1 Two elementary results on aggregation of technologies and preferences

In what follows we'll discuss "aggregation". What do we mean with this term? We say that an economy admits aggregation if the behavior of the aggregate equilibrium quantities (e.g., aggregate consumption, investment, wealth,...) and prices (e.g., wage, interest rate, ...) does not depend on the distribution of the individual quantities across agents. In other words, we can aggregate whenever we can define a fictitious "representative agent" that behaves, in equilibrium, as the sum of all individual consumers.

1.1 Aggregating firms with the same technology

Consider an economy with M firms, indexed by $i = 1, 2, ..., M$ which produce a homogeneous good with the same technology $zF(k^i, n^i)$ where z is aggregate productivity. Assume that F is strictly increasing, strictly concave, differentiable in both arguments and constant returns to scale. Can we aggregate these individual firms into a representative firm?

Suppose inputs markets are competitive. Then, each price-taking firm of type i solves

$$\max_{\{k^i, n^i\}} zF\left(k^i, n^i\right) - wn^i - (r+\delta)\,k^i,$$

with first-order conditions

$$\begin{aligned} zF_k\left(k^i, n^i\right) &= r+\delta, \qquad (1)\\ zF_n\left(k^i, n^i\right) &= w, \end{aligned}$$

Recall that by CRS, F_k and F_n are homogenous of degree zero, hence:

$$\frac{F_k\left(k^i, n^i\right)}{F_n\left(k^i, n^i\right)} = \frac{f_k\left(k^i/n^i\right)}{f_n\left(k^i/n^i\right)}.$$

Dividing through the two first-order conditions, we obtain

$$\frac{f_k\left(k^i/n^i\right)}{f_n\left(k^i/n^i\right)} = \frac{r+\delta}{w},$$

and using the fact that the left-hand side is a strictly decreasing function of (k^i/n^i), we obtain

$$\frac{k_i}{n_i} = g\left(\frac{r+\delta}{w}\right) = \frac{K}{N}, \text{ for every } i = 1, 2, ..., M$$

where capital letters denote averages: every firm chooses the same capital-labor ratio.

Aggregate production across all firms:

$$
\begin{aligned}
z\sum_{i=1}^{M} F\left(k^i, n^i\right) &= z\sum_{i=1}^{M}\left[F_k\left(k^i, n^i\right)k_i + F_n\left(k^i, n^i\right)n_i\right] \\
&= z\sum_{i=1}^{M}\left[f_K\left(K/N\right)k_i + f_N\left(K/N\right)n_i\right] \\
&= zf_K\left(K/N\right)K + zf_N\left(K/N\right)N \\
&= zF\left(K, N\right)
\end{aligned}
$$

where the last line uses the CRS property of F. This derivation proves the existence of a "representative" firm with technology $zF(K, N)$. Note that z is the same across firms. With CRS, if one firm is more productive than all the others, it gets all the inputs.

1.2 Aggregating consumers with the same preferences

Consider a version of the neoclassical growth model with N types of consumers indexed by $i = 1, 2, ..., N$ with the same endowments of capital $k_0^i = \kappa$ for all $i's$ and same preferences

$$
U\left(c_0^i, c_1^i, ...\right) = \sum_{t=0}^{\infty}\beta^t u\left(c_{it}\right).
$$

Assume that markets are competitive, so that every consumer faces the same prices. Then, one would think that since all the N consumers make the same decisions, we can aggregate them into a representative agent, right? Not so quickly... Unless the utility function u is strictly concave, agents may not make the same optimal choices of consumption and leisure.

Let's combine these two results on firms and consumers:

Result 1.0 (trivial aggregation): Suppose that (i) every firm has the same productivity z and the same CRS production function F, where F is strictly increasing and strictly concave; (ii) consumers have the same initial endowments, and same preferences, and their utility function u is strictly increasing and strictly concave. Then, the neoclassical growth model admits a formulation with one representative firm and one representative household.

2 Gorman aggregation

We now study a more interesting case of "demand aggregattion", i.e., aggregation of individual demand curves. Consider a static economy populated by N agents indexed by $i = 1, ..., N$. The commodity space comprises M consumption goods $\left\{c^1, ..., c^M\right\}$ whose price vector is $p = \{p_m\}_{m=1}^M$. Each consumer is endowed with a_i units of wealth, and has utility u_i, strictly increasing and concave over each of the M goods. All goods markets are competitive.

Consider a particular good c^m. Aggregate demand of good c^m is given by the sum of all individual demands, or

$$C^m(p, \{a_i\}) = \sum_{i=1}^{N} c_i^m(p, a_i),$$

which makes it clear that, in general, you need to know the entire distribution of assets across agents to determine aggregate quantities.

When can we, instead, write aggregate demand as $C^m(p, A)$ where $A = \sum_{i=1}^N a_i$? In other words, when does aggregate demand only depend on the aggregate endowment, not on its distribution. In order for this representation to be true, it must be that if we reallocate one dollar of wealth from consumer i to j, the total demand of i and j does not change, or

$$\frac{\partial c_i^m(p, a_i)}{\partial a_i} = \frac{\partial c_j^m(p, a_j)}{\partial a_j}|_{da_j = -da_i} \text{ for all } (i, j) \text{ and for all } m.$$

This condition is true if all agents have the same marginal propensity to consume out of wealth, i.e., if the individual decision rule for consumption (recall, an outcome of optimization) can be written as

$$c_i^m(p, a_i) = \kappa_i^m(p) + \pi^m(p)\, a_i. \tag{2}$$

Condition (2) means that individual Engel curves (individual expenditures as a function of wealth) are linear. Then, aggregate consumption of good c^m is

$$C^m(p, A) = \bar{\kappa}^m(p) + \pi^m(p)\, A$$

where $\bar{\kappa}^m(p) = \sum_{i=1}^I \kappa_i^m(p)$.

This demand aggregation result is due to Gorman (1961), who stated it in terms of indirect utility as follows:

Result 1.0.1 (Gorman aggregation): If (and only if) agents' indirect utility functions can be represented as $v_i(p, a_i) = \alpha_i(p) + \beta(p) a_i$, *then aggregate consumption can be expressed as the choice of a representative agent with indirect utility* $V(p, A) = \bar{\alpha}(p) + \beta(p) A$ *where* $\bar{\alpha}(p) = \sum_{i=1}^{I} \alpha_i(p)$.

That is, for Gorman aggregation what we want is an indirect utility function that can be separated into a term that depends on prices and the consumer's identity but not on her wealth, and a term that depends on a function of prices that is common to all consumers that is multiplied by that consumer's wealth. This indirect utility is said to be of the Gorman form. You can easily prove both directions (iff) of this result using Roy's identity to obtain the demand function from the indirect utility function. Recall that Roy's identity establishes that

$$c^m(p, a_i) = -\frac{\frac{\partial v_i(p,a_i)}{\partial p^m}}{\frac{\partial v_i(p,a_i)}{\partial a_i}}. \tag{3}$$

Gorman aggregation (or demand aggregation) is a very powerful result for a number of reasons. First, beyond giving conditions for aggregations, it also explains how to construct the preferences of the representative agent. Moreover, it requires only (somewhat strong) assumptions on the demand side, such as restrictions on u, but no restriction on financial markets or technology. For example, we don't need complete financial markets. In addition, it is only based on consumer optimization and does not require any equilibrium restrictions.

Quasilinear utility: We now consider an example of Gorman aggregation: agent i with quasilinear utility over two goods (c_1, c_2) solves

$$\begin{aligned} &\max_{\{c_i^1, c_i^2\}} u_i\left(c_i^1\right) + \beta c_i^2 \\ &s.t. \\ p_1 c_i^1 + c_i^2 &= a_i \end{aligned}$$

where p_1 is the relative price –we chose p_2 as the numeraire. The FOCs of this problem are:

$$
\begin{aligned}
u_{i,c}\left(c_i^1\right) &= \lambda_i p_1, \\
\beta &= \lambda_i,
\end{aligned}
$$

where λ_i is the multiplier on the budget constraint. Note the key property of quasi-linear utility. The demand for good 1 is determined by the relative price but not by the endowment level. The demand for good 2 is then determined residually from the budget constraint.

The solution is therefore: $c_i^1 = u_{i,c}^{-1}(\beta p_1)$ and, from the budget constraint, $c_i^2 = a_i - p_1 u_{i,c}^{-1}(\beta p_1)$. The key observation here is that both individual demand functions show constant marginal propensities (respectively zero and one) out of wealth: Gorman aggregation holds. Note that the indirect utility function becomes

$$
\begin{aligned}
v_i(p_1; a_i) &= u_i\left(u_{i,c}^{-1}(\beta p_1)\right) + \beta\left[a_i - p_1 u_{i,c}^{-1}(\beta p_1)\right] \\
&= \alpha_i(p_1) + \beta a_i
\end{aligned}
$$

which is linear in individual wealth a_i and has common coefficient β, so we show again that we satisfy Gorman aggregation.

Another example for which Gorman aggregation holds is that of a homothetic utility function. A utility function $u(x_1, x_2)$ is homothetic if $u(\alpha x_1, \alpha x_2)$ has the same MRS as $u(x_1, x_2)$. In a model where competitive consumers optimize with *homothetic* utility functions subject to a budget constraint, the ratios of any two goods demanded only depends only relative prices, not on income or scale, a helpful proprty for aggregation. In the next section, we explore a model with homothetic preferences.

2.1 Neoclassical growth model with complete markets, heterogeneity in endowments, and homothetic preferences.

We now study a version of the neoclassical growth model with complete markets where consumers are only different in terms of their initial endowments of wealth. There is no individual or aggregate uncertainty. We show that, with homothetic preferences, we obtain demand aggregation. This derivation folows Chatterjee (1994).

2.1.1 The economy

Demographics– The economy is inhabited by N types of infinitely lived agents, indexed by $i = 1, 2, ..., N$. Denote by μ_i the number of agents i and normalize the total number of agents to one, $\sum_{i=1}^{N} \mu_i = 1$, so that averages and aggregates are the same.

Preferences– Preferences are time separable, defined over streams of consumption, and common across agents:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_{it}),$$

where the period utility function u belongs to one of the following three classes: log, power, exponential, i.e.

$$u(c) = \begin{cases} \log(\bar{c} + c) & \text{with } \bar{c} + c > 0, \quad \bar{c} \leq 0 \\ \frac{(\bar{c}+c)^{1-\sigma}}{1-\sigma} & \text{with } \bar{c} + c > 0, \quad \bar{c} \leq 0 \\ -\bar{c}\exp(-\sigma c) & \text{with } \bar{c} > 0 \end{cases} \tag{4}$$

We impose $\bar{c} \leq 0$ for log and power utility to allow for a subsistence level for consumption, and we impose $\bar{c} > 0$ for the exponential case. Note that utility is the same for each type.[1] When period utility belongs to the families in (4), then preferences share a common property. They are *quasi-homothetic*, i.e., they have affine Engel curves in wealth: the wealth-expansion path is linear.[2]

Markets and property rights– There are spot markets for the final good (which can be used for both consumption and investment) and complete financial markets, i.e. there are no constraints on transfers of income across periods. We assign the property rights on capital to the firm and the ownership of the firm to the household.[3] This is a different arrangement of property rights from the one you are used to seeing. Here households own shares of the firm.

We will let the initial level of wealth, at date t, differ across agents. Let a_{it} be the individual wealth of type i at time t. Then, $a_{it} = s_{it}A_t$, where s_{it} is the share of the firm-value owned by consumer i at time t. By summing both sides of this equation

[1] At the end of this derivation think about what happens if $\bar{c}$ is indexed by i.

[2] When $\bar{c} = 0$, preferences are homothetic because the constant in the consumption function becomes zero and Engel curves start at the origin, i.e. they are linear. However, linearity of the wealth-expansion path is not affected by the constant $\bar{c}$.

[3] If we had chosen to model the firm's problem as static (i.e. the firm rents capital services from households), every argument in this lecture would still hold. You should check this, as well as every other claim I make without proving it!

over i and exploiting the fact that $\sum_{i=1}^{N} \mu_i s_{it} = 1$ for every t, we obtain that aggregate household wealth equals the value of the firm (note also that s_{it} can be larger than 1). Besides intertemporal trading, there will be no other securities traded among agents in equilibrium, since there is no risk.

Technology– The aggregate production technology is $Y_t = f(K_t)$ with f strictly increasing, strictly concave and differentiable.

Household's problem– Given complete markets, the maximization problem of household i at time t can therefore be stated as:

$$\begin{aligned}
&\max_{\{c_{i\tau}\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_{i\tau}) \\
&s.t. \\
&\sum_{\tau=t}^{\infty} p_\tau c_{i\tau} \leq p_t a_{it}
\end{aligned} \tag{5}$$

where a_{it} is the wealth of agent i in term of consumption units at time t. Let λ_{it} be the Lagrange multiplier on the individual i time t Arrow-Debreu budget constraint.

Solution– Consider the log-preferences case. From the FOC of the household problem at time t with respect to consumption at time τ, we have:

$$\beta^{\tau-t} u'(c_{i\tau}) = \lambda_{it} p_\tau \quad \Rightarrow \quad \beta^{\tau-t}\left(\frac{1}{\bar{c}+c_{i\tau}}\right) = \lambda_{it} p_\tau \quad \Rightarrow \quad c_{i\tau} = \frac{\beta^{\tau-t}}{\lambda_{it} p_\tau} - \bar{c}. \tag{6}$$

Substituting this FOC into the budget constraint of (5), we can derive an expression for the multiplier λ_{it}:

$$\begin{aligned}
\sum_{\tau=t}^{\infty} p_\tau \left(\frac{\beta^{\tau-t}}{\lambda_{it} p_\tau} - \bar{c}\right) &= p_t a_{it} \\
\frac{1}{\lambda_{it}(1-\beta)} - \bar{c} \sum_{\tau=t}^{\infty} p_\tau &= p_t a_{it} \\
\frac{1}{\lambda_{it}} &= (1-\beta)\left[p_t a_{it} + \bar{c} \sum_{\tau=t}^{\infty} p_\tau\right]
\end{aligned} \tag{7}$$

Let's now substitute the expression on the last line into equation (6) evaluated at $\tau = t$, i.e.

$$c_{it} = \frac{1}{\lambda_{it} p_t} - \bar{c},$$

in order to solve explicitly for c_{it}:

$$
\begin{aligned}
c_{it} &= \frac{1}{p_t}\left[(1-\beta)\, p_t a_{it} + (1-\beta)\,\bar{c}\sum_{\tau=t}^{\infty} p_\tau\right] - \bar{c} \\
&= \bar{c}\left[(1-\beta)\sum_{\tau=t}^{\infty}\left(\frac{p_\tau}{p_t}\right) - 1\right] + (1-\beta)\, a_{it} \qquad (8) \\
&= \Theta\left(p^t, \bar{c}\right) + (1-\beta)\, a_{it},
\end{aligned}
$$

where

$$
\Theta\left(p^t, \bar{c}\right) \equiv \bar{c}\left[(1-\beta)\sum_{\tau=t}^{\infty}\left(\frac{p_\tau}{p_t}\right) - 1\right] \qquad (9)
$$

is a function of the subsistence level and of the whole price sequence $p^t = \{p_t, p_{t+1}, ...\}$. Thus, we have the optimal individual consumption rule

$$
c_{it} = \Theta\left(p^t, \bar{c}\right) + (1-\beta)\, a_{it}, \qquad (10)
$$

which is an *affine function* of asset holdings at time t for each type i. We know that (10) implies that we can Gorman-aggregate demand functions.

Even though we have only derived it for the log-case, it is easy to check that this representation of the consumption function holds also for the other two classes of preferences (power and exponential utility).

2.1.2 Equilibrium aggregate dynamics

Denote aggregate variables with capital letters. From (10), we derive easily that aggregate consumption only depends on aggregate variables (prices and aggregate wealth), but it is independent of the distribution of wealth. By summing over i on the LHS and RHS of (10) with weights μ_i we arrive at:

$$
C_t = \Theta\left(p^t, \bar{c}\right) + (1-\beta)\, A_t. \qquad (11)
$$

Since equilibrium aggregate consumption C_t has the same form as individual optimal consumption choice c_{it}, it is clear that (11) can be obtained as the solution to the following representative agent problem:

$$
\begin{aligned}
&\max_{\{C_\tau\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} u\left(C_\tau\right) \\
&s.t. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(PP)} \\
&\sum_{\tau=t}^{\infty} p_\tau C_\tau \leq p_t A_t
\end{aligned}
$$

which is exactly as in (5) but we have replaced small letters with capital letters. From the FOCs

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = \frac{p_t}{p_{t+1}}. \tag{12}$$

Let's make some further progress on the solution. To do so, we need to solve for the representative firm's problem.

Firm's problem– The representative firm owns physical capital and makes the investment decision by solving the problem

$$\begin{aligned} A_t &= \max_{\{I_\tau\}} \sum_{\tau=t}^{\infty} \left(\frac{p_\tau}{p_t}\right) [f(K_\tau) - I_\tau] \qquad (13) \\ &s.t. \\ K_{\tau+1} &= (1-\delta) K_\tau + I_\tau, \end{aligned}$$

where p_t is the time t price of the final good. Let's define real profits $\pi_t \equiv f(K_t) - I_t$. Then, it is easy to see that A_t is the value of the firm, i.e. the present value of future profits discounted at rate (p_τ/p_t), the relative price of consumption between time τ and time t. Recall that $p_t/p_{t+1} = (1 + r_{t+1})$ where r is the interest rate.

It is useful to compute the first-order condition (FOC) of the firm problem with respect to K_{t+1} by substituting the law of motion for capital into (13). The problem becomes:

$$\max_{K_{t+1}} \left\{ f(K_t) - K_{t+1} + (1-\delta) K_t + \frac{p_{t+1}}{p_t} [f(K_{t+1}) - K_{t+2} + (1-\delta) K_{t+1}] + \frac{p_{t+2}}{p_{t+1}} ... \right\}$$

with FOC:

$$1 = \frac{p_{t+1}}{p_t} [f'(K_{t+1}) + (1-\delta)] \tag{14}$$

which, incidentally, is exactly the same FOC of a "static" firm who rents capital from households at every date.

From (12) and the FOC for the firm's problem (14), we obtain the familiar Euler equation of the neoclassical growth model

$$u'(C_t) = \beta u'(C_{t+1}) [f'(K_{t+1}) + (1-\delta)]. \tag{15}$$

We can state our first important result:

Result 1.1: If preferences are quasi-homothetic, and agents are heterogeneous in initial endowments, the neoclassical growth model with complete markets admit a single-agent *representation. The dynamics of aggregate quantities and prices do not depend on the distribution of endowments: they are the same as in the standard neoclassical growth model with representative agent.*

Two remarks are in order. First, equation (15) governs the dynamics of capital in the representative agent growth model where firms rent capital from households, instead of owning it. Therefore, we have discovered that in complete markets it is irrelevant whether we attribute property rights on capital to firms (and let households own shares of the firms) or to workers (and let firms rent capital from households). Second, equation (15) also governs the dynamics of capital in the social-planner problem. We are still in complete markets, and the Welfare Theorems hold.

Steady-state– The dynamics of the economy will converge to the steady-state values of capital stock satisfying the modified golden rule $f'(K^*) = 1/\beta - (1-\delta)$. Note now that in steady-state $p_{t+1}/p_t = \beta$ for all t, hence from the definition of $\Theta(p^t, \bar{c})$ in (9) we conclude that $\Theta(p^t, \bar{c}) = 0$ and $c_i = (1-\beta) a_i$. In other words, in steady-state, the average propensity to save is β, independently of wealth, for every type of household.

To conclude, in the neoclassical growth model with complete markets and where agents have heterogeneous wealth endowments, the dynamics of the aggregate variables do not depend on the evolution of the wealth distribution. But is the inverse statement true? Does the evolution of the wealth distribution across households (i.e., wealth inequality) depend on the dynamics of aggregate variables (prices and quantities)? We show below that the answer is: yes, it does.

2.1.3 Equilibrium dynamics of the wealth distribution

From the lifetime budget constraint of agent i at time t

$$\begin{aligned} p_t c_{it} + \sum_{\tau=t+1}^{\infty} p_\tau c_{i\tau} &= p_t a_{it} && (16) \\ c_{it} + \left(\frac{p_{t+1}}{p_t}\right) a_{i,t+1} &= a_{it} && (17) \\ \frac{a_{i,t+1}}{a_{it}} &= \left(\frac{p_t}{p_{t+1}}\right)\left(1 - \frac{c_{it}}{a_{it}}\right), && (18) \end{aligned}$$

which expresses the growth rate of wealth for type i as a function of her consumption-wealth ratio.

By aggregating over types in equation (16), we can obtain an equivalent equation at the aggregate level:

$$\begin{aligned}
\sum_i \mu^i c_{it} + \left(\frac{p_{t+1}}{p_t}\right) \sum_i \mu^i a_{i,t+1} &= \sum_i \mu^i a_{it} \\
C_t + \left(\frac{p_{t+1}}{p_t}\right) A_{t+1} &= A_t \\
\frac{A_{t+1}}{A_t} &= \left(\frac{p_t}{p_{t+1}}\right)\left(1 - \frac{C_t}{A_t}\right)
\end{aligned}$$

We want to establish conditions under which an individual's share of total wealth will grow over time, i.e. $s_{i,t+1} > s_{it}$. First of all, note that:

$$\frac{s_{i,t+1}}{s_{i,t}} > 1 \quad \Leftrightarrow \quad \frac{a_{i,t+1}}{a_{it}} > \frac{A_{t+1}}{A_t} \quad \Leftrightarrow \frac{c_{it}}{a_{it}} < \frac{C_t}{A_t} \tag{19}$$

Moreover, from equations (10) and (11),

$$\frac{c_{it}}{a_{it}} = \frac{\Theta\left(p^t, \bar{c}\right)}{a_{it}} + (1-\beta), \quad \text{and} \quad \frac{C_t^i}{A_t} = \frac{\Theta\left(p^t, \bar{c}\right)}{A_t} + (1-\beta)$$

and therefore

$$\frac{c_{it}}{a_{it}} < \frac{C_t}{A_t} \quad \Leftrightarrow \quad \frac{\Theta\left(p^t, \bar{c}\right)}{a_{it}} < \frac{\Theta\left(p^t, \bar{c}\right)}{A_t} \quad \Leftrightarrow \quad \Theta\left(p^t, \bar{c}\right)\left(a_{it} - A_t\right) > 0$$

and, thus, summarizing we have the following equivalence (i.e., "if and only if") condition:

$$\frac{s_{i,t+1}}{s_{it}} > 1 \quad \Leftrightarrow \quad \Theta\left(p^t, \bar{c}\right)\left(a_{it} - A_t\right) > 0,$$

which means that whether consumer's i wealth share is increasing or decreasing over time depends on 1) the sign of the constant Θ (equal for everyone) and 2) on her relative position in the distribution. For example, if $\Theta > 0$ then for a consumer whose initial wealth is above average, her share will grow, whereas for a consumer whose initial wealth is below average, her share will fall. And hence the distribution will become more unequal over time. Note that the dynamics of the wealth distribution depend on the entire sequence of prices, hence on the dynamics of aggregate variables in equilibrium.

Two results are immediate. First, in steady-state, $\Theta\left(p^t, \bar{c}\right) = 0$ and $s_{i,t+1} = s_{it}$ since every agent has the same average propensity to save β. Similarly, even out of steay-state, in absence of subsistence level, $\bar{c} = 0$ we still have $\Theta = 0$, and the same prediction is true. Thus, the neoclassical growth model with heterogeneous endowments and homothetic preferences has a sharp prediction for the evolution of inequality.

Result 1.2: In the neoclassical growth model with complete markets, homothetic preferences, heterogeneous endowments, but without subsistence level ($\bar{c} = 0$)*, the wealth distribution remains unchanged along the transition path, i.e., initial conditions in endowments (and inequality) persist forever.*

The intuition is that if $\bar{c} = 0$ then the average propensity to consume, and save, is the same for every agent. Every agent accumulates wealth at the same rate.

In presence of a subsistence level, the dynamics are more interesting. We now determine the sign of Θ, through:

Lemma 1.1 (Chatterjee, 1994): $\Theta(p^t, \bar{c})$ *is greater (less) than zero if and only if the economy is converging from below (above) to the steady-state, i.e. if* $K_\tau < (>) K^*$*.*

Proof: *Suppose the economy grows towards the steady-state, i.e.* $K_\tau < K^*$*. Then the sequence* $\{f'(K_\tau)\}$ *is decreasing and, from equation* (15)*, the sequence* $\{p_{\tau+1}/p_\tau\}$ *is increasing towards* β*, i.e.,* $p_{\tau+1}/p_\tau \leq \beta$ *for all* $\tau \geq t$ *where the strict inequality holds at least for some* τ*. It follows that*

$$p_\tau/p_t = (p_\tau/p_{\tau-1})(p_{\tau-1}/p_{\tau-2}) \dots (p_{t+2}/p_{t+1})(p_{t+1}/p_t) < \beta^{\tau-t}.$$

From the definition of $\Theta(p^t, \bar{c})$ *in* (8)*, use the above equation to obtain*

$$\Theta\left(p^t, \bar{c}\right) = \bar{c}\left[(1-\beta)\sum_{\tau=t}^{\infty}\left(\frac{p_\tau}{p_t}\right) - 1\right] > \bar{c}\left[(1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t} - 1\right] = 0$$

where the inequality follows from $\bar{c} < 0$*.* **QED**

The implications for the evolution of the wealth distribution in an economy growing towards the steady-state (the empirically interesting case) are easy to determine, at this point. In the presence of a subsistence level ($\bar{c} < 0$), $\Theta > 0$. From equation (19) this implies that the average propensity to consume (save) declines (increases) with wealth: poor agents must consume proportionately more out of their wealth to satisfy the subsistence level, so wealth inequality increases along the transition. In other words:

Result 1.3: In the neoclassical growth model with complete markets, homothetic preferences, heterogeneous endowments and subsistence level $\bar{c} < 0$*, as the economy grows towards the steady-state: (i) the wealth distribution becomes more unequal, as rich agents accumulate more than poor agents along the transition path, and (ii) there is no change* *in the ranking of households in the wealth distribution, i.e., initial conditions in wealth (and consumption) ranking persist forever.*

A bit more intuition about this result. First, inequality grows along the transition when the economy is growing. If K_t grows, then r_t falls along the transition. This means that, as the economy converges, households have a temporarily high capital income, but consumption smoothing suggest this temporarily high income should be saved. Who is going to save the most? Those with high initial wealth, and hence high capital income. This is the key force that makes the wealrhy save even more. The opposite logic holds when the economy converges from above.

Second, to see clearly why the initial ranking of shouseholds does not change, recall that from (18) growth in individual wealth is:

$$\frac{a_{i,t+1}}{a_{it}} = \left(\frac{p_t}{p_{t+1}}\right)\left(1 - \frac{c_{it}}{a_{it}}\right) = \left(\frac{p_t}{p_{t+1}}\right)\left[\beta - \frac{\Theta\left(p^t, \bar{c}\right)}{a_{it}}\right]$$

therefore the growth rate of wealth is ordered by a_{it}.

The main conclusion is that in this model there is no economic or social mobility. This is not a good model to understand why some individual are born poor and make it in life, while other are born rich and end up poor as rats. This is just a model of castes. Finally note that absence of economic mobility is sort of implicit in the fact that the CE allocations can be obtained as the result of a planner's problem with fixed individual weights (see below).

Robustness– We now discuss how robust this result is to three of the key assumptions made so far in the analysis: 1) all agents have same discount factor β, 2) all agents are equally productive, 3) markets are complete.

1. When agents have different discount factors, then none of the results hold any longer. Suppose that $\bar{c} = 0$ to simplify the analysis. Then, from (10)

$$c_{it} = (1 - \beta_i)\, a_{it},$$

   therefore the average propensity to save out of wealth is higher the more patient is the individual and from (19), wealth grows faster for the more patient individuals. In the limit, in steady-state, the most patient type holds all the wealth, and the distribution becomes degenerate.

2. Caselli and Ventura (2000) extend the Gorman aggregation result to a version of the neoclassical growth model where agents also differ in their endowments of efficiency units of labor.

3. In absence of markets and trade (autarky), every consumer has access to her own technology. Each agent i will solve his own maximization problem in isolation

$$\begin{array}{l} \max_{\{c_{i\tau}\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} u\left(c_{i\tau}\right) \\ s.t. \\ k_{i,\tau+1} = (1-\delta) k_{i\tau} + f\left(k_{i\tau}\right) - c_{i\tau} \\ k_{it} \text{ given} \end{array}$$

with different initial conditions k_{it}. It is easy to see that, independently of initial conditions, each agent will converge to the same capital stock k^*, hence in the long-run the distribution of wealth is perfectly equal. Interestingly, we conclude that less developed financial markets induce less wealth inequality, in the long-run.

2.1.4 Indeterminacy of the wealth distribution in steady-state

One very important implication of the aggregation Result 1.1 is that in steady-state the wealth distribution is indeterminate. From (13), (15) and (10), the set of equations characterizing the steady-state is:

$$\begin{aligned} c_i &= (1-\beta) a_i,\ i = 1, 2, ..., N \\ f'\left(K^*\right) &= 1/\beta - (1-\delta), \\ A^* &= \frac{1}{1-\beta}\left[f\left(K^*\right) - \delta K^*\right] \\ \sum_{i=1}^{N} \mu_i a_i &= A^*, \end{aligned}$$

We therefore have $(N+3)$ equations and $(2N+2)$ unknowns $\left(\{c_i, a_i\}_{i=1}^{N}, K^*, A^*\right)$. In other words, the multiplicity of the steady-state wealth distributions is of order $N-1$.[4]

However, suppose we start from a given wealth distribution at date $t=0$ when the economy has not yet reached its steady-state, then the dynamics of the model are uniquely determined by Results 1.2 and 1.3 and the final steady-state distribution is determined as well. So, let's restate our finding in:

[4]This means that, if $N=1$ (representative agent), the steady-state is unique. If $N=2$, there is a continuum of steady-states of dimension 1, and so on.

Result 1.4: In the steady-state of the neoclassical growth model with N agents, heterogeneous initial endowments and homothetic preferences, there is a continuum with dimension $(N-1)$ of steady-state wealth distributions. However, given an initial wealth distribution $\{a_{i0}\}_{i=1}^{N}$ at $t=0$, the equilibrium wealth distribution $\{a_{it}\}_{i=1}^{N}$ in every period t is uniquely determined, and so is the final steady-state distribution.

Example– Consider an economy where $N=2$, where the production technology is $zf(K)$. Then, the set of steady-state equations is

$$\begin{aligned} c_i &= (1-\beta)\, a_i,\ i=1,2, \\ zf'(K^*) &= 1/\beta - (1-\delta), \\ A^* &= \frac{1}{1-\beta}\left[zf(K^*) - \delta K^*\right], \\ \mu_1 a_1 + \mu_2 a_2 &= A^* \end{aligned}$$

So we have 5 equations, but 6 unknowns $\{a_1, a_2, K^*, A^*, c_1, c_2\}$. We can represent graphically all the possible equilibrium paths between two steady states that differ for their level of technological progress z , say (z_L, z_H). The figure shows that the model has a continuum of steady-state distributions of wealth of dimension one, all consistent with the uniquely determined aggregate capital stock K^*. If we pick an initial distribution in the initial steady-state with productivity z_L, the equilibrium path to the final steady-state with productivity level z_H is uniquely determined.

Finally, in terms of language, this whole section shows that it is important to distinguish "steady-state" from "equilibrium path". In this economy, the equilibrium path is always unique (given initial conditions), but the steady-state is not.

3 The Negishi Approach

Negishi (1960) suggested a method to calculate the competitive equilibrium (CE) prices and allocations of complete markets economies (in particular, economies for which the first welfare theorem holds) with heterogeneous households. This method is particularly useful for those economies where Gorman aggregation does not go through and, hence we do not know how to write the preferences of the representative agent.[5]

[5] In his original paper, Negishi (1960) used this equivalence result to propose a simple way to show existence of competitive equilibria.

From the first welfare theorem, we know that any CE is a Pareto optimum (PO), hence it can be found as the solution to a social planner problem with "some" Pareto weights given to each agent. Suppose we want to compute a particular CE of an economy where agents are initially endowed with heterogeneous shares $\{s_{i0}\}_{i=1}^{N}$ of the aggregate wealth. Can we use the planner problem for this purpose? Negishi showed that the key is to search for the "right" weights given to each type of agent in the social welfare function of the planner. Each set of weights corresponds to a Pareto efficient allocation, the key is to find the set of weights which correspond to our desired CE in the original economy.[6]

3.1 An Example

Consider our neoclassical growth model of section (2.1) with two types of consumers ($N = 2$). The agent's i problem in the decentralized Arrow-Debreu equilibrium can be written as

$$\max_{\{c_{it}\}} \sum_{t=0}^{\infty} \beta^t u(c_{it})$$
$$s.t.$$
$$\sum_{t=0}^{\infty} p_t c_{it} \leq p_0 a_{i0}$$

where $a_{i0} = s_{i0} A_0$ is the initial wealth endowment, given at $t = 0$. Let's assign the property rights on capital to the firm, so the firm's problem is exactly the one of the previous section.

From the FOC of the agent of type i, we obtain

$$FOC(c_{it}) \longrightarrow \beta^t u'(c_{it}) = \lambda_i p_t,$$

where λ_i is the multiplier on the time zero budget constraint. Thus, putting together the FOC's for the two types:

$$\frac{u'(c_{1t})}{u'(c_{2t})} = \frac{\lambda_1}{\lambda_2}. \tag{20}$$

[6] Incidentally, the notion of social welfare was introduced by Samuelson (1956).

Now, write down the following Negishi planner problem (NP) for our economy

$$\max_{\{c_{1t},c_{2t},K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 u(c_{1t}) + \alpha_2 u(c_{2t})\right] \tag{NP}$$

$$s.t.$$

$$c_{1t} + c_{2t} + K_{t+1} \leq f(K_t) + (1-\delta) K_t$$

$$K_0 \text{ given}$$

where (α_1, α_2) are the planner's weights for each type of household in the social welfare function.

The FOC's for this problem are

$$FOC(c_{it}) \longrightarrow \alpha_i \beta^t u'(c_{it}) = \theta_t, \tag{21}$$

$$FOC(K_{t+1}) \longrightarrow u'(c_{it}) = \beta u'(c_{i,t+1}) \left[f'(K_{t+1}) + (1-\delta)\right] \tag{22}$$

where θ_t is the Lagrange multiplier on the planner's resource constraint at time t. Note that putting together the first-order conditions for consumption for the two agents we arrive at

$$\frac{u'(c_{1t})}{u'(c_{2t})} = \frac{\alpha_2}{\alpha_1}, \tag{23}$$

which tells us that the planner allocates consumption proportionately to the weight it gives to each consumer (with strictly concave utility).[7]

If we want the NP to deliver the same solution as the CE, we need the PO allocations and the CE allocations to be the same. Given strict concavity of preferences, this implies that, putting together (23) and (20):

$$\frac{\alpha_2}{\alpha_1} = \frac{\lambda_1}{\lambda_2}$$

Hence, the relative weights of the planner must correspond to the inverse of the ratio of the Lagrange multipliers on the time-zero Arrow-Debreu budget constraint for the two agents in the CE.

In particular, for log preferences $u(c_{it}) = \log c_{it}$, we derived in equation (7) that

$$\left(\frac{1}{\lambda^i}\right) = (1-\beta) p_0 a_{i0} \Rightarrow \lambda_i = \left[\frac{1}{(1-\beta) p_0 A_0}\right] \frac{1}{s_{i0}},$$

[7] Note also that the ratio of marginal utility across agents is kept constant in every period (a key features of complete markets allocations, also called *full insurance*).

therefore we obtain that

$$\frac{\alpha_2}{\alpha_1} = \frac{\lambda_1}{\lambda_2} = \frac{s_{20}}{s_{10}}.$$

Imposing the natural and innocuous normalization $\alpha_1 + \alpha_2 = 1$, we can solve explicitly for the two weights: $\alpha_1 = s_{10}$ and $\alpha_2 = s_{20}$, i.e., the weights are exactly equal to the initial wealth shares ("exactly" is because of log utility, in general weights will be proportional to the initial shares). The higher is the initial wealth share s_{i0}, the lower is the multiplier λ_i and the larger is the Pareto weight α_i on the Negishi problem: the planner must deliver more to consumption to the agent who has a large initial share of wealth in the decentralized equilibrium.

Result 1.5: Consider an economy with agents heterogeneous in endowments where the First Welfare Theorem holds. Then, the competitive equilibrium allocations can be computed through an appropriate planner's problem where the relative weights on each agent in the social welfare function are proportional to the relative individual endowments: those agents who initially have more wealth will get a higher weight in the planner's problem.

Now, note that using equation (21) we obtain that

$$\frac{\alpha_i \beta^t u'(c_{it})}{\alpha_i \beta^{t+1} u'(c_{i,t+1})} = \frac{\theta_t}{\theta_{t+1}} \Longrightarrow \frac{u'(c_{it})}{\beta u'(c_{i,t+1})} = \frac{\theta_t}{\theta_{t+1}}.$$

Substituting this last expression into (22), we arrive at a relationship between the sequence of capital stocks and the sequence of Lagrange multipliers on the planner's resource constraint

$$\frac{\theta_t}{\theta_{t+1}} = f'(K_{t+1}) + (1 - \delta).$$

Recall that equation (15) dictating the optimal choice of capital for the firm in the CE stated that

$$\frac{p_t}{p_{t+1}} = f'(K_{t+1}) + (1 - \delta).$$

Hence, we have

$$\frac{\theta_t}{\theta_{t+1}} = \frac{p_t}{p_{t+1}}, \tag{24}$$

in other words, the Arrow-Debreu equilibrium prices can be uncovered as the sequence of Lagrange multipliers in the Pareto problem: intuitively, the multipliers gives us the shadow value of an extra unit of consumption and, in the CE, prices signal exactly this type of scarcity.[8] Note that equation (24) is true for any well behaved u.

In conclusion, we have uncovered a tight relation between weights of the NP problem and initial endowments in the CE and an equivalence between Lagrange multiplier on the resource constraint of the NP problem and prices in the CE. This strict relationship, that we have uncovered for the log utility case, is true more in general.

3.2 General application of the Negishi method

In general, without specific restrictions on preferences (e.g., CRRA utility), one may not have closed form solutions for the λ's in the CE, so the algorithm is a little more involved. The objective is to compute the CE allocations for an economy with N types of agents and endowment distribution $\{ai_0\}_{i=1}^{N}$. We can describe the algorithm in four steps:

1. In the Negishi social planner problem (NP), guess a vector of weights $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, ..., \alpha_N\}$. The normalization $\sum_{i=1}^{N} \alpha_i = 1$ means that $\boldsymbol{\alpha}$ belongs to the N-dimensional simplex

$$\Delta_N = \{\boldsymbol{\alpha} \in \mathbb{R}_+^N : \sum_{i=1}^{N} \alpha_i = 1\}$$

   and the simplex traces out the entire set of Pareto-optimal allocations.

2. From the NP problem compute the sequence of allocations $\left\{\{c_{it}\}_{i=1}^{N}, K_t\right\}_{t=0}^{\infty}$ and the implied sequence of multipliers $\{\theta_t\}_{t=0}^{\infty}$ on the resource constraint in each period t. In practice, at every t, one needs to solve the $N+2$ equations

$$\begin{aligned} \alpha^i \beta^t u'(c_{it}) &= \theta_t,\ i = 1, .., N \\ \frac{\theta_t}{\theta_{t+1}} &= f'(K_{t+1}) + (1-\delta) \\ \sum_{i=1}^{N} \mu_i c_{it} + K_{t+1} &= f(K_t) + (1-\delta) K_t \end{aligned}$$

   in $N+2$ unknowns $\left(\{c_{it}\}_{i=1}^{N}, K_{t+1}, \theta_{t+1}\right)$. At every t, (K_t, θ_t) are given, therefore the Negishi method simplifies enormously the computation of the equilibrium: the

[8] Using p_0 as the numeraire and imposing the normalizations $p_0 = \theta_0 = 1$, the above relationship implies that $p_t = \theta_t$ so equilibrium prices are exactly equal to the shadow prices of consumption in the planner's problem.

Negishi solution requires solving, for every time t, a small simultaneous system of equations. Recall that, instead, to solve for the CE allocations, at every time t one must set the excess demand function to zero and the excess demand function depends on the entire price sequence—an infinitely dimensional object. To understand, take another look at the consumption allocation (10) where $\Theta(\cdot)$ depends on the entire price sequence from t onward.

Instead of guessing (and iterating over) an infinite sequence of prices, one guesses and iterates over a finite set of weights. With a caveat: even though K_0 is given, θ_0 is not. One has also to guess a value for θ_0. The reason is that, unless you have the right value for θ_0, the system will not be on the saddle-path and capital will diverge. In other words, there is another condition that we need to satisfy in the growth model, the transversality condition:

$$\lim_{t\to\infty} \beta^t u'(c_{it}) K_t = 0 \Longrightarrow \lim_{t\to\infty} \frac{1}{\alpha^i}\theta_t \cdot K_t = 0$$

which states that, in the limit, the marginal value of a unit of capital is zero.

3. Exploit the equivalence between prices p_t and multipliers θ_t to verify whether the time-zero Arrow-Debreu budget constraint of each agent holds exactly at the guessed vector of weight $\boldsymbol{\alpha}$. Specifically, for each agent, compute the implicit transfer function $\tau_i(\boldsymbol{\alpha})$ associated with the assumed vector of weights

$$\tau_i(\boldsymbol{\alpha}) = \sum_{t=0}^{\infty} \theta_t c_{it}(\alpha_i) - \theta_0 a_{i0}, \text{ for every } i = 1, 2, ..., N \tag{25}$$

and if (25) holds for agent i with a "greater (smaller) than" sign, it means that the planner is giving too much (little) weight to agent i. So, in the next iteration reduce (increase) the weight α_i given to agent i. Note one useful property of the transfer functions:

$$\sum_{i=1}^{N} \tau_i(\boldsymbol{\alpha}) = \sum_{i=1}^{N} \left[\sum_{t=0}^{\infty} \theta_t c_{it}(\alpha_i) - \theta_0 a_{i0} \right] = \sum_{t=0}^{\infty} p_t C_t - p_0 A_0 = 0$$

since the discounted present value of resources of the economy cannot be greater than its current wealth. Put differently, recall that from the representative firm problem

$$A_0 = \sum_{t=0}^{\infty} \left(\frac{p_t}{p_0}\right) [f(K_t) - I_t] = \sum_{t=0}^{\infty} \left(\frac{p_t}{p_0}\right) C_t$$

which establishes the same result. To conclude, the individual transfer functions $\tau_i(\boldsymbol{\alpha})$ must sum to zero.

4. Iterate over $\boldsymbol{\alpha}$ until you find the vector of weights $\boldsymbol{\alpha}^*$ that sets *every* individual transfer function $\tau_i(\boldsymbol{\alpha}^*)$ to zero. This vector corresponds to the PO allocations that are affordable by each agent in the CE, given their initial endowment, without the need for any transfer across-agents. Thus, we are computing exactly the CE associated to initial conditions $\{a_{i0}\}_{i=1}^N$.

See also Ljungqvist-Sargent, section 8.5.3, for a discussion of the Negishi algorithm.

4 Aggregation with complete markets

We just learned that the equilibrium allocations of a complete market economy with heterogeneity can be obtained as the solution of a Negishi planner problem. The Negishi approach can be used to prove a more general aggregation result for complete markets economies due to Constantinides (1982) and then refined by Ogaki (2003). Consider the economy described above where we know that the Welfare Theorems hold, and therefore we can solve for the equilibrium allocations through the following Negishi planner problem

$$\begin{aligned} & \max_{\{c_{it},K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{N} \alpha_i u(c_{it}) \\ & s.t. \\ K_{t+1} + C_t &= f(K_t) + (1-\delta) K_t \\ C_t &= \sum_{i=1}^{N} \mu_i c_{it} \end{aligned}$$

Now, split the problem in two stages. First, given a sequence of aggregate consumption $\{C_t\}_{t=0}^{\infty}$, consider the static problem of how to allocate C_t across agents at every t

$$\begin{aligned} U(C_t) &= \max_{\{c_{it}\}} \sum_{i=1}^{N} \alpha_i u(c_{it}) \\ & s.t. \\ C_t &= \sum_{i=1}^{N} \mu_i c_{it} \text{ for all } t \end{aligned}$$

where $U(C_t)$ is the indirect utility function of the planner at date t. Second, consider the consumption/investment problem

$$\begin{aligned} & \max_{\{C_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t) \\ & s.t. \\ K_{t+1} + C_t &= f(K_t) + (1-\delta) K_t \\ & K_0 \text{ given} \end{aligned}$$

This second-stage problem shows that the aggregate dynamics of the model can be described by the solution to the problem of a representative agent (RA), but the RA's preferences are different from preferences of the individual consumer, i.e., $U \neq u$. Indeed, one can even allow for different utility functions u^i across agents and show that this result still holds.

The Constantinides aggregation theorem is a very general existence (i.e., existence of a representative agent) result: note that the only restriction on preferences is strict concavity of u, but homotheticity or quasilinearity are not required. However, it requires complete markets, while Gorman aggregation does not. It also suggests an algorithm for constructing the preferences of the RA, but in most cases an analytical solution for U is not attainable, even though we know U exists. In what follows, we show an example with closed-form solution.

4.1 An Example

This example is taken from Maliar and Maliar (2001, 2003). In their model, agents have non-homothetic preferences in consumption and leisure, and are subject to idiosyncratic, but insurable, shocks to labor endowment (i.e., markets are still complete). They show how to recover preferences of a fictitious representative agent whose optimization problem yields, as a solution, the equilibrium aggregate dynamics of the original economy. This representative agent ends up having a *different* (but not so different) utility function from that of individuals populating the economy.

Demographics– The economy is inhabited by a continuum of infinitely lived agents, indexed by $i \in I \equiv [0,1]$. Denote by μ^i the measure of agents i in the set I and normalize the total number of agents to one, $\int_I d\mu^i = 1$, so that averages and aggregates are the same. Initial heterogeneity is in the dimension of initial wealth endowments.

Uncertainty– Agents are subject to idiosyncratic productivity shocks to skills. Let ε_t^i be the shock of agent i, and suppose shocks are iid, with mean 1, and defined over the set E. This is not necessary, but it simplifies the notation.

Preferences– Preferences are time separable, defined over streams of consumption, given by

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_{it}, 1 - h_t^i\right).$$

where period utility is given by

$$u\left(c_{it}, 1 - h_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \phi \frac{\left(1 - h_t\right)^{1-\sigma} - 1}{1-\sigma} \tag{26}$$

and note that preferences are not quasi-homothetic, unless $\sigma = \gamma$.

Markets and property rights– There are spot markets for the final good (which can be used for both consumption and investment) whose price is normalized to one, and complete financial markets, i.e. agents can trade a full set of state-contingent claims. The agent's portfolio is composed by Arrow securities of the type $a_{t+1}^i(\varepsilon)$ which pay one unit of consumption at time $t+1$ if the individual's shock is ε and zero otherwise. Let $p_t(\varepsilon)$ the price of this security and $\int_E p_t(\varepsilon)\, a_{t+1}^i(\varepsilon)\, d\varepsilon$ the value of such portfolio for agent i.

Technology and firm's problem– The aggregate production technology is $Y_t = Z_t f(K_t, N_t)$ with f strictly increasing and strictly concave in both arguments and differentiable. The representative firm rents capital from households. N_t is aggregate labor input in efficiency units, i.e. $N_t = \int_I \varepsilon_t^i h_t^i d\mu^i$.

Household problem– For agent i:

$$\begin{aligned}
& \max_{\left\{c_{it}, k_{t+1}^i, a_{t+1}(\varepsilon)\right\}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_{it}, 1 - h_t^i\right) && (27)\\
& s.t. \\
c_{it} + k_{t+1}^i + \int_E p_t(\varepsilon)\, a_{t+1}^i(\varepsilon)\, d\varepsilon \quad = \quad & (1-\delta)\, k_t^i + w_t \varepsilon_t^i h_t^i + a_t^i\left(\varepsilon_t^i\right) \\
& \left\{k_0^i, a_0^i\right\} \text{ given}
\end{aligned}$$

Equilibrium– This is a complete markets economy. The First Welfare Theorem tells us that the equilibrium is Pareto optimal, so we can use a social planner problem to characterize the equilibrium by applying the Negishi method. The key, as usual, is to find the right weights that guarantee that allocations are affordable for each agent, given their initial endowments.

Aggregation?– Given that preferences are not homothetic, we know that Gorman's strong aggregation concept will not hold. But can we, nevertheless, obtain a RA whose choices describe the evolution of the aggregate economy? And how the preferences of the RA look like?

Letting θ_t be the multiplier on the aggregate feasibility constraint, from the FOC with respect to individual i in the Negishi planner problem:

$$\begin{aligned} \alpha^i \left(c_{it}\right)^{-\gamma} &= \theta_t \\ \alpha^i \phi \left(1 - n_t^i\right)^{-\sigma} &= \theta_t w_t \varepsilon_t^i \end{aligned} \tag{28}$$

Rearranging gives

$$\begin{aligned} c_{it} &= \left(\frac{\alpha^i}{\theta_t^i}\right)^{\frac{1}{\gamma}} \\ \left(1 - h_t^i\right) \varepsilon_t^i &= \left(\frac{\phi \alpha^i}{\theta_t w_t}\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{1-1/\sigma} \end{aligned}$$

and note that consumption of individual i is proportional to its weight α^i in the social welfare function. Leisure is directly proportional to its weight (a wealth effect) and inversely proportional to individual productivity: efficiency arguments induce the planner to make high-productivity individuals work harder.

Integrating the two FOCs across agents gives

$$\begin{aligned} C_t &= \int_I c_{it} d\mu^i = \int_I \left(\frac{\alpha^i}{\theta_t}\right)^{\frac{1}{\gamma}} d\mu^i \\ 1 - N_t &= 1 - \int_I \varepsilon_t^i h_t^i d\mu^i = 1 - \int_I \left(\frac{\phi \alpha^i}{\theta_t w_t}\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{1-1/\sigma} d\mu^i \end{aligned} \tag{29}$$

Now, note that

$$c_{it} = \frac{\left(\frac{\alpha^i}{\theta_t}\right)^{\frac{1}{\gamma}}}{\int_I \left(\frac{\alpha^i}{\theta_t}\right)^{\frac{1}{\gamma}} d\mu^i} C_t \Rightarrow c_{it} = \frac{\left(\alpha^i\right)^{\frac{1}{\gamma}}}{\int_I \left(\alpha^i\right)^{\frac{1}{\gamma}} d\mu^i} C_t \tag{30}$$

$$\left(1 - h_t^i\right) \varepsilon_t^i = \left(\frac{\phi \alpha^i}{\theta_t w_t}\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{1-1/\sigma} \Rightarrow \left(1 - h_t^i\right) = \left(\frac{\phi \alpha^i}{\theta_t w_t}\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{-1/\sigma}$$

$$\Rightarrow \left(1 - h_t^i\right) = \frac{\left(\alpha^i\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{-1/\sigma}}{\int_I \left(\alpha^i\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{1-1/\sigma} d\mu^i} \left(1 - N_t\right)$$

Now, consider the social welfare function for the Negishi planner who is using weights $\{\alpha^i\}$

$$\int_I \left[\frac{\left(c_{it}\right)^{1-\gamma} - 1}{1-\gamma} + \phi \frac{\left(1 - h_t^i\right)^{1-\sigma} - 1}{1-\sigma}\right] \alpha^i d\mu^i$$

and substitute the two expressions in (30) into the social welfare function:

$$\int_I \alpha^i \frac{\left[\frac{\left(\alpha^i\right)^{\frac{1}{\gamma}}}{\int_I \left(\alpha^i\right)^{\frac{1}{\gamma}} d\mu^i} C_t\right]^{1-\gamma} - 1}{1-\gamma} d\mu^i + \phi \int_I \alpha^i \frac{\left[\frac{\left(\alpha^i\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{-1/\sigma}}{\int_I \left(\alpha^i\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{1-1/\sigma} d\mu^i} \left(1 - N_t\right)\right]^{1-\sigma} - 1}{1-\sigma} d\mu^i$$

$$= \frac{\frac{\int_I \alpha^i \left(\alpha^i\right)^{\frac{1-\gamma}{\gamma}} d\mu^i}{\left[\int_I \left(\alpha^i\right)^{\frac{1}{\gamma}} d\mu^i\right]^{1-\gamma}} C_t^{1-\gamma} - 1}{1-\gamma} + \phi \frac{\frac{\int_I \alpha^i \left(\alpha^i\right)^{\frac{1-\sigma}{\sigma}} \left(\varepsilon_t^i\right)^{1-1/\sigma} d\mu^i}{\left[\int_I \left(\alpha^i\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{1-1/\sigma} d\mu^i\right]^{1-\sigma}} \left(1 - N_t\right)^{1-\sigma} - 1}{1-\sigma}$$

which yields the utility for the RA

$$\frac{C_t^{1-\gamma} - 1}{1-\gamma} + \phi \Phi \frac{\left(1 - N_t\right)^{1-\sigma} - 1}{1-\sigma}$$

where

$$\Phi = \frac{\left[\int_I \left(\alpha^i\right)^{\frac{1}{\sigma}} \left(\varepsilon_t^i\right)^{1-1/\sigma} d\mu^i\right]^{\sigma}}{\left[\int_I \left(\alpha^i\right)^{\frac{1}{\gamma}} d\mu^i\right]^{\gamma}}$$

is independent of t because shocks are iid. Therefore the RA problem which describes the aggregate allocations for this economy becomes:

$$\max_{\{C_t, H_t, K_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} + \phi \Phi \frac{\left(1 - N_t\right)^{1-\sigma} - 1}{1-\sigma} \tag{31}$$

$$s.t.$$

$$C_t + K_{t+1} = \left(1 - \delta\right) K_t + Z_t f\left(K_t, N_t\right)$$

$$K_0 \text{ given}$$

Some remarks are in order:

1. We have found a RA, but its preferences are not the ones of the individual agent. Note that preferences of the RA depend on N_t (aggregate efficiency-weighted hours) instead of H_t (aggregate hours), and note that they feature a new weight on leisure Φ. This is the first reason why this is not a Gorman aggregation type of result.

2. The preference shifter Φ, in general, *depends on the distribution of shocks and the initial distribution of endowments*, therefore aggregate quantities do depend on the distribution of exogenous shocks and initial wealth (but not on the time-varying distribution of wealth). This is the second reason why this is, technically, not Gorman's aggregation. Note, however, that because these distributions are exogenous, it is a simple problem. In particular, in the case of no initial wealth heterogeneity $\alpha^i = 1$ for all i, $\Phi = \left[\int_I (\varepsilon_t^i)^{1-1/\sigma} d\mu^i\right]^\sigma$. Suppose that log $\varepsilon \sim N(-v_\varepsilon/2, v_\varepsilon)$, then

$$\Phi = \exp\left(\sigma\left(\frac{\sigma-1}{\sigma}\right)\left(\frac{\sigma-1}{\sigma}-1\right)\frac{v_\varepsilon}{2}\right) = \exp\left(\frac{1-\sigma}{\sigma}\frac{v_\varepsilon}{2}\right)$$

   which shows how the variance of the shocks affects the taste for leisure of the fictitious representative agent.

3. Suppose $\gamma = \sigma$. Then utility is quasi-homothetic. If there are no skill shocks, but only differences in endowments, then $\Phi = 1$ and $H_t = N_t$ and the utility of the representative agent is the same as the individual agent. We are back to Gorman's aggregation. If there are idiosyncratic shocks, then $\Phi \neq 1$ and Gorman aggregation fails, which establishes that Gorman aggregation holds only if the unique source of heterogeneity across agents is in initial wealth.

4. Suppose $\gamma \neq \sigma$. Then utility is not quasi-homothetic. Even if there are no skill shocks, but only differences in endowments, then Φ depends on the distribution of endowments and Gorman's aggregation fails.

Finally, note that the assumption that agents can trade a full set of claims contingent on all possible realizations of idiosyncratic labor productivity shocks is not very realistic, as we will argue later in the course.