1 Constrained Efficiency in the Neoclassical Growth Model with Idiosyncratic Shocks

We have learned that the equilibrium allocations in the neoclassical growth model with idiosyncratic risk display *over-accumulation* of capital relative to the *first best* where $R\beta = 1$. The reason is that, given the lack of perfect insurance markets, agents save more for self-insurance, the capital stock goes up, the wage rises, and the interest rate falls below the discount rate, i.e. $R\beta < 1$.

In other words, incomplete-markets equilibrium allocations are Pareto inefficient, or, not first-best. An unconstrained planner achieves Pareto efficiency because it has more freedom in allocating resources than is provided by the system of incomplete markets: it can make state-contingent transfers across agents. Put it differently, the unconstrained planner effectively reintroduce the missing markets.

While first-best is an important and useful benchmark, one can argue that the interesting question is —not so much whether a new economic structure with state contingent transfer can do better— but whether the market performs efficiently relative to the set of allocations achievable with the this same economic structure. We want to assume that the planner cannot overcome directly the frictions imposed by the missing markets, but faces the same constraints on trade and transfer of resources across agents as in the original environment. This is the concept of *constrained efficiency*.

How do we investigate constrained optimality of the equilibrium allocations in the Aiyagari model? We must solve the problem of a planner that instructs consumption and saving decisions to each agent —by choosing a consumption policy function— while facing the same technology and asset structure (i.e., only a risk-free bond), and hence the same set of constraints, that agents face in the decentralized equilibrium.

We will find that the competitive equilibrium is constrained inefficient. The reason is the presence of a so called "pecuniary externality", i.e. each agents' decision has a negligible effect on prices that individual agents do not take into account. But by choosing a consumption policy for each agent in the economy, the planner can affect prices in the right way. In this sense, it is as if the planner has an additional instrument for redistributing income across states which is not available in the decentralized market.

We follow Davila, Hong, Krusell, and Rios-Rull (DHKR) in the exposition of the recursive problem. Let's start from the competitive equilibrium and let $a' = g^*(a, \varepsilon)$ be the decision rule of the agent. The necessary FOC of the agent in the steady state with invariant distribution λ^* is

$$u_c\left[R(\lambda^*)a + w(\lambda^*)\varepsilon - g^*(a,\varepsilon)\right] \geq \beta R(\lambda^*) \sum_{\varepsilon' \in E} u_c\left[R(\lambda^*) g^*(a,\varepsilon) + w(\lambda^*)\varepsilon' - g^*(g^*(a,\varepsilon), \varepsilon')\right]\pi(\varepsilon', \varepsilon),$$

which we can compactly rewrite as

$$u_c \geq \beta R(\lambda^*) \sum_{\varepsilon' \in E} u_c' \pi(\varepsilon', \varepsilon). \tag{1}$$

The problem of the planner who maximizes social welfare by choosing a saving policy $g(a, \varepsilon)$ (i.e., a saving level a' for every point in the state space) is

$$\begin{aligned}
\Omega(\lambda) &= \max_{g(a,\varepsilon)\in A} \int_{A\times E} u\left[R(\lambda)a + w(\lambda)\varepsilon - g(a,\varepsilon)\right]d\lambda + \beta\Omega(\lambda') \\
& \quad s.t. \\
R(\lambda) &= F_K(K,H) \text{ and } w(\lambda) = F_H(K,H) \\
H &= \int_{A\times E} \varepsilon d\lambda \\
K &= \int_{A\times E} a d\lambda \\
\lambda'(\mathcal{A}\times\mathcal{E}) &= \int_{A\times E} 1_{\{g(a,\varepsilon)\in\mathcal{A}\}}\pi(\varepsilon'\in\mathcal{E},\varepsilon)\, d\lambda(a,\varepsilon)
\end{aligned}$$

It is easy to see that the necessary FOC for the planner who chooses the level a' for a particular pair (a, ε) is:

$$u_c \geq \beta R(\lambda') \sum_{\varepsilon' \in E} u_c' \pi(\varepsilon', \varepsilon) + \beta \int_{A\times E} \left(a' F_{KK}' + \varepsilon' F_{HK}'\right) u_c' d\lambda'.$$

Compared to the competitive equilibrium Euler equation (1), we have an extra term which comes from the fact that the planner internalizes the effects that individual savings have on prices, so it "takes derivatives" also with respect to equilibrium prices that are, in turn, equal to marginal productivities of the factors of production.

In particular, the term in parenthesis under the integral captures the effect of an additional unit of savings on next-period individual labor income (ε') and on next-period individual capital income (a') of all agents through next-period price changes, F_{KH}' and F_{KK}'. More savings increase the capital stock, thus raise the marginal product of labor

F'_H, wages, and labor income, while decrease the marginal product of capital F'_K, the interest rate, and capital income. This effect is averaged across all agents through weights equal to their marginal utility of consumption. So, poor agents receive more weight.

This extra term can be either positive or negative since $F'_{HK} > 0$ but $F'_{KK} < 0$. Note that, in the representative agent case, this term is zero because if F is CRS, then F_K and F_H are homogenous of degree zero.

We can use the CRS assumption on F by rewriting the planner's Euler equation as

$$u_c \geq \beta R(\lambda') \sum_{\varepsilon' \in E} u'_c \pi(\varepsilon', \varepsilon) + \beta F'_{KK} K' \int_{A \times E} \left(\frac{a'}{K'} - \frac{\varepsilon'}{H} \right) u'_c d\lambda'.$$

This expression clarifies that if income of the poor agents (those with u'_c large) is labor-intensive, then the extra term will be positive since $F'_{KK} < 0$ and, for the poor agents who have the highest weight, $\frac{a'}{K'} < \frac{\varepsilon'}{H}$. The opposite is true if income of the poor is capital-intensive. Therefore the factor composition of income of the poor agents is key in determining the constrained-efficiency properties of this economy. In the first case, arguably the more plausible, the planner wants agents to save more than in the decentralized equilibrium, and hence equilibrium allocations display *under-accumulation* relative to the constrained optimum. This is a surprising result.

The intuition comes from the fact that the planner always wants to redistribute from rich to poor. If the poor have mostly labor income, then the way to redistribute is to increase equilibrium wages by inducing agents to save more than in equilibrium. Larger individual savings increase the aggregate capital stock and increase wages. Another way to understand this result is that in this economy wealth inequality is a symptom of inefficiency because it is generated by uninsurable shocks. Higher capital stock reduces the return on saving and reduces the wealth inequality induced by the missing markets.

Quantitatively, DHKR calibrate the model to the US economy (and the match wealth inequality using the strategy in Diaz-Gimenez, Castaneda, and Rios-Rull) and find that the constrained efficient capital stock is a staggering 3.5 times higher than the laissez-faire economy capital stock.