Basic DMP search model

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Setup

Basic DMP model

- Diamond-Mortensen-Pissarides model
- linear, very tractable, workhorse model in labor macro
- several drawback, we will discuss it
- equilibrium model
- steady state and out-of-steady state dynamics

Demographics and technology

- time is continuous
- agents: workers and firms
- risk-neutral, discount factor ρ
- workers
  - employed: fraction $1 - u$, wage w
  - unemployed: fraction u, benefits/leisure b
- firms
  - each firm employs a single worker
  - produces y units of output per worker
  - post vacancies, cost c per vacancy
- existing matches are destroyed at the rate δ

Matching technology

- matching function $m = m(U, V)$
  - U - number of unemployed
  - V number of vacancies
  - homogenous of degree 1, increasing, concave in both arguments
- number of matches in interval dt is $m(U, V)\,dt$
- market tightness: $\theta = \frac{V}{U}$
- random matching:
  - $p = \frac{m(U,V)}{U}$ - probability that a worker is contacted
  - $q = \frac{m(U,V)}{V}$ prob. that a vacancy is contacted
- implications of CRS:

$$p(\theta) = m(1, \theta), \quad q(\theta) = m\left(\frac{1}{\theta}, 1\right)$$

$$p(\theta) = \theta q(\theta)$$

Value functions for a worker

- W, U - value functions of being employed and unemployed
- let's first write a discrete-time version of the value functions

$$U = z\Delta t + e^{-\rho\Delta t}\left[p(\theta)\,\Delta t\cdot W + (1 - p(\theta)\,\Delta t)\,U\right]$$
$$W = w\Delta t + e^{-\rho\Delta t}\left[(1 - \delta\Delta t)\,W + \delta\Delta t\cdot U\right]$$

- rewrite

$$U\left(1 - e^{-\rho\Delta t}\right) = z\Delta t + e^{-\rho\Delta t}p(\theta)\,\Delta t\,(W - U)$$
$$W\left(1 - e^{-\rho\Delta t}\right) = w\Delta t - e^{-\rho\Delta t}\delta\Delta t\,(W - U)\,,$$

- divide by Δt, take a limit as $\Delta t \to 0$

$$\rho U = z + p(\theta)\,(W - U) \tag{1}$$
$$\rho W = w - \delta\,(W - U) \tag{2}$$

Value functions for firms

- V, J - value of a vacancy and a (filled) job

$$\rho J = y - w - \delta (J - V) \tag{3}$$

$$\rho V = -c + q(\theta)(J - V) \tag{4}$$

- free entry: firms post vacancies until $V = 0$

$$V = 0 \Rightarrow c = q(\theta) J \tag{5}$$

- asset value of a job: $J = c/q(\theta)$; $1/q(\theta)$ is an average time to fill a job
- simplify equation for J

$$\rho J = y - w - \delta J$$

- equation free_entry is the most important equation in the model

Wage determination

- first notice that

$$W - U = \frac{w - z}{\rho + \delta + p(\theta)}$$
$$J - V = \frac{y - w}{\rho + \delta}$$

- observation: $J > 0$ iff $y > w$, $W > U$ iff $w > z$
- workers are willing to work if $w > z$
- firms are willing to hire workers if $w < y$
- any wage $w \in [z, y]$ can be a solution
- to get a particular wage w^*, we need a rule: Nash bargaining

Nash bargaining

- γ – a bargaining power of a worker
- Nash bargaining: solution maximizes weighted product of firm's and worker's surplus

$$
\begin{aligned}
w^* &= \arg\max\left(W(w)-U\right)^{\gamma}\left(J(w)-V\right)^{1-\gamma} \\
&= \arg\max \gamma \log\left(W(w)-U\right)+(1-\gamma)\log\left(J(w)-V\right)
\end{aligned}
$$

- first order conditions

$$
\gamma\frac{W'(w)}{W(w)-U}+(1-\gamma)\frac{J'(w)}{J(w)-V}=0
$$

- notice that $W'(w)=-J'(w)$, hence we get

$$
\gamma\left(J(w)-V\right)=(1-\gamma)\left(W(w)-U\right)
$$

- the gain to the worker is proportional to the gain of the firm

Match surplus

- define match surplus: $S = J + W - U - V$
- we can rewrite the above equation as

$$\begin{aligned} W - U &= \gamma S \\ J - V &= (1-\gamma) S \end{aligned}$$

- match surplus does not depend on wage

$$\begin{aligned} \rho S &= \rho\left(J + W - U - V\right) \\ &= y - z - \delta\left(J + W - U - V\right) - p\left(\theta\right)\left(W - U\right) \\ &= y - z - \delta S - p\left(\theta\right)\gamma S \\ S &= \frac{y - z}{\rho + \delta + \gamma p\left(\theta\right)} \end{aligned}$$

Job creation condition

- combine $c = q(\theta) J$ and $J = \frac{y-w}{\rho+\delta}$

$$y - w - \frac{c(\rho+\delta)}{q(\theta)} = 0 \tag{6}$$

- this is a job creation curve in (θ, w) space
- use condition $c = q(\theta) J = q(\theta)(1-\gamma) S$ to get a *job creation curve (JC)*

$$\frac{c}{q(\theta)} = (1-\gamma) \frac{y-z}{\rho+\delta+\gamma p(\theta)} \tag{7}$$

- this is a job creation curve in (u, v) space
- LHS: cost of recruiting one worker since $1/q(\theta)$ is expected time to fill a vacancy
- RHS: value of a worker to the firm
- we used free entry to substitute away the equilibrium value of a job

Wage curve

- solve for the wage (some algebra involved)

$$w = (1 - \gamma)\, z + \gamma\, (y + c\theta)$$

- wage is a weighted average of worker's unemployment benefits z and output y and the hiring costs
- it is called a wage curve – trade-off between w and θ
- recall that $c\theta = cv/u$, where cv is the total hiring costs: a worker gets rewarded for saving firm hiring costs if the worker stays working for the firms

Unemployment dynamics

- law of motion for unemployment

$$\dot{u} = \delta \left(1 - u\right) - p\left(\theta\right) u$$

- steady state value of u

$$u = \frac{\delta}{\delta + p\left(\theta\right)}$$

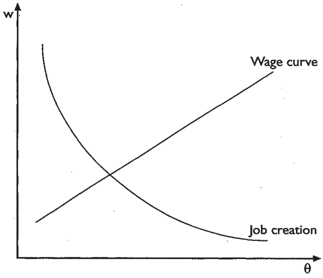
- downward sloping relationship between u and θ; or between u and v

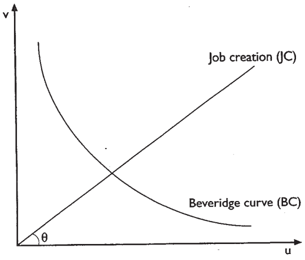
- called Beveridge curve (BC)

Comparative statics

Graphical representation

- 3 variables (u, v, w)
- 3 equations: wage curve, job creation curve, Beveridge curve





source: Pissarides (2000)

Comparative statics

- increase in y:
  - w increases, θ increases, v increase, u decreases
- increase in z
  - w increases, θ decreases, v falls, u increases
- increase in γ
  - similar to z
- increase in ρ
  - v falls, u increases

Dynamics

Fast dynamics

- consider discrete time version of the law of motion, assume θ si constant

$$\begin{aligned}
u_{t+1} &= u_t\left(1-p\left(\theta\right)\right)+\left(1-u_t\right)\delta \\
u_{t+1} &= u_t\left(1-\delta-p\left(\theta\right)\right)+x \\
u_{t+1}-u^* &= \left(u_t-u^*\right)\left(1-\delta-p\left(\theta\right)\right)+\delta-u^*\left(\delta+p\left(\theta\right)\right) \\
u_t-u^* &= \left(1-\delta-p\left(\theta\right)\right)^t\left(u_0-u^*\right)
\end{aligned}$$

- half-life: find t such that
$$\frac{u_t-u^*}{u_0-u^*}=\frac{1}{2}$$
- values for the U.S. (monthly): $\delta=0.034$, $p\left(\theta\right)=0.45$
$$t=\frac{-\log 2}{\log\left(1-\delta-p\left(\theta\right)\right)}\approx\frac{-\log 2}{\log\frac{1}{2}}=1$$
- hence half-life is 1 month!
- empirically, unemployment rate is much more persistent

Dynamics

- study dynamics of the model
- go back to the value functions, write them in a dynamic way

$$U_t = z\Delta t + e^{-\rho\Delta t}\left[p\left(\theta_t\right)\Delta t \cdot W_{t+\Delta t} + \left(1 - p\left(\theta_t\right)\Delta t\right)U_{t+\Delta t}\right]$$

- rewrite

$$\underbrace{U_t - e^{-\rho\Delta t}U_{t+\Delta t}}_{U_t - U_{t+\Delta t} + U_{t+\Delta t} - e^{-\rho\Delta t}U_{t+\Delta t}} = \left[z + e^{-\rho\Delta t}p\left(\theta_t\right)\left(W_{t+\Delta t} - U_{t+\Delta t}\right)\right]\Delta t$$

- to get

$$\rho U_t - \dot{U}_t = z + p\left(\theta_t\right)\left(W_t - U_t\right)$$

- and other

$$\rho W_t - \dot{W}_t = w_t - \delta\left(W_t - U_t\right)$$
$$\rho V_t - \dot{V}_t = -c + q\left(\theta_t\right)\left(J_t - V_t\right)$$
$$\rho J_t - \dot{J}_t = y - w_t - \delta\left(J_t - V_t\right)$$

Dynamics

- free entry condition in every period: $V_t = 0 \Rightarrow c = q(\theta_t) J_t$
- we assume that wage is continuously renegotiated

$$\begin{aligned} W_t - U_t &= \gamma (W_t - U_t + J_t - V_t) = \gamma S_t \\ J_t - V_t &= (1-\gamma) S_t \\ w_t &= \gamma y + (1-\gamma) z + \gamma c \theta_t \end{aligned}$$

- state variable: u_t; jump variable v_t (and also w_t)

Solving dynamic system

- combine equations

$$\begin{aligned} J &= \frac{c}{q(\theta)} \\ \dot{J} &= (\rho+\delta) J - (y-w) \\ w_t &= \gamma y + (1-\gamma) z + \gamma c \theta_t \end{aligned}$$

- to find a differential equation for θ_t

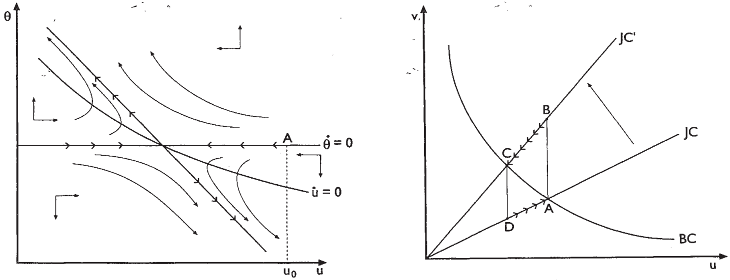
$$\underbrace{-\frac{c}{q(\theta)^2} q'(\theta)}_{\geq 0} \dot{\theta} = (\rho+\delta) \frac{c}{q(\theta)} - (1-\gamma)(y-z) + \gamma c \theta$$

- RHS: increasing in θ, LHS: positive coefficient
- differential equation for u_t

$$\dot{u}_t = \delta (1-u_t) - p(\theta_t) u_t$$

- system of 2 differential equations for (u_t, θ_t)

Dynamics



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Efficiency

Efficiency

- Is DMP equilibrium efficient?
- Are workers searching enough? Are firms posting enough vacancies?
- externality: if a firm posts a vacancy, it decreases matching probability for every other firm
- Can Nash bargaining internalize this externality?

Social planner

- social planner

$$\max_{\theta} \int e^{-\rho t} \left(y(1-u) + zu - c \underbrace{\theta u}_{v} \right) dt$$

$$s.t. \quad \dot{u} = \delta(1-u) - p(\theta) u$$

- write the current-value Hamiltonian

$$H = y(1-u) + zu - c\theta u + \lambda [\delta(1-u) - p(\theta) u]$$

- λ - co-state variable
- optimality conditions are

$$\begin{cases} \frac{\partial H}{\partial \lambda} & = \dot{u} \\ \frac{\partial H}{\partial u} & = -\dot{\lambda} + \rho\lambda \\ \frac{\partial H}{\partial \theta} & = 0 \end{cases}$$

- transversality condition: $\lim_{T \to \infty} e^{-\rho T} \lambda(T) u(T) = 0$

Social planner – cont.

- interpretation of λ: social value of having one additional worker unemployed
- define $\mu = -\lambda$, then μ is social value of a job

$$\frac{\partial H}{\partial u} = -y + z - c\theta + \mu\left(\delta + p\left(\theta\right)\right) = \dot{\mu} - \rho\mu$$
$$\frac{\partial H}{\partial \theta} = -cu + \mu p'\left(\theta\right)u = 0$$

Social planner – cont.

- define the elasticity of the matching function with respect to unemployment,

$$\varepsilon(u,v) \equiv \frac{\partial m(u,v)}{\partial u} \cdot \frac{u}{m(u,v)}.$$

- since $m(\cdot)$ is homogeneous of degree 1, we also have that

$$\varepsilon(u,v) = \varepsilon(\theta) = \frac{m_1\left(\theta^{-1},1\right)}{m(1,\theta)}$$

- simplify

$$\begin{aligned} p'(\theta) &= \frac{d}{d\theta}(\theta q(\theta)) = q(\theta) + \theta q'(\theta) \\ &= q(\theta)\left(1 + \theta \frac{q'(\theta)}{q(\theta)}\right) = q(\theta)\left(1 - \frac{1}{\theta}\frac{m_1\left(\theta^{-1},1\right)}{m\left(\theta^{-1},1\right)}\right) \\ &= q(\theta)(1-\varepsilon(\theta)) \end{aligned}$$

Social planner – cont.

- system of 2 differential equations (u, θ)
- we can derive an equation for $\dot{\theta}$ by combining

$$\dot{\mu} = -y + z - c\theta + \mu \left(\rho + \delta + p\left(\theta\right)\right) \tag{8}$$

$$\frac{c}{q\left(\theta\right)} = \left(1 - \varepsilon\left(\theta\right)\right)\mu \tag{9}$$

- we get a similar system as in a decentralized economy
- again, the only solution will be $\dot{\theta} = 0$
- $\dot{\theta} = 0$ implies $\dot{\mu} = 0$

Social planner – cont.

- we thus have

$$
\begin{aligned}
0 &= -y + z - c\theta + \mu\left(\rho + \delta + p\left(\theta\right)\right) \\
&= -y + z - c\theta + \mu\left(\rho + \delta + \varepsilon p\left(\theta\right) + \left(1 - \varepsilon\right) p\left(\theta\right)\right) \\
&= -y + z - c\theta + \mu\left(\rho + \delta + \varepsilon p\left(\theta\right)\right) + c\theta
\end{aligned}
$$

- we get a final expression for the costate

$$
\mu = \frac{y - z}{\rho + \delta + \varepsilon p\left(\theta\right)}
$$

- which almost the same as the expression for surplus

$$
S = \frac{y - z}{\rho + \delta + \gamma p\left(\theta\right)}
$$

Social planner – cont.

- plugging back we get

$$\frac{c}{q(\theta)} = (1 - \varepsilon(\theta)) \frac{y - z}{\rho + \delta + \varepsilon(\theta) p(\theta)},$$

which is similar to the JC condition

- now it is easy to see when these two solutions coincide – Hosios condition

$$\gamma = \varepsilon(\theta)$$

- if we have a Cobb-Douglas matching function

$$m(U, V) = BU^{1-\alpha} V^{\alpha}$$

then this condition simplifies to $\gamma = \alpha$

Efficiency

- equilibrium is inefficient unless the Hosios condition holds
- if $\gamma > \varepsilon(\theta)$
  - workers get a larger fraction of the total surplus, firms get a smaller share, leading to less firm entry
  - lower equilibrium tightness: firms do not search enough and unemployment is too high
- if $\gamma < \varepsilon(\theta)$
  - firms get a larger share of the surplus and enter too much
  - unemployment is low, but too many resources would be wasted in vacancy costs

Interpretation

- What does the Hosios condition exactly mean?
- consider a myopic planner which takes the contact rates as given

$$\max_{v} \int e^{-\rho t} \left(y \left(1 - u \right) + zu - cv \right) dt$$
$$s.t. \quad \dot{u} = \delta \left(1 - u \right) - \overline{q} v$$

- $\bar{q}$ is taken as given
- optimality condition with respect to v :

$$\frac{\partial H}{\partial v} = -c + \mu \overline{q} \;\Rightarrow c = \underbrace{\mu}_{\text{value of a job}} \times \underbrace{\bar{q}}_{\text{prob of match}}$$

- by analogy with a free entry condition, $c = (1 - \gamma) S \times q$, it follows that this planner wants to set $\gamma = 0$ and give firm the whole value of a match

Interpretation

- in a social planner, we had

$$\frac{\partial H}{\partial v} = -c + \mu \left(q\left(\frac{v}{u}\right) + \frac{v}{u} q'\left(\frac{v}{u}\right)\right)$$

- or

$$c = \left(1 \underbrace{-\varepsilon\left(\theta\right)}_{\text{negative crowding out externality}} \right) \underbrace{\mu}_{\text{value of a job}} \times \underbrace{q\left(\theta\right)}_{\text{probability of match}}$$

- the value of externality is exactly equal to the elasticity of a matching function