Heterogeneous agents and inequality
Session 2
Stationary equilibrium in economies with idiosyncratic risk and incomplete markets

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This session

- Theories of consumption and wealth inequality with exogenous income risk
  1. The income fluctuation problem
  2. **Stationary equilibrium in an economy with idiosyncratic risk**

This session

- Analyze equilibrium quantities and prices in an economy with many individuals who face idiosyncratic risk (but w/o aggregate risk)
- Questions to be answered
  - How much of the observed wealth and consumption heterogeneity can be attributed to consumption smoothing and precautionary savings?
  - How do steady state quantities (output, capital) and prices (interest rate) change in a framework with idiosyncratic risk?
  - Examples:
    - How strongly does the interest rate depend on the level of idiosyncratic risk?
    - How important was the tightened borrowing limits for the 2007-2009 fall in real rates and output?

Stationary equilibrium in an economy with idiosyncratic risk

- Three building blocks
    1. Income Fluctuations Problem
    2. Neoclassical Production Function
    3. Asset market equilibrium

Stationary equilibrium in an economy with idiosyncratic risk

- Three building blocks
  1. Income Fluctuations Problem
  2. Neoclassical Production Function
  3. Asset market equilibrium
- NB
  - 'Allocation' is now a sequence of distributions (or 'measures') λ_t of agents over combinations of assets and labor endowments a, ϵ
  - Optimal HH behaviour given w, r and process for ϵ implies a transition law for λ_t
  - 'Stationary equilibrium' implies a constant measure $\lambda^{\star}$

Outline of this session

1. Stationary Recursive Competitive Equilibrium (SRCE)
    - Recap: Dynamic Recursive CE in the Neoclassical Growth Model
    - SRCE with idosyncratic risk and incomplete markets
        - Technical Preliminaries
        - Definition
        - Algorithm
2. Implications of "Aiyagari/Bewley models"
3. Extensions to get model closer to empirical wealth distribution

Learning Points

- How to define a stationary recursive competitive equilibrium in economies with incomplete markets and idiosyncratic risk
- How to use a simple algorithm to compute equilibrium in Aiyagari (1994)-type economies
- Understand the concept of constrained efficiency, and why the competitive equilibrium in the heterogeneous economy is not constrained efficient
- Understand strengths and weaknesses of Aiyagari model

I. Stationary Recursive Competitive Equilibrium

1. **Dynamic Recursive Competitive Equilibrium in the Neoclassical Growth Model (SL Ch 12))**

The Neoclassical economy

- $t = 1, 2, ...$
- 1 perishable good, used for consumption and investment
- **Agents**: 1 representative firm, 1 representative HH
- **Preferences** $U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$
- CRS **Technology** $Y_t = z_t F(K_t, H_t)$, depreciation δ
- **Uncertainty** z_t follows a (well-behaved) First-order Markov process π
- **Market Structure**: All markets (for goods, capital, labour) competitive

Recursive Competitive Equilibrium: Discussion

- "Recursive":
  - Find a vector of "state variables" X that sufficiently summarises the economy
  - Express HH and Firm decision rules as policy functions of X
  - Express prices as functions of state variables X
- "Competitive":
  - Agents take prices today and price functions in the future as independent of their own decisions
  - Means agents neglect the effect of their current decisions on future **aggregate** quantities

HH Problem

$$\begin{aligned}
v(a,X) &= \max_{c,a'}\{u(c)+\beta Ev(a',X')\} \\
& \quad s.t. \\
c+a' &= w(z,K)\cdot 1+R(z,K)\,a \\
c,a' &\geq 0 \\
X' &= \Psi(X)
\end{aligned}$$

- Ψ is **perceived** law of motion for aggregate states

Recursive Competitive Equilibrium: Definition

... is a value function v; HH decision rules $a'(a, X)$, and $c(a, X)$; choice functions for the firm H and A; pricing functions $r(X)$ and $w(X)$; and a perceived law of motion Ψ such that:

- given the pricing functions r and w, and Ψ, a' and c solve the HH's problem and v is the associated value function
- given prices, the firm chooses optimally K and H, i.e.

  $r(z, K) + \delta = zF_K(K, H)$

  $w(z, K) = zF_H(K, H)$
- the labor market clears $H = 1$
- the asset market clears: $A = a$
- the goods market clears: $c + a' = zF(K, H) + (1 - \delta) K$
- **Rational Expectations**: the aggregate LOM Ψ is generated by π and the policy function a'

I. Stationary Recursive Competitive Equilibrium

1. Dynamic Recursive Competitive Equilibrium in the Neoclassical Growth Model (Recap)
2. **SRCE in an economy with idiosyncratic risk and incomplete markets**
    - Need to generalise previous equilibrium concept to include a distribution of agents across state variables.

The economy

- $t = 1, 2, ...$
- 1 perishable good, used for consumption and investment
- **Agents**: representative firm, continuum of measure 1 of infinitely-lived, ex-ante identical consumers
- **Preferences** $U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$
- **Endowment** of labour efficiency units for each household:
  - $\varepsilon_t \in E \equiv \left\{\varepsilon^1, \varepsilon^2, ..., \varepsilon^{N-1}, \varepsilon^N\right\}$ where N is number of gridpoints
  - ε_t follows i.i.d Markov process: $\pi(\varepsilon', \varepsilon) = \Pr\left(\varepsilon_{t+1} = \varepsilon' \mid \varepsilon_t = \varepsilon\right)$ with unique ergodic density distribution $\Pi_i, i = 1, .., N$
  - LLN applies and π well-behaved: $H_t = \sum_{i=1}^{N} \varepsilon_i \Pi^* \left(\varepsilon_i\right)$, for all t

The economy (cont.)

- **Incomplete markets** imply BC: $c_t + a_{t+1} = (1 + r_t)\, a_t + w_t \varepsilon_t$
- **Borrowing constraint**: $a_{t+1} \geq -b$
- CRS **Technology** $Y_t = F(K_t, H_t)$, depreciation δ - No aggregate risk
- **Market Structure**: All markets (for goods, capital, labour) competitive

Definition: Stationary Equilibrium

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- Stationary Equilibrium: Equilibrium such that aggregate quantities, prices <u>and</u> the cross-sectional distribution of individuals over states a, ε, denoted $\lambda_t(a, \varepsilon)$, are constant over time.
  - Idiosyncratic risk and incomplete markets: individual quantities fluctuate over time
  - But exogenous transitions of income and endogenous transitions in individual assets are such that cross-sectional distribution λ_t is constant and equal to some $\lambda*, \forall t$

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  - But exogenous transitions of income and endogenous transitions in individual assets are such that cross-sectional distribution λ_t is constant and equal to some $\lambda *, \forall t$
- Question: Does this λ^* exist? Is it unique?

HH Problem

$$
\begin{aligned}
v(a,\varepsilon;\lambda) &= \max_{c,a'}\left\{u(c)+\beta\sum_{\varepsilon'\in E} v(a',\varepsilon';\lambda)\pi(\varepsilon',\varepsilon)\right\} \qquad (1)\\
& s.t.\\
c+a' &= R(\lambda)\,a+w(\lambda)\,\varepsilon\\
a' &\geq -b
\end{aligned}
$$

- Note: dependence on stationary distribution λ over assets and income, but no time-variation in λ, aggregate quantities or prices

Stationary Equilibrium: Technicalities

To find a stationary distribution, need to...

- define the mathematical object "distribution"
- define transition function for distributions

Compact state space and measure λ

- Individual states are a, ε.
- Hint from last session that $R < \frac{1}{\beta}$
    - For such R, there is $\bar{a} : a'(a, \varepsilon) < \bar{a} \; \forall a \leq \bar{a}, \forall \varepsilon$. So can define compact state space as $S = A \times E$, for $A = [-b, \bar{a}]$.
- $\lambda(S) \in [0, 1]$ is the measure of agents in state S.

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- Given a joint distribution of wealth and income today λ, need to find λ' that is consistent with individual optimization and exogenous income transitions.

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- Define Q as probability that an individual with current state (a, ε) transits to the set $\mathcal{A} \times \mathcal{E}$ (say, a specific gridpoint of (a', ε')) next period, or

$$Q\left((a, \varepsilon), \mathcal{A} \times \mathcal{E}\right) = I_{\{a'(a,\varepsilon) \in \mathcal{A}\}} \sum_{\varepsilon' \in \mathcal{E}} \pi(\varepsilon', \varepsilon) \tag{2}$$

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- Then λ follows LOM

$$\lambda_{n+1}\left(\mathcal{A} \times \mathcal{E}\right) = T^*\left(\lambda_n\right) = \int_{A \times E} Q\left((a, \varepsilon), \mathcal{A} \times \mathcal{E}\right) d\lambda_n. \tag{3}$$

Stationary recursive competitive equilibrium

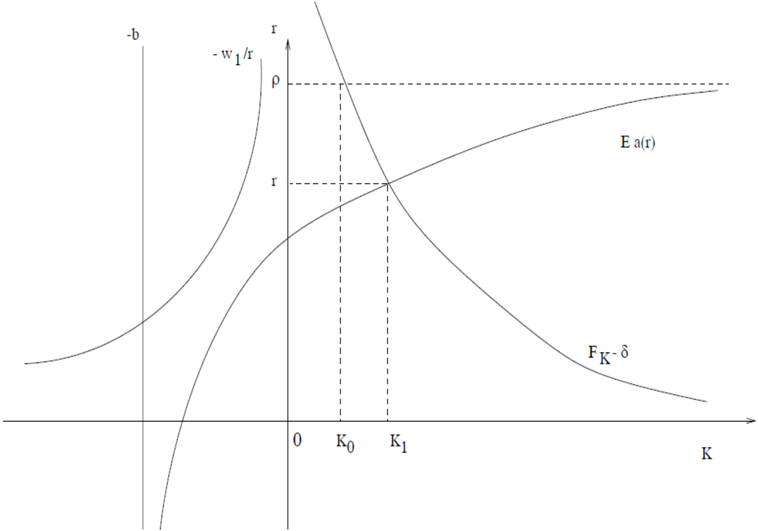
...is a value function $v : S \to \mathbb{R}$; policy functions for the household $a' : S \to \mathbb{R}$, and $c : S \to \mathbb{R}_+$; firm's choices H and K; prices r and w; & a stationary measure λ^* s.t.:

- given r and w, a' and c solve the HH problem and v is the associated value function,
- given r and w, the firm chooses optimally its capital K and its labor H, i.e. $r + \delta = F_K(K, H)$ and $w = F_H(K, H)$,
- the labor market clears: $H = \int_{A \times E} \varepsilon d\lambda^*$,
- the asset market clears: $K = \int_{A \times E} a'(a, \varepsilon) d\lambda^*$,
- (the goods market clears: $\int_{A \times E} c(a, \varepsilon) d\lambda^* + \delta K = F(K, H)$,)
- $\forall (\mathcal{A} \times \mathcal{E})$, the invariant probability measure λ^* satisfies $\lambda^* (\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q((a, \varepsilon), \mathcal{A} \times \mathcal{E}) \, d\lambda^*$ where the transition function Q is defined in (2)

Stationary recursive competitive equilibrium: Characterization and existence

- Equilibrium in labour market is trivial
- If we can show that at some r stationary capital supply and demand equal each other, **existence** follows (due to Walras' law)
- Capital Demand K(r), from inverting the FOC: $r + \delta = F_K(K, H)$
  1. decreasing, continuous
  2. $r \longrightarrow -\delta : K(r) \longrightarrow \infty$
  3. $r \longrightarrow \infty : K(r) \longrightarrow 0$
- Capital supply
  1. Capital supply is continuous in r
  2. Recall that $r \longrightarrow 1/\beta - 1 : E\{a(r)\} \longrightarrow \infty$
  3. also: $a'(a, \varepsilon; -1) = -b \ \forall(a, \varepsilon)$, so $E\{a(-1)\} = -b$
- So $E\{a(r)\}$ and $K(r)$ cross at least once!

Capital demand and supply in stationary equilibrium



Source: SL chapter 17

Stationary recursive competitive equilibrium: Uniqueness

- Uniqueness much more difficult to prove: $E\{a(r)\}$ may be non-monotone due to income and substitution effect of r

Calibration

- Similar to Rep Agent Economy:
  - Choose risk aversion "outside model"
  - Target $\frac{wL}{Y}$, $\frac{K}{Y}$, r choosing values for β, δ and α
- But:
  1. Need idiosyncratic income process and individual borrowing limit b
  2. Note: Steady State capital supply not infinitely elastic at $r = 1/\beta - 1$

Stationary recursive competitive equilibrium: Computational Algorithm

- For the dynamic equilibrium of the neoclassical growth model, could use 1st welfare theorem to compute economy from dynamic planner's problem:
  Reduces to a simple dynamic programming problem of maximising utility subject to aggregate feasibility
- In Aiyagari/Bewley models, need to
  1. solve dynamic HH problem and static firm problem separately given guess for r
  2. derive stationary distribution of a and thus aggregate capital supply
  3. calculate excess capital supply and update guess for r

Stationary recursive competitive equilibrium: Computational Algorithm

1. Choose $r_i : -\delta < r_i < 1/\beta - 1, i = 0$
2. Compute wages given r_i and inelastic labour supply
3. Solve the HH decision problem given r_i to obtain $a'(a, \varepsilon)$
4. Solve for λ_{r_i} by simulating the asset path of a large number N of agents for $t = 1, 2, ...$ until the moments of the cross-sectional distribution converge
5. Compute $E\{a(r)\}$ by averaging over $a_i, i = 1, ..., N$
6. Compute $K(r)$
7. If $E\{a(r)\} - K(r) > 0$ reduce r, otherwise increase it (bisection method)

Repeat 2-7 until capital market clears, i.e. $abs(A(r) - K(r)) \leq tolerance$

II. Constrained Efficiency

Constrained Efficiency: Definition

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- Intuition: Keep restrictions about technology, possibilities of transfers, etc. but choose allocation directly.
- Here: Keep market incompleteness (transfers between periods, not between agents), but let planner choose saving function $g(a, \varepsilon)$

Planner's problem: choose $g(a, \varepsilon)$

$$
\begin{aligned}
\Omega(\lambda) &= \max_{g(a,\varepsilon)\in A} \int_{A\times E} u\left(R(\lambda)\, a + w(\lambda)\,\varepsilon - g(a,\varepsilon)\right) d\lambda + \beta\Omega(\lambda') \\
& \quad s.t. \\
R(\lambda) &= F_K(K,H) \text{ and } w(\lambda) = F_H(K,H) \\
H &= \int_{A\times E} \varepsilon d\lambda \\
K &= \int_{A\times E} a d\lambda \\
\lambda'(\mathcal{A}\times\mathcal{E}) &= \int_{A\times E} 1_{\{g(a,\varepsilon)\in\mathcal{A}\}} \pi\left(\varepsilon' \in \mathcal{E}, \varepsilon\right) d\lambda(a,\varepsilon)
\end{aligned}
$$

Planner's problem: FOC

$$u_c \geq \beta R(\lambda') \sum_{\varepsilon' \in E} u'_c \pi(\varepsilon', \varepsilon) + \beta \int_{A \times E} (\varepsilon' F'_{HK} + a' F'_{KK}) u'_c d\lambda' \quad \forall (a, \varepsilon) \in \ S \qquad (4)$$

- Takes into account the effect of decisions on marginal factor returns F_K, F_H ("prices").
- Compare to Euler equation of consumer in competitive equilibrium: $u_c \geq \beta R(\lambda^*) \sum_{\varepsilon' \in E} u'_c \pi(\varepsilon', \varepsilon)$
- Since $F_{KK} < 0, F_{HK} > 0$, additional term $\beta \int_{A \times E} (\varepsilon' F'_{HK} + a' F'_{KK}) u'_c d\lambda'$ can be positive or negative.

The distribution of wealth

The distribution of wealth

- Model delivers wealth Gini of ≈ 0.4, much smaller than the ≈ 0.8 observed in the data
- Moreover, individuals at the bottom of the distribution have too much wealth, while those at the top have too little.
- Determinants of wealth inequality in the model
  - (+) Labor income risk, σ, ρ
  - Subjective discount factor, β
  - (+) degree of insurance (welfare state)
  - (-) taxes on capital
- Other theories/models: Piketty claims (r-g) drives wealth inequality
- Aspects related/**correlated** with wealth inequality in the data
  - Top tax brackets on (labor) income (over time)
  - Anglo-Saxon countries

Amendments to deliver more realistic wealth distributions

- Note: **major** simplification - agents in model *ex ante* homogenous, and only hit by one type of shock (labor earnings)
  1. Krusell and Smith (1997): heterogeneity in discount factors
  2. Reiter (2004), "Do the Rich Save Too Much? How to Explain the Top Tail of the Wealth Distribution"
     - Rich exposed to idiosyncratic return risk from closely held business
     - Utility from wealth (status)
- Other dimension of fitting data: Add mechanism to model to fit cross-country variation in wealth inequality

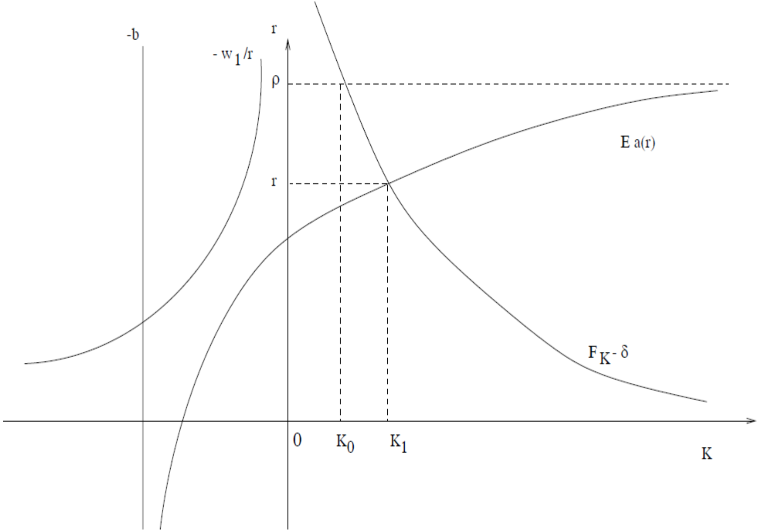
1. Precautionary Savings: Aiyagari (1994), Huggett (1993)

1. Precautionary Savings

- Define PS as difference between equilibrium capital stocks with idiosyncratic consumption risk (K^*) and without (K^{FI})
- Two interacting mechanisms generating PS:
  - $U''' > 0$ as discussed in previous lecture
  - Borrowing limit, $a_{t+1} \geq -b$
- With full insurance, get K^{FI} from $r = 1/\beta - 1 = f'(k^{FI}) - \delta$
- With Cobb-Douglas production, this yields steady state savings as

$$r + \delta = \alpha \left(\frac{Y}{K} \right) = \alpha\delta \left(\frac{Y}{\delta K} \right) = \frac{\alpha\delta}{s} \Rightarrow s = \frac{\alpha\delta}{r + \delta},$$

Capital demand and supply: Full insurance vs. Idios. Risk



Source: SL chapter 17

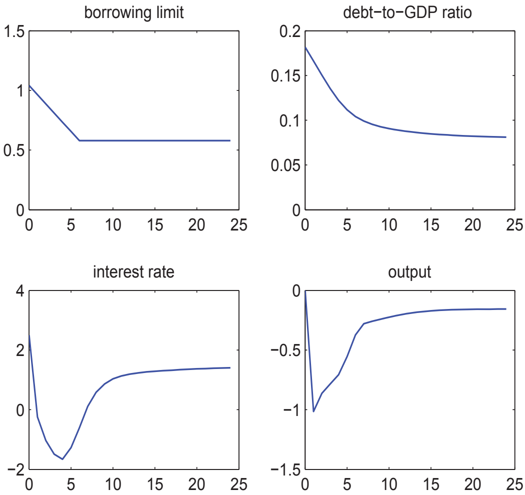
How much precautionary savings in the US? Aiyagari (1994)

- Depends on risk aversion and income process
- With log-utility and iid income shocks: PS ≈ 0
- With CRRA (+) of 5 and AR(1) income-autocorrelation (+) of 0.9: PS $= 14\%$ of output

Guerrieri and Lorenzoni: Implications of tightened borrowing limit

- Goal: Quantify importance of hhs credit crunch for recent recession
- Aiyagari model, but added endogenous labor supply
- Excercise performed: Compute transition dynamics resulting from tightening the borrowing limit
  - reduced b in $a' \geq -b$
- Results
  - r unambigiously decreases, and more initially than in LR
  - Output effect ambigious:
    - $(+)$ labor supply
    - (-) reduced consumption
    - (-) G.E. effect from r falling on labor supply and consumption
  - In their calibration output falls

Guerrieri and Lorenzoni: Transition dynamics



Summary

We learned ...

- ... to define a stationary recursive competitive equilibrium in economies with incomplete markets and idiosyncratic risk
- ... to use a simple algorithm to compute equilibrium in Aiyagari (1994)-type economies
- ... the concept of constrained efficiency, and why the competitive equilibrium in the heterogeneous economy is not constrained efficient
- ... strengths and weaknesses of Aiyagari type model

What we didn't look at...

- ... why can't agents trade contingent assets / why are markets incomplete?
  - Empirically: more insurance than self-insurance
  - Theoretically: want micro-foundation for market incompleteness, e.g. due to limited information (unobservable income shocks) or lack of contract enforcement
- ... economies with idiosyncratic **and** aggregate risk (Krusell and Smith 1998)

III. Extra material

III. Transitional Dynamics

- Question: How to evaluate welfare effect of policy reform, say an unexptected raise in labour income tax in $t = 1$ from $\tau_0 = \tau$ to $\tau_t = \tau'$, $t = 1, 2, ...$
- Answer: Important to take into account transitional dynamics. Total welfare effect is the sum of
  - "Mean consumption effect" of change in average c
  - "Uncertainty effect" of change in fluctuations in future c
  - "Distributional effect" of change in distribution of c
- Look at change between period 0 and period 1 in aggregate welfare (weighted average of group expected utility), or in conditional welfare at some a_{i0}, ε_{i0}

IV. Applications of "Bewley" Models

Applications of Bewley Models

- Look at applications of stationary economies with idiosyncratic income risk and complete markets
  1. Precautionary Savings and real interest rates
  2. The optimal quantity of debt
  3. Optimal income taxes

2. The effect of redistributive taxes in economies with incomplete markets and idiosyncratic risk

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- Can redistributive taxes increase welfare by increasing insurance?
- With exogenous labour supply, clearly redistribution with a 100 percent tax ($\tau = 1$) would be optimal. So need endogenous labour supply to analyse trade-off between insurance and disincentives to work

2. The effect of redistributive taxes in economies with incomplete markets and idiosyncratic risk

- Can redistributive taxes increase welfare by increasing insurance?
- With exogenous labour supply, clearly redistribution with a 100 percent tax ($\tau = 1$) would be optimal. So need endogenous labour supply to analyse trade-off between insurance and disincentives to work
- Look at Floden and Linde (2001)

The economy

- Standard Aiyagari (1994) economy from last sessions, plus endogenous LS and government fiscal policy
- Preferences over leisure and consumption $u(c, l)$
- Intensive margin of labour supply $h(a, \varepsilon) \in [0, 1]$: Agents can adjust their LS to their idiosyncratic productivity and their wealth levels (self-insurance by increasing h when her productivity is low and $a = -b$)
- HH BC $c + a' = (1 + r)\, a + (1 - \tau)\, w\varepsilon h + t$ where τ is a flat earnings tax, and t is the lump-sum transfer of the government.
- New labour market clearing condition $H = \int_{A \times E} \varepsilon h(a, \varepsilon) d\lambda^*$
- Government Budget Constraint $T = \tau w H$ (No-Debt), where T aggregate transfers ($= t$ in equilibrium).

Floden and Linde (2001)

- Ask: What is the optimal level of taxes $\tau^\star$ that maximises welfare in competitive equilibrium?
- For this, need welfare metric, or a **Social Welfare Function**. They use:

$$\max_{\tau} W(\tau) = \int_{A\times E} u(c(a,\varepsilon;\tau), 1 - h(a,\varepsilon;\tau))d\lambda^*(\tau),$$

Floden and Linde (2001): Results

- $\tau^{\star}_{US} = 0.27$ yields welfare gain of 5.6% relative to $\tau = 0$
- $\tau^{\star}_{SV} = 0.03$
- But: depends on flat tax and transfers, no other taxes, etc.

3. Optimal Quantity of Public Debt: Aiyagari and MacGrattan (1998)

- Add public debt to previous model, instead of balanced budget

$$T + (1 + r)\, B = B' + \tau w H,$$

- How does $B > 0$ change the equilibrium?
  1. In stationary equilibrium, increases distortionary taxes τ to pay for interest rB
  2. Public debt perfect substitute with (riskless) K, increases Assets available for risk-sharing
  3. Reduces capital demand, thus increasing interest rates
  4. Higher r makes capital holders better off, who at $B = 0$ hold assets with inefficiently low returns

Aiyagari and McGrattan (1998): Results

- For US economy, optimal quantity of debt $\approx 2/3$ of GDP