Macroeconomic Theory

Economics 702 and 704
Module 1

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Spring 2017

Organization

- **Times of Class:** Mon. and Wed. 9:00-Noon (MCNB 309)
- **Time of Recitations:** Fri. 9:30-11 (MCNB 286-7)
- **Instructor:** Dirk Krueger
- **Office:** 511 McNeil
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- **TA:** Sumedh Ambokar: office hours: Fri. 1:30-3.30 (MCNB 452)

Suggested Books

1. Lars Ljungqvist and Thomas J. Sargent, *Recursive Macroeconomic Theory, 3nd edition*, The MIT Press (2012) (LS)

2. Nancy L. Stokey and Robert E. Lucas, with Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press (1989) (SLP)

3. Daron Acemoglu, *Introduction to Modern Economic Growth*, Princeton University Press (2009)

4. Thomas Cooley, *Frontiers of Business Cycle Research,* Princeton University Press (1995)

Grading Policy

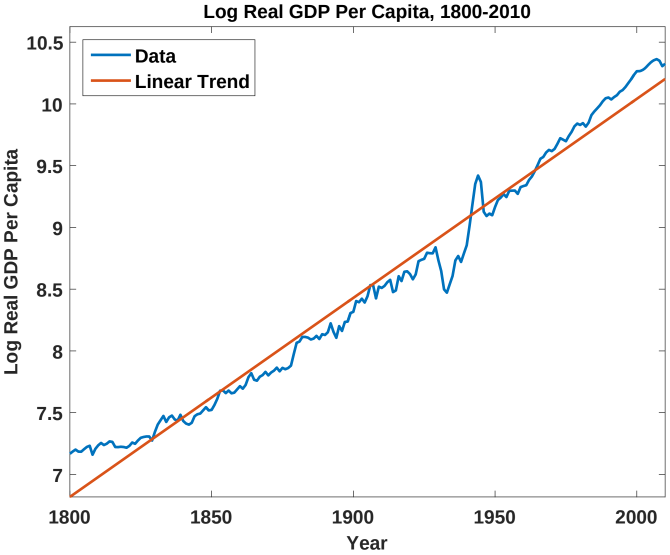
- 3 Homeworks (30%)
  - suggested answers will be available
  - cooperation is encouraged but should be acknowledged. Every student has to hand in her/his uniquely written assignment
- 1 Final (70%)

Topics of the Course

General Theme: Dynamic General Equilibrium Models -Tools and Applications

1. A Simple Model: Equilibrium and Optimality
2. The Neoclassical Growth Model, Calibration and Dynamic Programming
3. Models with Risk, Asset Pricing and the Real Business Cycle Model

Real GDP in the U.S. over Time



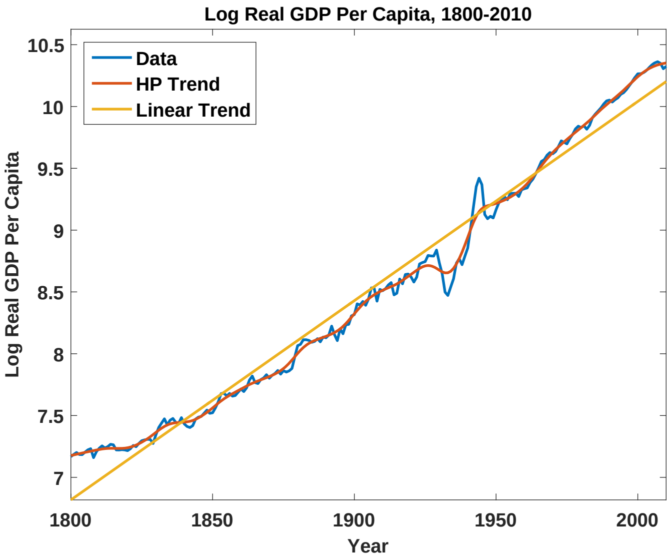
Real GDP in the U.S. over Time

- Constant growth rate

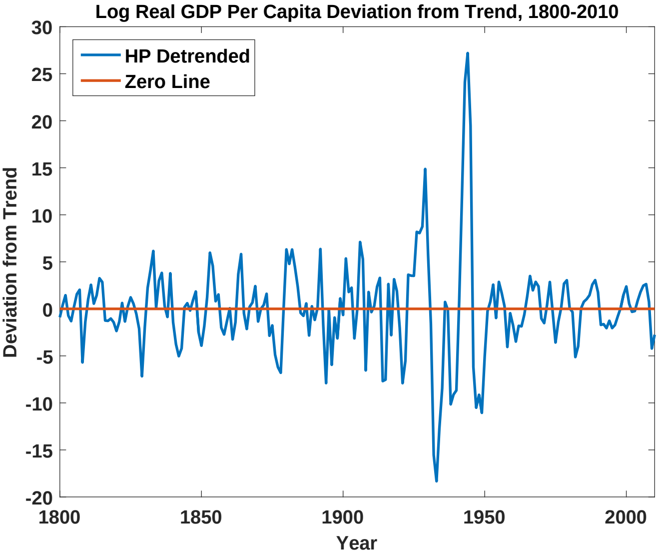
$$\begin{aligned} y_t &= y_{1800}(1+g)^{t-1800} \\ log(y_t) &= log(y_{1800}) + (t-1800) * log(1+g) \\ log(y_t) &\approx log(y_{1800}) + (t-1800) * g \end{aligned}$$

- From 1800 to 2010 (Maddison data) $g = 1.61\%$.
- Note: with $g = 1.61\%$ it takes $t = \frac{70}{g} = 43$ years to double living standards.

Real GDP in the U.S. over Time



Real GDP in the U.S. over Time: Deviation from Trend



Macroeconomics at a Glance

- Aggregate Production Function expresses output (GDP) as function of technology B, the capital stock K and labor L:

$$Y = BK^{\alpha}L^{1-\alpha} \tag{1}$$

- Expressed in terms of growth rates:

$$g_Y = g_B + \alpha g_K + (1-\alpha)g_L. \tag{2}$$

- Or in per capita terms: $y = Y/N, k = K/N, l = L/N$

$$g_y = g_B + \alpha g_k + (1-\alpha)g_l. \tag{3}$$

- Long-run growth in GDP: driven by technological progress (increase in B), increased capital stock (increase in K), and increase in population for some countries (increase in L).
- Short-run fluctuations in GDP: driven mainly by fluctuations in hours worked (fluctuations in l) and changes in the speed of technological progress (fluctuations in B).

General Principles for Specifying a Model

1. Households: **Preferences** over **commodities** and **endowments.** Maximize their preferences, subject to a constraint set.

2. Firms: **Technology**. Maximize (expected) profits, subject to their production plans being technologically feasible.

3. Government: **Policy.** Taken as given or optimally chosen, subject government budget constraint.

4. **Information**

5. **Equilibrium concept:** In this course: competitive equilibrium

An Example Economy

- Time is discrete and indexed by $t = 0, 1, 2, \ldots$
- 2 individuals, $i = 1, 2$, that live forever
- No firms or government
- Agents trade a nonstorable consumption good for each period

Definition

An allocation is a sequence $(c^1, c^2) = \{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ of consumption in each period for each individual.

An Example Economy

- Preferences

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

  with $\beta \in (0, 1)$.

- Endowment $e^i = \{e_t^i\}_{t=0}^{\infty}$

$$e_t^1 = \begin{cases} 2 \text{ if } t \text{ is even} \\ 0 \text{ if } t \text{ is odd} \end{cases}$$

$$e_t^2 = \begin{cases} 0 \text{ if } t \text{ is even} \\ 2 \text{ if } t \text{ is odd} \end{cases}$$

An Example Economy

- No risk
- Market Structure: At period 0 agents meet and trade all commodities
- Let p_t denote the price, in period 0, of one unit of consumption to be delivered in period t
- In all future periods deliveries of the consumption goods take place, no further trade after period t
- Perfect enforcement

Definition of Competitive Equilibrium: Household Probem

Given a sequence of prices $\{p_t\}_{t=0}^{\infty}$ households solve

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

$$\begin{aligned} &\text{s.t.} \\ \sum_{t=0}^{\infty} p_t c_t^i &\leq \sum_{t=0}^{\infty} p_t e_t^i \\ c_t^i &\geq 0 \text{ for all } t \end{aligned}$$

Definition of Competitive Equilibrium

Definition

A (competitive) Arrow-Debreu equilibrium are prices $\{\hat{p}_t\}_{t=0}^{\infty}$ and allocations $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ such that

1. Given $\{\hat{p}_t\}_{t=0}^{\infty}$, for $i = 1, 2$, $\{\hat{c}_t^i\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

$$\text{s.t.}$$

$$\begin{aligned} \sum_{t=0}^{\infty} \hat{p}_t c_t^i &\leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i \\ c_t^i &\geq 0 \text{ for all } t \end{aligned}$$

2. Goods markets clear:

$$\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 \text{ for all } t$$

Solving for the Equilibrium

- The first order necessary conditions

$$\begin{aligned} \frac{\beta^t}{c_t^i} &= \lambda_i p_t \\ \frac{\beta^{t+1}}{c_{t+1}^i} &= \lambda_i p_{t+1} \end{aligned}$$

  and hence

$$p_{t+1}c_{t+1}^i = \beta p_t c_t^i \text{ for all } t \tag{4}$$

  for $i = 1, 2$.

- Equations (4), together with the budget constraint can be solved for (Marshallian) demand function

$$c_t^i = c_t^i\left(\{p_t\}_{t=0}^{\infty}\right)$$

  or excess demand function

$$x_t^i = x_t^i\left(\{p_t\}_{t=0}^{\infty}\right) = c_t^i\left(\{p_t\}_{t=0}^{\infty}\right) - e_t^i$$

Solving for the Equilibrium

- Use goods market clearing conditions

$$c_t^1\left(\{p_t\}_{t=0}^{\infty}\right) + c_t^2\left(\{p_t\}_{t=0}^{\infty}\right) = e_t^1 + e_t^2 \text{ for all } t$$

  to solve for equilibrium prices

- In general a system of infinite equations (for each t one) in an infinite number of unknowns $\{p_t\}_{t=0}^{\infty}$

- Negishi's method very helpful

Solving for the Equilibrium

- For example
$$p_{t+1}c^i_{t+1} = \beta p_t c^i_t$$
- Sum over i
$$p_{t+1}\left(c^1_{t+1} + c^2_{t+1}\right) = \beta p_t(c^1_t + c^2_t)$$
- Using goods market clearing
$$p_{t+1}\left(e^1_{t+1} + e^2_{t+1}\right) = \beta p_t(e^1_t + e^2_t)$$
- Hence
$$p_{t+1} = \beta p_t$$
- Therefore equilibrium prices have to satisfy
$$p_t = \beta^t p_0$$
- Normalize $p_0 = 1$, i.e. make consumption at period 0 the numeraire. Then
$$\hat{p}_t = \beta^t$$

Solving for the Equilibrium

- From the first order condition

$$c_{t+1}^i = c_t^i = c_0^i$$

- Use the budget constraint: Left hand side

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i = c_0^i \sum_{t=0}^{\infty} \beta^t = \frac{c_0^i}{1-\beta}$$

- Right hand side: For agent 1

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^1 = 2 \sum_{t=0}^{\infty} \beta^{2t} = \frac{2}{1-\beta^2}$$

- For agent 2

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^2 = 2\beta \sum_{t=0}^{\infty} \beta^{2t} = \frac{2\beta}{1-\beta^2}$$

- Equilibrium allocation

$$\hat{c}_t^1 = \hat{c}_0^1 = (1-\beta)\frac{2}{1-\beta^2} = \frac{2}{1+\beta} > 1$$

Characteristics of Equilibrium

- Agent 1 consumes more in every period, just because she is rich first
- Consumption is smooth (in contrast to income)
- Intertemporal trade: in each even period agent 1 delivers $\frac{2\beta}{1+\beta}$ to the second agent and in all odd periods agent 2 delivers $\frac{2}{1+\beta}$ to the first agent.
- Trade is mutually beneficial: Without trade

$$u(e_t^i) = -\infty$$

- With trade

$$\begin{aligned}
u(\hat{c}^1) &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2}{1+\beta}\right) = \frac{\ln\left(\frac{2}{1+\beta}\right)}{1-\beta} > 0 \\
u(\hat{c}^2) &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2\beta}{1+\beta}\right) = \frac{\ln\left(\frac{2\beta}{1+\beta}\right)}{1-\beta} < 0
\end{aligned}$$

Pareto Optimality and the First Welfare Theorem

Definition

An allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ is feasible if

1. $$c_t^i \geq 0 \text{ for all } t, \text{ for } i = 1, 2$$

2. $$c_t^1 + c_t^2 = e_t^1 + e_t^2 \text{ for all } t$$

Definition

An allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ is Pareto efficient if it is feasible and if there is no other feasible allocation $\{(\tilde{c}_t^1, \tilde{c}_t^2)\}_{t=0}^{\infty}$ such that

$$\begin{aligned} u(\tilde{c}^i) &\geq u(c^i) \text{ for both } i = 1, 2 \\ u(\tilde{c}^i) &> u(c^i) \text{ for at least one } i = 1, 2 \end{aligned}$$

First Welfare Theorem

Proposition

Let $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ *be a competitive equilibrium allocation. Then* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ *is Pareto efficient.*

Proof of First Welfare Theorem

- Proof by contradiction: Suppose not. Then there exists another feasible allocation $(\{\tilde{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ such that

$$\begin{aligned} u(\tilde{c}^i) &\geq u(\hat{c}^i) \text{ for both } i = 1,2 \\ u(\tilde{c}^i) &> u(\hat{c}^i) \text{ for at least one } i = 1,2 \end{aligned}$$

- Without loss of generality assume that the strict inequality holds for $i = 1$.
- Step 1: Show that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 \tag{5}$$

- If not, then

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$

and

$$u(\tilde{c}^1) > u(\hat{c}^1)$$

- This cannot be since $\{\hat{c}_t^1\}_{t=0}^{\infty}$ is part of a competitive equilibrium.

Proof of First Welfare Theorem

- Step 2: Show that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2 \tag{6}$$

- If not, then

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 < \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

- Then there exists a $\delta > 0$ such that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + \delta \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

- Define a new allocation

$$\begin{aligned} \check{c}_t^2 &= \tilde{c}_t^2 \text{ for all } t \geq 1 \\ \check{c}_0^2 &= \tilde{c}_0^2 + \delta \text{ for } t = 0 \end{aligned}$$

Proof of First Welfare Theorem

- Obviously

$$\sum_{t=0}^{\infty} \hat{p}_t \check{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + \delta \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

- Furthermore

$$u(\check{c}^2) > u(\tilde{c}^2) \geq u(\hat{c}^2)$$

- But this can't be the case since $\hat{c}^2$ is by assumption part of the competitive equilibrium.

Proof of First Welfare Theorem

- Sum (5) and (6) to obtain

$$\sum_{t=0}^{\infty} \hat{p}_t(\tilde{c}_t^1 + \tilde{c}_t^2) > \sum_{t=0}^{\infty} \hat{p}_t(\hat{c}_t^1 + \hat{c}_t^2)$$

- But both allocations are feasible, hence

$$\tilde{c}_t^1 + \tilde{c}_t^2 = e_t^1 + e_t^2 = \hat{c}_t^1 + \hat{c}_t^2 \text{ for all } t$$

- Thus

$$\sum_{t=0}^{\infty} \hat{p}_t(e_t^1 + e_t^2) > \sum_{t=0}^{\infty} \hat{p}_t(e_t^1 + e_t^2),$$

  a contradiction. **QED**

Negishi's (1960) Method to Compute Equilibria

- Main idea: Compute Pareto-optimal allocations by solving simple social planners problem.
- Because of the first welfare theorem competitive equilibrium allocations are Pareto optimal.
- By solving for all Pareto optimal allocations we have solved for all potential equilibrium allocations.
- Then isolate Pareto efficient allocations who are competitive equilibrium allocations.

Social Planner Problem

$$\begin{aligned}
\max_{\{(c_t^1,c_t^2)\}_{t=0}^{\infty}} \alpha^1 u(c^1) + \alpha^2 u(c^2) &= \max_{\{(c_t^1,c_t^2)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\alpha^1 \ln(c_t^1) + \alpha^2 \ln(c_t^2)\right] \\
& \text{s.t.} \\
c_t^i &\geq 0 \text{ for all } i, \text{ all } t \\
c_t^1 + c_t^2 &= e_t^1 + e_t^2 \equiv 2 \text{ for all } t
\end{aligned}$$

for Pareto weights $\alpha = (\alpha^1, \alpha^2) \geq 0$.

Proposition

An allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ that solves the social planner problem for some $\alpha > 0$ is Pareto efficient.

Proposition

Any Pareto efficient allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ is the solution to the social planner problem for some $\alpha \geq 0, \alpha \neq 0$.

Characterizing the Solution of the Planner Problem

- Fix arbitrary $\alpha \geq 0$.
- Attach Lagrange multipliers $\frac{\mu_t}{2}$ to the resource constraints
- First order conditions

$$\begin{aligned} \frac{\alpha^1 \beta^t}{c_t^1} &= \frac{\mu_t}{2} \\ \frac{\alpha^2 \beta^t}{c_t^2} &= \frac{\mu_t}{2} \end{aligned}$$

- Combining yields

$$\begin{aligned} \frac{c_t^1}{c_t^2} &= \frac{\alpha^1}{\alpha^2} \\ c_t^1 &= \frac{\alpha^1}{\alpha^2} c_t^2 \end{aligned}$$

Characterizing the Solution of the Planner Problem

- Using the resource constraint

$$\begin{aligned} c_t^1 + c_t^2 &= 2 \\ \frac{\alpha^1}{\alpha^2} c_t^2 + c_t^2 &= 2 \\ c_t^2 &= 2/(1+\alpha^1/\alpha^2) = c_t^2(\alpha) \\ c_t^1 &= 2/(1+\alpha^2/\alpha^1) = c_t^1(\alpha) \end{aligned}$$

- Lagrange multipliers are given by

$$\mu_t = (\alpha^1 + \alpha^2)\beta^t$$

- Set of Pareto efficient allocations is given by

$$\begin{aligned} PO &= \{\{(c_t^1, c_t^2)\}_{t=0}^{\infty} : c_t^1 = 2/(1+\alpha^2/\alpha^1) \\ c_t^2 &= 2/(1+\alpha^1/\alpha^2) \text{ for some } \alpha^1/\alpha^2 \in [0,\infty)\} \end{aligned}$$

Identifying Equilibrium Allocations

- Compare the first order condition from equilibrium household maximization
$$\frac{\beta^t}{c_t^i} = \lambda_i p_t$$
to the first order necessary conditions from the social planners problem:
$$\frac{\alpha^i \beta^t}{c_t^i} = \frac{\mu_t}{2}.$$
- Conjecture $p_t = \mu_t$ (and $\lambda_1 = \frac{1}{2\alpha^i}$)
- With these prices FOC's in planner problem and in household maximization problem are identical.
- Now argue that these prices and allocations $(c^1(\alpha), c^2(\alpha))$ constructed from planner problem are competitive equilibria for the "right" α.

Identifying Equilibrium Allocations

- What requirements from competitive equilibrium definition not yet checked? Individual *budget constraints* have to hold!
- Construct transfer functions

$$t^i(\alpha) = \sum_t \mu_t \left[c_t^i(\alpha) - e_t^i\right]$$

  $t^i(\alpha)$ are the resources, valued at prices $\{\mu_t\}$, that agent i needs as transfer to afford the Pareto efficient allocation indexed by α.
- Competitive equilibrium allocation: find the Pareto weight α such that $t^1(\alpha) = t^2(\alpha) = 0$.

Identifying Equilibrium Allocations

- For example economy

$$
\begin{aligned}
t^1(\alpha) &= (\alpha^1 + \alpha^2) \sum_t \beta^t \left(\frac{2}{1 + \alpha^2/\alpha^1} - e_t^1 \right) \\
t^2(\alpha) &= (\alpha^1 + \alpha^2) \sum_t \beta^t \left(\frac{2}{1 + \alpha^1/\alpha^2} - e_t^2 \right)
\end{aligned}
$$

- Note that $t^i(\alpha)$ is homogeneous of degree 1. Thus if $t^i(\alpha) = 0$, then $t^i(\theta\alpha) = \theta t^i(\alpha) = 0$. Thus WLOG can normalize $\alpha^1 + \alpha^2 = 1$.
- Also note that $\sum_i t^i(\alpha) = 0$. Thus one equation is redundant.

Identifying Equilibrium Allocations

- Take first equation:

$$\begin{aligned} \frac{2}{(1-\beta)(1+\alpha^2/\alpha^1)} - \frac{2}{1-\beta^2} &= 0 \\ \alpha^2/\alpha^1 &= \beta \end{aligned}$$

- Corresponding consumption allocations are

$$\begin{aligned} c_t^1 &= \frac{2}{1+\alpha^2/\alpha^1} = \frac{2}{1+\beta} \\ c_t^2 &= \frac{2}{1+\alpha^1/\alpha^2} = \frac{2\beta}{1+\beta} \end{aligned}$$

- Equilibrium prices are given by the Lagrange multipliers $\mu_t = \beta^t$
- Note: Negishi method reduces the computation of equilibrium to a finite $(N-1)$ number of equations in a finite number of unknowns: need to find α's that make the transfer functions equal to zero.

Sequential Markets Equilibrium

- Arrow-Debreu market structure: trade takes place only at the beginning of time
- Now: Sequential Market structure with trade of consumption and one-period bonds in each period
- Key result: Set of equilibria is the same with both market structures

Sequential Markets Equilibrium

- Let r_{t+1} denote interest rate on one period bonds from period t to period $t+1$
- One period bond is promise to pay 1 unit of consumption in period $t+1$ in exchange for $\frac{1}{1+r_{t+1}}$ units of the consumption good in t.
- Interpret $q_t \equiv \frac{1}{1+r_{t+1}}$ as the relative price of one unit of the consumption good in period $t+1$ in terms of the period t consumption good.
- Let a_{t+1}^i denote the amount of such bonds
- Household i's budget constraint in period t

$$c_t^i + \frac{a_{t+1}^i}{(1+r_{t+1})} \leq e_t^i + a_t^i$$

or

$$c_t^i + q_t a_{t+1}^i \leq e_t^i + a_t^i$$

- Initial bond holdings a_0^i. We assume $a_0^i = 0$

Definition

A Sequential Markets equilibrium is allocations $\left\{\left(\hat{c}_t^i, \hat{a}_{t+1}^i\right)_{i=1,2}\right\}_{t=0}^{\infty}$, interest rates $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ such that

1. For $i = 1, 2$, given interest rates $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ $\{\hat{c}_t^i, \hat{a}_{t+1}^i\}_{t=0}^{\infty}$ solves

$$
\begin{aligned}
& \max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \\
& \text{s.t.} \\
c_t^i + \frac{a_{t+1}^i}{(1+\hat{r}_{t+1})} &\leq e_t^i + a_t^i \\
c_t^i &\geq 0 \text{ for all } t \\
a_{t+1}^i &\geq -\bar{A}^i
\end{aligned}
$$

2. For all $t \geq 0$

$$
\begin{aligned}
\sum_{i=1}^{2} \hat{c}_t^i &= \sum_{i=1}^{2} e_t^i \\
\sum_{i=1}^{2} \hat{a}_{t+1}^i &= 0
\end{aligned}
$$

Ponzi Schemes and Borrowing Constraints

- Constraint
$$a_{t+1}^i \geq -\bar{A}^i$$
rules out Ponzi schemes. Otherwise no equilibrium exists
- Suppose no constraint and suppose there would exist a SM-equilibrium $\{(\hat{c}_t^i, \hat{a}_{t+1}^i)_{i=1,2}\}_{t=1}^{\infty}, \{\hat{r}_{t+1}\}_{t=0}^{\infty}$.
- Agent i could always do better by setting
$$\begin{aligned} c_0^i &= \hat{c}_0^i + \frac{\varepsilon}{1+\hat{r}_1} \\ a_1^i &= \hat{a}_1^i - \varepsilon \\ a_2^i &= \hat{a}_2^i - (1+\hat{r}_2)\varepsilon \\ a_{t+1}^i &= \hat{a}_{t+1}^i - \prod_{\tau=1}^{t}(1+\hat{r}_{\tau+1})\varepsilon \end{aligned}$$
i.e. by borrowing $\varepsilon > 0$ more in period 0, consuming it and then rolling over the additional debt forever.
- Note: $\varepsilon > 0$ is arbitrarily large.

Equivalence of Arrow-Debreu and Sequential Market Structure

Proposition

Let allocations $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ *and prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *form an Arrow-Debreu equilibrium with*

$$\frac{\hat{p}_{t+1}}{\hat{p}_t} \leq \xi < 1.$$

Then there exist $(\bar{A}^i)_{i=1,2}$ *and a corresponding sequential markets equilibrium with allocations* $\{(\tilde{c}_t^i, \tilde{a}_{t+1}^i)_{i=1,2}\}_{t=0}^{\infty}$ *and interest rates* $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ *such that*

$$\tilde{c}_t^i = \hat{c}_t^i \text{ for all } i, \text{ all } t$$

Equivalence of Arrow-Debreu and Sequential Market Structure

Proposition

Reversely, let allocations $\{(\hat{c}_t^i, \hat{a}_{t+1}^i)_{i=1,2}\}_{t=0}^{\infty}$ *and interest rates* $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ *form a sequential markets equilibrium. Suppose that it satisfies*

$$\begin{aligned} \hat{a}_{t+1}^i &> -\bar{A}^i \text{ for all } i, \text{ all } t \\ \hat{r}_{t+1} &\geq \varepsilon > 0 \text{ for all } t \end{aligned}$$

Then there exists a corresponding Arrow-Debreu equilibrium $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$, $\{\tilde{p}_t\}_{t=0}^{\infty}$ *such that*

$$\hat{c}_t^i = \tilde{c}_t^i \text{ for all } i, \text{ all } t$$

Proof of Equivalence Result

- Step 1: Normalize $\hat{p}_0 = 1$ and relate equilibrium prices and interest rates by

$$1 + \hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$$

- Using the normalization $\hat{p}_0 = 1$ this implies

$$\prod_{j=1}^{t}(1 + \hat{r}_j) = \frac{\hat{p}_0}{\hat{p}_1} * \frac{\hat{p}_1}{\hat{p}_2} \cdots * \frac{\hat{p}_{t-1}}{\hat{p}_t} = \frac{1}{\hat{p}_t} \tag{7}$$

- Now show equivalence of budget sets in Arrow Debreu and SM equilibrium.

Proof of Equivalence Result

- SM budget constraints

$$\begin{aligned} c_0^i + \frac{a_1^i}{1+\hat{r}_1} &= e_0^i & (8) \\ c_1^i + \frac{a_2^i}{1+\hat{r}_2} &= e_1^i + a_1^i & (9) \\ &\vdots \\ c_t^i + \frac{a_{t+1}^i}{1+\hat{r}_{t+1}} &= e_t^i + a_t^i & (10) \end{aligned}$$

- Substituting for a_1^i from (9) in (8) one gets

$$c_0^i + \frac{c_1^i}{1+\hat{r}_1} + \frac{a_2^i}{(1+\hat{r}_1)(1+\hat{r}_2)} = e_0^i + \frac{e_1^i}{(1+\hat{r}_1)}$$

and thus

$$\sum_{t=0}^{T} \frac{c_t^i}{\prod_{j=1}^{t}(1+\hat{r}_j)} + \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = \sum_{t=0}^{T} \frac{e_t^i}{\prod_{j=1}^{t}(1+\hat{r}_j)}$$

Proof of Equivalence Result

- Let $T \to \infty$ and use (7):

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i + \lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$

- Given our assumptions on the equilibrium interest rates we have

$$\lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} \geq \lim_{T\to\infty} \frac{-\bar{A}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = 0$$

and hence

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$

Proof of Equivalence Result

- Suppose we have AD-equilibrium $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$, $\{\hat{p}_t\}_{t=0}^{\infty}$.
- Want to show that there exist SM equilibrium with $\tilde{c}_t^i = \hat{c}_t^i$ for all i, all t
- $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ satisfies market clearing.
- Define asset holdings

$$\tilde{a}_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - e_{t+\tau}^i\right)}{\hat{p}_{t+1}}$$

  and plug into SM budget constraint. Is satisfied (recall $1 + \tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$).

- Construct borrowing limit $\bar{A}^i$. By assumption $\frac{\hat{p}_{t+\tau}}{\hat{p}_{t+1}} \leq \xi^{\tau-1}$ and thus

$$\tilde{a}_{t+1}^i \geq -\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau} e_{t+\tau}^i}{\hat{p}_{t+1}} \geq -\sum_{\tau=1}^{\infty} \xi^{\tau-1} e_{t+\tau}^i > -\infty$$

  so that we can take

$$\bar{A}^i = 1 + \sup_t \sum_{\tau=1}^{\infty} \xi^{\tau-1} e_{t+\tau}^i < \infty$$

Proof of Equivalence Result

- Step 2: Show that $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ maximizes utility, subject to the SM budget constraints and the borrowing constraints.

- Suppose not. Any other allocation satisfying the SM budget constraints satisfies the AD budget constraint (step 1). If this alternative allocation would be better than $\{\tilde{c}_t^i = \hat{c}_t^i\}_{t=0}^{\infty}$ it would have been chosen as part of an AD-equilibrium, which it wasn't.

- Hence $\{\tilde{c}_t^i\}_{t=0}^{\infty}$ is optimal within the set of allocations satisfying the SM budget constraints at interest rates $1 + \tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$.

Proof of Equivalence Result

- Step 3: Now suppose $\{(\hat{c}_t^i, \hat{a}_{t+1}^i)_{i\in I}\}_{t=1}^{\infty}$ and $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ form a sequential markets equilibrium satisfying

$$\begin{aligned} \hat{a}_{t+1}^i &> -\bar{A}^i \text{ for all } i, \text{ all } t \\ \hat{r}_{t+1} &\geq \varepsilon > 0 \text{ for all } t \end{aligned}$$

- We want to show that there exists a corresponding Arrow-Debreu equilibrium $\{(\tilde{c}_t^i)_{i\in I}\}_{t=0}^{\infty}$, $\{\tilde{p}_t\}_{t=0}^{\infty}$ with

$$\hat{c}_t^i = \tilde{c}_t^i \text{ for all } i, \text{ all } t.$$

- $\{(\tilde{c}_t^i)_{i\in I}\}_{t=0}^{\infty}$ satisfies market clearing and the AD budget constraint (see step 1).
- Set $\tilde{p}_0 = 1$ and $\tilde{p}_{t+1} = \frac{\tilde{p}_t}{1+\hat{r}_{t+1}}$. Is $\{(\tilde{c}_t^i)_{i\in I}\}_{t=0}^{\infty}$ optimal among allocations that satisfy the AD budget constraint?

Proof of Equivalence Result

- For any other consumption allocation $\{c_t^i\}_{t=0}^{\infty}$satisfying the AD budget constraint, construct

$$a_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(c_{t+\tau}^i - e_{t+\tau}^i\right)}{\hat{p}_{t+1}}$$

- If $a_{t+1}^i \geq -\bar{A}^i$, then the $\{c_t^i\}_{t=0}^{\infty}$ cannot be better than $\{\hat{c}_t^i\}_{t=0}^{\infty}$ since it was available in the SM household problem.
- But suppose $a_{t+1}^i < -\bar{A}^i$. Since SM household maximization problem has a concave objective and convex constraint set, if its maximizer has $\hat{a}_{t+1}^i > -\bar{A}^i$ (constraint not binding), then no allocation with $a_{t+1}^i < -\bar{A}^i$ can be better.
- Hence $\{\tilde{c}_t^i\}_{t=0}^{\infty}$ is optimal for household i within the set of allocations satisfying her AD budget constraint.

Proof of Equivalence Result

- Note: Suppose we choose as borrowing constraint

$$\bar{A}^i = 1 + \sup_t \sum_{\tau=1}^{\infty} \xi^{\tau-1} e^i_{t+\tau}$$

- Then $a^i_{t+1} \geq -\bar{A}^i$ for all allocations satisfying the AD budget constraint (at equilibrium prices).

Interest Rates for the Example Economy

- Interest rate is constant and equals the subjective time discount rate.

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}} = \frac{1}{\beta}$$

- Thus

$$r_{t+1} = r = \frac{1}{\beta} - 1 = \rho$$

Some Utility Theory for Macroeconomists

- Utility function

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

  satisfies:

- Time separability
- Time discounting because $\beta < 1$. Indicates that agents are impatient.
  - Parameter β is referred to as (subjective) time discount factor.
  - Subjective time discount rate ρ is defined by $\beta = \frac{1}{1+\rho}$

Utility Theory: Properties of Lifetime Utility Function

- Homotheticity: Define the marginal rate of substitution between consumption at any two dates t and $t+s$ as

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}}$$

Definition

The function u is said to be homothetic if

$$MRS(c_{t+s}, c_t) = MRS(\lambda c_{t+s}, \lambda c_t)$$

for all $\lambda > 0$ and c.

- For u defined above

$$MRS(c_{t+s}, c_t) = \frac{\frac{\beta^{t+s}}{c_{t+s}}}{\frac{\beta^t}{c_t}} = \frac{\frac{\lambda\beta^{t+s}}{c_{t+s}}}{\frac{\lambda\beta^t}{c_t}} = MRS(\lambda c_{t+s}, \lambda c_t)$$

- Homotheticity crucial for existence of balanced growth path.

Utility Theory: Properties of Period Utility Function

- Period utility function or felicity function $U(c) = \ln(c)$ is continuous, twice continuously differentiable, strictly increasing (i.e. $U'(c) > 0$) and strictly concave (i.e. $U''(c) < 0$) and satisfies the Inada conditions

$$\begin{aligned} \lim_{c \searrow 0} U'(c) &= +\infty \\ \lim_{c \nearrow +\infty} U'(c) &= 0 \end{aligned}$$

- U is a of Constant Relative Risk Aversion (CRRA) form:

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

Note that

$$\ln(c) = \lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

Properties of CRRA Period Utility Function

- Define as $\sigma(c) = -\frac{U''(c)c}{U'(c)}$ the (Arrow-Pratt) coefficient of relative risk aversion.
- For CRRA utility functions

$$\sigma(c) = \sigma$$

- For our special case $\sigma = 1$

Properties of CRRA Period Utility Function

- Define intertemporal elasticity of substitution (IES) as

$$ies_t(c_{t+1}, c_t) = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\left[\frac{\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{d\left(\frac{c_{t+1}}{c_t}\right)}}{\frac{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}{\frac{c_{t+1}}{c_t}}}\right]^{-1}$$

- Inverse of the % change in MRS between consumption at t and $t+1$ in response to % change in consumption ratio $\frac{c_{t+1}}{c_t}$

Properties of CRRA Period Utility Function

- For the CRRA utility function note that

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}$$

- Thus

$$ies_t(c_{t+1}, c_t) = -\left[\frac{-\sigma\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma-1}}{\frac{\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}}{\frac{c_{t+1}}{c_t}}}\right]^{-1} = \frac{1}{\sigma}$$

Properties of CRRA Period Utility Function

- From the first order conditions of the household problem

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r_{t+1}}$$

- Thus the IES can alternatively be written as

$$ies_t(c_{t+1}, c_t) = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{1}{1+r_{t+1}}\right)}{\frac{1}{1+r_{t+1}}}\right]}$$

- Thus IES measures the percentage change in the consumption growth rate in response to a percentage change in the gross real interest rate, the intertemporal price of consumption.

Properties of CRRA Period Utility Function

- Note that for the CRRA utility function the Euler equation reads as
$$(1 + r_{t+1})\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} = 1.$$
- Taking logs on both sides and rearranging one obtains
$$\ln(1 + r_{t+1}) + \log(\beta) = \sigma \left[\ln(c_{t+1}) - \ln(c_t)\right]$$
or
$$\ln(c_{t+1}) - \ln(c_t) = \frac{1}{\sigma}\ln(\beta) + \frac{1}{\sigma}\ln(1 + r_{t+1}).$$
- Use this equation to estimate $\frac{1}{\sigma}$.

A Note on Balanced Growth

- Define balanced growth path (BGP) as a situation in which

$$\begin{aligned} c_t &= (1+g)^t c_0 \\ r_{t+1} &= r \text{ for all } t \end{aligned}$$

- Plugging into Euler equation

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) = \frac{1}{1+r} \text{ for all } t$$

- But for this equation to hold for all t we require that u is homothetic:

$$\begin{aligned} MRS(c_{t+1}, c_t) &= MRS((1+g)^t c_1, (1+g)^t c_0) \\ &= MRS(c_1, c_0) \end{aligned}$$

Neoclassical Growth Model: Overview

- Time is discrete, $t = 0, 1, 2, \ldots$
- In each period three goods traded
  - labor services n_t
  - capital services k_t
  - final output good y_t that can be either consumed, c_t or invested, i_t

Neoclassical Growth Model: Technology

- Aggregate production function F

$$y_t = F(k_t, n_t)$$

- Output can be consumed or invested

$$y_t = i_t + c_t$$

- Investment augments the capital stock which depreciates at a constant rate δ over time

$$k_{t+1} = (1-\delta)k_t + i_t$$

or

$$i_t = k_{t+1} - k_t + \delta k_t$$

- Gross investment i_t equals net investment $k_{t+1} - k_t$ plus depreciation δk_t. Will require that $k_{t+1} \geq 0$

Neoclassical Growth Model: Preferences and Endowments

- Large number of identical, infinitely lived households
- Utility function

$$u\left(\{c_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

- Endowments: $\bar{k}_0$ of initial capital and one unit of productive time in each period
- Information: no risk and perfect foresight
- Equilibrium: competitive (for later)

Pareto Optimal Allocations

Definition

An allocation $\{c_t, k_t, n_t\}_{t=0}^{\infty}$ is feasible if for all $t \geq 0$

$$
\begin{aligned}
F(k_t, n_t) &= c_t + k_{t+1} - (1-\delta)k_t \\
c_t &\geq 0, k_t \geq 0, 0 \leq n_t \leq 1 \\
k_0 &\leq \bar{k}_0
\end{aligned}
$$

Definition

An allocation $\{c_t, k_t, n_t\}_{t=0}^{\infty}$ is Pareto efficient if it is feasible and there is no other feasible allocation $\{\hat{c}_t, \hat{k}_t, \hat{n}_t\}_{t=0}^{\infty}$ such that

$$\sum_{t=0}^{\infty} \beta^t U(\hat{c}_t) > \sum_{t=0}^{\infty} \beta^t U(c_t)$$

Social Planner Problem in Sequential Formulation

$$
\begin{aligned}
w(\bar{k}_0) &= \max_{\{c_t, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
s.t. \qquad F(k_t, n_t) &= c_t + k_{t+1} - (1-\delta)k_t \\
c_t &\geq 0, k_t \geq 0, 0 \leq n_t \leq 1 \\
k_0 &\leq \bar{k}_0
\end{aligned}
$$

$w(\bar{k}_0) =$ *total lifetime* utility of the representative household if the social planner chooses $\{c_t, k_t, n_t\}_{t=0}^{\infty}$ *optimally* and the initial capital stock in the economy is $\bar{k}_0$

Assumptions

- Utility Function
  - U is continuously differentiable, strictly increasing, strictly concave and bounded.
  - U Satisfies the Inada conditions $\lim_{c \searrow 0} U'(c) = \infty$ and $\lim_{c \to \infty} U'(c) = 0$.
  - The discount factor β satisfies $\beta \in (0, 1)$
- Production Function
  - F is continuously differentiable and homogenous of degree 1, strictly increasing and strictly concave.
  - Furthermore $F(0, n) = F(k, 0) = 0$ for all $k, n > 0$.
  - F satisfies the Inada conditions $\lim_{k \searrow 0} F_k(k, 1) = \infty$ and $\lim_{k \to \infty} F_k(k, 1) = 0$. Also $\delta \in [0, 1]$

Consequences of Assumptions

- $n_t = 1$ for all t
- $k_0 = \bar{k}_0$
- Define
$$f(k) = F(k, 1) + (1 - \delta)k$$
The function f gives the total amount of the final good available for consumption or investment
- From assumptions it follows that f is continuously differentiable, strictly increasing and strictly concave, $f(0) = 0$, $f'(k) > 0$ for all k, $\lim_{k \searrow 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 1 - \delta$

Simplified Planners Problem

- Substitute for $c_t = f(k_t) - k_{t+1}$
- Rewrite problem as

$$\begin{aligned} w(\bar{k}_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\ 0 &\leq k_{t+1} \leq f(k_t) \\ k_0 &= \bar{k}_0 > 0 \text{ given} \end{aligned}$$

Questions about Social Planner Problem

1. Why do we want to solve such a hypothetical problem of an even more hypothetical social planner? Answer: WELFARE THEOREMS

2. How do we solve this problem? Answer: DYNAMIC PROGRAMMING

Basic Idea of Dynamic Programming

- Find a simpler maximization problem by exploiting the stationarity of the economic environment
- Demonstrate that the solution to the simpler maximization problem solves the original maximization problem.

$$
\begin{aligned}
w(k_0) &= \max_{\substack{\{k_{t+1}\}_{t=0}^{\infty} \text{ s.t.} \\ 0 \leq k_{t+1} \leq f(k_t),\ k_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\
&= \max_{\substack{k_1 \text{ s.t.} \\ 0 \leq k_1 \leq f(k_0),\ k_0 \text{ given}}} \left\{ U(f(k_0) - k_1) + \right. \\
&\quad \left. \beta \left[\max_{\substack{\{k_{t+1}\}_{t=1}^{\infty} \text{ s.t.} \\ 0 \leq k_{t+1} \leq f(k_t),\ k_1 \text{ given}}} \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1}) \right] \right\}
\end{aligned}
$$

- This suggests

$$
w(k_0) = \max_{0 \leq k_1 \leq f(k_0), k_0 \text{ given}} \{U(f(k_0) - k_1) + \beta w(k_1)\}
$$

Basic Idea of Dynamic Programming

- This suggests

$$w(k_0) = \max_{0 \leq k_1 \leq f(k_0), k_0 \text{ given}} \{U(f(k_0) - k_1) + \beta w(k_1)\}$$

- Two questions

1. Under which conditions is the suggestive discussion formally correct?
2. How do we solve *this* problem?

Recursive Formulation Planner's Problem

- Formulation of planners problem with a function on the left and right side of the maximization problem is called recursive formulation
- $v(.)$: value function for the recursive formulation of the problem.
- Interpretation of $v(k) =$: discounted lifetime utility of the representative agent from today onwards if social planner is given capital stock k at the beginning of the current period and allocates consumption across time optimally.
- $v(.)$ solves

$$v(k) = \max_{0 \leq k' \leq f(k)} \left\{ U(f(k) - k') + \beta v(k') \right\}$$

- Capital stock k that the planner brings into the current period completely determines what allocations are feasible from today onwards. It is called the "state variable"
- Variable k' is decided today by the social planner; it is called a "control variable".

Recursive Formulation Planner's Problem

- Equation above is a *functional equation* (the so-called Bellman equation): its solution is a function, rather than a number or a vector
- Interpretation: discounted lifetime utility of the representative agent, $v(k)$ equals current utility $u(f(k) - k')$, plus discounted lifetime utility from tomorrow onwards, $\beta v(k')$
- Solving functional equation means finding a value function v solving FE and an optimal policy function $k' = g(k)$ that describes the optimal k' as a function of k
- Problem: maximization has to be solved for every possible capital stock

Questions about the Functional Equation

1. Under what condition does a solution to FE exist?
2. If it exists, is it unique?
3. Is there a reliable algorithm that computes the solution?
4. Under what conditions can we solve FE and be sure to have solved the sequential planners problem i.e. under what conditions do we have $v = w$ and equivalence between the optimal sequential allocation $\{k_{t+1}\}_{t=0}^{\infty}$ and allocations generated by the optimal recursive policy $g(k)$.
5. Can we say something about the qualitative features of v and g?

An Example

- $$U(c) = \ln(c)$$

- $$F(k,n) = k^{\alpha} n^{1-\alpha} \text{ and } \delta = 1$$

- Then
$$f(k) = k^{\alpha}$$

- Functional equation
$$v(k) = \max_{0 \leq k' \leq k^{\alpha}} \left\{ \ln\left(k^{\alpha} - k'\right) + \beta v(k') \right\}$$

Method of Undetermined Coefficients

- Idea: Guess a particular functional form of a solution and then verify that the solution has in fact this form.

- Guess
$$v(k) = A + B\ln(k)$$
where A and B are coefficients that are to be determined.

- Three Step Procedure

Method of Undetermined Coefficients

- Step 1: Solve maximization problem on RHS, given guess for v

$$\max_{0 \leq k' \leq k^{\alpha}} \left\{ \ln\left(k^{a} - k'\right) + \beta\left(A + B\ln(k')\right)\right\}$$

First order condition

$$\begin{aligned} \frac{1}{k^{\alpha} - k'} &= \frac{\beta B}{k'} \\ k' &= \frac{\beta B k^{\alpha}}{1 + \beta B} \end{aligned}$$

- Step 2: Evaluate the right hand side at the optimum $k' = \frac{\beta B k^{\alpha}}{1+\beta B}$

$$\begin{aligned} \text{RHS} &= \ln\left(k^{a} - k'\right) + \beta\left(A + B\ln(k')\right) \\ &= \ln\left(\frac{k^{\alpha}}{1+\beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B k^{\alpha}}{1+\beta B}\right) \\ &= -\ln(1+\beta B) + \alpha\ln(k) + \\ &\quad \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B\ln(k) \end{aligned}$$

Method of Undetermined Coefficients

- Step 3: Verify. We have guessed LHS$= v(k) = A + B\ln(k)$. Equating LHS and RHS yields

$$\begin{aligned} A + B\ln(k) &= -\ln(1+\beta B) + \alpha\ln(k) \\ &\quad +\beta A + \beta B \ln\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B\ln(k) \end{aligned}$$

- Hence

$$\begin{aligned} B &= \alpha(1+\beta B) \\ B &= \frac{\alpha}{1-\alpha\beta} \end{aligned}$$

and

$$A = \frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right]$$

Method of Undetermined Coefficients

- Optimal policy function $k' = g(k)$

$$\begin{aligned} g(k) &= \frac{\beta B k^\alpha}{1+\beta B} \\ &= \alpha\beta k^\alpha \end{aligned}$$

- Construct a sequence $\{k_{t+1}\}_{t=0}^{\infty}$ from our policy function g: start from $k_0 = \bar{k}_0$ and

$$\begin{aligned} k_1 &= g(k_0) = \alpha\beta k_0^\alpha \\ k_2 &= g(k_1) = \alpha\beta k_1^\alpha = (\alpha\beta)^{1+\alpha} k_0^{\alpha^2} \end{aligned}$$

and so on.

Value Function Iteration: Analytical Approach

- Guess an arbitrary function $v_0(k)$, say $v_0(k) = 0$
- Solve

$$v_1(k) = \max_{0 \leq k' \leq k^\alpha} \left\{ \ln \left(k^\alpha - k' \right) + \beta v_0(k') \right\}$$

We can solve the maximization problem on the right hand side since we know v_0. Solution

$$k' = g_1(k) = 0 \text{ for all } k$$

Plugging back

$$v_1(k) = \ln \left(k^\alpha - 0 \right) + \beta v_0(0) = \ln k^\alpha = \alpha \ln k$$

- Now we can solve

$$v_2(k) = \max_{0 \leq k' \leq k^\alpha} \left\{ \ln \left(k^\alpha - k' \right) + \beta v_1(k') \right\}$$

since we know v_1 and so forth.

Value Function Iteration: Analytical Approach

- By iterating on the recursion

$$v_{n+1}(k) = \max_{0 \leq k' \leq k^\alpha} \left\{ \ln\left(k^\alpha - k'\right) + \beta v_n(k') \right\}$$

  we obtain a sequence of value functions $\{v_n\}_{n=0}^{\infty}$ and policy functions $\{g_n\}_{n=1}^{\infty}$

- Hopefully these sequences will converge to the solution v^* and associated policy g^* of the functional equation.

- Contraction Mapping Theorem guarantee that this in fact happens.

Numerical Value Function Iteration: Algorithm

- Key problem: Even a computer can carry out only a finite number of calculations and store finite-dimensional objects.
- Hence only approximation of the true value function
- Suppose

$$k, k' \in \mathcal{K} = \{0.04, 0.08, 0.12, 0.16, 0.20\}$$

- Value functions v_n

$$(v_n(0.04), v_n(0.08), v_n(0.12), v_n(0.16), v_n(0.20))$$

- Pick concrete values for the parameters α and β, say $\alpha = 0.3$ and $\beta = 0.6$

Numerical Value Function Iteration: Algorithm

- Initial guess $v_0(k) = 0$ for all $k \in \mathcal{K}$
- Solve

$$v_1(k) = \max_{\substack{0 \leq k' \leq k^{0.3} \\ k' \in \mathcal{K}}} \left\{ \ln\left(k^{0.3} - k'\right) + 0.6 * 0 \right\}$$

- Optimal policy $k'(k) = g_1(k) = 0.04$ for all $k \in \mathcal{K}$. Plugging this back in yields

$$\begin{aligned}
v_1(0.04) &= \ln(0.04^{0.3} - 0.04) = -1.077 \\
v_1(0.08) &= \ln(0.08^{0.3} - 0.04) = -0.847 \\
v_1(0.12) &= \ln(0.12^{0.3} - 0.04) = -0.715 \\
v_1(0.16) &= \ln(0.16^{0.3} - 0.04) = -0.622 \\
v_1(0.20) &= \ln(0.20^{0.3} - 0.04) = -0.550
\end{aligned}$$

Numerical Value Function Iteration: Algorithm

- One more iteration

$$v_2(k) = \left\{ \max_{0 \le k' \le k^{0.3}, k' \in \mathcal{K}} \ln\left(k^{0.3} - k'\right) + 0.6 v_1(k') \right\}$$

- Start with $k = 0.04$:

$$v_2(0.04) = \max_{0 \le k' \le 0.04^{0.3}, k' \in \mathcal{K}} \left\{ \ln\left(0.04^{0.3} - k'\right) + 0.6 v_1(k') \right\}$$

If $k' = 0.04$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.04\right) + 0.6(-1.08) = -1.72$$

If $k' = 0.08$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.08\right) + 0.6(-0.85) = -1.71$$

If $k' = 0.12$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.12\right) + 0.6(-0.72) = -1.77$$

If $k' = 0.16$, then

$$v_2(0.04) = \ln\left(0.04^{0.3} - 0.16\right) + 0.6(-0.62) = -1.88$$

Numerical Value Function Iteration: Summary of Second Iteration

- Table 1 below shows the value of

$$\left(k^{0.3} - k'\right) + 0.6 v_1(k')$$

for different values of k and k'

Table 1

k \ k'	0.04	0.08	0.12	0.16	0.2
0.04	-1.72	-1.71^*	-1.77	-1.88	-2.04
0.08	-1.49	-1.45^*	-1.48	-1.55	-1.64
0.12	-1.36	-1.31^*	-1.32	-1.37	-1.44
0.16	-1.27	-1.21^*	-1.21	-1.25	-1.31
0.2	-1.20	-1.13	-1.13^*	-1.16	-1.20

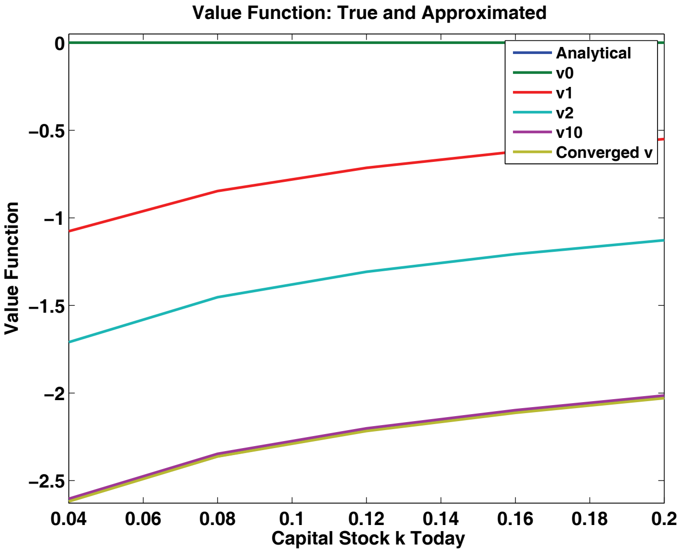
Numerical Value Function Iteration: Summary of Second Iteration

- Hence the value function v_2 and policy function g_2 are given by

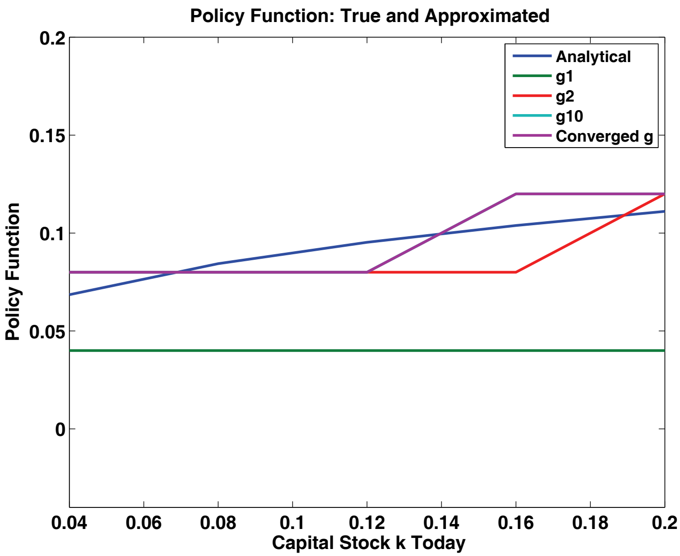
Table 2

k	$v_2(k)$	$g_2(k)$
0.04	-1.7097	0.08
0.08	-1.4530	0.08
0.12	-1.3081	0.08
0.16	-1.2072	0.08
0.2	-1.1279	0.12

Value Functions



Policy Functions



The Euler Equations and Transversality Condition: The Finite Horizon Case

- Social planner problem with time horizon T:

$$\begin{aligned} w^T(\bar{k}_0) &= \max_{\{k_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t U(f(k_t) - k_{t+1}) \\ 0 &\leq k_{t+1} \leq f(k_t) \\ k_0 &= \bar{k}_0 > 0 \text{ given} \end{aligned}$$

- Obviously $k_{T+1} = 0$.
- Finite-dimensional maximization problem with closed and bounded constraint set: by the Bolzano-Weierstrass a solution to the maximization problem exists, so that $w^T(\bar{k}_0)$ is well-defined
- Since constraint set is convex and U is strictly concave, the solution to the maximization problem is unique and the first order conditions are necessary and sufficient.

The Euler Equations

- Lagrangian

$$
\begin{aligned}
L &= U(f(k_0) - k_1) + \ldots \\
&\quad + \beta^t U(f(k_t) - k_{t+1}) + \beta^{t+1} U(f(k_{t+1}) - k_{t+2}) \\
&\quad + \ldots + \beta^T U(f(k_T) - k_{T+1})
\end{aligned}
$$

- First order conditions

$$
\begin{aligned}
\frac{\partial L}{\partial k_{t+1}} &= -\beta^t u'(f(k_t) - k_{t+1}) \\
&\quad + \beta^{t+1} u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \\
&= 0
\end{aligned}
$$

$$
\underbrace{u'(f(k_t) - k_{t+1})}_{\substack{\text{Cost in utility} \\ \text{for saving} \\ \text{1 unit more} \\ \text{capital for } t+1}} = \underbrace{\beta u'(f(k_{t+1}) - k_{t+2})}_{\substack{\text{Discounted} \\ \text{add. utility} \\ \text{from one more} \\ \text{unit of cons.}}} \underbrace{f'(k_{t+1})}_{\substack{\text{Add. prod.} \\ \text{possible with} \\ \text{one more unit} \\ \text{cap. in } t+1}}
$$

The Euler Equations

- Optimality conditions:

$$\beta U'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}) = U'(f(k_t) - k_{t+1})$$

- This intertemporal first order condition is called the Euler equation

- Second order difference equation, a system of T equations in the $T+1$ unknowns $\{k_{t+1}\}_{t=0}^{T}$ (with k_0 predetermined)

- Terminal condition $k_{T+1} = 0$. Can solve for the optimal $\{k_{t+1}\}_{t=0}^{T}$ uniquely (under appropriate conditions)

The Euler Equations: Example

- $U(c) = \ln(c)$ and $f(k) = k^{\alpha}$. Euler equation becomes

$$\begin{aligned}\frac{1}{k_t^{\alpha} - k_{t+1}} &= \frac{\beta \alpha k_{t+1}^{\alpha-1}}{k_{t+1}^{\alpha} - k_{t+12}} \\ k_{t+1}^{\alpha} - k_{t+2} &= \alpha\beta k_{t+1}^{\alpha-1}(k_t^{\alpha} - k_{t+1})\end{aligned}$$

- Trick: Define $z_t = \frac{k_{t+1}}{k_t^{\alpha}}$. The variable z_t is the saving rate of the social planner. Rewrite

$$\begin{aligned}1 - z_{t+1} &= \alpha\beta\left(\frac{1}{z_t} - 1\right) \\ z_{t+1} &= 1 + \alpha\beta - \frac{\alpha\beta}{z_t}\end{aligned}$$

- This is a first order difference equation with boundary condition $z_T = \frac{k_{T+1}}{k_T^{\alpha}} = 0$. Can solve backwards

$$z_t = \frac{\alpha\beta}{1 + \alpha\beta - z_{t+1}}$$

The Euler Equations: Example

- Hence

$$z_t = \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}$$

and therefore

$$\begin{aligned} k_{t+1} &= \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} k_t^{\alpha} \\ c_t &= \frac{1-\alpha\beta}{1-(\alpha\beta)^{T-t+1}} k_t^{\alpha} \end{aligned}$$

- Note that

$$\begin{aligned} & \lim_{T\to\infty} \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} k_t^{\alpha} \\ = \ & \alpha\beta k_t^{\alpha} \end{aligned}$$

Graphical Analysis

- Dynamics of Savings Rates

$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$$

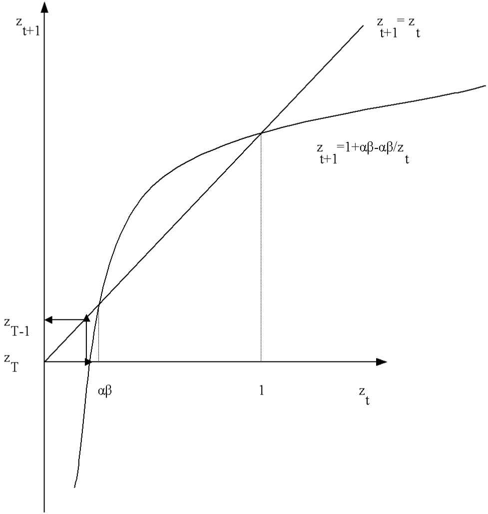
- Plot z_{t+1} against z_t. Properties:
  - Since $k_{t+1} \geq 0$, we have that $z_t \geq 0$
  - Limiting behavior

$$\begin{aligned} \lim_{z_t \searrow 0} \left(1 + \alpha\beta - \frac{\alpha\beta}{z_t}\right) &= -\infty \\ \lim_{z_t \nearrow \infty} \left(1 + \alpha\beta - \frac{\alpha\beta}{z_t}\right) &= 1 + \alpha\beta > 1 \end{aligned}$$

  - $$z_{t+1} = 0 \text{ for } z_t = \frac{\alpha\beta}{1 + \alpha\beta} < 1$$
  - Steady State: $z_{t+1} = z_t = z$. Two steady states in this example

$$\begin{aligned} z &= 1 + \alpha\beta - \frac{\alpha\beta}{z} \\ z &= 1 \text{ or } z = \alpha\beta \end{aligned}$$

Dynamics of Savings Rates



The Infinite Horizon Case

- Social planner problem:

$$
\begin{aligned}
w(\bar{k}_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\
0 &\leq k_{t+1} \leq f(k_t) \\
k_0 &= \bar{k}_0 > 0 \text{ given}
\end{aligned}
$$

- Euler equations

$$
\beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = U'(f(k_t) - k_{t+1})
$$

- Second order difference equation. But now only initial condition k_0, no terminal condition

Transversality Condition

- Statement of transversality condition:

$$\lim_{t\to\infty} \underbrace{\beta^t U'(f(k_t)-k_{t+1})f'(k_t)}_{\substack{\text{value in discounted} \\ \text{utility terms of one} \\ \text{more unit of capital}}} \underbrace{k_t}_{\substack{\text{Total} \\ \text{Capital} \\ \text{Stock}}} = 0$$

- Transversality condition substitutes for missing terminal condition. It is an optimality condition

- Shadow value of capital has to converge to zero

Alternative Statement of TVC

- $$\lim_{t\to\infty} \lambda_t k_{t+1} = 0$$
- Here λ_t is the Lagrange multiplier on the constraint
$$c_t + k_{t+1} = f(k_t)$$
- First order condition
$$\begin{aligned} \beta^t U'(c_t) &= \lambda_t \\ \beta^t U'(f(k_t) - k_{t+1}) &= \lambda_t \end{aligned}$$
- Hence
$$\begin{aligned} 0 &= \lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1})k_{t+1} \\ &= \lim_{t\to\infty} \beta^{t-1} U'(f(k_{t-1}) - k_t)k_t \\ &= \lim_{t\to\infty} \beta^{t-1} \beta U'(f(k_t) - k_{t+1}) f'(k_t) k_t \\ &= \lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t \end{aligned}$$

Sufficiency of Euler Equations and Transversality Conditions

Theorem

Let U, β and F (and hence f) satisfy assumptions 1. and 2. Then an allocation $\{k_{t+1}\}_{t=0}^{\infty}$ that satisfies the Euler equations and the transversality condition solves the sequential social planners problem, for a given k_0.

- Theorem states that under certain assumptions the Euler equations and the transversality condition are jointly sufficient
- Boundedness of U not required in the proof, so theorem applies to CRRA utility functions.
- Necessity? Yes for the log-case (Ekelund and Scheinkman (1985), Peleg and Ryder (1972), Weitzman (1973))

Example

- $U(c) = \ln(c)$ and $f(k) = k^\alpha$
- TVC becomes

$$\begin{aligned} & \lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t \\ = & \lim_{t\to\infty} \frac{\alpha\beta^t k_t^\alpha}{k_t^\alpha - k_{t+1}} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - \frac{k_{t+1}}{k_t^\alpha}} \\ = & \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t} \end{aligned}$$

- From the Euler equations

$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$$

- Solve forward with initial guess for z_0
- We show that only one guess for z_0 yields a sequence that does not violate the TVC or the nonnegativity constraint on capital or consumption.

Example

1. $z_0 < \alpha\beta$: in finite time $z_t < 0$, violating $k_{t+1} \geq 0$
2. $z_0 > \alpha\beta$. Then from Figure 3 we see that $\lim_{t\to\infty} z_t = 1$. All these paths violate the TVC.
3. $z_0 = \alpha\beta$. Then $z_t = \alpha\beta$ for all $t > 0$. This path satisfies the Euler equations and the TVC

$$\lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - \alpha\beta} = 0$$

- From theorem conclude that the sequence $\{z_t\}_{t=0}^{\infty}$ with $z_t = \alpha\beta$ is an optimal solution for the sequential social planners problem. Hence

$$k_{t+1} = \alpha\beta k_t^{\alpha}, \text{ with } k_0 \text{ given}$$

- How about $z_0 > \alpha\beta$? Then $\lim_{t\to\infty} z_t = 1$
- TVC

$$\lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t}$$

Example

- Linear approximation of z_{t+1} around the steady state $z = 1$

$$\begin{aligned}
z_{t+1} &= 1 + \alpha\beta - \frac{\alpha\beta}{z_t} = g(z_t) \\
z_{t+1} &\approx g(1) + (z_t - 1)g'(z_t)|_{z_t=1} \\
&= 1 + (z_t - 1)\left(\frac{\alpha\beta}{z_t^2}\right)|_{z_t=1} \\
&= 1 + \alpha\beta(z_t - 1) \\
(1 - z_{t+1}) &\approx \alpha\beta(1 - z_t) \\
&\approx (\alpha\beta)^{t-k+1}(1 - z_k) \qquad \text{for all } k
\end{aligned}$$

- Hence

$$\lim_{t\to\infty} \frac{\alpha\beta^{t+1}}{1 - z_{t+1}} \approx \lim_{t\to\infty} \frac{\beta^k}{\alpha^{t-k}(1 - z_k)} = \infty$$

as $0 < \alpha < 1$. Violation of TVC.

Steady States and the Modified Golden Rule

- Steady state: social optimum or competitive equilibrium with $c_t = c^*$ and $k_{t+1} = k^*$.
- Euler equations

$$\begin{aligned} \beta U'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}) &= U'(f(k_t) - k_{t+1}) \text{ or} \\ \beta U'(c_{t+1})f'(k_{t+1}) &= U'(c_t). \end{aligned}$$

- Steady state: $c_t = c_{t+1} = c^*$ and thus

$$\begin{aligned} \beta f'(k^*) &= 1 \\ f'(k^*) &= 1 + \rho \end{aligned}$$

- Recalling $f'(k) = F_k(k, 1) + 1 - \delta$ we obtain so-called modified golden rule

$$F_k(k^*, 1) - \delta = \rho$$

Steady States and the Modified Golden Rule

- For example above

$$\begin{aligned}\alpha\left(k^*\right)^{\alpha-1} &= \rho+1=\frac{1}{\beta}\\ k^* &= (\alpha\beta)^{\frac{1}{1-\alpha}}.\end{aligned}$$

- Note: can also be derived from

$$k'=\alpha\beta k^{\alpha}.$$

Setting $k'=k$ and solving we find again $k^*=(\alpha\beta)^{\frac{1}{1-\alpha}}$.

- From any initial capital stock k_0 the optimal sequence chosen by the social planner $\{k^*_{t+1}\}$ converges to $k^*=(\alpha\beta)^{\frac{1}{1-\alpha}}$.

Original Golden Rule

- Resource constraint in the steady state

$$c = f(k) - k.$$

- Capital stock that maximizes consumption per capita satisfies

$$\begin{aligned} f'(k^g) &= 1 \\ F_k(k^g, 1) - \delta &= 0 \end{aligned}$$

- Modified golden rule capital stock $k^* < k^g$.

A Remark About Balanced Growth

- Now: population growth as well as technological progress.
- At time t the size of the population is $N_t = (1+n)^t$.
- Output at date t is produced according to

$$F(K_t, N_t(1+g)^t)$$

- Objective function

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \text{ or}$$

$$\sum_{t=0}^{\infty} N_t \beta^t U(c_t)$$

- Resource constraint

$$(1+n)^t c_t + K_{t+1} = F(K_t, (1+n)^t(1+g)^t) + (1-\delta)K_t$$

A Remark About Balanced Growth

- Define growth-adjusted per capita variables as

$$\begin{aligned}\tilde{c}_t &= \frac{c_t}{(1+g)^t} \\ \tilde{k}_t &= \frac{k_t}{(1+g)^t} = \frac{K_t}{(1+n)^t(1+g)^t}\end{aligned}$$

- Divide the resource constraint by $(1+n)^t(1+g)^t$ to obtain

$$\tilde{c}_t + (1+n)(1+g)\tilde{k}_{t+1} = F(\tilde{k}_t, 1) + (1-\delta)\tilde{k}_t$$

- In order obtain balanced growth path we assume CRRA utility: $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Then

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\tilde{c}_t{}^{1-\sigma}}{1-\sigma}$$

where $\tilde{\beta} = \beta(1+g)^{1-\sigma}$.

A Remark About Balanced Growth

- Social planner problem becomes

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\left(f(\tilde{k}_t) - (1+g)(1+n)\tilde{k}_{t+1}\right)^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} 0 &\leq (1+g)(1+n)\tilde{k}_{t+1} \leq f(\tilde{k}_t) \\ \tilde{k}_0 &= k_0 \text{ given} \end{aligned}$$

  and all the analysis from above goes through completely unchanged.

- Balanced growth path is a socially optimal allocation (or a competitive equilibrium) where all variables grow at a constant rate.
- Balanced growth path in $\{c_t, k_{t+1}\}$ corresponds to steady state for $\{\tilde{c}_t, \tilde{k}_{t+1}\}$.

A Remark About Balanced Growth

- Euler equations

$$(1+n)(1+g)\left(\tilde{c}_t\right)^{-\sigma} = \tilde{\beta}\left(\tilde{c}_{t+1}\right)^{-\sigma}\left[F_k(\tilde{k}_{t+1}, 1) + (1-\delta)\right]$$

- Evaluated at the steady state $(\tilde{c}, \tilde{k})$

$$(1+n)(1+g) = \tilde{\beta}\left[F_k(\tilde{k}^*, 1) + (1-\delta)\right]$$

- Defining $\tilde{\beta} = \frac{1}{1+\tilde{\rho}}$ we have

$$\begin{aligned} (1+n)(1+g)(1+\tilde{\rho}) &= F_k(\tilde{k}^*, 1) + (1-\delta) \\ F_k(\tilde{k}^*, 1) - \delta &\approx n + g + \tilde{\rho}. \end{aligned}$$

- Original golden rule

$$F_k(\tilde{k}^g, 1) - \delta \approx n + g$$

A Remark About Balanced Growth

- Once optimal sequences $\{\tilde{c}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}$ have been computed, obtain variables of interest as

$$\begin{aligned} c_t &= (1+g)^t \tilde{c}_t \\ k_{t+1} &= (1+g)^t \tilde{k}_{t+1} \\ C_t &= (1+n)^t (1+g)^t \tilde{c}_t \\ K_{t+1} &= (1+n)^t (1+g)^t \tilde{k}_{t+1}. \end{aligned}$$

Competitive Equilibrium

- Goal: Decentralization of Pareto efficient allocation $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ as competitive equilibrium
- Arrow-Debreu market structure
- Perfect competition
- Ownership
  - Households own firms (residual claimants to their profits)
  - Households own capital stock (and rent it to firms)

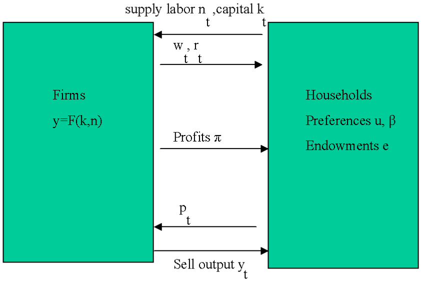
Traded Goods

1. Final output good, y_t that can be used for consumption c_t or investment. Let p_t denote the price of the period t final output good, quoted in period 0.

2. Labor services n_t. Let w_t be the price of one unit of labor services delivered in period t, quoted in period 0, in terms of the period t consumption good. Hence w_t is the real wage; it tells how many units of the period t consumption goods one can buy for the receipts for one unit of labor. The nominal wage is $p_t w_t$

3. capital services k_t. Let r_t be the rental price of one unit of capital services delivered in period t, quoted in period 0, in terms of the period t consumption good. r_t is the real rental rate of capital, the nominal rental rate is $p_t r_t$.

Flows of Goods and Payments



Competitive Equilibrium: Firms

- Firms behave competitively in output and factor markets
- The representative firm's problem is , given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$

$$
\begin{aligned}
\pi &= \max_{\{y_t, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t n_t) \\
s.t. \quad y_t &= F(k_t, n_t) \text{ for all } t \geq 0 \\
y_t, k_t, n_t &\geq 0
\end{aligned}
$$

Competitive Equilibrium: Households

- Households own the capital stock and have to decide how much labor and capital services to supply, how much to consume and how much capital to accumulate

- Taking prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ the representative consumer solves

$$\max_{\{c_t, i_t, x_{t+1}, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\begin{aligned} & \text{s.t.} \\ \sum_{t=0}^{\infty} p_t(c_t + i_t) & \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t n_t) + \pi \\ x_{t+1} & = (1-\delta)x_t + i_t \\ 0 & \leq n_t \leq 1, 0 \leq k_t \leq x_t \\ c_t, x_{t+1} & \geq 0 \qquad \text{all } t \geq 0 \\ & x_0 \text{ given} \end{aligned}$$

Definition of Competitive Equilibrium

Definition

A Competitive Equilibrium (Arrow-Debreu Equilibrium) consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ and the household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ such that

1. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ solves firm's problem
2. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ solves household's problem
3. Markets clear

$$\begin{aligned} y_t &= c_t + i_t \text{ (Goods Market)} \\ n_t^d &= n_t^s \text{ (Labor Market)} \\ k_t^d &= k_t^s \text{ (Capital Services Market)} \end{aligned}$$

Characterization of the Competitive Equilibrium

- Simplify notation

$$
\begin{aligned}
k_t &= k_t^d = k_t^s \\
n_t &= n_t^d = n_t^s
\end{aligned}
$$

- All prices strictly positive

$$p_t, r_t, w_t > 0$$

Characterization of the Competitive Equilibrium: Firms

- Firms maximization problem

$$\max_{k_t, n_t \geq 0} p_t \left(F(k_t, n_t) - r_t k_t - w_t n_t \right)$$

- Marginal Product Pricing

$$\begin{aligned} r_t &= F_k(k_t, n_t) \\ w_t &= F_n(k_t, n_t) \end{aligned}$$

- Profits of firms, using CRTS and Euler's theorem

$$\pi_t = p_t \left(F(k_t, n_t) - F_k(k_t, n_t) k_t - F_n(k_t, n_t) n_t \right) = 0$$

Implications of Constant Returns to Scale Technology

- F exhibits constant returns to scale (is homogeneous of degree 1)

$$F(\lambda k, \lambda n) = \lambda F(k, n) \text{ for all } \lambda > 0$$

- Euler's theorem: For any function that is homogeneous of degree k and differentiable at $x \in \mathbf{R}^L$ we have

$$kf(x) = \sum_{i=1}^{L} x_i \frac{\partial f(x)}{\partial x_i}$$

- Hence

$$F(k_t, n_t) = F_k(k_t, n_t)k_t + F_n(k_t, n_t)n_t$$

Proof of Euler's Theorem

- Since f is homogeneous of degree k we have for all $\lambda > 0$

$$f(\lambda x) = \lambda^k f(x)$$

- Differentiating both sides with respect to λ yields

$$\sum_{i=1}^{L} x_i \frac{\partial f(\lambda x)}{\partial x_i} = k\lambda^{k-1} f(x)$$

- Setting $\lambda = 1$ yields

$$\sum_{i=1}^{L} x_i \frac{\partial f(x)}{\partial x_i} = k f(x)$$

Indeterminacy of Number of Firms

- Constant returns to scale imply that marginal products are homogeneous of degree 0.
- To see this differentiate

$$F(\lambda k, \lambda n) = \lambda F(k, n)$$

with respect to one of the inputs, say k, to obtain

$$\begin{aligned} \lambda F_k(\lambda k, \lambda n) &= \lambda F_k(k, n) \\ F_k(\lambda k, \lambda n) &= F_k(k, n) \end{aligned}$$

- Now take $\lambda = \frac{1}{n}$ to obtain

$$F_k(k/n, 1) = F_k(k, n).$$

Indeterminacy of Number of Firms

- All firms operate with same capital-labor ratio

$$r_t = F_k(k_t, n_t) = F_k(k_t/n_t, 1)$$

- Total output could be produced by one representative firm or n_t firms with one worker:

$$F(k_t, n_t) = n_t F(k_t/n_t, 1)$$

- Both number of firms as well as output per firm is indeterminate and irrelevant in equilibrium. Only things that are determined are the common k/n and total output y.

Characterization of Equilibrium: Households

- Since utility is strictly increasing in c_t and independent of n_t

$$\begin{aligned} n_t &= 1, k_t = x_t \\ i_t &= k_{t+1} - (1-\delta)k_t \end{aligned}$$

- Budget constraint holds with equality. Rewrite problem as

$$\max_{\{c_t, k_{t=1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \text{ s.t.}$$

$$\begin{aligned} \sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) &= \sum_{t=0}^{\infty} p_t(r_t k_t + w_t) \\ c_t, k_{t+1} &\geq 0 \\ & k_0 \text{ given} \end{aligned}$$

Characterization of Equilibrium: Households

- Attaching μ as Lagrange multiplier to the budget constraint and taking FOC wrt c_t, c_{t+1} and k_{t+1}

$$\begin{aligned} \beta^t U'(c_t) &= \mu p_t \\ \beta^{t+1} U'(c_{t+1}) &= \mu p_{t+1} \\ \mu p_t &= \mu(1-\delta+r_{t+1})p_{t+1} \end{aligned}$$

- Combining yields

$$\begin{aligned} \frac{\beta U'(c_{t+1})}{U'(c_t)} &= \frac{p_{t+1}}{p_t} = \frac{1}{1-\delta+r_{t+1}} \\ \frac{(1-\delta+r_{t+1})\,\beta U'(c_{t+1})}{U'(c_t)} &= 1 \end{aligned}$$

Characterization of Equilibrium: Households

- Euler equation

$$\frac{(1-\delta+r_{t+1})\,\beta U'(c_{t+1})}{U'(c_t)} = 1$$

- Use the marginal pricing condition with $f(k_t) = F(k_t, 1) + (1-\delta)k_t$

$$r_t = F_k(k_t, 1) = f'(k_t) - (1-\delta)$$

  and goods market clearing

$$c_t = f(k_t) - k_{t+1}$$

- We obtain

$$\frac{f'(k_{t+1})\beta U'(f(k_{t+1}) - k_{t+2})}{U'(f(k_t) - k_{t+1})} = 1$$

- Exactly the same Euler equation as in the social planners problem

Characterization of Equilibrium: Households

- TVC? In social planner problem:

$$\begin{aligned} \lim_{t\to\infty} \lambda_t k_{t+1} &= 0 \text{ or} \\ \lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t &= 0 \end{aligned}$$

- For household problem:

$$\lim_{t\to\infty} p_t k_{t+1} = 0$$

Using the household first order condition yields

$$\begin{aligned} \lim_{t\to\infty} p_t k_{t+1} &= \frac{1}{\mu} \lim_{t\to\infty} \beta^t U'(c_t) k_{t+1} = \frac{1}{\mu} \lim_{t\to\infty} \beta^{t-1} U'(c_{t-1}) k_t \\ &= \frac{1}{\mu} \lim_{t\to\infty} \beta^{t-1} \beta U'(c_t)(1 - \delta + r_t) k_t \\ &= \frac{1}{\mu} \lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t = 0 \end{aligned}$$

- With $\{p_t\} = \{\mu_t\}$ SPP allocations satisfy household, firm optimality, equilibrium conditions. Have "proved" welfare thms.

Characterization: Rest of the Equilibrium

- Have determined equilibrium sequence of capital stocks $\{k_{t+1}\}_{t=0}^{\infty}$
- Allocations are given by

$$
\begin{aligned}
c_t &= f(k_t) - k_{t+1} \\
y_t &= F(k_t, 1) \\
i_t &= y_t - c_t \\
n_t &= 1
\end{aligned}
$$

- Factor equilibrium prices as

$$
\begin{aligned}
r_t &= F_k(k_t, 1) \\
w_t &= F_n(k_t, 1)
\end{aligned}
$$

- Prices of final output good: normalize $p_0 = 1$ and use Euler conditions to find

$$
p_{t+1} = \frac{\beta U'(c_{t+1})}{U'(c_t)} p_t = \frac{\beta^{t+1} U'(c_{t+1})}{U'(c_0)}
$$

Sequential Markets Equilibrium

- Household problem

$$
\begin{aligned}
& \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
& \text{s.t.} \\
c_t + k_{t+1} - (1-\delta)k_t &= w_t + r_t k_t \\
c_t, k_{t+1} &\geq 0 \\
& k_0 \text{ given}
\end{aligned}
$$

- Firms problem

$$\max_{k_t, n_t \geq 0} F(k_t, n_t) - w_t n_t - r_t k_t$$

Sequential Markets Equilibrium

Definition

A sequential markets equilibrium is prices $\{w_t, r_t\}_{t=0}^{\infty}$, allocations for representative household $\{c_t, k_{t+1}^s\}_{t=0}^{\infty}$ and for representative firm $\{n_t^d, k_t^d\}_{t=0}^{\infty}$ such that

1. Given k_0 and $\{w_t, r_t\}_{t=0}^{\infty}$, household allocations $\{c_t, k_{t+1}^s\}_{t=0}^{\infty}$ solve household maximization problem.
2. For each $t \geq 0$, given (w_t, r_t) firm allocation (n_t^d, k_t^d) solves firms' maximization problem.
3. Markets clear: for all $t \geq 0$

$$\begin{aligned} n_t^d &= 1 \\ k_t^d &= k_t^s \\ F(k_t^d, n_t^d) &= c_t + k_{t+1}^s - (1-\delta)k_t^s \end{aligned}$$

Recursive Competitive Equilibrium

- State variables (k, K). Control variables (c, k')
- Bellman equation

$$\begin{aligned} v(k, K) &= \max_{c, k' \geq 0} \left\{ U(c) + \beta v(k', K') \right\} \text{ s.t.} \\ c + k' &= w(K) + (1 + r(K) - \delta)k \\ K' &= H(K) \end{aligned}$$

- $K' = H(K)$ is called the aggregate law of motion.
- Solution is a value function v and policy functions $c = C(k, K)$ and $k' = G(k, K)$.
- Firms

$$\begin{aligned} w(K) &= F_l(K, 1) \\ r(K) &= F_k(K, 1) \end{aligned}$$

Recursive Competitive Equilibrium

Definition

A RCE is a value function $v : \mathbf{R}_+^2 \to \mathbf{R}$ and policy functions $C, G : \mathbf{R}_+^2 \to \mathbf{R}_+$ for the representative household, pricing functions $w, r : \mathbf{R}_+ \to \mathbf{R}_+$ and an aggregate law of motion $H : \mathbf{R}_+ \to \mathbf{R}_+$ such that

1. Given w, r and H, v solves the Bellman equation and C, G are the associated policy functions.
2. The pricing functions satisfy the firms FOC
3. Consistency
$$H(K) = G(K, K)$$
4. For all $K \in \mathbf{R}_+$
$$C(K, K) + G(K, K) = F(K, 1) + (1 - \delta)K$$

Calibration

- Purpose: choose parameters of the model so that model can be used for quantitative analysis of real world and counterfactual analysis
- Idea of calibration:
  1. Choose a set of empirical facts that the model should match
  2. Choose parameters so that equilibrium of model matches the facts
- Note: fact that model fits these facts can not be used as claim of success. Evaluation of success has to use other dimensions.

Calibration of the Neoclassical Growth Model

- Functional forms

$$
\begin{aligned}
U(c) &= \frac{c^{1-\sigma}-1}{1-\sigma} \\
F(K,N) &= K^{\alpha}\left[(1+g)^{t}N\right]^{1-\alpha}
\end{aligned}
$$

- Parameters: Technology (α, δ, g), Demographics n, Preferences (β, σ).
- Empirical Targets: Choose parameters such that balanced growth path (BGP) of model matches long-run (average) facts for U.S. economy.
- Need to decide on period length. Take period to last one year.

Parameters Directly Taken from Long Run Averages in the Data

- Population growth rate in model is n, in data $n = 1.1\%$
- Growth rate of per capita GDP in model is g, in data $g = 1.8\%$

Exploiting Equilibrium (BGP) Relationships

- Labor share

$$
\begin{aligned}
w_t &= F_N(K_t, (1+g)^t N_t) \\
&= (1-\alpha) K_t^{\alpha} \left[(1+g)^t\right]^{1-\alpha} N_t^{-\alpha} \\
\frac{w_t N_t}{Y_t} &= 1-\alpha
\end{aligned}
$$

- Thus choose α to match long run labor share. Yields $\alpha = 1/3$

Exploiting Equilibrium (BGP) Relationships

- Investment in BGP

$$\begin{aligned} I_t &= K_{t+1} - (1-\delta)K_t \\ &= (1+n)^{t+1}(1+g)^{t+1}\tilde{k}_{t+1} \\ &\quad -(1-\delta)(1+n)^t(1+g)^t\tilde{k}_t \\ &= \left[(1+n)(1+g) - (1-\delta)\right](1+n)^t(1+g)^t\tilde{k} \\ &= \left[(1+n)(1+g) - (1-\delta)\right]K_t \end{aligned}$$

- Thus in BGP

$$\frac{I_t}{K_t} = \frac{I_t/Y_t}{K_t/Y_t} = \left[(1+n)(1+g) - (1-\delta)\right] \approx n + g + \delta$$

- In the data long run average of $I/Y \approx 0.2$ and $K/Y \approx 3$. Using $g = 1.8\%$ and $n = 1.1\%$ then yields $\delta \approx 4\%$

Exploiting Equilibrium (BGP) Relationships

- Euler equation

$$(1+n)(1+g)\left(\tilde{c}_t\right)^{-\sigma} = (1+r_{t+1}-\delta)\tilde{\beta}\left(\tilde{c}_{t+1}\right)^{-\sigma}$$

- In BGP

$$\begin{aligned}(1+n)(1+g) &= (1+r-\delta)\beta(1+g)^{1-\sigma} \\ \beta(1+g)^{-\sigma} &= \frac{1+n}{1+r-\delta}\end{aligned}$$

- Equilibrium rental rate

$$r = \alpha\frac{Y}{K}$$

With $\alpha = 0.33$ and target $K/Y = 3$ we find $r = 11\%$ and $r-\delta = 7\%$.

- Plugging in $n = 1.1\%$ and $g = 1.8\%$ and $r-\delta = 7\%$, this is a relationship between the preference parameters β, σ:

$$\beta(1.018)^{-\sigma} = 0.944$$

Exploiting Equilibrium (BGP) Relationships

- BGP Euler equation
$$\beta(1.018)^{-\sigma} = 0.944$$
- If $g = 0$, then this relationship pins down β, but contains no information about σ.
- If $g > 0$, β and σ only jointly determined.
- Typical approach in models without risk: choose σ based on information outside the model
- Euler equation estimation with macro data: Hall (1982) finds $\frac{1}{\sigma} = 0.1$
- Euler equation estimation with micro data: Attanasio and co-authors (1993, 1995) find $\frac{1}{\sigma} \in [0.3, 0.8]$, possibly higher for particular groups.
- Lucas argues that cross-country differences in g are large, relative to cross-country differences in r (and n). Thus, conditional on all countries sharing same preference parameters, $\frac{1}{\sigma} \geq 1$.
- If we take $\sigma = 1$ (log-case), then $\beta = 0.961$ (i.e. $\rho = 3.9\%$).

Calibration: Summary

Param.	Value	Target
g	1.8%	g in Data
n	1.1%	n in Data
α	0.33	$\frac{wN}{Y}$
δ	4%	$\frac{I/Y}{K/Y}$
σ	1	Outside Evid.
β	0.961	K/Y

Functional Analysis for Macroeconomists

- Functional equation

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

  - x is the set of state variables, y is the set of control variables
  - F is the period return function
  - Γ is the constraint set

- For neoclassical growth model
  - x corresponds to k, y corresponds to k'
  - $F(k, k') = U(f(k) - k')$
  - $\Gamma(k) = \{k' \in \mathbf{R} : 0 \leq k' \leq f(k)\}$

Introducing Operator T

- Functional equation

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\}$$

- Operator T takes the function v as input and spits out a new function Tv
- Solution v^* to the functional equation is a fixed point of the operator T, i.e. satisfies

$$v^* = Tv^*$$

- Questions
  1. Under what conditions does the operator T have a fixed point v^* (existence of a solution)?
  2. Under what conditions is v^* unique?
  3. Under what conditions does the sequence of functions $\{v_n\}_{n=0}^{\infty}$, defined recursively by $v_0 = v_{guess}$ and $v_{n+1} = Tv_n$ converge to v^*?
- Answers: *Contraction Mapping Theorem*. But when is an operator T a contraction mapping? *Blackwell's sufficient conditions*!

Contraction Mapping Theorem

Definition

Let (S, d) be a metric space and $T : S \to S$ be a function mapping S into itself. The function T is a contraction mapping if there exists a number $\beta \in (0, 1)$ satisfying

$$d(Tx, Ty) \leq \beta d(x, y) \text{ for all } x, y \in S$$

The number β is called the modulus of the contraction mapping

Theorem

Let (S, d) be a complete metric space and let $v_n = T^n v_0$. Suppose that $T : S \to S$ is a contraction mapping with modulus β. Then a) the operator T has exactly one fixed point $v^* \in S$ and b) for any $v_0 \in S$, and any $n \in \mathbf{N}$ we have

$$d(T^n v_0, v^*) \leq \beta^n d(v_0, v^*)$$

Implications of the Contraction Mapping Theorem

- Part a): there is a $v^* \in S$ satisfying $v^* = Tv^*$ and there is only one such $v^* \in S$.

- Part b): from any starting guess v_0, the sequence $\{v_n\}_{n=0}^{\infty}$ as defined recursively above converges to v^* at a geometric rate of β.

Proof of Contraction Mapping Theorem

Lemma

Let (S, d) be a metric space and $T : S \to S$ be a function mapping S into itself. If T is a contraction mapping, then T is continuous.

Proof.

To show: for all $s_0 \in S$ and all $\varepsilon > 0$ there exists a $\delta(\varepsilon, s_0)$ such that if $s \in S$ and $d(s, s_0) < \delta(\varepsilon, s_0)$, then $d(Ts, Ts_0) < \varepsilon$. Fix arbitrary $s_0 \in S$ and $\varepsilon > 0$ and pick $\delta(\varepsilon, s_0) = \varepsilon$. Then

$$d(Ts, Ts_0) \leq \beta d(s, s_0) \leq \beta\delta(\varepsilon, s_0) = \beta\varepsilon < \varepsilon$$

$\square$

- Use this result to prove the Contraction Mapping Theorem

Proof of CMT: Existence and Uniqueness

- Start with an arbitrary v_0. Candidate for fixed point
$$v^* = \lim_{n\to\infty} v_n$$
- Step 1: Show that
$$v_n \to v^* \in S$$
- By assumption T is a contraction
$$\begin{aligned} d(v_{n+1}, v_n) &= d(Tv_n, Tv_{n-1}) \leq \beta d(v_n, v_{n-1}) \\ &= \beta d(Tv_{n-1}, Tv_{n-2}) \leq \beta^2 d(v_{n-1}, v_{n-2}) \\ &= \cdots = \beta^n d(v_1, v_0) \end{aligned}$$
- Triangle inequality implies that
$$\begin{aligned} & d(v_m, v_n) \\ \leq\ & d(v_m, v_{m-1}) + d(v_{m-1}, v_n) \\ \leq\ & d(v_m, v_{m-1}) + d(v_{m-1}, v_{m-2}) + \cdots + d(v_{n+1}, v_n) \\ \leq\ & \beta^m d(v_1, v_0) + \beta^{m-1} d(v_1, v_0) + \cdots \beta^n d(v_1, v_0) \\ =\ & \beta^n \left(\beta^{m-n-1} + \cdots + \beta + 1\right) d(v_1, v_0) \\ \leq\ & \frac{\beta^n}{1-\beta} d(v_1, v_0) \end{aligned}$$
- Hence $\{v_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since (S, d) is a complete metric space, the sequence converges in S and v^* is well-defined.

Proof of CMT: Existence and Uniqueness

- Step 2: Show $Tv^* = v^*$.
- From continuity of T

$$Tv^* = T\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} T(v_n) = \lim_{n\to\infty} v_{n+1} = v^*$$

- Step 3: Show that fixed point v^* is unique.
- Suppose not. Then there exists another $\hat{v} \in S$ such that $\hat{v} = T\hat{v}$ and $\hat{v} \neq v^*$. Then there exists $a > 0$ such that $d(\hat{v}, v^*) = a$. But

$$0 < a = d(\hat{v}, v^*) = d(T\hat{v}, Tv^*) \leq \beta d(\hat{v}, v^*) = \beta a$$

a contradiction.

Proof of CMT: Convergence at Geometric Rate

- Proof by Induction
- Step 1: For $n = 0$, using the convention that $T^0 v = v$ the claim automatically holds.
- Step 2: Suppose that

$$d(T^k v_0, v^*) \leq \beta^k d(v_0, v^*)$$

We need to show that

$$d(T^{k+1} v_0, v^*) \leq \beta^{k+1} d(v_0, v^*)$$

- But

$$\begin{aligned} d(T^{k+1} v_0, v^*) &= d(T\left(T^k v_0\right), Tv^*) \\ &\leq \beta d(T^k v_0, v^*) \leq \beta^{k+1} d(v_0, v^*) \end{aligned}$$

Useful Corollary to the Contraction Mapping Theorem

Corollary

Let (S, ρ) be a complete metric space, and let $T : S \to S$ be a contraction mapping with fixed point $v \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$, then $v \in S'$. If in addition $T(S') \subseteq S'' \subseteq S'$, then $v \in S''$.

- Why useful? Suppose want to prove that $v \in S''$ (e.g. that $v \in S''$, the set of strictly increasing continuous functions).
    - Pick closed subset S' of S (e.g. S is set of continuous functions, S' the set of continuous increasing functions).
    - Stick a value function $v \in S'$ on RHS of the Bellman equation. Prove what comes out on LHS is in S''
- Corollary assures that unique fixed point of Bellman equation $v \in S''$

Blackwell's Sufficient Conditions

Theorem

Let $X \subseteq \mathbf{R}^L$ and $B(X)$ be the space of bounded functions $f : X \to \mathbf{R}$ with the d being the sup-*norm. Let $T : B(X) \to B(X)$ be an operator satisfying*

1. *Monotonicity: If $f, g \in B(X)$ are such that $f(x) \leq g(x)$ for all $x \in X$, then $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.*
2. *Discounting: Let the function $f + a$, for $f \in B(X)$ and $a \in \mathbf{R}_+$ be defined by $(f + a)(x) = f(x) + a$ (i.e. for all x the number a is added to $f(x)$). There exists $\beta \in (0, 1)$ such that for all $f \in B(X), a \geq 0$ and all $x \in X$*

$$[T(f + a)](x) \leq [Tf](x) + \beta a$$

If these two conditions are satisfied, then the operator T is a contraction with modulus β.

Proof of Blackwell's Theorem

- Notation
$$f \leq g \;\Leftrightarrow f(x) \leq g(x) \text{ for all } x \in X$$
- Want to show that if T satisfies conditions 1. and 2. there exists $\beta \in (0,1)$ such that for all $f, g \in B(X)$, $d(Tf, Tg) \leq \beta d(f, g)$.
- Step 1: Fix $x \in X$.
Then $f(x) - g(x) \leq \sup_{y \in X} |f(y) - g(y)|$. But this is true for all $x \in X$. Hence
$$\begin{aligned} f(x) &\leq g(x) + d(f,g) \\ f &\leq g + d(f,g) \end{aligned}$$
- By monotonicity
$$Tf \leq T[g + d(f,g)]$$
and by discounting
$$\begin{aligned} Tf &\leq T[g + d(f,g)] \leq Tg + \beta d(f,g) \\ Tf - Tg &\leq \beta d(f,g) \end{aligned}$$

Proof of Blackwell's Theorem

- Step 2: Switch the roles of f and g

$$\begin{aligned} Tg - Tf &\leq \beta d(f,g) \\ -(Tf - Tg) &\leq \beta d(g,f) = \beta d(f,g) \end{aligned}$$

- Step 3: Combining steps 1 and 2 yields

$$\begin{aligned} (Tf)(x) - (Tg)(x) &\leq \beta d(f,g) \text{ for all } x \in X \\ (Tg)(x) - (Tf)(x) &\leq \beta d(f,g) \text{ for all } x \in X \end{aligned}$$

- Therefore

$$\sup_{x \in X} |(Tf)(x) - (Tg)(x)| = d(Tf, Tg) \leq \beta d(f,g)$$

Example: Neoclassical Growth Model

- Metric space $(B[0,\infty), d)$ the space of bounded functions on $[0,\infty)$ with d being the sup-norm.

- Operator

$$Tv(k) = \max_{0 \le k' \le f(k)} \left\{ U(f(k) - k') + \beta v(k') \right\}$$

- Verify that the operator T maps $B[0,\infty)$ into itself.

- Take v to be bounded, since we assumed that U is bounded, then Tv is bounded

Example: Neoclassical Growth Model

- Monotonicity: Suppose $v \leq w$. Let by $g_v(k)$ denote an optimal policy corresponding to v. Then for all $k \in [0, \infty)$

$$\begin{aligned} Tv(k) &= U(f(k) - g_v(k)) + \beta v(g_v(k)) \\ &\leq U(f(k) - g_v(k)) + \beta w(g_v(k)) \\ &\leq \max_{0 \leq k' \leq f(k)} \left\{ U(f(k) - k') + \beta w(k') \right\} = Tw(k) \end{aligned}$$

- Discounting:

$$\begin{aligned} T(v + a)(k) &= \max_{0 \leq k' \leq f(k)} \left\{ U(f(k) - k') + \beta(v(k') + a) \right\} \\ &= \max_{0 \leq k' \leq f(k)} \left\{ U(f(k) - k') + \beta v(k') \right\} + \beta a \\ &= Tv(k) + \beta a \end{aligned}$$

- Hence neoclassical growth model with bounded utility satisfies Blackwell's conditions and T is a contraction mapping with modulus β. Thus it has unique fixed point that can be computed from any starting guess v_0 by repeated application T.

Theorem of the Maximum

- Problem
$$h(x) = \max_{y \in \Gamma(x)} \{f(x, y)\}$$

- Define
$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$$

- Question: what can we say about the properties of h, G?

Continuity of Correspondences

- Let X, Y be arbitrary sets. A correspondence $\Gamma : X \rightrightarrows Y$ maps each element $x \in X$ into a subset $\Gamma(x)$ of Y.

Definition

- A compact-valued correspondence $\Gamma : X \rightrightarrows Y$ is upper-hemicontinuous (uhc) at a x if $\Gamma(x) \neq \varnothing$ and if for all sequences $\{x_n\}$ in X converging to $x \in X$ and all sequences $\{y_n\}$ in Y such that $y_n \in \Gamma(x_n)$ for all n, there exists a convergent subsequence of $\{y_n\}$ that converges to some $y \in \Gamma(x)$. A correspondence is uhc if it is uhc at all $x \in X$.
- A correspondence $\Gamma : X \rightrightarrows Y$ is lower-hemicontinuous (lhc) at x if $\Gamma(x) \neq \varnothing$ and if for every $y \in \Gamma(x)$ and every sequence $\{x_n\}$ in X converging to $x \in X$ there exists $N \geq 1$ and a sequence $\{y_n\}$ in Y converging to y such that $y_n \in \Gamma(x_n)$ for all $n \geq N$. A correspondence is lhc if it is lhc at all $x \in X$.
- A correspondence $\Gamma : X \rightrightarrows Y$ is continuous if it is both uhc and lhc.

Theorem of the Maximum

Theorem

Let $X \subseteq \mathbf{R}^L$ and $Y \subseteq \mathbf{R}^M$, let $f : X \times Y \to \mathbf{R}$ be a continuous function, and let $\Gamma : X \rightrightarrows Y$ be a compact-valued and continuous correspondence. Then $h : X \to \mathbf{R}$ is continuous and $G : X \to Y$ is nonempty, compact-valued and upper-hemicontinuous.

- Application to the Neoclassical Growth Model. Recursive formulation:
- Problem

$$Tv(k) = \max_{0 \leq k' \leq f(k)} \left\{ U(f(k) - k') + \beta v(k') \right\}$$

- $x = k$, $y = k'$, $X = Y = \mathbf{R}_+$, $f(x, y) = U(f(x) - y) + \beta v(y)$ and $\Gamma : X \Rightarrow Y$ is given by $\Gamma(x) = \{y \in \mathbf{R}_+ | 0 \leq y \leq f(x)\}$
- If v is continuous then Theorem of Maximum implies that $Tv(.)$ is a continuous function and that optimal policy $g(.)$ is an uhc correspondence. If $g(.)$ is a function, then it is continuous.

Principle of Optimality

- Suppose functional equation (FE)

$$v(x) = \sup_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\}$$

  has a unique solution v^* which is approached from any initial guess v_0

- Sequential problem (SP)

$$\begin{aligned} w(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ s.t.\ x_{t+1} &\in \Gamma(x_t) \\ x_0 &\in X \text{ given} \end{aligned}$$

Principle of Optimality: The Questions

- Under what conditions $v = w$
- Under what conditions $\{x_{t+1}\}_{t=0}^{\infty} \triangleq y = g(x)$, i.e.

$$x_{t+1} = g(x_t)$$

Principle of Optimality: Notation

- Let X be the set of possible values that the state x can take
- Correspondence $\Gamma : X \rightrightarrows X$ describes the feasible set of next period's states y, given that today's state is x.
- Graph of Γ, A is defined as

$$A = \{(x, y) \in X \times X : y \in \Gamma(x)\}$$

- Period return function $F : A \to \mathbf{R}$
- Fundamentals of analysis are (X, F, β, Γ). For neoclassical growth model F and β describe preferences and X, Γ describe technology.
- Any sequence of states $\{x_t\}_{t=0}^{\infty}$ is a *plan*
- For a given initial condition x_0, the set of feasible plans $\Pi(x_0)$ from x_0 is

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} : x_{t+1} \in \Gamma(x_t)\}$$

with generic element $\bar{x} \in \Pi(x_0)$

Principle of Optimality: Assumptions

- **Assumption 1:** $\Gamma(x)$ is nonempty for all $x \in X$
- **Assumption 2:** For all initial x_0 and all feasible plans $\bar{x} \in \Pi(x_0)$

$$\lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

exists (although it may be $+\infty$ or $-\infty$)
- Note that Assumption 2 holds if
  1. F is bounded and $\beta \in (0,1)$
  2. Let $F^+(x,y) = \max\{0, F(x,y)\}$ and $F^-(x,y) = \max\{0, -F(x,y)\}$. Assumption 2 is satisfied if for all $x_0 \in X$, all $\bar{x} \in \Pi(x_0)$, either

$$\begin{aligned} \lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F^+(x_t, x_{t+1}) &< +\infty \text{ or} \\ \lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F^-(x_t, x_{t+1}) &< +\infty \end{aligned}$$

  3. For every $x_0 \in X, \bar{x} \in \Pi(x_0)$ there are (possibly $x_0, \bar{x}$-dependent) $\theta \in (0, 1/\beta)$ and $0 < c < +\infty$ such that $\forall t, F(x_t, x_{t+1}) \leq c\theta^t$.

Principle of Optimality: Further Notation

- Define sequence $u_n : \Pi(x_0) \to \mathbf{R}$ by

$$u_n(\bar{x}) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

and $u : \Pi(x_0) \to \bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$

$$u(\bar{x}) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

which is well-defined since under assumption 2 the limit exists

- Note that

$$w(x_0) = \sup_{\bar{x}\in\Pi(x_0)} u(\bar{x})$$

- Note that, whenever w exists, it is unique.

Principle of Optimality: Equivalence of Value Functions

Theorem

Suppose (X, Γ, F, β) *satisfy assumptions 1. and 2. Then*

1. *the function* w *satisfies the functional equation* (FE)
2. *if for all* $x_0 \in X$ *and all* $\bar{x} \in \Pi(x_0)$ *a solution* v *to the functional equation* (FE) *satisfies*

$$\lim_{n \to \infty} \beta^n v(x_n) = 0$$

   then $v = w$

- Supremum function from the sequential problem solves the functional equation
- Result 2 is key. It states a condition under which a solution to the functional equation is a solution to the sequential problem

Example: A Consumption Problem

- Infinitely lived household has initial wealth $x_0 \in X = \mathbf{R}$
- Borrow or lend at a gross interest rate $1 + r = \frac{1}{\beta} > 1$. Price of one-period bond is $q = \beta$
- Sequential budget constraint

$$c_t + \beta x_{t+1} \leq x_t$$

- Sequential problem is

$$\begin{aligned} w(x_0) &= \sup_{\{(c_t, x_{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t c_t \\ s.t.\ 0 &\leq c_t \leq x_t - \beta x_{t+1} \\ & \quad x_0 \text{ given} \end{aligned}$$

- Obviously $w(x_0) = +\infty$ for all $x_0 \in X$

A Consumption Problem: Recursive Formulation

- State variable: current wealth x
- Control variable: next period's wealth y
- Consumption $c = x - \beta y$
- Functional equation

$$v(x) = \sup_{y \leq \frac{x}{\beta}} \{x - \beta y + \beta v(y)\}$$

- Function $w(x) = +\infty$ satisfies this functional equation
- Function $\check{v}(x) = x$ satisfies the functional equation, too. Plug in for $v(y) = y$ to obtain for all $x \in X$

$$\sup_{y \leq \frac{x}{\beta}} \{x - \beta y + \beta y\} = \sup_{y \leq \frac{x}{\beta}} x = x = \check{v}(x)$$

- Note: theorem does not apply for $\check{v}$. Look at sequence $\{x_n\}$ defined by $x_n = \frac{x_0}{\beta^n}$. This is a feasible plan but

$$\lim_{n \to \infty} \beta^n v(x_n) = \lim_{n \to \infty} \beta^n x_n = x_0 > 0.$$

Principle of Optimality: Equivalence of Policy Functions

- Optimal policy correspondence g from functional equation generates plan $\{\hat{x}_{t+1}\}_{t=0}^{\infty}$ by

$$
\begin{aligned}
x_0 &= \hat{x}_0 \\
\hat{x}_1 &\in g(\hat{x}_0) \\
\hat{x}_{t+1} &\in g(\hat{x}_t)
\end{aligned}
$$

- How does such generated plan $\{\hat{x}_{t+1}\}_{t=0}^{\infty}$ relate to solution $\{\bar{x}_{t+1}\}_{t=0}^{\infty}$ of sequential problem?

Principle of Optimality: Equivalence of Policy Functions

Theorem

Suppose (X, Γ, F, β) *satisfy assumptions 1. and 2.*

1. *Let* $\bar{x} \in \Pi(x_0)$ *be a feasible plan that attains the supremum in the sequential problem. Then for all* $t \geq 0$

$$w(\bar{x}_t) = F(\bar{x}_t, \bar{x}_{t+1}) + \beta w(\bar{x}_{t+1})$$

2. *Let* $\hat{x} \in \Pi(x_0)$ *be a feasible plan satisfying, for all* $t \geq 0$

$$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$$

   and additionally

$$\lim_{t\to\infty} \sup \beta^t w(\hat{x}_t) \leq 0$$

   Then $\hat{x}$ *attains the supremum in* (SP) *for the initial condition* x_0.

Equivalence of Policy Functions: Implications

- Any optimal plan in SP, together with supremum function w as value function satisfies the functional equation for all t.
- Second part is key: For the "right" fixed point of functional equation w the corresponding policy g generates plan $\hat{x}$ that solves sequential problem if it satisfies the additional limit condition
- If limit condition in first theorem is satisfied, then second limit condition is satisfied: if

$$\lim_{t\to\infty} \beta^t v(x_t) = 0$$

for any feasible $\{x_t\} \in \Pi(x_0)$, all x_0 then $v = w$. So for any plan $\{\hat{x}_t\}$ generated from a policy g associated with $v = w$ we have

$$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$$

and since $\lim_{t\to\infty} \beta^t v(\hat{x}_t)$ exists and equals to 0 we have

$$\limsup_{t\to\infty} \beta^t v(\hat{x}_t) = 0$$

- But first theorem may not apply, but second theorem still might.

Equivalence of Policy Functions: Example

- Sequential Problem

$$
\begin{aligned}
w(x_0) &= \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (x_t - \beta x_{t+1}) \\
s.t.\ 0 &\leq x_{t+1} \leq \frac{x_t}{\beta} \\
& \quad x_0 \text{ given}
\end{aligned}
$$

- Note that

$$
\begin{aligned}
w_0(x_0) &= (x_0 - \beta x_1) + \beta(x_1 - \beta x_2) + \ldots \\
&= x_0
\end{aligned}
$$

Equivalence of Policy Functions: Example

- Functional equation

$$v(x) = \max_{0 \leq x' \leq \frac{x}{\beta}} \left\{ x - \beta x' + \beta v(x') \right\}$$

- One solution of this functional equation is $v(x) = x$, since plugging in $v(x') = x'$ yields

$$x = v(x) = \max_{0 \leq x' \leq \frac{x}{\beta}} \left\{ x - \beta x' + \beta x' \right\} = x$$

- First Theorem, second part does not apply since feasible plan defined by

$$x_n = \frac{x_0}{\beta^n}$$

  has for all $x_0 > 0$

$$\lim_{n \to \infty} \beta^n v(x_n) = x_0 > 0$$

- Still can apply second theorem to show that certain plans are optimal plans

Equivalence of Policy Functions: Example

- Let $\{\hat{x}_t\}$ be defined by $\hat{x}_0 = x_0$ and $\hat{x}_t = 0$ all $t > 0$. Then

$$\limsup_{t\to\infty} \beta^t w(\hat{x}_t) = 0$$

and we can conclude the second theorem that this plan is optimal for the sequential problem. Other plans are, too.

- But consider the plan defined by $\hat{x}_t = \frac{x_0}{\beta^t}$. Obviously this is a feasible plan satisfying

$$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$$

but since for all $x_0 > 0$

$$\limsup_{t\to\infty} \beta^t w(\hat{x}_t) = x_0 > 0$$

- Second theorem does not apply and we can't conclude that $\{\hat{x}_t\}$ is optimal (it is not!)

Dynamic Programming with Bounded Returns

- Functional equation

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

with associated operator $T : C(X) \to C(X)$

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

- Now: Make stronger assumptions on (X, F, β, Γ) to be able to say more about qualitative features of v and g, where

$$g(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

is the policy correspondence associated with v

Uniqueness of Solution of FE

- **Assumption 3:** X is a convex subset of $\mathbf{R}^L$ and the correspondence $\Gamma : X \Rightarrow X$ is nonempty, compact-valued and continuous.
- **Assumption 4:** The function $F : A \to \mathbf{R}$ is continuous and bounded, and $\beta \in (0, 1)$
- Assumptions 1. and 2. hold, so previous theorems apply

Theorem

Under Assumptions 3. and 4. the operator T maps $C(X)$ into itself. T has a unique fixed point v and for all $v_0 \in C(X)$

$$d(T^n v_0, v) \leq \beta^n d(v_0, v)$$

The policy correspondence g is compact-valued and upper-hemicontinuous

Monotonicity of the Value Function

- **Assumption 5:** For fixed y, $F(., y)$ is strictly increasing in each of its L components
- **Assumption 6:** Γ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$

Theorem

Under Assumptions 3. to 6. the unique fixed point v of T is strictly increasing.

Strict Concavity of Value Function and Uniqueness of Optimal Policy

- **Assumption 7:** F is strictly concave, i.e. for all $(x,y),(x',y') \in A$ and $\theta \in (0,1)$

$$\begin{aligned} & F[\theta(x,y)+(1-\theta)(x',y')] \\ \geq \;& \theta F(x,y)+(1-\theta)F(x',y') \end{aligned}$$

  and the inequality is strict if $x \neq x'$
- **Assumption 8:** Γ is convex in the sense that for $\theta \in [0,1]$ and $x, x' \in X$, the fact $y \in \Gamma(x), y' \in \Gamma(x')$

$$\theta y+(1-\theta)y' \in \Gamma(\theta x+(1-\theta)x')$$

Theorem

Under Assumptions 3.-4. and 7.-8. the unique fixed point of v is strictly concave and the optimal policy g is a single-valued continuous function.

Differentiability of the Value Function and Benveniste-Scheinkman Condition

- **Assumption 9:** The period return function F is continuously differentiable on the interior of A.

Theorem

(Benveniste-Scheinkman): Under assumptions 3.-4. and 7.-9. if $x_0 \in int(X)$ *and* $g(x_0) \in int(\Gamma(x_0))$*, then the unique fixed point of* T, v *is continuously differentiable at* x_0 *with*

$$\frac{\partial v(x_0)}{\partial x^i} = \frac{\partial F(x_0, g(x_0))}{\partial x^i}$$

Benveniste and Scheinkman meet Euler

- Functional equation

$$v(k) = \max_{0 \leq k' \leq f(k)} U(f(k) - k') + \beta v(k')$$

- FOC with respect to k' gives

$$U'(f(k) - k') = \beta v'(k')$$

- Use Benveniste-Scheinkman to obtain

$$v'(k) = U'(f(k) - g(k))f'(k)$$

and hence

$$\begin{aligned} U'(f(k) - g(k)) &= \beta v'(k') = \beta U'(f(k') - g(k'))f'(k') \\ &= \beta f'(g(k))U'(f(g(k)) - g(g(k)) \end{aligned}$$

- Let $k = k_t$, $g(k) = k_{t+1}$ and $g(g(k)) = k_{t+2}$, then

$$U'(f(k_t) - k_{t+1}) = \beta f'(k_{t+1})U'(f(k_{t+1}) - k_{t+2})$$

Models with Risk

- Event $s_t \in S$. The set $S = \{1, 2, \ldots N\}$.

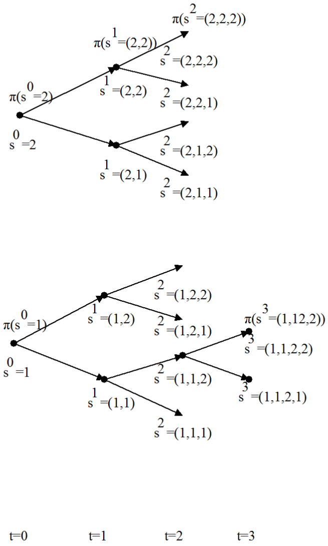
- Example: randomly fluctuating endowments

$$\begin{aligned} s_t &= 1 \Rightarrow \text{agent 1 has endowment 2} \\ s_t &= 2 \Rightarrow \text{agent 2 has endowment 2} \end{aligned}$$

- Event history $s^t = (s_0, s_1, \ldots s_t) \in S^t = S \times S \times \ldots \times S$

- Probability of a particular event history $\pi_t(s^t)$. Assume that $\pi_t(s^t) > 0$ for all $s^t \in S^t$, for all t.

Event Tree for Models with Risk



Endowments and Preferences

- Commodities indexed by time t and by event histories s^t. A consumption allocation is:

$$(c^1, c^2) = \{c_t^1(s^t), c_t^2(s^t)\}_{t=0, s^t \in S^t}^{\infty}$$

- Endowments

$$(e^1, e^2) = \{e_t^1(s^t), e_t^2(s^t)\}_{t=0, s^t \in S^t}^{\infty}$$

- For particular example

$$\begin{aligned} e_t^1(s^t) &= \begin{cases} 2 & \text{if } s_t = 1 \\ 0 & \text{if } s_t = 2 \end{cases} \\ e_t^2(s^t) &= \begin{cases} 0 & \text{if } s_t = 1 \\ 2 & \text{if } s_t = 2 \end{cases} \end{aligned}$$

- Preferences

$$u(c^i) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) U(c_t^i(s^t))$$

Arrow-Debreu Equilibrium

- Trade takes place at period 0, *before* any risk has been realized (in particular, before s_0 has been realized)

- Arrow-Debreu prices $p_t(s^t)$

Arrow-Debreu Equilibrium

Definition

A (competitive) Arrow-Debreu equilibrium are prices $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ and allocations $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,2}$ such that

1. Given $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$, for $i=1,2$, $\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ solves

$$\max_{\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi_t(s^t)U(c_t^i(s^t))$$

   s.t.

$$\begin{aligned}\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\hat{p}_t(s^t)c_t^i(s^t) &\leq \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\hat{p}_t(s^t)e_t^i(s^t)\\ c_t^i(s^t) &\geq 0 \text{ for all } t, \text{ all } s^t\in S^t\end{aligned}$$

2. $$\hat{c}_t^1(s^t)+\hat{c}_t^2(s^t)=e_t^1(s^t)+e_t^2(s^t) \text{ for all } t, \text{ all } s^t\in S^t$$

Arrow-Debreu Equilibrium: Remarks

- Only one budget constraint
- Market clearing has to hold date, by date, event history by event history
- Normalization of only one commodity to 1

Characterization of Equilibrium Prices

- First order conditions

$$\begin{aligned} \beta^t \pi_t(s^t) U'(c_t^i(s^t)) &= \mu p_t(s^t) \\ \pi_0(s_0) U'(c_0^i(s_0)) &= \mu p_0(s_0) \end{aligned}$$

- Combining yields

$$\frac{p_t(s^t)}{p_0(s_0)} = \beta^t \frac{\pi_t(s^t)}{\pi_0(s_0)} \frac{U'(c_t^i(s^t))}{U'(c_0^i(s_0))}$$

- This immediately implies that

$$\frac{U'(c_t^1(s^t))}{U'(c_0^1(s_0))} = \frac{U'(c_t^2(s^t))}{U'(c_0^2(s_0))}$$

or

$$\frac{U'(c_t^2(s^t))}{U'(c_t^1(s^t))} = \frac{U'(c_0^2(s_0))}{U'(c_0^1(s_0))}$$

Characterization of Equilibrium Prices

- Ratio of marginal utilities between the two agents is constant over time and across states of the world.
- With CRRA period utility

$$\left(\frac{c_t^2(s^t)}{c_t^1(s^t)}\right)^{-\sigma} = \left(\frac{c_0^2(s^0)}{c_0^1(s^0)}\right)^{-\sigma}$$

- Thus there exist $\theta^i \geq 0$ with $\sum_i \theta^i = 1$ such that

$$c_t^i(s^t) = \theta^i e_t(s^t)$$

  with $e_t(s^t) = \sum_i e_t^i(s^t)$.
- Using this and normalizing $p_0(s_0) = 1$ we find

$$\begin{aligned} p_t(s^t) &= \beta^t \frac{\pi_t(s^t)}{\pi_0(s_0)} \left(\frac{c_t^i(s^t)}{c_0^i(s_0)}\right)^{-\sigma} \\ &= \beta^t \frac{\pi_t(s^t)}{\pi_0(s_0)} \left(\frac{e_t(s^t)}{e_0(s_0)}\right)^{-\sigma} \end{aligned}$$

Empirical Implications of Equilibrium

- Perfect consumption insurance: individual consumption of all households responds proportionally to aggregate shocks $e_t(s^t)$, but not at all to individual household endowment shocks $e_t^i(s^t)$ that leave aggregate endowments unchanged.

$$c_t^i(s^t) = \theta^i e_t(s^t)$$

- Asset prices and the representative household:

$$p_t(s^t) = \beta^t \frac{\pi_t(s^t)}{\pi_0(s_0)} \left(\frac{e_t(s^t)}{e_0(s_0)} \right)^{-\sigma}$$

- Arrow Debreu prices (and thus all other asset prices) are the same in this economy and in a representative agent economy where the representative agent has endowment stream $\{e_t(s^t)\}$ and CRRA utility with RRA σ.

Pareto Optimal Allocations

Definition

An allocation $\{(c_t^1(s^t), c_t^2(s^t))\}_{t=0,s^t\in S^t}^{\infty}$ is feasible if

1. $$c_t^i(s^t) \geq 0 \text{ for all } t, \text{ all } s^t \in S^t, \text{ for } i = 1, 2$$

2. $$c_t^1(s^t) + c_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t) \text{ for all } t, \text{ all } s^t \in S^t$$

Definition

An allocation $\{(c_t^1(s^t), c_t^2(s^t))\}_{t=0,s^t\in S^t}^{\infty}$ is Pareto efficient if it is feasible and if there is no other feasible allocation $\{(\tilde{c}_t^1(s^t), \tilde{c}_t^2(s^t))\}_{t=0,s^t\in S^t}^{\infty}$ such that

$$\begin{aligned} u(\tilde{c}^i) &\geq u(c^i) \text{ for both } i = 1, 2 \\ u(\tilde{c}^i) &> u(c^i) \text{ for at least one } i = 1, 2 \end{aligned}$$

First Welfare Theorem

Proposition

Let $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,2}$ *be a competitive equilibrium allocation. Then* $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,2}$ *is Pareto efficient.*

Sequential Markets Equilibrium

- Previously: one-period IOU's
- Now: one period state-contingent IOU's.
- Let $q_t(s^t, s_{t+1} = j)$ denote price at period t of a contract that pays one unit of consumption in period $t+1$ if and only if $t+1$ event is $s_{t+1} = j$
- Let $a^i_{t+1}(s^t, s_{t+1})$ be the quantities Arrow securities bought at period t by agent i.
- Budget constraint of agent i

$$c^i_t(s^t) + \sum_{s_{t+1} \in S} q_t(s^t, s_{t+1}) a^i_{t+1}(s^t, s_{t+1}) \leq e^i_t(s^t) + a^i_t(s^t)$$

Sequential Markets Equilibrium

Definition

A SM equilibrium is allocations $(\hat{c}^i, \hat{a}^i)_{i=1,2}$ and prices $\hat{q}_t$ such that

1. For $i = 1, 2$, given $\hat{q}$, for all i, $(\hat{c}^i, \hat{a}^i)_{i=1,2}$ solves

$$\max_{c^i, a^i} u(c^i) \text{ s.t.}$$

$$\begin{aligned}
c_t^i(s^t) + \sum_{s_{t+1} \in S} \hat{q}_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) &\leq e_t^i(s^t) + a_t^i(s^t) \text{ for all } t, s^t \in S^t \\
c_t^i(s^t) &\geq 0 \text{ for all } t, s^t \in S^t \text{ and all } s_{t+1} \in S \\
a_{t+1}^i(s^t, s_{t+1}) &\geq -\bar{A}^i \text{ for all } t, s^t \in S^t \text{ and all } s_{t+1} \in S
\end{aligned}$$

2. For all $t \geq 0$

$$\begin{aligned}
\sum_{i=1}^{2} \hat{c}_t^i(s^t) &= \sum_{i=1}^{2} e_t^i(s^t) \text{ for all } t, s^t \in S^t \\
\sum_{i=1}^{2} \hat{a}_{t+1}^i(s^t, s_{t+1}) &= 0 \text{ for all } t, s^t \in S^t \text{ and all } s_{t+1} \in S
\end{aligned}$$

Asset Pricing

- Suppose we have AD Prices $\{p_t(s^t)\}$ or prices of Arrow securities, $\{q_t(s^t, s_{t+1})\}$.

- Can prove equivalence between AD and SM equilibrium, with mapping of prices being given as

$$\begin{aligned} q_t(s^t, s_{t+1}) &= \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} \\ p_t(s^t) &= p_0(s_0) * q_0(s_0, s_1) * \ldots q_{t-1}(s^{t-1}, s_t) \end{aligned}$$

Asset Pricing

- Time zero (cum dividend) price of any asset j with dividend stream $d^j = \{d_t^j(s^t)\}$

$$P_0^j(d) = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) d_t^j(s^t)$$

- Ex-dividend price of asset at node s^t in terms of period t consumption good

$$P_t^j(d; s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) d_\tau^j(s^\tau)}{p_t(s^t)}$$

- One-period gross realized real return of such an asset between s^t and s^{t+1}

$$R_{t+1}^j(s^{t+1}) = \frac{P_{t+1}^j(d; s^{t+1}) + d_{t+1}^j(s^{t+1})}{P_t^j(d; s^t)}$$

Asset Pricing: Examples

- Arrow security that pays in $\hat{s}^{t+1}$
  - Price in terms of consumption at s^t

$$P_t^A(d; s^t) = \frac{p_{t+1}(\hat{s}^{t+1})}{p_t(s^t)} = q_t(s^t, \hat{s}_{t+1})$$

  - Associated gross realized return between s^t and $\hat{s}^{t+1} = (s^t, \hat{s}_{t+1})$

$$\begin{aligned} R_{t+1}^A(\hat{s}^{t+1}) &= \frac{1}{p_{t+1}(\hat{s}^{t+1})/p_t(s^t)} \\ &= \frac{p_t(s^t)}{p_{t+1}(\hat{s}^{t+1})} = \frac{1}{q_t(s^t, \hat{s}_{t+1})} \end{aligned}$$

  and $R_{t+1}^A(s^{t+1}) = 0$ for all $s_{t+1} \neq s_{t+1}$.

Asset Pricing: Examples

- One-period risk free bond:
  - Price

$$P_t^B(d;s^t) = \frac{\sum_{s^{t+1}} p_{t+1}(s^{t+1})}{p_t(s^t)} = \sum_{s^{t+1}} q_t(s^t, s_{t+1})$$

  - Realized return

$$\begin{aligned} R_{t+1}^B(s^{t+1}) &= \frac{1}{P_t^B(d;s^t)} \\ &= \frac{1}{\sum_{s^{t+1}} q_t(s^t, s_{t+1})} = R_{t+1}^B(s^t) \end{aligned}$$

Asset Pricing: Examples

- Stock (Lucas tree) that pays as dividend the aggregate endowment in each period

$$P_t^S(d; s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) e_\tau(s^\tau)}{p_t(s^t)}$$

- Call option: buy one share of the Lucas tree at time T (at all nodes) for a price K has a price $P_t^{call}(s^t)$ at node s^t

$$P_t^{call}(s^t) = \sum_{s^T | s^t} \frac{p_T(s^T)}{p_t(s^t)} \max\left\{P_T^S(d; s^T) - K, 0\right\}$$

- Put option

$$P_t^{put}(s^t) = \sum_{s^T | s^t} \frac{p_T(s^T)}{p_t(s^t)} \max\left\{K - P_T^S(d; s^T), 0\right\}.$$

Markov Processes

- So far: no specification of exact nature of risk
- Problem: Number of event histories (and hence commodities) grows quickly with time
- Now: restriction to discrete time, discrete state, time homogeneous Markov processes
- Let

$$\pi(j|i) = \text{prob}(s_{t+1} = j|s_t = i)$$

  Conditional probabilities of s_{t+1} only depend on realization of s_t, not s_{t-1} or other past realizations
- Time homogeneity means that π is not indexed by time

Markov Processes

- Given that $s_{t+1} \in S$ and $s_t \in S$ and S is a finite set, $\pi(.|.)$ is an $N \times N$-matrix of the form

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \vdots & \cdots & \pi_{1N} \\ \pi_{21} & & & \vdots & & \vdots \\ \vdots & & & \vdots & & \vdots \\ \pi_{i1} & \cdots & \cdots & \pi_{ij} & \cdots & \pi_{iN} \\ \vdots & & & \vdots & & \vdots \\ \pi_{N1} & \cdots & \cdots & \vdots & \cdots & \pi_{NN} \end{pmatrix}$$

  with generic element $\pi_{ij} = \pi(j|i) =$prob$(s_{t+1} = j|s_t = i)$

- Since $\pi_{ij} \geq 0$ and $\sum_j \pi_{ij} = 1$ for all i, matrix π is called a stochastic matrix

Markov Processes

- Suppose probability distribution over states today is given by the N-dimensional column vector $P_t = (p_t^1, \ldots, p_t^N)^T$ with $\sum_i p_t^i = 1$.
- Probability of being in state j tomorrow is given by

$$p_{t+1}^j = \sum_i \pi_{ij} p_t^i$$

$$P_{t+1} = \pi^T P_t$$

- A *stationary* distribution Π of the Markov chain π satisfies

$$\Pi = \pi^T \Pi$$

- Fact: π has at least one stationary Π. It is the eigenvector (normalized to length 1) associated with eigenvalue $\lambda = 1$ of π^T.
- If there is only one such eigenvalue, then there is a unique stationary distribution. If more than one unit eigenvalue, then there are multiple stationary distributions
- With s_t's following a Markov chain, we have

$$\pi_{t+1}(s^{t+1}) = \pi(s_{t+1}|s_t) * \pi(s_t|s_{t-1}) \ldots \pi(s_1|s_0) * \Pi(s_0)$$

Markov Processes: Examples

- Suppose

$$\pi = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

  for some $p \in (0,1)$.Unique invariant distribution is $\Pi(s) = 0.5$ for both s.

- Suppose

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  then any distribution over the two states is an invariant distribution.

Stochastic Neoclassical Growth Model: Technology

- Technology

$$y_t = e^{z_t} F(k_t, n_t)$$

- z_t is a technology shock that has unconditional mean 0 and follows N-state Markov chain with state space $Z = \{z_1, z_2, \ldots z_N\}$ and transition matrix $\pi = (\pi_{ij})$, associated stationary distribution Π.
- Evolution of the capital stock

$$k_{t+1} = (1 - \delta)k_t + i_t$$

- Resource constraint

$$y_t = c_t + i_t$$

Stochastic Neoclassical Growth Model: Endowments and Preferences

- Preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi_t(z^t) U(c_t(z^t))$$

- Endowments: initial endowment of capital, k_0 and one unit of time in each period.

- Information: z_t is publicly observable. z_0 is drawn from the stationary distribution Π.

Recursive Formulation of Planners' Problem

- State variables (k, z)
- Control variable k'
- Bellman equation

$$\begin{aligned} v(k,z) &= \max_{k'} \left\{ U(e^z F(k,1) + (1-\delta)k - k') + \beta \sum_{z'} \pi(z'|z) v(k',z') \right\} \\ \text{s.t. } 0 &\leq k' \leq e^z F(k,1) + (1-\delta)k \end{aligned}$$

Real Business Cycle Model

- Main driver of output fluctuations are fluctuations in labor input.
- Add labor-leisure choice to the model

$$U(c_t, 1 - n_t)$$

- Bellman equation

$$\begin{aligned} v(k,z) &= \max_{k',n} \left\{ \begin{array}{c} U(e^z F(k,n) + (1-\delta)k - k', 1-n) \\ +\beta \sum_{z'} \pi(z'|z) v(k',z') \end{array} \right\} \\ 0 &\leq k' \leq e^z F(k,1) + (1-\delta)k, \ 0 \leq n \leq 1 \end{aligned}$$

- This is the benchmark model of modern business cycle research (see the Cooley, 1995 book).

Key Optimality Conditions

- Intratemporal optimality condition

$$e^z F_n(k,n) = \frac{U_2(c,1-n)}{U_1(c,1-n)}$$

- Intertemporal optimality condition

$$U_1(c,1-n) = \beta \sum_{z'} \pi(z'|z) v'(k',z')$$

- Envelope condition

$$v'(k,z) = (e^z F_k(k,n) + 1 - \delta)\, U_1(c,1-n)$$

- Using this in intertemporal optimality condition yields stochastic Euler equation

$$U_1(c,1-n) = \beta \sum_{z'} \pi(z'|z) \left(e^{z'} F_k(k',n') + 1 - \delta\right) U_1(c',1-n')$$

Recursive Competitive Equilibrium

Definition

A RCE is value, policy functions $v, c, n, g : \mathbf{R}^4_+ \to \mathbf{R}$ for households, labor demand functions for firms $N : \mathbf{R}^2_+ \to \mathbf{R}_+$, pricing functions $w, r : \mathbf{R}^2_+ \to \mathbf{R}_+$ and aggregate law of motion $H : \mathbf{R}^2_+ \to \mathbf{R}_+$ s.t.

1. Given w, r, H, value function v, policy functions c, n, g solve

$$\begin{aligned} v(k,z,K) &= \max_{c,k',n\geq 0}\left\{U(c,n)+\beta\sum_{z'\in Z}\pi(z'|z)v(k',z',K')\right\} \\ c+k' &= w(z,K)n+(1+r(z,K)-\delta)k \\ K' &= H(z,K) \end{aligned}$$

2. The labor demand and pricing functions satisfy

$$\begin{aligned} w(z,K) &= e^z F_n(K,N(z,K)) \\ r(z,K) &= e^z F_k(K,N(z,K)). \end{aligned}$$

3. Consistency

$$H(z,K) = g(K,z,K)$$

4. Market Clearing

$$\begin{aligned} c(K,z,K)+g(K,z,K) &= e^z F(K,N(z,K))+(1-\delta)K \\ N(z,K) &= n(K,z,K) \end{aligned}$$

Cyclical Variation of GDP and Labor Productivity

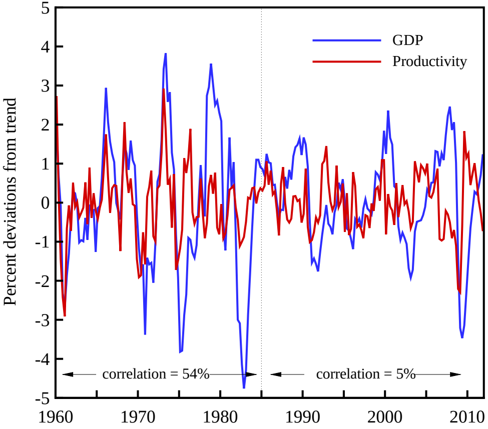


FIGURE 1

GDP and Aggregate Labor Productivity, 1960:1–2011:4,

Percent Deviations from HP-filtered Trend