1 Bewley Economies with Aggregate Uncertainty

So far we have assumed away aggregate fluctuations (i.e., business cycles) in our description of the incomplete-markets economies with uninsurable idiosyncratic risk à la Bewley-Aiyagari. In this section, the objective is to combine aggregate and idiosyncratic risk into an equilibrium model.

The good news is that we can use the recursive language to do this. The bad news is that solving exactly for the equilibrium allocations of this economy is *impossible*. The reason is that the measure of agents across states becomes an aggregate state of the economy, since households need to know it to forecast future prices. Future prices are a function of the future capital stock that, in turn, is the aggregation of individual saving decisions under the equilibrium distribution. However, a distribution is an infinitely dimensional object: how do we keep track of such a monster? The answer is that we will approximate the exact equilibrium and argue that the approximation is very good for standard parameterizations.

Aggregate and Idiosyncratic Risk– We introduce aggregate fluctuations through an aggregate productivity shock z that shifts the production function, i.e.

$$Y = zF(K, H),$$

and assume that the aggregate shock can take only two values, $z \in Z = \{z_b, z_g\}$ with $z_b < z_g$. To keep things simple, we also assume only two values for the individual productivity shock, $\varepsilon \in E = \{\varepsilon_b, \varepsilon_g\}$ with $\varepsilon_b < \varepsilon_g$. For example, if $\varepsilon_b = 0$, then it's as if the worker is unemployed for a period.

Let

$$\pi(z', \varepsilon' | z, \varepsilon) = \Pr(z_{t+1} = z', \varepsilon_{t+1} = \varepsilon' | z_t = z, \varepsilon_t = \varepsilon)$$

be the Markov chain that describes the joint evolution of the exogenous shocks. This notation allows the transition probabilities for ε to depend on z (the dependence of z on ε is also allowed in principle but does not make sense!). For example, one should expect that

$$\pi(z_g, \varepsilon_g | z_b, \varepsilon_b) > \pi(z_b, \varepsilon_g | z_b, \varepsilon_b), \text{ and } \pi(z_b, \varepsilon_b | z_g, \varepsilon_g) > \pi(z_g, \varepsilon_b | z_g, \varepsilon_g)$$

i.e. finding a job is easier if the economy is exiting from a recession, and losing a job is more likely when the economy is entering a recession.

State variables– The two individual states are $(a, \varepsilon) \in S$ and the two aggregate states are $(z, \lambda) \in Z \times \Lambda$ where $\lambda(a, \varepsilon)$ is the measure of households across states. The individual states are directly budget relevant, whereas the aggregate states are needed to compute and forecast prices.[1]

Household Problem– The household problem can be written in recursive form as:

$$\begin{aligned} v(a, \varepsilon; z, \lambda) &= \max_{c,a'} \left\{ u(c) + \beta \sum_{\varepsilon' \in E, z' \in Z} v(a', \varepsilon'; z', \lambda')\pi(z', \varepsilon' | z, \varepsilon) \right\} \quad (1) \\ & \quad s.t. \\ c + a' &= w(z, K(\lambda))\varepsilon + R(z, K(\lambda)) a \\ a' &\geq 0 \\ \lambda' &= \Psi(z, \lambda, z') \end{aligned}$$

where $\Psi(z, \lambda, z')$ is the law of motion of the endogenous aggregate state, and depends on z'. This dependence is inherited from π since the fraction of agents with $\varepsilon' = \varepsilon_b$ and $\varepsilon' = \varepsilon_g$ next period, given that the current aggregate productivity level is z, depends on z'.

The key complication is that the value function v depends on λ which is a distribution. Where is this dependence coming from? To solve their problem, households need to compute current prices and, most importantly, forecast prices next period. Prices depend on aggregate capital, and aggregate capital this period and next period, K and K', depends on how assets are distributed in the population, through λ, because in equilibrium

$$K = \int_{A \times E} a d\lambda \text{ and } K' = \int_{A \times E} a' d\lambda$$

Let's be more specific about why agents need to know λ. Consider the Euler equation associated to the problem above. Let the saving policy be denoted by g, and drop the depndence of K and K' on λ to ease the notation:

$$u_c(R(z, K) a + w(z, K)\varepsilon - g(a, \varepsilon; z, K)) \geq$$
$$\beta E[R(z', K') u_c(R(z', K') g(a, \varepsilon; z, K) + w(z', K')\varepsilon' - g(g(a, \varepsilon; z, K), \varepsilon'; z', K'))].$$

It is clear that to solve for g, households need to forecast prices next period, and next period prices depend on K' which, in turn, depends on λ. Since λ is a state variable,

[1] Note that, although it seems reasonable that λ is enough to complete the description of the state (i.e. a recursive equilibrium exists), this is not obvious at all and there are counterexamples of economies for which one needs to keep track of a longer history of distributions.

agents need to know its equilibrium law of motion Ψ which is a complicated mapping of distributions into distributions. A note: prices depend on the K/H ratio, not just on K, but the dynamics of H can be perfectly forecasted, conditional on z', through π because labor supply is exogenous. H could be time varying, but we know how to forecats it.

A **Recursive Competitive Equilibrium** for this economy is a value function v; decision rules for the household a', and c; choice functions firm H and K; pricing functions r and w; and, a law of motion Ψ such that:

- given the pricing functions $r(z,K)$ and $w(z,K)$ and the law of motion Ψ, the decision rules a' and c solve the household's problem (1) and v is the associated value function,
- given the pricing functions $r(z,K)$ and $w(z,K)$, the firm chooses optimally its capital K and its labor H, i.e.

$$\begin{aligned} r(z,K)+\delta &= zF_K(K,H), \\ w(z,K) &= zF_H(K,H), \end{aligned} \tag{2}$$

- the labor market clears: $H=\int_{A\times E}\varepsilon d\lambda$,
- the asset market clears: $K=\int_{A\times E} a d\lambda$,
- the goods market clears:

$$\int_{A\times E} c(a,\varepsilon;z,\lambda)d\lambda+\int_{A\times E} a'(a,\varepsilon;z,\lambda)d\lambda = zF(K,H)+(1-\delta)K,$$

- For every pair (z,z'), the aggregate law of motion Ψ is generated by the exogenous Markov chain π and the policy function a' as follows:

$$\lambda'(\mathcal{A}\times\mathcal{E})=\Psi_{(\mathcal{A}\times\mathcal{E})}(z,\lambda,z')=\int_{A\times E} Q_{z,z'}((a,\varepsilon),\mathcal{A}\times\mathcal{E})\,d\lambda, \tag{3}$$

  where $Q_{z,z'}$ is the transition function between two periods where the aggregate shock goes from z to z' and is defined by

$$Q_{z,z'}((a,\varepsilon),\mathcal{A}\times\mathcal{E})=I_{\{g(a,\varepsilon;z,\lambda)\in\mathcal{A}\}}\sum_{\varepsilon'\in\mathcal{E}}\pi_\varepsilon(\varepsilon'|z,\varepsilon,z'), \tag{4}$$

  where I is the indicator function, $g(a,\varepsilon;z,\lambda)$ is the optimal saving policy, and $\pi_\varepsilon(\varepsilon'|z,\varepsilon,z')$ is the conditional transition probability for ε which can be easily derived from π.

1.1 Computation of an Approximate Equilibrium

The state space of the problem of the household is, technically, infinite-dimensional because it contains a distribution. The problem is to find an efficient way to compute the law of motion

$$\lambda' = \Psi(z, \lambda, z').$$

Krusell and Smith (1998) contains the insight that, since we cannot work with an infinitely dimensional distribution, we need to approximate the distribution with a finite-dimensional object. Any distribution can be represented by its entire (in general, infinite) set of moments. Let $\bar{m}$ be a M dimensional vector of the first M moments (mean, variance, skewness, kurtosis,...) of the *wealth distribution*, i.e., the marginal of λ with respect to a. Our new state is exactly the vector $\bar{m} = \{m_1, m_2, ..., m_M\}$ with law of motion

$$\bar{m}' = \Psi_M(z, \bar{m}) = \begin{cases} m_1' = \psi_1(z, \bar{m}) \\ \quad\cdots \\ m_M' = \psi_M(z, \bar{m}) \end{cases}. \tag{5}$$

Note that we lost the dependence on z' since we are only interested in the wealth distribution. The law of motion for the marginal wealth distribution does not depend on z' since capital is pre-determined.

This method is based on the idea that households have *partial information* about λ. They don't know every detail about that measure, but only a set of moments, e.g. its mean, its variance, the Gini coefficient, the share held by the top 5% and so on. Hence, they use these M statistics to approximate the true distribution and form forecasts.

To make this approach operational, one needs to: 1) fix M and 2) specify a functional form for Ψ_M. Krusell and Smith (and this is their main finding) show that one obtains an excellent forecasting rule by simply setting $M = 1$ and by specifying a law of motion of the form:

$$\ln K' = b_z^0 + b_z^1 \ln K,$$

where only the first moment $m^1 = K$ would matter to predict the first moment next period.

The new partial-information problem of the agent becomes

$$
\begin{aligned}
v(a,\varepsilon;z,K) &= \max_{c,a'} \left\{ u(c) + \beta \sum_{\varepsilon',z'} v(a',\varepsilon';z',K')\pi(z',\varepsilon'|z,\varepsilon) \right\} \qquad (6) \\
& \quad s.t. \\
c + a' &= w(z,K)\varepsilon + R(z,K)a \\
a' &\geq 0 \\
\ln K' &= b_z^0 + b_z^1 \ln K.
\end{aligned}
$$

Note that this state space is definitely manageable: we collapsed an infinitely dimensional distribution λ into one variable, K. Now, we also know how to solve this problem. It will be a fixed-point algorithm over the law of motion (i.e., a function) for K: recall that in equilibrium the law of motion used by the agents has to be consistent with the aggregation of the optimal individual decisions ("aggregate consistency" of rational expectation equilibrium).[2]

1.1.1 Algorithm

The algorithm to solve this problem (and the associated equilibrium) is the following:

1. Guess the coefficients of the law of motion $\{b_z^0, b_z^1\}$

2. Solve the household problem and obtain the decision rules $a'(a,\varepsilon;z,K), c(a,\varepsilon;z,K)$. Note that with the law of motion for K in hand, we have all we need to solve for decision rules. For example, if we iterate on the Euler equation, then we need to compute the rule $a' = g(a,\varepsilon;z,K)$ that solves

$$
\begin{gathered}
u_c(R(z,K)a + w(z,K)\varepsilon - g(a,\varepsilon;z,K)) \geq \\
\beta E\{R(z',K')u_c(R(z',K')g(a,\varepsilon;z,K) + w(z',K')\varepsilon' - g(a',\varepsilon';z',K'))\}.
\end{gathered}
$$

   Thus we can use standard methods to obtain $g(\cdot)$.

3. Simulate the economy for N individuals and T periods. For example, $N = 10,000$ and $T = 2,000$. Draw first a random sequence for the aggregate shocks. Next one

[2] Recall that to solve for the stationary equilibrium, since the distribution is time-invariant, we guess only one value for the capital stock (or the interest rate). To compute the equilibrium transitional dynamics, we guess a deterministic sequence of capital stocks. With aggregate uncertainty, we need to guess a law of motion for the aggregate capital stock.

for the individual productivity shocks for each $i = 1, ..N$, conditional on the time-path for the aggregate shocks. Use the decision rules to generate sequences of asset holdings $\{a_t^i\}_{t=1,i=1}^{T,N}$ and in each period compute the average capital stock

$$A_t = \frac{1}{N}\sum_{i=1}^{N} a_t^i.$$

4. Discard the first T^0 periods (e.g. $T^0 = 500$) to avoid dependence from the initial conditions. Using the remaining sequence, run the regression

$$\ln A_{t+1} = \beta_z^0 + \beta_z^1 \ln A_t \tag{7}$$

and estimate the coefficients $\left(\beta_z^0, \beta_z^1\right)$. Note that this step requires running two regressions, one for each state z. Since the law of motion is time-invariant, we can separate the dates t in the sample where the state is z_b from those where the state is z_g and record (A_t, A_{t+1}) and run the regressions on these two separate samples.

5. If $\left(\beta_z^0, \beta_z^1\right) \neq (b_z^0, b_z^1)$, then try a new guess and go back to step 1. If the two pairs are equal for each $z \in \{z_g, z_b\}$, then it means that the approximate law of motion used by the agents is consistent with the one generated in equilibrium by aggregating individual choices. Notice that, once you reached the fixed point, you are sure that market clearing in the asset market holds: next period prices are determined by K' induced by the law of motion and this law of motion induces individual choices a' that aggregate into $A' = K'$.

6. Recall that this equilibrium computation is approximate: we still need to verify how good this approximation is to the fully rational-expectation equilibrium. For this purpose, compute a measure the fit of the regression in step 4), for example by using R^2. Next, try augmenting the state space with another moment, for example using $m^2 = E\left(a_i^2\right)$. Repeat steps 1)-5) until convergence. If the R^2 of the new equation (7) has improved significantly, keep adding moments until R^2 is large and does not respond to addition of new explanatory moments. Otherwise, stop: it means that additional moments do not add new useful information in forecasting prices.

1.2 A Near-Aggregation Result in the Krusell-Smith Economy

Krusell and Smith's main finding is that a law of motion based only on the mean, i.e.,

$$\ln K' = \begin{cases} 0.095 + 0.962 \ln K, & \text{for } z = z_g \\ 0.085 + 0.965 \ln K, & \text{for } z = z_b \end{cases}$$

delivers an $R^2 = 0.999998$ which means that the agents with this simple forecasting rule make very small errors, for example the maximal error in forecasting the interest rate 25 years into the future is around 0.1%. This result is called *near-aggregation* in the sense that in equilibrium, the evolution of aggregate quantities and prices does not depend on the distribution but, approximately, depends only on the aggregate shock and aggregate capital. Hence, it is almost like in a complete-markets economy, where aggregation of heterogenous individuals holds perfectly.

What is the intuition for the fact that keeping track of the mean of the distribution of assets is enough? Recall that if policy functions are linear, i.e.,

$$a'(a, \varepsilon, z, \lambda) = b_z^0 + b_z^1 a + b_z^2 \varepsilon,$$

then

$$K' = \int_{A \times E} a'(a, \varepsilon, z, \lambda) d\lambda = b_z^0 + b_z^1 K + b_z^2 H_z = \tilde{b}_z^0 + b_z^1 K$$

which would explain why the mean is a sufficient statistic. But saving functions are in general, not linear with uninsurable idiosyncratic shocks. They're exactly linear only with complete markets (recall the exact aggregation result of the Chatterjee economy with homothetic preferences?), where we showed that the distribution does not affect the dynamics of aggregate variables.

So, why do we get *near-aggregation* in practice? For three reasons. First, the saving functions $a'(a, \varepsilon, z, \lambda)$ for this class of problems usually display lots of curvature for low levels of ε and low levels of assets a, but beyond this region they're *almost linear*. Second, the agents with this high curvature are few and have low wealth, so they matter very little in determining aggregate wealth. What matters for the determination of the aggregate capital stock are the ones who hold a lot of capital, i.e., the rich, not the poor! Third, aggregate productivity shocks move the asset distribution only very slightly, and the mass of the distribution is always where the saving functions are linear.

But why do agents have linear saving functions, i.e. a constant marginal propensity to save out of wealth, for a very wide range of the asset space? After all, if they save for precautionary reasons (as they do in these economies) they should do so more when they hold few assets and less when they hold large assets, so the saving function should be nonlinear. The answer is that most of the consumers in this economy can smooth consumption very effectively through self-insurance, by cumulating a relatively small amount of wealth. Thus their saving behavior is guided mostly by their intertemporal motive rather than their insurance motive, like in complete markets.

To understand why the risk-free asset is such a good vehicle of self-insurance in the Krusell-Smith model, we discuss here two theoretical results in the literature that can help explain it.

First, Yaari (1976) analyzed the optimal consumption path of a perfectly impatient household ($\beta = 1$) with general concave preferences (hence with prudence) who lives for T periods, faces *iid* endowment shocks and saves and borrows at rate $r = 0$. Yaari shows that as $T \to \infty$, the optimal consumption plan converges to that of a consumer who eats a constant fraction of his wealth every period. In this sense, these households behave like certainty-equivalent consumers who are not concerned about future risk. Recall that an agent with quadratic preferences has consumption determined by

$$c_t = \frac{r}{1+r}\left[a_t + E_0 \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right] = \frac{r}{1+r} a_t + H(\mathbf{y},r),$$

where $\mathbf{y}$ is its whole future income sequence. His assets next period are determined by

$$\begin{aligned} a_{t+1} &= (1+r)(a_t - c_t + y_t) \\ &= (1+r)\left[a_t - \frac{r}{1+r} a_t - H(\mathbf{y},r)\right] + (1+r) y_t \\ &= a_t + \tilde{H}(\mathbf{y}, r) \end{aligned}$$

and note that the coefficient on past wealth is exactly one, which is very close to the one computed by Krusell-Smith.

Second, Levine and Zame (2001) analyze an economy populated by infinitely-lived consumers with standard preferences satisfying $u''' > 0$ who face stationary individual endowment shocks (i.e., not random-walk) and trade a risk-free asset in zero net supply. They prove that, as $\beta \to 1$ and the individuals become perfectly patient, "market incompleteness will not matter" in the sense that the welfare of the optimal consumption plan in this economy tends to the welfare of a complete markets economy where every agent consumes her average endowment every period. In other words, a great deal of risk-sharing may take place even in absence of a complicated structure of financial markets.[3]

Finally, at this point it is not surprising that Krusell and Smith find that the cyclical properties of aggregates in their model economy (i.e. volatility of output, consumption, investment, cross-correlations, etc...) are very similar to those of the standard representative agent model.

References

[1] Krusell P., and A. Smith (1998);. "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, vol. 106(5), 867-896.

[2] Yaari, M. (1976); "A Law of Large Numbers in the Theory of Consumer's Choice under Uncertainty," *Journal of Economic Theory*, 12, 202-217.

[3] Levine, D., and W. Zame (2001); "Does Market Incompleteness Matter?," *Econometrica*, vol. 70(5), 1085-1839.

[3] Note however, that Levine and Zame show that their result holds in the presence of aggregate uncertainty only if markets are complete with respect to aggregate risk. In the Krusell-Smith economy there is no insurance against aggregate risk, but aggregate fluctuations are quantitatively small.