_{ij\tau} \right) = j \cdot var \left(v_{ij\tau} \right).$$

If wage inequality grows steeply over the life cycle, then hours inequality must grow too, unless σ is very large, i.e., labor supply elasticity is small. But if the Frisch elasticity is close to zero, then the variance of hours worked would be close to zero too, while in the data is positive and quite large (even net of measurement error).

We conclude that the full insurance model has counterfactual implications about the path of inequality over the life cycle. Before examining the incomplete markets model, let's take a look at the other extreme benchmark, autarky.

1.2 Autarky

In autarky, i.e. no insurance and no storage, consumption for every individual is equal to its earnings, or:

$$c_{i,j,\tau} = \kappa_{j\tau} \varepsilon_{i,j,\tau} h_{ij\tau} \tag{3}$$

and hours worked satisfy the standard intratemporal first-order condition:

$$c_{i,j,\tau}^{-\gamma} \kappa_{j\tau} \varepsilon_{i,j,\tau} = A h_{ij\tau}^{\sigma}. \tag{4}$$

Combining these two equations, we obtain

$$\begin{aligned} c_{i,j,\tau}^{-\gamma} \kappa_{j\tau} \varepsilon_{i,j,\tau} &= A c_{i,j,\tau}^{\sigma} \left(\kappa_{j\tau} \varepsilon_{i,j,\tau} \right)^{-\sigma} \\ c_{i,j,\tau} &= \left(\frac{1}{A} \right)^{\frac{1}{\sigma+\gamma}} \left(\kappa_{j\tau} \varepsilon_{i,j,\tau} \right)^{\frac{1+\sigma}{\sigma+\gamma}} \end{aligned}$$

which implies that in the cross section

$$var_j(\log c_{ij\tau}) = \left(\frac{1+\sigma}{\sigma+\gamma}\right)^2 var_j(\log \varepsilon_{ij\tau})$$

and consumption dispersion grows over the life cycle, indeed, potentially even more than wage dispersion, if $\gamma < 1$. For consumption inequality to grow slower than wage inequality, γ must be higher than one.

We now turn to hours inequality. Using (3) into the first-order condition (4):

$$\begin{aligned} (\kappa_{j\tau}\varepsilon_{i,j,\tau}h_{ij\tau})^{-\gamma}\kappa_{j\tau}\varepsilon_{i,j,\tau} &= Ah^{\sigma}_{ij\tau} \\ h_{ij\tau} &= \left(\frac{1}{A}\right)^{\frac{1}{\sigma+\gamma}}(\kappa_{j\tau}\varepsilon_{i,j,\tau})^{\frac{1-\gamma}{\sigma+\gamma}} \end{aligned}$$

which implies that, in the cross-section,

$$var_j(\ln h_{ij\tau}) = \left(\frac{1-\gamma}{\sigma+\gamma}\right)^2 var_j(\ln \varepsilon_{ij\tau}).$$

Therefore, depending how far γ is from 1, hours can rise, fall or be flat. If γ is sufficiently higher than 1, then consumption inequality will grow less than wage inequality and hours inequality can also be relatively flat.

Clearly autarky is an extreme. The key question is: does a plausibly calibrated incomplete-markets model (with a financial market structure in between autarky and complete markets) generate the observed increase in consumption inequality?

1.3 Lifecycle incomplete markets economy

This section is based on Storesletten, Telmer and Yaron (2004). Consider the following economy. Time is discrete and indexed by $t = 0, 1, ...$ and the economy is stationary.

Demographics: The economy is populated by overlapping cohorts of individuals. Each cohort of newborn agents has measure one. Individuals of a cohort are born at age $j = 0$ and die for sure upon reaching age J. They survive up to age j with probability φ^j. It is also useful to denote the conditional survival probability between age j and age $j+1$ with $\varphi_j = \varphi^j/\varphi^{j-1}$.

Preferences: Intra-period utility is given by $u(c_j)$ with $u' > 0, u'' < 0$. We abstract from labor supply.

Endowments: Individual productivity endowments are the sum (in log) of a deterministic age component, plus an innate ability component, plus two stochastic components:

$$\begin{aligned} \log y_{ij} &= \kappa_j + \varepsilon_{ij} \\ \varepsilon_{ij} &= \rho\varepsilon_{i,j-1} + \omega_{ij} \end{aligned}$$

where ε_{ij} is iid over time and η_{ij} follows an AR(1). Let's discretize all pieces and let $\pi(\varepsilon', \varepsilon)$ be the relevant conditional distribution of the shosck. Agents go through two phases of the life-cycle: work and retirement. Assume that

$$\varepsilon_{ij} = 0 \text{ for } j \geq J^{ret}$$

where age J^{ret} denotes mandatory retirement. New cohorts of agents are born with zero initial wealth.

Technology: Output is produced through the aggregate production function:

$$C + \delta K = Y = F(K, N)$$

Government: It taxes labor income at rate τ and finances a pay-as-you-go social security system. The social security system pays retirement benefits which depend on average lifetime earnings $\bar{y}^{ret}$, where

$$\bar{y}_i^{ret} = \frac{1}{J^{ret}} \sum_{j=0}^{J_{ret}-1} y_{ij},$$

based on a given formula given by the function $P(\bar{y}^{ret})$, with $P' > 0$ and $P'' < 0$.

The government expropriates accidental bequests of agents who die at ages $j < J$ and redistributes them across all living agents equally through a lump sum transfer ϕ.

Markets: Only one-period non state contingent bonds are traded (a mix of claims to physical capital and private IOUs). Workers can borrow up to $-\underline{a}$ and retirees cannot borrow. Asset and goods market are competitive.

1.3.1 Household problem

The problem of the household during working age can be written in recursive formulation as:

$$\begin{aligned} V_j(\varepsilon_j, a_j, \bar{y}_j) &= \max_{\{c_j, a_{j+1}\}} u(c_j) + \beta\varphi_{j+1} E_j [V_{j+1}(\varepsilon_{j+1}, a_{j+1}, \bar{y}_{j+1})] \\ & \quad s.t. \\ c_j + a_{j+1} &= Ra_j + (1-\tau) w \exp(\kappa_j + \varepsilon_j) + \phi \\ a_{j+1} &\geq -\underline{a} \\ \bar{y}_{j+1} &= \bar{y}_j + \frac{w \exp(\kappa_{j+1} + \varepsilon_{j+1})}{J^{ret}} \end{aligned}$$

where here we abuse a bit notation and we set, implicitly, the continuation value

$$V_{J^{ret}}(0, 0, 0, a_{J^{ret}}, \bar{y}_{J^{ret}}) = \tilde{V}_{J^{ret}}\left(a_{J^{ret}}, \bar{y}^{ret}\right).$$

During retirement, the household problem becomes

$$\begin{aligned} \tilde{V}_j\left(a_j, \bar{y}^{ret}\right) &= \max_{\{c_j, a_{j+1}\}} u(c_j) + \beta\varphi_{j+1} \tilde{V}_{j+1}\left(a_{j+1}, \bar{y}^{ret}\right) \\ & \quad s.t. \\ c_j + a_{j+1} &= Ra_j + P\left(\bar{y}^{ret}\right) + \phi \\ a_{j+1} &\geq 0 \end{aligned}$$

Also, note that since there is no uncertainty for the retirees, there is no expectation in the Bellman equation.

1.3.2 Stationary equilibrium

Let $s \equiv (\varepsilon_j, a_j, \bar{y})$ be the vector of states for the worker and $\tilde{s} \equiv (a_j, \bar{y}^{ret})$ the vector of states for the retiree. A stationary equilibrium is a collection of: 1) decision rules $\{c_j(s), a_{j+1}(s)\}$ for workers and $\{\tilde{c}_j(\tilde{s}), \tilde{a}_{j+1}(\tilde{s})\}$ for retirees, 2) value functions $\left\{V_j(s), \tilde{V}_j(\tilde{s})\right\}$, 3) prices $\{w, R\}$, 4) aggregate quantities $\{K, N\}$, 5) tax rate τ (e.g., given the lump-sum transfer ϕ), and 6) stationary measures $\left\{\mu_j, \tilde{\mu}_j\right\}$ such that:

- The decision rules are the solution to the household problem and satisfy the associated value functions.

- Input prices equal marginal products of capital and labor.

- The labor market clears

$$N = \sum_{j=0}^{J^{ret}-1} \varphi^j \int \exp\left(\kappa_j + \varepsilon_j\right) d\mu_j.$$

- The capital market clears:

$$K = \sum_{j=0}^{J^{ret}-1} \varphi^j \int a_{j+1}\left(s\right) d\mu_j + \sum_{j=J^{ret}}^{J} \varphi^j \int \tilde{a}_{j+1}\left(\tilde{s}\right) d\tilde{\mu}_j.$$

- The government budget is balanced

$$\tau w N = \sum_{j=J^{ret}}^{J} \varphi^j \int P\left(\bar{y}^{ret}\right) d\tilde{\mu}_j$$

  and the government rebates all the assets of the deceased

$$\sum_{j=0}^{J^{ret}-1} \varphi^j \left(1 - \varphi_{j+1}\right) \int a_{j+1}\left(s\right) d\mu_j + \sum_{j=J^{ret}}^{J} \varphi^j \left(1 - \varphi_{j+1}\right) \int \tilde{a}_{j+1}\left(\tilde{s}\right) d\tilde{\mu}_j = \phi \sum_{j=0}^{J} \varphi^j$$

- The goods market clears

$$\sum_{j=0}^{J^{ret}-1} \varphi^j \int c_j\left(s\right) d\mu_j + \sum_{j=J^{ret}}^{J} \varphi^j \int \tilde{c}_j\left(\tilde{s}\right) d\tilde{\mu}_j + \delta K = F\left(K, N\right)$$

- The distributions $\left\{\mu_j, \tilde{\mu}_j\right\}$ are stationary. For example, for $j < J^{ret}$:

$$\mu_{j+1} = \int_S Q_j\left(s, \mathcal{S}\right) d\mu_j$$

  where

$$Q_j\left(\left(\varepsilon, a, \bar{y}\right), \mathcal{E} \times \mathcal{A} \times \bar{\mathcal{Y}}\right) = I_{\{a_{j+1}(s) \in \mathcal{A}\}} \cdot \sum_{\varepsilon_{j+1} \in \mathcal{E}} \pi_\varepsilon\left(\varepsilon_{j+1}, \varepsilon_j\right) \cdot I_{\left\{\bar{y}_j + \frac{w \exp\left(\kappa_{j+1} + \varepsilon_{j+1}\right)}{J^{ret}} \in \bar{\mathcal{Y}}\right\}}$$

1.3.3 Solution method

We first guess $\{R^0, \phi^0\}$. From R^0 we obtain the age w^0, the capital stock K^0 and the aggregate labor input N^0 (exogenous). We don't need to guess τ^0 as well because the distribution of $\bar{y}_j$ across agents is exogenous, so from the balanced budget of the government, given our guesses, we can recover τ^0.

Once we have all we need in the agent's budget constraint, we can solve the household problem. We do it backward, starting from the last period in retirement which is a static problem, all the way to age $j = 0$. Then we use the $j + 1$ decision rule in the Euler equation for age j (like in the transitional dynamics).

The simulation step is the same as always, with the caveat that we need to respect the demographic structure. From the aggregation of wealth among agents alive at date t, we obtain A^0 (and the implied wealth of the deceased) which we need to compare to K^0 and ϕ^0.

1.3.4 What matters for the rise of consumption inequality over the life cycle?

Storesletten, Telmer and Yaron (2004) show that this model does a good job in matching the lifecycle profile of consumption inequality, given wage inequality as an exogenous input. In particular, the key determinants of risk sharing in this economy are:

- **Financial wealth:** the amount of financial wealth held in the economy is key because the larger is financial wealth, the smaller is the impact of earnings shocks on consumption. The intuition is that agents consume out of human wealth and financial wealth and, the larger is the latter relative to the former, the less earnings shocks impact consumption. Usually β is set so that $K/Y = 3.5$ to reproduce this same ratio for the US economy. For example, reducing β such that K/Y is 1.5 increases the rise of consumption inequality over the life cycle substantially.

- **Social security**: Social security redistributes across generations, but also within generations because P is concave, so it provides some additional insurance. More social insurance means less growth in consumption inequality.

1.4 Aggregate shocks in life cycle economies

This section is based on Krueger and Kubler (2003). Suppose we add to the model an aggregate productivity shock, i.e.

$$Y = zF(K, N)$$

with $z \in Z$ stochastic which follows the Markov chain $\pi(z', z)$.

Let's get rid of all idiosyncratic uncertainty in ε so that we have one type of agent for each age. Then, the full aggregate state of the economy is z together with the $(J+1)$-dimensional wealth vector $\bar{a} = (a_0, a_1, ..., a_J)$. Note that now $\bar{y}$ is a deterministic function of age, as labor income is deterministic.

The Euler equation for an agent of age j is

$$u_c\left(w(z,K)\kappa_j + R(z,K)a_j - a_{j+1}(a_j; z, \bar{a})\right) \geq$$
$$\beta \sum_{z' \in Z} u_c\left(w(z',K')\kappa_{j+1} + R(z',K')a_{j+1}(a_j; z, \bar{a}) - a_{j+2}\left(a_{j+1}(a_j; z, \bar{a}); z', \bar{a}'\right)\right)\pi(z', z)$$

with

$$\begin{aligned} K &= \sum_{j=0}^{J} a_j \\ K' &= \sum_{j=0}^{J} a_{j+1}(a_j, z, \bar{a}) \\ \bar{a}' &= \begin{bmatrix} a_0 \\ a_1(a_0, z, \bar{a}) \\ \ldots \\ a_J(a_{J-1}, z, \bar{a}) \end{bmatrix} = \Gamma(\bar{a}, z) \end{aligned}$$

and so we have all the pieces to compute the Euler equation.

Can we use the Krusell-Smith approach implemented as in the infinite horizon model? I.e., can we instead of keeping track of the entire J dimensional vector $\bar{a}$ just keep track of average capital K and use an approximate law of motion for K only of the type

$$\ln K' = b_0^z + b_1^z \ln K$$

as in the original Krusell-Smith economy? Recall that, in that economy with infinite horizon, most of the agents have the same marginal propensity to save and consume, i.e., the saving decision rules are linear for a wide range. In a life-cycle economy, instead, because of the finite horizon (and no bequest motive), elderly agents have much lower propensity to save than young agents who have to save to finance their retirement. Therefore, keeping track of the mean of the asset distribution alone leads to large forecasting errors, i.e. in some cases R^2 can be as low as 0.66 and quasi aggregation fails badly. The key is the enormous heterogeneity in saving rates across agents of different ages.

This calls for a simplification: instead of keeping track of a $J+1$ dimensional vector, since we know that wealth is hump shaped by age, we can approximate the wealth distribution by, for example, a cubic polynomial of age, i.e., 4 parameters instead of $J+1$.