An Introduction to Macroeconomics with Household Heterogeneity

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Part I

Introduction

Chapter 1

Overview over the Monograph

This monograph is meant to be an introduction into the research field of macroeconomics with household heterogeneity. It is concerned with constructing, computing and empirically evaluating dynamic stochastic macroeconomic models in which individual households differ according their abilities, wages, incomes, preferences or market opportunities in a way such that the household sector cannot be aggregated into a representative agent. In these models, therefore, aggregate allocations and prices will depend on the cross-sectional household distribution of household characteristics. We will argue below that the macroeconomic implications of these models differ, often (but not always) in a quantitatively significant way, from their representative agent counterpart. In addition, these models are in principle suitable to ask and answer positive and normative distributional questions about which the representative household paradigm is silent by construction.

In the next chapter we will briefly summarize selected empirical observations on wages, earnings, income, consumption and wealth from cross-sectional, household level data sets that have motivated this literature, and briefly describe the data sources from which these observations are drawn. We will also document some empirical puzzles that we aim at explaining with the models to be developed below. Note that to call an empirical finding a puzzle requires to take a stand on the standard economic theory relative to which the finding is puzzling; on of the goals of the course is to develop extensions or, if needed, radical departures, of standard theory to explain the empirical puzzles. The remainder of the monograph is then devoted to the construction, analysis and applications of theoretical models aimed at explaining these facts that also can be used for applications (e.g. to the

positive or normative analysis of fiscal policy).

In order to provide a taxonomy of the models discussed in this monograph, consider a world in which an infinitely-lived households is faced with income risk and aims at maximizing expected lifetime utility. Let $s \in S$ denote a state of nature, where S is a finite set. For ease of exposition[1] let the shock s be *iid* over time, with probabilities $\pi(s)$. Household income is given by $y(s)$. Households desire smooth consumption in the presence of stochastic income fluctuations, and the key distinguishing feature of the models discussed in this monograph is the set of financial assets that households have access to, and the extent to which they can go short in these assets. On one extreme, households have no access to any formal or informal ways to smooth income fluctuations, and thus their consumption fully inherits the stochastic properties of the income process. I will call these households hand-to-mouth consumers, and the associated (non-) market structure financial autarchy.

On the other extreme households may be able to trade a full set of state-contingent short-term bonds, so that their budget constraint reads as

$$c + \sum_{s' \in S} q(s')a'(s') \leq y(s) + a(s) \tag{1.1}$$

where c is current consumption, $y(s)$ is current income, $a'(s')$ is the number of bonds bought today that pay of one unit of consumption in state s' tomorrow, and $q(s')$ is its price. If households face no other trading constraints[2] I will call the resulting model the standard complete markets model. The next chapter is devoted to the review of its theoretical predictions and empirical tests. With this market structure (and under suitable assumptions on household preferences) household consumption is fully insured against household-specific (idiosyncratic) income shocks and the household sector can be represented by a representative consumer. As a result, with this market structure macroeconomics can proceed by ignoring underlying household heterogeneity induced by income shocks. This model therefore serves as an important point of departure and comparison for models in which household heterogeneity does matter for macroeconomic outcomes.

The remainder of this monograph is then devoted to models whose market structure lies in between those of financial autarchy and complete markets,

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[1]In this introduction, as in most of this monograph, my focus is not on utmost generality, but rather on clarity of exposition without (hopefully) compromising on rigor.

[2]One obviously needs some constraints on short-sales to rule out Ponzi schemes the presence of which shall always be assumed in this monograph even if not explicitky stated.

and in which households will be able to partially, but not fully (self-)insure against random income fluctuations. In part II I will discuss the perhaps most important workhorse model in this literature. In the standard incomplete markets model (SIM) households have access only to a single, one period risk-free bond, so that their period budget constraint reads as

$$c + qa' \leq y(s) + a \tag{1.2}$$

and the shortsale of bonds a' might be limited by a constraint of the form $a' \geq -\bar{A}$. In chapter 5 I will discuss the theoretical properties of various versions (with alternating assumptions on the household utility function, the stochastic nature of the income process, and the tightness of the borrowing constraint) of the model in which the price of the bond q and thus the interest rate $1 + r = 1/q$ is exogenously given. (and as a consequence we can analyze the behavior of one household in isolation). After a brief digression in chapter 6 that discusses the properties of the main driving of this class of models, the stochastic process for earnings, I will then incorporate the decision problems from chapter 5 into a dynamic general equilibrium model in which interest rates and wages are determined endogenously in the labor and capital market. This is done in chapter 7. There, individual households' consumption and saving decisions are aggregated to obtain aggregate labor and capital supply, firms' decisions are aggregated to obtain aggregate labor and capital demand, and wages and interest rates move to clear both markets. Depending on whether individual uncertainty averages out in the aggregate (no aggregate uncertainty) wages and interest are constant over time or are themselves stochastic processes (presence of aggregate uncertainty), leading to severe computational problems when computing these models. The aggregation of individual decisions also leads to (possibly time-varying) cross-sectional consumption and wealth distributions; thus these models are possibly useful for the study of the effects of redistributive and social insurance policies such as progressive taxation, unemployment insurance, welfare or social security.

Common to all these models is the assumption of the absence of explicit insurance arrangements in an environment in which mutual insurance is potentially quite beneficial. In part III I will discuss a strand of the literature that aims to explain the stylized facts from chapter 3with models that depart directly from the complete markets model, without a priori ruling out explicit insurance contracts (as the standard incomplete market model does). Chapter 8 discusses models in which imperfect consumption insurance arises due to the assumed imperfect enforceability of insurance contracts. If households

are given the option to default, with the punishment, say, of being expelled into financial autarchy forever after (and thus becoming the hand-to-mouth consumers discussed above), full consumption insurance might not be incentive compatible. I will argue that this results in a model in which households face a budget constraint of the form (1.1), exactly as in the standard complete market model, but now in addition face state-contingent shortsale constraints on the state contingent bonds $a'(s') \geq \bar{A}(s')$ that are potentially very tight and do not permit the implementation of the full consumption insurance allocation. Finally I will briefly discuss, in chapter 9 models in which perfect consumption insurance might be impossible to be implemented since individual incomes are not publicly observable. Consequently a mechanism designer of consumption insurance contracts will have to respect the incentives of households to mis-report their incomes and to construct a partial-insurance consumption insurance contract that induces all households to report their incomes truthfully.

A short, necessarily subjective assessment of where this area of research stands and where it might be headed will conclude this monograph.

Chapter 2

Why Macro with Heterogeneous Households (or Firms)?

- Empirical Fact: lots of observed and unobserved heterogeneity among households (Heckman, Wolpin and many other micro labor people)

- Models: Household heterogeneity affects:

  - Aggregate quantities (Aiyagari '94) and prices (Huggett '93) and their evolution (Krusell and Smith '98). But how much?
  - Answers to normative questions (e.g. Lucas '87 vs Krusell-Smith '99 and Krebs '07 on the cost of business cycles)
  - Answers to counterfactual policy (effect of temporary tax cuts, see Heathcote '05, Kaplan and Violante '12, '13)

- Many questions involve distributional aspects that cannot be studied in rep. agent world

  - What explains aggregate trends in inequality (e.g. Krueger and Perri '06)
  - Aggregate and redistributive consequences of change in tax progressivity, social security (e.g. Auerbach and Kotlikoff '87)

- Note: strong overlap in questions with empirical micro, especially labor economics.

Chapter 3

Some Stylized Facts and Some Puzzles

In this chapter we will discuss the main stylized facts that heterogeneous agent macro models are designed to explain. We are mainly interested in four main economic variables, labor earnings, income, wealth and consumption

3.1   Household Level Data Sources

3.1.1   CEX

For the US, the only household level data set with extensive information about a wide range of consumption expenditures is the Consumer Expenditure Survey (CEX) or (CES).[1] The CEX is conducted by the U.S. Bureau of the Census and sponsored by the Bureau of Labor statistics. ¿From 1980 onwards the survey is carried out on a yearly basis. The CEX is a so-called rotating panel: each household in the sample is interviewed for four consecutive quarters and then rotated out of the survey. Hence in each quarter 20% of all households is rotated out of the sample and replaced by new households. In each quarter about 3000 to 5000 households are in the sample, and the sample is representative of the U.S. population. The main advantage of the CEX is that it contains very detailed information about

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[1]In class I will also discuss some European data sources, such as the British Family Expenditure Survey (FES) and the German Socio-Economic Panel (SOEP) and the Einkommens- und Verbrauchsstichprobe.

consumption expenditures. Information about income and wealth is inferior to the Survey of Consumer Finances (SCF), the Current Population survey (CPS) and the Panel Study of Income Dynamics (PSID), also the panel dimension is significantly shorter than for the PSID (one household is only followed for 4 quarters). For further information and the complete data set see http://www.stats.bls.gov/csxhome.htm.

3.1.2   SCF

With respect to income, the PSID as well as the SCF contains data that are supposedly of higher quality than the income data from the CEX. The SCF is conducted in three year intervals; the four available surveys are for the years 1989, 1992, 1995 and 1998. It is conducted by the National Opinion Research center at the University of Chicago and sponsored by the Federal Reserve system. It contains rich information about U.S. households' income and wealth. In each survey about 4,000 households are asked detailed questions about their labor earnings, income and wealth. One part of the sample is representative of the U.S. population, to give an accurate description of the entire population. The second part over-samples rich households, to get a more precise idea about the precise composition of this groups' income and wealth composition. As we will see, this group accounts for the majority of total household wealth, and hence it is particularly important to have good information about this group. The main advantage of the SCF is the level of detail of information about income and wealth. The main disadvantage is that it is not a panel data set, i.e. households are not followed over time. Hence dynamics of income and wealth accumulation cannot be documented on the household level with this data set. For further information and some of the data see http://www.federalreserve.gov/pubs/oss/oss2/98/scf98home.html.

3.1.3   PSID

The Panel Study of Income Dynamics (PSID) is conducted by the Survey Research Center of the University of Michigan and mainly sponsored by the National Science Foundation. The PSID is a panel data set that started with a national sample of 5,000 U.S. households in 1968. The same sample individuals are followed over the years, barring attrition due to death or non-response. New households are added to the sample on a consistent basis, making the total sample size of the PSID about 8700 households. The income

and wealth data are not as detailed as for the SCF, but its panel dimension allows to construct measures of income and wealth dynamics, since the same households are interviewed year after year. Also the PSID contains data on consumption expenditures, albeit only food consumption. In addition, in 1990, a representative national sample of 2,000 Latinos, differentially sampled to provide adequate numbers of Puerto Rican, Mexican-American, and Cuban-Americans, was added to the PSID database. This provides a host of information for studies on discrimination. For further information and the complete data set see http://www.isr.umich.edu/src/psid/index.html

3.1.4 CPS

The Current Population Survey (CPS) is conducted by the U.S. Bureau of the Census and sponsored by the Bureau of Labor Statistics. In its annual March supplement detailed information about household income is collected. The survey started to gather information about household income in 1948, but comprehensive information about household income and income of its members is available only since 1970's. The main advantage of the CPS is its sample size: in each year it contains a representative sample of 40,000 to 60,000 households. However, no information about consumption or wealth information is collected. Also, this survey, like the SCF is a purely cross-sectional data set without panel dimension as it does not follow individual families over time. Fore more details see http://www.bls.census.gov/cps.

3.1.5 Data Sources for Other Countries

- Family Expenditure Survey (FES) for the U.K. Very similar in scope to the American CEX.

- Socio-Economic Panel (SOEP) for Germany. Very similar in scope to the American PSID.

- Einkommens- und Verbrauchsstichprobe for Germany. Similar to CEX, but less frequent.

- Italian Survey of Household Income and Wealth (SHIW). Biannual panel data for income, consumption and wealth.

- Eurosystem Household Finance and Consumption Survey (by ECB)

- Many other national data sets for other countries. See RED 2011, Vol. 1.

- Luxembourg Income Study as well as the Luxembourg Wealth Study

3.2   Main Findings

3.2.1   Organization of Facts

- Variables of interest: w, h, wh, y, c, a

- Dimensions of the data

	μ	σ	$\Delta\sigma$
age a	means (1)	See next box $\rightarrow$	var. over lc (2)
time t	RA macro	ineq. levels (3)	Ineq. trends (4)

- Follow a cohort over time as she ages: Life cycle means (1) and cross-cohort variances (2)

- Cross-section of population at point in time (3)

- Evolution of the cross-section over time (inequality trends), (4)

- Household debt and default

3.2.2   Aggregate Time Series: Means over Time

The focus of this monograph is to explain household behavior and its implications for the macro economy. The main economic choices individual households make are how much to work, and how much of the resulting income to consume and save.

In figure **??** we plot real GDP and real total household consumption expenditures, on a logarithmic scale, for the U.S. from 1964 to 2012. We observe that 60-70% of GDP is used for private consumption expenditure. This fraction is fairly stable, but displays an upward trend starting in the 80's and continuing through 90's. Associated with this increase is a decline in the personal savings rate s_p, which is defined as

$$s_p = 1 - \frac{C_p}{Y_{pdi}}$$
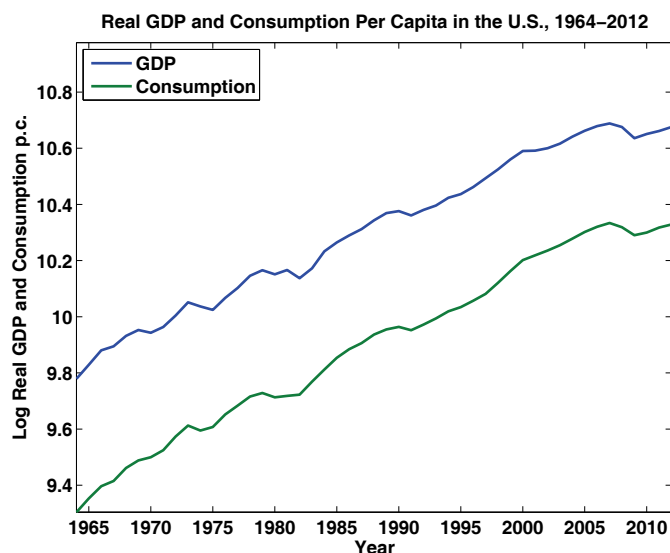
Figure 3.1: Aggregate Consumption over Time

where C_p is total private consumption expenditures and Y_{pdi} is personal disposable income.

Consumer durables make up about 13% of total nominal consumer expenditures, fraction slightly increasing. Fraction devoted to nondurables declined, now about 30%. Expenditures for consumer services about 60% of total consumer expenditures now, increasing. BUT: use of nominal data misleading because of changes in relative prices. Now: deflate components by their relative price (data from 1987). Consumers seem to reallocate consumption towards durables, away from nondurables. Reconciliation with first graph: relative price of services has increased. Consumption is smoother over the business cycle than GDP; Indicates households' (limited) ability to isolate their consumption path from income fluctuations induced by the business cycle. Nondurable consumption least volatile, then services; purchases of consumer durables fluctuate most over the cycle.

Summary Statistics on Consumption Growth (1959-96)

	Nodur. & Serv.	Dur.
Annual Growth Rate *p.a.*	2.3%	4.8%
Std. Dev. of Gr. Rate	1.8%	6.9%
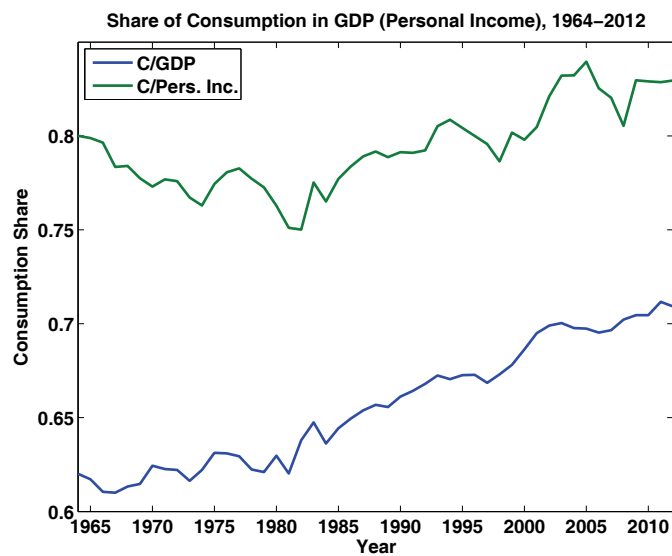
Figure 3.2: Consumption Share over Time
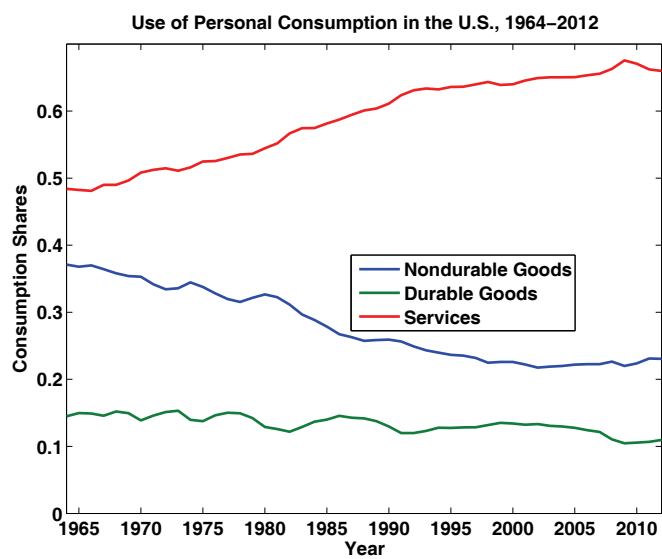
Figure 3.3: Components of Household Consumption over Time

Real Consumption in the U.S., 1995–2012
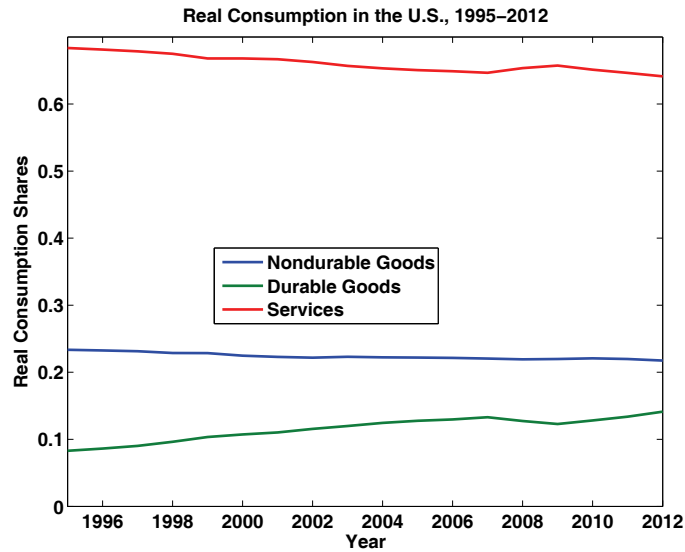
Figure 3.4: Components of Household Consumption over Time, Goods-Specific Prices

Personal Saving Rate, 1964–2012
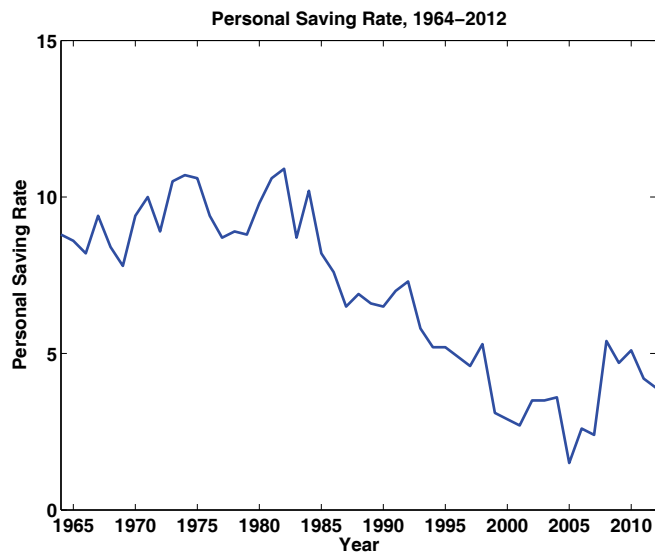
Figure 3.5: Household Saving Rate over Time

Figure: Labor Income and Net Worth by Age, SCF 2007 ($1,000)
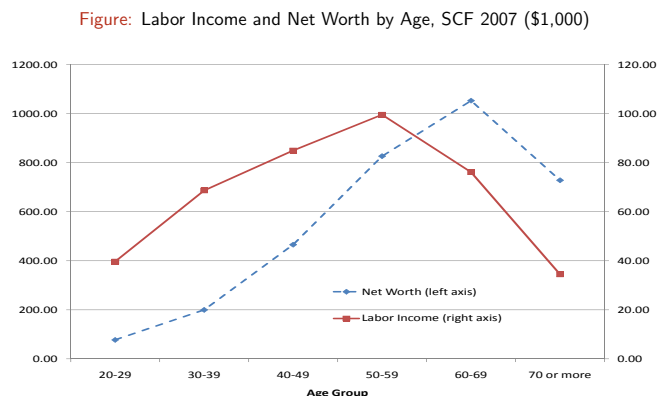
Figure 3.6: Income and Net Worth over the Life Cycle

3.2.3   Means over the Life Cycle

In this section we document mean life cycle profiles of labor earnings, wealth (net worth) and consumption. In figure 3.2.3 we plot the average (across households of a specific age) labor income and net worth against household age.[2]

As we see from this figure, in the data, mean labor income (right scale) has a hump shape over the life cycle and peaks at age of around 45 to 50. The hump is very significant in size: average earnings at age 45 is almost 2.5 times as at age 25. And at age 65 households on average earn only about 60-70% of the earnings of the typical household age 45. W also observe that net worth (left scale) is accumulated over the life cycle, and then is decumulated in retirement. However, average net worth is still very significantly positive at old age.

In figure 3.2.3 we plot consumption over the life cycle. The figure contains two plots, both based on average consumption across households of a given age. One plot measures household consumption as the total consumption expenditure of a household, the other aims at constructing per capita consumption in the household, by dividing household consumption expenditures

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[2]This plot is derived from data from the 2007 SCF, and reproduced from Glover et al., 2014.
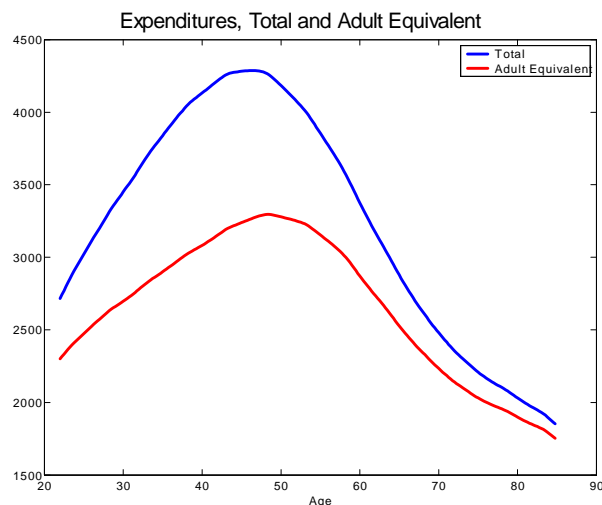
Figure 3.7: Consumption over the Life Cycle

by a household equivalence scale that controls for household size.[3]

We observe that consumption follows qualitatively the same hump shape over the life cycle as labor income does. It appears that consumption tracks income over the life cycle.

Excess sensitivity puzzle: consumption excessively sensitive to predicted changes in income:

However, in the data family size is also hump-shaped over the life cycle, and the figure shows that about half of the hump is gone after family size is controlled for, leaving a smaller (but still significantly positive) hump to be explained by deviations from the most basic life cycle model.

If one follows an average household over its life cycle[4], two main stylized facts emerge (see Attanasio (1999) for the detailed figures). First, disposable income follows a hump over the life cycle, with a peak around the age of 45 (the age of the household is defined by the age of the household head). This finding is hardly surprising, given that at young ages households tend to

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[3]This figure is taken from Fernandez-Villaverde and Krueger (2007). Similar findings are presented by Attanasio et al. (1999) and Gourinchas and Parker (2002), among others.

[4]How to construct such an average household in the absence of panel data sets which lack a sufficient time series dimension is a challenging problem. The solution, the so-called pseudo panel method, will be discussed in the second half of the course.

obtain formal education or training on the job and labor force participation
of women is low because of child bearing and rearing. As more and more
agents finish their education and learn on the job as well as promotions occur,
average wages within the cohort increase. Average personal income at age
45 is almost 2.5 times as high as average personal income at age 25. After
the age of 45 personal income first slowly, then more rapidly declines as more
and more people retire and labor productivity (and thus often wages) fall.
The average household at age 65 has only 60% of the personal income that
the average household at age 45 obtains.

The second main finding is the surprising finding. Not only personal
income, but also consumption follows a hump over the life cycle. In other
words, consumption seems to track income over the life cycle fairly closely.
This is one statement of the so-called **excess sensitivity** puzzle: consump-
tion appears to be excessively sensitive to predicted changes in income. In
fact, the two standard theories of intertemporal consumption allocation we
will consider in the next section both predict that (under specific assumptions
spelled out explicitly below) consumption follows a martingale and current
income does not help to forecast future consumption. The hump-shaped con-
sumption age profile apparently seems to contradict this hypothesis. Later
in the course we will investigate in detail whether, once we control for house-
hold size (which also happens to follow a hump shape), the hump-shape in
consumption disappears or whether the puzzle persists. Figure **??** (taken
from Fernandez-Villaverde and Krueger, 2007) documents the life cycle pro-
file of consumption, with and without adjustment for family size by household
equivalence scales. The figure is derived using a synthetic cohort analysis, a
technique from Panel data econometrics that allows us to construct average
life cycle profiles for households that we do not observe over their entire life
(we will talk about this technique in detail below). The key observation from
this figure is that consumption displays a hump over the life cycle, and that
this hump persists, even after controlling for family size. The later observa-
tion is not entirely noncontroversial, and we will discuss below the different
positions on this issue.

3.2.4   Dispersion over the Life Cycle

Deaton and Paxson (1994) and Storesletten et al. (2004a) document how
inequality of income and consumption evolves *over the life cycle*. Figure 1 of
Storesletten et al. (2004a) plots, against age of the households, the variance

of log earnings and log consumption. The figure shows that as a cohort ages, the distribution of earnings and consumption within a cohort fans out.[5] Note that the increase in consumption inequality is substantially less pronounced than the increase in the dispersion of earnings.

Cross-sectional dispersion (or inequality) has also changed dramatically over time. Heathcote et al. (2004) and many others have documented a strong upward trend in wage and even more pronounced in household earnings inequality over the last 30 years. The inequality in hours worked has remained fairly stable, and consumption inequality has increased by substantially less than earnings inequality (see Krueger and Perri (2006) and the discussion in Attanasio et al. (2007)). Finally, it appears that wealth inequality has increased as well in the 1980's and since the remained fairly constant, to the extent the wealth data are available and can be trusted (see Favilukis, 2007). A substantial body of literature has sprung up in the last few years trying to explain these trends with structural macro models with heterogenous agents, of the type studied in these notes.

Heathcote, Perri and Violante (2009)

3.2.5 Dispersion (Inequality) at a Point in Time

Minneapolis FED QR pieces by Victor and Vincenzo. Show Lorenz curves and tables with inequality statist

3.2.6 Changes in Dispersion over Time

Wages, earnings and income: Heathcote, Perri and Violante (2011), consumption: Krueger and Perri (2006, wealth more speculative: Favilukis (2007)

3.2.7 Other Interesting Facts (or Puzzles)

Other intersting puzzles in this literature include:

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[5]When working with household data that does not contain a long panel dimension nd thus one cannot literally follow the same cohort and uses repeated cross-sections of households, oe runs into the well-known identfication problem that time, cohort and age effects are not separately identified (since age, time and cohort birth year are prefectly colinear). There are varios ways to deal with this problem which I will discuss in class.

Excess Smoothness Puzzle

If the stochastic income process of households is only difference stationary (say, it follows a random walk) then a shock to current income translates (more than) one to one into a shock to permanent income and hence should induce a large shock to consumption. With difference-stationary income processes consumption should be as volatile as current income, but we saw that, at least on the aggregate level, consumption is smoother than income. Is in fact consumption "too smooth". This is the excess smoothness puzzle.

Lack of Decumulation Puzzle

Household level wealth data show that a significant fraction of very old households still hold a large portfolio of financial and real estate assets. Why don't these households decumulate their assets and enhance their consumption, as standard life-cycle theory predicts?

Drop of Consumption Puzzle

As people retire, their consumption drops by about 15% on average

Portfolio Allocation Puzzle

The median wealth US household does not own stocks, but holds its major fraction of the wealth portfolio concentrated in its own home and the rest in low-return checking or savings accounts. This is despite the fact that returns to equity and returns to human capital (i.e. the households' wage) are roughly uncorrelated for this fraction of the population. In addition, Gross and Souleles (2000) document that a significant fraction of the population has simultaneously high-interest credit card debt and liquid, low return assets such as a significantly positive checking account balance. Tobacman (2007) documents that a substantial fraction of the U.S. population repeatedly borrows using payday loans, which carry an annualized interest rate of about 7000%.

Debt and Default

The US legal system allows private households to file for personal bankruptcy under Chapter 7 or Chapter 11 of the US Bankruptcy Code. In particular,

under Chapter 7 households are discharged of all their debts, are not required to use any of their future labor income to repay the debt and can even keep their assets (financial or real estate) below a state-dependent exemption level. Whereas about 1% of all households per year file for personal bankruptcy, White (1998) computes that currently at least 15% of all US households would financially benefit from filing for bankruptcy. The fraction of U.S. households filing for bankruptcy has increased sharply over the last decade. So has the extent of uncollateralized debt, as a fraction of disposable household income (the same is very much true for collateralized debt). Also, charge-off rates of lenders (the fraction of loans the lender does not recover), have increased substantially (see Livshits et al., 2007). Show pictures on trends of debt and default (both uncollateralized and collateralized) from Livshits et al.

Any given model will likely not be able to resolve all these puzzles at once, and some models will abstract from some of the issues altogether, but the "stylized facts" of this section should be kept in mind in order to guide extensions of the models presented next.

Chapter 4

The Standard Complete Markets Model

Let the economy be populated by N individuals, indexed by $i \in I = \{1, 2, \ldots N\}$. Each individual lives for T periods, where $T = \infty$ is allowed. In each period there is one nonstorable consumption good. Each individual household has a stochastic endowment process $\{y_t^i\}$ of this consumption good. Risk in this economy is represented by a current aggregate shock (or "event") s_t. We will assume that $s_t \in S$, where S is a finite set of cardinality K, and denote by $s^t = (s_0, \ldots s_t)$ the event history of this economy, respectively. Each event history s^t has a probability $\pi_t(s^t) > 0$ of occurring, and all households share common and correct beliefs about these probabilities. Using slight abuse of notation let by $S^t = S \times S \times \ldots \times S$ denote the $t+1$-fold Cartesian product of S such that $s^t \in S^t$. Also, in order to make some of the results a little easier to derive, take the event s_0 in period 0 as given (i.e. not random); obviously which event the economy starts with is irrelevant for the discussion to follow.

Individual endowments are then functions of the underlying event histories[1], that is $y_t^i = y_t^i(s^t)$. Note that so far we did not make any assumptions about the stochastic process governing $\{s_t\}$; in particular the process need not be a Markov processes, and the individual income processes implied by our formulation need not be *iid* across agents or time.

A consumption allocation $(c^i)_{i \in I} = \{(c_t^i(s^t))_{i \in I}\}_{t=0, s^t \in S^t}^T$ maps aggregate

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[1] The assumption that S is finite is to avoid having to worry about measurability of the consumption allocations below.

Note that we could as well have taken as fundamental sources of risk the individual income processes $\{y_t^i\}$ and defined s_t as $s_t = (y_t^1, \ldots, y_t^N)$.

event histories s^t into consumption of agents $i \in I$ at time t. Agents have preferences over consumption allocations that are assumed to permit an expected utility representation

$$u^i(c^i) = \sum_{s^t \in S^t} \pi_t(s^t) U_t^i(c^i, s^t) \tag{4.1}$$

Furthermore, unless otherwise noted, we will assume that the utility function is additively time-separable and that agents discount the future at common subjective time discount factor $\beta \in (0,1)$, so that the period utility function takes the form $U_t^i(c^i, s^t) = \beta^t U^i(c_t^i(s^t), s^t)$ and thus expected lifetime utility is given by:

$$u^i(c^i) = \sum_{t=0}^{T} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) U^i(c_t^i(s^t), s^t) \tag{4.2}$$

By $\rho = \frac{1}{\beta} - 1$ let denote the subjective time discount rate (i.e. $\beta = \frac{1}{1+\rho}$). The presence of s^t allows for the potential presence of preference shocks to individual agents' utility from consumption. For example, the arrival of a new child (modeled as an exogenous stochastic process driven by $\{s_t\}$) may reduce utility from a given household consumption level, provided that members care about per capita consumption (and $c_t^i(s^t)$ denotes consumption expenditures of household i). For future reference we now define a feasible allocation and a Pareto-efficient allocation.

Definition 1 *A consumption allocation $\{(c_t^i(s^t))_{i \in I}\}_{t=0, s^t \in S^t}^{T}$ is feasible if*

$$c_t^i(s^t) \geq 0 \text{ for all } i, t, s^t \tag{4.3}$$

$$\sum_{i=1}^{N} c_t^i(s^t) = \sum_{i=1}^{N} y_t^i(s^t) \text{ for all } t, s^t \tag{4.4}$$

Definition 2 *A consumption allocation is Pareto efficient if it is feasible and there is no other feasible consumption allocation $\{(\hat{c}_t^i(s^t))_{i \in I}\}_{t=0, s^t \in S^t}^{T}$ such that*

$$u^i(\hat{c}^i) \geq u^i(c^i) \text{ for all } i \in I \tag{4.5}$$

$$u^i(\hat{c}^i) > u^i(c^i) \text{ for some } i \in I \tag{4.6}$$

4.1 Theoretical Results

The main distinction between the complete markets model and the standard incomplete markets model is the set of assets that agents can trade to hedge against individual income risk, and hence the individual budget constraints. In the standard complete markets model we suppose that there is a complete set of contingent consumption claims that are traded at time 0, prior to the realization of income (or any other type of) risk. The individual Arrow Debreu budget constraints take the form

$$\sum_{t=0}^{T} \sum_{s^t \in S^t} p_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{T} \sum_{s^t \in S^t} p_t(s^t) y_t^i(s^t) \tag{4.7}$$

where $p_t(s^t)$ is the period 0 price of one unit of period t consumption, delivered if event history s^t has realized.

4.1.1 Arrow-Debreu Equilibrium

Definition 3 *An Arrow Debreu competitive equilibrium consists of allocations $\{(c_t^i(s^t))_{i \in I}\}_{t=0, s^t \in S^t}^T$ and prices $\{p_t(s^t)\}_{t=0, s^t \in S^t}^T$ such that*

1. *Given $\{p_t(s^t)\}_{t=0, s^t \in S^t}^T$, for each $i \in I$, $\{c_t^i(s^t)\}_{t=0, s^t \in S^t}^T$ maximizes (4.2) subject to (4.3) and (4.7)*

2. *$\{(c_t^i(s^t))_{i \in I}\}_{t=0, s^t \in S^t}^T$ satisfies (4.4) for all t, s^t.*

Now let us make the following assumption

Assumption 4 *The period utility functions U^i are twice continuously differentiable, strictly increasing, strictly concave in its first argument and satisfy the Inada conditions*

$$\lim_{c \to 0} U_c^i(c, s^t) = \infty \tag{4.8}$$

$$\lim_{c \to \infty} U_c^i(c, s^t) = 0 \tag{4.9}$$

where U_c^i is the derivative of U with respect to its first argument.

It is then straightforward to argue that in this economy any competitive equilibrium allocation is the solution to the social planners problem of maximizing

$$\max_{\{(c_t^i(s^t))_{i\in I}\}_{t=0,s^t\in S^t}^T} \sum_{i=1}^N \alpha^i u^i(c^i) \tag{4.10}$$

subject to (4.3) and (4.4), for *some* Pareto weights $(\alpha^i)_{i=1}^N$ satisfying $\alpha^i \geq 0$ and $\sum_{i=1}^N \alpha^i = 1$. First, observe that the assumption is (more than) sufficient to establish the first welfare theorem, and thus we know that any competitive equilibrium allocation is Pareto efficient. Second, any Pareto efficient allocation is the solution to the social planner problem in (4.10), for *some* Pareto weights (see, e.g. MasColell et. al., chapter 16; this result requires other parts of assumption 1, especially concavity).

Characterizing Efficient Allocations: Perfect Consumption Insurance

Characterizing the solutions to the maximization problem in (4.10), and thus characterizing the properties of all efficient (and thus competitive equilibrium) allocations is straightforward. Attaching Lagrange multipliers $\lambda(s^t)$ to the resource constraint and ignoring the non-negativity constraints on consumption we obtain as first order necessary conditions for an optimum

$$\alpha^i \beta^t \pi_t(s^t) U_c^i(c_t^i(s^t), s^t) = \lambda_t(s^t) \tag{4.11}$$

for all $i \in I$. Hence for $i, j \in I$

$$\frac{U_c^i(c_t^i(s^t), s^t)}{U_c^j(c_t^j(s^t), s^t)} = \frac{\alpha^j}{\alpha^i} \tag{4.12}$$

for all dates t and all states s^t. Hence in *any* efficient allocation (and thus in a market economy with a complete set of contingent consumption claims being traded) the ratio of marginal utilities of consumption of any two agents is constant across time and states. Also those agents, ceteris paribus (i.e. if they had the same utility function and the same preference shocks), with higher relative Pareto weights will consume more in every state of the world because the utility function is assumed to be strictly concave.

Thus the benevolent social planner whose aim to insure consumption of all households over time and across states of nature finds it optimal to keep

the ratio of marginal utilities of consumption of any two agents constant across time and states. Since the social planner faces no constraints other than the resource constraint it is natural to define as perfect consumption insurance of an allocation the property (4.12) of a constant ratio of marginal utilities:

Definition 5 *A consumption allocation $\{(c_t^i(s^t))_{i\in I}\}_{t=0,s^t\in S^t}^T$ is said to satisfy perfect consumption insurance if the ratio of marginal utilities of consumption between any two agents is constant across time and states of the world.*

Thus, the complete markets model exhibits perfect consumption insurance, in the sense defined above. With additional assumptions we obtain an even stronger (and hence easier to test empirically) implication of the complete markets model.

Assumption 6 *All agents have identical CRRA utility, and either preference shocks are absent or utility is separable consumption and preference shocks, i.e.*

$$U^i(c, s^t) = \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma} + v^i(s^t) & \text{if } \sigma \neq 1 \\ \log(c) + v^i(s^t) & \text{if } \sigma = 1 \end{cases} \tag{4.13}$$

Here $\sigma \geq 0$ is the coefficient of relative risk aversion.

We discuss the properties of the CRRA utility function further in Appendix 4.6

With this assumption equation (4.12) becomes

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}} \tag{4.14}$$

i.e. the ratio of consumption between any two agents is constant across time and states. This, in particular, implies that there exist weights $(\theta^i)_{i\in I}$ with $\theta^i \geq 0$ and $\sum_{i\in I}\theta^i = 1$ such that

$$c_t^i(s^t) = \theta^i \sum_{i\in I} y_t^i(s^t) \equiv \theta^i y_t(s^t) = \theta^i c_t(s^t) \tag{4.15}$$

where $y_t(s^t) = \sum_{i\in I} y_t^i(s^t)$ is aggregate income in the economy and $c_t(s^t) = \sum_{i\in I} c_t^i(s^t)$ is aggregate consumption. The weights are given in closed form by

$$\theta_i = \frac{\alpha_i^{\frac{1}{\sigma}}}{\sum_{j=1}^N \alpha_j^{\frac{1}{\sigma}}} \geq 0 \tag{4.16}$$

Note that with logarithmic preferences ($\sigma = 1$) we have that $\theta_i = \alpha_i$, i.e. the share of aggregate consumption an agent i receives from the social planner corresponds to the Pareto weight the planner attaches to this agent.

That is, with separable CRRA utility any efficient consumption allocation (and thus any complete markets competitive equilibrium allocation) has the feature that individual consumption at each date, in each state of the world is a constant fraction of aggregate income (or consumption). Note that it does *not* imply that individual consumption is constant across time and states of the world, because it still varies with aggregate income (a variation against which the agents cannot be mutually insured by the social planner). It also does not imply that consumption among agents is equalized. ¿From (4.14) we see that the level of consumption of agent i will depend positively on the Pareto weight of that agent.

Determining the "Right" Pareto Weights

So far we have characterized efficient allocations, given arbitrary Pareto weights $(\alpha^i)_{i \in I}$ of the social planner, and we have argued that all competitive equilibrium allocations share this characterization. Although not necessary for the empirical tests of these implications we now briefly describe how to construct competitive equilibrium consumption allocations, that is, how to find the the "right" Pareto weights. This section describes Negishi's (1960) method for doing so.[2] Start from an arbitrary efficient consumption allocation $c_t^i(s^t, \alpha)$, indexed by Pareto weights $\alpha = (\alpha^i)_{i \in I}$. For given α there is no reason to believe that each household can afford the corresponding allocation

$$c_t^i(s^t) = \theta^i c_t(s^t) = \frac{\alpha_i^{\frac{1}{\sigma}}}{\sum_{j=1}^N \alpha_j^{\frac{1}{\sigma}}} c_t(s^t) = c_t^i(s^t, \alpha).$$

This was irrelevant in the social planner problem, but is required for a competitive equilibrium allocation. But in order to compute how much such

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[2]Negishi's goal was to prove existence of equilibrium. He did so by first arguing that all solutions to the social planner problem satisfy the elements of the definition of a competitive equilibrium apart from household optimality (with prices equal to Lagrange multipliers on the resource constraints of the social planner problem). He then showed that there exist Pareto weights such that the associated solution to the social planner problem solves the household maximization problem (again with prices given by the Lagrange multipliers on the resource constraint). These Pareto weights are given by the solution to the system of equations (4.18) below.

an allocation costs we need the appropriate prices. It turns out that the Lagrange multipliers (that is, the shadow prices) $\lambda_t(s^t, \alpha)$ on the resource constraints from the social planner problem are appropriate. So for each agent i define the *transfer functions* as

$$t^i(\alpha) = \sum_t \sum_{s^t \in S^t} \lambda_t(s^t) \left[c_t^i(s^t, \alpha) - y_t^i(s^t) \right]. \tag{4.17}$$

Thus $t^i(\alpha)$ is the value of lifetime consumption net of the value of lifetime income, where income and consumption at each history s^t is valued at its social value $\lambda_t(s^t, \alpha)$. For our economy, from equation (4.11) the Lagrange multipliers are given by

$$
\begin{aligned}
\lambda_t(s^t, \alpha) &= \alpha^i \beta^t \pi_t(s^t) U_c^i(c_t^i(s^t), s^t) \\
&= \alpha^i \beta^t \pi_t(s^t) c_t^i(s^t, \alpha)^{-\sigma} \\
&= \alpha^i \beta^t \pi_t(s^t) \left(\theta^i c_t(s^t) \right)^{-\sigma} \\
&= \left[\sum_{j=1}^N \alpha_j^{\frac{1}{\sigma}} \right]^\sigma \beta^t \pi_t(s^t) c_t(s^t)^{-\sigma}
\end{aligned}
$$

In a competitive equilibrium the value of each agent's consumption allocation has to not exceed (and with strictly increasing utility, has to equal) the value of that agent's income stream. The "right" α's are then those that insure that

$$t^i(\alpha) = 0 \text{ for all } i. \tag{4.18}$$

Note that since we normalized $\sum \alpha^i = 1$ there are only $N - 1$ unknowns. But it is easy to show that $\sum t^i(\alpha) = 0$, and thus the system (4.18) has only have $N - 1$ independent equations as well.

Once one has solved for α^* from these equations, the competitive equilibrium allocations are given by $c_t^i(s^t, \alpha^*)$ and equilibrium prices are given by $\lambda_t(s^t, \alpha^*)$, as we will confirm below. Note that solving for competitive equilibria using the Negishi approach then amounts to solving the social planner problem for arbitrary weights α, and solving a system of $N - 1$ equations in $N - 1$ unknowns, which may be substantially easier than solving for the competitive equilibrium directly. But also note that in general equations (4.18) can be quite messy, and in particular, can be highly nonlinear in the α's, so that typically no analytical solution to the system (4.18) is available.

Equation (4.17) also makes clear that the Pareto weights that make the transfer functions $t^i(\alpha)$ equal to zero depend on the properties of the individual income processes $\{y_t^i\}$. As a consequence, the equilibrium *level* and thus share θ^i of consumption of each agent i depends crucially on the value of her endowment process.

Competitive Equilibrium Prices and the Representative Agent

After this detour, let's turn back to the characterization of efficient (and hence equilibrium) allocations. The growth rate of consumption between any two dates and states is given from (4.15) as

$$\log\left(\frac{c_{t+1}^i(s^{t+1})}{c_t^i(s^t)}\right) = \log\left(\frac{c_t(s^{t+1})}{c_t(s^t)}\right) \tag{4.19}$$

that is, if agents have CRRA utility that is separable in consumption, then in an efficient (competitive equilibrium) allocation individual consumption growth is perfectly correlated with and predicted by aggregate consumption growth. In particular, individual income growth should not help to predict individual consumption growth once aggregate consumption (income) growth is accounted for. This result is the starting point of the most basic empirical tests of perfect consumption insurance, see e.g. Mace (1991), Cochrane (1991), among others.

To obtain Arrow Debreu prices associated with equilibrium allocations we obtain from the consumer problem of maximizing (4.2) subject to (4.7) that

$$\frac{p_{t+1}(s^{t+1})}{p_t(s^t)} = \beta\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}\frac{U_c^i(c_{t+1}^i(s^{t+1}), s^{t+1})}{U_c^i(c_t^i(s^t), s^t)} \tag{4.20}$$

Under assumption 2 this becomes

$$\frac{p_{t+1}(s^{t+1})}{p_t(s^t)} = \beta\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}\left(\frac{c_{t+1}^i(s^{t+1})}{c_t^i(s^t)}\right)^{-\sigma} \tag{4.21}$$

$$= \beta\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}\left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\right)^{-\sigma} \tag{4.22}$$

and hence equilibrium Arrow Debreu prices can be written as functions of aggregate consumption only. We then have the following

Proposition 7 *Suppose allocations $\{(c_t^i(s^t))_{i \in I}\}_{t=0,s^t \in S^t}^T$ and prices $\{p_t(s^t)\}_{t=0,s^t \in S^t}^T$ form an Arrow-Debreu equilibrium. Then under assumption 2 the allocation $\{c_t(s^t)\}_{t=0,s^t \in S^t}^T$ defined by*

$$c_t(s^t) = \sum_{i \in I} c_t^i(s^t) \tag{4.23}$$

and prices $\{p_t(s^t)\}_{t=0,s^t \in S^t}^T$ form an Arrow Debreu equilibrium for the representative agent economy in which the representative agent has an endowment process given by

$$y_t(s^t) = \sum_{i \in I} y_t^i(s^t) \tag{4.24}$$

and CRRA preferences with parameter σ.

The fact that the equilibrium of the representative agent economy has consumption allocations given by (4.23) is of course trivial and follows directly from the market clearing condition. The content of this proposition lies in the statement that in the representative agent economy (with the representative household having the same CRRA utility function as agents in the heterogeneous agent economy) has the same equilibrium Arrow-Debreu prices as the economy with I consumers that might differ in their income processes in an arbitrary way. Thus, in order to derive Arrow-Debreu prices (and hence all other asset prices) in an economy with complete markets, it is sufficient to study the corresponding representative agent economy. Most of the consumption based asset pricing literature since Lucas (1978) (and indeed most of model-based macroeconomics) has indeed employed models with a representative consumer. As the previous result suggests, if financial markets are complete (and sufficiently strong assumptions on agents' utility functions are made), then the abstraction from household heterogeneity is innocuous from the perspective of macroeconomic research. It should also be noted that for this proposition assumption 2 can be weakened[3], although

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[3]We have assumed idential CRRA utility functions and time discount factors across all agents. Koulovatianos (2007), building on the large literature on aggregation in dynamic models, such as Chatterjee (1994) and Caselli and Ventura (2000), gives necessary and sufficient conditions on individual utility functions to obtain a representative consumer. See his Theorem 1 and 2.

Maintaining identical time discount factors the period utility function has to be either of power or exponential form (with heterogeneity in the CARA, but not in the CRRA possible). If the time discount factor is allowed to vary across agents, only exponential utility gives rise to the result stated here.

the construction of the utility function of the representative agent may more involved.

Endogenous Labor Supply

The previous discussion took as exogenous a stochastic income process and thus abstracted from endogenous labor supply. Now we will argue that under the appropriate assumption on the utility function the results from the previous section go through essentially unchanged if agents make endogenous labor supply decisions. However, we will also present an example that shows that nonseparabilities between consumption and leisure in the utility function may alter the empirical predictions of the complete markets model in a qualitatively important way.

Instead of stochastic income shocks let households now face stochastic productivity (wage) shocks $w_t^i(s^t)$. We assume that a household produces $w_t^i(s^t)$ units of the consumption good per unit of time worked. In a competitive equilibrium individual productivity and real wages will coincide, and thus $w_t^i(s^t)$ will also denote the stochastic wage process that household i faces in a competitive equilibrium.

As long as markets are complete, equilibrium allocations can still be determined by solving a social planner problem with appropriate social welfare weights. Now the resource constraints read as

$$\sum_{i=1}^N c_t^i(s^t) = \sum_{i=1}^N w_t^i(s^t) l_t^i(s^t)$$

and the period utility function is given by $U(c_t^i(s^t), l_t^i(s^t))$, where $l_t^i(s^t) \in [0, 1]$ is the fraction of the time a household works.

The key efficiency conditions now read as

$$\frac{U_c(c_t^i(s^t) l_t^i(s^t))}{U_c(c_t^j(s^t), l_t^j(s^t))} = \frac{\alpha^j}{\alpha^i} \tag{4.25}$$

$$-\frac{U_l(c_t^i(s^t) l_t^i(s^t))}{U_c(c_t^i(s^t), l_t^i(s^t))} = w_t^i(s^t) \tag{4.26}$$

The first condition is the familiar efficient risk sharing condition across households. The second condition governs the efficient allocation of consumption

and leisure for an arbitrary household i. We can divide equations (4.26) for any two agents to arrive at

$$\frac{U_l(c_t^i(s^t)l_t^i(s^t))}{U_l(c_t^j(s^t), l_t^j(s^t))} = \frac{\alpha^j}{\alpha^i}\frac{w_t^i(s^t)}{w_t^j(s^t)} \tag{4.27}$$

For given welfare weights α equations (4.25) and (4.27) determine relative consumption and leisure allocations between households i and j.

Let us consider two important classes of period utility functions $U(.)$ in the following two examples.

Example 8 *Suppose $U(c_t^i(s^t), l_t^i(s^t))$ is additively separable between consumption and leisure (e.g. $U(c_t^i(s^t), l_t^i(s^t)) = v(c_t^i(s^t)) - g(l_t^i(s^t))$ where v is strictly concave and g is strictly convex. In this case the implications for efficient consumption risk sharing are exactly the same as in the case with exogenous labor supply:*

$$\frac{v'(c_t^i(s^t))}{v'(c_t^j(s^t))} = \frac{\alpha^j}{\alpha^i}.$$

In particular, if v is of CRRA form, then as before consumption of each agent i equals a constant fraction θ^i of aggregate production. The efficient allocation of labor supply, in this case, is characterized by

$$\frac{g'(l_t^i(s^t))}{g'(l_t^j(s^t))} = \frac{\alpha^j}{\alpha^i}\frac{w_t^i(s^t)}{w_t^j(s^t)}.$$

Thus, since g is strictly convex more productive households work harder. But although labor supply does respond to idiosyncratic productivity shocks $w_t^i(s^t)$, as before the share of consumption of agent i does not.

Example 9 *$U(c_t^i(s^t), l_t^i(s^t))$ is nonseparable. Then in general also consumption shares respond to idiosyncratic wage shocks (which in turn suggests that one has to be cautious when testing perfect consumption insurance). We demonstrate this through a second example. Suppose the period utility function is of familiar Cobb-Douglas form*

$$U(c_t^i(s^t), l_t^i(s^t)) = \frac{[c_t^i(s^t)^\nu(1 - l_t^i(s^t))^{1-\nu}]^{1-\sigma} - 1}{1 - \sigma}.$$

Exploiting the efficiency conditions (4.25) and (4.27) we find that

$$
\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}} \left(\frac{w_t^i(s^t)}{w_t^j(s^t)}\right)^{(1-\nu)\left(1-\frac{1}{\sigma}\right)}
$$

$$
\frac{1 - l_t^i(s^t)}{1 - l_t^j(s^t)} = \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}} \left(\frac{w_t^j(s^t)}{w_t^i(s^t)}\right)^{1-(1-\nu)\left(1-\frac{1}{\sigma}\right)}.
$$

Since $\sigma > 0$ and $\nu > 0$ the second equation implies that more productive households (holding Pareto weights constant) consume less leisure and work more, that is, if w^i goes up relative to w^j in a particular node, then household i's labor supply increases, relative to that of household j. The first equation implies that more productive households also consume more if and only if $\sigma > 1$. If this condition is satisfied, it is easy to verify from the utility function that $U_{c,1-l} < 0$, that is, then the marginal utility of consumption falls with higher leisure. Recall from equation (4.25) that it is efficient to keep the ratio of marginal utilities of consumption constant across time and states. If leisure of household i goes down relative to j and $U_{c,1-l} < 0$, then for fixed consumption U_c^i / U_c^j would rise. Thus to keep U_c^i / U_c^j constant, consumption of household i has to rise relative to household j in that node: it is efficient to compensate household i for her higher labor supply with larger consumption. If $\sigma < 1$ marginal utility of consumption increases with leisure, and the reverse logic is true. Finally, if $\sigma \to 1$ then the utility function is additively separable in consumption and leisure

$$
U(c_t^i(s^t), l_t^i(s^t)) = \nu \log\left(c_t^i(s^t)\right) + (1-\nu) \log\left(1 - l_t^i(s^t)\right)
$$

and case 1 applies. Regardless of the size of σ, with nonseparabilities between consumption and leisure (that is, as long as $\sigma \neq 1$), in the efficient perfect risk sharing allocation individual consumption of household i responds to idiosyncratic productivity shocks $\{w_t^i(s^t)\}$ for that household, a fact that has to be accounted for in the empirical tests of the perfect consumption insurance hypothesis.

4.1.2   Sequential Markets Equilibrium

In the Arrow Debreu market structure trade in state-contingent consumption claims takes place only once, at the beginning of time, prior to the resolution of any risk. With this market structure it is most straightforward to

establish the connection between equilibrium and efficient allocations, and to characterize the former. However, this market structure might appear somewhat unrealistic and does not capture the trade of real-world financial assets (such as bonds and stocks) over time.[4] Therefore I now present a second market structure and associated equilibrium concept, which I will call sequential markets equilibrium, in which consumption and a carefully chosen set of financial assets are traded at each node s^t of the event tree. Fortunately, as discussed below, the set of equilibrium allocations in both market structures will coincide, and thus the perfect risk sharing characterization of consumption derived above will also apply to sequential markets equilibrium allocations.

We assume that the set of financial assets consists of one period contingent bonds, that is, financial contracts bought in period t, that pay out one unit of the consumption good in $t+1$, but only if a *particular* state $s_{t+1} \in S$ is realized tomorrow.[5] Let $q_t(s^t, s_{t+1})$ denote the price, at node s^t, of a contract that pays out one unit of consumption in period $t+1$ if and only if tomorrow's event is s_{t+1}. These contracts are often called Arrow securities, contingent claims or one-period state contingent bonds. Denote by $a_{t+1}^i(s^t, s_{t+1})$ the quantity of the Arrow security bought (or sold) at node s^t by agent i that pays off in node $s^{t+1} \equiv (s^t, s_{t+1})$.

The period t, event history s^t budget constraint of agent i is given by

$$c_t^i(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \leq y_t^i(s^t) + a_t^i(s^t) \qquad (4.28)$$

Note that agents purchase Arrow securities $\{a_{t+1}^i(s^t, s_{t+1})\}_{s_{t+1} \in S}$ for all contingencies $s_{t+1} \in S$ that can happen tomorrow, but that, once s_{t+1} is realized, only the $a_{t+1}^i(s^{t+1})$ corresponding to the particular realization of s_{t+1} pays off and thus determines her asset position at the beginning of the next period. We assume that all agents start their life with an asset position of zero, that is, $a_0^i(s_0) = 0$. We can now state the following:

---

[4]Admittedly, such assets could be introduced into the Arrow Debreu market structure and could be priced in a straightforward manner, once equilibrium prices $\{p_t(s^t)\}$ for state-contingent consumption claims have been determined.

[5]A full set of one-period Arrow securities is sufficient to make markets "sequentially complete", in the sense that any (nonnegative) state-contingent consumption allocation $\{c_t^i(s^t)\}$ is attainable with an appropriate sequence of Arrow security holdings $\{a_{t+1}^i(s^t, s_{t+1})\}$ satisfying all sequential markets budget constraints.

Definition 10 *A SM equilibrium is allocations* $\{ \left(\hat{c}^i_t(s^t), \{ \hat{a}^i_{t+1}(s^t, s_{t+1}) \}_{s_{t+1} \in S} \right)_{i \in I} \}^T_{t=0, s^t \in S^t}$, *and prices for Arrow securities* $\{ \hat{q}_t(s^t, s_{t+1}) \}^T_{t=0, s^t \in S^t, s_{t+1} \in S}$ *such that*

1. *For* $i \in I$ *given* $\{ \hat{q}_t(s^t, s_{t+1}) \}^T_{t=0, s^t \in S^t, s_{t+1} \in S}$, *for all* i, $\{ \hat{c}^i_t(s^t), \{ \hat{a}^i_{t+1}(s^t, s_{t+1}) \}_{s_{t+1} \in S} \}^T_{t=0, s^t \in S^t}$ *maximizes* (4.2) *subject to* (4.28) *and the constraints* $c^i_t(s^t) \geq 0$ *and* $a^i_{t+1}(s^t, s_{t+1}) \geq -\bar{A}^i(s^{t+1}, \hat{q}, c^i)$

2. *For all* $t, s^t \in S^t$

$$\sum^I_{i=1} \hat{c}^i_t(s^t) = \sum^I_{i=1} y^i_t(s^t) \tag{4.29}$$

$$\sum^I_{i=1} \hat{a}^i_{t+1}(s^t, s_{t+1}) = 0 \text{ for all } s_{t+1} \in S \tag{4.30}$$

Several comments about this equilibrium definition are in order. First, note that since at each node s^t as many different state-contingent bonds are traded as there are events in S, we require a market clearing condition for each of these Arrow securities. Second, we need to restrict the negative position of state contingent bonds (that is, the amount of state-contingent debt) by some upper bound $\bar{A}^i(s^{t+1}, q, c^i)$ which for now I allow to be a general function of the node s^{t+1}, the equilibrium prices and the chosen consumption allocation. In the absence of these constraints an infinitely lived household would run Ponzi schemes: take out initial debt and roll it over indefinitely. As a consequence the household budget set would be unbounded, the household maximization problem would have no solution and thus no sequential markets equilibrium would exist.[6]

Let us now discuss the No Ponzi scheme condition $a^i_{t+1}(s^t, s_{t+1}) \geq -\bar{A}^i(s^{t+1}, \hat{q}, c^i)$ in greater detail. The objective is to come up with a specification for $\bar{A}^i(s^{t+1}, \hat{q}, c^i)$ such that Ponzi schemes are ruled out, but that otherwise does not impose borrowing constraints that are binding in equilibrium.[7] With tighter, potentially binding borrowing constraints the equivalence between the associated sequential markets equilibria and Arrow Debreu equilibria

---

[6]If agents have a finite life span T I will require them to die without debt, that is, impose $a_{T+1}(s^T, s_{T+1}) \geq 0$ for all s_{T+1}. Thus in this case $\bar{A}^i(s^{T+1}) \equiv 0$.

[7]This discussion follows Wright (1987). For those interested in a generalization of the result in the presence of long-lived assets, see Huang and Werner (2004).

would break down. Appendix 4.4 discusses the choice of the No Ponzi condition in greater detail; here we simply present a specification that is easy to motivate, commonly used and gives rise to the desired equivalence result.

For a given Arrow securities price process $q = \{q_t(s^t, s_{t+1})\}$ define date zero prices of state contingent consumption claims $\{v_0(s^t)\}$ implied by q as follows:

$$
\begin{aligned}
v_0(s_0) &= 1 \\
v_0(s^{t+1}) &= v_0(s^t)q_t(s^t, s_{t+1}) = q_0(s^0, s_1) * \ldots * q_t(s^t, s_{t+1}) * 1. \quad (4.31)
\end{aligned}
$$

Using these prices derived from q we now state the No Ponzi scheme condition as

$$
-v_0(s^T)a_T(s^{T-1}, s_T) \leq W(s^T) \quad (4.32)
$$

where

$$
W(s^T) = \sum_{t=T}^{\infty} \sum_{s^t | s^T} v_0(s^t)y_t^i(s^t) \quad (4.33)
$$

is the Arrow-Debreu future wealth of the agent. Thus

$$
\bar{A}^i(s^{t+1}, q) = \frac{W(s^{t+1})}{v_0(s^{t+1})} \quad (4.34)
$$

We will assume that $\bar{A}^i(s^{t+1}, q)$ is finite for all t, s^{t+1}, which is a joint restriction on the income process $\{y_t^i\}$ and the Arrow security prices q under consideration. The no Ponzi condition, stated this way, is sometimes called the natural debt limit. It is easier to check as it only involves the endowment process, but also prices (either Arrow securities prices or Arrow Debreu prices).

We now have the following

Proposition 11 *If (c, a, q) is a sequential markets equilibrium, then (c, p) is an Arrow-Debreu equilibrium where prices are given as*

$$
p_t(s^t) = v_0(s^t) \quad (4.35)
$$

Reversely, if (c, p) is an Arrow-Debreu equilibrium, then there exists asset holdings a such that (c, a, q) is a sequential markets equilibrium, with

$$
q_t(s^t, s_{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} \quad (4.36)
$$

Proof. See Appendix 4.5 ∎

This proposition shows that each Arrow-Debreu equilibrium can be implemented as a sequential markets equilibrium with a full set of one-period Arrow securities (and vice versa, each sequential markets equilibrium is implementable as an Arrow Debreu equilibrium).

Recursive Formulation of Household Problem

In order to facilitate comparisons with the standard incomplete markets model (and for the purpose of the numerical solutions of equilibria of complete markets models without use of the social planner problem) it is useful to present a recursive formulation of the household problem from a sequential markets equilibrium.

Suppose the stochastic process driving the economy $\{s_t\}$ is a Markov chain, and let $\pi(s'|s)$ denote the transition probabilities of the chain. Then the consumer problem, under the assumption that Arrow security prices take the form[8] $q_t(s^t, s_{t+1}) = q(s_{t+1}|s_t)$, can be written recursively as

$$v(a, s) = \max_{\{a'(s')\}_{s' \in S}} \left\{ U\left(y(s) + a - \sum_{s'} a'(s')q(s'|s) \right) + \beta \sum_{s'} \pi(s'|s)v(a'(s'), s') \right\}$$

$$(4.37)$$

---

[8]Under assumption 2 and the assumption that $\{s_t\}$ is Markov, this assumption is valid. Using (4.36) and (4.22) we find

$$
\begin{aligned}
q(s^t, s_{t+1}) &= \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} \\
&= \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma} \\
&= \beta \pi(s_{t+1}|s_t) \left(\frac{y_{t+1}(s_{t+1})}{y_t(s_t)} \right)^{-\sigma} \\
&= q(s_{t+1}|s_t)
\end{aligned}
$$

The first order and envelope condition are

$$q(s'|s)U'\left(y + a - \sum_{s'} a'(s')q(s'|s)\right) = \beta\pi(s'|s)v_a(a'(s'), s')$$

$$v_a(a, s) = U'\left(y + a - \sum_{s'} a'(s')q(s'|s)\right)$$

$$(4.38)$$

Combining yields a recursive version of the Euler equation, to be solved for the policy function $a'(a, s; s')$

$$q(s'|s)U'\left(y(s) + a - \sum_{s'} a'(a, s; s')q(s'|s)\right)$$

$$= \beta\pi(s'|s)U'\left(y(s') + a'(a, s; s') - \sum_{s''} a'(a'(a, s; s'), s'; s'')q(s''|s')\right)$$

We will use these equations in our discussions of complete markets models with enforcement frictions in later chapters of this monograph.

4.2 Empirical Implications for Asset Pricing

In this section we briefly discuss the asset pricing implications of the complete markets model.[9] The previous sections derived Arrow Debreu prices and prices of Arrow securities as a function of aggregate consumption $c_t(y^t)$ or endowment $y_t(s^t) = c_t(y^t)$. These were given as:

$$q_t(s^t, s_{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} = \beta\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}\left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\right)^{-\sigma} \qquad (4.39)$$

as long as households have CRRA utility function with common risk aversion σ. That is, together with the normalization $p_0(s_0) = 1$ (remember we took the initial state s_0 as given) the exogenous endowment process $\{y_t(s_t)\}$ completely determines Arrow-Debreu prices and prices of Arrow securities. Equipped with these we can price any other asset, an asset j being defined by

---

[9]Parts of this section are based on Kocherlakota (1996) and Campbell (1999, 2003) who provide an extensive review of the Consumption Based Asset Pricing Model (CCAPM).

the stream of dividends $d^j = \{d_t^j(s^t)\}$ where $d_t^j(s^t)$ is the amount of consumption goods asset j delivers at node s^t of the event tree. The period zero (cum dividend) price of an asset is then simply the value of all the consumption goods it delivers, that is

$$P_0^j(d) = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) d_t^j(s^t)$$

and the ex-dividend price of an asset at node s^t, in terms of the s^t consumption good, that pays a remaining dividend stream $\{d_\tau(s^\tau)\}_{\tau \geq t+1}$ is given by

$$P_t^j(d; s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) d_\tau^j(s^\tau)}{p_t(s^t)}.$$

Finally, define the one-period gross realized real return of such an asset between s^t and s^{t+1} as

$$R_{t+1}^j(s^{t+1}) = \frac{P_{t+1}^j(d; s^{t+1}) + d_{t+1}^j(s^{t+1})}{P_t^j(d; s^t)}$$

Example 12 *The simplest examples are one-period assets such as the Arrow securities discussed in the previous section. Such an asset, purchased at s^t pays a dividend of one unit of consumption in a particular state $\hat{s}_{t+1}$. Thus its price at time t is given by*

$$P_t^A(d; s^t) = \frac{p_{t+1}(\hat{s}^{t+1})}{p_t(s^t)}$$

which is noting else but $q_t(s^t, \hat{s}_{t+1})$. The associated gross realized return between s^t and $\hat{s}^{t+1} = (s^t, \hat{s}_{t+1})$ is given by

$$R_{t+1}^A(\hat{s}^{t+1}) = \frac{0+1}{p_{t+1}(\hat{s}^{t+1})/p_t(s^t)} = \frac{p_t(s^t)}{p_{t+1}(\hat{s}^{t+1})} = \frac{1}{q_t(s^t, \hat{s}_{t+1})}$$

and $R_{t+1}^j(s^{t+1}) = 0$ for all $s_{t+1} \neq \hat{s}_{t+1}$.

Example 13 *Now consider a one-period risk free bond, that is, a promise to pay one unit of consumption in every node s^{t+1} tomorrow that can follow a given node s^t. The price of such an asset is given by*

$$P_t^B(d; s^t) = \frac{\sum_{s^{t+1}} p_{t+1}(s^{t+1})}{p_t(s^t)} = \sum_{s^{t+1}} q_t(s^t, s_{t+1}).$$

The associated realized return for all nodes s^{t+1} is given by

$$R_{t+1}^B(s^{t+1}) = \frac{1}{P_t^B(d; s^t)} = \frac{1}{\sum_{s^{t+1}} q_t(s^t, s_{t+1})} = R_{t+1}^B(s^t)$$

and thus is nonstochastic, conditional on s^t. This justifies calling this asset a risk-free bond.

Example 14 *A stock (sometimes called a Lucas tree) that pays as dividend the aggregate endowment in each period has a price (per share, if the total number of shares outstanding is one)*

$$P_t^S(d; s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) y_\tau(s^\tau)}{p_t(s^t)}$$

Example 15 *An option to buy one share of the Lucas tree at time T (at all nodes) for a price K has a price $P_t^{call}(s^t)$ at node s^t given by*

$$P_t^{call}(s^t) = \sum_{s^T | s^t} \frac{p_T(s^T)}{p_t(s^t)} \max\left\{ P_T^S(d; s^T) - K, 0 \right\}$$

Such an option is called a call option. A put option is the option to sell the same asset, and is given by

$$P_t^{put}(s^t) = \sum_{s^T | s^t} \frac{p_T(s^T)}{p_t(s^t)} \max\left\{ K - P_T^S(d; s^T), 0 \right\}.$$

The price K is called the strike price (and easily could depend on s^T, too).

We will now relate returns of assets in our model to aggregate consumption (equivalently, aggregate income) in our model. Since both asset returns as well as aggregate consumption can be measured in the data, this relation will equip us with important empirically testable implications of our model.

Definition 16 *A stochastic discount factor is a stochastic process $\{m_{t+1}(s^{t+1})\}$ that satisfies*

$$E\left(m_{t+1}(s^{t+1}) R_{t+1}^j(s^{t+1}) | s^t \right) = 1 \text{ for all } t, s^t \tag{4.40}$$

and for all assets j in the economy.

A stochastic discount factor is also often called a pricing kernel, since it incorporates all the essential information of asset prices in the model. We now want to derive the unique stochastic discount factor for the complete markets model. Arguably the easiest way to do this is to employ the Euler equation from the sequential markets household maximization problem. Since we have already argued that for all pricing purposes we can restrict ourselves to the representative household we ignore household subscripts i from now on, until further notice. Generalizing the household budget constraint to allow for an arbitrary set of assets (with a generic asset denoted by j), which includes, but need not be restricted to, the Arrow securities, we have

$$c_t(s^t) + \sum_j P_t^j(d; s^t) a_{t+1}^j(s^t) \leq y_t(s^t) + \sum_j a_t^j(s^{t-1}) \left[d_t^j(s^t) + P_t^j(d; s^t) \right]$$

Attach Lagrange multiplier $\lambda_t(s^t)$ to this constraint and $\lambda_{t+1}(s^{t+1})$ to the corresponding constraints (there are as many as there are possible shocks s_{t+1} tomorrow). The first order conditions with respect to $c_t(s^t), c_{t+1}(s^{t+1})$ and $a_{t+1}^j(s^t)$ read as (remember we assume that asset choices are not constrained beyond the non-binding No Ponzi conditions)

$$
\begin{aligned}
\beta^t \pi_t(s^t) c_t(s^t)^{-\sigma} &= \lambda_t(s^t) \\
\beta^{t+1} \pi_{t+1}(s^{t+1}) c_{t+1}(s^{t+1})^{-\sigma} &= \lambda_{t+1}(s^{t+1}) \\
\lambda_t(s^t) P_t^j(d; s^t) &= \sum_{s_{t+1}} \lambda_{t+1}(s^{t+1}) \left[d_{t+1}^j(s^{t+1}) + P_{t+1}^j(d; s^{t+1}) \right]
\end{aligned}
$$

Substituting the first two equations into the third and re-arranging yields

$$
\begin{aligned}
1 &= \sum_{s_{t+1}} \frac{\beta^{t+1} \pi_{t+1}(s^{t+1}) c_{t+1}(s^{t+1})^{-\sigma} \left[d_{t+1}^j(s^{t+1}) + P_{t+1}^j(d; s^{t+1}) \right]}{\beta^t \pi_t(s^t) c_t(s^t)^{-\sigma} P_t^j(d; s^t)} \\
&= \sum_{s_{t+1}} \beta \pi_{t+1}(s^{t+1} | s^t) \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma} \frac{\left[d_{t+1}^j(s^{t+1}) + P_{t+1}^j(d; s^{t+1}) \right]}{P_t^j(d; s^t)} \\
&= \sum_{s_{t+1}} \beta \pi_{t+1}(s^{t+1} | s^t) \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma} R_{t+1}^j(s^{t+1}) \\
&= E \left(\beta \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma} R_{t+1}^j(s^{t+1}) | s^t \right) \tag{4.41}
\end{aligned}
$$

for all assets j. Thus the unique stochastic discount factor process $\{m_{t+1}(s^{t+1})\}$ is given by

$$m_{t+1}(s^{t+1}) = \beta \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma}.$$

Note that $m_{t+1}(s^{t+1})$ is simply a function of aggregate consumption growth and the two preference parameters (σ, β). Thus the equation

$$E \left(\beta \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma} R_{t+1}^j(s^{t+1}) | s^t \right) = 1$$

is the basic empirical restriction on asset returns $\{R_{t+1}^j\}$ and consumption growth $\{c_{t+1}/c_t\}$ implied by the complete markets model with CRRA preferences. This equation forms the basis of a large literature that investigates the asset pricing implications of the complete markets model (or the consumption capital asset pricing model, CCAPM, as it is sometimes called).

Now let us use and interpret condition (4.40) in the definition of a stochastic discount factor further. Note that (4.40) has to hold for all assets j traded in this economy, and thus for the risk free bond, in particular. Thus, for the risk-free bond we have[10]

$$
\begin{aligned}
E \left(m_{t+1}(s^{t+1}) R_{t+1}^B(s^t) | s^t \right) &= 1 \\
R_{t+1}^B(s^t) &= \frac{1}{E \left(m_{t+1}(s^{t+1}) | s^t \right)} \\
P_t^B(d; s^t) &= \frac{1}{R_{t+1}^B(s^t)} = E \left(m_{t+1}(s^{t+1}) | s^t \right)
\end{aligned}
$$

that is, the price of a risk-free bond equals the conditional expectation of the stochastic discount factor, which perhaps justifies the alternative name "pricing kernel" more directly.

For ease of notation for what follows, let $E_t = E(.|s^t)$ denote the conditional expectation. The stochastic discount factor then satisfies, for all assets j,

$$E_t(m_{t+1} R_{t+1}^j) = 1 \qquad (4.42)$$

---

[10]Using the fact that $R_{t+1}^B(s^t)$ is nonstochastic conditional on s^t, and employing the relation between asset prices and returns as the fact that we are considering a one-period bond that only pays off in period $t + 1$.

We can rewrite (4.42) as

$$1 = E_t(m_{t+1})E_t(R_{t+1}^j) + Cov_t(m_{t+1}, R_{t+1}^j)$$

and thus

$$E_t(R_{t+1}^j) = \frac{1 - Cov_t(m_{t+1}, R_{t+1}^j)}{E_t(m_{t+1})}.$$

Using the fact that $E_t(m_{t+1}) = 1/R_{t+1}^B$ we obtain

$$\frac{E_t(R_{t+1}^j)}{R_{t+1}^B} = 1 - Cov(m_{t+1}, R_{t+1}^j) \tag{4.43}$$

Defining net returns as $r_{t+1}^j = R_{t+1}^j - 1$ and using the approximation

$$\begin{aligned}
\frac{E_t(R_{t+1}^j)}{R_{t+1}^B} &= \frac{1 + E_t(r_{t+1}^j)}{1 + r_{t+1}^B} = \frac{1 + r_{t+1}^B + E_t(r_{t+1}^j) - r_{t+1}^B}{1 + r_{t+1}^B} \\
&= 1 + \frac{E_t(r_{t+1}^j) - r_{t+1}^B}{1 + r_{t+1}^B} \approx 1 + E_t(r_{t+1}^j) - r_{t+1}^B
\end{aligned}$$

in (4.43) we obtain

$$E_t(r_{t+1}^j) - r_{t+1}^B \approx -Cov_t(m_{t+1}, R_{t+1}^j) = -\rho_t(m_{t+1}, R_{t+1}^j)std_t(m_{t+1})std_t(R_{t+1}^j) \tag{4.44}$$

for any asset j, the risk-free bond B, and any stochastic discount factor m. Here ρ is the correlation coefficient and $std(.)$ denotes the standard deviation. That is, the (conditional) expected excess return of asset j over the risk-free rate r_{t+1}^B equals the (negative of) the covariance of the (gross) return of that asset j with the stochastic discount factor, as long as the real return on the risk free bond r_{t+1}^B between period t and $t + 1$ is close to zero, and thus the approximation used above is accurate.

Although (4.44) holds true for any asset j and any utility function and implied stochastic discount factor, the main implications of this equations have been studied for asset j being stocks, and thus $E_t(r_{t+1}^j) - r_{t+1}^B$ being the excess return on equity, and with the utility function of time-separable CRRA form. In this case equation (4.44) becomes:

$$E_t(r_{t+1}^S) - r_{t+1}^B = -\rho_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}, R_{t+1}^S \right] * std_t \left(\beta \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma} \right) * std_t \left(R_{t+1}^S \right). \tag{4.45}$$

One quick way to assess whether the representative agent model has a chance to rationalize the observed excess returns on equity is the following. From the data the annual average premium to be explained is roughly 7%. The annual standard deviation of stock returns is roughly 15.5% in the data. We also observe the time series of aggregate consumption growth $\left\{ \frac{c_{t+1}}{c_t} \right\}$ and returns $\left\{ R_{t+1}^{stock} \right\}$ in the data. One simple way to ask whether (or, to what extent) the complete markets model can rationalize the equity premium is simply to calculate the right hand side for given (σ, β), replacing theoretical with sample moments. The results of this exercise, which are carried out by Kocherlakota (1996) and many others since are disastrous for the model, so let us explain why by interpreting (4.45).

First, the is no hope in explaining *any* premium if $\rho_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}, R_{t+1}^S \right] >$ 0, that is, if (loosely speaking) consumption growth and stock returns are negatively correlated (since c_{t+1}/c_t is raised to a negative power). Any asset that pays well in $t+1$ (i.e. has high returns between period t and $t+1$) in bad states of the world (low consumption growth c_{t+1}/c_t) is a good hedge against consumption risk, and thus does not command any risk premium (in fact a negative premium). Thus it is the positive *correlation* between consumption growth and asset returns[11] that generate the equity premium, rather than the fact that stock returns are very risky. Of course, conditional on a given positive correlation between c_{t+1}/c_t and R_{t+1}^S, the *size* of the explained equity premium is increasing in the risk of stock returns and the volatility of consumption growth. In addition, high values of σ will, for a given dispersion of consumption growth, help to explain a larger share of the equity premium, by raising $\frac{1}{c_{t+1}/c_t}$ to larger powers. Given the empirically weak, but positive correlation between c_{t+1}/c_t and R_{t+1}^S and the low volatility of c_{t+1}/c_t, the asset pricing literature was led to conclude that the empirically observed excess returns on stocks can only be rationalized with implausibly large values of σ. This is a version of Mehra and Prescott's (1985) statement of the equity premium puzzle.[12]

---

[11]Again, the precise statement is: the negative correlation between $\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}$ and R_{t+1}^S.

[12]Mehra and Prescott (1985) considered values for σ above 10 as implausible and showed that for values of σ below that threshold the model-implied equity premium is smaller than the target in the data by factor of at least 10.

An alternative way to state the puzzle is to realize that $-\rho_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}, R_{t+1}^S \right] \leq$ 1, since correlations are bounded between -1 and 1. Then equation (4.45) implies that

$$\frac{E_t(r_{t+1}^S) - r_{t+1}^B}{std_t\left(R_{t+1}^S\right)} = -\rho_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}, R_{t+1}^S \right] * std_t \left(\beta \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma} \right)$$

$$\frac{E_t(r_{t+1}^S) - r_{t+1}^B}{std_t\left(r_{t+1}^S\right)} \leq std_t(m_{t+1})$$

The entity on the left hand side is the so-called Sharpe ratio, which gives the excess return a given asset (here stocks) commands per unit of risk (as measured by the standard deviation of the return). The equation gives an upper bound for that ratio: it cannot be larger than the volatility of the stochastic discount factor. This bound, derived in various forms by Shiller (1982) and Hansen and Jagannathan (1991), is valid for any asset (and any stochastic discount factor). It indicates that a model that does not generate a volatile enough stochastic discount factor has no chance in explaining the equity premium puzzle. For annual data, the Sharpe ratio for stocks is in the order of about $0.3 - 0.5$, whereas the standard deviation of consumption growth is not larger than 0.035 (the exact value depends somewhat on the sample period). Thus without large values for σ (and perhaps for all possible σ) the Hansen-Jagannathan bound is clearly violated by the unique stochastic discount factor implied by the complete markets model: consumption growth is simply too smooth in the data to generate a large enough equity premium, even if it was perfectly correlated with stock returns (which it is not, either).

But was is the problem with a large σ, absent direct empirical evidence[13] on its value? First, one may argue that households endowed with such large risk aversion would make other choices (e.g. buy large amounts of all kinds of insurance) that are at odds with the data. what is the problem with assuming a large value for σ? But perhaps more importantly, such high assumed values would raise a second asset pricing puzzle. To see why, note that the risk-free

---

[13]Work by Barsky et al. (1997) and Kimball, Sahm and Shapiro (2008, 2009) has used survey responses from the HRS and PSID to shed some light on the empirical distribution of risk aversion in the U.S. population.

rate satisfies

$$R_{t+1}^B = \frac{1}{E_t(m_{t+1})} = \frac{1}{E_t\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right)} = \frac{1}{\beta E_t\left(\frac{1}{\left(\frac{c_{t+1}}{c_t}\right)^{\sigma}}\right)}$$

Since in the data consumption is growing over time, thus c_{t+1}/c_t tends to be positive and on average around 1.02 in U.S. data. Therefore making σ large makes $E_t(m_{t+1})$ small, and thus R_{t+1}^B large. Why is this the case? A large σ implies small $IES = \frac{1}{\sigma}$. With a small IES households desire a smooth consumption profile. But consumption grows in the data. Thus a large interest rate is needed to persuade them to postpone consumption. In the data, however, the real risk free rate is small, around 1% on average. This is Weil's (1989) risk free rate puzzle.

One possible resolution to this puzzle is simply to make β large (larger than 1!).[14] As long as $\beta(1 + g_c)^{1-\sigma} < 1$ (on average), there is no problem for lifetime utility to converge. But $\beta > 1$ and the associated degree of patience might be considered implausible. Alternatively one might contemplate utility functions in which a households' attitude towards risk (variation of consumption across states of the world) and towards intertemporal smoothing (variation of consumption over time) is *not* governed by the same parameter. Epstein and Zin (1989. 1991) propose such a class of utility functions:

Remark 17 *Epstein and Zin (1989, 1991), extending earlier work by Kreps and Porteus (1978) propose a class of utility functions that deviate from expected utility and that can be written as*

$$u(c, s^t) = \left\{ c_t(s^t)^{1-\frac{1}{\gamma}} + \beta \left[\sum_{s^{t+1}} \pi_t(s^{t+1}|s^t) u(c, s^{t+1})^{1-\sigma} \right]^{\frac{1-\frac{1}{\gamma}}{1-\sigma}} \right\}^{\frac{1}{1-\frac{1}{\gamma}}}$$

where $u(c, s^t)$ is lifetime (continuation) utility from a consumption allocation c from node s^t onwards, and $u(c, s^{t+1})$ has the same interpretation at node

---

[14]Suppose that consumption grows at a constant rate of 2%, then to rationalize $R^B = 1.01$ in the presence of large risk aversion (say $\sigma = 10$, for an IES of 0.1) we would need

$$\beta = \frac{1}{\frac{R^B}{\left(\frac{c_{t+1}}{c_t}\right)^{\sigma}}} = 1.21.$$

s^{t+1}. *With these preferences the IES is controlled by the parameter γ, and risk aversion is controlled by the separate parameter σ. These preferences have become extremely popular in modern consumption-based asset pricing because they help to jointly explain low risk-free rates as well as high excess returns on stocks.*

4.2.1   An Example

In this section we present an example for which we can solve for the risk free rate and the equity premium in closed form.[15] This is useful since in the example it becomes fully transparent which parameters determine the risk-free rate and the equity premium. The economy is populated by representative consumers and the income of these households comes from dividends. Households trade risk-free bonds in zero net supply and shares of the stock which entitles them to the dividends of the stock.[16] Also suppose that dividends follow the process

$$\log(d_{t+1}) = \log(d_t) + g + u_{t+1}$$

where u_{t+1} is *iid* normally distributed with zero mean and variance σ_u^2. Since dividends are the only form of income to the representative household we have $c_t = d_t$ for all t and all states of the world. Thus consumption satisfies $\frac{c_{t+1}}{c_t} = e^{g+u_{t+1}}$. We can now use this in our asset pricing equation $E_t(m_{t+1}R_{t+1}^j) = 1$. For the risk-free rate we obtain

$$R_{t+1}^B E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \right] = 1$$

$$R_{t+1}^B E_t \left[\beta e^{-\sigma(g+u_{t+1})} \right] = 1$$

$$R_{t+1}^B \beta e^{-\sigma g} E_t \left[e^{-\sigma u_{t+1}} \right] = 1$$

Now we note that because u_{t+1} is normal with zero mean and variance σ_u^2 we have $E_t \left[e^{-\sigma u_{t+1}} \right] = e^{\frac{1}{2}\sigma^2 \sigma_u^2}$ and thus the risk-free rate is given by

$$R^B = \frac{1}{\beta} e^{\sigma \left[g - \frac{1}{2}\sigma \sigma_u^2 \right]}$$

---

[15]This is a simplified version of the example presented in Barro (2009) which excludes disaster risk.

[16]Whether or not households also trade a full set of Arrow securities is not crucial in this representative agent economy since there is no trade in equilibrium.

or

$$r^B = \log(R^B) = \rho + \sigma g - \frac{1}{2}\sigma^2\sigma_u^2 \tag{4.46}$$

where we used the definition of the time discount rate ρ as $\beta = \frac{1}{1+\rho}$ and the approximation $\rho \approx \log(1+\rho)$. We see that the more patient households are (the larger is ρ), the higher is the risk free rate. A higher consumption (dividend) growth rate g and a lower intertemporal elasticity of substitution raise the risk-free rate as well, for the reason discussed above. Finally, the last term shows that the risk-free rate falls with the amount of consumption risk. As we will discuss below, with CRRA preferences households have a precautionary savings motive (whose size is governed by the parameter σ as well) that is more potent the larger is consumption risk. To insure that households are willing to consume their dividends in the presence of the extra incentive to save, the risk free rate has to fall. This rationalizes the negative last term in equation (4.46).

Solving for the expected return on equity is harder since we don't know the endogenous price of stock P_t^S. But note that

$$R_{t+1}^S = \frac{d_{t+1} + P_{t+1}^S}{P_t^S} = \left(\frac{d_{t+1}}{P_{t+1}^S} + 1\right)\frac{P_{t+1}^S}{P_t^S}.$$

Now conjecture that the price-dividend ratio $\frac{P_{t+1}^S}{d_{t+1}}$ is a time-and state invariant constant so that $P_{t+1}^S = \kappa d_{t+1}$. Then

$$R_{t+1}^S = \left(\frac{d_{t+1}}{P_{t+1}^S} + 1\right)\frac{P_{t+1}^S}{P_t^S} = \left(\frac{1}{\kappa} + 1\right)\frac{d_{t+1}}{d_t} = \theta\frac{c_{t+1}}{c_t}$$

where $\theta = \left(\frac{1}{\kappa} + 1\right)$ is a yet to be determined constant. Again using the asset pricing equation, this time for stocks

$$E_t(m_{t+1}R_{t+1}^S) = 1$$

$$E_t\left[\beta\theta\left(\frac{c_{t+1}}{c_t}\right)^{1-\sigma}\right] = 1$$

$$\beta\theta E_t\left[e^{(1-\sigma)(g+u_{t+1})}\right] = 1$$

Again using $E_t\left[e^{(1-\sigma)u_{t+1}}\right] = e^{\frac{1}{2}(1-\sigma)^2\sigma_u^2}$ we have

$$\theta = \frac{1}{\beta}e^{(1-\sigma)\left[-g-\frac{1}{2}(1-\sigma)\sigma_u^2\right]}$$

Thus

$$
\begin{aligned}
R_{t+1}^S &= \frac{1}{\beta} e^{(1-\sigma)\left[-g-\frac{1}{2}(1-\sigma)\sigma_u^2\right]} e^{(g+u_{t+1})} \\
&= \frac{1}{\beta} e^{g\sigma - \frac{1}{2}(1-\sigma)^2 \sigma_u^2} e^{u_{t+1}} \\
E_t R_{t+1}^S &= \frac{1}{\beta} e^{g\sigma - \frac{1}{2}\sigma_u^2\left[(1-\sigma)^2-1\right]} = \bar{R}^S \\
\bar{r}^S &= \log(\bar{R}^S) = \rho + g\sigma + \frac{1}{2}\sigma(\sigma+2)\sigma_u^2
\end{aligned}
$$

Finally, the equity premium is given by

$$
\begin{aligned}
\bar{r}^S - r^B &= \rho + g\sigma + \frac{1}{2}\sigma(\sigma+2)\sigma_u^2 - \left[\rho + \sigma g - \frac{1}{2}\sigma^2\sigma_u^2\right] \\
&= \frac{1}{2}\sigma(\sigma+2-\sigma)\sigma_u^2 = \sigma\sigma_u^2
\end{aligned}
$$

and thus is simply an increasing function of risk aversion and the volatility of consumption (dividends). For similar analyses using Epstein-Zin preferences, see e.g. Weil (1989), Tallarini (2000) or Barro (2009).

4.3   Tests of Complete Consumption Insurance

As we have seen in section 4.1 a efficient consumption allocation features perfect consumption insurance. Since equilibrium allocations in the complete markets model are efficient, they share this property as well. In this section we discuss empirical tests of this empirical prediction of the complete markets model. Important papers implementing such tests are Altug and Miller (1990), Mace (1991), Cochrane (1991), Nelson (1994), Townsend (1994), Attanasio and Davis (1996), Hayashi, Altonji and Kotlikoff (1996), Schulhofer-Wohl (2011) and Mazzocco and Saini (2012). Here we restrict attention on the paper by Mace, whose test using U.S. data follows perhaps most directly from the theory, and to the most recent extension by Mazzocco and Saini (2012) using Indian village data.

4.3.1 Derivation of Mace's (1991) Empirical Specifications

The paper by Mace presents the most straightforward test of the complete consumption insurance hypothesis. The main implication of perfect risk sharing is that individual consumption should respond to aggregate income (or consumption) shocks, but not to idiosyncratic income shocks, no matter whether these idiosyncratic shocks are unanticipated or not anticipated (this is important in discriminating the complete markets model from the standard incomplete markets models to be discussed in latter chapters).

Recall that under assumptions 1 and 2, equation (4.19) implies that

$$\Delta \ln c_t^i = \Delta \ln c_t \qquad (4.47)$$

that is, the growth rate of individual consumption $\Delta \ln c_t^i \equiv \ln c_t^i(s^t) - \ln c_{t-1}^i(s^{t-1}) \approx \frac{c_t^i - c_{t-1}^i}{c_{t-1}^i}$ should equal the growth rate of aggregate consumption $\Delta \ln c_t$, and be independent of individual income growth and other individual-specific shocks. That is, in the regression

$$\Delta \ln c_t^i = \alpha_1 \Delta \ln c_t + \alpha_2 \Delta \ln y_t^i + \epsilon_t^i \qquad (4.48)$$

under the null hypothesis of complete consumption insurance (and with CRRA utility) $\alpha_1 = 1$ and $\alpha_2 = 0$. In this regression the error term ϵ_t^i captures (thus far unmodeled) individual preference shocks as well as measurement error in income and or consumption growth.

We can generalize our assumption 2 and still arrive at an empirical specification similar to equation (4.48) that can be used to test for complete consumption insurance. Suppose that household preferences are of the form

$$U^i(c_t^i(s^t), s^t) = e^{b^i(s^t)} \cdot \frac{c_t^i(s^t)^{1-\sigma} - 1}{1 - \sigma} \qquad (4.49)$$

where $b^i(s^t)$ capture preference shocks such as changes in family size (which are taken to be exogenous). Taking first order conditions in the planners problem of maximizing

$$\sum_i \sum_t \sum_{s^t} \alpha^i \beta^t \pi_t(s^t) U^i(c_t^i(s^t), s^t)$$

$$s.t.$$

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t)$$

and then taking logs of these conditions yields

$$\ln(c_t^i(s^t)) = \frac{1}{\sigma}b^i(s^t) + \frac{1}{\sigma}\ln(\alpha^i) - \frac{1}{\sigma}\ln\left(\frac{\lambda_t(s^t)}{\beta^t \pi_t(s^t)}\right) \qquad (4.50)$$

Here $\lambda_t(s^t)$ is the Lagrange multiplier on the resource constraint, as before.

By taking averages of (4.50) over all agents we obtain

$$\frac{1}{N}\sum_j \ln(c_t^j(s^t)) = \frac{1}{\sigma N}\sum_j b^j(s^t) + \frac{1}{\sigma N}\sum_j \ln(\alpha^j) - \frac{1}{\sigma}\ln\left(\frac{\lambda(s^t)}{\beta^t \pi_t(s^t)}\right) \quad (4.51)$$

and using this equation to substitute out $\frac{1}{\sigma}\ln\left(\frac{\lambda(s^t)}{\beta^t \pi_t(s^t)}\right)$ in equation (4.50) yields:

$$
\begin{aligned}
\ln(c_t^i(s^t)) &= \frac{1}{\sigma}\left(b^i(s^t) - \frac{1}{N}\sum_j b^j(s^t)\right) + \frac{1}{\sigma}\left(\ln(\alpha^i) - \frac{1}{N}\sum_j \ln(\alpha^j)\right) \\
&\quad + \frac{1}{N}\sum_j \ln(c_t^j(s^t)) \\
&= \frac{1}{\sigma}\left(b^i(s^t) - b^a(s^t)\right) + \frac{1}{\sigma}\left(\ln(\alpha^i) - \ln(\alpha^a)\right) + \ln(c_t^a(s^t)) \quad (4.52)
\end{aligned}
$$

where

$$b^a(s^t) \equiv \frac{1}{N}\sum_j b^j(s^t)$$

$$\ln(\alpha^a) \equiv \frac{1}{N}\sum_j \ln(\alpha^j)$$

$$\ln(c_t^a(s^t)) \equiv \frac{1}{N}\sum_j \ln(c_t^j(s^t)) \qquad (4.53)$$

are population averages. Note that we have abused notation somewhat in the last two expressions, which are sums of logs rather than logs of sums. Finally, by taking first differences of (4.52) to get rid off the term $\frac{1}{\sigma}\left(\ln(\alpha^i) - \ln(\alpha^a)\right)$ we arrive at an equation that can be brought to the data (and again suppressing dependence on s^t):

$$\Delta \ln(c_t^i) = \Delta \ln(c_t^a) + \frac{1}{\sigma}\left(\Delta b_t^i - \Delta b_t^a\right) \qquad (4.54)$$

Thus, with this specification of preferences we can test the complete insurance hypothesis with the empirical specification

$$\Delta \ln(c_t^i) = \alpha_1 \Delta \ln(c_t^a) + \alpha_2 \Delta \ln(y_t^i) + \epsilon_t^i \tag{4.55}$$

where the error ϵ_t^i now captures individual and aggregate changes in preference shocks as well as potentially measurement error.[17] The null hypothesis of complete consumption insurance implies that $\alpha_1 = 1$ and $\alpha_2 = 0$. Thus the regression equation used to test for complete consumption insurance remains substantially unchanged under the more general utility specification with preference shocks, keeping in mind that in the definition above, $\ln(c_t^a(s^t)) \neq \ln(c_t)$. Note that the basic prediction of the theory is that individual household i's consumption should not respond to idiosyncratic shocks once aggregate consumption is accounted for. Idiosyncratic *income* shocks are just one, arguably important, but not the only source of idiosyncratic risk. Including other measures of idiosyncratic shocks in regression (4.55) and testing the complete consumption insurance by investigating whether the corresponding regression coefficient(s) equals zero is equally valid, from the perspective of the theory presented so far. See especially Cochrane (1991) on this point.

Finally, we can derive a similar specification under the assumption that preferences take a Constant Absolute Risk Aversion (CARA) form. Suppose the period utility function is given by:

$$U^i(c_t^i(s^t), s^t) = -\frac{1}{\gamma} e^{-\gamma \left(c_t^i(s^t) - b^i(s^t)\right)} \tag{4.56}$$

where γ is the coefficient of absolute risk aversion, then, using the same manipulations as before one obtains

$$\Delta c_t^i = \Delta c_t^m + \left(\Delta b_t^i - \Delta b_t^m\right) \tag{4.57}$$

---

[17]Note that, defining

$$\epsilon_t^i = \frac{1}{\sigma} \left(\Delta b_t^i - \Delta b_t^a\right)$$

then the cross-sectional expectation of ϵ_t^i across households i equals zero, by definition of b_t^a.

where

$$c_t^m = \frac{1}{N} \sum_j c_t^i$$

$$b_t^m = \frac{1}{N} \sum_j b_t^i \qquad (4.58)$$

Thus, under exponential utility the empirical specification used to test complete consumption insurance is

$$\Delta c_t^i = \alpha_1 \Delta c_t^m + \alpha_2 \Delta y_t^i + \epsilon_t^i \qquad (4.59)$$

with the null hypothesis being that $\alpha_1 = 1$ and $\alpha_2 = 0$.

4.3.2   Results of the Tests

Mace uses CEX data from 1980 to 1983 to estimate equations (4.55) and (4.59). We focus our discussion on the specifications in which individual income growth is used as measure of idiosyncratic risk, rather than changes in unemployment status. In order to implement the regressions household level data on consumption and income are needed. For each household, since a specification in first differences is employed (in order to remove the individual fixed effects), we require at least two observations on both income and consumption. Thus, although the U.S. Consumer Expenditure Survey (CEX) has no extended panel dimension, the fact that it contains income and consumption data for four quarters for each household makes this data set an appropriate (if not ideal) source for Mace's application.[18][19]

The total number of households in the CEX sample that Mace employs is $10,695$. Each household contributes one differenced observation; Mace chooses to use the information from the first and the forth interview of each household (remember that each household stays in the sample for only 4 quarters) to obtain differenced consumption and income observations. For the growth rate specification households that report non-positive income or

---

[18]For results using more recent CEX data that are broadly consistent with Mace's findings, see Krueger and Perri (2005).

[19]Apart from the issue of measurement error in the income and consumption data, as Gervais and Klein point out (and then address), the fact that income and consumption are measured for different and only partially overlapping time intervals creates additional problems when implementing the risk sharing regressions using CEX data.

consumption in any of the two quarters have to be excluded, so that the sample size for this specification is smaller. Mace employs several measures of consumption in her analysis; we will report results for total consumption expenditures, for nondurables, for services and for food expenditures. The income variable is disposable income, defined as before-tax income minus income taxes, deductions for social security and pension plans as well as occupational expenses (e.g. union dues).

In Table 1 we present the results from the estimation of equation (4.55); in addition Mace includes a constant to the regression. The first three columns present the OLS estimates (with standard errors in parentheses) for the constant, the coefficient on aggregate consumption growth and on individual income growth. The forth column gives the test statistic of an F-test[20] of the joint hypothesis that $\alpha_1 = 1$ and $\alpha_2 = 0$ and the last column shows the R^2 of the regression.

Table 1: Growth Rate Specification					
Cons. Measure	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	F-test	R^2
Total Consumption	−0.04 (0.01)	1.06 (0.08)	0.04 (0.007)	14.12*	0.021
Nondurables	−0.02 (0.01)	0.97 (0.07)	0.04 (0.006)	22.69*	0.027
Services	−0.02 (0.01)	0.93 (0.10)	0.04 (0.01)	12.44*	0.011
Food	−0.02 (0.01)	0.91 (0.07)	0.04 (0.006)	18.67*	0.020

The results in Table 1 are, for the most part, inconsistent with the complete insurance hypothesis. For all consumption groups the F-test is decisively rejected at the 5% confidence level (in fact it would be rejected at the 1% confidence level). Loosely speaking, this is mostly due to the fact that individual income growth *does* help to explain individual consumption growth,

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[20]The test statistic for the F-test is distributed as an $F(2, N − 3)$, where N is the number of usable household observations in the sample. For the difference specification $N = 10,695$ whereas for the growth rate specification it varies by consumption category because households with nonpositive consumption in either interview have to be excluded.

A ∗ after the test statistic indicates that the null hypothesis can be rejected at the 95% confidence level.

even when aggregate consumption growth is accounted for, in contrast to what is implied by the complete consumption insurance hypothesis. Note that for all consumption measures, based on a simple t-test the null hypothesis that $\alpha_1 = 1$ cannot be rejected at conventional confidence levels, but the hypothesis that $\alpha_2 = 0$ can be soundly rejected for all consumption measures. Also note that the constant is on the edge of being significant for most specifications and that the R^2 is quite low for all specifications. Although the complete consumption insurance hypothesis can be statistically rejected, note that the response of consumption to changes in income is quantitatively small. Thus one might be lead to conclude that the hypothesis provides a useful approximation to the data. As we will discuss below, measurement error in income might question this assessment. Furthermore we will argue in chapter 5 that the standard *incomplete* markets model (which does not predict perfect consumption insurance) would imply consumption responses to income shocks of a rough magnitude reported here *if* most income shocks are of transitory nature.

In Table 2 we report the results for the specification relying on CARA utility, i.e. the estimates of equation (4.59), again including an intercept.

Table 2: First Difference Specification					
Cons. Measure	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	F-test	R^2
Total Consumption	-77.87 (19.32)	1.06 (0.11)	0.03 (0.02)	1.27	0.008
Nondurables	-13.97 (3.33)	0.99 (0.06)	0.01 (0.003)	7.71^*	0.023
Services	-30.47 (16.47)	1.01 (0.10)	0.01 (0.007)	1.14	0.009
Food	-7.46 (2.12)	1.01 (0.08)	0.005 (0.002)	2.52	0.020

For this specification the results are more supportive of perfect consumption smoothing. In particular, the F-test is rejected only for nondurable consumption expenditures, again primarily due to the fact that individual consumption growth is sensitive to individual income growth. Note that here the intercept is significant for almost all measures of consumption (in theory it should be zero). It is these results, in addition to the findings that the estimated value of α_2 is quantitatively small, that lead Mace to the conclusion

that perfect risk sharing is a good first approximation of the data, at least for exponential utility.

However, as Nelson (1994) argues, if one excludes households classified as "incomplete income reporters" by the CEX (households that either did not respond to certain income questions or gave inconsistent answers) and uses expenditures for the entire quarter as the relevant measure of consumption (as opposed to Mace who used only the expenditure in the month preceding the interview) one finds a strong rejection of complete consumption insurance even for the CARA utility specification.

4.3.3 The Problem of Measurement Error

Mace (1991) discusses several econometric issues arising in her regression analysis. Somewhat surprisingly the paper does not include an extended discussion of measurement error, which is a serious issue in micro data sets in general, and for the CEX in particular. Given the survey design of the CEX measurement error in the income variable is most likely the biggest concern; as we will show it may bias the test of perfect risk sharing in favor of the null hypothesis of perfect consumption insurance.

To make things simple let us suppose we want to estimate the equation

$$\Delta c_t^i = \alpha_2 \Delta y_t^i + \epsilon_t^i \tag{4.60}$$

i.e. for the moment we ignore the presence of aggregate consumption growth. Note that if we had a single cross-section of first differenced household observations, one would exactly run a regression of the from (4.60).[21] For simplicity also assume that in the cross-section $E\left(\Delta y_t^i\right) = 0$, that is, individual income changes are measured in deviation from aggregate income changes.[22] The null hypothesis of perfect risk sharing implies $\hat{\alpha}_2 = 0$. For ease of exposition we abstract from a constant in the regression and assume $\alpha_0 = 0$.

Suppose we have N household observations for one period t, and suppose that the income variable is measured with error

$$\Delta z_t^i = \Delta y_t^i + \Delta v_t^i \tag{4.61}$$

---

[21]See Cochrane (1991) for a discussion why the pooling of cross-sectional and time series data is not free of problems and for an implementation of a test of complete risk sharing using purely cross-sectional data.

[22]This assumption is not central for the argument but makes the algebraic expressions slighly easier.

where Δz_t^i is the measured income change, Δy_t^i is the true income change and Δv_t^i is an additive measurement error satisfying $E\left(\Delta v_t^i\right) = 0$, where $E(.)$ is the cross-sectional (across households) expectation, recalling that we hold t fixed. Furthermore assume that $Var\left(\Delta v_t^i\right) = \sigma_v^2$, that $Var\left(\Delta y_t^i\right) = \sigma_y^2$ and that $\Delta v_t^i, \epsilon_t^i$ and Δy_t^i are all mutually independent.[23]

The equation we estimate is then

$$\Delta c_t^i = \alpha_2 \Delta z_t^i + \epsilon_t^i \tag{4.62}$$

and the OLS estimate for α_2 is

$$\hat{\alpha}_2 = \frac{N^{-1}\sum_{j=1}^{N}\Delta c_t^j \Delta z_t^j}{N^{-1}\sum_{j=1}^{N}\left(\Delta z_t^j\right)^2} \tag{4.63}$$

We now show that $\hat{\alpha}_2$ is an inconsistent estimator of α_2, with persistent bias towards zero. We have that, multiplying out and using Slutsky's theorem

$$
\begin{aligned}
\operatorname{plim}_{N\to\infty}\hat{\alpha}_2 &= \operatorname{plim}_{N\to\infty}\frac{N^{-1}\sum_{j=1}^{N}\Delta c_t^j \Delta z_t^j}{N^{-1}\sum_{j=1}^{N}\left(\Delta z_t^j\right)^2} \\
&= \operatorname{plim}_{N\to\infty}\frac{N^{-1}\sum_{j=1}^{N}\left(\alpha_2\Delta y_t^i + \epsilon_t^i\right)\left(\Delta y_t^j + \Delta v_t^j\right)}{N^{-1}\sum_{j=1}^{N}\left(\Delta y_t^j + \Delta v_t^j\right)^2} \\
&= \alpha_2\frac{\operatorname{plim}_{N\to\infty}N^{-1}\sum_{j=1}^{N}\left(\Delta y_t^i\right)^2}{\operatorname{plim}_{N\to\infty}N^{-1}\sum_{j=1}^{N}\left(\Delta y_t^j\right)^2 + \operatorname{plim}_{N\to\infty}N^{-1}\sum_{j=1}^{N}\left(\Delta v_t^j\right)^2} \\
&= \alpha_2 \cdot \frac{\sigma_y^2}{\sigma_y^2 + \sigma_v^2} = \alpha_2 \cdot \frac{1}{1 + \frac{\sigma_v^2}{\sigma_y^2}}
\end{aligned}
$$

Thus, without measurement error ($\sigma_v^2 = 0$) $\hat{\alpha}_2$ estimates α_2 consistently. However, the higher the ratio between measurement noise and signal, σ_v^2/σ_y^2 the larger is the "attenuation bias" and the more is the estimator $\hat{\alpha}_2$ asymptotically biased towards zero. Therefore, in the presence of substantial measurement error in income growth or income changes we may not reject the null of perfect risk sharing even when it should be rejected had we observed the data without measurement error. Furthermore we might conclude that

---

[23]The assumption that ϵ_t^i and Δy_t^i are independent is already needed to make the OLS estimate for α_2 consistent even in the absence of measurement error.

the deviations from complete consumption insurance are quantitatively small in the presence of a small estimate $\hat{\alpha}_2$ when in fact that low estimate is due to measurement error in income. This discussion is meant to give an indication for why even the results in Mace supporting perfect risk sharing should be taken with caution; they are also meant to motivate similar analyses like the one in Cochrane (1991) that focus on measures of idiosyncratic income risk which may be less prone to measurement error and correlation with the error term than changes in or growth rates of individual income.

4.3.4 Other Empirical Issues and Solutions

In principle all variables that capture idiosyncratic shocks to a households' consumption possibility set could be used as regressor on the right hand side of the consumption insurance regressions. As Cochrane (1991) points out, a good regressor should have the property that it is uncorrelated with household preference shocks (and thus the error term) and should be measured precisely and if it has measurement error, this measurement error should not be correlated with measurement error in consumption. Household income is problematic in these regards.

Cochrane therefore proposes other regressors that are less prone to measurement error and can be more plausibly taken as exogenous with respect to household preference shocks. The variables he considers are days of work lost due to sickness, due to strike, days looking for a job etc. His empirical results, derived for food consumption from the PSID are mixed; perfect consumption insurance is rejected for some variables, but not for others. An alternative approach to the problem of household income growth being correlated with the error term is to use an IV estimator. For this strategy to be successful one needs to find an instrument that is correlated with income but not correlated with household preference shocks (and thus the error term in the regressions above).

4.3.5 Self-Selection into Occupations with Differential Aggregate Risk

If households are heterogeneous with respect to their risk aversion (heterogeneity that will show up in the error term) and self-select into occupation that are differentially affected by aggregate risk, then individual income will

be correlated with the error term and the standard consumption insurance regressions will yield inconsistent result. Future versions of these notes will discuss the papers by Schulhofer-Wohl (2011) and Mazzocco and Saini (2012) in greater detail.

Consider the following example based on Mazzocco and Saini (2012):

$$\max_{\{c_t^i(s^t), l_t^i(s^t)\}} \sum_{i=1}^{2} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \alpha^i \beta^t \pi_t(s^t) U^i(c_t^i(s^t))$$

$$s.t.$$

$$\sum_{i=1}^{N} c_t^i(s^t) = \bar{Y}_t(s^t)$$

$$c_t^i(s^t) \geq 0$$

- Standard efficiency condition

$$\alpha^1 U_c^1(c_t^1(s^t)) = \alpha^2 U_c^2(c_t^2(s^t))$$

  which together with resource constraint determines efficient split of $\bar{Y}_t(s^t)$.

- Let U^1, U^2 be CRRA with $\sigma_1 < \sigma_2$.

- Plotting $\alpha^i U_c^i(c_t^i(s^t))$ against $c_t^i(s^t)$, the curves for $i = 1$ and $i = 2$ intersect once (and only once). If $\sigma_1 = \sigma_2$, they are parallel to each other (and on top of each other if $\alpha^1 = \alpha^2$).

- Plotting the efficient $c_t^i(s^t)$ against $\bar{Y}_t(s^t)$, the curves (expenditure functions) intersect once (and only once) if $\sigma_1 < \sigma_2$. Let intersection point be denoted by $\bar{Y}^*$.

- Consumption of household 1 efficiently varies more with $\bar{Y}_t(s^t)$ than consumption of household 2. Less risk averse household bears more of the consumption risk.

- Test of whether expenditure functions of households intersect forms basis of test of preference heterogeneity.

- Absent preference shocks, with identical CRRA preferences

$$\Delta \log c_{t+1}^i = \Delta \log c_{t+1}^a$$

where

$$\log c_{t+1}^a = \frac{1}{2} \sum_{i=1}^{2} \log c_{t+1}^i$$

- Now suppose $\bar{Y}_t < \bar{Y}^* < \bar{Y}_{t+1}$, then with efficient risk sharing and $\sigma_1 < \sigma_2$ we have

$$\Delta \log c_{t+1}^2 = \Delta \log c_{t+1}^a < \Delta \log c_{t+1}^1$$

which a researcher that presupposes $\sigma_1 = \sigma_2$ has to interpret as rejection of efficient risk sharing.

- This shows potential problem of risk sharing regressions with preference heterogeneity.

- **Signing the Bias** Suppose households share risk efficiently

- Suppose households have CRRA utility with $\sigma_i \neq \sigma_j$.

- Suppose that

  - if $\Delta \bar{Y}_{t+1} \geq 0$, then $\Delta \log y_{t+1}^i$ is decreasing in σ_i (in good aggregate times the least risk averse do particularly well)
  - if $\Delta \bar{Y}_{t+1} < 0$, then $\Delta \log y_{t+1}^i$ is increasing in σ_i (in bad aggregate times the most risk averse do particularly well)

- Then in the regression

$$\Delta \log c_{t+1}^i - \Delta \log c_{t+1}^a = \xi \Delta \log y_{t+1}^i + \varepsilon_{t+1}^i$$

we have $E(\hat{\xi}) > 0$, despite the fact that households share risk efficiently.

- **Intuition**

$$E(\hat{\xi}) = \xi + \frac{Cov(\Delta \log y_{t+1}^i, \varepsilon_{t+1}^i)}{Var(\Delta \log y_{t+1}^i)}$$

- Since households share risk efficiently, the true $\xi = 0$. But $Cov(\Delta \log y_{t+1}^i, \varepsilon_{t+1}^i) > 0$. Why?

- In bad times $\Delta \bar{Y}_{t+1} < 0$, $\Delta \log c_{t+1}^i$ is particularly low for low σ households (thus $\varepsilon_{t+1}^i < 0$) and $\Delta \log y_{t+1}^i$ is low for these households. Reversely for high σ households

- In good times the reverse is true.

- Thus $Cov(\Delta \log y_{t+1}^i, \varepsilon_{t+1}^i) > 0$

- Key problem: unobserved σ_i is an omitted variable correlated with $\Delta \log y_{t+1}^i$ and with error term ε_{t+1}^i. Results in biased regression result.

- IV may be hard because need an instrument that is uncorrelated with σ_i and correlated with income. Better not be an endogenous choice.

- Note: Schulhofer-Wohl (2011) argues that in the data households with low σ indeed self-select into jobs with income growth more correlated with aggregate income.

- **Tests of Efficient Risk Sharing with Heterogeneity in Risk Preferences** Social Planner Problem

$$\max_{\{c_t^i(s^t), l_t^i(s^t)\}} \sum_{i=1}^N \sum_{t=0}^\infty \sum_{s^t \in S^t} \alpha^i \beta^t \pi_t(s^t) U^i(c_t^i(s^t), l_t^i(s^t))$$

$$s.t.$$
$$\sum_{i=1}^N c_t^i(s^t) = \sum_{i=1}^N w_t^i(s^t) l_t^i(s^t) + Y_t(s^t)$$
$$c_t^i(s^t) \geq 0, l_t^i(s^t) \in [0,1]$$

where $Y_t(s^t)$ is nonlabor endowment in economy. Note: relative to Mazzocco and Saini we suppress preference shocks.

- Rewrite resource constraint as

$$\sum_{i=1}^N \left(c_t^i(s^t) + w_t^i(s^t)(1 - l_t^i(s^t)) \right)$$
$$= \sum_{i=1}^N w_t^i(s^t) + Y_t(s^t) \equiv \bar{Y}_t(s^t)$$

- Define "expenditure" of household i as

$$\rho_t^i(s^t) = c_t^i(s^t) + w_t^i(s^t)(1 - l_t^i(s^t))$$

- Split social planner problem into three steps

- Step 3: Conditional on given $\rho_t^i(s^t)$, what is the efficient split between consumption and leisure

$$
\begin{aligned}
V^i(\rho_t^i(s^t), w_t^i(s^t)) &= \max_{c_t^i(s^t), l_t^i(s^t)} U^i(c_t^i(s^t), l_t^i(s^t)) \\
&\text{s.t.} \\
\rho_t^i(s^t) &= c_t^i(s^t) + w_t^i(s^t)(1 - l_t^i(s^t)) \\
c_t^i(s^t) &\geq 0, l_t^i(s^t) \in [0, 1]
\end{aligned}
$$

- Familiar optimality condition

$$\frac{U_l^i(c_t^i(s^t), l_t^i(s^t))}{U_c^i(c_t^i(s^t), l_t^i(s^t))} = -w_t^i(s^t)$$

together with constraint determines indirect utility function $V^i(\rho_t^i(s^t), w_t^i(s^t))$.

- Note: if labor supply is exogenous, this step is trivial and $V^i(\rho_t^i(s^t), w_t^i(s^t)) = U^i(c_t^i(s^t))$

- Step 2: Optimal allocation of expenditures across two arbitrary households i, j. Note: this is the crucial step for the tests.

$$
\begin{aligned}
&V^{i,j}(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t)) \\
&= \max_{\rho_t^i(s^t), \rho_t^j(s^t)} \alpha^i V^i(\rho_t^i(s^t), w_t^i(s^t)) \\
&\quad + \alpha^j V^j(\rho_t^j(s^t), w_t^j(s^t)) \\
&\text{s.t.} \\
\rho_t^i(s^t) + \rho_t^j(s^t) &= \rho_t^{i,j}(s^t)
\end{aligned}
$$

- Optimal solution $\rho^i(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t))$, $\rho^j(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t))$

- Note: if $i = 1, j = 2$ and labor is exogenous, we have executed step 2 before.

- Step 1: Optimal split of overall resources $\bar{Y}_t(s^t)$ between groups of households (if $N = 2$, this point is mute).

- Assume that number of households is even (if not, need to group three households into one group).

$$\max_{\{\rho_t^{2i-1,2i}(s^t)\}} \sum_{i=1}^{N/2} V^{2i-1,2i}(\rho_t^{2i-1,2i}(s^t), w_t^{2i-1}(s^t), w_t^{2i}(s^t))$$

$$\text{s.t.} \sum_{i=1}^{N/2} \rho_t^{2i-1,2i}(s^t) = \bar{Y}_t(s^t).$$

- Theorem: Take an arbitrary partition of households (say the one proposed in step 1). A consumption-leisure allocation that solves step 1 - step 3 solves the social planner problem.

- **Test of Risk Tolerance Heterogeneity:** Suppose two households i, j share risk efficiently (i.e. solve subproblem 2; subproblem 3 deals with efficient allocation of consumption and leisure within a household, subproblem 1 with efficient allocation of resources across groups).

- Suppose there are two histories $s^t, \hat{s}^\tau$ such that

$$\begin{aligned} w^i(s^t) &= w^i(\hat{s}^\tau) \\ w^j(s^t) &= w^j(\hat{s}^\tau) \end{aligned}$$

- Suppose that expenditure functions cross:

$$\begin{aligned} \rho^i(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t)) &> \rho^j(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t)) \\ \rho^i(\rho_t^{i,j}(\hat{s}^t), w_t^i(s^t), w_t^j(s^t)) &< \rho^j(\rho_t^{i,j}(\hat{s}^t), w_t^i(s^t), w_t^j(s^t)) \end{aligned}$$

- Then $U^i \neq U^j$. If U^i, U^j are CRRA, then $\sigma^i \neq \sigma^j$

- **Remarks:** One maintained assumption: efficient risk sharing. Thus tests only useful in context of risk sharing tests (or if efficient risk sharing can be ascertained independently). Also: what if later tests of efficient risk sharing reject that hypothesis?

- Second maintained assumption: wages held constant across $s^t, \hat{s}^\tau$. Need to find a way to do it in the data.

- Consequence of result: if test not rejected, can proceed as before. If homogeneity of risk preferences rejected, need to device tests that are robust to preference heterogeneity.

- **Tests of Efficient Risk Sharing:** If the U^i are strictly concave and if i, j share risk efficiently, then ρ^i, ρ^j are strictly increasing in $\rho^{i,j}$.

- Besides w^i, w^j and $\rho^{i,j}$, no idiosyncratic variable should enter ρ^i, ρ^j. Note: if period utility affected by observable heterogeneity (e.g. family size and composition) or unobservable heterogeneity, this can enter ρ^i, ρ^j as well. But not e.g. nonlabor income y^i.

- Note: if consumption and leisure are separable or wages are constant over time and across states, the above two conditions are also sufficient for efficient risk sharing.

- **Implementing the Tests:** For each pair i, j, estimate the expenditure functions $\rho^i(\rho_t^{i,j}, w_t^i, w_t^j), \rho^j(\rho_t^{i,j}, w_t^i, w_t^j)$.

- Test 1: Holding wages fixed, compute

$$g_t^{i,j} = \rho_t^i - \rho_t^j$$

  and test whether always positive/negative. If not, evidence for preference heterogeneity.

- Test 2: Test whether nonlabor income y^i of household i enters significantly in ρ^i. If yes evidence against efficient risk sharing.

- Test 3: Test whether slope of ρ^i, ρ^j with respect to $\rho^{i,j}$ is positive. If not, evidence against efficient risk sharing.

- **Data:** INCRISAT (International Crops Research Institute for the Semi-Arid Tropics) VLS (Village Level Studies) on Indian villages.

- From 1975 6 villages, from 1981 10 villages in rural India.

- From each village 40 households (10 landless laborers, 10 small farmers, 10 medium size farmers, 10 large farmers).

- Key: weather is very important for these villages, and has lots of annual and seasonal variation. Life is risky there.

- 3 villages selected (Aurepale, Shirapur, Kanzara)

- Monthly data from 1975 - 1985. About 120 observations for about 30 households in each village.

- Observe: labor supply, labor income, assets (used to construct nonlabor income), price of goods, monetary and nonmonetary transactions (from which consumption is constructed), demographics of household (including caste).

- **Results (Summary):** Standard risk sharing tests (adding income changes to regression of consumption change) strongly rejected efficient risk sharing.

- Tests of preference heterogeneity: homogeneous preferences rejected for 25% of pairs in Aurepale, 16% in Shirapur, 35% in Kanzara.

- Efficient risk sharing tests: results fairly uniform across the two tests. For few (less than 5%) of pairs efficient risk sharing rejected. But enough to still formally reject efficient risk sharing on village level.

- Very few rejections across pairs of efficient risk sharing on village-caste level.

4.4 Appendix A: Discussion of the No Ponzi Condition

In this appendix we further discuss the exact form of the no Ponzi condition, following Wright (1987). To rule out Ponzi schemes he imposes a condition of the form

$$-v_0(s^T)a_T^i(s^{T-1}, s_T) \leq \liminf_{N\to\infty} S_N(s^T) \qquad (4.64)$$

where

$$S_N(s^T) = \sum_{t=T}^{N} \sum_{s^t|s^T} v_0(s^t)\left(y_t^i(s^t) - c_t^i(s^t)\right) \qquad (4.65)$$

is the time zero value of future income net of consumption (that is, the time zero value of period savings) from node s^T up to date N. Note that for a fixed s^T, the sequence $\{S_N(s^T)\}$ need not converge[24], but if it does, then

$$\liminf_{N\to\infty} S_N(s^T) = \sum_{t=T}^{\infty} \sum_{s^t|s^T} v_0(s^t)\left(y_t^i(s^t) - c_t^i(s^t)\right) \qquad (4.66)$$

Intuitively, the no Ponzi scheme condition

$$\begin{aligned} a_{t+1}^i(s^t, s_{t+1}) &\geq& -\bar{A}^i(s^{t+1}, q, c^i) \qquad (4.67) \\ &=& -\frac{\liminf S_N(s^{t+1})}{v_0(s^{t+1})} \end{aligned}$$

---

[24]Remember that for any sequence $\{x_n\}_{n=0}^{\infty}$ the number

$$x^* = \liminf_{n\to\infty} x_n$$

is the smallest cluster point of the sequence, i.e.

$$x^* = \sup\{w|x_n < w \text{ for at most finite number of } n\text{'s}\}$$

Note that if $\{x_n\}$ converges, then

$$x^* = \lim_{n\to\infty} x_n$$

and if $\{x_n\}$ is unbounded below, then

$$x^* = -\infty$$

rules out a sequence of asset holdings and consumption for which, at any node s^{t+1}, the beginning of the period debt $a^i_{t+1}(s^t, s_{t+1})$ exceeds the node s^{t+1} of future savings, $-\frac{\liminf S_N(s^{t+1})}{v_0(s^{t+1})}$.

Let us interpret the No Ponzi scheme condition a bit further. Writing out (4.64) explicitly using (4.65) we obtain:

$$-v_0(s^T)a^i_T(s^{T-1}, s_T) \leq \lim_{N\to\infty} \inf \sum_{t=T}^{N} \sum_{s^t|s^T} v_0(s^t) \left(y^i_t(s^t) - c^i_t(s^t)\right) \tag{4.68}$$

Now suppose that the sequential budget constraint holds with equality (which it will as an optimum as long as the utility function is strictly increasing). Then

$$
\begin{aligned}
y^i_t(s^t) - c^i_t(s^t) &= \sum_{s_{t+1}} q_t(s^t, s_{t+1})a^i_{t+1}(s^t, s_{t+1}) - a^i_t(s^t) \tag{4.69}\\
&\equiv q_t(s^t) \cdot a^i_{t+1}(s^t) - a^i_t(s^t)
\end{aligned}
$$

where we define (in slight abuse of notation):

$$q_t(s^t) \cdot a^i_{t+1}(s^t) = \sum_{s_{t+1}} q_t(s^t, s_{t+1})a^i_{t+1}(s^t, s_{t+1}) \tag{4.70}$$

as the value of the asset portfolio bought in node s^t. Using equation (4.70), equation (4.68) becomes

$$
\begin{aligned}
-v_0(s^T)a^i_T(s^{T-1}, s_T) \leq{}& \lim_{N\to\infty} \inf \sum_{t=T}^{N} \sum_{s^t|s^T} v_0(s^t) \left[q_t(s^t) \cdot a^i_{t+1}(s^t) - a^i_t(s^t)\right]\\
={}& \lim_{N\to\infty} \inf \Big\{ v_0(s^T)q_T(s^T) \cdot a^i_{T+1}(s^T) - v_0(s^T)a^i_T(s^T)\\
&+ \sum_{s^{T+1}|s^T} v_0(s^{T+1})q_{T+1}(s^{T+1}) \cdot a^i_{T+2}(s^{T+1}) - v_0(s^{T+1})a^i_{T+1}(s^{T+1})\\
&+ \ldots + \sum_{s^N|s^T} v_0(s^N)q_N(s^N) \cdot a^i_{N+1}(s^N) - v_0(s^N)a^i_N(s^N) \Big\}\\
={}& \lim_{N\to\infty} \inf \Big\{ -v_0(s^T)a^i_T(s^T) + \sum_{s^N|s^T} v_0(s^N)q_N(s^N) \cdot a^i_{N+1}(s^N) \Big\}
\end{aligned}
$$

$$\tag{4.71}$$

The last equality is due to the fact that all intermediate terms cancel out. We demonstrate this for the first terms

$$v_0(s^T)q_T(s^T) \cdot a^i_{T+1}(s^T) = v_0(s^T) \sum_{s_{T+1}} q_T(s^T, s_{T+1})a^i_{T+1}(s^T, s_{T+1})$$

(4.72)

$$\sum_{s^{T+1}|s^T} v_0(s^{T+1})a^i_{T+1}(s^{T+1}) = \sum_{s^{T+1}|s^T} v_0(s^T)q_T(s^T, s_{T+1})a^i_{T+1}(s^{T+1})$$

$$= v_0(s^T) \sum_{s_{T+1}} q_T(s^T, s_{T+1})a^i_{T+1}(s^T, s_{T+1})$$

(4.73)

Simplifying equation (4.71) we can therefore conclude that the no Ponzi scheme condition, as stated in (4.64), can equivalently be written as

$$\lim_{N \to \infty} \inf \left\{ \sum_{s^N|s^T} v_0(s^N)q_N(s^N) \cdot a^i_{N+1}(s^N) \right\} \geq 0 \qquad (4.74)$$

and thus rules out asset sequences for which the present value of future debt is bounded above zero. Using this No Ponzi condition Wright (1987) then states and proves proposition 11 in the main text.

4.5 Appendix B: Proofs

In this appendix we provide a sketch of a proof of proposition 11 in the main text. See Wright (1987) for a complete proof.

Proof. It is sufficient to show that the budget sets described by the Arrow Debreu budget constraint (4.7) and the sequential markets budget constraint (4.28) plus the no Ponzi condition (4.64) (or alternatively (4.74)) contain the same possible consumption choices.

(1) Suppose (c^i, a^i) satisfies (4.28) and (4.64). Then

$$-v_0(s^T)a^i_T(s^{T-1}, s_T) \leq \lim_{N \to \infty} \inf \sum_{t=T}^{N} \sum_{s^t|s^T} v_0(s^t)\left(y^i_t(s^t) - c^i_t(s^t)\right) \qquad (4.75)$$

Using the definition of Arrow Debreu prices and setting $T = 0$ yields

$$-p_0(s_0)a^i_0(s_0) \leq \lim_{N \to \infty} \inf \sum_{t=0}^{N} \sum_{s^t|s_0} p_t(s^t)\left(y^i_t(s^t) - c^i_t(s^t)\right) \qquad (4.76)$$

Noting that $a_0^i(s_0) = 0$ for all s_0 we find that

$$
\begin{aligned}
0 &\leq \liminf_{N \to \infty} \sum_{t=T}^{N} \sum_{s^t \mid s_0} p_t(s^t) \left(y_t^i(s^t) - c_t^i(s^t) \right) \\
&= \liminf_{N \to \infty} \sum_{t=0}^{N} \sum_{s^t} p_t(s^t) \left(y_t^i(s^t) - c_t^i(s^t) \right) \\
&= \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) \left(y_t^i(s^t) - c_t^i(s^t) \right)
\end{aligned}
\tag{4.77}
$$

and thus c^i satisfies the Arrow-Debreu budget constraint.[25]

(2) Suppose c^i satisfies the Arrow Debreu budget constraint (4.7). We want to construct asset holdings a such that (c^i, a) satisfy (4.28) and (4.64). Fix N and define

$$
a^N(s^N) = 0 \text{ for all } s^N
\tag{4.78}
$$

and recursively

$$
a^N(s^t) = c_t^i(s^t) - y_t^i(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1}) a^N(s^t, s_{t+1})
\tag{4.79}
$$

---

[25]The last equality is valid whenever the series converges. One can circumvent the problem of possible nonconvergence by adjusting the definition of Arrow Debreu equilibrium: one defines the Arrow debreu budget constraint as

$$
p'(c^i - y^i) \leq 0
$$

where

$$
p'c = \liminf_{T \to \infty} \sum_{t=0}^{T} \sum_{s^t} p_t(s^t) c_t(s^t)
$$

Then all arguments go through even if, for a given price process p, the infinite sum does not converge for some process c. For details see Wright (1987), in particular footnote 6.

Multiplying by $v_0(s^t)$ yields

$$
\begin{aligned}
v_0(s^t)a^N(s^t) &= v_0(s^t)\left[c_t^i(s^t) - y_t^i(s^t)\right] + \sum_{s_{t+1}} v_0(s^t)q_t(s^t, s_{t+1})a^N(s^t, s_{t+1}) \\
&= v_0(s^t)\left[c_t^i(s^t) - y_t^i(s^t)\right] + \sum_{s^{t+1}|s^t} v_0(s^{t+1})a^N(s^t, s_{t+1}) \\
&= v_0(s^t)\left[c_t^i(s^t) - y_t^i(s^t)\right] \\
&\quad + \sum_{s^{t+1}|s^t} v_0(s^{t+1})\left[c_{t+1}^i(s^{t+1}) - y_{t+1}^i(s^{t+1})\right] + v_0(s^{t+1})q_t(s^t) \cdot a^N(s^t)
\end{aligned}
$$

(4.80)

With $a^N(s^N) = 0$ for all s^N we have, via continuing substitution

$$
v_0(s^T)a^N(s^T) = \sum_{t=T}^{N} \sum_{s^t|s^T} v_0(s^t)\left[c_t^i(s^t) - y_t^i(s^t)\right] \tag{4.81}
$$

But for each s^t, $a^N(s^t)$ is a sequence in N. Now define

$$
a_t(s^t) = \lim_{N\to\infty} \inf a^N(s^t) \tag{4.82}
$$

Taking $\liminf'$s on both sides of (4.79) yields

$$
a_t(s^t) = c_t^i(s^t) - y_t^i(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1})a_{t+1}(s^t, s_{t+1}) \tag{4.83}
$$

and hence the sequential budget constraint. Taking $\liminf'$s on both sides of (4.81) yields

$$
-v_0(s^T)a_T(s^{T-1}, s_T) = \lim_{N\to\infty} \inf \sum_{t=T}^{N} \sum_{s^t|s^T} v_0(s^t)\left[y_t^i(s^t) - c_t^i(s^t)\right] \tag{4.84}
$$

and hence (c^i, a) as constructed above satisfies the no Ponzi scheme condition.
∎

4.6 Appendix C: Properties of CRRA Utility

In this appendix we discuss the basic properties of the CRRA utility function, given by

$$
U(c) = \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma \neq 1 \\ \ln(c) & \text{if } \sigma = 1 \end{cases}
$$

with $\sigma > 0$. Note that $\lim_{\sigma \to 1} \frac{c^{1-\sigma}-1}{1-\sigma} = \ln(c)$ which justifies the second line in the definition of the CRRA utility function above. First we note that U satisfies the "usual" properties: U is continuous, three times continuously differentiable, strictly increasing (i.e. $U'(c) > 0$), strictly concave (i.e. $U''(c) < 0$) and satisfies the Inada conditions

$$\lim_{c \searrow 0} U'(c) = +\infty$$
$$\lim_{c \nearrow +\infty} U'(c) = 0$$

4.6.1   Constant Relative Risk Aversion

Define as

$$\sigma(c) = -\frac{U''(c)c}{U'(c)}$$

the Arrow-Pratt coefficient of relative risk aversion. Hence $\sigma(c)$ indicates a household's attitude towards risk, with higher $\sigma(c)$ representing higher risk aversion. The (relative) risk premium measures the household's willingness to pay (and thus reduce safe consumption $\bar{c}$) to avoid a proportional consumption gamble in which a household can win, but also lose, a fraction of $\bar{c}$. See figure **??** for a depiction of the risk premium.

Arrow-Pratt's theorem states that this risk premium is proportional (up to a first order approximation) to the coefficient of relative risk aversion $\sigma(\bar{c})$. This coefficient is thus a quantitative measure of the willingness to pay to avoid consumption gambles. Typically this willingness depends on the level of consumption $\bar{c}$, but for a CRRA utility function it does not, in that the entity $\sigma(c)$ is constant for all c, and equal to the parameter σ. This explains the name of this class of period utility functions.[26]

Add CRRA.wmf pic

---

[26]The CRRA utility function belongs to the more general class of hyperbolic absolute risk aversion utility functions, given by the general form

$$U(c) = \frac{1-\mu}{\mu}\left(\frac{\alpha c}{1-\mu} + \omega\right)^{\mu}$$

where μ, α, ω are parameters. It is easy to show that, up to irrelavant constants, CRRA utility is a special case of this general form (as are the CARA and quadratic utility functions).

4.6.2 Constant Intertemporal Elasticity of Substitution

Now consider the lifetime utility function (for ease of exposition, in the absence of risk)

$$u(c) = \sum_{t=0}^{\infty} \beta^t U(c_t).$$

Define the intertemporal elasticity of substitution (IES) as $ies_t(c_{t+1}, c_t)$ as

$$ies_t(c_{t+1}, c_t) = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{d\left(\frac{c_{t+1}}{c_t}\right)} \frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right]^{-1}$$

that is, as the inverse of the percentage change in the marginal rate of substitution MRS between consumption at t and $t+1$ in response to a percentage change in the consumption ratio $\frac{c_{t+1}}{c_t}$. For the CRRA utility function note that

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}$$

and thus

$$ies_t(c_{t+1}, c_t) = -\left[\frac{-\sigma\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma-1}}{\frac{\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}}{\frac{c_{t+1}}{c_t}}}\right]^{-1} = \frac{1}{\sigma}$$

and the intertemporal elasticity of substitution is constant, independent of the level or growth rate of consumption, and equal to $1/\sigma$. Graphically, the IES measures the curvature of the utility function. If $\sigma = 0$ consumption in two adjacent periods are perfect substitutes and the IES equals $ies = \infty$. If $\sigma \to \infty$ the utility function converges to a Leontieff utility function, consumption in adjacent periods are prefect complements and $ies = 0$.

The IES has a useful interpretation in terms of observable behavior. From the first order conditions of the household problem we obtain

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{p_{t+1}}{p_t} = q_{t+1} = \frac{1}{1 + r_{t+1}} \tag{4.85}$$

where r_{t+1} is the real interest rate between periods t and $t+1$. Thus, for any model in which the intertemporal Euler equation holds with equality the IES can alternatively be written as:

$$ies_t(c_{t+1}, c_t) = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{1}{1+r_{t+1}}\right)}{\frac{1}{1+r_{t+1}}}\right]}$$

that is, the IES measures the percentage change in the consumption growth rate in response to a percentage change in the gross real interest rate, the intertemporal price of consumption.

Note that for the CRRA utility function the Euler equation reads as

$$(1 + r_{t+1})\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} = 1.$$

Taking logs on both sides and rearranging one obtains

$$\ln(1 + r_{t+1}) + \log(\beta) = \sigma \left[\ln(c_{t+1}) - \ln(c_t)\right]$$

or

$$\ln(c_{t+1}) - \ln(c_t) = \frac{1}{\sigma}\ln(\beta) + \frac{1}{\sigma}\ln(1 + r_{t+1}). \tag{4.86}$$

This equation can then be used to obtain empirical estimates of the IES. With time series data on consumption growth and real interest rates the IES $\frac{1}{\sigma}$ can be estimated from a regression of the former on the later.[27]

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[27]Note that in order to interpret (4.86) as a regression one needs a theory where the error term comes from. In models with risk this error term can be linked to expectational errors, and (4.86) with error term arises as a first order approximation to the stochastic version of the Euler equation. We will discuss the stochastic Euler equation at length in part II of this monograph.

4.6.3 Homotheticity and Balanced Growth

Finally, the lifetime utility function u is said to be homothetic if $MRS(c_{t+s}, c_t) = MRS(\lambda c_{t+s}, \lambda c_t)$ for all $\lambda > 0$ and c, that is, if scaling consumption in all periods up leaves the marginal rate of substitution of consumption between any two periods unaffected. It is easy to verify that for a period utility function U of CRRA variety the lifetime utility function u is homothetic, since

$$MRS(c_{t+s}, c_t) = \frac{\beta^{t+s} (c_{t+s})^{-\sigma}}{\beta^{t} (c_t)^{-\sigma}} = \frac{\beta^{t+s} (\lambda c_{t+s})^{-\sigma}}{\beta^{t} (\lambda c_t)^{-\sigma}} = MRS(\lambda c_{t+s}, \lambda c_t) \quad (4.87)$$

Homotheticity of the lifetime utility function in turn is a desirable property in many macroeconomic applications. It implies that, if an agent's lifetime income doubles, optimal consumption choices will double in *each* period (income expansion paths are linear).[28] It also means that consumption allocations are independent of the units in which income and consumption are measured. This property of the utility function is also crucial for the existence of a balanced growth in models with growth in income or endowments (or in models with technological progress in the production function). Define a balanced growth path as a situation in which consumption grows at a constant rate, $c_t = (1+g)^t c_0$ and the real interest rate is constant over time, $r_{t+1} = r$ for all t.

Plugging in for a balanced growth path, equation (4.85) yields, for all t

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) = \frac{1}{1+r}.$$

But for this equation to hold *for all* t we require that

$$MRS(c_{t+1}, c_t) = MRS((1+g)^t c_1, (1+g)^t c_0) = MRS(c_1, c_0)$$

and thus requires that u is homothetic (where $\lambda = (1+g)^t$ in equation (4.87)). Thus homothetic lifetime utility is a necessary condition for the existence of a balanced growth path in growth models. Above we showed that CRRA period utility implies homotheticity of lifetime utility u. Without proof here we state that CRRA utility is the *only* period utility function such that lifetime utility is homothetic. Thus (at least in the class of time separable

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[28]In the absense of borrowing constraints and other frictions that will be discussed later in this monograph.

lifetime utility functions) CRRA period utility is a necessary condition for the existence of a balanced growth path, which in part explains why this utility function is used in a wide range of macroeconomic applications.[29]

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[29]Note that the class of (non-time separable) Epstein-Zin utility functions discussed in remark 17 above inherit the homotheticity property from the time separable lifetime utility with CRRA period utility.

Part II

The Standard Incomplete Markets Model (SIM)

In this part of the monograph we discuss the standard incomplete markets model in which households can lend (and perhaps borrow) at a risk-free rate, with the goal of insulating consumption against fluctuations in labor income. This literature starts with the famous work by Friedman (1957) on the permanent income hypothesis and the life cycle hypothesis of Modigliani and Brumberg (1954). Common to both studies is the focus on a single consumer (with interest rates are exogenously given and not derived endogenously in general equilibrium) and the absence of explicit insurance arrangements against individual income risk (these are ruled out in ad hoc fashion). These early and seminal contributions sparked important theoretical work in the 1970's on what became known as the "income fluctuation problem".

In chapter 5, after a brief introduction by means of a two period toy model, we then present a formal SIM model (with quadratic period utility and nonbinding borrowing constraints) that implies both certainty equivalence (households make consumption savings decisions that are -ex ante- identical to those that are optimal in a world without income risk) as well as results in Friedman's permanent income hypothesis. We then discuss extensions of the basic theory that relax the assumption of quadratic utility and/or slack borrowing constraints and show how this can lead to precautionary saving behavior (these concepts will be made precise below). We also discuss how versions of the model can be solved numerically for which no analytical solution exists (as is the norm).

The key quantitative ingredient into SIM is the stochastic labor income or wage process to be fed into the household consumption-saving problem. Thus, in chapter 6 we digress from the main theme of this part of the monograph and briefly discuss the empirical properties of these process, with specific focus on the recent discussion in the literature about how persistent income shocks are that hit individuals or households.

Finally, in chapter 7 we present general equilibrium versions of the SIM model. We first discuss stationary equilibria, characterized by an endogenous cross-sectional distribution over household characteristics (the relevant aggregate state variable in this class of models) that is time invariant. although individual households move up and down in that distribution. We will then consider deterministic transitions in which the cross-sectional distribution of household characteristics evolves over time in a deterministic fashion, before concluding with a business cycle version of the model in which the economy is hit by recurrent aggregate shocks and thus the cross-sectional distribution follows a stochastic law of motion.

Chapter 5

The SIM in Partial Equilibrium

Permanent income type models assume that agents do not have access to a complete set of contingent consumption claims. The main difference between this type of models and the complete markets model thus manifests itself in the budget constraints. Whereas the complete markets model in sequential formulation has budget constraints of the form (4.28), for the SIM model we have[1]

$$c_t(s^t) + q_t(s^t)a_{t+1}(s^t) = y_t(s^t) + a_t(s^{t-1}) \qquad (5.1)$$

Here $q_t(s^t)$ is the price at date t, event history s^t, of one unit of consumption delivered in period $t+1$ regardless of what event s_{t+1} is realized. Notation is crucial here: a_{t+1} is the quantity of risk-free one period bonds being purchased in period t and paying off in period $t+1$. Thus $a_{t+1}(s^t)$ is a function only of s^t and not of s^{t+1} whereas in the complete markets model with a full set of Arrow securities these assets are indexed by $(s^t, s_{t+1}) = s^{t+1}$ and each asset pays off only at a particular event history s^{t+1}. Obviously each of these Arrow securities has a potentially different price $q_t(s^t, s_{t+1})$, as discussed in the previous chapter.

The basic income fluctuation problem is to maximize

$$u(c) = \sum_{t=0}^{T} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) U(c_t(s^t), s^t) \qquad (5.2)$$

---

[1]In principle the budget constraints have to hold only as inequalities. At the optimal allocations they will always hold as strict equality as long as household utility is strictly increasing in consumption, an assumption we maintain throughout.

subject to the sequence of period budget constraints given in (5.1) and a short-sale constraint on assets $a_{t+1}(s^t) \geq -\bar{A}(s^t; q)$ which rules out Ponzi schemes[2], but is assumed to be sufficiently wide so as to never bind at the optimal allocation chosen by the household. In later parts of this chapter we will consider borrowing limits that are more stringent and potentially binding. Let the initial wealth of an agent be denoted by $a_0(s_{-1}) = a_0$. At this point it is not required to make further assumptions on a_0. Unless noted explicitly we assume that U satisfies Assumption 1 of the previous chapter (U is twice continuously differentiable, strictly increasing, strictly concave and satisfies the Inada conditions). Note that for the discussion in this chapter time and household age can be used interchangeably since one household is considered in isolation. Thus the index t will be used for a generic time period or age of the household.

5.1 A Simple 2 Period Toy Model

To develop intuition for the solution of the general model we first look at the simplest example of a SIM with two periods and without risk.

5.1.1 Household Decision Problem

The problem of the household then becomes

$$\max_{c_c, c_1, a_1} U(c_0) + \beta U(c_1) \qquad (5.3)$$

$$\text{s.t.}$$

$$c_0 + qa_1 = y_0 + a_0 \qquad (5.4)$$

$$c_1 = y_1 + a_1 \qquad (5.5)$$

$$c_0, c_1 \geq 0 \qquad (5.6)$$

One can consolidate the budget constraints to a lifetime budget constraint which reads as:

$$c_0 + \frac{c_1}{1+r} = a_0 + y_0 + \frac{y_1}{1+r} \qquad (5.7)$$

---

[2]In the case of an infinite planning horizon; for the finite horizon case we continue to impose $a_{T+1}(s^T) \geq 0$.

where $1 + r = \frac{1}{q}$ is the gross real interest rate. Equation (5.7) together with the Euler equation

$$U'(c_0) = \beta(1 + r)U'(c_1) \tag{5.8}$$

uniquely determine the optimal consumption allocation (c_0, c_1) over the life cycle of the household, as a function of the model parameters (y_0, y_1, r, β) and any parameters characterizing the period utility function (e.g. the parameter σ in the case of CRRA utility).

5.1.2 Comparative Statics

The comparative statics with respect to most model parameters can be signed easily. Consumption in both periods depends positively on the present discounted value of lifetime income (including initial wealth a_0)

$$W_0(y_0, y_1, r, a_0) = a_0 + y_0 + \frac{y_1}{1 + r} \tag{5.9}$$

but is independent of the timing of income and thus the income profile (y_0, y_1). An increase in the time discount factor β (turning the household more patient) induces a shift from early consumption c_0 to late consumption c_1. However, even in this very simple model signing the effect of changes in the interest rate r on consumption allocations is not straightforward. An increase in the interest rate r has three effects on consumption allocations:

1. An increase in r reduces the relative price of consumption in period 1, relative to consumption in period 0, and thus reduces current consumption and relatively less expensive compared to period 1 consumption and increases saving and future consumption. This is the (intertemporal) substitution effect.

2. An increase in r reduces the price of period 1 consumption in absolute terms, acting like an increase in income. This is the income effect which, ceteris paribus, increases consumption in both periods.

3. Provided that the household has income in the second period, an increase in r reduces the present value of lifetime income $W_0(y_0, y_1, r, a_0)$ and hence reduces current and future consumption. This is the human capital or wealth effect.

Which effect dominates depends on the particular form of the utility func-
tion and the income profile. We can at least partially rank the magnitudes
of these effects for the case in which households have $CRRA$ period utility,
discussed in the previous chapter. Recall that the parameter $1/\sigma$ measures
the intertemporal elasticity of substitution as discussed in the appendix of
the previous chapter and hence the potency of the substitution effect.[3]

In the case of CRRA utility the explicit solution of the model is given by:

$$c_0 \;=\; \frac{W_0(y_0, y_1, r, a_0)}{1 + \frac{\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}}{1+r}} \tag{5.10}$$

$$c_1 \;=\; \frac{\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}W_0(y_0, y_1, r, a_0)}{1 + \frac{\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}}{1+r}} \tag{5.11}$$

The dependence of W_0 on $1+r$ captures the wealth effect, the term $(1+r)^{\frac{1}{\sigma}}$
captures the substitution effect (absent if $1/\sigma = 0$) and the term $1+r$ in the
second denominator encompasses the income effect. Since

$$\frac{\partial W_0(y_0, y_1, r, a_0)}{\partial(1+r)} = -\frac{y_1}{(1+r)^2} \leq 0$$

the human wealth effect is negative for consumption in both periods, and is
arbitrarily small or large depending on the size of future income y_1, and thus
can either dominate, or is being dominated by the other two effects.

Now suppose the wealth effect is absent (e.g. because $y_1 = 0$). Then

$$\frac{\partial c_0}{\partial(1+r)} = W_0(y_0, y_1, r, a_0) * \left(1 + \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}\right)^{-2} * \left(\frac{\beta^{\frac{1}{\sigma}}}{(1+r)^2}\left((1+r)^{\frac{1}{\sigma}} - \frac{1}{\sigma}(1+r)^{\frac{1}{\sigma}}\right)\right)$$

$$= W_0(y_0, y_1, r, a_0) * \left(1 + \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}\right)^{-2} * \left(\frac{\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}}{(1+r)^2}\left(1 - \frac{1}{\sigma}\right)\right) \tag{5.12}$$

The first term in the last bracket is the positive income effect, the second term
the negative intertemporal substitution effect. expression the human capital
effect, the second term is the combination between income and substitution
effect. For CRRA utility the relative magnitude of these two effects only

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[3]In the absence of risk, risk aversion (as measured by σ) of households is irrelevant for
the interpretation of the solution.

depends on the size of the intertemporal elasticity of substitution, as given by the parameter $\frac{1}{\sigma}$. It is immediate that we can distinguish the following three cases:

1. *If $1/\sigma = 1$ (log-case, unit intertemporal elasticity of substitution), income and substitution effect exactly cancel out and current consumption does not respond to changes in interest rates, absent a negative human wealth effect.*

2. *If $1/\sigma > 1$ (high intertemporal elasticity of substitution), then the substitution effect dominates the income effect and hence current consumption declines as reaction to an increase in the interest rate.*

3. *If $1/\sigma < 1$ (low intertemporal elasticity of substitution), then the income effect dominates the substitution effect and, absent a human wealth effect, current consumption increases (and current saving decreases) with the interest rate.*

Tedious but straightforward algebra also reveals that for consumption in period 1 both the income and the substitution effect from an increase in the interest rate are positive, and thus, absent the negative wealth effect, future consumption rises with an increase in the interest rate. To summarize, in the absence of income risk the only motive when making the intertemporal consumption decision is to insulate the timing of consumption from income variation over time, by means of (dis-saving) using the one-period risk free bonds. In the next sections we will discuss how this behavior is affected by the presence of income risk and the potential desire to hedge against this income risk through precautionary saving.

5.2 The General Model with Certainty Equivalence

Now we turn attention to the study of the general model in which households live for multiple periods, face a potentially stochastic income process and maximize (5.2) subject to (5.1). In the next section we will study versions of the model that give rise to certainty equivalence: households will make consumption savings decision in the presence of income risk that are (ex ante, prior to the resolution of income risk) identical to the choices they

would make in the absence of risk. The following section will then relax the assumptions leading to certainty equivalence and study versions of the model in which households engage in precautionary saving behavior.[4]

For the rest of this chapter we impose the following assumption on the exogenous prices (interest rates) of the one period risk-free bond.

Assumption 18 *The price of the one period bond is nonstochastic and constant over time,*

$$q_t(s^t) = q = \frac{1}{1+r}.$$

5.2.1   Nonstochastic Income

In order to set the stage for a comparison with the situation in which income is stochastic, first consider a version of the model in which household income is given by the deterministic sequence $y = \{y_t\}_{t=0}^T$. Also assume that the process $\{s_t\}$ that potentially drives household preference shifts is a deterministic sequence. Define as

$$W_0 = W_0(a_0, y, r) = a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} \tag{5.13}$$

the present value of all future labor income, including initial wealth. We assume that W_0 is finite (otherwise the household maximization problem does not have a solution).[5] Forming the Lagrangian, taking first order conditions and combining yields the standard Euler equation

$$U_c(c_t, s^t) = \beta(1+r)U_c(c_{t+1}, s^{t+1}) \tag{5.15}$$

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[4]These terms will be defined more formally below.

[5]When $T = \infty$ we impose a short sale constraint on the bond that prevents Ponzi schemes, but is loose enough to allow optimal consumption smoothing. Defining as the present discounted value of remaining lifetime income

$$W_{t+1} = \sum_{\tau=t+1}^{\infty} \frac{y_\tau}{(1+r)^{\tau-t}}$$

and assuming that the sum is finite for all t, we require that for all t bond holdings adhere to the "natural borrowing limit"

$$a_{t+1} \geq -W_{t+1} \tag{5.14}$$

Note that W_{t+1} is finite if $r > 0$ and the sequence $\{y_t\}$ is bounded from above.

and hence, for all t,

$$U_c(c_t, s^t) = \left(\frac{1+\rho}{1+r}\right)^t U_c(c_0, s_0) \tag{5.16}$$

where time discount factor β and time discount rate ρ are related through the definition $\beta = \frac{1}{1+\rho}$. As before in the two-period model, the sequence of Euler equations (5.16) and the intertemporal budget constraint

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = W_0 \tag{5.17}$$

jointly determine the optimal consumption choice of the household. Inspecting equations (5.16) we obtain the following predictions from the deterministic version of the income fluctuation problem. These predictions are sharpest under the assumption that the rate r at which the market discounts the future equals the subjective time discount rate ρ of households.

Specific Case $\rho = r$

If $\rho = r$ then the right hand side of (5.16) is constant over time and thus $U_c(c_t, s^t)$ is time-invariant. In periods in which the preference shifter s^t makes marginal utility high, consumption also has to be high, since marginal utility is decreasing in consumption. This may provide us with a nontrivial theory of life-cycle consumption profiles even in the absence of risk, and even under the assumption $\rho = r$.

Example 19 *If U is separable between consumption and preference shifters s^t, then we immediately obtain that household consumption is constant over time (over the life cycle)*

$$c_t = c_{t+1} \tag{5.18}$$

something that seems counterfactual[6] in light of the discussion in chapter 3. We can combine () with the intertemporal budget constraint () to solve explicitly for the level of consumption. This yields, for all t,

$$c_0 = c_t = c_t = \begin{cases} \theta^{-1}\frac{rW_0}{1+r} & \text{if } T < \infty \\ \frac{r}{1+r}W_0 & \text{if } T = \infty \end{cases} \tag{5.19}$$

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[6]Note that under the separability assumption, even if $\rho \neq r$, consumption does not display a life-cycle hump shape, but rather monotonically trends upwards (if $r > \rho$, i.e. if incentives to postpone consumption dominate impatience) or monotonically downwards (if $\rho > r$, i.e. if impatience dominates incentives to postpone consumption).

*where $\theta = \left(1 - \frac{1}{(1+r)^{T+1}}\right)$ is an adjustment factor needed if the lifetime hori-
zon T of the household is finite. Thus consumption at each date is equal to
"permanent income"*

$$c_t = \frac{rW_0}{\theta\left(1+r\right)}$$

*At each date households consume annuity value of their lifetime wealth W_0,
composed of initial financial wealth and the present discounted value of life-
time labor income. This is the purest version of the permanent income hy-
pothesis: consumption at each date should equal permanent income.*

Example 20 *For more realistic assumptions on the time (age) profile of
preference shifters it need not be the case that household consumption is con-
stant or trending monotonically upwards or downwards over time. Suppose
that s_t stands in for the (exogenous) number of individual members present in
the household. In the data, household size first increases and then decreases
with the age t of the household head (since this person first tends to get mar-
ried and have children, then the children leave the household and finally the
household head or spouse dies). Now suppose that the period utility function
takes the form*

$$U(c_t, s^t) = \exp(s_t) * \frac{c_t^{1-\sigma}}{1 - \sigma} \tag{5.20}$$

*so that marginal utility from household consumption expenditure c_t is higher if
there are more members of the households that have to share the benefits from
a given expenditure c_t. With this preference specification the Euler equation
reads as*

$$\frac{c_t}{c_0} = \exp(s_t - s_0)^{\frac{1}{\sigma}} \tag{5.21}$$

*As family size increases (i.e. $s_t > s_0$), so will optimal household consump-
tion. Thus a hump-shaped family size profile, at least qualitatively, can ac-
count for a hump-shaped household consumption expenditure profile in the
data, even if households follow the simple consumption-savings model out-
lined in this section.*[7]

Example 21 *A similar argument can be made if consumption and leisure
are non-separable in the utility function. Suppose s_t reflects the amount*

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[7]Attanasio et. al. (1999) and Fernandez-Villaverde and Krueger (2007) investigate this
point from a quantitative perspective.

worked (as a share of total available non-sleeping hours) in period t, again taken as exogenous for now. Let the period utility function be given by

$$U(c_t, s_t) = \frac{\left(c_t^\gamma (1 - s_t)^{1-\gamma}\right)^{1-\sigma} - 1}{1 - \sigma} \quad \textit{with } \sigma > 1 \textit{ and } \gamma \in (0, 1) \quad (5.22)$$

where γ is a share parameter measuring the importance of consumption relative to leisure $1 - s_t$ in the utility function. The Euler equation becomes

$$\frac{c_t}{c_0} = \left(\frac{1 - s_t}{1 - s_0}\right)^{\frac{(1-\gamma)(1-\sigma)}{1-\gamma+\gamma\sigma}} \quad (5.23)$$

Thus, in periods where labor supply s_t is high, so should consumption. Therefore a hump-shaped life cycle profile of hours worked can explain, at least qualitatively, a hump-shaped life cycle consumption profile.[8] Also note that since, at retirement, the number of hours worked decreases sharply, consumption should fall at retirement as it does in the data. Thus this argument is the starting point for a potential resolution of the consumption retirement puzzle.[9]

General Case $\rho \neq r$

For a fixed sequence of preference shifters $\{s_t\}$, a decline in the interest rate r makes the life cycle consumption profile less steep, as future consumption is substituted in favor of current consumption, ceteris paribus. The reverse is true for an increase in the interest rate. Note that this is not a statement about the *level* of the consumption *profile* but rather its slope, because the effect of interest rate changes on consumption levels also depends on the size of the income and human wealth effects, as discussed in the previous section. Thus the Euler equations only capture the substitution effect and make predictions about the shape of the consumption profile over the life

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[8]See Heckman (1974) for the original source of this point. A hump-shaped profile of hours worked in turn would be part of the optimal household labor supply choice in the presence of a hump-shaped life cycle wage profile (at least under the appropriate assumptions on household preferences).

[9]Aguiar and Hurst (2005, 2007) argue that older households (especially those in retirement) pay lower prices for the same consumption goods because they spend more time shopping for good deals. This results in a decline in life cycle consumption *expenditures* towards the end of life, without necessarily implying a fall in utility-generating consumption services that enter the utility function.

cycle. To determine consumption levels, in addition the intertemporal budget constraint 5.17 has to be invoked.

Before turning to the model with a stochastic income process, in the example below we provide an explicit solution to the household problem when households have CRRA utility.

Example 22 *Suppose households have preferences represented by (4.13). Then the Euler equation reads as*

$$(c_{t+1})^{-\sigma} = \left(\frac{1+\rho}{1+r}\right)(c_t)^{-\sigma}.$$

Taking logs on both sides and rearranging yields

$$\Delta \ln c_{t+1} = \frac{1}{\sigma}\left[\ln(1+r) - \ln(1+\rho)\right] \tag{5.24}$$

Using the Euler equations and the intertemporal budget constraint one can solve for the optimal consumption allocation, which, for all $t \geq 0$, is given by:

$$c_t = \left(\frac{1+r}{1+\rho}\right)^{\frac{t}{\sigma}}\left(\frac{1-\gamma}{1-\gamma^{T+1}}\right)W_0 \tag{5.25}$$

$$\equiv MPC(t,T) * W_0 \tag{5.26}$$

where γ is defined[10] as $\gamma = \left(\frac{[1+r]^{1-\sigma}}{1+\rho}\right)^{\frac{1}{\sigma}}$. Holding lifetime income W_0 fixed we find that an increase in the lifetime horizon T reduces the marginal propensities to consume out of lifetime wealth $MPC(t,T)$ at all ages t.

5.2.2   Stochastic Income and Quadratic Preferences

Now let us consider maximizing (5.2) subject to (5.1), but now permit the income process to be stochastic.[11] Attaching Lagrange multiplier $\lambda_t(s^t)$ to

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[10]In order to insure that lifetime utility is finite at the optimal allocation, in addition to assuming $W_0 < \infty$, we require that $\gamma < 1$.

[11]Again, to make the problem well-defined we need a No-Ponzi condition. For finite T we require

$$a_{T+1}(s^T) \geq 0 \tag{5.27}$$

the event history s^t budget constraint and taking first order conditions with respect to $c_t(s^t)$ and $c_{t+1}(s^{t+1})$ yields

$$\beta^t \pi_t(s^t) U_c(c_t(s^t), s^t) = \lambda_t(s^t) \tag{5.30}$$

$$\beta^{t+1} \pi_{t+1}(s^{t+1}) U_c(c_{t+1}(s^{t+1}), s^{t+1}) = \lambda_{t+1}(s^{t+1}) \tag{5.31}$$

Taking first order conditions with respect to $a_{t+1}(s^t)$ yields

$$\lambda_t(s^t) q = \sum_{s^{t+1}|s^t} \lambda_{t+1}(s^{t+1}) \tag{5.32}$$

Combining yields

$$\beta^t \pi_t(s^t) U_c(c_t(s^t), s^t) = \frac{1}{q} \sum_{s^{t+1}|s^t} \beta^{t+1} \pi_{t+1}(s^{t+1}) U_c(c_{t+1}(s^{t+1}), s^{t+1}) \tag{5.33}$$

for all s^T.

For the infinite horizon case, $T = \infty$, we impose

$$a_{t+1}(s^t) \geq - \inf_{\{s^{\tau+1}\}|s^t} \sum_{\tau=t+1}^{\infty} \frac{y_{\tau+1}(s^{\tau+1})}{(1+r)^{\tau-(t+1)}} \tag{5.28}$$

$$\equiv - \inf_{\{s^{\tau+1}\}|s^t} W_{t+1}(s^t) \equiv -\bar{A}_{t+1}(s^t) \tag{5.29}$$

where the right hand side is the minimum (infimum, if the minimum does not exist) realized present discount value of future income over all infinite histories $\{s^{\tau+1}\}$ that can follow node s^t. We assume that the process $\{\bar{A}_{t+1}(s^t)\}$ is bounded above, which is true as long as $r > 0$ and current income only depends on the current shock: $y_\tau(s^\tau) = y(s_\tau)$.

In that case define

$$y_{\min} = \min_{s_t \in S} y(s_t)$$

and

$$\begin{aligned}
\bar{A}_{t+1}(s^t) &= \inf_{\{s^{\tau+1}\}|s^t} \sum_{\tau=t+1}^{\infty} \frac{y_{\tau+1}(s^{\tau+1})}{(1+r)^{\tau-(t+1)}} \\
&= y_{\min} \sum_{\tau=t+1}^{\infty} \frac{1}{(1+r)^{\tau-(t+1)}} \\
&= \frac{y_{\min}}{1 - \frac{1}{1+r}} = \frac{(1+r)y_{\min}}{r}
\end{aligned}$$

This No Ponzi constraint, the natural extension of the concept of the "natural borrowing constraint" to a stochastic income process, insures that households having incurred the maximal amount of debt can repay it with probability 1 (by setting consumption to zero in all future periods). As long as the utility function satisfies the Inada conditions this constraint will never be binding.

or

$$U_c(c_t(s^t), s^t) \;=\; \left(\frac{1+r}{1+\rho}\right) \sum_{s^{t+1}|s^t} \pi_{t+1}(s^{t+1}|s^t) U_c(c_{t+1}(s^{t+1}), s^{t+1})$$

$$=\; \left(\frac{1+r}{1+\rho}\right) E\left(U_c(c_{t+1}(s^{t+1}), s^{t+1})|s^t\right) \tag{5.34}$$

or, more compactly,

$$U_c(c_t, s^t) = \left(\frac{1+r}{1+\rho}\right) E_t\left(U_c(c_{t+1}, s^{t+1})\right) \tag{5.35}$$

where E_t is the expectation of s^{t+1} conditional on s^t. This is the standard stochastic Euler equation for the standard *incomplete* markets (SIM) model. We immediately see that if $\rho = r$, then the marginal utility of consumption process $\{U_c(c_t, s^t)\}$ follows a martingale, i.e. its period t expectation of the $t+1$ variable equals the t variable.[12]

Remark 23 *Recall that for the standard complete markets (SCM) model, under the assumption[13] that Arrow securities prices satisfy* $q_t(s^{t+1}) = \frac{\pi_{t+1}(s^{t+1}|s^t)}{1+r}$

---

[12]A martingale is a stochastic process $\{x_t\}$ that satisfies, with probability 1,

$$E_t(x_{t+1}) = x_t.$$

A process that satisfies, with probability 1,

$$E_t(x_{t+1}) \geq x_t$$

is called a submartingale, and a process for which

$$E_t(x_{t+1}) \leq x_t$$

holds with probability 1 is called a supermartingale. Thus the relationship between ρ and r determines whether marginal utility in the SIM model without binding borrowing constraints is a sub-, super-, or standard martingale. The same statement applies to consumption itself, for the case of quadratic utility discussed below.

[13]¿From equation (4.39) we observe that this assumption is warranted as long as aggregate income (endowment) and thus aggregate consumption is constant over time. In chapter 7 we will construct general equilibrium models with a continuum of households in which this will be true. Note that for the SCM model

$$\frac{1}{1+r} = \beta.$$

delivers a set of Euler equations of the form

$$U_c(c_t(s^t), s^t) = \left(\frac{1+r}{1+\rho}\right) U_c(c_{t+1}(s^{t+1}), s^{t+1}) \text{ for all } s^{t+1} \qquad (5.36)$$

Thus the key difference between the SCM and in the SIM models is that in the former the Euler equation holds state by state s_{t+1}, whereas in the SIM model it only holds in (conditional) expectation. This is due to the fact that in the SCM model households have access to a full set of Arrow securities, which allows households to smooth marginal utility over time state by state, whereas the best a household in the SIM model can do is to use the uncontingent bond to smooth marginal utility over time on average (in expectation).

As in the deterministic case, the Euler equations (5.35) determine consumption profiles over time and will also allow us to make statements about how consumption responds to income shocks, but they are, by themselves, insufficient to deduce the level of consumption in every period. For that we need, in addition, the sequence[14] of period budget constraints (5.1). In general, no closed form solution for the stochastic consumption process exists, and one has to resort to the numerical techniques discussed in section 5.6.

However, if households have quadratic period utility, then a sharp characterization of the optimal consumption choices can be obtained, and these choices obey "Certainty Equivalence". To demonstrate this, we now make

Assumption 24 *The period utility function is quadratic and separable in consumption*

$$U(c_t(s^t), s^t) = -\frac{1}{2}\left(c_t(s^t) - \bar{c}\right)^2 + v(s^t) \qquad (5.37)$$

where $\bar{c}$ is the bliss level of consumption, assumed to be large relative to an agents' stochastic income process.[15]

---

[14]As we will see below, under the assumption that interest rates are nonstochastic and households have access only to an uncontingent short-term bond, we can consolidate the sequence of budget constraints into an intertemporal budget constraint, exactly as in the case without risk. This is by no means true in incomplete markets models with more complex asset market structure and asset price processes.

[15]We assume that $\bar{c}$ is so large that the agent cannot afford, given his income process, to obtaim $c_t(s^t) = \bar{c}$ for all s^t. Note that it is easy to pick $\bar{c}$ large enough for T finite, since $a_{T+1}(s^T) = 0$ is required. For $T = \infty$ it must be chosen large enough so that the consumption allocation $c_t(s^t) = \bar{c}$ leads to required asset holdings that eventually violate the no Ponzi scheme condition.

Under assumptions 18 and 24 the Euler equations in (5.35) become

$$E_t c_{t+1} = \alpha_1 + \alpha_2 c_t \tag{5.38}$$

with $\alpha_1 = \bar{c}\left(1 - \frac{1+\rho}{1+r}\right)$ and $\alpha_2 = \frac{1+\rho}{1+r}$. If furthermore $\rho = r$, then equation (5.38) turns into:

$$E_t c_{t+1} = c_t \tag{5.39}$$

i.e. not only marginal utility but consumption itself follows a martingale. We summarize the most important implications of (5.38) or (5.39) as:

1. The agent's optimal consumption decisions obey certainty equivalence; comparing the rules for optimally allocating consumption over time in the certainty and the case with income risk, (5.18) and (5.39), we see that they are identical.[16] Of course the *realized* consumption path under income risk will deviate, ex post, from the path chosen by the household in the absence of risk.

2. From equation (5.38) it follows that the optimal stochastic consumption process chosen by the household therefore obeys the regression equation

$$c_{t+1} = \alpha_1 + \alpha_2 c_t + u_{t+1} \tag{5.40}$$

   where u_{t+1} is a random variable satisfying $E_t u_{t+1} = 0$. The main empirical implication of the SIM model in its most basic, certainty equivalence form is that period $t+1$ consumption c_{t+1} is perfectly predicted by period t consumption c_t and no other variables that are in the households' information set at period t should help predict it. In particular, once c_t is included in the regression, current income y_t, current assets a_t, past consumption c_{t-1} or any other variable should not enter with a significant coefficient if included in regression 5.40. Following the literature we call this the consumption "martingale" hypothesis.

3. Note, however, that the income *realization* in period y_{t+1} can affect the choice c_{t+1}, but only through that part that constitutes a deviation

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[16]Of course, the assumption $\rho = r$ is not required for the result, as it is straightforward to show that, with quadratic utility, the consumption dynamics in the no-risk case is given by

$$c_{t+1} = \alpha_1 + \alpha_2 c_t$$

compared to equation (5.38) in the main text.

(a "shock") from its period t expectation $y_{t+1} - E_t(y_{t+1})$. The component of income y_{t+1} that is already predictable in period t (that is, whatever of y_{t+1} is in the information set at time t) should not affect consumption in period t. The fact that consumption responds to income shocks strongly sets the SIM apart from the complete markets model where even unexpected changes in (individual) income do not affect consumption, since agents can perfectly insure (using the purchase of Arrow securities) against these unexpected changes.[17]

Hall's (1978) Empirical Tests of the Martingale Hypothesis Using Macroeconomic Data

The implications discussed under point 2. above are empirically tested by a large literature[18], starting from the seminal paper by Robert Hall (1978). He used aggregate consumption and income data to run the two regressions, motivated by equation (5.40)

$$
\begin{aligned}
c_{t+1} &= \alpha_1 + \alpha_2 c_t + \alpha_3 c_{t-1} + \alpha_4 c_{t-2} + \alpha_5 c_{t-3} + u_{t+1} &\quad (5.41)\\
c_{t+1} &= \beta_1 + \beta_2 c_t + \beta_3 y_t + v_{t+1} &\quad (5.42)
\end{aligned}
$$

Under the null of the martingale hypothesis, $\alpha_3 = \alpha_4 = \alpha_5 = 0$ and $\beta_3 = 0$. Hall (1978) used per capita nondurable consumption expenditures (including services) in constant 1972 dollars from the NIPA as his measure of c_t. His income measure for y_t is total nominal disposable income per capita from the NIPA, divided by the implicit deflator for nondurable consumption and services.[19] Hall's basic results from regression for quarterly data from 1948-1977 are summarized in Table 3

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[17]Recall that in our discussion of perfect consumption insurance in chapter 4 we never made a distinction between predicted and unpredicted changes in income; such distinction is unnecessary in the complete markets model.

[18]Most of this literature starts from the Euler equation with CRRA utility, obtains a first order approximation and then implements a regression similar to equation (5.42), but in logs, on household level consumption and income data. We will derive the approximation and discuss the relevant literature in the next section.

[19]To stress this point again, Hall's (1978) original test used aggregate data, although the theory envisions a single household as its unit of analysis. As long as households have linear marginal utility, equation (5.38) can be aggregated perfectly across households and the theory makes the same predictions for aggregate consumption and income data. Such aggregation fails for the SIM whenever we deviate from the quadratic utility case, as demonstrated below.

Table 3: Test of Martingale Hypothesis

Regression	Parameter Estimates and Standard Errors (in Parenthesis)					R^2
(5.41)	$\alpha_1 = 8.2$ (8.3)	$\alpha_2 = 1.13$ (.092)	$\alpha_3 = -.04$ (.142)	$\alpha_4 = .03$ (.142)	$\alpha_5 = -.113$ (.093)	.9988
(5.42)	$\beta_1 = -16$ (11)	$\beta_2 = 1.024$ (.044)	$\beta_3 = -.01$ (.032)			.9988

The basic message of Table 3 is that the martingale hypothesis cannot be rejected. The coefficients on lagged consumption and current income are not significantly different from 0, and F-tests indicate that the hypothesis that they are jointly equal to zero cannot be rejected at standard confidence levels. The inclusion of further lags in consumption or income does not change the result. This is strong evidence in favor of the martingale hypothesis.

However, when Hall current stock prices (the real value of the S&P 500) are included into the regression, as a proxy for current wealth a_t, the coefficient is highly significant, formally rejecting the martingale hypothesis. Thus the empirical evidence using aggregate data is mixed, but with perhaps surprisingly large support for the hypothesis from regressions using consumption and income data.[20] The more recent literature using household level data tends to find stronger re

The Consumption Function under Certainty Equivalence

The previous analysis only used the Euler equation to derive the empirical test of the model. Equations (5.38) and (5.39) do not constitute close-from solutions of the model, as the right hand side of both equations contains the endogenous (and stochastic) variable c_t. We now assume $\rho = r$ and combine (5.39) with the household budget constraints (5.1) to solve for the optimal consumption allocation in closed form, and to investigate how consumption responds to income shocks, under the assumption of quadratic utility.[21]

In appendix 5.8 we show how to consolidate the sequence of constraints (5.1) over time, in order to arrive at an intertemporal budget constraint

---

[20]Note that Hall's study was conducted prior to the development of the co-integration literature in time series econometrics (see e.g. Engle and Granger, 1987) that raised concerns about the interpretation of regressions using aggregate nonstationary data.

[21]We could use (5.38) and derive similar results for the case $\rho \neq r$, but at the expense of substantial algebra, and without obtaining additional insights.

similar to equation (5.17). For a fixed node s^t, this intertemporal budget constraint reads as

$$\sum_{\tau=0}^{T-t} \sum_{s^{t+\tau}|s^t} \frac{\pi_{t+\tau}(s^{t+\tau}|s^t)c_{t+\tau}(s^{t+\tau})}{(1+r)^\tau} \leq a_t(s^{t-1}) + \sum_{\tau=0}^{T-t} \sum_{s^{t+\tau}|s^t} \frac{\pi_{t+\tau}(s^{t+\tau}|s^t)y_{t+\tau}(s^{t+\tau})}{(1+r)^\tau}$$

(5.43)

$$E\left(\sum_{\tau=0}^{T-t} \frac{c_{t+\tau}}{(1+r)^\tau}\middle| s^t\right) \leq a_t(s^{t-1}) + E\left(\sum_{\tau=0}^{T-t} \frac{y_{t+\tau}}{(1+r)^\tau}\middle| s^t\right) \equiv W_t(s^t)$$

(5.44)

or, in short,

$$E_t \sum_{\tau=0}^{T-t} \frac{c_{t+\tau}}{(1+r)^\tau} \leq E_t \sum_{\tau=0}^{T-t} \frac{y_{t+\tau}}{(1+r)^\tau} + a_t \equiv W_t \qquad (5.45)$$

where the dependence of all variables on s^t is understood and the expectation is conditional on s^t.

Now, using equation (5.39) and the law of iterated expectations, we have[22]

$$E_t c_{t+\tau} = c_t \text{ for all } \tau \geq 0.$$

Using this result in the intertemporal budget constraint (5.45), which at the optimal allocation holds with equality, we can solve for consumption in closed

---

[22]Since

$$E_t c_{t+1} = c_t$$
$$E_t c_{t+2} = E_t E_{t+1} c_{t+2} = E_t c_{t+1} = c_t$$

form[23]:

$$c_t = \begin{cases} \theta_t^{-1} \frac{rW_t}{1+r} & \text{if } T < \infty \\ \frac{r}{1+r} W_t & \text{if } T = \infty \end{cases} \qquad (5.46)$$

where $\theta_t = \left(1 - \frac{1}{(1+r)^{T-t+1}}\right)$.

Remark 25 *More explicitly,*

$$c_t(s^t) = \begin{cases} \theta_t^{-1} \frac{rW_t(s^t)}{1+r} & \text{if } T < \infty \\ \frac{r}{1+r} W_t(s^t) & \text{if } T = \infty \end{cases} \qquad (5.47)$$

Note that this equation holds for the fixed initial node s_0 as well, so that (say, for $T = \infty$),

$$c_0(s_0) = \frac{r}{1+r}\left(a_0 + E_0 \sum_{\tau=0}^{\infty} \frac{y_{t+\tau}}{(1+r)^\tau}\right)$$

where a_0 is the exogenous initial asset position the household starts her life with (whereas for future periods $W_t(s^t)$ is a function of the endogenous choice $a_t(s^{t-1})$). Compare this to the consumption function under certain income in (5.19): in period 0 households make identical consumption (and thus saving) choices in the presence and absence of income risk: the SIM model with quadratic utility (and nonbinding borrowing constraints) exhibits certainty equivalence. Of course, in the stochastic case future consumption and asset holdings respond to future income shocks (which are absent in the deterministic case). However, for a given amount of assets carried into period t, households at that time period would make identical consumption-savings choices for period t in the deterministic and the stochastic income case.

---

[23]More explicitly,

$$c_t(s^t) = \begin{cases} \theta_t^{-1} \frac{rW_t(s^t)}{1+r} & \text{if } T < \infty \\ \frac{r}{1+r} W_t(s^t) & \text{if } T = \infty \end{cases}$$

Note that this equation holds for the fixed initial node s_0 as well, so that (say, for $T = \infty$),

$$c_0(s_0) = \frac{r}{1+r}\left(a_0 + E_0 \sum_{\tau=0}^{\infty} \frac{y_{t+\tau}}{(1+r)^\tau}\right)$$

where a_0 is the *exogenous* initial asset position the household starts her life with (whereas for future periods $W_t(s^t)$ is a function of the endogenous choice $a_t(s^{t-1})$).

Consumption Responses to Income Shocks[24]

Given the consumption function in equation (5.46) we can also determine how *realized* consumption changes in response to income shocks. We have already argued that expected changes in income have no effect on consumption; here we are interested in the impact of income shocks. Defining $\Delta c_t = c_t - c_{t-1}$ to be the realized change in consumption between period $t-1$ and period t, in Appendix 5.9 we show that

$$\theta_t \Delta c_t = \frac{r}{1+r} \sum_{s=0}^{T-t} \frac{(E_t - E_{t-1})y_{t+s}}{(1+r)^s} \equiv \eta_t \tag{5.48}$$

Here $(E_t - E_{t-1})y_{t+s} = E_t y_{t+s} - E_{t-1}y_{t+s}$ is the revision of the expectation about period $t+s$ income between period $t-1$ and period t, and thus η_t is the annuity value of the revisions of expectations, between $t-1$ and t, of all future incomes. Again, the adjustment factor $\theta_t = \left(1 - \frac{1}{(1+r)^{T-t+1}}\right)$ controls for the length of remaining lifetime if T is finite. Thus the realized change in consumption between two periods equals the annuity value of the revision in expectations about the present discounted value of future income, η_t. Its exact value depends on the specifics of the stochastic income process that households face. We now consider several concrete examples.

Example 26 *Suppose the income process of a household is specified as*

$$y_t = y_t^p + u_t \tag{5.49}$$
$$y_t^p = y_{t-1}^p + v_t \tag{5.50}$$

where y_t^p is the "permanent" part of current income, u_t is the transitory part and v_t is the innovation to the permanent part of income.[25] We assume that u_t and v_t are uncorrelated iid random variables with $E_t u_{t+s} = E_t v_{t+s} = 0$ for $s > 0$, where we adopt the timing convention that u_t, v_t are known when E_t is taken. This process can be rewritten as

$$y_t = y_{t-1} + u_t - u_{t-1} + v_t \tag{5.51}$$

---

[24]This subsection is based on Blundell and Preston (1998).

[25]Income processes with persistent (often permanent) and transitory shocks are very commonly estimated by labor economists and used by quantitative macroeconomists in their heterogeneous household models. Predominantly the permanent-transitory distinction is applied to log-income, however, see chapter 5.9 for a more detailed discussion.

and thus

$$y_{t+s} = y_{t-1} + u_{t+s} - u_{t-1} + \sum_{\tau=t}^{t+s} v_\tau \tag{5.52}$$

We now demonstrate that agents behaving according to the SIM model with quadratic utility react quite differently to permanent income shocks v_t and transitory income shocks u_t. We then have that

$$E_t y_{t+s} = y_{t-1} - u_{t-1} + v_t + \begin{cases} u_t \ if \ s = 0 \\ 0 \ if \ s > 0 \end{cases} \tag{5.53}$$

$$E_{t-1} y_{t+s} = y_{t-1} - u_{t-1} + 0 + 0 \tag{5.54}$$

and hence

$$(E_t - E_{t-1}) y_{t+s} = \begin{cases} u_t + v_t & if \ s = 0 \\ v_t & if \ s > 0 \end{cases} \tag{5.55}$$

Thus we can calculate η_t in equation (5.48) as

$$\eta_t = \frac{r}{1+r} u_t + \theta_t v_t$$

and thus we find that, for the specific income process with permanent and transitory shocks, the realized change in consumption is given by

$$\theta_t \Delta c_t = \frac{r}{1+r} u_t + \theta_t v_t \tag{5.56}$$

$$\Delta c_t = \frac{r\theta_t^{-1}}{1+r} u_t + v_t. \tag{5.57}$$

Thus households adjust their consumption one for one with a permanent income shock v_t, but only change their consumption mildly, by $\frac{r\theta_t^{-1}}{1+r}$ in response to a purely temporary shock u_t.

Example 27 *The last example assumes that income shocks are either permanent or transitory. Now suppose that households are only subject to one income shock, and that a fraction γ of the shock is mean-reverting whereas a fraction $1 - \gamma$ is permanent. Thus the income process takes the form*

$$y_t = y_{t-1} + \varepsilon_t - \gamma \varepsilon_{t-1} \tag{5.58}$$

with $\gamma \in [0, 1]$. Comparing (5.58) to (5.51) we see that the case of $\gamma = 0$ corresponds to a process with only permanent shocks, whereas $\gamma = 1$ corresponds to a process with only transitory shocks. Under such a process we find that

$$(E_t - E_{t-1}) y_{t+s} = \begin{cases} \varepsilon_t & if \ s = 0 \\ (1 - \gamma)\varepsilon_t & if \ s > 0 \end{cases} \tag{5.59}$$

and thus

$$\theta_t \Delta c_t = \frac{r\gamma}{1 + r}\varepsilon_t + (1 - \gamma)\theta_t \varepsilon_t \tag{5.60}$$

Therefore the household's consumption response to the ε_t shock is a convex combination (with weights γ and $1 - \gamma$) of the response to a transitory and to a permanent shock.

Example 28 *Finally, for a simple AR(1) income process of the form*

$$y_t = \delta y_{t-1} + \varepsilon_t \tag{5.61}$$

with $0 < \delta < 1$ we find that

$$\theta_t \Delta c_t = \frac{r\varepsilon_t}{1 + r} \sum_{s=0}^{T-t} \left(\frac{\delta}{1 + r}\right)^s. \tag{5.62}$$

Comparing this result to example 26 above we see that the change in consumption in reaction to a persistent (but not permanent) shock ε_t falls quantitatively in between that induced by a purely transitory shock (which equals $\frac{r\theta_t^{-1}}{1+r}$), since $\sum_{s=0}^{T-t} \left(\frac{\delta}{1+r}\right)^s > 1$ and that of a permanent shock (which equals 1), since $\delta < 1$.

5.3 Prudence

So far we have analyzed the SIM model in partial equilibrium, under the assumptions that household have quadratic utility and that constraints on borrowing were not binding. We showed that household behavior exhibit certainty equivalence: ex ante, prior to the realizations of risk, consumption and savings choices are identical without and with risk. We also studied how consumption and thus saving responds to income shocks ex post.

In the next two sections we will in turn relax both key assumptions that gave risk to certainty equivalence, and show that the relaxation of either assumption can give rise to precautionary saving behavior such that an increase

in income risk induces households to save more and consume less. In this section[26] we demonstrate that households engage in precautionary saving if they have convex marginal utility, $U'''(c) > 0$. In section 5.4 it is shown that in the presence of liquidity constraints agents may exhibit the same behavior even under quadratic utility.[27]

To set the stage for this section, recall that with quadratic utility and absent binding liquidity constraints the consumption function was given by (5.46) and exhibited certainty equivalence. Specifically, the optimal consumption rule was only a function of an annuity factor $\frac{r}{1+r}$ (adjusted by an additional term if T is finite) and otherwise only depends on the expected present discounted value of lifetime income W_t. From the definition of W_t we see that only the conditional (on period t information) *first moment* of future incomes $E_t y_{t+s}$ matters for the consumption choice, but not the extent of income risk (the conditional variance of future labor income) or higher moments of the random variables $\{y_{t+s}\}_{s \geq 1}$. Thus a change in the riskiness of future labor income (with unchanged mean) or any other higher moment of the distribution of future labor income leaves the current consumption (and thus saving) choice completely unaltered.

The key steps in deriving this result were to exploit quadratic utility to obtain equation (5.38) as a special case of the general stochastic Euler equation (5.35) for the SIM model, reproduced here for completeness, but with preference shocks suppressed:

$$U_c(c_t) = \left(\frac{1+r}{1+\rho}\right) E_t \left(U_c(c_{t+1}, s^{t+1}) \right) \tag{5.63}$$

and then to combine this Euler equation with the intertemporal budget constraint (5.45), which was derived from the sequential budget constraints, a derivation that in turn required the absence of binding borrowing constraints. In this section we will continue to use the intertemporal budget constraint, but will investigate the implications of the intertemporal Euler equation of utility is not quadratic (and thus marginal utility is not linear in consumption, which was required for obtaining (5.38).

---

[26] For this part the key references include Kimball (1990), Barsky, Mankiw and Zeldes (1986), Deaton (1991) and Carroll (1997).

[27] See Zeldes (1989a, 1989b) and again Deaton (1991) for the classic references.

5.3.1 A Simple Model and a General Result

In order to derive the result that relates the sign of the third derivative of the utility function to the presence of precautionary saving in the most transparent form we first consider a simple two period example, similar in spirit to the one studied for the certainty case in section 5.1 above.[28] We assume that income in period 0 is known and equal to y_0, but period 1 income y_1 is stochastic. For convenience[29] we also shall assume that $\rho = r = a_0 = 0$. It is convenient to decompose period 1 income into a deterministic and a stochastic component:

$$y_1 = \bar{y}_1 + \tilde{y}_1 \tag{5.64}$$

where $\bar{y}_1 = E_0 y_1$ is the (conditional) expectation of y_1 at period 0 and $\tilde{y}_1$ is a random variable with $E_0(\tilde{y}_1) = 0$. As before we define

$$W_0 = y_0 + \bar{y}_1 \tag{5.65}$$

as expected present discounted value of lifetime labor income. Under these assumptions the budget constrains in both periods read as

$$
\begin{aligned}
c_0 + a_1 &= y_0 \tag{5.66} \\
c_1 &= a_1 + \bar{y}_1 + \tilde{y}_1 \tag{5.67}
\end{aligned}
$$

and substituting out a_1 in the second equation yields

$$
\begin{aligned}
c_1 &= y_0 - c_0 + \bar{y}_1 + \tilde{y}_1 \\
&= W_0 - c_0 + \tilde{y}_1 \tag{5.68}
\end{aligned}
$$

Note that c_1 is a random variable that varies with the realization of the stochastic component of period 1 income $\tilde{y}_1$.

Finally define

$$
\begin{aligned}
s &= W_0 - c_0 \tag{5.69} \\
&= a_1 + \bar{y}_1 = E_0\left(c_1\right)
\end{aligned}
$$

as saving out of lifetime wealth (which equals expected consumption in period 1). This concept of saving, a simple shift of a_1 by the constant $\bar{y}_1$, will be

---

[28]The discussion of the simple model follows Barsky, Mankiw and Zeldes (1986).
[29]The results go through unchanged with $\rho \neq 0$ and/or $r \neq 0$ as well as $a_0 \neq 0$, but the algebra becomes substantially more messy.

useful later on when stating the general precautionary result. With these definitions the stochastic Euler equation (5.63), ignoring preference shocks, becomes

$$U_c(c_0) = E_0 U_c(W_0 - c_0 + \tilde{y}_1) \qquad (5.70)$$

Note that the only endogenous choice in this equation is the deterministic number c_0, the consumption choice for period 0, since W_0 is an exogenous constant and $\tilde{y}_1$ is an exogenous random variable. We now want to analyze how the optimal choice of c_0 depends on the parameters of the model, and especially, how it varies with the characteristics of income risk, measured by the variance of labor income in period 1, i.e. with $\sigma_y^2 = Var_0(\tilde{y}_1)$.

Intuition for the General Result

Before turning to the general result that shows under which conditions consumption c_0 is decreasing (and thus saving s and a_1 are increasing) in income risk σ_y^2 we consider a special income process where this result can be derived using simple algebra. Thus suppose that the stochastic part of income in period 1 can only take two values:

$$\tilde{y}_1 = \begin{cases} -\varepsilon & \text{with prob. } \frac{1}{2} \\ \varepsilon & \text{with prob. } \frac{1}{2} \end{cases} \qquad (5.71)$$

with $0 < \varepsilon < \bar{y}_1$. Therefore $\sigma_y^2 = \varepsilon^2$ and ε measures the amount of income risk the household faces. We wish to determine under what condition $c_0(\varepsilon)$ is a strictly decreasing function. Writing out equation (5.70) and using the assumption that $\tilde{y}_1$ follows a two-point distribution yields:

$$U_c(c_0) = \frac{1}{2} \left[U_c(W_0 - c_0 + \varepsilon) + U_c(W_0 - c_0 - \varepsilon) \right]. \qquad (5.72)$$

Totally differentiating (5.72) with respect to ε delivers:

$$U_{cc}(c_0)\frac{dc_0}{d\varepsilon} = \frac{1}{2} \left[U_{cc}\left(W_0 - c_0 + \varepsilon\right)\right) \left(-\frac{dc_0}{d\varepsilon} + 1\right) + U_{cc}\left(W_0 - c_0 - \varepsilon\right)\right) \left(-\frac{dc_0}{d\varepsilon} - 1\right) \right]$$

$$(5.73a)$$

and thus

$$\frac{dc_0(\varepsilon)}{d\varepsilon} = \frac{\frac{1}{2}\left[U_{cc}(W_0 - c_0 + \varepsilon) - U_{cc}(W_0 - c_0 - \varepsilon)\right]}{U_{cc}(c_0) + \frac{1}{2}\left[U_{cc}\left(W_0 - c_0 + \varepsilon\right)\right) + U_{cc}\left(W_0 - c_0 - \varepsilon\right)\right)]} \qquad (5.74)$$

The denominator of this expression is unambiguously negative (since we assume that U is strictly concave). The nominator is positive if and only if

$$U_{cc}\left(W_0 - c_0 + \varepsilon\right)) - U_{cc}\left(W_0 - c_0 - \varepsilon\right)) \tag{5.75}$$

is positive.

But this is true for arbitrary $\varepsilon > 0$ if and only if $U_{ccc}(c) > 0$. Hence consumption in period 0 strictly declines in reaction to a marginal increase in period 1 income risk σ_y^2 if and only if the third derivative of the utility function is positive. Using equation (5.66) and (5.69) it then follows immediately that a_1 and s are strictly increasing functions of σ_y^2 for all levels of σ_y^2 if and only if $U_{ccc}(c) > 0$ for all c. Thus a sufficient (and necessary) condition for the household to exhibit precautionary saving behavior (i.e. to increase saving in response to increased income risk) in the absence of binding borrowing constraints is strictly convex *marginal* utility U_c. Of course it is a simple corollary of this result that under the assumption $U_{ccc}(c) > 0$ households deviate from certainty equivalence behavior; simply realize that $c_0(\varepsilon = 0) > c_0(\varepsilon > 0)$.

An Application

The model above has been used by Barsky, Mankiw and Zeldes (1986) to show that the presence of income risk alone can invalidate the Ricardian equivalence hypothesis. Recall that this hypothesis states that for a given process of government spending a change in the timing of taxation does not affect household behavior and macroeconomic aggregates. Thus if Ricardian equivalence holds it should not matter whether current government expenditures are financed via current taxes of by government debt that is redeemed later (using future taxes).

A simple example is sufficient to make this argument. Suppose that government spending satisfies $G_0 = G_1 = 0$ and that the benchmark policy is one of no taxation in either period (and thus the situation analyzed in the previous subsection). Now consider the Ricardian experiment of lowering taxes in period 0, say, in order to stimulate consumption in the economy in a Keynesian-style fiscal expansion. Thus in period 0 households receive a lump-sum subsidy of size t, and in period 1 have to pay positive taxes in order to finance the repayment of the debt the government incurred for paying the transfers. Let τ denote the tax rate on labor income in period 1. We assume that the government deals with a continuum population of measure

1, each member of which faces the income process specified in (5.71). Thus government tax revenues from the income tax defined by τ are given by[30]

$$\tau E_0(y_1) = \tau \bar{y}_1$$

and the size of the period 0 transfers thus has to satisfy (recall that we assumed $r = 0$)

$$t = \tau \bar{y}_1$$

in order for the intertemporal government budget constraint to hold.

Therefore we note that, after the policy innovation, the expected present discounted value of lifetime labor income for each household equals

$$W_0(t, \tau) = y_0 + t + (1-\tau)E_0\left(\bar{y}_1 + \tilde{y}_1\right) = y_0 + t + (1-\tau)\bar{y}_1 = y_0 + \bar{y}_1 = W_0(t = \tau = 0),$$

and the fiscal policy reform leaves the expected present discounted value of lifetime income unchanged. If households were certainty equivalence consumers, their consumption behavior would therefore be unaffected by the Ricardian tax policy experiment. We note, however, that after-tax income in the second period now equals $(1 - \tau)(\bar{y}_1 + \tilde{y}_1)$, with associated variance $(1 - \tau)^2 \sigma_y^2 < \sigma_y^2$. Therefore, from the result in the previous section (as along as the size of τ, t is small), the household consumes more and saves less in response to the small temporary, debt-financed tax cut. Ricardian equivalence fails, despite the fact that the household behaves fully rational and forward looking, faces no borrowing constraints, and the tax is (or at least looks like) a lump-sum tax. The tax change *does* affect the extent of income risk, however, and thus induces a decline in precautionary saving. With Barsky, Mankiw and Zeldes' (1986) words, the household is a "Ricardian consumer with Keynesian propensities" to consume out of current income.

The General Theory

Kimball (1990) has defined the term "prudence" to mean the "propensity to prepare and forearm oneself in the face of uncertainty" and hence the *intensity* of a precautionary saving motive induced by the preferences of

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[30]We invoke a law of large numbers here that guarantees that the *deterministic* tax revenues from the entire population of identical households equals the expected tax revenue from each individual household. We will discuss potenial conerns with the applicability of the law of large numbers in chapter 7 and refer the reader to Feldman and Gilles (1985) and Judd (1985) for the mathematical details of the problem and one potential solution.

households (although the concept of prudence applies to other decisions in uncertain environments as well, and not just to saving behavior). From this definition it is clear that the term prudence characterizes preferences, whereas precautionary saving characterizes behavior: prudence leads to precautionary saving, but both concepts should be kept separately.

Also note that, as should be clear already from the results in the previous sections, that prudence and risk aversion are very distinct concepts (although they both describe preferences). Risk aversion is controlled by the concavity of the utility function ($U_{cc}(c) < 0$) whereas prudence (the precautionary savings *motive*) is controlled by the convexity of the marginal utility function ($U_{ccc}(c) > 0$). This distinction is most transparent for households with quadratic utility: these households dislike the additional consumption risk induced by additional income risk (in the SIM model) and would be willing to pay a positive insurance premium to get rid off the additional risk, but what they won't do is to increase their savings in order to hedge against that additional income risk (as we have demonstrated in section 5.2.2 that households with quadratic utility, absent binding borrowing constraints display certainty equivalence behavior).

Now turning to the general theory of prudence and precautionary saving, the result derived in section 5.3.1 merely demonstrates, via an example, that $U_{ccc}(c) > 0$ implies precautionary saving behavior, but it makes no statement about the *magnitude* of the effect of income risk on optimal household consumption and saving choices. The ingenious insight of Kimball (1990) is that a quantitatively meaningful theory of precautionary saving based on preferences that exhibit prudence can be derived as a mathematically straightforward extension of the quantitative theory of risk aversion developed by Pratt (1964) and Arrow (1965).

Pratt (1964) shows that the (equivalent[31]) risk premium $\theta(c, \tilde{y}_1)$, for a small, zero mean absolute risk (that is, a gamble that adds or takes away the random (zero mean) quantity $\tilde{y}_1$ for c), which is defined implicitly from the

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[31]The compensating risk premium $\theta^*(c, \tilde{y}_1)$ is defined as

$$E_0 U(c + \tilde{y}_1 + \theta^*(c, \tilde{y}_1)) = U(c). \tag{5.76}$$

equation[32]

$$E_0 U(c + \tilde{y}_1) = U(c - \theta(c, \tilde{y}_1)) \tag{5.77}$$

is explicitly characterized by

$$\theta(c, \tilde{y}_1) = -\frac{1}{2}\sigma_y^2 \frac{U_{cc}(c)}{U_c} + o(\sigma_y^2), \tag{5.78}$$

where $o(\sigma_y^2)$ is a term that converges to zero faster than does σ_y^2. This result justifies the coefficient of absolute risk aversion

$$r(c) = -\frac{U_{cc}(c)}{U_c(c)} \tag{5.79}$$

as an appropriate measure to *quantify* the willingness of a household to pay to avoid a small risk. Correspondingly, the coefficient of relative risk aversion

$$\sigma(c) = -\frac{c U_{cc}(c)}{U_c(c)} \tag{5.80}$$

measures the willingness to pay to avoid small relative gambles in which percentages of c are at risk. Of course, for the special cases of CRRA utility $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and CARA utility $U(c) = -\frac{1}{\gamma}e^{-\gamma c}$ these entities are given, correspondingly, by

$$
\begin{aligned}
r(c) &= \frac{\sigma}{c} \text{ and } \sigma(c) = \sigma \\
r(c) &= \gamma \text{ and } \sigma(c) = \gamma c.
\end{aligned}
$$

The key insight of Kimball (1990), building on earlier work by Leland (1968), Sandmo (1970) and Rothschild and Stiglitz (1970) was to notice that if one simply adds one derivative to all expressions obtained from equation (5.77) one immediately obtains a quantitative theory of precautionary saving. To see this, recall from equation (5.70) that the optimal consumption-saving choice (c_0, a_1), or equivalently, (c_0, s) is determined by the Euler equation

$$E_0 U_c(W_0 - c_0 + \tilde{y}_1) = U_c(c_0)$$

or, by the definition of s from equation (5.69):

$$E_0 U_c(s + \tilde{y}_1) = U_c(W_0 - s).$$

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[32]In general this premium is a function of the starting point c and the characteristics of the random variable $\tilde{y}_1$. The notations $\theta(c, \tilde{y}_1)$ and $\theta^*(c, \tilde{y}_1)$ are meant to convey this.

Now define the equivalent precautionary saving premium $\psi(s, \tilde{y}_1)$ implicitly by the equation

$$E_0 U_c(s + \tilde{y}_1) = U_c(s - \psi(s, \tilde{y}_1)) \tag{5.81}$$

Comparing equations (5.77) and (5.81) it follows immediately that all results (and especially the result summarized in equation (5.78) above) derived by Pratt (1964) for the equivalent risk premium $\theta(c, \tilde{y}_1)$ immediately apply to the equivalent precautionary saving premium $\psi(s, \tilde{y}_1)$, with all expressions in the original Pratt (1964) results replaced by one higher derivative.

Similarly, Kimball (1990) defines the compensating precautionary saving premium by $\psi^*(s, \tilde{y}_1)$ as

$$E_0 U_c(s + \tilde{y}_1 + \psi^*(s, \tilde{y}_1)) = U_c(s), \tag{5.82}$$

that is, the amount of extra saving (starting from the saving level s without risk) the household would find optimal in the presence of risk, relative to the risk-free case. Since the compensating precautionary saving premium is easier to interpret (and, at least locally, equal to the equivalent premium) we will focus on this measure of precautionary saving from now on, referring the reader to Kimball (1990) for a complete treatment. Again comparing equations (5.82) and (5.76) allows for a straightforward application of the results in Pratt (1964) to the precautionary saving premium. Consequently, we have the following

Proposition 29 *(Kimball 1990). The compensating precautionary saving premium is given by*

$$\psi^*(s, \tilde{y}_1) = -\frac{1}{2} \sigma_y^2 \frac{U_{ccc}(s)}{U_{cc}(s)} + o(\sigma_y^2)$$

The residual term $o(\sigma_y^2)$ satisfies

$$\lim_{\sigma_y^2 \to 0} \frac{o(\sigma_y^2)}{\sigma_y^2} = 0. \tag{5.83}$$

This result shows that the compensating precautionary saving premium is positive if and only if $U_{ccc}(s) > 0$, since $U_{cc} < 0$ by assumption. Also note that U_{ccc} is evaluated at $s = E_0(c_1)$. Perhaps more importantly, the proposition states that the *magnitude* of the precautionary saving premium

is exclusively determined, to a first order, by the amount of income risk σ_y^2 and the index of absolute prudence

$$p(s) = -\frac{U_{ccc}(s)}{U_{cc}(s)}$$

which justifies this index as a quantitative measure of prudence (and thus as an index of the intensity of the precautionary savings motive). Also note that for the special case of CRRA and CARA utility functions we find that:

$$p(s) \quad = \quad \frac{\sigma + 1}{c} \tag{5.84}$$

$$p(s) \quad = \quad \gamma \tag{5.85}$$

and in both cases risk aversion and prudence is controlled by a single parameter (σ and γ, respectively).[33]

Now that we have fully characterized the precautionary saving premium through proposition 29 we want to further interpret what this entity $\psi^*(s, \tilde{y}_1)$ actually measures, in terms of observable behavior. To do so, we first recall that the Euler equation for the case without risk reads as

$$U_c(W_0 - s) = U_c(s) \tag{5.87}$$

Combining equations (5.82) and (5.87) yields

$$EU_c(s + \tilde{y}_1 + \psi^*(s, \tilde{y}_1)) = U_c(W_0 - s) \tag{5.88}$$

For future reference define $c_0(W_0, \tilde{y}_1)$ as the optimal consumption choice associated with lifetime wealth W_0 and income in the second period determined

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[33] As for risk aversion and the risk premium, one can also define the index of relative prudence as

$$pr(c) = -\frac{cU_{ccc}(c)}{U_{cc}(c)} \tag{5.86}$$

and establish the same results as above if income risk is proportional to wealth W_0. For CRRA and CARA utility the coefficient of relative prudence is given, respectively, by

$$pr(c) \quad = \quad \sigma + 1$$
$$pr(c) \quad = \quad \gamma c.$$

by the random variable $\tilde{y}_1$, and as $s(W_0, \tilde{y}_1)$ the optimal saving (out of lifetime wealth) function. The corresponding optimal policy functions in the absence of income risk are defined as $c_0(W_0, 0)$ and $s(W_0, 0)$, which are of course related by

$$c_0(W_0, 0) + s(W_0, 0) = W_0.$$

Furthermore define the lifetime wealth W_0 needed to make a given consumption c_0 optimal as $W_0(c_0, \tilde{y}_1)$. Note that $W_0(c, \tilde{y}_1)$ is defined as the quantity x that satisfies

$$W_0(c_0(x, \tilde{y}_1), \tilde{y}_1) = x.$$

It is then easy to show[34] that $\psi^*(s, \tilde{y}_1)$ equals the additional wealth required to keep consumption at a given level c_0 if a small income risk is introduced, that is

$$\psi^*(W_0(c_0, 0) - c_0, \tilde{y}_1) = W_0(c_0, \tilde{y}_1) - W_0(c_0, 0) \tag{5.89}$$

In other words, $\psi^*(s, \tilde{y}_1)$ can also be interpreted as the *magnitude* of the rightward shift of the consumption function $c_0(W_0, 0)$ with wealth W_0 on the x-axis, at a particular level of consumption c_0, as income risk is turned on.

We now collect further important results about precautionary saving contained in Kimball (1990).

Remark 30 *The previous results were local in the sense that we considered a specific value for c_0 or s, and small risks. However, Kimball (1990) also*

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[34]By the definition of $W_0(c_0, \tilde{y}_1)$ we have

$$U_c(c_0) = E_0 U_c(W_0(c_0, \tilde{y}_1) - c_0 + \tilde{y}_1) = U_c(W_0(c_0, 0) - c_0).$$

where the last equality follows from equation (5.88). By definition of $\psi^*(W_0(c_0, 0) - c_0, \tilde{y}_1)$

$$E_0 U_c(W_0(c_0, 0) - c_0 + \tilde{y}_1 + \psi^*(W_0(c_0, 0) - c_0, \tilde{y}_1)) = U_c(W_0(c_0, 0) - c_0).$$

Equating the left hand sides of the previous two equations yields

$$\begin{aligned} &E_0 U_c(W_0(c_0, 0) - c_0 + \tilde{y}_1 + \psi^*(W_0(c_0, 0) - c_0, \tilde{y}_1)) \\ = {}& E_0 U_c(W_0(c_0, \tilde{y}_1) - c_0 + \tilde{y}_1). \end{aligned}$$

But since $U_{cc} < 0$ for all c, this last equation implies that, since both $W_0(c_0, 0) + \psi^*(W_0(c_0, 0) - c_0, \tilde{y}_1)$ and $W_0(c_0, \tilde{y}_1)$ are deterministic numbers:

$$W_0(c_0, 0) + \psi^*(W_0(c_0, 0) - c_0, \tilde{y}_1) = W_0(c_0, \tilde{y}_1).$$

provides a global when he shows that, for any two utility functions U, V, if

$$-\frac{U_{ccc}(s)}{U_{cc}(s)} > -\frac{V_{ccc}(s)}{V_{cc}(s)}$$

for all s then

$$\psi_U^*(s, \tilde{y}_1) > \psi_V^*(s, \tilde{y}_1)$$

for all s and all nondegenerate random variables $\tilde{y}_1$. This result is particularly useful for two utility functions U, V that yield the same consumption and savings functions in the absence of risk (e.g. if both U and V are of CRRA form with $\sigma_U > \sigma_V$; recall that $\rho = r = 0$). Then this result implies that for the same introduction of risk $\tilde{y}_1$ households with utility function U shift the consumption function to the right uniformly more than households with V when income risk is introduced.

Remark 31 *Often the situation of interest is one where the initial choice situation already contains some risk, and then additional risk is added, and the question arises what happens to consumption and saving in response to this additional risk. As long as the new risk is independently distributed to the already present risk, the results go through unchanged. However, if the additional risk is, for example, a mean-preserving spread of the old risk (and thus the additional risk is not independent of the initial risk), then the results stated above do not carry over.*

Remark 32 *Finally it is possible to determine how, at a given initial level of consumption $c_0 = c_0(W_0, 0)$ the propensity to consume out of wealth $\frac{\partial c_0(W_0, 0)}{\partial W_0}$ is affected by the introduction of income risk. From equation (5.89) we find*

$$W_0(c_0, \tilde{y}_1) - W_0(c_0, 0) = \psi^*(s(c_0, 0), \tilde{y}_1).$$

where $s(c_0, 0) \equiv W_0(c_0, 0) - c_0$. Suppose all terms are differentiable[35] in c_0, then

$$\frac{\partial W_0(c_0, \tilde{y}_1)}{\partial c_0} - \frac{\partial W_0(c_0, 0)}{\partial c_0} = \frac{\partial \psi^*(s(c_0, 0), \tilde{y}_1)}{\partial s(c_0, 0)} \frac{\partial s(c_0, 0)}{\partial c_0}$$

Because of time separability consumption c_0 in period zero and consumption in period one (which is equal to $s(c_0, 0)$ without risk) are both normal goods and thus increase together as W_0 increases. Thus $\frac{\partial s(c_0, 0)}{\partial c_0} > 0$ and the sign of

---

[35]See footnote 19 of Kimball (1990) how to proceed if that assumption is violated.

the left hand side is determined exclusively by the sign of $\frac{\partial \psi^*(s(c_0,0),\tilde{y}_1)}{\partial s(c_0,0)}$, which from proposition 29 is exclusively determined by the sign of

$$p'(s) = \frac{\partial \left(-\frac{U_{ccc}(s)}{U_{cc}(s)} \right)}{\partial s}$$

that is, the sign of $\frac{\partial W_0(c_0,\tilde{y}_1)}{\partial c_0} - \frac{\partial W_0(c_0,0)}{\partial c_0}$ depends solely on whether absolute prudence is strictly increasing, constant (such as for the CARA utility function) or strictly decreasing (such as for the CRRA utility function) in s. Since the inverse of $\frac{\partial W_0(c_0,.)}{\partial c_0}$ is the marginal propensity to consume out of lifetime wealth, we find that the marginal propensity to consume out of lifetime wealth in period zero (that is, the slope of the consumption function), at a given consumption level c_0 is strictly declining (increasing) with the introduction of income risk if absolute prudence is strictly increasing (decreasing) at the s associated with c_0.

Example 33 *Assume that $U(c) = \log(c)$, which has positive and strictly decreasing absolute prudence, and also assume that $\tilde{y}_1$ follows the process from subsection 5.3.1. We can then determine the consumption function in closed form as:*

$$c_0(W_0;\varepsilon) = \frac{3}{4}w - \frac{1}{4}\left(8\varepsilon^2 + (W_0)^2\right)^{\frac{1}{2}}$$

where $c_0(W_0;\varepsilon)$ is well-defined for wealth levels $W_0 \geq \varepsilon$. We note that

$$c_0(W_0;\varepsilon) < c_0(W_0;\varepsilon = 0) = \frac{1}{2}W_0 \text{ for all } W_0 \geq \varepsilon$$

and that for all $\varepsilon > 0$ the function $c_0(W_0;\varepsilon)$ is strictly increasing and strictly concave, with

$$\lim_{W_0 \to \infty} (c_0(W_0;\varepsilon) - c_0(W_0;\varepsilon = 0)) = 0$$

for all $\varepsilon > 0$. Furthermore

$$\frac{1}{2} = \frac{\partial c_0(W_0;\varepsilon = 0)}{\partial W_0} < \frac{\partial c_0(W_0;\varepsilon)}{\partial W_0} = \frac{3}{4} - \frac{1}{16\left(\left(\frac{\varepsilon}{w}\right)^2 + \frac{1}{16}\right)^{\frac{1}{2}}}$$

and

$$\lim_{w \to \infty} \frac{\partial c_0(W_0;\varepsilon)}{\partial W_0} = \frac{1}{2} = \frac{\partial c_0(W_0;\varepsilon = 0)}{\partial W_0}.$$

Figure **??** *shows the consumption function (for the case $\varepsilon = 0.2$) and the precautionary saving premium at $c_0 = 0.2$. As predicted by the previous remark the slope of the consumption function with risk is larger than the slope of the consumption function without risk, for all wealth levels W_0.*

Add Kimball.eps picture

5.3.2   A Parametric Example for the General Model

Now let us consider the general model with many periods. In general nothing analytical can be said about the optimal consumption profile, and how it responds to income shock. That is, a fully analytical counterpart to the case with quadratic utility carried out in section 5.2.2 is not available if marginal utility is not linear and thus households deviate from certainty equivalence behavior. However, in this section we show that, using CRRA utility and an *approximation* (whose accuracy can then evaluated numerically) we can derive a closed form expression for consumption growth (log-changes) and thus the consumption profile. In the next section we present a similar result (but without having to resort to approximations) for changes in consumption when households have CARA utility.

Now let us again start with the stochastic Euler equation from equation (5.35):

$$U_c(c_t, s^t) = \left(\frac{1+r}{1+\rho}\right) E_t \left(U_c(c_{t+1}, s^{t+1})\right) \tag{5.90}$$

Now we assume separability between consumption and preference shocks as well as CRRA utility, and the equation becomes

$$\left(\frac{1+r}{1+\rho}\right) E_t \left[\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right] = 1 \tag{5.91}$$

$$E_t \left[e^{\ln\left(\left(\frac{1+r}{1+\rho}\right)\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right)}\right] = 1$$

$$E_t \left(e^{-\sigma \ln(c_{t+1})} e^{\sigma \ln(c_t) + \ln(1+r) - \ln(1+\rho)}\right) = 1$$

$$e^{\sigma \ln(c_t) + \ln(1+r) - \ln(1+\rho)} E_t \left(e^{-\sigma \ln(c_{t+1})}\right) = 1. \tag{5.92}$$

We derived equation (5.91) from (5.92) since it will be useful in the analysis below. We note that since c_t is known when expectations E_t are taken, the

term $E_t\left[\left(\frac{1}{c_t}\right)^{-\sigma}\right] = \left(\frac{1}{c_t}\right)^{-\sigma}$ does not create any problem for obtaining a closed-form solution. However, the term $E_t\left[(c_{t+1})^{-\sigma}\right]$ does, since it is the expectation of a term that is nonlinear in c_{t+1}.

Thus, in order to arrive at a regression equation similar to the one Hall (1978) derives and estimates with quadratic utility (see equation (5.38)) most of the empirical literature proceeds by taking a first order Taylor approximation of the function $f(c_{t+1}) = c_{t+1}^{-\sigma}$ around the point $c_{t+1} = c_t$ in (5.91). This yields

$$\begin{aligned} c_{t+1}^{-\sigma} &\approx c_t^{-\sigma} - \sigma(c_{t+1} - c_t)c_t^{-\sigma-1} \\ &= c_t^{-\sigma}\left(1 - \sigma\frac{c_{t+1} - c_t}{c_t}\right). \end{aligned}$$ (5.93)

Using this in (5.91) delivers

$$\left(\frac{1+r}{1+\rho}\right) E_t\left[1 - \sigma\frac{c_{t+1} - c_t}{c_t}\right] = 1$$

and thus

$$E_t\frac{c_{t+1} - c_t}{c_t} = \frac{1}{\sigma}\left(\frac{r - \rho}{1+r}\right).$$ (5.94)

which relates expected consumption growth to the gap between the interest rate and the subjective time discount rate, with the size of the response determined by the intertemporal elasticity of substitution $\frac{1}{\sigma}$. Although one could easily bring (5.94) to the data, the equation most frequently estimated is derived from using the additional approximations $\frac{c_{t+1}-c_t}{c_t} \approx \Delta \ln c_{t+1}$ and $1 - \frac{1+r}{1+\rho} = \frac{r-\rho}{1+r} \approx \ln(1+r) - \ln(1+\rho)$ in (5.94) to obtain

$$E_t\Delta \ln c_{t+1} = \frac{1}{\sigma}\left[\ln(1+r) - \ln(1+\rho)\right].$$ (5.95)

This equation can then be used to estimate, via OLS regression the intertemporal elasticity of substitution $\frac{1}{\sigma}$, and to test the hypothesis implied by the model that any variable X_t in the information set of the household at time t should not help predict consumption growth. Thus the literature on Euler equation estimation (see e.g. Attanasio and Weber (1993, 1995) or Attanasio and Browning (1995) has used data on household consumption and real interest rates to run the regression

$$\Delta \ln c_{t+1} = \alpha_0 + \alpha_1 \ln(1 + r_{t+1}) + \beta X_t + \varepsilon_{t+1}$$

to obtain estimates of $\hat{\alpha}_1 = \widehat{\frac{1}{\sigma}}$ and to test whether $\hat{\beta} = 0$, where the variables X_t often include household income in period t.

Although estimating the approximated stochastic Euler equation has been the predominant approach to determine the parameters and test the main implication of the partial equilibrium SIM model, critiques of this approach (see e.g. Carroll, 2001) question whether the approximation (5.93) used to derive equation (5.95) is accurate. Related to this point, by using this approximation equation (5.95) ignores the importance of precautionary saving (despite the fact that with CRRA utility the household is prudent and has preference-induced precautionary saving motive).

To see this point, now assume that $\ln(c_{t+1})$ is normally distributed[36] with mean $\mu = E_t \ln(c_{t+1})$ and variance σ_c^2. Although it is in general not desirable to make distributional assumptions on endogenous variables (such as c_{t+1}), we do it here to demonstrate the potential problem with the Euler equation estimation approach, in an analytically tractable way. With this assumption we find that

$$
\begin{aligned}
E_t(c_{t+1}^{-\sigma}) &= E_t\left(e^{-\sigma \ln(c_{t+1})}\right) \\
&= \int_{-\infty}^{\infty} e^{-\sigma u} \frac{e^{-\frac{(u-\mu)^2}{2\sigma_c^2}}}{\sqrt{2\pi}\sigma_c} du \\
&= e^{\frac{(\mu-\sigma\sigma_c^2)^2-\mu^2}{2\sigma_c^2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{\left[u-(\mu-\sigma\sigma_c^2)\right]^2}{2\sigma_c^2}}}{\sqrt{2\pi}\sigma_c} du \\
&= e^{\frac{(\mu-\sigma\sigma_c^2)^2-\mu^2}{2\sigma_c^2}} \\
&= e^{\frac{1}{2}\sigma^2\sigma_c^2-\mu\sigma} \\
&= e^{\frac{1}{2}\sigma^2\sigma_c^2-\sigma E_t \ln(c_{t+1})} \quad\quad\quad\quad (5.96)
\end{aligned}
$$

Inserting this expression into equation (5.92) yields

$$
e^{-\sigma E_t \Delta \ln(c_{t+1})+\ln(1+r)-\ln(1+\rho)+\frac{1}{2}\sigma^2\sigma_c^2} = 1. \quad\quad\quad\quad (5.97)
$$

---

[36]This in turn requires appropriate assumptions on the underlying stochastic income process.For the two period model above, this requires that the random variable

$$
Z = \ln(\kappa + \tilde{y}_1)
$$

is normally distributed, where $\kappa = w - c_0$ is a constant.

Finally, taking logs on both sides and rearranging gives

$$E_t \Delta \ln(c_{t+1}) = \frac{1}{\sigma} \left[\ln(1 + r) - \ln(1 + \rho] + \frac{1}{2} \sigma \sigma_c^2 \right. \tag{5.98}$$

Remember that by linearizing the Euler equation became (see (5.95)):

$$E_t \Delta \ln c_{t+1} = \frac{1}{\sigma} \left[\ln(1 + r) - \ln(1 + \rho) \right] \tag{5.99}$$

and thus under the approximation expected consumption growth is chosen in exactly the same way as in the absence of any risk (compare equation (5.99) to the Euler equation in the absence of risk, equation (5.24)). From equations (5.98) and (5.99) we observe that:

1. The "correct" consumption allocation does not obey certainty equivalence. As CRRA utility exhibits prudence and thus induces a precautionary savings motive to households, from (5.98) we see that future consumption risk, captured in the last term of (5.98) tilts the consumption profile upward in expectation, relative to the consumption growth rule (5.99) derived under the linearization (which is in turn identical, in expectation, to the certainty case). Consumption growth is higher in the presence of risk since households find it optimal to postpone consumption for precautionary motives.

2. The degree to which consumption is postponed (i.e. the degree to which there is precautionary saving) in response to future consumption (income) risk is determined by the parameter σ that controls the magnitude of prudence for the CRRA utility function, as shown above for the two period model.

3. All variables X_t that, at period t, help to predict the variability of future consumption σ_c^2, will help to predict expected consumption growth, according to the exact Euler equation 5.98 (but not according to the approximated Euler equation (5.99)). For example, agents with a higher level of current assets or income may have lower future consumption variability and thus, according to equation (5.98), lower consumption growth. This point was made, among others, by Carroll (1992). Thus it may be flawed to run the regression

$$\ln c_{t+1} - \ln c_t = \alpha_1 + \beta X_t + \varepsilon_t \tag{5.100}$$

where X_t may be current wealth, and interpret a statistically significant estimate of β as evidence against the SIM model with CRRA utility.

5.4   Liquidity Constraints

So far we have assumed that the household can borrow up to some arbitrarily large amount, up to the no-Ponzi scheme condition which was assumed to be generous enough never to be binding. In this section we investigate how the analysis from the previous section (that mainly exploited the stochastic Euler equation) has to be adjusted if households instead face potentially binding borrowing constraints. These constraints seem empirically plausible and formal econometric tests seem to indicate not only their presence, but also their effect on consumption allocations (see Zeldes (1989) and the literature originating from that study).

Before investigating the impact of borrowing constraints on the Euler equation, we first want to demonstrate under what condition such a constraint is likely binding. To do so we need a sharp characterization of the optimal stochastic process of asset holdings. Such a characterization is typically not available for general utility functions, but with quadratic utility and the resulting certainty equivalent behavior. We recall that for this case, the optimal consumption rule of an infinitely lived consumer is given by

$$c_t = \frac{r}{1+r}\left[E_t\sum_{s=0}^{\infty}\frac{y_{t+s}}{(1+r)^s} + a_t\right]. \tag{5.101}$$

From the budget constraint we find that

$$a_{t+1} = (1+r)(y_t - c_t) + (1+r)a_t \tag{5.102}$$

and plugging in optimal consumption c_t yields:

$$\begin{aligned}
a_{t+1} &= (1+r)y_t + (1+r)a_t - r\left[E_t\sum_{s=0}^{\infty}\frac{y_{t+s}}{(1+r)^s} + a_t\right]\\
&= a_t + y_t - E_t\sum_{s=1}^{\infty}\frac{ry_{t+s}}{(1+r)^s}\\
&= a_t + y_t - E_t\sum_{s=1}^{\infty}\left[\frac{y_{t+s}}{(1+r)^{s-1}} - \frac{y_{t+s}}{(1+r)^s}\right] \tag{5.103}
\end{aligned}$$

as the optimal asset allocation decision. Thus the realized change in asset holdings is given by

$$\Delta a_{t+1} = a_{t+1} - a_t = -\sum_{s=1}^{\infty}\frac{E_t\Delta y_{t+s}}{(1+r)^{s-1}}$$

The exact properties of wealth $\{a_{t+1}\}$ depend on the stochastic income process. For example, if the income process is of the form (5.49)-(5.50) with transitory and permanent income shocks (u_t and v_t respectively), then

$$E_t \Delta y_{t+s} = \begin{cases} -u_t & \text{if } s = 1 \\ 0 & \text{if } s > 1 \end{cases}$$

and thus

$$\Delta a_{t+1} = u_t \tag{5.104}$$

i.e. assets follow a random walk. Permanent income shocks v_t are fully absorbed by consumption and thus do not induce changes in the asset position of the household. Transitory income shocks in contrast trigger a consumption response of only $\frac{ru_t}{1+r}$, and thus the remaining part of the shock $\frac{u_t}{1+r}$ is absorbed by $\frac{a_{t+1}}{1+r}$ and thus assets change by the whole transitory shock u_t.

This discussion shows that a certainty equivalent consumer will violate any fixed borrowing limit $a_{t+1} \geq -\bar{A}$ with probability 1 (since assets follow a random walk). The asset process might not may not violate the no Ponzi condition if we specify it carefully[37], but calls into question the plausibility of the assumption that liquidity constraints are never binding, a maintained assumption so far.

5.4.1 The Euler Equation with Liquidity Constraints

Now let us assume that there exist borrowing constraints, and for simplicity let us follow Deaton (1991), Schechtman (1976), Aiyagari (1994) and many others and assume that agents cannot borrow at all[38], i.e. face the constraint $a_{t+1}(s^t) \geq 0$ for all s^t. These constraints may or may not be binding, depending on the realizations of the labor income shock, but we have to take these constraints into account explicitly when deriving the stochastic Euler equation. Let us attach Lagrange multiplier $\mu_t(s^t)$ to the borrowing constraint $a_{t+1}(s^t) \geq 0$ at event history s^t. The first order conditions with respect to consumption, (5.30) and (5.31) remain unchanged. The first order condition

---

[37]For quadratic utility the natural borrowing constraint, assumed so far, is still too tight since negative consumption is permissible with this utility specification.

[38]If there is a positive probability of zero income in every period then the constraint $a_{t+1}(s^t) \geq 0$ is the natural borrowing limit.

with respect to assets $a_{t+1}(s^t)$ now becomes

$$\frac{\lambda_t(s^t)}{1+r} - \mu_t(s^t) = \sum_{s^{t+1}|s^t} \lambda_{t+1}(s^{t+1}) \tag{5.105}$$

with complementary slackness conditions

$$
\begin{aligned}
a_{t+1}(s^t), \mu_t(s^t) &\geq 0 \\
a_{t+1}(s^t)\mu_t(s^t) &= 0
\end{aligned} \tag{5.106}
$$

Combining the first order condition yields

$$U_c(c_t(s^t)) - \frac{\mu_t(s^t)(1+r)}{\beta^t \pi_t(s^t)} = (1+r)\beta \sum_{s^{t+1}|s^t} \pi_{t+1}(s^{t+1}|s^t)U_c(c_{t+1}(s^{t+1}), s^{t+1}) \tag{5.107}$$

or in short

$$
\begin{aligned}
U_c(c_t) &\geq \frac{1+r}{1+\rho}E_t U_c(c_{t+1}) \\
&= \frac{1+r}{1+\rho}E_t U_c(c_{t+1}) \text{ if } a_{t+1} > 0
\end{aligned} \tag{5.108}
$$

For the analysis below it is useful to further rewrite the Euler equation (5.108). Note from the budget constraint that

$$
\begin{aligned}
c_t &= y_t + a_t - \frac{a_{t+1}}{1+r} \\
&\leq y_t + a_t
\end{aligned} \tag{5.109}
$$

where the last inequality follows from the liquidity constraint $a_{t+1} \geq 0$. Thus either $a_{t+1} = 0$ and therefore $c_t = y_t + a_t$ and

$$U_c(y_t + a_t) = U_c(c_t) \geq \frac{1+r}{1+\rho}E_t U_c(c_{t+1})$$

or $a_{t+1} > 0$, therefore $c_t < a_t + y_t$ and thus (using strict concavity of the utility function)

$$U_c(y_t + a_t) < U_c(c_t) = \frac{1+r}{1+\rho}E_t U_c(c_{t+1})$$

Hence the Euler equation can be compactly written as

$$U_c(c_t) = \max\left\{ U_c(y_t + a_t), \frac{1+r}{1+\rho}E_t U_c(c_{t+1}) \right\} \tag{5.110}$$

5.4.2 Precautionary Saving Due to Liquidity Constraints

In section 5.2.2 we showed that in the absence of binding liquidity constraints households with quadratic utility obey certainty equivalence and thus do not engage in precautionary saving behavior. Now we demonstrate that the presence of potentially binding liquidity constraints induces precautionary saving behavior *even with* quadratic utility.

With quadratic preferences equation (5.110) becomes

$$-(c_t - \bar{c}) = \max\left\{-(y_t + a_t - \bar{c}), -\frac{1+r}{1+\rho}\left(E_t c_{t+1} - \bar{c}\right)\right\} \qquad (5.111)$$

To simplify the algebra assume $\rho = r$. Then (5.111) can be rewritten as[39]

$$
\begin{aligned}
c_t &= \min\{y_t + a_t, E_t c_{t+1}\} & (5.112)\\
&= \min\{y_t + a_t, E_t \min\{y_{t+1} + a_{t+1}, E_{t+1} c_{t+2}\}\} & (5.113)
\end{aligned}
$$

From equation (5.113) we make the following observations:

1. If the certainty equivalence solution (5.101) (or its finite horizon counterpart) has associated asset holdings (5.103) that satisfy $a_{t+1} \geq 0$ with probability 1, then it is the optimal consumption allocation even in the presence of borrowing constraints. Whether or not this is the case obviously depends crucially on the stochastic income process. If the income process takes the form (5.49)-(5.50) and thus asset holdings follow a random walk, then the constraint $a_{t+1} \geq 0$ is violated with probability one.

2. In the absence of borrowing constraints we know from our discussion of certainty equivalence that

$$c_t = E_t(c_{t+s})$$

   for all $s > 0$. Suppose now there exists an $s > 0$ (e.g. $s = 1$) such that for some realization of the income shock y_{t+s} (with positive probability) the household is borrowing constrained, and thus $y_{t+s} + a_{t+s} < E_{t+s} c_{t+s}$. Then equation (5.112) implies that (using the law of iterated expectations) $c_t < E_t(c_{t+s})$. Thus, even if the liquidity constraint is not binding in period t, *future* binding constraints affect the *current* consumption choice c_t.

---

[39]Remember that $\max\{-x, -y\} = \min\{x, y\}$.

3. The previous argument also implies that any change in the environment that affects the incidence of future binding borrowing constraints affects current consumption. Importantly, suppose the variance of future income increases (say, for y_{t+1}), making lower realizations of y_{t+1} possible or more likely. If consequently the set of y_{t+1} values for which the borrowing constraint binds becomes larger, then

$$E_t \min\{y_{t+1} + a_{t+1}, E_t c_{t+2}\} \qquad (5.114)$$

declines and so does c_t, since in more instances the minimum is the first of the two objects in equation (5.114). Saving increases in reaction to increases in future income risk, because households, afraid of future contingencies of low consumption (agents with quadratic utility *are* risk-averse) and aware of their inability to smooth low income shocks via borrowing, increase their precautionary savings. Note that we obtain precautionary savings behavior without prudence (i.e. without a precautionary saving motive induced by preference), and purely from the existence of liquidity constraints (and risk aversion). Hence when observing increases in saving in response to increased income risk, this may have a preference-based explanation (agents are prudent: $U_{ccc} > 0$) or an (incomplete) financial market-based interpretation (credit markets prevent or limit borrowing).

5.4.3   Empirical Tests of Liquidity Constraints

[TBC Discuss the empirical literature on liquidity constraints]

   This Euler equation, which sometimes holds with inequality, depending on whether the liquidity constraint is binding, is the basis for empirical tests for liquidity constraints. In particular, Zeldes (1989) subdivides the sample of households into two groups, depending on their current wealth positions, with one group composed of households whose liquidity constraint is most likely not binding (high wealth households) and the other group composed of households whose constraint is most likely binding (low wealth households). The empirical test then consists of running a regression like Hall (1978), but for both groups separately (and with micro data rather than macro data) and ask whether current income helps forecast future consumption growth. for both groups As evidence for the presence of liquidity constraints would be interpreted as a finding which shows that current income helps predict

consumption growth for the low-wealth group, but not for the high-wealth group. Zeldes (broadly) finds such evidence and concludes that borrowing constraints are important in shaping consumption choices, at least for the currently wealth-poor. In evaluating his result, however, you should keep in mind that we argued above that with preferences that exhibit prudence, current income may help predict consumption growth, if it contains information about the extent of future income uncertainty. While it is not entirely obvious why this should be the case for low-wealth households and not for high-wealth households, note that wealth accumulation is an endogenous choice, so the low wealth-group sample is not simply a random sample. Therefore it is not inconceivable that low-wealth households have income processes with very different stochastic properties that high-wealth households, including predictability of future income risk by current income. [TBC]

5.5 Prudence and Liquidity Constraints: Theory

Finally we will discuss optimal consumption choices with both a precautionary saving motive and binding borrowing constraints. As we will see in the next section, most of the analysis relies on numerical approximations of the consumption or the savings function, as analytical solutions for this problem, in contrast to the certainty equivalence case, are usually not available. However, the theoretical literature has provided qualitative characterizations of the optimal consumption-savings choices, under varying assumptions about the stochastic income process and the relative size of the interest rate r the and time discount rate ρ. As it will turn out, the theoretical results depend crucially on the assumptions about the relative magnitudes of r and ρ; this in turn have important implications for the relationship between r and ρ in the general equilibrium models in the next chapter, in which the interest rate is determined endogenously. The important references from this literature include Schechtman (1976), Schechtman and Escudero (1977), Yaari (1977), Sotomayor (1984), Clarida (1987), Caballero (1990, 1991), Deaton (1991), Huggett (1993) and Chamberlain and Wilson (2000), Carroll and Kimball (1996, 2001).

In the class of SIM models we study consumption-savings decisions are determined by two main motives. First, the degree of impatience ρ of

households, relative to the market interest rate, determines whether, ceteris paribus, consumption is rising, falling and constant over time. Second, the precautionary savings motive calls for the postponement of consumption in favor of saving, and thus, again ceteris paribus, an upward sloping consumption profile. Below we will show, roughly speaking, that with $\rho \leq r$ the (im)patience motive and the precautionary savings motive reinforce each other and consumption as well as assets rise over time without bounds, at least as long as households are infinitely lived.[40] In contrast, if $\rho > r$ then the precautionary and the impatience motives compete, and under appropriate conditions asset holdings and consumption remain bounded even if the household lives forever, and consumption and asset possess an ergodic distribution. Thus we need to consider the cases $\rho > r$, $\rho = r$ and $\rho < r$ separately. We briefly discuss the situation where the planning horizon of the household is finite and thus follows a meaningful life cycle. However, sharp theoretical results are only available for an infinite planning horizon when households face a time (age) invariant income process.

5.5.1   Finite Lifetime

If T is finite, obviously the stochastic processes for consumption and assets c_t and a_{t+1} remain bounded, independent of the relationship between r and ρ (recall that the income process was assumed to be bounded).[41] The life cycle profile of consumption is determined by the relative strength of the impatience and the precautionary savings motive. Attanasio et al. (1999) and Gourinchas and Parker (2003), and many others since, have shown that as long as the interest rate and the time discount factor satisfy $r < \rho$ and are of "appropriate" size, then for realistic income processes (and thus realistic income risk) estimated from micro data such as the PSID, early in the life cycle remaining lifetime income risk is high and thus the precau-

---

[40]This result will also imply that, since aggregate asset demand is infinite with $\rho < r$ in the long, no steady state general equilibrium (to be defined in chapter 7) will exist with $r > \rho$.

[41]This also implies that aggregate asset demand is finite even if $r > \rho$ and thus this case cannot be ruled out to occur (and does occur frequently in applications) in general equilibrium life cycle models. General equilibrium models with many overlapping generations, each of which faces a income fluctuation problem with finite horizon, have become popular tools to analyze policy reforms, from social security reform to fundamental tax reform. See e.g. Conesa and Krueger (1999, 2006) and Conesa et al. (2009) for representative examples.

tionary saving motive dominates and consumption initially grows. Later in life the consumer has accumulated an asset buffer against that risk; in addition remaining lifetime income risk declines towards retirement. Thus the impatience motive starts to dominate and consumption declines (especially if mortality risk that increases the effective time rate is factored in). Combining both observations delivers a model-generate (expected) consumption profile that is hump-shaped as in the data. The asset life cycle profile is dominated by the precautionary saving early in life; in addition the life cycle household saves for retirement so that assets increase over the life cycle towards retirement and are then decumulated in order to finance old age consumption. The quantitative magnitude of saving for retirement is in turn crucially affected by the generosity of the social security system. These results are derived from numerical simulations of the model since in general no analytical solutions or qualitative characterizations of the finite horizon life cycle model with uninsurable income risk exist.

5.5.2 Infinite Horizon

If households live forever and the stochastic income process they face is time-invariant as well, the only two forces governing the both the consumption and savings behavior are the impatience motive and the precautionary savings motive as the life cycle savings motive is absent without a time (age) dependent income process. Since the quantitative importance of the impatience motive is determined by the size of the interest rate r, relative to the time discount factor ρ, it is intuitive that the theoretical properties of the optimal consumption-savings allocation depend crucially on the relative size of r and ρ. We therefore distinguish three cases.

Case $\rho < r$

In this case both the (im-)patience motive and the precautionary saving motive point towards postponing consumption in favor of saving. In fact, it is optimal to accumulate assets without bounds as we will demonstrate next.

Proposition 34 *[Sotomayor (1984), Chamberlain and Wilson (200)]: Let $r > \rho > 0$ and assume that either U is bounded or that the income process is iid with support $y_t \in [a, A]$ where $a \geq 0$ and $A < \infty$. Then*

$$\lim_{t \to \infty} c_t = \infty \ almost \ surely \tag{5.115}$$

and

$$\lim_{t\to\infty} a_t = \infty \ \text{almost surely} \qquad (5.116)$$

The basic intuition for this result can be derived from the Euler equation with liquidity constraints, equation (5.108). Under the assumption that $\rho < r$ we obtain that

$$
\begin{aligned}
U_c(c_t) &\geq \frac{1+r}{1+\rho} E_t U_c(c_{t+1}) \\
&> E_t U_c(c_{t+1}).
\end{aligned}
\qquad (5.117)
$$

Hence the stochastic process for marginal utility $\{U_c(c_t)\}$, which is strictly positive, follows a supermartingale.[42] We now can invoke the martingale convergence theorem (which is reviewed in remark 35 below) it follows that the sequence of random variables $\{U_c(c_t)\}$ converges almost surely to some limit random variable $U_c(c)$. Iterating on the Euler equation, for all t

$$E_0 U_c(c_{t+1}) \leq \left(\frac{1+\rho}{1+r}\right)^{t+1} U_c(c_0) \qquad (5.118)$$

and thus $E_0 U_c(c_{t+1}) \to 0$. But since $U_c(.)$ is strictly positive, it must be the case that not only $\{U_c(c_t)\}$ converges almost surely to some limit random variable $U_c(c)$, which is what the martingale convergence theorem guarantees, but it converges almost surely to $U_c(c) = 0$. From the strict concavity and the Inada conditions of the utility function it follows that consumption converges to ∞ with probability one, that is, equation (5.115) is satisfied. With a bounded labor income process, to finance diverging consumption requires diverging asset income and thus equation (5.116) follows. will exist. Although we refer the reader to Sotomayor (1984) and Chamberlain and Wilson (2000) for proofs of this result, note that the assumptions in the previous proposition are used to insure that lifetime utility of the consumer remains finite under the optimal consumption allocation.

---

[42]In fact, for the seqence of random variables $\{U_c(c_t)\}$ to be a supermartingale it is required that

$$E\left|U_c(c_t)\right| < \infty$$

which is assured if the endowment process $\{y_t\}$ is such that positive consumption is is possible with probability 1 (or alternatively, if $U_c(0) < \infty$, but this would violate the Inada condition).

Remark 35 *The martingale convergence theorem is due to the American Mathematician Joseph Leo Doob. In order to formally state the theorem in our context, it is required to describe the probability space on which our stochastic consumption processes are defined. Let F denote the set of all infinite sequences[43] $s = (s_0, s_1 \ldots, s_t, \ldots)$, let $\mathcal{F}$ denote a σ-algebra on F and $\pi : \mathcal{F} \to [0, 1]$ denote a probability measure defined over the measurable space $(F, \mathcal{F})$. A random variable $X : F \to \mathbf{R}$ is a measurable function with respect to $\mathcal{F}$, i.e. a function X such that for all $a \in \mathbf{R}$ the set*

$$\{s \in F : X(s) \leq a\} \in \mathcal{F}.$$

A filtration is a sequence of σ-algebras $\{\mathcal{F}_t\}$ with $\mathcal{F}_t \subset \mathcal{F}$ that satisfy

$$\mathcal{F}_t \subset \mathcal{F}_{t+1}$$

for all t. For our purposes we can interpret $\mathcal{F}_t$ as capturing the information the household has at time t. Imagine that at time zero nature draws an infinite sequence $s \in F$, but which s was drawn is unknown to the household (she just knows the probability measure π over all possible sequences). As time unfolds the household learns more and more about what infinite history s was drawn. For example, suppose that $s_t \in \{1, 2\}$ for all t. Then $s \in F$ is an infinite history of 1's and 2's. Then

$$\mathcal{F}_0 = \{\oslash, F, F_0^1, F_0^2\}$$

where

$$F_0^i = \{\hat{s} \in F : \hat{s}_0 = i\},$$

that is, at time zero the household can distinguish between infinite histories whose first entry differs (but not among histories with identical first entries). Similarly

$$\mathcal{F}_1 = \{\oslash, F, F_1^{11}, F_1^{12}, F_1^{21}, F_1^{22}, \ldots\}$$

where the set

$$F_1^{ij} = \{\hat{s} \in F : \hat{s}_0 = i, \hat{s}_1 = j\}.$$

Now households can distinguish between infinite histories that differ in one of the first two entries. A stochastic process is a sequence $\{X_t, \mathcal{F}_t\}_{t=0}^{\infty}$ such that

---

[43]For ease of notation we treat s_0 as random here too, rather than as the fixed intial node (as we did so far).

$\{\mathcal{F}_t\}_{t=0}^{\infty}$ *is a filtration and* $\{X_t\}_{t=0}^{\infty}$ *is a sequence of random variables such that* X_t *is measurable with respect to* $\mathcal{F}_t$ *for all t. We say that* $\{X_t\}_{t=0}^{\infty}$ *is adapted to the filtration* $\{\mathcal{F}_t\}_{t=0}^{\infty}$. *Our notation* $\{c_t(s^t), \pi_t(s^t)\}$ *can now be related to this rigorous way of describing the stochastic structure of the economy. First, we can construct the probability* $\pi_t(s^t)$ *of event history* s^t *from the probability measure* π *over infinite histories as*

$$\pi_t(s^t) = \int \mathbf{1}_{A(s^t)} d\pi$$

where

$$A(s^t) = \{\hat{s} \in F : \hat{s}^t = s^t\}.$$

Note that formally $\pi_t(.)$ *is a probability measure on the measurable space* $(S^t, \mathcal{F}_t)$. *Finally, although consumption, defined as a stochastic process, is a sequence of functions* $c_t : F \to \mathbf{R}$ *mapping infinite histories into the real numbers, the measurability of* c_t *with respect to* $\mathcal{F}_t$ *requires that for any two infinite histories* $s, \hat{s} \in F$ *with* $s^t = \hat{s}^t$ *we have*

$$c_t(s) = c_t(\hat{s}).$$

Thus we can write, in short, $c_t(s^t)$ *for all infinite histories for which the finite history until period t is given by* s^t. *Finally, with our definition of what exactly a stochastic process is we can now define a martingale as a stochastic process* $\{X_t, \mathcal{F}_t\}_{t=0}^{\infty}$ *such that for all t*

$$X_t = E\left[X_{t+1}|\mathcal{F}_t\right]$$

almost surely.[44] *Similar a submartingale is defined as a stochastic process such that for all t*

$$X_t < E\left[X_{t+1}|\mathcal{F}_t\right]$$

almost surely, and a a supermartingale is defined as a stochastic process such that for all t

$$X_t > E\left[X_{t+1}|\mathcal{F}_t\right]$$

---

[44]In addition the stochastic process has to satisfy the regularity condition

$$E\left(|X_t|\right) < \infty$$

for all t.

almost surely. The martingale convergence theorem then states that a super-martingale $\{X_t, \mathcal{F}_t\}_{t=0}^{\infty}$ that satisfies

$$K = \sup_t E(|X_t|) < \infty$$

converges to a limit random variable X^* almost surely and that the limit random variable X^* satisfies $E(|X^*|) \leq K$. Note that X^* in general is a nondegenerate random variable, rather than a fixed number.

The result in proposition 34 also implies that, in a model with many infinitely lived households each of which solving an income fluctuation, there will not be a long-run stationary asset distribution if $r > \rho$, since the asset holdings of all households diverge with probability one. Consequently no stationary general equilibrium exists[45] in such a model (which we will study in chapter 7) that satisfies $r > \rho$. We therefore now turn to the cases $r \leq \rho$.

Case $\rho = r$

Sotomayor (1984) and Chamberlain and Wilson (2000) show that the previous result (that consumption and assets diverge to ∞ almost surely) goes through even for $\rho = r$, if the income process is sufficiently stochastic, in a sense to be made precise below.

Deterministic Labor Income   That a sufficiently risky income process is required for the result can easily be seen by considering the limiting case in which labor income follows a deterministic process. Therefore, now assume that the endowment process is a deterministic sequence $\{y_t\}_{t=0}^{\infty}$. Absent the tight borrowing constraint this is of course a trivial case where consumption is constant over time, see subsection 5.2.1. However, with the tight borrowing constraints $a_{t+1} \geq 0$ even the deterministic case is not straightforward to tackle. If income is falling over time, then the household can realize constant consumption at the optimal level by saving early and dissaving once income falls. But if labor income tends to grow with time, in order to implement a constant consumption stream the household would need to borrow against future higher labor income which the tight borrowing constraint prevents.

---

[45]One possibility to circumvent this problem yet remain within the class of models in which household age is not a state variable (and thus life cycle considerations are abstracted from) is to introduce a constant and sufficiently large probability of death of each household, as in Yaari (1965).

Then we have the following proposition, due to Chamberlain and Wilson (2000):

Proposition 36 *Define*

$$x_t = \frac{r}{1+r} \sum_{\tau=t}^{\infty} \frac{y_\tau}{(1+r)^{\tau-t}} \tag{5.119}$$

Then

$$\bar{c} := \lim_{t\to\infty} c_t = \sup_t x_t =: \bar{x} \tag{5.120}$$

Proof. See Sargent and Ljungquist (2nd edition), chapter 16.3.1 Note that for this result U need not be bounded, a maintained assumption of Chamberlain and Wilson (2000) who of course are mainly interested in the stochastic labor income case. ∎

The number x_t is the period t annuity value of future labor income. Since the income process $\{y_t\}$ is bounded from above (because y_t can take only finitely many values) the sequence $\{x_t\}$ is bounded from above and thus $\bar{x}$ is a finite number which gives the maximal (over time) annuity value of income.

The intuition for this result is easiest to see if there is a *finite* date T at which the borrowing constraint binds for the last time. In that case $a_T = 0$ and $a_{T+\tau} > 0$ for $\tau > 0$. Since $\rho = r$ and there is no income risk the Euler equation then implies (see equation (5.108)) that:

$$c_T = c_{T+1} = c_{T+\tau} = \hat{c} \text{ for all } \tau > 0 \tag{5.121}$$
$$c_t \leq c_T \text{ for all } t < T \tag{5.122}$$

The budget constraints from period T onwards read as

$$c_T + \frac{a_{T+1}}{1+r} = y_T$$
$$c_{T+\tau} + \frac{a_{T+\tau+1}}{1+r} = y_{T+\tau} + a_{T+\tau}$$

Consolidating these into a lifetime budget constraint (from T onwards) yields[46]

$$\sum_{\tau=0}^{\infty} \frac{c_{T+\tau}}{(1+r)^\tau} = \sum_{\tau=0}^{\infty} \frac{y_{T+\tau}}{(1+r)^\tau}$$

---

[46]This follows as long as

$$\lim_{\tau\mapsto\infty} \frac{a_{T+\tau+1}}{(1+r)^{\tau+1}} = 0.$$

and thus

$$\sum_{\tau=0}^{\infty} \frac{c_{T+\tau}}{(1+r)^{\tau}} = \hat{c} \sum_{\tau=0}^{\infty} \frac{1}{(1+r)^{\tau}} = \frac{(1+r)\hat{c}}{r} = \sum_{\tau=0}^{\infty} \frac{y_{T+\tau}}{(1+r)^{\tau}}$$

and thus, for all t and τ,

$$c_t \leq c_T = c_{T+\tau} = \hat{c} = \frac{r}{1+r} \sum_{\tau=0}^{\infty} \frac{y_{T+\tau}}{(1+r)^{\tau}} = x_T.$$

_____

The fact that

$$\lim_{\tau \mapsto \infty} \frac{a_{T+\tau+1}}{(1+r)^{\tau+1}} \geq 0$$

is implied by the borrowing constraint $a_{T+\tau+1} \geq 0$, and a consumption-savings plan that has

$$\lim_{\tau \mapsto \infty} \frac{a_{T+\tau+1}}{(1+r)^{\tau+1}} > 0$$

cannot be optimal, intuitively, because the household "leaves resources on the table" rather than consuming it. The condition

$$\lim_{\tau \mapsto \infty} \frac{a_{T+\tau+1}}{(1+r)^{\tau+1}} \leq 0$$

is called the transversality condition. It is an optimality condition (like the Euler equation), that, under appropriate conditions on the utility function, the endowment process and the interest process, is jointly *sufficient*, together with the Euler equations, for an optimal consumption-savings plan. Under alternative conditions on the fundamentals the transversality condition is a *necessary* condition for an optimal allocation.

Note the fundamental distinction between a No-Ponzi condition and the transversality condition. The No-Ponzi condition is required for the household maximization problem to have a solution and in the current context would read as

$$\lim_{\tau \mapsto \infty} \frac{a_{T+\tau+1}}{(1+r)^{\tau+1}} \geq 0.$$

The transversality condition is an optimality condition and states

$$\lim_{\tau \mapsto \infty} \frac{a_{T+\tau+1}}{(1+r)^{\tau+1}} \leq 0.$$

Taking the No Ponzi condition and the transversality condition (and asserting its necessity) together, any candidate for an optimal consumption-savings plan of a well defined deterministic consumption-savings problem has to satisfy:

$$\lim_{\tau \mapsto \infty} \frac{a_{T+\tau+1}}{(1+r)^{\tau+1}} = 0.$$

That is, in this case consumption is weakly rising over time until the date the borrowing constraint binds for the last time, and then constant and equal to the annuity value of income from that date onwards. The proposition generalizes this result to the case in which the borrowing constraint binds infinitely often and thus consumption ceases to rise only in the time limit.

Stochastic Labor Income   Now let's turn to the more interesting case in which the endowment process is stochastic. Here the result that consumption and asset holdings diverge as time extends to infinity is restored, provided that, for any event history s^t the present discounted value of future income is sufficiently stochastic. The next proposition, again proved under various assumptions by Sotomayor (1984) and Chamberlain and Wilson (2000), states this formally.

Proposition 37 *Suppose that either (a) U is bounded and $\{y_t\}$ is a stochastic process that satisfies the following condition: there exists $\varepsilon > 0$ such that for all $\alpha \in \mathbf{R}$ and all s^t*

$$prob\left\{\alpha \le \sum_{\tau=t}^{\infty} \frac{y(s^\tau)}{(1+r)^{\tau-t}} \le \alpha + \varepsilon \left| s^t \right. \right\} < 1 - \varepsilon \qquad (5.123)$$

or (b) $\{y_t\}$ is a sequence of nondegenerate iid random variables with support $y_t \in [a, A]$ with $a \ge 0$ and $A < \infty$. Then, almost surely,

$$\lim_{t \to \infty} c_t = \infty \qquad (5.124)$$

$$\lim_{t \to \infty} a_t = \infty \qquad (5.125)$$

Note from (b) that Sotomayor (1984) needs no assumptions on the boundedness of the utility and the level of variability of the income process (other that it has to be nondegenerate) to prove the result, but needs the *iid* assumption. Dispensing with the *iid* comes at the cost of having to make U bounded and the income process "sufficiently" stochastic so that present discounted value of future income leaves set $[\alpha, \alpha + \varepsilon]$ with probability of at least ε. So unfortunately the theorem does not apply to an economy with both serially correlated shocks and standard utility functions ($CRRA$ or $CARA$).

Note that one can show that condition (5.123) holds for income following a finite-state Markov chain with $\Pi(y) > 0$ for all y and $\pi(y'|y) > 0$ for all y, y'. The previous theorem implies that, under the appropriate conditions, the

state space for assets a_t is unbounded. In particular, since with probability 1 asset holdings explode, the demand for assets in a steady state of an economy composed of agents with $\rho = r$ (and facing the income fluctuation problem discussed here) is infinite and thus no steady state equilibrium can exist in which the interest rate, endogenously determined in general equilibrium, will satisfy $\rho = r$.

Case $\rho > r$

At this stage it will prove helpful to formulate the income fluctuation problem with borrowing constraints recursively. For the general problem in which income follows a stationary, finite state Markov chain[47] the Bellman equation is

$$v(a, y) = \max_{a', c \geq 0} \left\{ U(c) + \frac{1}{1 + \rho} \sum_{y'} \pi(y'|y) v(a', y') \right\} \quad (5.126)$$

$$\text{s.t. } c + \frac{a'}{1 + r} = y + a \quad (5.127)$$

As first order condition we obtain

$$U_c(c) \geq \frac{1 + r}{1 + \rho} \sum_{y'} \pi(y'|y) v(a', y') \quad (5.128)$$

$$= \quad \text{if } a' > 0$$

The envelope condition reads as

$$v'(a, y) = U_c(c) \quad (5.129)$$

so that the Euler equation once again becomes

$$U_c(c) \geq \frac{1 + r}{1 + \rho} \sum_{y'} \pi(y'|y) U_c(c') \quad (5.130)$$

$$= \quad \text{if } a' > 0 \quad (5.131)$$

or

$$U_c(c) = \max \left\{ U_c(y + a), \frac{1 + r}{1 + \rho} \sum_{y'} \pi(y'|y) U_c(c') \right\} \quad (5.132)$$

---

[47]We will deal with the case of nonstationary income below.

Both equations (5.126) and (5.132) can be used to compute optimal policy functions $a'(a, y)$ and $c(a, y)$ as well as the value function $v(a, y)$ using exactly the same iterative procedures as described in the last section, just with the modification that the Bellman equation has the additional constraint $a' \geq 0$ and the Euler equation has two parts now.

Income Process IID   If the income process is *iid* we can reduce the state space from two to one dimension by introducing the variable cash at hand $x = a + y$. The Bellman equation becomes

$$v(x) = \max_{0 \leq a' \leq (1+r)x} \left\{ u \left(x - \frac{a'}{1 + r} \right) + \frac{1}{1 + \rho} \sum_{y'} \pi(y') v(a' + y') \right\} \quad (5.133)$$

with Euler equation

$$U_c \left(x - \frac{a'(x)}{1 + r} \right) = \max \left\{ U_c(x), \frac{1 + r}{1 + \rho} \sum_{y'} \pi(y') U_c \left(a'(x) + y' - \frac{a'(x')}{1 + r} \right) \right\} \quad (5.134)$$

$$= \max \left\{ U_c(x), \frac{1 + r}{1 + \rho} \sum_{y'} \pi(y') U_c \left(a'(x) + y' - \frac{a' \left[a'(x) + y' \right]}{1 + r} \right) \right\}$$

Schechtman and Escudero (1977) provide the following characterizations of the optimal policy function $a'(x)$ for next periods' asset holdings and for consumption $c(x)$. We first consider the deterministic case.

Proposition 38 *Let $T = \infty$ and $\Pi(y) = 1$ (no uncertainty). Then there exists an $\tilde{x}$ such that if $x \geq \tilde{x}$, then*

$$x' = a'(x) + y < x \quad (5.135)$$

A liquidity constrained agent remains so forever, i.e. $a'(y) = 0$ and $c(y) = y$. Consumption is strictly increasing in cash at hand, or

$$\frac{dc(x)}{dx} > 0 \quad (5.136)$$

There exists an $\bar{x} > y$ such that $a'(x) = 0$ for all $x \leq \bar{x}$ and $a'(x) > 0$ for all $x > \bar{x}$. Finally $\frac{dc(x)}{dx} \leq 1$ and $\frac{da'(x)}{dx} < 1 + r$.

Proof. If $a'(x) > 0$ then from envelope and FOC

$$
\begin{aligned}
v'(x) &= \frac{1+r}{1+\rho} v'(x') \\
&< v'(x') \tag{5.137}
\end{aligned}
$$

since $\frac{1+r}{1+\rho} < 1$ by our maintained assumption. Since v is strictly concave (v inherits all the properties of U) we have $x > x'$. Obviously, if $a'(x) = 0$ and $x > y$, then $x' = y < x$. Thus pick any $\tilde{x} > y$.

For second part, suppose that $a'(y) > 0$. Then from the first order condition and strict concavity of the value function

$$
\begin{aligned}
v'(y) &= \frac{1+r}{1+\rho} v'(a'(y) + y) \\
&< v'(a'(y) + y) \\
&< v'(y) \tag{5.138}
\end{aligned}
$$

a contradiction. Hence $a'(y) = 0$ and $c(y) = y$.

The last part we also prove by contradiction. Suppose $a'(x) > 0$ for all $x > y$. Pick arbitrary such x and define the sequence $\{x_t\}_{t=0}^{\infty}$ recursively by

$$
\begin{aligned}
x_0 &= x \tag{5.139} \\
x_t &= a'(x_{t-1}) + y \geq y \tag{5.140}
\end{aligned}
$$

If there exists a smallest T such that $x_T = y$ then we found a contradiction, since then $a'(x_{T-1}) = 0$ and $x_{T-1} > 0$. So suppose that $x_t > y$ for all t. But then $a'(x_t) > 0$ by assumption. Hence

$$
\begin{aligned}
v'(x_0) &= \frac{1+r}{1+\rho} v'(x_1) \\
&= \left(\frac{1+r}{1+\rho}\right)^t v'(x_t) \\
&< \left(\frac{1+r}{1+\rho}\right)^t v'(y) \\
&= \left(\frac{1+r}{1+\rho}\right)^t u'(y) \tag{5.141}
\end{aligned}
$$

where the inequality follows from the fact that $x_t > y$ and the strict concavity of v. the last equality follows from the envelope theorem and the fact that $a'(y) = 0$ so that $c(y) = y$.

But since $v'(x_0) > 0$ and $u'(y) > 0$ and $\frac{1+r}{1+\rho} < 1$, we have that there exists finite t such that $v'(x_0) > \left(\frac{1+r}{1+\rho}\right)^t u'(y)$, a contradiction.

The proof that the derivatives of the policy functions have the asserted properties follow from the strict concavity of the value function and the first order condition. ∎

This last result bounds the optimal asset holdings (and hence cash at hand) from above for $T = \infty$. Since computational techniques usually rely on the finiteness of the state space we want to make sure that for our theory the state space can be bounded from above. For the finite lifetime case there is no problem. The most an agent can save is by consuming 0 in each period and hence

$$a_{t+1}(x_t) \leq x_t \leq (1+r)^{t+1}a_0 + \sum_{j=0}^{t}(1+r)^j y_{\max} \qquad (5.142)$$

which is bounded for any finite lifetime horizon $T < \infty$.

The last theorem says that cash at hand declines over time or is constant at y, in the case the borrowing constraint binds. The theorem also shows that the agent eventually becomes credit-constrained: there exists a finite τ such that the agent consumes his endowment in all periods following τ. This follows from the fact that marginal utility of consumption has to increase at geometric rate $\frac{1+r}{1+\rho}$ if the agent is unconstrained (and thus consumption to decline) and from the fact that once he is credit-constrained, he remains credit constrained forever. This can be seen as follows. First $x \geq y$ by the credit constraint. Suppose that $a'(x) = 0$ but $a'(x') > 0$. Since $x' = a'(x) + y = y$ we have that $x' \leq x$. Thus from the previous proposition $a'(x') \leq a'(x) = 0$ and hence the agent remains credit-constrained forever.

For the infinite lifetime horizon, under deterministic and constant income we have a full qualitative characterization of the allocation: If $a_0 = 0$ then the consumer consumes his income forever from time 0. If $a_0 > 0$, then cash at hand and hence consumption is declining over time, and there exists a time $\tau(a_0)$ such that for all $t > \tau(a_0)$ the consumer consumes his income forever from thereon, and consequently does not save anything.

We now consider the stochastic case with income being *iid* over time. The proof of the following proposition is identical to the deterministic case. Remember the assumption that the minimum income level $y_1 \geq 0$.

Proposition 39 *Consumption is strictly increasing in cash at hand, i.e.*

$c'(x) \in (0,1]$. *Optimal asset holdings are either constant at the borrowing limit or strictly increasing in cash at hand, i.e. $a'(x) = 0$ or $\frac{da'(x)}{dx} \in (0, 1+r)$*

It is obvious that $a'(x) \geq 0$ and hence $x'(x, y') = a'(x) + y' \geq y_1$ so we have $y_1 > 0$ as a lower bound on the state space for x. We now show that there is a level $\bar{x} > y_1$ for cash at hand such that for all $x \leq \bar{x}$ we have that $c(x) = x$ and $a'(x) = 0$

Proposition 40 *There exists $\bar{x} > y_1$ such that for all $x \leq \bar{x}$ we have $c(x) = x$ and $a'(x) = 0$*

Proof. Suppose, to the contrary, that $a'(x) > 0$ for all $x \geq y_1$. Then, using the first order condition and the envelope condition we have for all $x \geq y_1$

$$v(x) = \frac{1+r}{1+\rho}Ev'(x') \leq \frac{1+r}{1+\rho}v'(y_1) < v'(y_1) \qquad (5.143)$$

Picking $x = y_1$ yields a contradiction. ∎

Hence there is a cutoff level for cash at hand below which the consumer consumes all cash at hand and above which he consumes less than cash at hand and saves $a'(x) > 0$. So far the results are strikingly similar to the deterministic case. Unfortunately here it basically ends, and therefore our analytical ability to characterize the optimal policies. In particular, the very important proposition showing that there exists $\tilde{x}$ such that if $x \geq \tilde{x}$ then $x' < \tilde{x}$ does not go through anymore, which is obviously quite problematic for computational considerations. In fact we state, without a proof, a result due to Schechtman and Escudero (1977).

Proposition 41 *Suppose the period utility function is of constant absolute risk aversion form $u(c) = -e^{-c}$, then for the infinite life income fluctuation problem, if $\Pi(y = 0) > 0$ we have $x_t \to +\infty$ almost surely.*

Proof. See Schechtman and Escudero (1977), Lemma 3.6 and Theorem 3.7 ∎

Fortunately there are fairly general conditions under which one can, in fact, prove the existence of an upper bound for the state space. Again we will refer to Schechtman and Escudero for the proof of the following results. Intuitively why would cash at hand go off to infinity even if the agents are impatient relative to the market interest rate, i.e. even if $\beta(1+r) < 1$? If

agents are very risk averse, face borrowing constraints and a positive proba-
bility of having very low income for a long time, they may find it optimal to
accumulated unbounded funds over time to self-insure against the eventual-
ity of this unlikely, but very bad event to happen. It turns out that if one
assumes that the risk aversion of the agent is sufficiently bounded, then one
can rule this out.

Proposition 42 *Suppose that the marginal utility function has the property
that there exist finite $e_{u'}$ such that*

$$\lim_{c \to \infty} (\log_c u'(c)) = e_{u'} \tag{5.144}$$

Then there exists a $\tilde{x}$ such that $x' = a'(x) + y_N \leq x$ for all $x \geq \tilde{x}$.

Proof. See Schechtman and Escudero (1977), Theorems 3.8 and 3.9 ∎
The number $e_{u'}$ is called the asymptotic exponent of u'. Note that if the
utility function is of CRRA form with risk aversion parameter σ, then since

$$\log_c c^{-\sigma} = -\sigma \log_c c = -\sigma \tag{5.145}$$

we have $e_{u'} = -\sigma$ and hence for these utility function the previous proposition
applies. Also note that for CARA utility function

$$\log_c e^{-c} = -c \log_c e = -\frac{c}{\ln(c)} \tag{5.146}$$

$$-\lim_{c \to \infty} \frac{c}{\ln(c)} = -\infty \tag{5.147}$$

and hence the proposition does not apply.
So under the proposition of the previous theorem we have the result that
cash at hand stays in the bounded set $X = [y_1, \tilde{x}]$.[48] Consumption equals cash
at hand for $x \leq \bar{x}$ and is lower than x for $x > \bar{x}$, with the rest being spent on
capital accumulation $a'(x) > 0$. Figure **??** below shows the situation for the
case in which income can take only two possible realizations $Y = \{y_1, y_N\}$.
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[48]If $x_0 = a_0 + y_0$ happens to be bigger than $\tilde{x}$, then pick $\tilde{x}' = x_0$.

Income Serially Correlated, but Stationary Now consider the case where income is correlated over time and follows a Markov chain with transition π. Now the trick of reducing the state to the single variable cash at hand does not work anymore. This was only possible since current income y and past saving a entered additively in the constraint set of the Bellman equation, but neither variable appeared separately. With serially correlated income, however, current income influences the probability distribution of future income. There are several possibilities of choosing the state space for the Bellman equation. One can use cash at hand and current income, (x, y), or asset holdings and current income (a, y). Obviously both ways are equivalent and I opted for the later variant, which leads to the functional equation (5.133). What can we say in general about the properties of the optimal policy functions $a'(a, y)$ and $c(a, y)$? Huggett (1993) proves a proposition similar to the ones above showing that $c(a, y)$ is strictly increasing in a and that $a'(a, y)$ is constant at the borrowing limit or strictly increasing (which implies a cutoff $\bar{a}(y)$ as before, which now will depend on current income y). What turns out to be very difficult to prove is the existence of an upper bound of the state space, $\tilde{a}$ such that $a'(a, y) \leq a$ if $a \geq \tilde{a}$. Huggett proves this result for the special case that income can only take two states $y \in \{y_l, y_h\}$ with $0 < y_l < y_h$ and $\pi(y_h|y_h) \geq \pi(y_h|y_l)$, and CRRA utility. See his Lemmata 1-3 in the appendix. I am not aware of any more general result for the non-iid case. With respect to computation in more general cases, we have to cross our fingers and hope that $a'(a, y)$ eventually (i.e. for finite a) crosses the 45^0-line for all y.

5.6 Prudence and Liquidity Constraints: Computation

The analysis in the previous section is as far as one can push the model analytically. Apart from very special cases (see Caballero (1990) for an explicit solution when utility is CARA, agents life for finite time, the interest rate is zero and income follows a random walk with normally distributed *iid* disturbances) the model cannot be solved analytically.

For computational purposes we want to make the problem a consumer faces recursive. From now on, for the rest of this chapter, unless otherwise noted, we assume that the stochastic process governing labor in-

come is described by a finite state, stationary Markov process with domain $y \in Y = \{y_1, \ldots, y_N\}$ and transition probabilities $\pi(y'|y)$, where we assume that $y_1 \geq 0$ and $y_{i+1} > y_i$. As state variables we choose current asset holdings and the current labor income shock (a, y). In order to make the problem well-behaved we have to make sure that agents don't go into debt so much that they can't pay at least the interest on that debt and still have non-negative consumption. Let $\bar{A}$ be the maximum amount an agent is allowed to borrow. Since consumption equals $c = y + a - \frac{a'}{1+r}$, we have, for an agent that borrowed to the maximum amount $a = -\bar{A}$, received the worst income shock y_1 and just repays interest (i.e. $a' = -\bar{A}$):

$$c = y_1 + \bar{A} - \frac{\bar{A}}{1+r} \geq 0 \tag{5.148}$$

Non-negativity of consumption implies the borrowing limit

$$\bar{A} = \frac{1+r}{r}y_1 \tag{5.149}$$

which we impose on the consumer. Since $\bar{A}$ also equals the present discounted value of future labor income in the worst possible scenario of always obtaining the lowest income realization y_1, we may call this borrowing limit the "natural debt limit" (see Aiyagari (1994)). Note that, since borrowing up to the borrowing limit implies a positive probability of zero consumption next period (if $\pi(y_1|y) > 0$ for all $y \in Y$), this borrowing constraint is not going to be binding, as long as the utility function satisfies the Inada conditions.

For any $a \in [-\bar{A}, \infty)$ and any $y \in Y$ we then can write Bellman's equation as

$$v_t(a, y) = \max_{-\bar{A} \leq a' \leq (1+r)(a+y)} \left\{ U\left(y + a - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y'|y)v_{t+1}(a', y') \right\} \tag{5.150}$$

The first order conditions to this problem is

$$\frac{1}{1+r}U'\left(y + a - \frac{a'}{1+r}\right) = \beta \sum_{y'} \pi(y'|y)v'_{t+1}(a', y') \tag{5.151}$$

where v'_{t+1} is the derivative of v_{t+1} with respect to its first argument. The envelope condition is

$$v'_t(a, y) = U'\left(y + a - \frac{a'}{1+r}\right) \tag{5.152}$$

Combining both conditions yields

$$U'\left(y + a - \frac{a'}{1+r}\right) = \beta(1+r)\sum_{y'} \pi(y'|y)U'\left(y' + a' - \frac{a''}{1+r}\right) \quad (5.153)$$

Defining

$$c_t = y + a - \frac{a'}{1+r} \quad (5.154)$$

$$c_{t+1} = y' + a' - \frac{a''}{1+r} \quad (5.155)$$

we obtain back our stochastic Euler equation (5.35). There are several important special cases that merit a brief discussion.

5.6.1 IID Income shocks

If labor income shocks are *iid* over time with probability density π, then one can reduce the state space to a single variable, so called cash at hand $x = a + y$. The budget constraint then becomes

$$c + \frac{a'}{1+r} = x \quad (5.156)$$

and since

$$x' = a' + y' \quad (5.157)$$

we have the associated Bellman equation

$$v_t(x) = \max_{-\bar{A} \leq a' \leq (1+r)x} \left\{ U\left(x - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y')v_{t+1}(a' + y') \right\}$$

5.6.2 Serially Correlated, Mean-Reverting Income Shocks

With serially correlated income shocks the state space consists of (a, y), asset holdings and the current income shock. Independent of whether the state space is (a, y) or only cash at hand x, we can either work on the value function directly or use the Euler equation to solve for the optimal policies. Depending on whether the time horizon of the household is finite or infinite, different iterative procedures can be applied:

Finite Time Horizon $T < \infty$

1. Value function iteration: We want to find sequences of value functions $\{v_t(a, y)\}_{t=0}^{T}$ and associated policy functions $\{c_t(a, y), a'_t(a, y)\}_{t=0}^{\infty}$. Given that the agent dies at period T we can normalize $v_{T+1}(a, y) \equiv 0$. Then we can iterate backwards on the equation (5.150): at period t we know the function $v_{t+1}(., .)$, hence can solve the maximization problem to find functions $a'_t(., .), v_t(., .)$ and $c_t(., .)$. Note that for each t there are as many maximization problems to solve as there are admissible (a, y)-pairs.

2. Policy function iteration on the Euler equation. We are looking for sequences of policy functions $\{c_t(a, y), a'_t(a, y)\}_{t=0}^{\infty}$. Again, given that the agent dies at period $T+1$ we know that at period T all income will be consumed and nothing we be saved, thus

$$c_T(a, y) = a + y \qquad\qquad (5.158)$$
$$a'_T(a, y) = 0 \qquad\qquad (5.159)$$

The Euler equation between period t and $t + 1$ reads as

$$U'\left(y + a - \frac{a'_t(a, y)}{1 + r}\right) = \beta(1+r)\sum_{y'} \pi(y'|y)U'\left(y' + a'_t(a, y) - \frac{a'_{t+1}(a_t(a, y), y')}{1 + r}\right)$$

$$(5.160)$$

where $a'_{t+1}(., .)$ is a known function from the previous step and we want to solve for the function $a'_t(., .)$. Note that for a given (a, y) no maximization is needed to find $a'_t(a, y)$ as in the previous procedure, one just has to find a solution to a (potentially highly nonlinear) single equation.

Infinite Time Horizon $T = \infty$

1. Value function iteration: now we look for a time invariant value function $v(a, y)$ and associated policy functions $a'(a, y)$ and $c(a, y)$. We need to find a fixed point to Bellman's equation, since there is no final period to start from. Thus we make an initial guess for the value function,

$v^0(a, y)$ and then iterate on the functional equation

$$v^n(a, y) = \max_{-\bar{A} \leq a' \leq (1+r)(y+a)} \left\{ U\left(y + a - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y'|y) v^{n-1}(a', y') \right\}$$

(5.161)

until convergence, i.e. until

$$\left\| v^n - v^{n-1} \right\| \leq \varepsilon \tag{5.162}$$

where ε is the desired precision of the approximation. Note that at each iteration we generate policy functions $a'^n(a, y)$ and $c^n(a, y)$. We know that under appropriate assumptions ($\beta < 1$, U bounded) the operator defined by Bellman's equation is a contraction mapping and hence convergence of the iterative procedure to a unique fixed point is guaranteed.

2. Policy function iteration on the Euler equation. Again we have no final time period, so we guess an initial policy $a'^0(a, y)$ or $c^0(a, y)$ and then iterate on

$$U'\left(y + a - \frac{a'^n(a, y)}{1+r}\right) = \beta(1+r) \sum_{y'} \pi(y'|y) U'\left(y' + a'^n(a, y) - \frac{a'^{n-1}(a'^n(a, y), y')}{1+r}\right)$$

(5.163)

until

$$\left\| a'^n - a'^{n-1} \right\| \leq \varepsilon_a \tag{5.164}$$

Deaton (1992) argues that under the assumption $\beta(1 + r) < 1$ the operator defined by the Euler equation is a contraction mapping, so that convergence to a unique fixed point is guaranteed.

5.6.3 Permanent Shocks

Deaton (1991) and Carroll (1997), among many others, demonstrate how to tackle problems in which the level of income of households follow a random walk. For agents living forever, there is no hope to obtain finite bounds on the amount of assets that are potentially accumulated, hence the state space is unbounded, which poses problems for the computation of optimal policies (and the value function). It should also be noted that in such an economy there is in general no hope for a stationary general equilibrium,

since the income distribution is nonstationary and thus in general the wealth distribution will be for any fixed interest rate r. Hence when dealing with general equilibrium models with infinitely lived agents, one of the maintained assumptions is stationarity of the individual income processes.[49]

The partial equilibrium problem with nonstationary income process and infinite horizon can be tackled, however, through an appropriate change of variables. For this trick to work it is crucial to assume that households have $CRRA$ period utility. First assume that, following Deaton (1991) that the labor income process of the household features iid growth rates, that is, assume that

$$z_{t+1} = \frac{y_{t+1}}{y_t} \tag{5.165}$$

is a sequence of iid random variables. This implies that

$$\log(y_{t+1}) - \log(y_t) \tag{5.166}$$

is equal to an iid random variable, and thus implies that the natural logarithm of income follows a random walk, potentially with drift

$$\log(y_{t+1}) = \mu + \log(y_t) + \varepsilon_t \tag{5.167}$$

where $\{\varepsilon_t\}$ is a sequence of iid random variables with $E(\varepsilon_t) = 0$. Note that this specification implies positive income in all periods, with probability one. In order to overcome the problem that the state space for assets is unbounded, which makes the dynamic programming problem of the household difficult to tackle, normalize all variables by the current level of income

$$\theta_t = \frac{c_t}{y_t} \tag{5.168}$$

$$w_t = \frac{x_t}{y_t} = \frac{a_t + y_t}{y_t} \tag{5.169}$$

---

[49]Note that the same comment does not apply for economies in which agents die with probability 1 in finite time in which case typically a stationary general equilibrium can be found.

The budget constraint then becomes

$$
\begin{aligned}
c_t + \frac{a_{t+1}}{1+r} &= y_t + a_t \\
c_t + \frac{x_{t+1} - y_{t+1}}{1+r} &= x_t \\
\theta_t + \frac{w_{t+1} - 1}{1+r} z_{t+1} &= w_t \\
w_{t+1} &= \frac{(1+r)(w_t - \theta_t)}{z_{t+1}} + 1 \qquad (5.170)
\end{aligned}
$$

and, by dividing both sides of the Euler equation by $y_t^{-\sigma}$ we obtain

$$
\begin{aligned}
\theta_t^{-\sigma} &= \max \left\{ w_t^{-\sigma}, \left(\frac{1+r}{1+\rho}\right) E_t \left[\left(\frac{c_{t+1}}{y_t}\right)^{-\sigma} \right] \right\} \\
&= \max \left\{ w_t^{-\sigma}, \left(\frac{1+r}{1+\rho}\right) E_t \left[(\theta_{t+1} z_{t+1})^{-\sigma} \right] \right\} \qquad (5.171)
\end{aligned}
$$

We are searching for a time-invariant policy function $\theta(w)$ solving this Euler equation. Note that, written in recursive formulation θ_t corresponds to $\theta(w)$ and θ_{t+1} to $\theta(w')$ where $w' = \frac{(1+r)(w-\theta(w))}{z'}$. Assuming that $z' = z_{t+1}$ can take on only a finite number of values we obtain

$$
\theta_t^{-\sigma} = \left\{ \max w^{-\sigma}, \left(\frac{1+r}{1+\rho}\right) \sum_{z'} \pi(z') \left(\theta \left(\frac{(1+r)(w-\theta(w))}{z'} + 1 \right) z' \right)^{-\sigma} \right\}
$$
$$(5.172)$$

This Euler equation can be solved numerically for the optimal policy function $\theta(w)$ of consumption per current income (the average propensity to consume) by doing policy iteration on[50]

$$
\theta(w)^{-\sigma} = \max \left\{ w^{-\sigma}, \left(\frac{1+r}{1+\rho}\right) \sum_{z'} \left(\theta \left(\frac{(1+r)(w-\theta(w))}{z'} + 1 \right) z' \right)^{-\sigma} \right\}
$$
$$(5.174)$$

---

[50]Deaton and Laroque (1992) show that with nonstationary income the operator associated with the Euler equation is a contraction if

$$
\frac{1+r}{1+\rho} E(z^{-\sigma}) < 1 \qquad (5.173)
$$

Note that Deaton (1991) is not able to develop a condition under which the state space for w is bounded above. It is obviously bounded below by 1 since assets a_t are restricted to be nonnegative. In accordance to the model with stationary income he establishes a threshold $\bar{w}$ such that $\theta(w) = w$ and $w'(w) = 1$ for all $w \leq \bar{w}$; i.e. for low levels of normalized cash at hand it is optimal for the consumer to eat all cash at hand today and save nothing for tomorrow.

As discussed in the next chapter, a very popular specification of the household labor income process postulates that log-labor income is composed of a permanent component that follows a random walk and a transitory shock. That is, suppose labor income follows the process[51]

$$\begin{aligned}
\ln(y_{t+1}) &= p_{t+1} + \varepsilon_{t+1} \\
p_{t+1} &= p_t + \eta_{t+1}
\end{aligned}$$

Here η_{t+1} and ε_{t+1} are permanent and transitory income shocks that are independently (over time and of each other) and normally distributed with variances $(\sigma_\varepsilon^2, \sigma_\eta^2)$ and means $(-\frac{\sigma_\varepsilon^2}{2}, -\frac{\sigma_\eta^2}{2})$. Denote the cumulative distribution function of the Normal distribution by N.

As before one can make the problem stationary by expressing all variables relative to the permanent component of income, p_t, see e.g. Carroll (1997). Now the state space consists of assets a, the permanent component of log-income p and the transitory shock e. The dynamic programming problem can be written as

$$\begin{aligned}
v(a, p, \varepsilon) &= \max_{c, a' \geq 0} \left\{ \frac{c^{1-\sigma}}{1 - \sigma} + \beta \int_{\varepsilon'} \int_{\eta'} v(a', p + \eta', \varepsilon') dN(\varepsilon') dN(\eta') \right\} \\
&\quad s.t. \\
c + \frac{a'}{1 + r} &= \exp(p + \varepsilon) + a
\end{aligned}$$

With the usual definition of cash at hand

$$\begin{aligned}
x &= y + a \\
&= \exp(p + \varepsilon) + a
\end{aligned}$$

[51]In example 26 we studied a similar process, but there the level of labor income (rather than its log) was subject to permanent and transitory shocks (and the period utility function was quadratic rather than of CRRA form).

we can reduce the state space and re-write the recursive problem as

$$v(x, p) = \max_{0 \le c \le x} \left\{ \frac{c^{1-\sigma}}{1 - \sigma} + \beta \int_{\varepsilon'} \int_{\eta'} v(\exp(p' + \varepsilon') + (1 + r)(x - c), p + \eta') dN(\varepsilon') dN(\eta') \right\}$$

(5.175)

where we have used the fact that by definition of x' and the budget constraint

$$
\begin{aligned}
x' &= \exp(p' + \varepsilon') + a' \\
&= \exp(p' + \varepsilon') + (1 + r)(x - c)
\end{aligned}
$$

Note, however, that since p follows a random walk, the problem above is not stationary. Therefore we attempt to make it stationary by defining variables relative to the permanent level of income:

$$\tilde{c} = \frac{c}{\exp(p)} \quad \text{and} \quad w = \frac{x}{\exp(p)}$$

We now conjecture that, due to our assumption of CRRA utility (which guarantees homotheticity of the lifetime utility function), the value function takes the form

$$v(x, p) = \tilde{v}(w) \exp(p)^{1-\sigma}$$

and that we can find associated policy functions $\tilde{c}(w)$ and $w'(w)$ that solve the dynamic programming problem associated with the "deflated" value function $\tilde{v}(w)$. Dividing (5.175) by $\exp(p)^{1-\sigma}$ and using the conjectured form of the value function yields

$$\tilde{v}(w) = \max_{0 \le \tilde{c} \le w} \left\{ \frac{\tilde{c}^{1-\sigma}}{1 - \sigma} + \beta \int_{\varepsilon'} \int_{\eta'} \exp(p' - p)^{1-\sigma} \tilde{v}(w') dN(\varepsilon') dN(\eta') \right\}$$

with

$$
\begin{aligned}
w' &= \frac{x'}{\exp(p')} = \frac{\exp(p' + \varepsilon') + (1 + r)(x - c)}{\exp(p')} \\
&= \exp(\varepsilon') + \frac{(1 + r)(x - c)}{\exp(p + \eta')} = \exp(\varepsilon') + \frac{(1 + r)(w - \tilde{c})}{\exp(\eta')}
\end{aligned}
$$

and thus the Bellman equation becomes

$$\tilde{v}(w) = \max_{0 \le \tilde{c} \le w} \left\{ \frac{\tilde{c}^{1-\sigma}}{1 - \sigma} + \beta \int_{\varepsilon'} \int_{\eta'} \exp(\eta')^{1-\sigma} \tilde{v} \left(\exp(\varepsilon') + \frac{(1 + r)(w - \tilde{c})}{\exp(\eta')} \right) dN(\varepsilon') dN(\eta') \right\}$$

which we need to solve numerically for the value function $\tilde{v}(w)$ and the associated policy function $\tilde{c}(w)$. Once we have determined those, consumption and asset levels are given by

$$
\begin{aligned}
c(x,p) &= \exp(p)\tilde{c}\left(\frac{x}{\exp(p)}\right) \\
a'(x,p) &= (x - c(x,p))(1+r).
\end{aligned}
$$

Alternatively one can use the Euler equation (and policy function iteration) to determine the optimal consumption policy. In original form, the Euler equation associated with Bellman equation (5.175) reads as

$$
c(a,p,\varepsilon)^{-\sigma} = \max\left\{(\exp(p+\varepsilon)+a)^{-\sigma}, \beta(1+r)\int_{\varepsilon'}\int_{\eta'}c(a',p+\eta',\varepsilon')dN(\varepsilon')dN(\eta')\right\}
$$

or, in terms of cash at hand

$$
\begin{aligned}
c(x,p)^{-\sigma} &= \max\left\{x^{-\sigma}, \beta(1+r)\int_{\varepsilon'}\int_{\eta'}c(x',p+\eta')dN(\varepsilon')dN(\eta')\right\} \\
\text{with } x' &= \exp(p+\eta'+\varepsilon') + (1+r)(x-c(x,p))
\end{aligned}
$$

As before, conjecturing that $c(x,p) = \exp(p)\tilde{c}(w)$ with $w = x/\exp(p)$ and dividing the Euler equation by $\exp(p)^{-\sigma}$ yields

$$
\tilde{c}(w)^{-\sigma} = \max\left\{w^{-\sigma}, \beta(1+r)\int_{\varepsilon'}\int_{\eta'}\left(\exp(\eta')\tilde{c}\left[\exp(\varepsilon') + \frac{(1+r)(w-\tilde{c}(w))}{\exp(\eta')}\right]\right)^{-\sigma}dN(\varepsilon')dN(\eta')\right.
$$

which is a nonlinear functional equation in the unknown function $\tilde{c}$ with the one state variable w. Obviously very similar transformations yield the appropriate Bellman equations and Euler equations for the finite horizon case.

5.6.4   From Policy Functions to Simulated Time Series

Once one has solved for the policy functions one can simulate time paths for consumption and asset holdings for a particular agent. Start at some asset level a_0, possibly equal to zero, then draw a sequence of random income numbers $\{y_i\}_{t=0}^{M}$ according to the specified income process. This will usually require first drawing random numbers from a uniform distribution and then

mapping these numbers into the appropriate labor income realizations. Now one can generate a time series for consumption and asset holdings (for the infinite horizon case) by

$$c_0 = c(a_0, y_0) \tag{5.176}$$
$$a_1 = a'(a_0, y_0) \tag{5.177}$$

and recursively

$$c_i = c(a_i, y_i) \tag{5.178}$$
$$a_{i+1} = a'(a_i, y_i) \tag{5.179}$$

The same applies to the finite time horizon case, but here the policy functions being applied also vary with time.

5.6.5   Implementation of Specific Algorithms

All algorithms for the different cases require the approximation of policy or value functions over the state space of potential incomes and assets, (a, y), the latter in principle being a continuous state variable.[52] There is a good number of alternative approaches of how to approximate these functions, ranging from a full discretization of the problem, to approximations that divide the state space into smaller pieces and then approximate the functions under consideration by simple functions (these are so-called finite element methods) to approaches that approximate the unknown function over the entire state space by a weighted sum of simple functions (e.g. polynomials), so called projection methods. For a complete treatment of these methods, see chapters 10-12 of Judd (1998), McGrattan (1999) or Herr and Maussner (2009).[53]

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[52]In the canonical consumption-savings problem the sate space is typically small. For problems with multiple continuous state variables and smooth functions to be approximated Krueger and Kubler (2004) propose a sparse grid method for approximation of multidimensional (but smooth) functions. Recently, Judd, Maliar and Maliar (2012) developed a related algorithm that endogenizes the choice of the sparse grid.

[53]Recently Carroll (2006) proposed an endogenous grid method that in practice speeds up computation of the class of problems discussed in this section tremendously.

5.7   Prudence and Liquidity Constraints: Empirical Implications

In this section we discuss the implications of the SIM with respect to two important empirical dimensions and review the relevant empirical and quantitative literature. First, in the next subsection we investigate how strongly (if at all) consumption is expected to respond to income shocks, or reversely, how well households that only have access to a simple state-uncontingent asset can self-insure. Second, we derive the implications for the evolution of the cross-sectional consumption distribution of a life-cycle version of the SIM and confront it with the data.

5.7.1   Time Series Implications: How Does Consumption Respond to Income Shocks

Recall from section 5.2.2 that the certainty equivalence version of the SIM with $\beta(1 + r) = 1$ and an income process with transitory and permanent shocks implies that the realized change in household consumption is given by

$$\Delta c_t = \frac{r\theta_t^{-1}}{1 + r}u_t + v_t$$

where v_t is a permanent income shock, u_t is a transitory income shock and $\theta_t = \left(1 - \frac{1}{(1+r)^{T-t+1}}\right)$ is an adjustment factor that depends on how close the household is towards the end of life T. If $T = \infty$ the household is infintely lived and $\theta_t = 1$. Thus, according to this version of the model, consumption responds one for one to permanent income shocks and almost not at all to transitory income shocks if the household is both young (t is small relative to T) and the interest rate r is small.[54]

Closed form Solution for CARA Utility

Wang (2003) for a closed form solution with CARA utility. Same upshot as with quadratic utility, but with constant precautionary savings term

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[54]Of course, if $t = T$ we find $\frac{r\theta_t^{-1}}{1+r} = 1$ and household consumption responds strongly to "temporary" income shocks since at the end of life a temporary income shock is a permanent shock.

Empirical Estimates of Insurance Coefficients

With certainty equivalence and a specific income process the question how strongly consumption responds to income shocks with different persistence has a precise answer, as demonstrated above. More broadly, Blundell, Pistaferri and Preston (2008) propose as measure for how strongly consumption responds (or better, does not respond) the following concept of consumption insurance coefficients. Suppose that the stochastic part of log-labor earnings can be written as

$$\log(y_{it}) = \sum_{\tau=0}^{t} \sum_{n=1}^{N} \alpha_\tau^n x_{it-\tau}^n \tag{5.180}$$

where the $(x_{it}^1, x_{it}^2, \ldots, x_{it}^N)$ are N income shocks (think of them as shocks with different persistence), assumed to be independently distributed across time, households and across each other ($cov(x_{it}^n, x_{j\tau}^m) = 0$ for all i, j, all t, τ, all m, n).

Example 43 *One popular stochastic earnings process (see section 5.6.3 and also the next chapter) models the **logarithm** of earnings as the sum of a permanent and transitory shock, as did equations* (5.49) *and* (5.50) *for the level of earnings:*

$$\log(y_{it}) = p_{it} + \varepsilon_{it} \tag{5.181}$$
$$p_{it} = p_{it-1} + \eta_{it} \tag{5.182}$$

where ε_{it} is the transitory shock and η_{it} is the permanent shock. This process fits into the general structure of equation (5.180) *by letting $N = 2$, $(x_{it}^1, x_{it}^2) = (\varepsilon_{it}, \eta_{it})$ and $\alpha_0^1 = 1, \alpha_\tau^1 = 0$ for all $\tau > 0$, and $\alpha_\tau^2 = 1$ for all $\tau \geq 0$. Recall that this process also implies*

$$\Delta \log(y_{it}) = \eta_{it} + \Delta \varepsilon_{it} \tag{5.183}$$

The consumption insurance coefficient for income shock n is then defined as

$$\phi_t^n = 1 - \frac{Cov_i(\Delta \log(c_{it}), x_{it}^n)}{Var_i(x_{it}^n)} \tag{5.184}$$

where Cov_i and Var_i are taken with respect to the cross-section of household observations indexed by i observed at time (or age) t. The number ϕ_t^n measures what share of stochastic labor income variability due to shock

n does *not* translate into corresponding consumption changes, that is, how well household consumption is insured about the n-th component of income risk. Thus these measures of consumption insurance have become important "summary statistics".

Equipped with this definition we can now ask how much consumption insurance is there in the data, and how well do our theoretical models capture this degree of consumption insurance. The model statistics are easy to compute, either by hand (if the model has an analytical solution) or through straightforward simulations. To estimate the empirical counterpart is significantly harder since the income shocks $\{x_{it}^n\}$ are not directly observed in the data (but they are observed in simulations of the theoretical models), even in the ideal situation where we have panel data on income and consumption, $\{c_{it}, y_{it}\}$.

To estimate the denominator $Var_i(x_{it}^n)$ of equation (5.184) one needs to make specific assumption of the form of the income process. To estimate the numerator $Cov_i(\Delta \log(c_{it}), x_{it}^n)$ one in addition needs to know (or at least make assumptions about) the stochastic process household consumption growth follows. Blundell et al. (2008) assume that log-earnings follow the process in equations (5.181)-(5.182), and they postulate that unexplained[55] consumption growth is determined by the process[56]

$$\Delta \log(c_{it}) = \pi_t^\eta \eta_{it} + \pi_t^\varepsilon \varepsilon_{it} + \xi_{it}$$

where the remaining error term ξ_{it} is uncorrelated with the permanent and transitory income shock at all leads and lags. One can loosely interpret π_t^η as the (potentially time- or age-varying) marginal propensity to consume out of a permanent income shock, with π_t^ε possessing the same interpretation for a transitory shock, but should keep in mind that $\Delta \log(c_{it})$ is the change in log-consumption (i.e. consumption growth) rather than the change in consumption levels. In fact, ignoring this distinction (both for income and consumption), the canonical PIH model with certainty equivalence and infinite horizon has the implication that $\pi_t^\eta = 1$ and $\pi_t^\varepsilon = \frac{r}{1+r}$, as shown in

---

[55]That is, after consumption growth has been purged from its predictable components (that are due to, e.g. life cycle effects and family size and composition as well as other observable household characteristics).

[56]Blundell et al. (2008) provide an appendix where they argue that this consumption growth process is consistent with a first order approximation of the Euler equation (of course, with non-binding borrowing constraints) in a model where income follows the permanent-transitory process stipulated and households have CRRA period utility.

section 5.6.3.

Note that with this postulated consumption process

$$\phi_t^\varepsilon = 1 - \frac{Cov_i(\pi_t^\eta \eta_{it} + \pi_t^\varepsilon \varepsilon_{it} + \xi_{it}, \varepsilon_{it})}{Var_i(\varepsilon_{it})} = 1 - \pi_t^\varepsilon$$

$$\phi_t^\eta = 1 - \frac{Cov_i(\pi_t^\eta \eta_{it} + \pi_t^\varepsilon \varepsilon_{it} + \xi_{it}, \eta_{it})}{Var_i(\eta_{it})} = 1 - \pi_t^\eta$$

and thus the insurance coefficients are just one minus the marginal propensities to consume out of the various income shocks. But again, as long as $(\varepsilon_{it}, \eta_{it})$ are not directly observable, neither the ϕ_t^x nor the π_t^x can be empirically estimated.

However, given the assumed income and consumption growth processes, recognize that, using equation (5.183),

$$Cov_i(\Delta \log(y_{it}), \Delta \log(y_{it+1})) = Cov_i(\eta_{it} + \varepsilon_{it} - \varepsilon_{it-1}, \eta_{it+1} + \varepsilon_{it+1} - \varepsilon_{it}) = -Var_i(\varepsilon_{it})$$

$$Cov_i(\Delta \log(c_{it}), \Delta \log(y_{it+1})) = Cov_i(\pi_t^\eta \eta_{it} + \pi_t^\varepsilon \varepsilon_{it} + \xi_{it}, \eta_{it+1} + \varepsilon_{it+1} - \varepsilon_{it})$$

$$= -\pi_t^\varepsilon Var_i(\varepsilon_{it}) = -Cov_i(\Delta \log(c_{it}), \varepsilon_{it}).$$

Therefore we can estimate

$$\phi_t^\varepsilon = 1 - \frac{Cov_i(\Delta \log(c_{it}), \varepsilon_{it})}{Var_i(\varepsilon_{it})} = 1 - \frac{Cov_i(\Delta \log(c_{it}), \Delta \log(y_{it+1}))}{Cov_i(\Delta \log(y_{it}), \Delta \log(y_{it+1}))}$$

which only requires household consumption and income data with small panel dimension (two periods for consumption, three for income). Note that the last ratio has a nice interpretation, it is the regression coefficient of a cross-sectional regression of consumption growth on income growth, using one period ahead income growth as an instrument.

A similar, but more data-demanding calculation yields

$$Cov_i(\Delta \log(y_{it}), \Delta \log(y_{it-1}) + \Delta \log(y_{it}) + \Delta \log(y_{it+1})) = Var_i(\eta_{it})$$

$$Cov_i(\Delta \log(c_{it}), \Delta \log(y_{it-1}) + \Delta \log(y_{it}) + \Delta \log(y_{it+1})) = Cov_i(\Delta \log(c_{it}), \eta_{it})$$

and thus

$$\phi_t^\eta = 1 - \frac{Cov_i(\Delta \log(c_{it}), \eta_{it})}{Var_i(\eta_{it})}$$

$$= 1 - \frac{Cov_i(\Delta \log(c_{it}), \Delta \log(y_{it-1}) + \Delta \log(y_{it}) + \Delta \log(y_{it+1}))}{Cov_i(\Delta \log(y_{it}), \Delta \log(y_{it-1}) + \Delta \log(y_{it}) + \Delta \log(y_{it+1}))}$$

This expression has the same instrumental variable interpretation as above, but now with $\Delta \log(y_{it-1}) + \Delta \log(y_{it}) + \Delta \log(y_{it+1})$ as instrument for $\Delta \log(y_{it})$. Also note that now we need for subsequent income observations per household to estimate the consumption insurance coefficient for permanent shocks.

Next discuss empirical findings, also Blundell, Pistaferri and Saporta-Eksten (2013)

Next discuss Kaplan and Violante (2011) for quantitative results in OLG model

5.7.2   Cross-Sectional Implications: How Does Consumption Inequality Evolve over the Life Cycle

Review Deaton-Paxson and Storesletten et al. (2004) and Blundell and Preston (1996)

$$c_t = c_{t-1} + \frac{r\theta_t^{-1}}{1+r}u_t + v_t$$

and thus

$$Var_i(c_t) = Var_i(c_{t-1}) + \frac{r\theta_t^{-1}}{1+r}\sigma_u^2 + \sigma_v^2$$

since by assumption $Corr_i(c_{t-1}, u_t) = Corr_i(c_{t-1}, v_t) = Corr_i(v_t, u_t) = 0$. With only permanent shocks consumption inequality should increase linearly with the age of the cohort (and at the same rate as income inequality).

5.8   Appendix A: The Intertemporal Budget Constraint

In this appendix we derive the intertemporal budget constraint (5.43) from the sequence of budget constraints (5.17). For a specific node s^t the latter reads as

$$c_t(s^t) + \frac{a_{t+1}(s^t)}{1+r} = y_t(s^t) + a_t(s^{t-1}). \qquad (5.185)$$

In the next period, for node s^{t+1} it reads as

$$c_{t+1}(s^{t+1}) + \frac{a_{t+2}(s^{t+1})}{1+r} = y_{t+1}(s^{t+1}) + a_{t+1}(s^t) \qquad (5.186)$$

Solving (5.186) for $a_{t+1}(s^t)$ and multiplying by $\frac{\pi_{t+1}(s^{t+1}|s^t)}{1+r}$ yields:

$$\frac{\pi_{t+1}(s^{t+1}|s^t)a_{t+1}(s^t)}{1+r} = \pi_{t+1}(s^{t+1}|s^t)\left(\frac{c_{t+1}(s^{t+1}) + \frac{a_{t+2}(s^{t+1})}{1+r} - y_{t+1}(s^{t+1})}{1+r}\right)$$
(5.187)

Since $a_{t+1}(s^t)$ is only a function of s^t (but not s_{t+1}), equation (5.187) holds for every node s^{t+1} following s^t. Thus we can sum (5.187) over all nodes $s^{t+1}|s^t$ to obtain

$$\frac{a_{t+1}(s^t)}{1+r} = a_{t+1}(s^t)\sum_{s^{t+1}|s^t}\frac{\pi_{t+1}(s^{t+1}|s^t)}{1+r} = \sum_{s^{t+1}|s^t}\pi_{t+1}(s^{t+1}|s^t)\left(\frac{c_{t+1}(s^{t+1}) + \frac{a_{t+2}(s^{t+1})}{1+r} - y_{t+1}(s^{t+1})}{1+r}\right)$$
(5.188)

Substituting (5.188) back into (5.185) yields

$$c_t(s^t) + \sum_{s^{t+1}|s^t}\frac{\pi_{t+1}(s^{t+1}|s^t)c_{t+1}(s^{t+1})}{1+r} + \sum_{s^{t+1}|s^t}\frac{\pi_{t+1}(s^{t+1}|s^t)a_{t+2}(s^{t+1})}{(1+r)^2}$$

$$= y_t(s^t) + \sum_{s^{t+1}|s^t}\frac{\pi_{t+1}(s^{t+1}|s^t)y_{t+1}(s^{t+1})}{1+r} + a_t(s^{t-1}) \qquad (5.189)$$

Repeating this procedure until period T yields

$$c_t(s^t) + \sum_{\tau=1}^{T-t}\sum_{s^{t+\tau}|s^t}\frac{\pi_{t+\tau}(s^{t+\tau}|s^t)c_{t+1}(s^{t+1})}{(1+r)^\tau} + \sum_{s^T|s^t}\frac{\pi_T(s^T|s^t)a_{T+1}(s^T)}{(1+r)^{T-t+1}}$$

$$= y_t(s^t) + \sum_{\tau=1}^{T-t}\sum_{s^{t+\tau}|s^t}\frac{\pi_{t+\tau}(s^{t+\tau}|s^t)y_{t+\tau}(s^{t+\tau})}{(1+r)^\tau} + a_t(s^{t-1}). \qquad (5.190)$$

Thus as long as the term

$$\sum_{s^T|s^t}\frac{\pi_T(s^T|s^t)a_{T+1}(s^T)}{(1+r)^{T-t+1}} \geq 0 \qquad (5.191)$$

we obtain equation (5.43). But equation (5.191) follows from the No-Ponzi conditions imposed in (5.27) and (5.28). For $T < \infty$ this is trivial since $a_{T+1}(s^T) \geq 0$ for all s^T immediately implies (5.191). For $T = \infty$ we note that the constraint

$$a_{t+1}(s^t) \geq -\bar{A}_{t+1}(s^t)$$

together with the assumption that the process $\{\bar{A}_{t+1}(s^t)\}$ is bounded above implies that

$$\lim_{T \to \infty} \sum_{s^T | s^t} \frac{\pi_T(s^T | s^t) a_{T+1}(s^T)}{(1+r)^{T-t+1}} = 0.$$

5.9   Appendix B: Derivation of Consumption Response to Income Shocks

Our goal in this appendix is to derive equation (5.48). Rewrite (5.46) as

$$\theta_t c_t = \frac{r W_t}{1+r} \tag{5.192}$$

$$\theta_{t-1} c_{t-1} = \frac{r W_{t-1}}{1+r}. \tag{5.193}$$

Expanding equation (5.192) by using the definition for W_t yields

$$\theta_t c_t = \frac{r W_t}{1+r} = \frac{r a_t}{1+r} + \frac{r}{1+r} E_t \sum_{s=0}^{T-t} \frac{y_{t+s}}{(1+r)^s} \tag{5.194}$$

Plugging in for a_t from the period $t-1$ budget constraint

$$a_t = (1+r)\left(a_{t-1} + y_{t-1} - c_{t-1}\right) \tag{5.195}$$

yields

$$\theta_t c_t = r\left(a_{t-1} + y_{t-1} - c_{t-1}\right) + \frac{r}{1+r} E_t \sum_{s=0}^{T-t} \frac{y_{t+s}}{(1+r)^s} \tag{5.196}$$

Bringing the term rc_{t-1} to the left and dividing both sides by $(1+r)$ delivers

$$\frac{\theta_t c_t + r c_{t-1}}{1+r} = \frac{r}{1+r}\left(a_{t-1} + y_{t-1}\right) + \frac{r}{(1+r)^2} E_t \sum_{s=0}^{T-t} \frac{y_{t+s}}{(1+r)^s} \tag{5.197}$$

Subtracting $\theta_{t-1}c_{t-1}$ from both sides, using (5.193) and the definition of W_{t-1} one gets

$$\frac{\theta_t c_t + r c_{t-1}}{1+r} - \theta_{t-1}c_{t-1} = \frac{r}{(1+r)^2}E_t\sum_{s=0}^{T-t}\frac{y_{t+s}}{(1+r)^s} + \frac{r}{1+r}y_{t-1} - \frac{r}{1+r}E_{t-1}\sum_{s=0}^{T-t+1}\frac{y_{t-1+s}}{(1+r)^s} \tag{5.198}$$

But now we note that the left hand side of equation (5.198) equals:

$$
\begin{aligned}
\frac{\theta_t c_t + r c_{t-1}}{1+r} - \theta_{t-1}c_{t-1} &= \frac{\theta_t c_t}{1+r} - \left(1 - \frac{1}{(1+r)^{T-t+2}} - \frac{r}{1+r}\right)c_{t-1}\\
&= \frac{\theta_t c_t}{1+r} - \frac{1}{1+r}\left(1 + r - \frac{1}{(1+r)^{T-t+1}} - r\right)c_{t-1}\\
&= \frac{\theta_t \Delta c_t}{1+r} \tag{5.199}
\end{aligned}
$$

Plugging (5.199) back into equation (5.198) and multiplying both sides by $(1+r)$ finally yields

$$
\begin{aligned}
\theta_t \Delta c_t &= \frac{r}{1+r}E_t\sum_{s=0}^{T-t}\frac{y_{t+s}}{(1+r)^s} + r y_{t-1} - r E_{t-1}\sum_{s=0}^{T-t+1}\frac{y_{t-1+s}}{(1+r)^s}\\
&= \frac{r}{1+r}E_t\sum_{s=0}^{T-t}\frac{y_{t+s}}{(1+r)^s} - r E_{t-1}\sum_{s=1}^{T-t+1}\frac{y_{t-1+s}}{(1+r)^s}\\
&= \frac{r}{1+r}E_t\sum_{s=0}^{T-t}\frac{y_{t+s}}{(1+r)^s} - \frac{r}{1+r}E_{t-1}\sum_{s=1}^{T-t+1}\frac{y_{t-1+s}}{(1+r)^{s-1}}\\
&= \frac{r}{1+r}E_t\sum_{s=0}^{T-t}\frac{y_{t+s}}{(1+r)^s} - \frac{r}{1+r}E_{t-1}\sum_{s=0}^{T-t}\frac{y_{t+s}}{(1+r)^s}\\
&= \frac{r}{1+r}\sum_{s=0}^{T-t}\frac{(E_t - E_{t-1})y_{t+s}}{(1+r)^s} = \eta_t
\end{aligned}
$$

which is equation (5.48) in the main text.

Chapter 6

A Digression: Stochastic Earnings or Wage Processes

Before turning to general equilibrium versions of standard incomplete markets models we briefly want to discuss the specification of the key quantitative ingredient into the model, the stochastic labor income process (or wage process, if labor supply is a choice variable). The discussion in this section will follow Guvenen (2007) who in turn builds upon papers by Liliard and Weiss (1979), MaCurdy (1982), Abowd and Card (1989), Baker (1997), Haider (2001), Meghir and Pistaferri (2004) and many others.

In most of the literature labor earnings or income e_{iht} of individual or household i with actual or potential labor market experience (or age) h at time t is assumed to follow a process of the form

$$\log(e_{iht}) = g(\theta_t, X_{it}) + f(\gamma_i, X_{it}) + \log(y_{it}) \qquad (6.1)$$

where θ_t, γ_i are parameter vectors, X_{it} are observable characteristics of household (individual) i at time t and $\log(y_{it})$ is the stochastic part of earnings. We will now discuss the exact specification of each of these components.

1. Deterministic determinants of earnings whose effects are not allowed to vary across households, $g(\theta_t, X_{it})$: the parameter vector can potentially change over time, but is restricted to be identical across households. Typical observables X_{it} that are used in the literature are age and (potential or actual) labor market experience of the individual/household,

dummies for family composition or education[1] or birth cohort, or a unit vector (so that this component includes time dummies). Guvenen (2007) specifies this term as a cubic polynomial in potential experience h_{it} of the household

$$g(\theta_t, X_{it}) = \theta_{0t} + \theta_{1t}h_{it} + \theta_{2t}h_{it}^2 + \theta_{3t}h_{it}^3.$$

The key restriction is that the parameter vector θ is allowed to vary with time t, but not with household identity i. Time dependence of the constant θ_{0t} allows for time fixed effects impacting earnings (e.g. business cycle effects). The time varying coefficients on the experience (or age) polynomial reflect potentially time varying returns to labor market experience.

2. Household-specific deterministic determinants of the life cycle earnings profile, $f(\gamma_i, X_{it})$: the parameter vector γ_i is assumed to be constant over time, but can potentially vary across households. There are two main contenders for the form of f, nested in the general formulation specified by Guvenen (2007):

$$f(\gamma_i, X_{it}) = \alpha_i + \beta_i h_{it} \qquad (6.2)$$

where (α_i, β_i) are household-specific parameters, drawn from a population distribution that is normal with zero mean, variances $\sigma_\alpha^2, \sigma_\beta^2$ and covariance $\sigma_{\alpha\beta}$. There is almost no disagreement that one should allow for $\sigma_\alpha^2 > 0$, that is, there are initial and permanent differences in the *levels* of household lifetime earnings processes. Note, however, that the more heterogeneity is already taken out of the data through part 1. (either education, family composition dummies, or separate analysis for different group), the lower is the estimate of σ_α^2 going to be. The real controversy (and one whose resolution turns out to matter a lot for the estimates of the entire earnings process, and thus the quantitative properties of our models) is about whether one allows heterogeneity across individuals/household in the slope of the life cycle earnings profile, that is, whether one allows for $\sigma_\beta^2 > 0$. Early papers allowed for $\sigma_\beta^2 > 0$ and estimated this variance to be significantly different from zero. In a very influential paper MaCurdy (1982) argued for $\sigma_\beta^2 = 0$ and

---

[1]Alternatively, the analysis is done separately for different groups, where group identity is often determined by the level of education attained by the household.

then estimated a stochastic earnings process of the form (6.1) imposing this restriction. We will return to MaCurdy's argument below, and here just mention that most of the literature since MaCurdy followed his lead (e.g. Abowd and Card (1989), Meghir and Pistaferri (2004), Storesletten et al. (2004) and many others). Guvenen (2007) challenges this position and argues that the evidence that leads MaCurdy to assume $\sigma_\beta^2 = 0$ is weak. He then allows for $\sigma_\beta^2 > 0$, estimates it to be significantly positive and argues that omitting slope heterogeneity may bias the estimates governing the stochastic part of the earnings process $\log(y_{it})$. So the first key question we will address is whether one should allow for $\sigma_\beta^2 > 0$.

3. The stochastic part of the earnings process $\log(y_{it})$: this is the part we will approximate as a Markov chain below and take as input into our quantitative models. An infinite horizon model abstracts from the life cycle aspects of earnings as specified in part 1. and 2. Thus the key model input in this class of models is the stochastic component of earnings. Even in quantitative life cycle models where parts 1. and 2. are modelled as well, the stochastic part of the earnings process is a key ingredient of the model. Typically, the stochastic part of log-earnings is modeled as the sum of a transitory and a persistent shock of the form

$$\log(y_{it}) = z_{it} + \phi_t \varepsilon_{it} \tag{6.3}$$

where

$$z_{it} = \rho z_{it-1} + \pi_t \eta_{it} \tag{6.4}$$

and the innovations are uncorrelated standard normal random variables, that are also uncorrelated with the (α_i, β_i). Note that the persistent part has to be initialized with some z_{i0}, often this initial condition is simply set to zero. Thus the stochastic part of log-earnings is modeled as a sum of a transitory shock with (potentially time varying) variance ϕ_t^2 and a persistent shock whose conditional variance is given by π_t^2 and the persistence is given by the (time-invariant) parameter ρ. Sometimes a low-order MA term is allowed for the transitory shock, and some authors a priori impose $\rho = 1$, estimating the stochastic part as a combination of a transitory and a fully permanent shock. The key question for this part of the process is how persistent income shocks are, that is, how big is ρ and how large is π_t^2 relative to ϕ_t^2. Since we know

from our previous discussion that more persistent shocks are harder to self-insure against, the empirical answer to this question is evidently of great importance for the quantitative properties of our models.

6.1 Estimation

The parameters to be estimated are the θ_t's, the variances and the covariance of (α_i, β_i) and the (ϕ_t^2, π_t^2) as well as ρ. In a first stage the θ_t's are estimated by regressing individual or household log-earnings on observable household characteristics, in Guvenen's (2007) case a cubic polynomial in potential experience h_{it}. In a second stage the stochastic part of the earnings equation is estimated (if α_i, β_i are random coefficients their variances are typically estimated jointly with the remaining stochastic part of the process). While there are various methods to do this, the most popular method is to use a minimum distance estimator that, by choice of the parameters, minimizes the weighted distance between specific cross-sectional moments measured in the data and implied by the model.

The moments used in estimation are cross-sectional variances and auto-covariances of $\log(e_{iht})$, that is,

$$E(\log(e_{iht})^2)$$
$$E(\log(e_{iht})\log(e_{ih+n,t+n}))$$

for $n \geq 1$. For the data, $E(.)$ denotes cross-sectional sample averages, for the statistical model given jointly by (6.2)-(6.4) we can compute these cross-sectional moments explicitly (alternatively one could simulate them easily). Both in the data and in the model these moments could be computed separately for different experience levels h, in practice small sample sizes in the data leads most researchers to combine (i.e. take some average of) the data for different experience groups. Guvenen (2007) provides the formulas for these moments from the model and argues that from these moments the parameters of the statistical model are identified. In particular, the moments from the model are given by

$$E(\log(e_{iht})^2) = \sigma_\alpha^2 + 2\sigma_{\alpha\beta}h + \sigma_\beta^2 h^2 + \phi_t^2 + E(z_{iht}^2)$$
$$E(\log(e_{iht})\log(e_{ih+n,t+n})) = \sigma_\alpha^2 + 2\sigma_{\alpha\beta}(2h+n) + \sigma_\beta^2 h(h+n) + \rho^n E(z_{iht}^2)$$

where $E(z_{iht}^2)$ in turn is a function of ρ and the π_t^2.

6.2   Results

Guvenen (2007) estimates the process above on US PSID data from 1968 to 1993. The following table summarizes his results.

Estimates for Earnings Process from PSID

Sample	ρ	σ_α^2	σ_β^2	$corr_{\alpha\beta}$	$\bar{\pi}^2$	$\bar{\phi}^2$
All	0.988 (.024)	0.058 (.011)	–	–	0.015 (.007)	0.061 (.010)
College	0.979 (.055)	0.031 (.021)	–	–	0.0099 (0.013)	0.047 (.020)
High School	0.972 (.023)	0.053 (.015)	–	–	0.011 (.007)	0.052 (.008)
All	0.821 (.030	0.022 (.074)	0.00038 (.00008)	−0.23 (.43)	0.029 (.008)	0.047 (.007)
College	0.805 (.061)	0.023 (.112)	0.00049 (.00014)	−0.70 (1.22)	0.025 (.015)	0.032 (.017)
High School	0.829 (.029)	0.038 (.081)	0.00020 (.00009)	−0.25 (.59)	0.022 (.008)	0.034 (.007)

The first three rows display the results if the stochastic earnings process is estimated under the restriction that $\sigma_\beta^2 = 0$ (and consequently $\sigma_{\alpha\beta} = 0$). We observe that with this restriction the persistent shock is very persistent as the estimate of ρ is close to, and statistically indistinguishable from 1. This result is very much consistent with the previous literature that has imposed $\sigma_\beta^2 = 0$. There is substantial heterogeneity in the intercept of life cycle earnings profiles ($\hat{\sigma}_\alpha^2$ is significantly different from zero) and this heterogeneity is decreasing with educational attainment. Both the estimates of the size of transitory and persistent shocks is consistent with the previous literature as well (the table displays time averages of π_t^2 and ϕ_t^2).

Once slope heterogeneity is allowed (the last three rows of the table), three findings stand out. First, the estimated heterogeneity across households in the slope of life cycle earnings profile is significantly different from zero in a statistical sense, and is substantial from an economic point of view. Guvenen (2007) calculated that of the entire cross-sectional earnings dispersion of a cohort that has reached age 55, 75% of that variance for individuals with a college degree and 55% of those with high school degree is accounted for

by heterogeneity in slopes of life cycle earnings profiles (rather than shocks or initial differences in levels). Not surprisingly this share is higher for college graduates since they, according to the estimates from the table, face significantly higher profile heterogeneity. Even though the estimates for σ_β^2 are small in magnitude, as long as earnings of households grow at permanently different rates across households, by later ages in the life cycle these different slopes generate substantial dispersion across household in earnings. Second, if we allow for slope heterogeneity, the estimated degree of heterogeneity in earnings levels σ_α^2 declines substantially and becomes statistically insignificant. Third, the estimated persistence of the income process declines massively, from a process that is essentially a random walk to a strongly mean-reverting process.[2] Again these findings are typical for the literature that allows for slope heterogeneity in earnings.

Thus the results in the table essentially contrast two views of how the stochastic earnings process of individuals or households look like. According to what Guvenen (2007) calls the Restricted Income Profile (RIP) view, life cycle earnings are characterized by substantial heterogeneity in initial levels, no heterogeneity in slopes and highly persistent shocks over the life cycle. In contrast, according to the Heterogeneous Income Profiles (HIP) view, the key heterogeneity across households is in the slopes of life cycle earnings profiles which are realized at labor market entry. Subsequent earnings shocks are not very persistent (and thus, as we have seen above, should be quite easy to self-insure against). We will now discuss further where the initial evidence (mainly by MaCurdy, 1982) in favor of RIP came from and why $\sigma_\beta^2 = 0$ and $\rho \approx 1$ seem to go hand in hand, and $\sigma_\beta^2 \gg 0$ and $\rho \ll 1$ seem to go hand in hand.

---

[2]The variances of the transitory component declines somewhat in size, while that of the persistent component increases (but remember, the persistent component is not nearly as persistent anymore). Also note that the correlation between α_i and β_i is estimated to be negative (albeit insignificantly so). This may suggest that households face trade-offs between careers with low initial earnings levels but subsequently higher earnings growth, and those with higher levels and lower growth.

6.3 Interpretation

6.3.1 RIP vs. HIP

Why did MaCurdy (1982) and many other following his lead conclude that $\sigma_\beta^2 = 0$? Suppose for simplicity[3] that the true process has $\phi_t = 0$ (only persistent shocks) and that the variance of the persistent shock is not time varying, $\pi_t = \pi$. Then the stochastic part of the earnings process becomes

$$
\begin{aligned}
\log(\hat{e}_{iht}) &= \alpha_i + \beta_i h_{it} + z_{it} \\
z_{it} &= \rho z_{it-1} + \pi \eta_{it}.
\end{aligned}
$$

Now let us consider year-by-year earnings growth rates

$$
\Delta \log(\hat{e}_{iht}) = \log(\hat{e}_{iht}) - \log(\hat{e}_{ih-1t-1}) = \beta_i + \Delta z_{it},
$$

according to the model. Let us also assume that the process started at $t = -\infty$ so that

$$
z_{it} = \pi \sum_{\tau=0}^{\infty} \rho^{t-\tau} \eta_{i\tau}
$$

and hence

$$
\begin{aligned}
\Delta z_{it} &= z_{it} - z_{it-1} \\
&= (\rho - 1) z_{it-1} + \pi \eta_{it} \\
&= \pi \left[(\rho - 1) \sum_{\tau=0}^{\infty} \rho^{t-1-\tau} \eta_{i\tau} + \eta_{it} \right]
\end{aligned}
$$

Therefore if we calculate the so-called autocovariance function between earnings growth at time t and experience h and earnings growth $n \geq 2$ periods

---

[3]Simplicity means that the basic argument does not depend on these assumptions (but the exact algebra does).

from now we find

$$
\begin{aligned}
& Cov\left[\Delta \log(\hat{e}_{iht}), \Delta \log(\hat{e}_{ih+nt+n})\right] \\
= \ & E\left[\Delta \log(\hat{e}_{iht}) \cdot \Delta \log(\hat{e}_{ih+nt+n})\right] \\
= \ & \sigma_\beta^2 + \pi^2 E\left[\left((\rho-1)\sum_{\tau=0}^{\infty}\rho^{t-1-\tau}\eta_{i\tau} + \eta_{it}\right)\left((\rho-1)\sum_{\tau=0}^{\infty}\rho^{t+n-1-\tau}\eta_{i\tau} + \eta_{it+n}\right)\right] \\
= \ & \sigma_\beta^2 + \pi^2\left[(\rho-1)\rho^{n-1} + (\rho-1)^2\rho^n\sum_{\tau=0}^{\infty}\rho^{2(t-1-\tau)}\right] \\
= \ & \sigma_\beta^2 - \pi^2\rho^{n-1}(1-\rho)\left[1 - (1-\rho)\rho\sum_{\tau=0}^{\infty}\rho^{2(t-1-\tau)}\right] \\
= \ & \sigma_\beta^2 - \frac{\pi^2\rho^{n-1}(1-\rho)}{1+\rho}.
\end{aligned}
\tag{6.5}
$$

The key observation is that the second term exponentially declines with n, so that for large enough leads n we should see empirically $Cov\left[\Delta \log(\hat{e}_{iht}), \Delta \log(\hat{e}_{ih+nt+n})\right] > 0$ if $\sigma_\beta^2 > 0$. Intuitively, if the slopes of life cycle earnings profiles vary across individuals, in the cross sections earnings growth rates today and in the future (at least the far future) have to be positively correlated.

When MaCurdy (and the subsequent literature) computed the autocovariance function from the data, he found that they were all negative for $n \leq 10$ (he did not compute it for longer leads, partially because the data was not available in 1982), and insignificantly different from zero for $n \geq 2$. He interpreted this as inconsistent with $\sigma_\beta^2 > 0$. Furthermore the fact that for $n \geq 2$ the autocovariances are statistically zero requires, in light of (6.5), that $\rho = 1$. The subsequent literature has then proceeded as MaCurdy: compute the autocovariance function, see whether it all negative, conclude that $\sigma_\beta^2 = 0$, and from it being zero at leads $n \geq 2$ conclude that $\rho \approx 1$ (which comes out of the estimation when imposing $\sigma_\beta^2 = 0$ also). Note that in the presence of transitory shocks the autocovariance function of the model is negative rather than zero for $n = 1$ even if $\rho = 1$, so the significantly negative autocovariance function at lead $n = 1$ is easily rationalized with the presence of transitory shocks..

Guvenen's (2007) key argument against MaCurdy's argument is the low power of the test. he shows that with his empirical estimates for $\sigma_\beta^2 > 0$ the first n for which the model autocovariance is significantly positive is large, in the order of $n = 12$ or above. He also shows that the model-simulated

autocovariances (with the estimated parameters) are not inconsistent with the data. While Guvenen's evidence is likely not the last word in this debate, it casts some doubt on the arguments against HIP advanced so far in the literature.

6.3.2 Size of ρ

The last point we want to discuss is that why, if the truth is HIP but we estimate the earnings process under the restriction $\sigma_\beta^2 = 0$, we may end up with an estimate of ρ that is biased upward. Take two households with the same initial income and deterministic but different earnings growth

$$\log(e_{it}) = \alpha + \beta_i t \tag{6.6}$$

with $\beta_1 < \beta_2$. Suppose the econometrician estimates

$$\log(e_{it}) = \rho \log(e_{it-1}) + \eta_{it}. \tag{6.7}$$

with $\log(e_{i0}) = \alpha/\rho$. In the first period the econometrician observes

$$\begin{aligned}
\log(e_{11}) &= \alpha + \beta_1 \\
\log(e_{21}) &= \alpha + \beta_2
\end{aligned}$$

and has to interpret this, from the perspective of (6.7), as a negative shock for household 1 and a positive shock for household 2. Since under the truth (6.6) earnings of both households deviate more and more from the common average over time, the econometrician has to interpret these larger and larger deviations as repeated and very persistent negative shocks for household 1 and positive shocks for household 2. Guvenen's (2007) figure 1 provides an instructive visualization of this point. Thus when one estimates the earnings process without allowing for slope heterogeneity when in fact it is present one tends to find significantly higher values for ρ than when estimating it with σ_β^2 permitted. Note that the difference between an estimate (on annual basis) of $\rho = 0.98$ (RIP) and $\rho = 0.82$ (HIP) is huge: under the RIP estimate after 20 years still 2/3's of a shock is present in current earnings, whereas under the HIP estimate only 2% of the shock 20 years ago is present. Obviously self-insurance will do very well dealing with the latter shock, and not well at all dealing with the former shock.

Chapter 7

The SIM in General Equilibrium

In this section we will look at general equilibrium versions of the PILCH model discussed in the last chapter. In general equilibrium the interest rate(s) and real wage(s) are determined endogenously within the model. Three reasons for considering general equilibrium come to my mind

1. It imposes theoretical discipline. As we saw above, the behavior of consumption and saving over time depends crucially on the relative size of the interest rate r and the time discount factor ρ. In the partial equilibrium analysis so far *both* r and ρ were exogenous parameters, to be chosen by the model builder. In general equilibrium, once ρ is chosen, r will be determined endogenously. The relationship between ρ and r shifts from being arbitrarily chosen to an equilibrium relationship; as we will see, only a stationary equilibrium with $r < \rho$ can exist in these models.

2. It gives rise to an endogenously determined consumption and wealth distribution and hence provides a theory of wealth inequality. To be successful, this model should be able to reproduce stylized facts of empirical wealth distributions. Note that it is not a theory of the income distribution, as the stochastic income process is an input to the model (and needs to be chosen by the model builder), rather than a result of the model.

3. It enables meaningful policy experiments. Partial equilibrium models,

by keeping interest rates and wages fixed when analyzing the change in household behavior due to changes in particular policies (tax reform, social security reform, welfare reform), may over- or understate the full effects of such a reform. A claim, e.g., that a reform of the social security system towards a system that invests more funds in the stock market is welfare improving may ignore the fact that, once large additional funds are invested in the stock market, the excess return of stocks over bonds (or the population growth rate plus the growth rate of productivity), may diminish or vanish altogether. Only in cases in which one can reasonably expect relative prices to remain uninfluenced by policy reforms (for a small open economy, say) or one can convincingly argue that price effects are quantitatively unimportant[1] would a partial equilibrium analysis yield unbiased results.

7.1 A Model Without Aggregate Uncertainty

7.1.1 The Environment

But let us leave the realm of ideology and describe the model. The economy is populated by a continuum of measure 1 of individuals that all face an income fluctuation problem of the sort described above. Each individual has the same stochastic labor endowment process $\{y_t\}_{t=0}^{\infty}$ where $y_t \in Y = \{y_1, y_2, \ldots y_N\}$. A households' labor income in a particular period is given by $w_t y_t$ where w_t is the real wage that one unit of labor commands in the economy, which will be constant in a stationary equilibrium. Hence y_t can be interpreted as the number of efficiency units of labor a household can supply in a given time period. The labor endowment process is assumed to follow a stationary Markov process. Let $\pi(y'|y)$ denote the probability that tomorrow's endowment takes the value y' if today's endowment takes the value y. We assume a law of large numbers to hold: not only is $\pi(y'|y)$ the probability of a particular agent of a transition from y to y' but also the deterministic fraction of the population that has this particular transition.[2]

---

[1]It is not clear to me, though, how one would arrive at such a conclusion without actually first doing the general equilibrium analysis.

[2]The validity of such a law of large numbers what subject to some debate in the 1980's. See Judd (1985), Feldman and Gilles (1985) and Uhlig (1996) for representative papers. I will briefly discuss the main result in Feldman and Gilles (1985).

Let Π denote the stationary distribution associated with π, assumed to be unique. We assume that at period 0 the income of all agents, y_0, is given, and that the distribution of incomes across the population is given by Π. Given our assumptions, then, the distribution of income in all future periods is also given by Π. In particular, the total labor endowment in the economy (in efficiency units) is given by

$$\bar{L} = \sum_y y\Pi(y) \tag{7.2}$$

As before, let denote by $\pi_t(y^t|y_0)$ the probability of event history y^t, given initial event y_0. We have

$$\pi_t(y^t|y_0) = \pi(y_t|y_{t-1}) * \ldots * \pi(y_1|y_0) \tag{7.3}$$

Hence, although there is substantial idiosyncratic uncertainty about a particular individual's labor endowment and hence labor income, the aggregate

---

What one would think and like to be true is the following: define as set of agents $I = (0, 1]$ with associated measure space $(I, \mathcal{B}(I), m)$ where m is the Lebesgue measure (essentially measuring length or mass of an interval of agents) and $\mathcal{B}(I)$ is the Borel sigma algebra. Define a probability space on which the joint stochastic income process for the continuum of agents is defined as $(\Omega, \mathcal{F}, P)$. For simplicity we only consider the case where all agents draw income in only one period. Finally define a continuum of random variables as $y(i, .) : \Omega \to \{0, 1\}$ where we assume for simplicity that income can only take the values 0 and 1 and does so with equal probability for each household. What one would hope for is that for all sets $G = (c, d] \in I$ and almost all $\omega \in \Omega$

$$\int_G y(i, \omega)m(di) = \frac{1}{2}m(G). \tag{7.1}$$

However, if one insists that the $y(i, .)$ are pairwise independent for any two agents, then Feldman and Gilles (1985) and Judd (1985) show that for all ω either $y(., \omega)$ is not m-measurable or there exists a set G such that (7.1) does not hold. However, for our purposes pairwise independence is not necessary: we just would like all random variables $y(i, .)$ to have a desired distribution Π and the we would like to have a population distribution over incomes $y(i, .)$ that equals Π for all ω, that is, for aggregates to be nonstochastic.

Luckily Feldman and Gilles show that for a probability space $(Y, \mathcal{B}(Y), \Pi)$ there exists a continuum of random variables $y(i, .) : \Omega \to Y$ such that for all i the random variable is distributed according to Π and that for all $\omega \in \Omega$ and all $D \in \mathcal{B}(Y)$

$$m(i \in I : \{y(i, \omega) \in D\}) = \Pi(D)$$

that is, the population income distribution is nonstochastic and given by Π. Evidently, in light of the first proposition by Feldman and Gilles the continuum of random variables cannot be pairwise independent.

labor endowment and hence labor income in the economy is constant over time, i.e. there is no aggregate uncertainty. Without this assumption there would be no hope for the existence of a *stationary* equilibrium in which wages w and interest rates r are constant over time. In the next section we will consider a model that is similar to this one, but will include one source of aggregate uncertainty, in addition to the idiosyncratic labor income uncertainty present in this model.

Each agent's preferences over stochastic consumption processes are given by

$$u(c) = E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{7.4}$$

with $\beta = \frac{1}{1+\rho}$ and $\rho > 0$. As before the agent can self-insure against idiosyncratic labor endowment shocks by purchasing at period t uncontingent claims to consumption at period $t+1$ at a price $q_t = \frac{1}{1+r_{t+1}}$. Again, in a stationary equilibrium r_{t+1} will be constant across time. The agents budget constraint is given by

$$c_t + a_{t+1} = w_t y_t + (1 + r_t) a_t \tag{7.5}$$

Note that we slightly change the way assets are traded: instead of zero coupon bonds being traded at a discount $q = \frac{1}{1+r}$ we now consider a bond that trades at price 1 today and earns gross interest rate $(1+r_{t+1})$ tomorrow. We do this for the following reason: the asset being traded will be physical capital, with the interest rate being determined by the marginal product of capital. As long as the interest rate is constant as in Aiyagari (1994) both formulations are equivalent (if you derive the corresponding Euler equations for both formulations, you'll find that they're identical). With aggregate uncertainty as in Krusell and Smith (1998), however, it would make a substantial difference whether agents can trade a risk free bond or, as Krusell and Smith assume, risky capital. Thus, in order to be consistent with their formulation, I opted for changing the budget constraint. Evidently all theoretical results from the last section about the income fluctuation problem still apply (as long as the interest rate is nonstochastic), because the optimality conditions for the households are identical across both formulations.

We impose an exogenous borrowing constraint on asset holdings: $a_{t+1} \geq 0$. Aiyagari (1994) considers several alternative borrowing constraints, but uses the no-borrowing constraint in his applications. The agent starts out with initial conditions (a_0, y_0). His consumption at period t after endowment

shock history y^t has been realized is denoted by $c_t(a_0, y^t)$ and his asset holdings by $a_{t+1}(a_0, y^t)$. Let $\Phi_0(a_0, y_0)$ denote the initial measure over (a_0, y_0) across households. In accordance with our previous assumption the marginal distribution of Φ_0 with respect to y_0 is assumed to be Π. At each point of time an agent is characterized by her current asset position a_t and her current income y_t. These are her individual state variables. What describes the aggregate state of the economy is the cross-sectional distribution over individual characteristics $\Phi_t(a_t, y_t)$. This concludes the description of the household side of the economy.

On the production side we assume that competitive firms, taking as given wages w_t and interest rates r_t, have access to a standard neoclassical production technology[3]

$$Y_t = F(K_t, L_t) \tag{7.6}$$

where $F \in C^2$ features constant returns to scale and positive but diminishing marginal products with respect to both production factors. We also assume the Inada conditions to hold. Firms choose labor inputs and capital inputs to maximize the present discounted value of profits. As usual with constant returns to scale and perfect competition the number of firms is indeterminate and without loss of generality we can assert the existence of a single, representative firm. In this economy the only asset is the physical capital stock. Hence in equilibrium the aggregate capital stock K_t has to equal the sum of asset holdings of all individuals, i.e. the integral over the a_t's of all agents. We assume that capital depreciates at rate $0 < \delta < 1$. The aggregate resource constraint is then given as

$$C_t + K_{t+1} - (1 - \delta)K_t = F(K_t, L_t) \tag{7.7}$$

where C_t is aggregate consumption at period t. We are now ready to define an equilibrium. We first define a sequential markets equilibrium, then a recursive equilibrium and finally a stationary recursive equilibrium. The first definition is done in order to stress similarities and differences with the complete markets model, the third because this is what we will compute numerically, and the second is presented because the third is a special case, and with aggregate uncertainty we will need the more general definition anyway.

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[3]In the next section aggregate uncertainty is introduced by incorporating stochastic productivity into the aggregate production function, in the tradition of real business cycles.

Definition 44 *Given* Φ_0, *a sequential markets competitive equilibrium is allocations for households* $\{c_t(a_0, y^t), a_{t+1}(a_0, y^t)\}_{t=0, y^t \in Y^t}^{\infty}$, *allocations for the representative firm* $\{K_t, L_t\}_{t=0}^{\infty}$, *and prices* $\{w_t, r_t\}_{t=0}^{\infty}$ *such that*

1. *Given prices, allocations maximize* (7.4) *subject to* (7.5) *and subject to the nonnegativity constraints on assets and consumption and the nonnegativity condition on assets*

2.

$$r_t = F_k(K_t, L_t) - \delta \qquad (7.8)$$
$$w_t = F_L(K_t, L_t) \qquad (7.9)$$

3. *For all t*

$$K_{t+1} = \int \sum_{y^t \in Y^t} a_{t+1}(a_0, y^t) \pi(y^t | y_0) d\Phi_0(a_0, y_0) \qquad (7.10)$$

$$L_t = \bar{L} = \int \sum_{y^t \in Y^t} y_t \pi(y^t | y_0) d\Phi_0(a_0, y_0) \qquad (7.11)$$

$$\int \sum_{y^t \in Y^t} c_t(a_0, y^t) \pi(y^t | y_0) d\Phi_0(a_0, y_0) + K_{t+1}$$
$$= F(K_t, L_t) + (1 - \delta) K_t \qquad (7.12)$$

The last three conditions are the asset market clearing, the labor market clearing and the goods market clearing condition, respectively

Now let us define a recursive competitive equilibrium. We have already conjectured what the correct state space is for our economy, with (a, y) being the individual state variables and $\Phi(a, y)$ being the aggregate state variable. First we need to define an appropriate measurable space on which the measures Φ are defined. Define the set $A = [0, \infty)$ of possible asset holdings and the set Y of possible labor endowment realizations. Define by $\mathcal{P}(Y)$ the power set of Y (i.e. the set of all subsets of Y) and by $\mathcal{B}(A)$ the Borel σ-algebra of A. Let $Z = A \times Y$ and $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(A)$. Finally define by $\mathcal{M}$ the set of all probability measures on the measurable space $M = (Z, \mathcal{B}(Z))$. Why all this? Because our measures Φ will be required to be elements of $\mathcal{M}$. Now we are ready to define a recursive competitive equilibrium. At the heart of any RCE is the recursive formulation of the household problem. Note that

we have to include all state variables in the household problem, in particular the aggregate state variable, since the interest rate r will depend on Φ. Hence the household problem in recursive formulation is

$$v(a, y; \Phi) \quad = \quad \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y'|y)v(a', y'; \Phi') \qquad (7.13)$$

$$\text{s.t. } c + a' \quad = \quad w(\Phi)y + (1 + r(\Phi))a \qquad (7.14)$$

$$\Phi' \quad = \quad H(\Phi) \qquad (7.15)$$

The function $H : \mathcal{M} \to \mathcal{M}$ is called the aggregate "law of motion". Now let us proceed to the equilibrium definition.

Definition 45 *A recursive competitive equilibrium is a value function v : $Z \times \mathcal{M} \to R$, policy functions for the household $a' : Z \times \mathcal{M} \to R$ and $c : Z \times \mathcal{M} \to R$, policy functions for the firm $K : \mathcal{M} \to R$ and $L : \mathcal{M} \to R$, pricing functions $r : \mathcal{M} \to R$ and $w : \mathcal{M} \to R$ and an aggregate law of motion $H : \mathcal{M} \to \mathcal{M}$ such that*

1. *v, a', c are measurable with respect to $\mathcal{B}(Z)$, v satisfies the household's Bellman equation and a', c are the associated policy functions, given $r()$ and $w()$*

2. *K, L satisfy, given $r()$ and $w()$*

$$r(\Phi) \quad = \quad F_k(K(\Phi), L(\Phi)) - \delta \qquad (7.16)$$

$$w(\Phi) \quad = \quad F_L(K(\Phi), L(\Phi)) \qquad (7.17)$$

3. *For all $\Phi \in \mathcal{M}$*

$$K'(\Phi') \quad = \quad K(H(\Phi)) = \int a'(a, y; \Phi)d\Phi \qquad (7.18)$$

$$L(\Phi) \quad = \quad \int y d\Phi \qquad (7.19)$$

$$\int c(a, y; \Phi)d\Phi + \int a'(a, y; \Phi)d\Phi \qquad (7.20)$$
$$= \quad F(K(\Phi), L(\Phi)) + (1 - \delta)K(\Phi)$$

4. *The aggregate law of motion H is generated by the exogenous Markov process π and the policy function a' (as described below)*

Now let us specify what it means that H is generated by π and a'. H basically tells us how a current measure over (a, y) translates into a measure Φ' tomorrow. So H has to summarize how individuals move within the distribution over assets and income from one period to the next. But this is exactly what a transition function tells us. So define the transition function $Q_\Phi : Z \times \mathcal{B}(Z) \to [0, 1]$ by[4]

$$Q_\Phi((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \left\{ \begin{array}{c} \pi(y'|y) \text{ if } a'(a, y; \Phi) \in \mathcal{A} \\ 0 \text{ else} \end{array} \right. \qquad (7.21)$$

for all $(a, y) \in Z$ and all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$. $Q_\Phi((a, y), (\mathcal{A}, \mathcal{Y}))$ is the probability that an agent with current assets a and current income y ends up with assets a' in $\mathcal{A}$ tomorrow and income y' in $\mathcal{Y}$ tomorrow. Suppose that $\mathcal{Y}$ is a singleton, say $\mathcal{Y} = \{y_1\}$. The probability that tomorrow's income is $y' = y_1$, given today's income is $\pi(y'|y)$. The transition of assets is non-stochastic as tomorrows assets are chosen today according to the function $a'(a, y)$. So either $a'(a, y)$ falls into $\mathcal{A}$ or it does not. Hence the probability of transition from (a, y) to $\{y_1\} \times \mathcal{A}$ is $\pi(y'|y)$ if $a'(a, y)$ falls into $\mathcal{A}$ and zero if it does not fall into $\mathcal{A}$. If $\mathcal{Y}$ contains more than one element, then one has to sum over the appropriate $\pi(y'|y)$.

How does the function Q_Φ help us to determine tomorrow's measure over (a, y) from today's measure? Suppose Q_Φ were a Markov transition matrix for a finite state Markov chain and Φ_t would be the distribution today. Then to figure out the distribution Φ_t tomorrow we would just multiply Q by Φ_t, or

$$\Phi_{t+1} = Q_{\Phi_t}^T \Phi_t \qquad (7.22)$$

where here T stands for the transpose of a matrix. But a transition function is just a generalization of a Markov transition matrix to uncountable state spaces. Hence we need integrals:

$$\Phi'(\mathcal{A}, \mathcal{Y}) = (H(\Phi))(\mathcal{A}, \mathcal{Y}) = \int Q_\Phi((a, y), (\mathcal{A}, \mathcal{Y}))\Phi(da \times dy) \qquad (7.23)$$

The fraction of people with income in $\mathcal{Y}$ and assets in $\mathcal{A}$ is that fraction of people today, as measured by Φ, that transit to $(\mathcal{A}, \mathcal{Y})$, as measured by Q_Φ.

In general there no presumption that tomorrow's measure Φ' equals today's measure, since we posed an arbitrary initial distribution over types,

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[4]Note that, since a' is also a function of Φ, Q is implicitly a function of Φ, too.

Φ_0. If the sequence of measures $\{\Phi_t\}$ generated by Φ_0 and H is not constant, then obviously interest rates $r_t = r(\Phi_t)$ are not constant, decision rules vary with Φ_t over time, and the computation of equilibria is difficult in general. We would have to compute the function H explicitly, mapping measures (i.e. infinite-dimensional objects) into measures. This is exactly what we are going to discuss in the next section. In this section we are interested in stationary Recursive Competitive Equilibria.

Definition 46 *A stationary recursive competitive equilibrium is a value function $v : Z \to R$, policy functions for the household $a' : Z \to R$ and $c : Z \to R$, policies for the firm K, L, prices r, w and a measure $\Phi \in \mathcal{M}$ such that*

1. *v, a', c are measurable with respect to $\mathcal{B}(Z)$, v satisfies the household's Bellman equation and a', c are the associated policy functions, given r and w*

2. *K, L satisfy, given r and w*

$$r = F_K(K, L) - \delta \tag{7.24}$$
$$w = K_L(K, L) \tag{7.25}$$

3.

$$K = \int a'(a, y) d\Phi \tag{7.26}$$

$$L = \int y d\Phi \tag{7.27}$$

$$\int c(a, y) d\Phi + \int a'(a, y) d\Phi = F(K, L) + (1 - \delta)K \tag{7.28}$$

4. *For all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$*

$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi \tag{7.29}$$

where Q is the transition function induced by π and a' as described above (not indexed by Φ anymore)

Note the big simplification: value functions, policy functions and prices are not any longer indexed by measures Φ, all conditions have to be satisfied only for the equilibrium measure Φ. The last requirement states that the measure Φ reproduces itself: starting with measure over incomes and assets Φ today generates the same measure tomorrow. In this sense a stationary RCE is the equivalent of a steady state, only that the entity characterizing the steady state is not longer a number (the aggregate capital stock, say) but a rather complicated infinite-dimensional object, namely a measure.[5]

7.1.2   Theoretical Results: Existence and Uniqueness

In this section we want to summarize what we know about the existence and uniqueness of a stationary RCE. We first argue that the problem of existence boils down to the question of whether one equation in one unknown, namely the interest rate, has a solution. To see this, first note that by Walras' law one of the market clearing conditions is redundant; so let's ignore the goods market equilibrium condition. The labor market equilibrium is easy and gives us $L = \bar{L}$ and $\bar{L}$ is exogenously given, specified by the labor endowment process. It remains the asset market clearing condition

$$K = K(r) = \int a'(a,y)d\Phi \equiv Ea(r) \tag{7.30}$$

where $Ea(r)$ are the average asset holdings in the economy. This condition requires equality between the demand for capital by firms and the supply of capital by households (last period's demand for assets, with physical capital being the only asset in the economy).

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[5]If we restrict attention to a finite set of capital stocks, $A = \{a_1, \ldots, a_M\}$ then Φ is an $M * N \times 1$ column vector and the Markov transition function Q is an $M * N \times M * N$ matrix with generic element $q_{ij,kl}$ giving the probability of going from $(a,y) = (a_i, y_l)$ to $(a',y') = (a_k, y_l)$ tomorrow. Using the convention that rows index states today and columns index states tomorrow, an invariant measure Φ has to satisfy the matrix equation

$$\Phi = Q^T \Phi.$$

That is, Φ is the eigenvector (rescaled to have the sum of its rows equal to 1) associated with an eigenvalue $\lambda = 1$ of the matrix Q^T. Since Q^T is a stochastic matrix (every row sums to 1 and all entires are nonnegative) it has at least one unit eigenvalue and thus at least one stationary measure Φ. But if Q^T has multiple unit eigenvalues there are multiply statioanry measures (in fact, a continuum of them, since convex combinations of stationary measures are themselves stationary measures.

¿From (7.24) it is clear that the capital demand of the firm $K(r)$ is a function of r alone, defined implicitly as

$$r = F_k(K(r), \bar{L}) - \delta \tag{7.31}$$

since labor supply $L = \bar{L} > 0$ is exogenous. Furthermore we know from the assumptions on the production function that $K(r)$ is a continuous, strictly decreasing function on $r \in (-\delta, \infty)$ with

$$\lim_{r \to -\delta} K(r) = \infty \tag{7.32}$$

$$\lim_{r \to \infty} K(r) = 0 \tag{7.33}$$

It is obvious that the average capital supply (asset demand) by households $Ea(r)$ satisfies $Ea(r) \in [0, \infty]$ for all r in $(-\delta, \infty)$. It is our goal to characterize $Ea(r)$. In particular, if $Ea(r)$ is continuous and satisfies

$$\lim_{r \to -\delta} Ea(r) < \infty \tag{7.34}$$

$$\lim_{r \to \infty} Ea(r) > 0 \tag{7.35}$$

then a stationary recursive competitive equilibrium exists. Furthermore, if $Ea(r)$ is strictly increasing in r (the substitution effects outweighs the income effect), then the stationary RCE is unique.

The first thing we have to argue is that average capital supply (asset demand) is in fact a function of r alone and that it is well defined on the appropriate range for r. First we note that the wage rate w is a function solely of r via

$$w(r) = F_L(K(r), \bar{L}) \tag{7.36}$$

As r increases, $K(r)$ decreases, the capital-labor ratio declines and with it the wage rate. Hence, once r is known, the recursive problem of the household can be solved, because all prices are known.

The first step is to ask under what conditions the recursive problem of the household has a solution and under which conditions the support of asset holdings is bounded from above (it is bounded from below by 0). Various assumptions give us these results, but here are the most important ones (note that Aiyagari (1994) does not prove anything for the case of serially correlated income and assumes bounded utility, Huggett (1993) does, but with additional assumptions on the structure of the Markov process).

Proposition 47 *(Huggett 1993) For $\rho > 0, r > -1$ and $y_1 > 0$ and CRRA utility with $\sigma > 1$ the functional equation has a unique solution v which is strictly increasing, strictly concave and continuously differentiable in its first argument. The optimal policies are continuous functions that are strictly increasing (for $c(a, y)$) or increasing or constant at zero (for $a'(a, y)$)*

Similar results can be proved for the *iid* case and arbitrary bounded U with $\rho > r$ and $\rho > 0$, see Aiyagari (1994).

With respect to the boundedness of the state space, as seen in the previous section we require $\rho > r$ and additional assumptions. Under these assumptions there exists an $\bar{a}$ s.t. $a'(\bar{a}, y_N) = \bar{a}$ and $a'(a, y) \leq \bar{a}$ for all $y \in Y$ and all $a \in [0, \bar{a}] = A$. From now on we will restrict the state space to $Y \times A$ and it will be understood that $Z = A \times Y$. Thus, under the maintained assumptions by Huggett or Aiyagari there exists an optimal policy $a'_r(a, y)$ indexed by r. The next step is to ask what more is needed to make aggregate asset demand

$$Ea(r) = \int a'_r(a, y) d\Phi_r \qquad (7.37)$$

well-defined, where Φ_r has to satisfy

$$\Phi_r(\mathcal{A}, \mathcal{Y}) = \int Q_r((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi_r \qquad (7.38)$$

and Q_r is the Markov transition function defined by a_r as

$$Q_r((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \left\{ \begin{array}{c} \pi(y'|y) \text{ if } a'_r(a, y) \in \mathcal{A} \\ 0 \text{ else} \end{array} \right. \qquad (7.39)$$

Hence all is needed for $Ea(r)$ to be well-defined is to establish that the operator $T^*_r : \mathcal{M} \to \mathcal{M}$ defined by

$$\left(T^*_r(\Phi)\right)(\mathcal{A}, \mathcal{Y}) = \int Q_r((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi \qquad (7.40)$$

has a unique fixed point (that T^*_r maps $\mathcal{M}$ into itself follows from SLP, Theorem 8.2). To show this Aiyagari (in the working paper version, and quite loosely described) draws on a theorem in SLP and Huggett on a similar theorem due to Hopenhayn and Prescott (1992). In both theorems the key condition is a monotone mixing condition that requires a positive probability

to go from the highest asset level $\bar{a}$ to a intermediate asset level in N periods and an evenly high probability to go from 0 assets to an intermediate asset level also in N periods. More precisely stated, the theorem by Hopenhayn and Prescott states the following. Define the order " $\geq$ " on Z as

$$z \geq z' \text{ iff } [(z_1 \geq z'_1 \text{ and } z_2 = z'_2) \text{ or } (z' = c = (0, y_1)) \text{ or } (z = d = (\bar{a}, y_N)) \tag{7.41}$$

Under this order it is easy to show that $(Z, \geq)$ is an ordered space, Z together with the Euclidean metric is a compact metric space, $\geq$ is a closed order, $c \in Z$ and $d \in Z$ are the smallest and the largest elements in Z (under order $\geq$) and $(Z, \mathcal{B}(Z))$ is a measurable space. Then we have (see Hopenhayn and Prescott (1992), Theorem 2)

Proposition 48 *If*

1. *Q_r is a transition function*

2. *Q_r is increasing*

3. *There exists $z^* \in Z$, $\varepsilon > 0$ and N such that*

$$P^N(d, \{z : z \leq z^*\}) > \varepsilon \text{ and } P^N(c, \{z : z \geq z^*\}) > \varepsilon \tag{7.42}$$

Then the operator T_r^* has a unique fixed point Φ_r and for all $\Phi_0 \in \mathcal{M}$ the sequence of measures defined by

$$\Phi_n = (T^*)^n \Phi_0 \tag{7.43}$$

converges weakly to Φ_r

Here $P^N\{z, \mathcal{Z})$ is the probability of going from state z to set $\mathcal{Z}$ in N steps. Instead of proving this result (which turns out to be quite tough) we will explain the assumptions, heuristically verify them and discuss what the theorem delivers for us. Assumption 1 requires that Q_r is in fact a transition function, i.e. $Q_r(z, .)$ is a probability measure on $(Z, \mathcal{B}(Z))$ for all $z \in Z$ and $Q_r(., \mathcal{Z})$ is a $\mathcal{B}(Z)$-measurable function for all $\mathcal{Z} \in \mathcal{B}(Z)$. Given that $a'(a, y)$ is a continuous function, the proof of this is not too hard. The assumption that Q_r is increasing means that for any nondecreasing function $f : Z \to \mathbf{R}$ we have that

$$(Tf)(z) = \int f(z') Q_r(z, dz') \tag{7.44}$$

is also nondecreasing. The proof that Q_r satisfies monotonicity is straight-forward, given that $a'(a,y)$ is increasing in both its arguments[6], so that bigger z's make $Q_r(z,.)$ put more probability mass on bigger z'. Together with f being nondecreasing the result follows. Finally, why is the monotone mixing condition 3. satisfied? Pick (this is going to be very heuristic here) $z^* = \left(\frac{1}{2}\left(a'(0,y_N)+\bar{a}\right),y_1\right)$. Starting at d, with a sequence of bad shocks y_1 one converges to assets 0 monotonically and from c one converges to $\bar{a}$ with a sequence of good shocks y_N. Hence with probability bigger than zero one goes with finite steps N_1 or less from d to something below z^* and with finite steps N_2 or less from c to something above z^*. Take $N = \max\{N_1, N_2\}$.

The conclusion of the theorem then assures the existence of a unique invariant measure Φ_r which can be found by iterating on the operator T_r^*. Convergence is in the weak sense, that is, a sequence of measures $\{\Phi_n\}$ converges weakly to Φ_r if for every continuous and bounded real-valued function f on Z we have

$$\lim_{n\to\infty} \int f(z)d\Phi_n = \int f(z)d\Phi_r \qquad (7.45)$$

The argument in the preceding section demonstrated that the function $Ea(r)$ is well-defined on $r \in [-\delta, \rho)$. Since $a_r'(a,y)$ is a continuous function jointly in (r,a), see SLP, Theorem 3.8 and Φ_r is continuous in r (in the sense of weak convergence), see SLP, Theorem 12.13, the function $Ea(r)$ is a continuous function of r on $[-\delta, \rho)$. Note that the real bottleneck in the argument is in establishing an upper bound of the state space for assets, as Z needs to be compact for the theorem by Hopenhayn and Prescott to work. To bound the state space an interest rate $r < \rho$ is needed, as the previous sections showed.

The remaining things to be argued are that $\lim_{r\to-\delta} Ea(r) < \infty$ and $\lim_{r\to\rho} Ea(r) > K(\rho)$. The first condition is obviously satisfied whenever a bound on the state space for assets, $\bar{a}$, can be found. So this part of the problem is easily resolved. We also know from the previous section that for $\rho = r$ all consumers accumulate infinite assets eventually, so that, loosely speaking $Ea(\rho) = \infty$. What is asserted (but not proved) by Aiyagari is that as r approaches ρ from below, $Ea(r)$ goes to ∞ by some form of continuity argument (but note that, strictly speaking, $Ea(\rho)$ is not well-defined).

This is a so far missing step in the existence proof. Otherwise we have es-

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[6]For $a'(a,y)$ to be increasing in y we need to assume that the exogenous Markov chain does not feature negatively correlated income.

tablished that both $K(r)$ and $Ea(r)$ are continuous functions on $r \in (-\delta, \rho)$, that for low r we have that $Ea(r) < K(r)$ and for high $r < \rho$ we have $Ea(r) > K(r)$. These arguments together then guarantee the existence of r^* such that

$$K(r^*) = Ea(r^*) \tag{7.46}$$

and hence the existence of a stationary recursive competitive equilibrium.

With respect to uniqueness the verdict is negative, as we can't prove the monotonicity of the function $Ea(r)$ which is due to offsetting income and substitution effects of saving with respect to the interest rate directly, as well as the indirect effect that the interest rate has on the wage rate. Even harder is the question about the stability of the stationary equilibrium. We know that, provided the economy starts with initial distribution $\Phi_0 = \Phi_{r^*}$, then the economy remains there forever. The question arises whether, for an arbitrary initial distribution $\Phi_0 \neq \Phi_{r^*}$ it is the case that $\Phi_t \to \Phi_{r^*}$ and $r_t \to r^*$ (either locally or globally). Note that to assess this question one has to examine either the sequential equilibrium or the law of motion H in the full-blown RCE. To the best of my knowledge no stability result has been established for these types of economies as of today.

7.1.3 Computation of the General Equilibrium

Since finding a stationary RCE really amounts to finding an interest rate r^* that clears the asset market, consider the following algorithm

1. Fix an $r \in (-\delta, \rho)$. For a fixed r we can solve the household's recursive problem (e.g. by value function iteration or policy function iteration). This yields a value function v_r and decision rules a'_r, c_r, which obviously depend on the r we picked.

2. The policy function a'_r and π induce a Markov transition function Q_r. Compute the unique stationary measure Φ_r associated with this transition function from (7.29)

3. Compute excess demand for capital

$$d(r) = K(r) - Ea(r) \tag{7.47}$$

   Note that $d(r)$ is just a number. If this number happens to equal zero, we are done and have found a stationary RCE. If not we update

our guess for r and start from 1. anew. If $Ea(r)$ is increasing in r (which means that the substitution effect outweighs the income effect and potential indirect income effects from changes in wages) we know that $d(r)$ is a strictly decreasing function. The updating should then be such that if $d(r) > 0$, increase the guess for r, if $d(r) < 0$ decrease the guess for r.

7.1.4 Qualitative Results

One general qualitative result of these models is excess saving and the over-accumulation of capital, compared to a complete markets model in which a full array of insurance contracts against idiosyncratic income uncertainty is at the disposal of households. Remember that equilibria in this model are Pareto efficient, from which it readily follows that the stationary equilibrium in the present model (or any equilibrium, for that matter) is suboptimal (in an ex-ante welfare sense, of course). As seen in Chapter 2, without aggregate uncertainty consumption of all agents would be constant in any steady state of a complete markets model. From the complete markets Euler equation it follows right away that the stationary equilibrium interest rate with complete markets satisfies $r^{CM} = \rho$. But this means, since $r^* < \rho$ that the steady state capital stock under complete markets satisfies $K^{CM} < K^*$, i.e. there is overaccumulation of capital in this economy, compared to the first-best. Put another way, since total gross investment (and total saving) in the steady state of both economies equal

$$S^{CM} = \delta K^{CM} \tag{7.48}$$
$$S^* = \delta K^* > \delta K^{CM} = S^{CM} \tag{7.49}$$

i.e. agents in this model oversave because of precautionary reasons. Only potentially binding liquidity constraints are needed for this result (in non of the discussion above did we ever assume $U_{ccc} > 0$), with prudence strengthening the quantitative importance of the result. One of the main objectives of Aiyagari's quantitative analysis is to assess whether this excess precautionary saving is quantitatively important, i.e. whether PILCH people, in general equilibrium, accumulate significantly too many assets compared to the benchmark complete markets model. As metric for this the aggregate savings rate and real interest rate is used. Note for future reference that the

aggregate saving rate in a stationary equilibrium is given by

$$s = \frac{S}{Y} = \delta\frac{K}{Y} \tag{7.50}$$

7.1.5 Numerical Results

Calibration

Before solving the model Aiyagari has to specify functional forms of preferences and technology. Period utility is assumed to be $CRRA$ and Aiyagari experiments with values $\sigma = \{1, 3, 5\}$. The model period length is taken to be one year, and he picks $\rho = 0.0416$ ($\beta = 0.96$). For the complete markets model this yields an (empirically reasonable) annual real interest rate of 4.16%, and we know that in the present model the interest rate will be lower.

The aggregate production function is assumed to take Cobb-Douglas form with share parameter $\alpha = 0.36$. Since α equals the capital share of income and capital income as share of GDP is roughly 36% on average for US postwar data (the number is somewhat sensitive to how the public capital stock and consumer durables are treated). The annual depreciation rate is set at $\delta = 8\%$, values usually used in these types of calibration exercises (Cooley and Prescott (1995) provide a careful description of the calibration procedure for a standard RBC model).

Compared to standard RBC theory the only nonstandard input the model requires is the stochastic labor income process, where we note that labor income is just labor endowment scaled by a constant. Aiyagari poses the following $AR(1)$ process for the natural logarithm of labor earnings

$$\log(y_{t+1}) = \theta \log(y_t) + \sigma_\varepsilon \left(1 - \theta^2\right)^{\frac{1}{2}} \varepsilon_{t+1} \tag{7.51}$$

Note that for this process we find

$$corr(\log(y_{t+1}), \log(y_t)) = \frac{Cov(\log(y_{t+1}), \log(y_t))}{\sqrt{Var(\log(y_t)) * Var(\log(y_{t+1}))}} = \theta \tag{7.52}$$

$$Var(\log(y_{t+1})) = \sigma_\varepsilon^2 \tag{7.53}$$

For this income process $\log(y_t)$ can take any value in $(-\infty, \infty)$. Our theory developed so far has assumed a finite state space for $y_t \in Y$ or equivalently $Y^{\log}$ for $\log(y_t)$. The transformation of processes with continuous state space into finite state Markov chains was pioneered in economics by George

Tauchen (1986) and roughly goes like this. First we pick the finite set $Y^{\log}$.
Aiyagari picks the number of states to be 7. Since $\log(y_t)$ can take any value
between $(-\infty, \infty)$, first subdivide the real line into 7 intervals

$$
\begin{aligned}
I_1 &= (-\infty, -\frac{5}{2}\sigma_\varepsilon) \\
I_2 &= [-\frac{5}{2}\sigma_\varepsilon, -\frac{3}{2}\sigma_\varepsilon) \\
I_3 &= [-\frac{3}{2}\sigma_\varepsilon, -\frac{1}{2}\sigma_\varepsilon) \\
I_4 &= [-\frac{1}{2}\sigma_\varepsilon, \frac{1}{2}\sigma_\varepsilon) \\
I_5 &= [\frac{1}{2}\sigma_\varepsilon, \frac{3}{2}\sigma_\varepsilon) \\
I_6 &= [\frac{3}{2}\sigma_\varepsilon, \frac{5}{2}\sigma_\varepsilon) \\
I_7 &= [\frac{5}{2}\sigma_\varepsilon, \infty)
\end{aligned}
\tag{7.54}
$$

and take as state space for log-income the set of "midpoints" of the intervals

$$
Y^{\log} = \{-3\sigma_\varepsilon, -2\sigma_\varepsilon, -\sigma_\varepsilon, 0, \sigma_\varepsilon, 2\sigma_\varepsilon, 3\sigma_\varepsilon\}
\tag{7.55}
$$

The matrix of transition probabilities is then determined as follows. Fix a
state $s_i = \log(y) \in Y^{\log}$ today. The conditional probability of a particular
state $s_j = \log(y') \in Y^{\log}$ tomorrow is then computed by

$$
\pi(\log(y') = s_j | \log(y) = s_i) = \int_{I_j} \frac{e^{-\frac{(x - \theta s_i)^2}{2\sigma_y^2}}}{\sqrt{2\pi}\sigma_y} dx
\tag{7.56}
$$

where $\sigma_y = \sigma_\varepsilon (1 - \theta^2)^{\frac{1}{2}}$. That is, we integrate the Normal distribution with
mean $\log(y)$ and variance $\sigma_\varepsilon^2 (1 - \theta^2)$ over the interval I_j. Doing this for all
states today and tomorrow (this integration has to be done either numeri-
cally or one has to use tables for the pdf of a Normal distribution) yields
the transition matrix π. Now we can find the stationary distribution of π,
hopefully unique, by solving the matrix equation

$$
\Pi = \pi^T \Pi
\tag{7.57}
$$

for Π. Given that the states for $\log(y)$ are given by $Y^{\log}$, we find the state space for levels of income as

$$\tilde{Y} = \{e^{-3\sigma_\varepsilon}, e^{-2\sigma_\varepsilon}, e^{-\sigma_\varepsilon}, 1, e^{\sigma_\varepsilon}, e^{2\sigma_\varepsilon}, e^{3\sigma_\varepsilon}\} \tag{7.58}$$

Now we compute the average labor endowment as

$$\bar{y} = \sum_{y \in \tilde{Y}} y\Pi(y) \tag{7.59}$$

and normalize all states by $\bar{y}$ to arrive at

$$Y = \{y_1, \ldots, y_7\} \tag{7.60}$$
$$\{\frac{e^{-3\sigma_\varepsilon}}{\bar{y}}, \frac{e^{-2\sigma_\varepsilon}}{\bar{y}}, \frac{e^{-\sigma_\varepsilon}}{\bar{y}}, \frac{1}{\bar{y}}, \frac{e^{\sigma_\varepsilon}}{\bar{y}}, \frac{e^{2\sigma_\varepsilon}}{\bar{y}}, \frac{e^{3\sigma_\varepsilon}}{\bar{y}}\}$$

and now the average (and aggregate) labor endowment equals

$$\sum_{y \in Y} y\Pi(y) = 1. \tag{7.61}$$

Of course, this normalization of aggregate labor supply to 1 only fixes the units of account for this economy and thus is completely innocuous. How good the approximation of the continuous process with its finite discretization is depends on the parameter values that one chooses for the persistence θ and dispersion σ_ε (and of course on the number of states allowed in the discretization). Although Aiyagari's Table 1 seems to indicate that the approximation works fine for the parameter values he considers, it is my experience that for high values of θ (close to 1) the discretization has a harder and harder time getting the persistence correct, in addition to the dispersion being somewhat off already for $\theta = 0.9$.

Aiyagari computes stationary equilibria for parameter values in the range

$$\theta \in \{0, 0.3, 0.6, 0.9\} \tag{7.62}$$
$$\sigma_\varepsilon \in \{0.2, 0.4\} \tag{7.63}$$

Quite a few labor economists these days believe, however, that the persistence parameter should be higher than $\theta = 0.9$, closer to 1 (what happens if $\theta = 1$?). But then there is substantial debate whether the simple $AR(1)$ process for the log of labor earnings is the appropriate stochastic process to model this variable. Instead let us turn to the results.

Results

First note that for the Cobb-Douglas production function and $\bar{L} = 1$ we have $Y = K^\alpha$ and

$$r + \delta = \alpha K^{\alpha-1} \tag{7.64}$$

Thus

$$r + \delta = \frac{\alpha Y}{K} = \frac{\alpha \delta}{s} \tag{7.65}$$

where s is the aggregate saving rate. Thus

$$s = \frac{\alpha \delta}{r + \delta} \tag{7.66}$$

and there is a one-to-one mapping between equilibrium interest rates r and equilibrium aggregate savings rates s. For the benchmark case of complete markets we have $r^{CM} = \rho = 4.16\%$ and thus, given the other parameters chosen, $s = 23.7\%$. Aiyagari's Table II contains aggregate savings rates and interest rates for the incomplete markets general equilibrium model under various assumptions about risk aversion (also prudence) σ, persistence θ and dispersion σ_ε. The most important findings are:

- Keeping prudence and dispersion fixed, an increase in the persistence of the income shock leads to increased precautionary saving and bigger overaccumulation of capital, compared to the complete markets benchmark.

- Keeping fixed persistence and dispersion in income, an increase in prudence σ leads to more precautionary saving and more severe overaccumulation of capital

- Keeping prudence and income persistence constant, an increase in the dispersion of the income process leads to more precautionary saving and more severe overaccumulation of capital.

- Quantitatively, the biggest effect on precautionary saving seem to stem from the persistence parameter. In particular, for high income persistence (and high dispersion) the aggregate saving rate is 14 percentage points higher than in the complete markets benchmark.

The other implications of the model that Aiyagari discusses, the welfare cost of idiosyncratic income fluctuations (as well as the welfare benefit from access to self-insurance), as well as the predictions of the model with respect to the wealth and consumption distribution will be addressed once the model is enriched by aggregate fluctuations.

7.2 An Incomplete Markets Model with Unsecured Debt and Equilibrium Default

Basic idea: incorporate household default on uncollateralized debt into standard SIM model. Institutional motivation: chapter 7 of U.S. bankruptcy code. Discussion based on simplified version of Chatterjee et. al., (EC, 2007). Focus is on explaining main mechanism of the model, not to present the most general version of Chatterjee et al. (2007).

7.2.1 Model Overview

- Starting point: standard SIM GE model as in Aiyagari (1994).

- Production firms completely standard.

- Continuum (of measure 1) of infinitely lived households solve standard income fluctuation problem, can borrow at a loan-size dependent interest rate schedule, can default.

- Competitive financial intermediaries extend loans, break even after accounting for losses from default.

- Loan interest schedule determined endogenously in equilibrium.

7.2.2 Institutional Details of Bankruptcy

- Bankruptcy flag $h \in \{0, 1\}$. Summarizes past default behavior. h is a state variable.

- Current bankruptcy decision $d \in \{0, 1\}$.

- Consequence of $d = 1$: Can't save (and of course can't borrow) in current period.

- Consequence of $h = 1$:

  - Can't borrow.
  - Fraction $\gamma \in (0, 1)$ of income lost.

State Transitions

- If $h = 0$, then household can choose $d \in \{0, 1\}$. If $d = 0$, then $h' = 0$. If $d = 1$, then $h' = 1$.

- If $h = 1$, then $d \in \{0\}$. With probability λ, $h' = 1$. With probability $1 - \lambda$, $h' = 0$.

7.2.3   Household Problem in Recursive Formulation

- Individual State variables (a, y, h).

- Assets $a \in A = \mathbb{R}$.

- Exogenous idiosyncratic income $y \in Y = [y_{\min}, y_{\max}]$, *iid* over time, identically distributed across households, with measure $\pi(.)$. Assume $y_{\min} > 0$ and $y_{\max} < \infty$ and a law of large numbers.

- Bankruptcy flag $h \in H = \{0, 1\}$.

- Population distribution over states (a, y, h) described by measure Φ over measurable space $(Z, \mathcal{B}(Z))$ where $Z = A \times Y \times Z$ and $\mathcal{B}(Z) = \mathcal{B}(A) \times \mathcal{B}(Y) \times \mathcal{P}(Z)$.

Household Budget Sets

- Household take as given a wage w and loan price function $q(a, y, h; a')$.

- Households choose consumption c, assets (loans if negative) a' as well as d.

- Budget set of admissible (c, a') pairs depends on (a, y, h) and on $d \in \{0, 1\}$.

- If $h = 0$ and $d = 0$:

$$B(a, y, h = 0; d = 0) = \{c \geq 0, a' \in A :$$
$$c + q(a, y, h; a')a' \leq wy + a\}.$$

Note that for some $(a, y) \in A \times Y$ this set might be empty (and the household will need to default).

- If $h = 0$ and $d = 1$:

$$B(a, y, h = 0; d = 1) = \{c \geq 0, a' = 0 : \; c \leq wy\}.$$

- If $h = 1$:

$$B(a, y, h = 1) = \{c \geq 0, a' \geq 0 :$$
$$c + q(a, y, h; a')a' \leq (1 - \gamma)wy + a\}$$

Dynamic Programming Problem of Households

- Value function $v(a, y, h)$.

- For $h = 0$: If $B(a, y, h = 0; d = 0) = \varnothing$, then "involuntary default and

$$v(a, y, h = 0) = \left\{ u(wy) + \beta \int v(0, y', h' = 1)d\pi(y') \right\}$$

If $B(a, y, h = 0; d = 0) \neq \varnothing$, then decision between repayment and "voluntary" (strategic) default:

$$v(a, y, h = 0) = \max_{d \in \{0,1\}}$$

$$\max_{(c,a') \in B(a,y,h=d=0)} u(c) + \beta \int v(a', y', h' = 0)d\pi(y')$$
$$u(wy) + \beta \int v(0, y', h' = 1)d\pi(y')$$

- For $h = 1$:

$$v(a, y, h = 1) = \max_{(c,a') \in B(a,y,h=1;d=0)}$$

$$\left\{ u(c) + \beta \left[\begin{array}{c} \lambda \int v(a', y', h' = 1)d\pi(y') + \\ (1 - \lambda) \int v(a', y', h' = 0)d\pi(y') \end{array} \right] \right\}$$

- Solution is a value function $v(a, y, h)$ and optimal policy functions $c(a, y, h)$, $a'(a, y, h)$ and $d(a, y, h)$.

7.2.4   Production Firms

- Measure 1 of identical, competitive firms.

- Constant returns to scale technology $F(K, L)$

- Profit maximization implies

$$
\begin{aligned}
w &= F_L(K, L) \\
r &= F_K(K, L) - \delta
\end{aligned}
$$

7.2.5   Financial Intermediaries

- Representative financial intermediary owns the capital stock K and buys new capital K'. Earns net return r on capital.

- They issue loans (deposits). Perfect competition loan by loan type $(a, y, h; a')$

- Consider a loan of size $a' < 0$ to a type (a, y, h) household (if $a' \geq 0$ call it a deposit).

  - Price $q(a, y, h; a')$. If $a' < 0$, intermediary gives household $q(a, y, h; a')a'$ today for promise to repay a' tomorrow.
  - (Equilibrium) default probability $p(a, y, h; a')$
  - Number $n(a, y, h; a')$ of loans of type $(a, y, h; a')$.

- Maximization problem

$$
\max_{n(a,y,h;a') \geq 0} \int_{(a,y,h;a')} n(a, y, h; a') *
$$
$$
\left[q(a, y, h; a')a' - \frac{[1 - p(a, y, h; a')]\, a'}{1 + r} \right]
$$

7.2.6   Stationary Recursive Competitive Equilibrium

Definition 49 *A Stationary RCE is a value function $v(a, y, h)$ and policy functions $c(a, y, h), a'(a, y, h), d(a, y, h)$ for the households, aggregate capital and labor (K, L), quantities of loan/deposit contracts $n(a, y, h; a')$, default frequencies $p(a, y, h; a')$, loan prices $q(a, y, h; a')$, factor prices (w, r) and a probability measure Φ such that:*

1. *Given (w, q), v solves the household Bellman equation and c, a', d are the optimal policy functions.*

2. *Given (w, r), (K, L) satisfy the production firms' first order condition.*

3. *Given (q, p, r), n solves the financial intermediary's problem.*

4. *Consistency of default probabilities: For all $(a, y, h; a')$*

$$p(a, y, h; a') = \int d(a', y', h' = 0)\pi(y'|y)dy'$$

5. *Loan markets clearing: For all $(a, y, h; a')$*

$$n(a, y, h; a') = \mathbf{1}_{\{a'(a,y,h)=a'\}}\Phi(a, y, h)$$

6. *Labor market clearing:*

$$L = \int yd\Phi = \int yd\pi$$

7. *Capital market clearing:*

$$K = \int \int q(a, y, h; a')a'n(a, y, h; a')da'd\Phi$$

8. *Goods market clearing:*

$$\int c(a, y, h)d\Phi + \delta K = F(K, L) - \gamma w \int y\Phi(da, dy, h = 1)$$

9. *Φ is an invariant probability measure of the associated Markov transition function induced by π, a' and d.*

Remark 50 *Paper proves existence of Stationary RCE by restricting $a' \in A$, with A a finite set.*

7.2.7   Characterization of Household Default Decision

- Assume that a household indifferent between defaulting or nor decides to default (CCNR call this the maximum default set). Recall: default policy $d(a, y, h)$.

- Along h-dimension: for $h = 1$, default is not an option. For $h = 0$, there is a choice.

- Along the y-dimension: for a given loan level a, define as default set
$$D(a) = \{y \in Y : d(a, y, h = 0) = 1\}$$

Proposition: $D(a)$ is either empty or a closed interval.

- Along the a-dimension:

  - If $a \geq 0$, then $D(a)$ is empty [it is never optimal to "default" on positive assets].
  - Let $\tilde{a} < a < 0$, then $D(a) \subseteq D(\tilde{a})$ [default set is expanding in indebtedness].

- Implied default probability on a loan a' for a type $(a, y, h = 0)$:
$$p(a, y, h; a') = \pi(D(a')) = p(a')$$

Proposition: Since y' is iid, the default probability is just a function of loan size a'. Furthermore $p(a') = 0$ for all $a' \geq 0$.

7.2.8   Characterization of Equilibrium Loan Interest Rate Function

- Intermediaries' problem

$$\max_{n(a,y,h;a') \geq 0} \sum_{(a,y,h;a')} n(a, y, h; a') *$$
$$\left[q(a, y, h; a')a' - \frac{[1 - p(a, y, h; a')]\, a'}{1 + r} \right]$$
$$= \max_{n(a,y,h;a') \geq 0} \sum_{(a,y,h;a')} n(a, y, h; a')a' *$$
$$\left[q(a, y, h; a') - \frac{[1 - p(a, y, h; a')]}{1 + r} \right]$$

- Thus if $n(a, y, h; a') > 0$ then

$$q(a, y, h; a') = \frac{1 - p(a, y, h; a')}{1 + r}$$

  and we assume that this is also the price for contracts not traded in equilibrium (for $n(a, y, h; a') = 0$).

The previous characterization of the household default set implies that (see their theorem 6):

1. Since $p(a, y, h; a') = p(a')$ we have $q(a, y, h; a') = q(a')$.

2. Since for all $a' \geq 0$ we have $p(a') = 0$, for those a'

$$q(a') = \frac{1}{1 + r}$$

3. Since for $\tilde{a}' < a' < 0$, $p(\tilde{a}') > p(a')$ we have

$$q(\tilde{a}') \leq q(a')$$

   that is, loan interest rates are increasing in loan size.

4. There exists $\bar{a}' < 0$ small enough such that

$$q(\bar{a}') = 0.$$

   The loan size $\bar{a}'$ is an effective borrowing limit.

- **Augmenting the Model:** Constant probability of death ρ.

- Introduce persistent type s of households. Assume that s follows Markov chain with transitions $\pi(s'|s)$.

- Make income persistent by assuming and $y \sim \pi_s(y)$. Interpret s as partially capturing socioeconomic characteristics: model blue collar vs. white collar households.

- In model households default because of bad earnings shocks. In data they also default because of

    - Large medical bills: introduce nondiscretionary health spending shocks $\zeta(s)$.
    - Divorce: introduce preference shocks $\eta(s)$.

7.2.9 Bringing the Model to the Data: Calibration and Estimation

- Challenge is to simultaneously account for high debt levels and high default probabilities. Medical and divorce shocks necessary.

- Calibration to

  - Aggregate statistics: standard since production side is neoclassical growth model.
  - Earnings and wealth distributions: choose the appropriate earnings process.
  - Debt and bankruptcy statistics: choose medical and divorce shocks appropriately.

7.2.10   Quantitative Predictions of the Model

- Table 1: Reasons for filing for bankruptcy in data.

- Table 2: Extended version of the model is consistent with incidence of bankruptcy filings, average debt of those in debt.

- Figure 3: Model matches U.S. wealth distribution well.

- Figure 5, table 5: who defaults?

- Figure 6: implied loan prices (interest rates).

Add private information (Narajabad, Athreya et al.)
Long term contracts (Drozd and Nosal)
Collateralized debt (mortgages) and default (foreclosures): Jeske and Krueger, Garriga and Schlagenhauf

7.3   Unexpected Aggregate Shocks and Transition Dynamics

In this section we consider hypothetical thought experiments of the following form. Suppose the economy is in a stationary equilibrium, with a given

government policy and all other exogenous elements that define preferences, endowments and technology. Now, unexpectedly, either government policy or some exogenous elements of the economy (such as the labor productivity process) change. This change was completely unexpected by all agents of the economy (a zero probability event), so that no anticipation actions were taken by any agent. The exogenous change may be either transitory or permanent; for the general discussion to follow this does not make a difference. We want to study the transition path induced by the exogenous change, from the old stationary equilibrium to a new stationary equilibrium (which may coincide with the old stationary equilibrium in case the exogenous change is of transitory nature, or may differ from it in case the exogenous change is permanent.

For concreteness, suppose that the economy to start with is the standard Aiyagari economy we studied in the previous section. As an example, we consider as exogenous unexpected change the sudden permanent introduction of a capital income tax at rate τ. The receipts are rebated lump-sum to households as government transfers T. The initial policy is characterized by $\tau = T = 0$. Obviously, since a tax on capital income changes households' savings decisions, we expect that, due to the policy change, individual behavior and thus aggregate variables such as the interest rate, wage rate and the capital stock change. We would also hope that, over time, the economy settles down to its new stationary equilibrium associated with the capital income tax. But since the economy starts, pre-reform, with an aggregate state (aggregate capital, wealth distribution) not equal to the final stationary equilibrium, one would expect that it requires time for the economy to settle down to its new stationary equilibrium. In other words, there will be a nontrivial transition path induced by the reform.

7.3.1 Definition of Equilibrium

We now want to define an equilibrium that allows for such transition path and then outline how one would possibly compute such a transition path. As should be clear from the previous discussion that no analytical characterization of the transition path is available in general, so that we have to rely on computational analysis.

Since, on the aggregate level, the transition path is characterized by a deterministic sequence of prices, quantities and distributions we will cast the definition and solution of the model in sequential notation, with the

household decision problem still being formulated recursively. Let $Z = Y \times \mathbf{R}_+$ be the set of all possible (y_t, a_t). Let $\mathcal{B}(\mathbf{R}_+)$ be the Borel σ-algebra of $\mathbf{R}_+$ and $\mathcal{P}(Y)$ be the power set of Y. Finally let $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(\mathbf{R}_+)$ and $\mathbf{M}$ be the set of all finite measures on the measurable space $(Z, \mathcal{B}(Z))$.

First let's write down the household problem

$$
\begin{aligned}
v_t(a, y) &= \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) v_{t+1}(a', y') \\
\text{s.t. } c + a' &= w_t y + (1 + (1 - \tau_t) r_t) a + T_t
\end{aligned}
$$

Note that value functions are now functions of time also, since aggregate prices and policies may change over time.

Definition 51 *Given the initial capital stock K_0 and initial distribution Φ_0, and fiscal legislation $\{\tau_t\}_{t=0}^{\infty}$ a competitive equilibrium is a sequence of individual functions for the household $\{v_t, c_t, a_{t+1} : Z \to \mathbf{R}\}_{t=0}^{\infty}$, sequences of production plans for the firm $\{L_t, K_t\}_{t=0}^{\infty}$, factor prices $\{w_t, r_t\}_{t=0}^{\infty}$, government transfers $\{T_t\}_{t=0}^{\infty}$, and a sequence of measures $\{\Phi\}_{t=1}^{\infty}$ such that, for all t,*

1. *(Maximization of Households): Given $\{w_t, r_t\}$ and $\{T_t, \tau_t\}$ the functions $\{v_t\}$ solve Bellman's equation for period t and $\{c_t, a_{t+1}\}$ are the associated policy functions*

2. *(Marginal Product Pricing): The prices w_t and r_t satisfy*

$$w_t = F_L(K_t, L_t) \tag{7.67}$$

$$r_t = F_K(K_t, L_t) - \delta. \tag{7.68}$$

3. *(Government Budget Constraint):*

$$T_t = \tau_t r_t K_t \tag{7.69}$$

   for all $t \geq 0$.

4. *(Market Clearing):*

$$\int c_t(a_t, y_t) d\Phi_t + K_{t+1} = F(K_t, L_t) + (1 - \delta) K_t \tag{7.70}$$

$$L_t = \int y_t d\Phi_t \tag{7.71}$$

$$K_{t+1} = \int a_{t+1}(a_t, y_t)d\Phi_t \tag{7.72}$$

5. *(Aggregate Law of Motion):*[7]

$$\Phi_{t+1} = \Gamma_t(\Phi_t) \tag{7.75}$$

Definition 52 *A stationary equilibrium is an equilibrium such that all elements of the equilibrium that are indexed by t are constant over time.*

7.3.2 Computation of the Transition Path

We are interested in computing the following equilibrium. At time 0 the economy is in the stationary equilibrium associated with $\tau_0 = 0$ and associated distribution Φ_0 (from this distribution all other aggregate variables, such as K_t, r_t, w_t can easily be derived) and associated value function v_0 and policy functions c_0, a_1. Now at time $t = 1$ the policy changes permanently, $\tau_t = \tau > 0$ for all $t \geq 1$. Let the new stationary equilibrium associated with τ be denoted by Φ_∞, with associated value function v_∞ and policy functions c_∞, a_∞ In the previous section we discussed how to compute such stationary equilibria. Now we want to compute the entire transition path and trace out the welfare consequences of such a policy innovation.

The key idea is to *assume* that after T periods the transition from the old to the new stationary equilibrium is completed. We will discuss below how to choose T, but heuristically it should be large enough so that the economy

---

[7]The functions Γ_t can be written explicitly as follows. Define Markov transition functions $Q_t : Z \times \mathcal{B}(Z) \to [0,1]$ induced by the transition probabilities π and the optimal policy $a_{t+1}(a, y)$ as

$$Q_t((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) \text{ if } a_{t+1}(a, y) \in \mathcal{A} \\ \quad\quad 0 \text{ else} \end{cases} \tag{7.73}$$

for all $(y, a) \in Z$ and all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$. Then

$$\Phi_{t+1}(\mathcal{A}, \mathcal{Y}) = [\Gamma_t(\Phi_t)](\mathcal{A}, \mathcal{Y}) = \int Q_t((a, y), (\mathcal{A}, \mathcal{Y}))d\Phi_t \tag{7.74}$$

for all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$.

had enough time to settle down to the new stationary equilibrium. The key insight then is to realize that if $v_T = v_\infty$, then for a given sequence of prices $\{r_t, w_t\}_{t=1}^T$ the household problem can be solved backwards (note that this is true independent of wether people live forever or not). This suggests the following algorithm.

Algorithm 53   *1. Fix T*

    2. Compute the stationary equilibrium $\Phi_0, v_0, c_0, a_0, r_0, w_0, K_0$ associated with $\tau = \tau_0 = 0$

    3. Compute the stationary equilibrium $\Phi_\infty, v_\infty, c_\infty, a_\infty, r_\infty, w_\infty, K_\infty$ associated with $\tau_\infty = \tau$. Assume that

$$\Phi_T, v_T, c_T, a_T, r_T, w_T, K_T = \Phi_\infty, v_\infty, c_\infty, a_\infty, r_\infty, w_\infty, K_\infty$$

    4. Guess a sequence of capital stocks $\{\hat{K}_t\}_{t=1}^{T-1}$ Note that since the capital stock at time $t = 1$ is determined by decisions at time 0, $\hat{K}_1 = K_0$. Also note that $L_t = L_0 = \bar{L}$ is fixed. Thus with the guesses on the capital stock we also obtain $\{\hat{r}_t, \hat{w}_t\}_{t=1}^{T-1}$ determined by

$$\begin{aligned} \hat{w}_t &= F_L(\hat{K}_t, \bar{L}) \\ \hat{r}_t &= F_K(\hat{K}_t, \bar{L}) - \delta \\ \hat{T}_t &= \tau_t \hat{r}_t \hat{K}_t. \end{aligned}$$

    5. Since we know $v_T(a, y)$ and $\{\hat{r}_t, \hat{w}_t, \hat{T}_t\}_{t=1}^{T-1}$ we can solve for $\{\hat{v}_t, \hat{c}_t, \hat{a}_{t+1}\}_{t=1}^{T-1}$ backwards.

    6. With the policy functions $\{\hat{a}_{t+1}\}$ we can define the transition laws $\{\hat{\Gamma}_t\}_{t=1}^{T-1}$. But since we know $\Phi_0 = \Phi_1$ from the initial stationary equilibrium, we can iterate the distributions forward

$$\hat{\Phi}_{t+1} = \hat{\Gamma}_t(\hat{\Phi}_t)$$

    for $t = 1, \ldots, T - 1$.

    7. With $\{\hat{\Phi}_t\}_{t=1}^T$ we can compute

$$\hat{A}_t = \int a \, d\hat{\Phi}_t$$

    for $t = 1, \ldots, T$.

8. *Check whether*

$$\max_{1 \leq t < T} \left| \hat{A}_t - \hat{K}_t \right| < \varepsilon$$

If yes, go to 9. If not, adjust your guesses for $\{\hat{K}_t\}_{t=1}^{T-1}$ in 4.

9. *Check whether $\left\| \hat{\Phi}_T - \Phi_T \right\| < \varepsilon$. If yes, the transition converges smoothly into the new steady state and we are done and should save $\{\hat{v}_t, \hat{a}_{t+1}, \hat{c}_t, \hat{\Phi}_t, \hat{r}_t, \hat{w}_t, \hat{K}_t\}$. If not, go to 1. and increase T.*

This procedure determines all the variables we are interested in along the transition path: aggregate variables such as r_t, w_t, Φ_t, K_t and individual decision rules c_r, a_{t+1}. It turns out that the value functions v_t enable us to make statements about the welfare consequences of a tax reform for which we just have computed the transition path.

7.3.3 Welfare Consequences of the Policy Reform

¿From our previous algorithm we obtain the sequence of value functions $\{v_t\}_{t=0}^{T}$. Remember the interpretation of the value functions: $v_0(a, y)$ is the expected lifetime utility of an agent with assets a and productivity shock y at time 0 in the initial stationary equilibrium, that is, for a person that thinks he will live in the stationary equilibrium with $\tau = 0$ forever. Similarly $v_1(a, y)$ is the expected lifetime utility of an agent with assets a and productivity y that has just been informed that there is a permanent tax change. This lifetime utility takes into account all the transition dynamics through which the agent is going to live. Finally $v_T(a, y) = v_\infty(a, y)$ is the lifetime utility of an agent with characteristics (a, y) born in the new stationary equilibrium (i.e. of an agent that does not live through the transition).

So in principle we could use v_0, v_1 and v_T to determine the welfare consequences from the reform. The problem, of course, is that utility is an ordinal concept, so that comparing $v_0(a, y)$ with $v_1(a, y)$ we can only determine whether agent (a, y) gains or loses from the reform, but we cannot meaningfully discuss how big these gains or losses are.

Now suppose that the period utility function is of $CRRA$ form

$$U(c) = \frac{c^{1-\sigma}}{1 - \sigma}$$

Consider the optimal consumption allocation in the initial stationary equilibrium, in sequential formulation, $\{c_s\}_{s=0}^\infty$, where for simplicity we have suppressed the explicit dependence of the history of productivity shocks. Then

$$v_0(a,y) = E_0 \sum_{s=0}^\infty \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

Now suppose we increase consumption in each date, in each state, in the old stationary equilibrium, by a fraction g, so that the new allocation is $\{(1+g)c_s\}_{s=0}^\infty$. The lifetime utility from that consumption allocation is

$$v_0(a,y;g) = E_0 \sum_{s=0}^\infty \beta^t \frac{[(1+g)c_t]^{1-\sigma}}{1-\sigma} = (1+g)^{1-\sigma} E_0 \sum_{s=0}^\infty \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = (1+g)^{1-\sigma} v_0(a,y)$$

Obviously $v_0(a,y;g=0) = v_0(a,y)$. If we want to quantify the welfare consequences of the policy reform for an agent of type (a,y) we can ask the following question: by what percent g do we have to increase consumption in the old stationary equilibrium, in each date and state, for the agent to be indifferent between living in the old stationary equilibrium and living through the transition induced by the policy reform.[8] This percent g solves

$$v_0(a,y;g) = v_1(a,y)$$

or

$$
\begin{aligned}
(1+g)^{1-\sigma} v_0(a,y) &= v_1(a,y) \\
g(a,y) &= \left[\frac{v_1(a,y)}{v_0(a,y)}\right]^{\frac{1}{1-\sigma}} - 1
\end{aligned}
$$

Evidently, this number is bigger than zero if and only if $v_1(a,y) > v_0(a,y)$, in which case the agent benefits from the reform, and $g(a,y)$ measures by how much, in consumption terms. Note that the number $g(a,y)$ depends on an agents' characteristics (one would expect households with a lot of assets to lose badly, households with little assets to lose not much or to even gain -remember that taxes are lump-sum redistributed). Also note that we only need to know $v_0(a,y)$ and $v_1(a,y)$ to compute this number, but our computation of the transition path gives us the value functions v_0, v_1 anyhow.

---

[8]Lucas (1978) proposed this consumption equivalent variation measure in order to assess the welfare costs of business cycles.

The computation of consumption equivalent variation, explicitly taking into account the transition path, is the theoretically correct way to assess the welfare consequences of a reform. Often studies try to assess the steady state welfare consequences of a policy reform, in particular if these studies do not explicitly compute transition paths. Even though I think that these welfare numbers are often not very meaningful, let us quickly describe the procedure. Obviously one can compute

$$g_{ss}(a,y) = \left[\frac{v_T(a,y)}{v_0(a,y)} \right]^{\frac{1}{1-\sigma}} - 1$$

with the interpretation that this number is the welfare gain of an agent being born with characteristics.(a,y) into the new as opposed to the old stationary equilibrium. If we want to study welfare consequences of a policy reform for households before their identity is revealed, we may define as

$$g_{ss} = \left[\frac{\int v_T(a,y)d\Phi_T}{\int v_0(a,y)d\Phi_0} \right]^{\frac{1}{1-\sigma}} - 1$$

the expected steady state welfare gain. Here $\int v_T(a,y)d\Phi_T$ is the expected lifetime utility of an agent in the new stationary equilibrium, before the agent knows with what characteristics he or she will be born (behind the veil of ignorance). The quantity $\int v_0(a,y)d\Phi_0$ is defined correspondingly. Even though these measures are not hard to compute (in fact, one does not need to compute the transition path to compute these numbers), they do ignore the fact that it takes time to go from the old stationary equilibrium to the new one. For the policy example at hand, the increase of the capital tax is likely to induce a decline in the capital stock, thus lower aggregate consumption and thus steady state welfare losses. What these losses ignore is that along the transition path part of the capital stock is being eaten, with associated consumption and welfare derived from it. Therefore welfare measures based on steady state comparisons may give only fairly limited information about the true welfare consequences of policy reforms.

7.4 Aggregate Uncertainty and Distributions as State Variables

In section 7.1 we discussed a model where agents faced significant amounts of idiosyncratic uncertainty, which they could, by assumption, only self-insure

against by precautionary saving via risk-less bonds. In the aggregate, the economy was stationary in that output, investment and aggregate consumption were constant over time. Thus by construction the model could not speak to how aggregate consumption, saving and investment fluctuates over the business cycle.

In this section we introduce aggregate uncertainty, in addition to idiosyncratic uncertainty, into the model. In contrast to idiosyncratic uncertainty, which is in principle insurable (but insurance against which we ruled out), there is no mutual insurance against aggregate shock, unless one analyzes a two-country world where the two countries have aggregate shocks which are imperfectly correlated.

7.4.1 The Model

In the spirit of real business cycle theory aggregate shocks take the form of productivity shocks to the aggregate production function

$$Y_t = s_t F(K_t, L_t) \tag{7.76}$$

where $\{s_t\}$ is a sequence of random variables that follows a finite state Markov chain with transition matrix π. We will follow Krusell and Smith (1998) and assume that the aggregate productivity shock can take only two values

$$s_t \in \{s_b, s_g\} = S \tag{7.77}$$

with $s_b < s_g$ and denote by $\pi(s'|s)$ the conditional probability of the aggregate state transiting from s today to s' tomorrow. Krusell and Smith (1998) interpret s_b as an economic recession and s_g as an expansion.

The idiosyncratic labor productivity y_t is assumed to take only two values

$$y_t \in \{y_u, y_e\} = Y \tag{7.78}$$

where $y_u < y_e$ stands for the agent being unemployed (having low labor productivity) and y_e stands for the agent being employed. Krusell and Smith attach the employed-unemployed interpretation to the idiosyncratic labor productivity variable, an interpretation which will be important in the calibration section. Obviously the probability of being unemployed is higher during recessions than during expansions, and the Markov chain governing idiosyncratic labor productivity should reflect this dependence of idiosyncratic uncertainty on the aggregate state of the economy.

In particular, let denote by

$$\pi(y', s'|y, s) \geq 0 \tag{7.79}$$

the conditional probability of an agent having individual productivity tomorrow of y' and the aggregate state being s' tomorrow, conditional on the individual and aggregate state y and s today. For example $\pi(y_u, s_b|y_e, s_g)$ is the probability of getting laid off tomorrow in a recession if the economy is booming today and the individual is employed. Consistency with aggregate transition probabilities requires that

$$\sum_{y' \in Y} \pi(y', s'|y, s) = \pi(s'|s) \text{ for all } y \in Y \text{ and all } s, s' \in S \tag{7.80}$$

i.e. the probability $\sum_{y' \in Y} \pi(y', s'|y, s)$ of going from a particular individual state y and aggregate state s to *some* individual state y' and a particular aggregate state s' equals the aggregate transition probability $\pi(s'|s)$.

We again assume a law of large numbers, so that idiosyncratic uncertainty averages out, and only aggregate uncertainty determines the number of agents in states $y \in Y$. Obviously this number will vary with today's aggregate state s. Krusell and Smith *assume* that the share of unemployed people only depends on s (but not on past aggregate states). Then let by $\Pi_s(y)$ denote the fraction of the population in idiosyncratic state y if aggregate state is s; e.g. $\Pi_{s_b}(y_u)$ is the deterministic number of unemployed people if the economy is in a recession. The assumption that $\Pi_s(y)$ only depends on s imposes further restriction on the Markov transition matrix π:

$$\Pi_{s'}(y') = \sum_{y \in Y} \frac{\pi(y', s'|y, s)}{\pi(s'|s)} \Pi_s(y) \text{ for all } s, s' \in S \tag{7.81}$$

Suppose the economy is in a boom today, $s = s_g$. The fraction of people unemployed in a recession tomorrow, $\Pi_{s'=s_b}(y' = y_u)$ equals the fraction of employed people today $\Pi_{s=s_g}(y = y_e)$ who get laid off in the recession, $\pi(y_u|s_b, y_e, s_g) = \frac{\pi(y_u, s_b|y_e, s_g)}{\pi(s_b|s_g)}$, plus the fraction of unemployed people today $\Pi_{s=s_g}(y = y_u)$ who remain unemployed, $\pi(y_u|s_b, y_u, s_g) = \frac{\pi(y_u, s_b|y_u, s_g)}{\pi(s_b|s_g)}$. The same restriction applies for all other states.

The exogenous Markov chain driving the economy is therefore described by the two sets of cardinality 2, S and Y and the joint 4×4 transition matrix

for idiosyncratic and aggregate productivity π

$$\pi = \begin{pmatrix} \pi(y_u, s_b | y_u, s_b) & \pi(y_u, s_b | y_e, s_b) & \pi(y_u, s_b | y_u, s_g) & \pi(y_u, s_b | y_e, s_g) \\ \pi(y_e, s_b | y_u, s_b) & \pi(y_e, s_b | y_e, s_b) & \pi(y_e, s_b | y_u, s_g) & \pi(y_e, s_b | y_e, s_g) \\ \pi(y_u, s_g | y_u, s_b) & \pi(y_u, s_g | y_e, s_b) & \pi(y_u, s_g | y_u, s_g) & \pi(y_u, s_g | y_e, s_g) \\ \pi(y_e, s_g | y_u, s_b) & \pi(y_e, s_g | y_e, s_b) & \pi(y_e, s_g | y_u, s_g) & \pi(y_e, s_g | y_e, s_g) \end{pmatrix}$$

$$(7.82)$$

In the previous section we defined, proved existence of and computed a stationary equilibrium, in which prices and the cross-sectional distribution of assets (i.e. all macroeconomic variables of interest) were constant over time. Obviously, with aggregate shocks there is no hope of such an equilibrium to exist. This makes our extension interesting in the first place, because it allows business cycle fluctuations, but also will impose crucial complications for the computation of the economy.

Since the cross-sectional distribution Φ of assets will vary with the aggregate shock, we have to include it as a state variable in our formulation of a recursive equilibrium. The aggregate state variables also include the aggregate productivity shock s, since this shock determines aggregate productivity and hence aggregate wages and interest rates.

For the recursive formulation of the household problem we note that the individual state variable is composed of individual asset holdings and income shocks (a, y), whereas the aggregate state variables include the aggregate shock and the distribution of assets (s, Φ). Hence the households' problem in recursive formulation is

$$v(a, y, s, \Phi) = \max_{c, a' \geq 0} \left\{ U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi(y', s' | y, s) v(a', y', s', \Phi') \right\}$$

$$s.t. \tag{7.83}$$

$$c + a' = w(s, \Phi) y + (1 + r(s, \Phi)) a \tag{7.84}$$

$$\Phi' = H(s, \Phi, s') \tag{7.85}$$

Note that s' is a determinant of Φ' since it specifies how many agents have idiosyncratic shock $y' = y_u$ and how many agents have $y' = y_e$. We have the following definition of a recursive competitive equilibrium

Definition 54 *A recursive competitive equilibrium is a value function* $v :$ $Z \times S \times \mathcal{M} \to R$, *policy functions for the household* $a' : Z \times S \times \mathcal{M} \to R$ *and* $c : Z \times S \times \mathcal{M} \to R$, *policy functions for the firm* $K : S \times \mathcal{M} \to R$ *and*

$L : S \times \mathcal{M} \to R$, pricing functions $r : S \times \mathcal{M} \to R$ and $w : S \times \mathcal{M} \to R$ and an aggregate law of motion $H : S \times \mathcal{M} \times S \to \mathcal{M}$ such that

1. *v, a', c are measurable with respect to $\mathcal{B}(S)$, v satisfies the household's Bellman equation and a', c are the associated policy functions, given $r()$ and $w()$*

2. *K, L satisfy, given $r()$ and $w()$*

$$
\begin{aligned}
r(s, \Phi) &= F_K(K(s, \Phi), L(s, \Phi)) - \delta & (7.86) \\
w(s, \Phi) &= F_L(K(s, \Phi), L(s, \Phi)) & (7.87)
\end{aligned}
$$

3. *For all $\Phi \in \mathcal{M}$ and all $s \in S$*

$$
K(H(s, \Phi)) = \int a'(a, y, s, \Phi) d\Phi \tag{7.88}
$$

$$
L(s, \Phi) = \int y d\Phi \tag{7.89}
$$

$$
\int c(a, y, s, \Phi) d\Phi + \int a'(a, y, s, \Phi) d\Phi \tag{7.90}
$$

$$
= F(K(s, \Phi), L(s, \Phi)) + (1 - \delta) K(s, \Phi) \tag{7.91}
$$

4. *The aggregate law of motion H is generated by the exogenous Markov process π and the policy function a' (as described below)*

Again define the transition function $Q_{\Phi, s, s'} : Z \times \mathcal{B}(Z) \to [0, 1]$ by

$$
Q_{\Phi, s, s'}((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y', s'|y, s) & \text{if } a'(a, y, s, \Phi) \in \mathcal{A} \\ 0 & \text{else} \end{cases} \tag{7.92}
$$

The aggregate law of motion is then given by

$$
\Phi'(\mathcal{A}, \mathcal{Y}) = (H(s, \Phi, s'))(\mathcal{A}, \mathcal{Y}) = \int Q_{\Phi, s, s'}((a, y), (\mathcal{A}, \mathcal{Y})) \Phi(da \times dy) \tag{7.93}
$$

Before describing the computational strategy for the recursive equilibrium some comments are in order:

1. One should first define a sequential markets equilibrium. Such an equilibrium will in general exist, but it is impossible to compute such an equilibrium directly. No claim of uniqueness of a sequential markets equilibrium can be made.

2. For the recursive equilibrium defined above Krusell and Smith assert
   that the current cross-sectional asset distribution and the current shock
   are sufficient aggregate state variables. This is not formally proved,
   and such a proof would be very hard. What one has to prove is that a
   recursive equilibrium generates a sequential equilibrium in the following
   sense: Cross-sectional distributions are generated, starting from the
   initial condition (s_0, Φ_0) as follows

$$\Phi_1(s^1) = H(s_0, \Phi_0, s_1) \tag{7.94}$$
$$\Phi_{t+1}(s^{t+1}) = H(s_t, \Phi_t(s^t), s_{t+1}) \tag{7.95}$$

   prices are generated by

$$r_t(s^t) = r(s_t, \Phi_t(s^t)) \tag{7.96}$$
$$w_t(s^t) = w(s_t, \Phi_t(s^t)) \tag{7.97}$$

   and, starting from initial conditions (a_0, y_0), an individual households'
   allocation is generated by

$$c_0(a_0, y_0, s_0) = c(a_0, y_0, s_0, \Phi_0) \tag{7.98}$$
$$a_1(a_0, y_0, s_0) = a'(a_0, y_0, s_0, \Phi_0) \tag{7.99}$$

   and in general recursively

$$c_t(a_0, y^t, s^t) = c(a_t(a_0, y^{t-1}, s^{t-1}), y_t, s_t, \Phi_t) \tag{7.100}$$
$$a_{t+1}(a_0, y^t, s^t) = a'(a_t(a_0, y^{t-1}, s^{t-1}), y_t, s_t, \Phi_t) \tag{7.101}$$

   Similarly optimal choices of the firm can be generated. Thus, a given
   recursive equilibrium generates sequential allocations and prices; it re-
   mains to be verified that these prices are in fact a sequential equilib-
   rium.

3. Finally, the issue of existence of a *recursive* equilibrium arises. We
   know that, since a sequential equilibrium in general exists, there is a
   state space large enough such that a recursive equilibrium (recursive
   in that state space) exists. So the issue is whether a recursive equilib-
   rium in which the aggregate state only contains the current shock and
   the current wealth distribution does exist. Although this state space
   seems "natural" in some sense, there is no guarantee of existence of
   such a recursive equilibrium. The analysis of this economy is purely
   computational in spirit as neither the existence, uniqueness, stability or
   qualitative features of the equilibrium can be theoretically established.

7.4.2 Computation of the Recursive Equilibrium

The key computational problem is the size of the state space. The aggregate wealth distribution Φ is an infinite-dimensional object, the aggregate law of motion therefore maps an infinite-dimensional space into itself. What we look for is a low-dimensional approximation of the wealth distribution. Why do agents need to keep track of the aggregate wealth distribution? In order to figure out today's interest and wage rate the aggregate (average) capital stock (i.e. the first moment of the wealth distribution) is sufficient. Thus, to forecast tomorrow's factor prices, all the agent needs to forecast is tomorrow's aggregate capital stock. The need to keep track of the current wealth distribution stems from the fact that the entire wealth distribution Φ today, not only its first moment K, determines tomorrow's aggregate capital stock via

$$K' = \int a'(a, y, s, \Phi) d\Phi \tag{7.102}$$

Another way of putting it, the average capital stock tomorrow is *not* equal to the saving function of some average, representative consumer, evaluated at today's average capital stock. If the optimal policy function for tomorrow's assets, would feature a constant propensity to save out of current assets and income (a' being linear in a, with same slope for all $y \in Y$), then exact aggregation would occur and in fact the average capital stock today would be a sufficient statistic for the average capital stock tomorrow. This insight is important for the quantitative results to come.

The computational strategy that Krusell and Smith (and many others since) follow is to approximate the distribution Φ with a finite set of moments. Remember that Φ is the distribution over (a, y). Obviously, since y can only take two values, the second dimension is not the problem, so in what follows we focus on the discussion of the distribution over assets a. Let the n-dimensional vector m denote the first n moments of the asset distribution (i.e. the marginal distribution of Φ with respect to its first argument).

We now posit that the agents use an approximate law of motion

$$m' = H_n(s, m) \tag{7.103}$$

mapping the first n moments of the asset distribution today, m, into the first n moments of the asset distribution tomorrow, m'. Note that by doing so agents are boundedly rational in the sense that moments of higher order than n of the current wealth distribution may help to more accurately forecast the

first n moments tomorrow. It is the hope that, according to some metric to be discussed later, agents don't make severe forecasting errors of tomorrow's average capital stock (the only variable they care about for forecasting future prices), and that the resulting approximate equilibrium is in some sense close to the rational expectations equilibrium in which agents use the entire wealth distribution to forecast tomorrow's prices.

The next step in the computation of an approximate equilibrium is to choose the number of moments and the functional form of the function H_n. Since the agents only need to know next periods average capital stock, Krusell and Smith first pick $n = 1$ and pose the following log-linear law of motion (remember the law of motion for the stochastic neoclassical growth model with log-utility and Cobb-Douglas production)

$$\log(K') = a_s + b_s \log(K) \tag{7.104}$$

for $s \in \{s_b, s_g\}$. Here (a_s, b_s) are parameters that need to be determined. The recursive problem of the household then becomes

$$
\begin{aligned}
v(a, y, s, K) &= \max_{c, a' \geq 0} \left\{ U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi(y', s'|y, s) v(a', y', s', K') \right\} \\
&\quad s.t. \tag{7.105} \\
c + a' &= w(s, K)y + (1 + r(s, K))a \tag{7.106} \\
\log(K') &= a_s + b_s \log(K) \tag{7.107}
\end{aligned}
$$

Note that we have reduced the state space from something that includes the infinite-dimensional space of measures to a four dimensional space $(a, y, s, K) \in \mathbf{R} \times Y \times S \times \mathbf{R}$. The algorithm for computing an approximate recursive equilibrium is then as follows:

1. Guess (a_s, b_s)

2. Solve the households problem to obtain decision rules $a'(a, y, s, K)$

3. Simulate the economy for a large number of T periods for a large number N of households (in their exercises Krusell and Smith pick $N = 5000$ and $T = 11000$: start with initial conditions for the economy (s_0, K_0) and for each household (a_0^i, y_0^i). Draw random sequences $\{s_t\}_{t=1}^T$ and $\{y_t^i\}_{t=1,i=1}^{T,N}$ and use the decision rule $a'(a, y, s, K)$ and the

perceived law of motion for K to generate sequences of $\{a_t^i\}_{t=1,i=1}^{T,N}$. Aggregate to find the implied sequence of aggregate capital stocks

$$K_t = \frac{1}{N} \sum_{i=1}^{N} a_t^i \qquad (7.108)$$

Discard the first τ periods, because of dependence on initial conditions.

4. With the remaining observations run the regressions

$$\log(K') = \alpha_s + \beta_s \log(K) \qquad (7.109)$$

to estimate (α_s, β_s) for $s \in S$. If the R^2 for this regression is high and $(\alpha_s, \beta_s) \approx (a_s, b_s)$ stop. An approximate equilibrium is found. Otherwise update guess for (a_s, b_s). If guesses for (a_s, b_s) converge, but R^2 remains low, try to add higher moments to the aggregate law of motion and/or experiment with a different functional form for the aggregate law of motion.

7.4.3 Calibration

In this economy no analytical results can be proved and one has to resort to numerical analysis. Krusell and Smith take the model period to be one quarter (note that this is a business cycle model). As preferences they assume $CRRA$ utility with rather low risk aversion of $\sigma = 1$ (i.e. log-utility). The time discount factor is chosen to be $\beta = 0.99^4 = 0.96$ on an annual basis, which implies a yearly subjective time discount rate of $\rho = 4.1\%$ as in Aiyagari. Also similar to Aiyagari they take the capital share to be $\alpha = 0.36$. As annual depreciation rate they choose $\delta = (1-0.025)^4 - 1 = 9.6\%$, slightly higher than Aiyagari, but within the range of values commonly used in real business cycle studies. The remaining parameters describe the joint aggregate-idiosyncratic labor productivity process. Unfortunately the paper itself does not contain a precise discussion of the parameterization, so that the discussion here relies partly on Krusell and Smith's information from the paper, partly on Imrohoroglu's (1989) paper to which they refer to and partly on the FORTRAN code posted on Tony Smith's web site.

The calibration strategy is to first calibrate the aggregate component of the productivity process, i.e. the set S and the 2×2 matrix $\pi(s'|s)$. Remember

that the aggregate state represents recessions and expansions. With respect to S they choose

$$S = \{0.99, 1.01\} \tag{7.110}$$

I would think that the standard deviation of the technology shock is a bit on the small side with 0.01. In fact, using Cooley and Prescott's (1995) continuous state process and discretizing into a two state chain yields a standard deviation of the shock of about 0.02. Krusell and Smith claim that with their aggregate process they are able to generate aggregate fluctuations of output similar to US data, which, given the information in the paper I wasn't able to verify. Given that Cooley and Prescott need sufficiently more variance in the technology shock to generate business cycles of reasonable size, this must mean that the economy with heterogenous agents and uninsurable idiosyncratic risk, for a given aggregate shock variance, is more volatile than its representative agent counterpart.

As for the transition matrix for the aggregate shock the first assumption made is symmetry, so that $\pi(s_g|s_g) = \pi(s_b|s_b)$. Krusell and Smith choose $\pi(s_g|s_g)$ such that, conditional on being in the good state today, the expected time in the good state are 8 quarters, or

$$8 = [1 - \pi(s_g|s_g)] \left[1 + 2\pi(s_g|s_g) + 3\pi(s_g|s_g)^2 + \ldots\right] \tag{7.111}$$

or

$$
\begin{aligned}
8 &= \frac{1}{1 - \pi(s_g|s_g)} \\
\pi(s_g|s_g) &= \frac{7}{8}
\end{aligned}
\tag{7.112}
$$

so that

$$\pi(s'|s) = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix} \tag{7.113}$$

It follows that, conditional on being in the bad state today, the expected time of staying there is also 8 quarters.

With respect to idiosyncratic labor productivity, the state space is chosen as

$$Y = \{0.25, 1\} \tag{7.114}$$

Remember that the idiosyncratic states are meant to represent employment and unemployment, so that it is assumed that an unemployed person makes

25% of the labor income of an employed person. Imrohoroglu (1989) argues that, although the average level of unemployment compensation is higher, a large fraction of the unemployed do not receive benefits (Imrohoroglu quotes a number of 61%), so that the 25% figure is argued as reasonable. Krusell and Smith adopt this value. The level of productivity for an employed person is free for normalization, and for the sake of the discussion I set it to 1 (doubling both states in Y obviously only changes units, but does not affect the results).

With respect to the transition probabilities for the idiosyncratic productivity we note that

$$\pi(y'|s', y, s) = \frac{\pi(y', s'|y, s)}{\pi(s'|s)} \tag{7.115}$$

or

$$\pi(y', s'|y, s) = \pi(y'|s', y, s) * \pi(s'|s) \tag{7.116}$$

So in order to specify $\pi(y', s'|y, s)$, given our previous work we have to specify the four (for each pair (s', s)) 2×2 matrices $\pi(y'|s', y, s)$ indicating, conditional on an aggregate transition from s to s', what the individual's probabilities of transition from employment to unemployment are. These should vary with the aggregate transition (s', s).

The calibration of these matrices are governed by the following observations. In an expansion the average time of unemployment, conditional on being unemployed today, is equal to 1.5 quarters. This implies that

$$
\begin{aligned}
1.5 \quad &= \quad [1 - \pi(y' = y_u|s' = s_g, y = y_u, s = s_g)] \sum_{i=1}^{\infty} i * \pi(y' = y_u|s' = s_g, y = y_u, s = s_g)^{i-1} \\
&= \quad \frac{1}{1 - \pi(y' = y_u|s' = s_g, y = y_u, s = s_g)} \\
\pi(y' \quad &= \quad y_u|s' = s_g, y = y_u, s = s_g) = 1 - \frac{1}{1.5} = \frac{1}{3}
\end{aligned} \tag{7.117}
$$

and hence $\pi(y' = y_e|s' = s_g, y = y_u, s = s_g) = \frac{2}{3}$. Similarly, to match the fact that in bad times the average time of unemployment, conditional on unemployed, equals 2.5 quarters we find $\pi(y' = y_u|s' = s_b, y = y_u, s = s_b) = 0.6$ and hence $\pi(y' = y_e|s' = s_b, y = y_u, s = s_b) = 0.4$. The authors then assume that the probability of remaining unemployed when times switch from expansion to recession is 1.25 times the same probability when the economy was already in a recession, implying $\pi(y' = y_u|s' = s_b, y = y_u, s = s_g) = 0.75$ and thus $\pi(y' = y_e|s' = s_b, y = y_u, s = s_g) = 0.25$. Finally, the

probability of remaining unemployed when times switch from recession to expansion is assumed to be 0.75 times the same probability when times were already good. This gives $\pi(y' = y_u | s' = s_g, y = y_u, s = s_b) = 0.25$ and $\pi(y' = y_e | s' = s_g, y = y_u, s = s_b) = 0.75$.

The final transition probabilities are derived from the facts that in recessions the unemployment rate is $\Pi_{s_b}(y_u) = 10\%$ and in expansions it is $\Pi_{s_g}(y_u) = 4\%$. With the probabilities already derived and equation (7.81) this uniquely pins down the remaining probabilities

$$
\begin{aligned}
\pi(y' &= y_u | s' = s_g, y = y_e, s = s_g) = 0.028 \\
\pi(y' &= y_e | s' = s_g, y = y_e, s = s_g) = 0.972 \\
\pi(y' &= y_u | s' = s_b, y = y_e, s = s_b) = 0.04 \\
\pi(y' &= y_e | s' = s_b, y = y_e, s = s_b) = 0.96 \\
\pi(y' &= y_u | s' = s_b, y = y_e, s = s_g) = 0.079 \\
\pi(y' &= y_e | s' = s_b, y = y_e, s = s_g) = 0.921 \\
\pi(y' &= y_u | s' = s_g, y = y_e, s = s_b) = 0.02 \\
\pi(y' &= y_e | s' = s_g, y = y_e, s = s_b) = 0.98
\end{aligned}
\tag{7.118}
$$

In short, the best times for finding a job are when the economy moves from a recession to an expansion, the worst chances are when the economy moves from a boom into a recession. Combining the aggregate transition probabilities with the idiosyncratic probabilities, conditional on the aggregate transitions, finally yields as transition matrix (7.82)

$$
\pi = \begin{pmatrix}
0.525 & 0.035 & 0.09375 & 0.0099 \\
0.35 & 0.84 & 0.03125 & 0.1151 \\
0.03125 & 0.0025 & 0.292 & 0.0245 \\
0.09375 & 0.1225 & 0.583 & 0.8505
\end{pmatrix}
\tag{7.119}
$$

With this parameterization we are ready to report results on computed approximate equilibria.

7.4.4   Numerical Results

The model generates three basic entities of interest: an aggregate law of motion

$$
m' = H_n(s, m),
\tag{7.120}
$$

individual decision rules $a'(a, y, s, m)$ and time-varying cross-sectional wealth distributions $\Phi(a, y)$. First let's focus on the aggregate law of motion. Remember that we allowed agents only to use the first n moments of the current wealth distribution to forecast the first n moments of tomorrow's wealth distribution, i.e. agents are boundedly rational. In particular, the aggregate law of motion perceived by agents may not coincide with the actual law of motion, which is just another way of saying that we are NOT computing a rational expectations equilibrium.

Since agents only have to forecast tomorrow's aggregate (average) capital stock (i.e. the first moment of Φ'), Krusell and Smith start out with $n = 1$. After repeated updating of the coefficients in (7.109) the coefficients converge and one obtains as perceived law of motion of agents

$$\log(K') = 0.095 + 0.962 \log(K) \text{ for } s = s_g \qquad (7.121)$$
$$\log(K') = 0.085 + 0.965 \log(K) \text{ for } s = s_b \qquad (7.122)$$

How irrational are agents? We would have computed a perfectly legitimate rational expectations equilibrium if the actual law of motion for the capital stock follows (7.121) to (7.122) exactly. Remember that at each step, to update the coefficients in (7.121) to (7.122), a simulated time series for the aggregate capital stock in conjunction with a sequence of aggregate shocks $\{(s_t, K_t)\}_{t=0}^T$ is generated. By dividing the sample into periods with $s_t = s_b$ and $s_t = s_g$ one then can run the regressions

$$\log(K_{t+1}) = \alpha_g + \beta_g \log(K_t) + \varepsilon_{t+1}^g \text{ for periods with } s_t = s_g \, (7.123)$$
$$\log(K_{t+1}) = \alpha_b + \beta_b \log(K_t) + \varepsilon_{t+1}^b \text{ for periods with } s_t = s_b \, (7.124)$$

In fact, given the computational algorithm, for the last step of the iteration on the perceived law of motion they did obtain $\hat{\alpha}_g = 0.095$, $\hat{\alpha}_b = 0.085$ and $\hat{\beta}_g = 0.962$, $\hat{\beta}_b = 0.965$. From econometrics we remember that with regression errors

$$\hat{\varepsilon}_{t+1}^j = \log(K_{t+1}) - \hat{\alpha}_j - \hat{\beta}_j \log(K_t) \text{ for } j = g, b \qquad (7.125)$$

we define the standard deviation of the regression error as

$$\sigma_j = \sqrt{\frac{1}{T_j} \sum_{t \in \tau_j} \left(\hat{\varepsilon}_t^j\right)^2} \qquad (7.126)$$

where τ_j is the set of time indices for which $s_t = s_j$ and T_j is the cardinality of that set. Note that, since the regression is on log-income, σ_j can be interpreted as the average percentage error for the prediction of the capital stock

made by the regression. The R^2 of the regression, as alternative measure of fit, is defined as that fraction of the variation in tomorrow's log-capital stock that is explained by the variation of today's log-capital stock, or

$$R_j^2 = 1 - \frac{\sum_{t \in \tau_j} \left(\hat{\varepsilon}_t^j \right)^2}{\sum_{t \in \tau_j} \left(\log K_{t+1} - \overline{\log K} \right)^2} \tag{7.127}$$

If $\sigma_j = 0$ for $j = g, b$ or equivalently, if $R_j^2 = 1$ for $j = g, b$ then households' perceived aggregate law of motion is exactly correct for all periods, i.e. agents do not make forecasting errors and are perfectly rational. Low σ_j and high R_j^2 are taken as evidence that the actual law of motion generated by individual behaviors and aggregation does not depart much from the perceived law of motion, i.e. that agents make only small forecasting errors.

Krusell and Smith obtain $R_j^2 = 0.999998$ for $j = b, g$ and $\sigma_g = 0.0028$, $\sigma_b = 0.0036$, i.e. extremely low average forecasting errors made by agents. They calculate maximal forecasting errors for interest rates 25 years into the future of 0.1% for their simulations. Unfortunately they do not report magnitudes of corresponding utility losses from forecasting errors (say, in consumption equivalent variation), but assert that given the small magnitude of forecasting errors even for the far future these utility losses from bounded rationality are negligible.

Before giving intuition for the results a word of caution is in order. In all of computational economics, since computer precision is limited, the best one can achieve is to compute an approximate equilibrium, i.e. allocations and prices in which markets almost clear (excess demand is ε away from zero). This approximate equilibrium may be arbitrarily far away from a true equilibrium (i.e. prices and allocations for which markets clear exactly). The study by Krusell and Smith is not unusual in this respect. But even if this general problem were absent, Krusell and Smith's equilibrium is at best an approximation to a rational expectations equilibrium and the true rational expectations equilibrium may be arbitrarily far away from the computed equilibrium. Krusell and Smith's agents make small forecasting errors (according to their σ_j, R_j^2 metric), but the aggregate law of motion at which agents make no forecasting errors (i.e. a rational expectations equilibrium) will in general involve higher moments and may look very different from the one they found.

7.4.5 Why Quasi-Aggregation?

Suppose all agents, for all interest rates $r(K)$ and wages $w(K)$ have linear savings functions with the same marginal propensity to save, so that

$$a'(a, y, s, K) = a_s + b_s a + c_s y \qquad (7.128)$$

Then

$$
\begin{aligned}
K' &= \int a'(a, y, s, K) d\Phi = a_s + b_s \int a d\Phi + c_s \bar{L} \\
&= \tilde{a}_s + b_s K \qquad (7.129)
\end{aligned}
$$

since $K = \int a d\Phi$. Therefore exact aggregation obtains and the first moment of the current wealth distribution (which equals the current capital stock) is in fact a sufficient statistic for Φ for forecasting the aggregate capital stock tomorrow.

In their Figure 2, Krusell and Smith plot savings functions (note: for a particular current aggregate stock and a particular shock s). We see that they are almost linear with same slope for $y = y_u$ and $y = y_e$. The only exceptions are unlucky agents $(y = y_u)$ with little assets which are liquidity constrained and hence have a low (zero) marginal propensity to save. However, since these (few) agents hold a negligible fraction of aggregate wealth, they don't matter for the aggregate capital dynamics. All other agents have almost identical propensities to save, thus individual saving decisions almost exactly aggregate, and the current aggregate capital stock is almost a sufficient statistic when forecasting tomorrow's capital stock: quasi-aggregation obtains.

The key question is why individual savings functions a' are almost linear in current assets a at just about all current asset levels. ¿From Figure 2 of Krusell and Smith we see that the slope of a' when plotted against a is roughly equal to 1 for all but very low asset levels. Remember from the PILCH model with certainty equivalence that optimal consumption was given by

$$c_t = \frac{r}{1+r} \left(E_t \sum_{s=0}^{T-t} \frac{y_{t+s}}{(1+r)^s} + a_t \right) \qquad (7.130)$$

and hence agents consume a fraction $\frac{r}{1+r}$ of current assets. That is, they save out of current assets for tomorrow

$$\frac{a_{t+1}}{1+r} = \left(1 - \frac{r}{1+r} \right) a_t + G(y) \qquad (7.131)$$

where $G(y)$ is a function of the stochastic income process. Thus under certainty equivalence

$$a_{t+1} = a_t + H(y) \tag{7.132}$$

and thus the saving function a' has slope 1 under certainty equivalence (and $\rho = r$). In Krusell and Smith's economy agents are prudent and face liquidity constraints, but almost act as if they are certainty equivalence consumers. Why? A few reasons come to my mind:

1. Agents are prudent, but not all that much. A $\sigma = 1$ is at the lower end of the empirical estimates for risk aversion. As we saw from Aiyagari's (1994) paper, the amount of precautionary saving increases significantly with increases in σ, and so should the deviation of agents decision rules from certainty equivalence.

2. The unconditional standard deviation of individual income is roughly 0.2, at the lower end of the estimates used by Aiyagari. As we saw, higher variability, induces more quantitatively important precautionary saving.

3. Probably most important, negative income shocks (unemployment) are infrequent and not very persistent, so that they don't force a large departure in behavior from certainty equivalence. Krusell and Smith's agents never become permanently disabled and don't face permanent wage declines after being laid off (once they find a new job, their relative wage is as high as before they were laid off)

Krusell and Smith consider various sensitivity tests with respect to these and other dimensions (change in time discount rates, endogenous labor-leisure choice etc.), and claim that the result of quasi-aggregation (only the mean capital stock matters for forecasting tomorrows mean capital stock) is robust to changes in the model parameterization.

7.4.6   Rich People are Not Rich Enough

The model generates an endogenous consumption and wealth distribution, something that all of representative agents macroeconomics is silent about. Note that the income distribution is, by specifying the income process that households face, an input into the model. To the extent that the income

process and hence the cross-sectional income distribution is realistic, one would hope that the resulting wealth distributions (remember that this distribution changes over time) is on average consistent with the cross-sectional wealth distribution in the data. Unfortunately, the model does a fairly bad job reproducing the US wealth distribution, in particular it fails to generate the high concentration of wealth at the upper end of the distribution. In the data, the richest 1% of the US population holds 30% of all household wealth, the top 5% hold 51% of all wealth. For the model described above the corresponding numbers are 3% and 11%, correspondingly. In the model people save to buffer their consumption against unemployment shocks, but since these shocks are infrequent and of short duration, they don't save all that much.

There are several ways of making a small fraction of the population save a lot, and hence making them accumulating a large fraction of overall wealth. The repair job that Krusell and Smith propose is to make some agents (stochastically) more patient than others. In particular, they assume that the time discount factor of agents β is stochastic and follows a three state Markov chain with $\beta \in B$ and transition probabilities $\gamma(\beta'|\beta)$. The dynamic programming problem of agents then becomes

$$v(a, y, \beta, s, K) \tag{7.133}$$

$$= \max_{c, a' \geq 0} \left\{ U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \sum_{\beta' \in B} \pi(y', s'|y, s)\gamma(\beta'|\beta)v(a', y', \beta', s', K') \right\}$$

$$s.t.$$

$$c + a' = w(s, K)y + (1 + r(s, K))a \tag{7.134}$$

$$\log(K') = a_s + b_s \log(K) \tag{7.135}$$

They pick $B = \{0.9858, 0.9894, 0.993\}$. Hence annual discount rates differ between 2.8% for the most patient agents and 5.6% for the most impatient agents. As transition matrix Krusell and Smith propose (remember that the model period is one quarter)

$$\gamma = \begin{pmatrix} 0.995 & 0.005 & 0 \\ 0.000625 & 0.99875 & 0.000625 \\ 0 & 0.005 & 0.995 \end{pmatrix} \tag{7.136}$$

which implies that 80% of the population has discount factor in the middle and 10% are on either of the two extremes (in the stationary distribution).

It also implies that patient and impatient agents remain so for an expected period of 50 years. De facto this parameterization creates three deterministic types of agents. The patient agents are the ones that will accumulate most of the wealth in this economy. With this trick of stochastic discount factors Krusell and Smith are able to approximate the US wealth distribution to a reasonably accurate degree (see their Figure 3): in the modified economy the richest 1% of the population holds 24% of all wealth, the richest 5% hold 55% percent of all wealth. Also note that the quasi-aggregation result extends to the economy with stochastic discount factors.

Part III

Complete Market Models with Frictions

In this chapter we will consider models that, in spirit, build on the complete markets model considered in Chapter 3, in the same fashion the models in Chapter 4 and 5 built on the simple Pilch model in Chapter 3. Most empirical studies reject the complete insurance hypothesis and thus cast doubt upon the complete markets model as a reasonable description of reality.

Therefore models are desired that predict some, but not perfect risk sharing. The Pilch model (in its general equilibrium form) generates some "risk sharing" via self-insurance: agents smooth part of their income fluctuation by asset accumulation and decumulation, with the part being determined by preferences and the nature of the income shocks. But remember that there are no formal risk-sharing arrangements in the Pilch model, as explicit contingent insurance contracts, which agents in the model would have an incentive to trade and financial intermediaries would have an incentive to offer, are ruled out by assumption, without any good reason from *within* the model.[9]

The models we study in this chapter will allow a full set of contingent consumption claims being traded (in the decentralized version) or being allocated by the social planner (in the centralized version of the model). That full insurance does not arise as optimal and equilibrium allocation is due to informational and/or enforcement frictions, which are explicitly modeled. These models adhere to the principle, most forcefully articulated by Townsend that the world is constrained efficient, and that is up to the researcher/modeler to find the right set of constraints that give rise to model allocations which are in line with empirical consumption allocations.

We will study two such sets of constraints: the first stemming from the fact that in actual economies financial contracts are only imperfectly enforceable (because there exist explicit bankruptcy provisions in the legal code or it is too costly to always enforce repayment), the second deriving from the fact that individual incomes and/or actions are imperfectly observable, so that contingent claims payments cannot be directly conditioned on these. Both types of models will deliver allocations characterized by some, but (depending on parameterizations) imperfect insurance, which is due explicitly to these frictions.

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[9]Townsend (1985) and Cole and Kocherlakota (1997) study environments with private information and show that the optimal contract between agents and financial intermediaries is a simple uncontingent debt contract, closely resembling the one-period uncontingent bonds that we let agents trade in the Pilch model.

Chapter 8

Limited Enforceability of Contracts

We start with models in which perfect insurance is prevented because at each point of time every agent can default on her financial obligations, with the ensuing punishment being modeled as exclusion from future credit market participation. The seminal contributions to this literature include Kehoe and Levine (1993), Kocherlakota (1996) and Alvarez and Jermann (2000). Kehoe and Levine (1993), building on earlier work of Eaton and Gersowitz (1981), develop a competitive equilibrium model with a small number of agents incorporating imperfect enforceability of contracts (in the standard Arrow Debreu model one of the crucial implicit assumptions is perfect enforcement of contracts); their 2001 Econometrica paper compares this model to a standard Pilch model in a fairly accessible way. Kocherlakota studies the same environment, but lets agents interact strategically; the no-default constraints become restrictions derived from the requirement of subgame perfection and constrained efficiency. Finally, Alvarez and Jermann (2000) show how to, under weak assumptions, decentralize constrained-efficient allocations arising in Kehoe and Levine and Kocherlakota as sequential markets equilibrium with a full set of Arrow securities and state-contingent borrowing constraints that are not "too tight", in the sense that agents can borrow up to the amount in which they are indifferent between repaying and defaulting (and being excluded from future intertemporal trade).

Lately some authors have extended these models to settings with a continuum of agents (the counterpart of Aiyagari's model), see Krueger (1999) and used them to address empirical questions (Krueger and Perri, 2005,

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2006a,b). Lustig (2001) develops a method to handle aggregate uncertainty in these models (as in Krusell and Smith), and uses it to study asset prices.[1]

In this chapter we will first study how to characterize (and compute) constrained efficient allocations for a model with 2 agents, how to decentralize it as a competitive Arrow-Debreu, a sequential markets and a subgame perfect equilibrium. We then use it to study the relationship between the variability of income and the resulting consumption distribution. We then will discuss how to deal with a continuum of agents, both in terms of theory and in terms of computation, and discuss other applications.

8.1　The Model

The model is populated by 2 (types of) infinitely lived households, $i = 1, 2$, that, in each period, consume a single perishable consumption good. Let by $e_t^i = e_t^i(s_t) > 0$ denote the stochastic symmetric endowments governed by the random variable $s_t \in S$, where the set S is finite. As before let denote by $s^t = (s_t, \dots s_1) \in S^t$ denote a history of shocks. The stochastic process is assumed to be Markov with transition probabilities $\pi(s_{t+1}|s_t)$ and invariant distribution Π. For given initial shock s_0, whose distribution is given by Π, the probabilities of endowment shock histories are given by

$$\pi(s^t) = \pi(s_t|s_{t-1}) * \dots \pi(s_1|s_0) \tag{8.1}$$

Assume that the endowment processes are symmetric in that if $e_t^1(s_t) = e_t^1(s^t) = e$, then there exists a $\hat{s}^t$ with $\pi(s^t) = \pi(\hat{s}^t)$ and $e_t^2(s_t) = e_t^2(s^t) = e$. That is, both agents face identical stochastic endowment processes.

A consumption allocation is denoted by $(c^1, c^2) = \{c_t^1(s^t), c_t^2(s^t)\}_{t=1, s^t \in S^t}^{\infty}$ and agents are assumed to have preferences over consumption streams given by

$$U(c) = (1 - \beta) \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t)) \tag{8.2}$$

where $\beta \in (0, 1)$ and u is strictly increasing, strictly concave and C^2 and satisfies INADA conditions.

---

[1]An OLG-version of the Kehoe and Levine model is studied by Azariadis and Lambertini (2001) and applied to the study of social security by Andolfatto and Gervais (2000). Development economists applied these class of models to study formal and informal risk-sharing arrangements in rural Africa (see Udry 1995) and India (see Ligon, Thomas and Worrall 2000).

The definition of a Pareto efficient allocation is standard, but we repeat it here for the sake of refreshing memory.

Definition 55 *An allocation (c^1, c^2) is Pareto efficient if it is resource feasible*

$$c^1 + c^2 = e^1 + e^2 \tag{8.3}$$

for all t, all s^t, and there is no other feasible allocation $(\hat{c}^1, \hat{c}^2)$ such that $U(\hat{c}^i) \geq U(c^i)$, with at least one inequality strict.

Note from Chapter 2 that an allocation is Pareto efficient if and only if it is resource feasible and satisfies

$$\frac{u'(c_t^1(s^t))}{u'(c_t^2(s^t))} = \gamma \text{ for all } t, \text{ all } s^t, \tag{8.4}$$

for some $\gamma > 0$.

In this economy there are strong incentives to share the endowment risk both agents face. The first question we want to pose and answer is what is the extent of risk sharing possible if agents cannot commit to long term contracts that are not individually rational? In order to do so we will compute constrained efficient allocations using recursive contract techniques, in which "promised utility" acts as a state variable. Then we decentralize these allocation within a

1. Dynamic transfer game: individual rationality translates into subgame perfection (Kocherlakota 1996)

2. Arrow Debreu competitive market. Individual rationality constraints enter individual consumption sets (Kehoe and Levine 1993, 2001)

3. Sequential market where agents enter (long-term) relationship with financial intermediaries that set appropriate borrowing constraints for Arrow securities (Alvarez and Jermann, 2000).

Now let us formalize the individual rationality constraints that agents face in the light of their inability to commit to repaying their debt. First define the continuation lifetime expected utility from allocation (c^1, c^2) for agent i in node s^t of the event tree as

$$\begin{aligned} U(c^i, s^t) &= (1-\beta)u(c_t^i(s^t)) + \\ &\quad (1-\beta)\sum_{\tau=t+1}^{\infty}\sum_{s^\tau|s^t}\beta^{\tau-t}\pi(s^\tau|s^t)u(c_\tau^i(s^\tau)) \end{aligned} \tag{8.5}$$

We now impose the following individual rationality constraints on allocations

$$U(c^i, s^t) \geq U(e^i, s^t) \equiv U^{i,Aut}(s_t) \tag{8.6}$$

These constraints on allocations say that, at no point of time, no contingency any agent would prefer to walk away from the allocation (c^1, c^2), with the consequence of living in financial autarky and consuming her own endowment from that node onward.

8.2   Constrained Efficient Allocations

Let Γ denote the set of all allocations that satisfy the resource constraints (8.3) and the individual rationality constraints (8.6). Also define $\bar{U}^i(s_0) = \max_{(c^1,c^2)\in\Gamma} U(c^i, s_0)$ and $\underline{U}(s_0) = U^{i,Aut}(s_0)$ as the upper and lower bounds of lifetime utility agent i can obtain from any allocation that is feasible and satisfies the individual rationality constraints (remember that endowments are symmetric). We have the following definition.

Definition 56 *An allocation $(c^1, c^2) \in \Gamma$ is constrained efficient if there is no other feasible allocation $(\hat{c}^1, \hat{c}^2) \in \Gamma$ such that $U(\hat{c}^i) \geq U(c^i)$, with at least one inequality strict.*

For given initial s_0, the constrained Pareto Frontier $W_{s_0} : [\underline{U}^1(s_0), \bar{U}^1(s_0)] \to [\underline{U}^2(s_0), \bar{U}^2(s_0)]$ is defined as

$$
\begin{aligned}
W_{s_0}(U) &= \max_{(c^1,c^2)\in\Gamma} U(c^2, s_0) \\
s.t.\ U(c^1, s_0) &\geq U
\end{aligned}
\tag{8.7}
$$

The number $W_{s_0}(U)$ is interpreted as the maximal lifetime utility (conditional on s_0 having been realized) agent 2 can obtain from any constrained-feasible allocation if agent 1 is guaranteed at least lifetime utility $[\underline{U}^1(s_0), \bar{U}^1(s_0)]$. We immediately have the following

Proposition 57 *An allocation (c^1, c^2) is constrained efficient if and only if it solves the above maximization problem, for some $U \in [\underline{U}^1(s_0), \bar{U}^1(s_0)]$.*

8.3 Recursive Formulation of the Problem

We now want to construct the constrained efficient consumption allocation (sometimes called a recursive contract, because it will, as we will see, have the nature of a long-term risk sharing contract. The basic idea for doing so goes back to Spear and Srivastava (1987) and Abreu (1988): since the individual rationality constraints constrain continuation utilities of allocations, make continuation utility a state variable in the recursive problem.

The constrained Pareto frontier was defined as maximizing the lifetime utility of the second agent, subject to guaranteeing the first agent at least a certain lifetime utility U. For the recursive problem this lifetime utility becomes a state variable (together with the current exogenous state of the world s). In order to notationally distinguish between the recursive and the sequential problem let the lifetime utility of agent 1 in the recursive formulation be denoted by w (obviously we can switch the roles of agent 1 and 2).

Thus state variables for the recursive problem are (w, s) and the Bellman equation reads as[2]

$$V(w, s) = \max_{\left(c_1, c_2, \{w'(s')\}_{s' \in S}\right)} \left\{ (1 - \beta) u(c_2) + \beta \sum_{s' \in S} \pi(s'|s) V(w'(s'), s') \right\}$$

$$\text{s.t.} \tag{8.8}$$

$$c_1 + c_2 = e^1(s_t) + e^2(s_t) \tag{8.9}$$

$$V(w'(s'), s') \geq U^{2,Aut}(s') \tag{8.10}$$

$$w'(s') \geq U^{1,Aut}(s') \tag{8.11}$$

$$w'(s') \leq \bar{U}^1(s') \tag{8.12}$$

$$w = (1 - \beta) u(c_1) + \beta \sum_{s' \in S} \pi(s'|s) w'(s') \tag{8.13}$$

The state space is $(w, s) \in [\underline{U}^1(s_0), \bar{U}^1(s_0)] \times S$. This is a standard dynamic programming problem that can be solved with standard techniques. Note that first the values of autarky have to be determined, which is done by

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[2]I deviate from the timing convention in Kocherlakota, who lets the economy start after an initial s_0 has been drawn, and analyzes allocations from period 1 onward. I formulate the problem so that consumption also takes place in period 0, after the period 0 shock has been realized. This is more consistent with my discussion in earlier chapters and will help to simplify the discussion of the continuum economy.

solving the equations

$$U^{i,Aut}(s) = (1 - \beta)u(e^i(s)) + \beta \sum_{s'} \pi(s'|s)U^{i,Aut}(s') \qquad (8.14)$$

These are $N = \text{card}(S)$ equations in N unknowns $U^{i,Aut}(s)$, $s \in S$. Once we have the values of autarky, we can solve the dynamic programming problem[3] for policy functions $c^i(w, s)$ and $w'(w, s; s')$.

Sequential consumption allocations are derived from the recursive policy functions $c^i(w, s)$ and $w'(w, s; s')$ as usual. The initial conditions for the sequential problem are (U, s_0). Then we derive the consumption allocation as follows:

$$
\begin{aligned}
c_0^i(s_0) &= c^i(U, s_0) \\
w_1^1(s^1) &= w'(U, s_0; s_1) \\
w_1^2(s^1) &= V(w_1^1(s^1), s_1)
\end{aligned}
\qquad (8.15)
$$

and recursively

$$
\begin{aligned}
w_t^1(s^t) &= w'(w_{t-1}^1(s^{t-1}), s_{t-1}; s_t) \\
w_t^2(s^t) &= V(w_t^1(s^t), s_t) \\
c_t^i(s^t) &= c^i(w_t^i(s^t), s_t)
\end{aligned}
\qquad (8.16)
$$

The remaining question is whether these allocations in fact solve the sequential planning problem and whether the function W_{s_0} satisfies $W_{s_0}(U) = V(U, s_0)$ for all $s_0 \in S$. But this is nothing else but proving Bellman's principle of optimality, which in fact has been done by Thomas and Worrall (1988). Kocherlakota provides a partial characterization of constrained efficient allocations; I will defer the discussion of this to the paper presentation and will discuss a simple example below.

8.4   Decentralization

Since both the papers by Kocherlakota and by Kehoe and Levine will be presented in class, I will skip the discussion of how constrained efficient allocations can be decentralized within a subgame perfect equilibrium of a

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[3]Strictly speaking one also needs the upper bound on utilities $\bar{U}(s)$. One guesses these bounds and then iterates between guesses and solutions of the Bellman equation, until one attains $V(\bar{U}(s), s) = U^{2,Aut}(s)$ for all s.

transfer game or a competitive equilibrium with enforcement constraints. I will instead present a decentralization of constrained-efficient allocations as a sequential markets equilibrium with state-contingent borrowing constraints that is due to Alvarez and Jermann (2000).

This sequential markets equilibrium is almost identical to the one discussed in Chapter 3; there, however, we picked borrowing constraints that were very "loose" in that they just prevented Ponzi schemes, but did allow first-best consumption smoothing. Now we are looking for borrowing constraints that have more bite, in that they resemble exactly the incentive constraints from the social planners problem.

Let by $q_t(s^t, s_{t+1})$ denote the price of the Arrow security at node s^t that pays out one unit of the consumption good if tomorrow's shock is s_{t+1}. Also denote by a_0^i the initial conditions of asset holdings for agent i, where of course $\sum_i a_0^i = 0$. Note that there exists a one-to-one mapping between the (a_0^1, a_0^2) and the position on the Pareto frontier U. By $\tilde{V}(a_t^i(s^t), s_t)$ let denote the continuation utility of agent i when entering node s^t with assets $a_t^i(s^t)$. It satisfies

$$\tilde{V}(a_t^i(s^t), s_t) = \max_{c^i, a^i} U(c^i, s^t) \qquad (8.17)$$

s.t.

$$c_t^i(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \leq e_t^i(s_t) + a_t^i(s^t) \qquad (8.18)$$

$$a_{t+1}^i(s^{t+1}) \geq \bar{A}^i(s^{t+1}) \qquad (8.19)$$

Evidently $\tilde{V}(a_0^i, s_0)$ is the lifetime utility of agent i with initial conditions (a_0^i, s_0) which faces the borrowing constraints $\bar{A}^i(s^{t+1})$. Alvarez and Jermann define borrowing constraints that are not too tight as satisfying

$$V(\bar{A}_t^i(s^t), s_t) = U^{i, Aut}(s_t) \qquad (8.20)$$

That is, the borrowing limits are such that agents that have borrowed up the maximum are exactly indifferent between repaying their debt and defaulting, with the punishment of default being specified as financial autarky. The definition of a sequential market equilibrium with borrowing constraints that are not too tight is standard and hence omitted.

Define Arrow securities prices as

$$q_t(s^t, s_{t+1}) = \max_{i=1,2} \left\{ \beta \pi(z'|z) \frac{u'(c_{t+1}^i(s^t, s_{t+1}))}{u'(c_t^i(s^t))} \right\} \qquad (8.21)$$

and implied Arrow-Debreu prices as

$$Q(s^t|s_0) = q_0(s_0, s_1) * q_1(s^1, s_2) * \ldots * q_{t-1}(s^{t-1}, s_t) \qquad (8.22)$$

For any allocation (c^1, c^2), the implied interest rate are said to be high if

$$\sum_{t \geq 0} \sum_{s^t} Q(s^t|s_0) * \left(c_t^1(s^t) + c_t^2(s^t)\right) < \infty \qquad (8.23)$$

Alvarez and Jermann then prove the following results:

Proposition 58 *Suppose a constrained efficient consumption allocation (c^1, c^2) has high implied interest rates. Then it can be decentralized as sequential markets equilibrium with some initial conditions and borrowing constraints that are not too tight.*

Proposition 59 *Suppose that a constrained efficient allocation (c^1, c^2) has some risk sharing, that is, for all t and all s^t there exists an i such that*

$$U(c^i, s^t) > U(e^i, s^t) \qquad (8.24)$$

Then the implied interest rates are high.

Combining these two results we have the following

Corollary 60 *Any nonautarkic constrained efficient consumption allocation can be decentralized as a sequential markets equilibrium with borrowing constraints that are not too tight.*

The main implication of this result is that one can solve for the entire set of potential equilibrium allocations by solving the recursive planning problem and then find Arrow securities prices and borrowing constraints from (8.20) and (8.21) that, together with the consumption allocation, form a sequential markets equilibrium.

8.5   An Application: Income and Consumption Inequality

In this section we will consider a simple example of the model below and use it to study how, in the model, an increase in income inequality affects

the consumption distribution. This example follows the one used in Krueger
and Perri's (2006) application of limited commitment models to income and
consumption inequality.

Let us assume that the aggregate state of the world can take two values
$s_t \in S = \{1, 2\}$. Let individual endowments be given by the functions

$$e^1(s_t) = \begin{cases} 1 + \varepsilon & \text{if } s_t = 1 \\ 1 - \varepsilon & \text{if } s_t = 2 \end{cases} \tag{8.25}$$

$$e^2(s_t) = \begin{cases} 1 - \varepsilon & \text{if } s_t = 1 \\ 1 + \varepsilon & \text{if } s_t = 2 \end{cases} \tag{8.26}$$

That is, if $s_t = 1$ then agent 1 has currently high endowment $1 + \varepsilon$, if $s_t = 2$
then agent 2 has currently high endowment. Here ε measures the variability
of individual endowment and also turns out be the (unconditional) standard
deviation of the cross-sectional income distribution.

Which agent is rich follows a stochastic process that is assumed to be
Markov with transition probabilities

$$\pi = \begin{pmatrix} \delta & 1 - \delta \\ 1 - \delta & \delta \end{pmatrix} \tag{8.27}$$

where $\delta \in (0, 1)$ denotes the conditional probability of agent 1 being rich
tomorrow, conditional on being rich today (and symmetrically for agent 2).
Thus δ is a measure of persistence of the income process, with $\delta = \frac{1}{2}$ denoting
the *iid* case and $\delta = 1$ reflecting a deterministic income process (that is,
permanent shocks). The stationary distribution associated with π, for any
given $\delta \in (0, 1)$ is given by $\Pi(s) = \frac{1}{2}$ for all $s \in S$, and we assume that
$\Pi(s_0) = \frac{1}{2}$ for all $s_0 \in S$. This assumption makes agents ex ante identical and
allows us to restrict attention to symmetric equilibrium allocations.

The good thing about this model is that we can solve for the continuation
expected discounted utility from autarky analytically. Since there are only
two aggregate states at each point of time, the continuation value from the
autarkic allocation can take two values, one for the currently rich agent,
denoted by $U(1+\varepsilon)$, and one for the currently poor agent, denoted by $U(1-\varepsilon)$.
These two values solve the following recursion

$$\begin{aligned} U(1 + \varepsilon) &= (1 - \beta)u(1 + \varepsilon) + \delta U(1 + \varepsilon) + (1 - \delta)U(1 - \varepsilon) \\ U(1 - \varepsilon) &= (1 - \beta)u(1 - \varepsilon) + \delta U(1 - \varepsilon) + (1 - \delta)U(1 + \varepsilon) \end{aligned} \tag{8.28}$$

Solving these two equations in two unknowns yields

$$U(1+\varepsilon) = \frac{1}{D}\{(1-\beta)\,u(1+\varepsilon) + \beta(1-\delta)\,[u(1+\varepsilon) + u(1-\varepsilon)]\}$$

$$U(1-\varepsilon) = \frac{1}{D}\{(1-\beta)\,u(1-\varepsilon) + \beta(1-\delta)\,[u(1-\varepsilon) + u(1+\varepsilon)]\} \quad (8.29)$$

where

$$D = \frac{(1-\beta\delta)^2 - (\beta - \beta\delta)^2}{1-\beta} > 0 \qquad (8.30)$$

The utility from autarky is the weighted sum of the utility from consuming the endowment today, $u(1 + \varepsilon)$ (or $u(1 - \varepsilon)$ for the currently poor agent) and the expected future utility, which is proportional to $u(1 + \varepsilon) + u(1 - \varepsilon)$. A change in ε thus changes current consumption and the risk of future consumption.

We are interested in how the variability of the income process ε affects the consumption distribution. Since the only reason perfect insurance is not possible is the presence of the individual rationality constraints, we first want to characterize the right hand side of these constraints $U(1+\varepsilon)$ and $U(1-\varepsilon)$, as functions of ε.

Let by $U^{FB} = u(1)$ denote the lifetime utility from the first best, perfect risk sharing allocation. This lifetime utility would be obtained by both agents if the individual rationality constraints were absent, i.e. with complete markets and no commitment problem.

By simple inspection of (8.29) we can prove the following

Lemma 61 *The continuation utilities from the autarkic allocation satisfy the following properties*

1. *$U(1 + \varepsilon)|_{\varepsilon=0} = U(1 - \varepsilon)|_{\varepsilon=0} = u(1) \equiv U^{FB}$*

2. *$U(1 + \varepsilon)$ and $U(1 - \varepsilon)$ are strictly concave and differentiable for all $\varepsilon \in [0, 1)$*

3. *$\frac{dU(1-\varepsilon)}{d\varepsilon} < 0$ for all $\varepsilon \in [0, 1)$*

4. *$\frac{dU(1+\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} > 0$ and $\lim_{\varepsilon\to 1} \frac{dU(1+\varepsilon)}{d\varepsilon} < 0$*

5. *There exists a unique $\varepsilon_1 = \arg\max_{\varepsilon\in[0,1)} U(1+\varepsilon)$. For all $\varepsilon \in [0, \varepsilon_1)$ we have $\frac{dU(1+\varepsilon)}{d\varepsilon} > 0$ and for all $\varepsilon \in (\varepsilon_1, 1)$ we have $\frac{dU(1+\varepsilon)}{d\varepsilon} < 0$*

6. *There exists at most one $\varepsilon_2 \in (0,1)$ such that $U(1 + \varepsilon_2) = u(1)$. If ε_2 exists, it satisfies $\varepsilon_2 > \varepsilon_1$.*

The proof of this lemma is straightforward and hence omitted. The interpretation is also fairly simple: without income fluctuations the autarkic allocation is first-best; since an increase in income variability reduces current consumption and increases future consumption risk, it reduces the continuation utility for the currently poor agent; for the rich agent initially the direct effect of currently higher consumption dominates the risk effect, but as ε becomes large the risk effect becomes dominant; the last two properties follow from strict concavity of $U(1 + \varepsilon)$ and the signs of the derivatives at $\varepsilon = 0$ and $\varepsilon \to 1$. The first figure at the end of this chapter graphically summarizes the lemma.

We now want to characterize constrained-efficient consumption allocations in this model and analyze how they change with changes in ε. For this we note that in any insurance mechanism as the one described in this model it is efficient to transfer resources from the currently rich to the currently poor agent. Therefore the constrained-efficient consumption distribution features maximal insurance, subject to delivering at least the continuation utility of autarky to the currently rich agent. This argument motivates the following proposition, which is due to Kehoe and Levine (2001):[4]

Proposition 62 *The constrained-efficient consumption distribution is completely characterized by a number $\varepsilon_c(\varepsilon) \geq 0$. The agent with income $1 + \varepsilon$ consumes $1 + \varepsilon_c(\varepsilon)$ and the agent with income $1 - \varepsilon$ consumes $1 - \varepsilon_c(\varepsilon)$ regardless of her past history. The number $\varepsilon_c(\varepsilon)$ is the smallest non-negative solution of the following equation*

$$\max(U^{FB}, U(1 + \varepsilon)) = U(1 + \varepsilon_c(\varepsilon)) \tag{8.31}$$

The interpretation of this result goes as follows: if $\max(U^{FB}, U(1+\varepsilon)) = U^{FB}$, then $\varepsilon_c(\varepsilon) = 0$ and the resulting allocation is the first-best, complete risk sharing allocation. From the previous lemma we know that this case applies to all $\varepsilon \geq \varepsilon_2$, if such ε_2 exists. Now suppose that $\max(U^{FB}, U(1 + $

---

[4]This proposition is less general than Kehoe and Levine's original proposition, because their environment also contains a long-lived asset. It is more general since, making agents ex ante identical enables us to establish the characterization of the constrained-efficient consumption distribution from the initial model period onward, and not just after a transition period after which initial conditions cease to matter, as in Kehoe and Levine.

$\varepsilon)) = U(1 + \varepsilon)$. Obviously one solution for $\varepsilon_c(\varepsilon) = \varepsilon$, i.e. the autarkic consumption allocation. We note from the previous lemma that if $\varepsilon \leq \varepsilon_1$, then this is in fact the resulting consumption allocation. In this case, for small ε, there is no risk sharing possible whatsoever, since at any level of risk sharing the rich agent would have an incentive to default. However, if $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ then there exists an $\varepsilon_c(\varepsilon) < \varepsilon_1$ such that $U(1 + \varepsilon) = U(1 + \varepsilon_c(\varepsilon))$. In this case the resulting consumption allocation features some risk sharing, but not complete risk sharing. The second figure at the end of this chapter shows a sample path for the income process and the resulting constrained-efficient sample path for the consumption allocation.

How does the dispersion of the consumption distribution $\varepsilon_c(\varepsilon)$ vary with an increase in income dispersion. From the previous discussion we immediately obtain the following proposition (see Krueger and Perri, 2006a):

Proposition 63 *For given δ, starting from a given income dispersion ε a marginal increase in ε leads to a decrease in consumption inequality if and only if $\varepsilon_c(\varepsilon) < \varepsilon$ (in the initial equilibrium there is positive risk sharing). The decrease is strict if and only if $0 < \varepsilon_c(\varepsilon) < \varepsilon_0$ (in the initial equilibrium there is positive, but not complete risk sharing).*

This discussion is summarized in the third figure at the end of this chapter. Similarly one can establish how the consumption distribution varies with changes in the persistence of the income process. See Kehoe and Levine (2001) or Krueger and Perri (2006a) for a proof.

Proposition 64 *For a given income dispersion ε a marginal increase in persistence δ leads to an increase in consumption inequality. The increase is strict if and only if $0 < \varepsilon_c(\varepsilon, \delta) < \varepsilon$ (in the initial equilibrium there is positive, but not complete risk sharing).*

So for this simple example we have a complete characterization of the cross-sectional consumption distribution and how it varies with the parameters of the model. However, the resulting consumption distribution is somewhat trivial. since the model consists of only two agents. For serious quantitative work we therefore would like a similar model, but with a large number of agents, in some sense a counterpart of Aiyagari's (1994) or Huggett's (1993) model for a world with limited commitment. The formulation and numerical solution of such model is discussed next.

8.6 A Continuum Economy

In this section we formulate a version of the above model with a continuum of infinitely lived agents. Let $\{y_t\}$ denote an agent's idiosyncratic income process, which is assumed to be a finite state Markov chain with transition probabilities π and associated invariant distribution Π. As before an endowment shock history is denoted by y^t, with probability of that history occurring, conditional on the initial income shock y_0, equal to $\pi(y^t|y_0)$. We again assume a law of large numbers so that $\pi(y^t|y_0)$ is also the deterministic fraction of the population that started with y_0 and has experienced history y^t. It is also assumed that the initial income distribution (and hence the income distribution at any future date) is given by Π. Agents in this continuum economy face the same commitment problem as the agents in the simple economy before.

Let $\Phi(a_0, y_0)$ denote the initial distribution over assets and income. We first want to compute and characterize constrained-efficient allocations and then decentralize them as equilibria with borrowing constraints that are not too tight, in the spirit of Alvarez and Jermann (2000). Note that an agents' initial wealth level (and initial income realization) determines how much expected discounted lifetime utility this agent can obtain. Let this utility level be given by w_0; we will formally establish the mapping between (a_0, y_0) and (w_0, y_0) below. Let the initial distribution of utility promises and income be denoted by $\Psi(w_0, y_0)$.

Previously we solved for the constrained-efficient consumption allocation by maximizing one agent's expected utility, subject to the promise-keeping constraint of delivering at least a certain minimum utility level to the other agent (and, of course, subject to the individual rationality constraints). With a continuum of agents this is evidently impossible, since there would be a continuum of promise-keeping constraints. Atkeson and Lucas (1992, 1995) propose a dual approach to overcome this problem: instead of maximizing someone's utility, subject to enforcement constraints, the resource constraint and individual rationality constraints one minimizes the cost of delivering a given level of promised utility to a particular agent; the distribution of utility promises is then adjusted so as to preserve resource feasibility.

Atkeson and Lucas first formally define constrained efficiency (in their dual sense), then formulate a sequential social planners problem that solves for constrained-efficient allocations and then make this problem recursive and finally prove that the policy functions from the recursive problem induce

constrained-efficient sequential consumption allocations $\{c_t(w_0, y^t)\}$. We will directly go to the recursive formulation; readers interested in the details may consult Krueger (1999).

In order to make the cost minimization operative we first have to endow the social planner with a time discount factor that measures the relative "price" of resources today versus tomorrow. Let this shadow interest rate be denoted by $R \in (1, \frac{1}{\beta})$. We will see later that this R will correspond to the risk-free interest rate in the decentralization of the allocations. Also define the function $C \equiv u^{-1}$ denote the inverse of the period utility function u. The entity $C(u)$ is interpreted as the amount of current consumption needed to generate period utility u. The key idea of Atkeson and Lucas is to realize that, since the individual rationality constraints involve continuation utilities, one may let the planner allocate utilities instead of consumption and use promises of expected discounted utility as state variables.[5] The function C may then be used to translate utility allocations back into consumption allocations.

So let the individual state variables in the dynamic programming problem be (w, y), that is, the current promise of expected discounted future utility an agent enters the period with and the current income realization y. In order to satisfy this utility promise the planner can either give the agent current utility h or expected utility from tomorrow onward, conditional on tomorrow's income shock y', $g(y')$. The dynamic programming problem the planner solves is then given by

$$V(w, y) = \min_{h, g(y')} \left(1 - \frac{1}{R}\right) C(h) + \frac{1}{R} \sum_{y' \in Y} \pi(y'|y) V(g(y'), y') \quad (8.32)$$

$$\text{s.t.}$$

$$w = (1 - \beta)h + \beta \sum_{y' \in Y} \pi(y'|y) g(y') \quad (8.33)$$

$$g(y') \geq U^{Aut}(y') \quad (8.34)$$

Here $V(w, y)$ is the total (normalized) resource cost the planner has to minimally spend in order to fulfill his utility promises w, without violating the individual rationality constraint of the agent: he can't promise less from tomorrow onwards, in any state of the world, than the agent would obtain from the autarkic allocation. As before, multiplying the current cost by the

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[5]Of course this is not their idea; it goes back at least to Abreu (1986) and Spear and Srivastava (1987).

factor $1 - \frac{1}{R}$ expresses total cost in the same units as current cost; it is an innocuous normalization that just changes the units in which $V(w, y)$ is measured. Note that if the income process is *iid*, the variable y does not appear in the minimization problem and hence the value function V does not depend on y.

It is a standard exercise to show that the operator induced by this functional equation is a contraction, that the resulting unique fixed point V is differentiable and strictly convex (strict convexity is not that straightforward, but note that since u is strictly concave, C is strictly convex) and that the resulting policies $h(w, y)$ and $g(w, y; y')$ are single-valued continuous functions. Furthermore h is strictly increasing in w and g is either constant at $U^{Aut}(y')$ or strictly increasing in w as well.

Attach Lagrange multiplier λ to the promise-keeping constraint (8.33) and multipliers $\pi(y'|y)\mu(y')$ to the individual rationality constraints (8.34). The first order conditions and envelope conditions read as

$$\left(1 - \frac{1}{R}\right) C'(h) = \lambda(1 - \beta) \tag{8.35}$$

$$\frac{1}{R}V'(g(y'), y') = \beta\lambda - \mu(y') \tag{8.36}$$

$$V'(w, y) = \lambda \tag{8.37}$$

Combining equations (8.36) and (8.37) and the complementary slackness conditions yields

$$V'(g(y'), y') = R\beta V'(w, y) - \mu(y') \tag{8.38}$$

or

$$\begin{aligned} V'(g(y'), y') &\leq R\beta V'(w, y) \\ &= \quad \text{if } g(y') > U^{Aut}(y') \end{aligned} \tag{8.39}$$

Now suppose that income is *iid* and that $R\beta < 1$, something that was true in general equilibrium in the standard incomplete markets model in the last chapter and is true (under weak conditions) in this model as well (see again Krueger (1999) or Krueger and Perri (2006b)). Then the Euler equation (8.39) becomes

$$\begin{aligned} V'(g(y')) &< R\beta V'(w) \\ &= \quad \text{if } g(y') > U^{Aut}(y') \end{aligned} \tag{8.40}$$

Since V is strictly convex, this implies that either $g(w; y') < w$ or $g(w; y') = U^{Aut}(y')$. That is, for states in which the individual rationality constraints are not binding utility promises for tomorrow are lower than for today; for states in which the constraint is binding utility promises are raised sufficiently in order to prevent the agent from reneging. Define $\underline{w} = \min_{y \in Y} U^{Aut}(y)$ and $\bar{w} = \max_{y \in Y} U^{Aut}(y)$. It is clear that for all w we must have $g(w; y') \geq \underline{w}$. On the other hand we have that for $y_{\max} = \arg \max_{y \in Y} U^{Aut}(y)$ we have $g(U^{Aut}(y_{\max}), y_{\max}) = U^{Aut}(y_{\max})$ and for all $w > U^{Aut}(y_{\max})$ we have $g(w, y') \leq g(w, y_{\max}) \leq U^{Aut}(y')$. Thus for *iid* income shocks the state space for promised utility w, denoted by W, is bounded: $W = [\underline{w}, \bar{w}]$. For 2 income shocks the fourth figure at the end of this chapter demonstrates the situation; for a formal proof of the assertions above see Krueger and Perri (2006b). You may want it instructive to follow an agent with given initial utility promise w_0 through his life with a sequence of income shocks: one positive income shock brings utility to $w = \underline{w}$, with a sequence of bad income shock making the agent move down in promised utility, until he hits $w = \underline{w}$, with a single good shock putting him back at $w = \bar{w}$.

So far resource feasibility and the determination of the distribution of promised utilities has not been discussed; in fact, the beauty of Atkeson and Lucas' (1992, 1995) approach is that it separates the dynamic programming problem the planner has to solve for a given individual (w, y) from the problems solved for other individuals in society. We want to find a gross real shadow interest rate and associated invariant measure over utility promises such that the resources available to the planner equals the resources given out by the planner.

The remaining discussion of this model has strong similarities with the discussion in Aiyagari (1994) and Huggett (1993) for standard incomplete markets models. First, for a given interest rate R, the dynamic programming problem (**??**) delivers value function $V(w, y)$ and policy functions $h(w, y)$ and $g(w, y; y')$. In order to find the associated stationary distribution over utility promises and income shocks Ψ_R we first determine the Markov transition function Q_R induced by π and $g(w, y; y')$. First one has to prove that there is an upper bound for utility promises w, denoted by $\bar{w}$ (this is straightforward for π being *iid*, but not for the general case). Then denote the state space for utility promises by $W = [\underline{w}, \bar{w}]$, define $Z = W \times Y$ and $\mathcal{B}(Z) = \mathcal{B}(W) \times \mathcal{P}(Y)$.

Then the transition function $Q_R : Z \times \mathcal{B}(Z) \to [0,1]$ is given by

$$Q_R((w,y),(A,\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \left\{ \begin{array}{cc} \pi(y'|y) & \text{if } g(w,y;y') \in A \\ 0 & \text{else} \end{array} \right. \tag{8.41}$$

The invariant measure Ψ_R over (w,y) then satisfies

$$\Psi_R(A,\mathcal{Y}) = \int Q_R((w,y),(A,\mathcal{Y}))d\Psi_R \tag{8.42}$$

Of course one has to prove that such invariant measure exists and is unique, the proof of which for the *iid* case is contained in Krueger (1999). For the general, non-iid case I'm not aware of any such result, which is similar to the status of theoretical results available for the standard incomplete markets model discussed in the previous chapter.

The last figure at the end of this chapter shows the stationary consumption distribution associated with $\Psi_R(w,y)$, again for the *iid* case with only two possible income shocks. Since consumption $C(h(w,y))$ is strictly increasing in utility promises w, the consumption distribution mimics the utility distribution Ψ_R. In particular, all agents with currently high shock consume the same, and the maximum in the consumption distribution, with agents with a sequence of bad shocks working themselves down through the consumption distribution until they have hit rock bottom.

So for a given intertemporal shadow interest rate R we now know how determine the associated stationary utility promise distribution. So far nothing assures that the social planner can satisfy this distribution of utility promises with the aggregate resources available to society. Total resources available are

$$\int y d\Psi_R = \sum_y y \Pi(y) \tag{8.43}$$

How about total resources needed by the planner? An agent that enters this period with utility promises w and current income y is awarded current utility $h(w,y)$. Thus requires, in terms of resources, $C(h(w,y))$. Thus total resources required to satisfy utility distribution Ψ_R in the current period (and by stationarity in each period) are

$$\int C(h(w,y))d\Psi_R \tag{8.44}$$

and therefore the excess resource requirements, as a function of the gross shadow interest rate R, are given by

$$d(R) = \int \left(C(h(w,y)) - y \right) d\Psi_R \tag{8.45}$$

To finish the determination of a stationary constrained-efficient allocation requires to find an R^* such that $d(R^*) = 0$. Computationally this can be done by searching over $R \in (1, \frac{1}{\beta})$ for such R^*. Theoretically, for the *iid* case one can show that $d(R)$ is a continuous and increasing function in R on $(1, \frac{1}{\beta})$. With additional assumptions on u and the income process one can show that $\lim_{R \to \frac{1}{\beta}} d(R) > 0$ and that $\lim_{R \to 1} d(R) \leq 0$, proving existence of a stationary constrained-efficient consumption allocation. Since it is hard to prove that $d(R)$ is strictly increasing, uniqueness is hard to establish (obviously the set of efficient $R's$ is convex). Again see Krueger (1999) or Krueger and Perri (2006b) for the details.

Finally we can decentralize a constrained efficient stationary consumption allocation as a sequential markets equilibrium with borrowing constraints that are not too tight, in the spirit of Alvarez and Jermann. Agents trade a full set of Arrow securities; in this model these are securities that pay out conditional on *individual* income realizations, rather than aggregate realizations. The determination of the borrowing constraints that are not too tight is as before and hence omitted. Finally, the price of an Arrow security bought/sold today by an agent with current individual income shock y that pays out one unit of consumption if tomorrow's income shock for that agent is y' turns out to be

$$q(y'|y) = q\pi(y'|y) = \frac{\pi(y'|y)}{R} \tag{8.46}$$

which justifies that R is in fact called a shadow interest rate: it turns out to be the equilibrium risk free interest rate in the corresponding equilibrium with borrowing constraints that are not too tight.

Finally we want to discuss the relationship between initial promised utilities w_0 and initial assets a_0. So suppose we have found a stationary constrained-efficient utility distribution $\Psi(w,y)$, a corresponding R^* and associated value and policy functions $V(w,y)$, $h(w,y)$ and $g(w,y;y')$. First, analogously to (8.15) and (8.16) from the recursive policy functions we can construct sequential constrained-efficient consumption allocations $\{c_t(w_0,y_0)\}$ for an agent

with initial conditions (w_0, y_0). What is missing is the connection between initial conditions (w_0, y_0) and (a_0, y_0) (and the corresponding relationship between $\Psi(w_0, y_0)$ and $\Phi(a_0, y_0)$). Define Arrow-Debreu prices associated with (8.46) as

$$
\begin{aligned}
Q(y_0) &= \Pi(y_0) \\
Q(y^t) &= q(y_t | y_{t-1}) * \ldots q(y_1 | y_0) \Pi(y_0)
\end{aligned} \tag{8.47}
$$

Then initial assets associated with (w_0, y_0), denoted by $a_0(w_0, y_0)$, are given as

$$
a_0(w_0, y_0) = \sum_{t=0}^{\infty} \sum_{y^t | y_0} Q(y^t) \left(c_t(w_0, y^t) - y_t \right) \tag{8.48}
$$

and the associated equilibrium consumption allocations are given by

$$
c_t(a_0, y^t) = c_t(a_0^{-1}(a_0, y_0), y^t) \tag{8.49}
$$

where $w_0 = a_0^{-1}(a_0, y_0)$ is the inverse function of (8.48) with respect to the first argument (note that this function is well-defined because $a_0(w_0, y_0)$ is strictly increasing in w_0). The distribution $\Phi(a_0, y_0)$ is then determined as

$$
\Phi(a_0, y_0) = \Psi(a_0^{-1}(a_0, y_0), y_0) \tag{8.50}
$$

What one then can prove (see Krueger (1999) is that for an initial distribution $\Phi(a_0, y_0)$ given by (8.50), the allocation determined by (8.49) and prices (8.46) are a competitive equilibrium with borrowing constraints that are not too tight (or alternatively, form a competitive equilibrium in the spirit of Kehoe and Levine (1993), where equilibrium prices are given by (8.47)).

A few final comments: the method of using utility promises as state variables to make dynamic contracting problems recursive has been used in a number of applications, for example in the study of optimal unemployment insurance (Shavell and Weiss 1979, Hopenhayn and Nicolini 1997, Zhao 2000), asset pricing and the equity premium (Alvarez and Jermann 2000, Lustig 2001), sovereign debt (Atkeson 1991, Kletzer and Wright 2000), redistributive taxation (Krueger and Perri 2006b, Attanasio and Rios-Rull 2001), time-consistent monetary policy (Chang 1998), time-consistent fiscal policy (Phelan and Stacchetti 1999, Sleet 1998) and risk sharing in village economies (Udry 1994, Ligon, Thomas and Worrall 2001).

Also, instead of using utility promises Marcet and Marimon (1999) have developed a recursive method that use (cumulative) Lagrange multipliers as

state variables. As before, the same problems with the curse of dimensionality when the number of heterogeneous agents becomes large arises.

So what next: on the methodological side one would like to figure out how to handle models with a large number of agents and aggregate uncertainty, both theoretically and numerically. Lustig (2001) takes a big step in that direction. On the substantial side a careful study of redistribution and insurance over the business cycle, of optimal insurance of large income risks in the future, both by government policies and private arrangements, and of the dynamics of the income, consumption and wealth distribution seems to be fruitful avenues for future research.

Chapter 9

Private Information

In this section we consider another friction that may prevent perfect risk sharing from occurring as a result of efficient or equilibrium consumption allocations. As in standard Arrow Debreu theory now households and financial intermediaries can write legally binding and enforceable contracts. However, we now assume that individual income realizations (and individual consumption) is private information of the agent. Financial intermediaries or the social planner have to rely on reports of income by agents. Consumption allocations therefore have to structured in such a way that households find it optimal to tell the truth about their income realization, rather than to lie about it. We will first consider the problem of a financial intermediary dealing with a single agent in isolation, before discussing a model with many agents, an aggregate resource constraint and an endogenous interest rate

9.1 Partial Equilibrium

Our treatment of the partial equilibrium case is motivated by the seminal papers by Green (1987) and Thomas and Worrall (1990), which in turn is nicely discussed in Ljungqvist and Sargent's book. Consider a risk-neutral financial intermediary that lives forever and discounts the future at time discount factor $\beta \in (0, 1)$. This financial intermediary (sometimes called principal) engages in a long-term relationship with a risk averse household (sometimes called agent) that also discounts the future at factor β. The agent faces a stochastic income process $\{y_t\}$, assumed to be *iid* with finite support $Y = \{y_1, y_2, \ldots y_N\}$ and probabilities $(\pi_1, \pi_2, \ldots, \pi_N)$; and seeks insurance

from the risk-neutral principal. Both parties can commit to long-term contracts, so that in the absence of private information the optimal consumption allocation for the agent, subject to the financial intermediary breaking even, is

$$c_t(y^t) = E(y_t) = E(y) = 1$$

where the last equality is by assumption (that is, we normalize mean income to 1). Thus the agent hands over his realized income in every period and receives constant consumption equal to mean income back from the financial intermediary. His lifetime utility equals

$$U(c) = (1 - \beta) \sum_{t=0}^{\infty} \sum_{y^t} \pi_t(y^t) \beta^t u(c_t(y^t)) = u(1)$$

and expected utility of the financial intermediary (or profits) equal

$$
\begin{aligned}
W(c) &= (1 - \beta) \sum_{t=0}^{\infty} \sum_{y^t} \pi_t(y^t) \beta^t \left(y_t - c_t(y^t) \right) \\
&= (1 - \beta) \sum_{t=0}^{\infty} \beta^t \sum_{y^t} \pi_t(y^t) \left(y_t - E(y_t) \right) \\
&= 0
\end{aligned}
$$

The big problem with this consumption allocation is that the agent's income realizations are private information. If the agent is promised constant consumption independent of his income report, then he would always report $y_t = y_1$, keep the difference between his true income and the report, receive 1 from the financial intermediary and do strictly better. Then, however, the principal would lose money, since his profits would be

$$
\begin{aligned}
W &= (1 - \beta) \sum_{t=0}^{\infty} \sum_{y^t} \pi_t(y^t) \beta^t \left(y_1 - 1 \right) \\
&= y_1 - 1 < 0.
\end{aligned}
$$

Thus the tension in the current environment is between the provision of insurance (without the informational frictions it is efficient for the financial intermediary to insure the agent, since he is better taking risk because of his risk-neutrality) and the provision of incentives to tell the truth (and we saw

above that without providing these incentives the principal is going to lose money).

What we want to construct in the following is the efficient long-term insurance contract between the two parties, explicitly taking into account the informational frictions in this environment. We will again immediately proceed to the recursive formulation of the problem, understanding that in principle one should first write down the sequential problem, then the recursive problem and then prove the principle of optimality.[1] The recursive formulation of the contracting problem again makes use of promised lifetime utility w as a state variable. Let us first pose the dynamic programming problem and then explain it:

$$V(w) = \min_{\{t_s, w_s\}_{s=1}^{N}} \sum \pi_s \left[(1-\beta)t_s + \beta V(w_s)\right] \quad (9.1)$$

$$\text{s.t.}$$

$$w = \sum \pi_s \left[(1-\beta)u(t_s + y_s) + \beta w_s\right] \quad (9.2)$$

$$(1-\beta)u(t_s + y_s) + \beta w_s \geq (1-\beta)u(t_k + y_s) + \beta w_k \ \forall s, k \quad (9.3)$$

$$t_s \in [-y_s, \infty) \quad (9.4)$$

$$w_s \in [\underline{w}, \bar{w}] \quad (9.5)$$

We cast the optimal contracting problem as the problem of minimizing cost of the contract for the principal, subject to delivering lifetime utility w to the agent (as assured by the promise-keeping constraint (9.2)), subject to the agent having the correct incentives to report income truthfully (see constraints (9.3)), and subject to the domain restrictions (9.4) and (9.5) which we will discuss below.[2] The choice variables are the state-dependent transfers the principal makes to the agent today, t_s, and the state-dependent continuation utility promises w_s from tomorrow onwards.

---

[1]In fact, the incentive constraints in the sequential formulation are much more complex. That they can be simplyfied to period-by-period incentive constraints (so-called temporary incentive compatibility constraints, in Green's (1987) language) is one of the major results proved in Green's paper. Only because of this result do we obtain a simple recursive formulation of the problem.

[2]We can restrict ourselves to contracts that induce truthtelling behavior because the revelation principle assures that whatever insurance can be provided with an arbitrary mechanism can also be provided with a mechanism (contract) in which agents report their income directly (a direct mechanism) and are provided with the appropriate incentives to report income truthfully.

The constraints (9.3) say the following: suppose state s with associated income realization y_s is realized. If the agent reports the truth, he receives transfers t_s and continuation utility w_s, for total lifetime utility

$$u(t_s + y_s) + \beta w_s.$$

If, on the other hand, he falsely reports y_k, he receives transfers t_k and continuation utility w_k, for total lifetime utility

$$u(t_k + y_s) + \beta w_k.$$

Note that even when mis-reporting income, his true lifetime utility depends on his true, rather than his reported income y_s. The constraints (9.3) simply state that it has to be in the agents' own interest to truthfully reveal his income. The domain restriction on transfers is self-explanatory: the principal can make arbitrarily large transfers, but cannot force the agent to make payments bigger than their current (truthfully reported) income y_s. With respect to the bounds on promised utility we note the following. Let $\bar{w} = \sup_c u(c)$ and $\underline{w} = \inf_c u(c)$; evidently it is not possible to deliver more lifetime utility that $\bar{w}$ even with infinite resources and it is not possible to deliver less lifetime utility than $\underline{w}$ even without giving the agent any consumption even. We explicitly allow $\underline{w} = -\infty$ and $\bar{w} = +\infty$, but will make more restrictive assumptions later. For now we assume that

Assumption: *The utility function $u : [0, \infty) \to \mathbf{R}$ is strictly increasing, strictly concave, at least twice differentiable and satisfies the Inada conditions.*

So far is has been left unclear what determines the w the agent will start the contract with. A higher initial w means more lifetime utility for the agent, but less lifetime utility for the principal. The expected lifetime utility for the principal, conditional on delivering w to the agent, is given by

$$W(w) = 1 - V(w)$$

since $V(w)$ measures the lifetime expected costs from the contract and 1 measures the expected revenues from the agent, equal to his expected per period and (by normalization of $1 - \beta$) lifetime revenue. Thus varying the w traces out the constrained utility possibility frontier between principal and agent. One particularly important w is that w that solves $V(w) = 1$, that is, that lifetime utility of the agent that yields 0 profits for the principal. If V is strictly increasing in w, such w is necessarily unique.

9.1.1 Properties of the Recursive Problem

By the standard contraction mapping arguments a unique solution V to the functional equation exists Let us first bound $V(w)$. The cheapest way to provide lifetime utility w is by delivering constant consumption. This may not be incentive compatible, in fact it will not be, but surely provides a lower bound on $V(w)$. So define $c^{fb}(w)$ as

$$u(c) = w$$

or $c^{fb}(w) = u^{-1}(w)$. The value of the transfers needed to deliver the constant consumption stream $c^{fb}(w)$ are given by

$$\underline{V}(w) = c^{fb}(w) - 1.$$

Obviously

$$\underline{V}(w) \leq V(w)$$

with strict inequality if any of the incentive constraints is binding. On the other hand, the principal can decide to give constant transfers $t_s = t$ (and constant continuation utility $w_s = w$) Surely, since transfers are independent of reported income, this induces truth-telling. In order to obtain the utility promise w, the transfer has to satisfy

$$\sum_s \pi_s u(y_s + t) = w$$

Denote this transfer by $\bar{t}(w)$. The cost of this policy is given by

$$\bar{V}(w) = \bar{t}(w)$$

and evidently

$$V(w) \leq \bar{V}(w)$$

with strict inequality whenever the distribution of income shocks is not degenerate.

Observe the following facts. Since $c^{fb}(w)$ is strictly increasing and strictly convex, so is $\underline{V}(w)$. Furthermore, as long as u satisfies the Inada conditions we have $\underline{V}'(\underline{w}) = 0$ and $\underline{V}'(\bar{w}) = \infty$, since $u(.)$ and $c^{fb}(.)$ are inverse functions of each other. Finally we have $\underline{V}(\underline{w}) = -1$ and $\underline{V}(\bar{w}) = \infty$. Similarly, the function $\bar{t}(w)$ and thus $\bar{V}(w)$ are strictly increasing and strictly convex,

with $\bar{V}(\bar{w}) = \infty$. While this does not necessarily mean that $V(w)$ is strictly increasing (it is straightforward to show that it is) and strictly convex (it is harder to show that it is), this discussion gives us a fairly tight bound on $V(w)$.

The fact that $V(w)$ is strictly increasing in w is intuitive: delivering higher lifetime utility to the agent requires higher transfers on average by the principal. Since marginal utility is declining, a given additional unit of transfers increases utility by less and less, thus one would expect the cost function to be strictly convex. We will go ahead and assert this without proof.

Now we discuss further properties of the optimal contract. Consider the constraints

$$(1 - \beta)u(t_s + y_s) + \beta w_s \;\geq\; (1 - \beta)u(t_{s-1} + y_s) + \beta w_{s-1} \quad (9.6)$$
$$(1 - \beta)u(t_{s-1} + y_{s-1}) + \beta w_{s-1} \;\geq\; (1 - \beta)u(t_s + y_{s-1}) + \beta w_s \quad (9.7)$$

Adding them results in

$$u(t_s + y_s) + u(t_{s-1} + y_{s-1}) \geq u(t_{s-1} + y_s) + u(t_s + y_{s-1})$$

or

$$u(t_s + y_s) - u(t_{s-1} + y_s) \geq u(t_s + y_{s-1}) - u(t_{s-1} + y_{s-1})$$

Since u is strictly concave and $y_s > y_{s-1}$ we have $t_{s-1} \geq t_s$ (it helps to draw a picture to convince you of this). From (9.6) it then easily follows that $w_s \geq w_{s-1}$. That is, it is efficient to give households with lower income realizations higher transfers. But in order to provide the incentives of not always claiming to have had low income realizations, these households are "punished" with lower continuation utilities. We summarize this in the following

Proposition 65 *In an efficient contract, lower income reports are rewarded with higher current transfers, but lower continuation utilities: $t_{s-1} \geq t_s$ and $w_s \geq w_{s-1}$ for all $s \in S$*

Second we claim that the principal only has to worry about the agent lying locally, that is, if income is y_s, the principal has to only worry about the agent reporting y_{s-1} or y_{s+1}. As long as the agent does not have an incentive to lie locally, he does not want to lie and report y_{s-2} or y_{s+2} or even more extreme values. Formally

Proposition 66 *Suppose that the true income state is s and that all local constraints of the form*

$$(1 - \beta)u(t_s + y_s) + \beta w_s \geq (1 - \beta)u(t_{s+1} + y_s) + \beta w_{s+1} \qquad (9.8)$$
$$(1 - \beta)u(t_s + y_s) + \beta w_s \geq (1 - \beta)u(t_{s-1} + y_s) + \beta w_{s-1} \qquad (9.9)$$

hold. Then all other incentive constraints are satisfied.

Proof. We will show that, if these constraints are satisfied, then a household would not want to report $s - 1$ when the true state is $s + 1$. A simple repetition of the argument will then show that the household would not want to report $s - 1$ for any state $k \geq s$. The case of mis-reporting upwards is, of course, symmetric and hence omitted.

Since $s > s - 1$, we have, from the previous result, $t_s \leq t_{s-1}$ and thus (again by concavity of u)

$$u(t_s + y_{s+1}) - u(t_s + y_s) \geq u(t_{s-1} + y_{s+1}) - u(t_{s-1} + y_s) \qquad (9.10)$$

Multiplying (9.10) by $(1 - \beta)$ and adding it to (9.9) yields

$$(1 - \beta)u(t_s + y_{s+1}) + \beta w_s \geq (1 - \beta)u(t_{s-1} + y_{s+1}) + \beta w_{s-1}$$

But

$$(1 - \beta)u(t_{s+1} + y_{s+1}) + \beta w_{s+1} \geq (1 - \beta)u(t_s + y_{s+1}) + \beta w_s$$

(use (9.9), but for $s + 1$) and thus

$$(1 - \beta)u(t_{s+1} + y_{s+1}) + \beta w_{s+1} \geq (1 - \beta)u(t_s + y_{s+1}) + \beta w_s$$
$$\geq (1 - \beta)u(t_{s-1} + y_{s+1}) + \beta w_{s-1}$$

That is, if the household does not want to lie one state down from s to $s - 1$, he does not want to lie two states down, from $s + 1$ to $s - 1$ either. ∎

This result reduces the set of incentive constraints substantially. It turns out that we can reduce it even further. Without proof we state the following proposition (if you are interested in the proof, consult Thomas and Worrall (1990) or the discussion of the same paper in Sargent and Ljungqvist's book; note that the proof requires strict convexity of the cost function V).

Proposition 67 *The local downward constraints (9.9) are always binding, the local upward constraints (9.8) are never binding.*

Proof. Omitted ∎

Intuitively, this result makes a lot of sense. Remember, the purpose of the contract is for the principal to insure the agent against bad income shocks, with high transfers in states with bad income realizations. This, of course, triggers the incentive to report lower income than actually realized, rather than higher income. Thus the lying-down constraints are the crucial constraints. With this result the dynamic programming problem becomes more manageable. In particular, if $N = 2$, there is only one incentive constraint and the promise keeping constraint. It is then quite feasible to characterize the qualitative properties of the optimal contract in more detail, which we leave to the discussion of Atkeson and Lucas (1992) in class.

Let us consider instead a simple example with $N = 2$ income states. From our characterization of the binding pattern of the constraints the dynamic programming problem becomes

$$V(w) = \min_{\{t_1, t_2, w_1, w_2\}} \pi\left[(1-\beta)t_1 + \beta V(w_1)\right] + (1-\pi)\left[(1-\beta)t_2 + \beta V(w_2)\right]$$

$$\text{s.t.}$$

$$w = \pi\left[(1-\beta)u(t_1 + y_1) + \beta w_1\right] + (1-\pi)\left[(1-\beta)u(t_2 + y_2) + \beta w_2\right]$$

$$(1-\beta)u(t_2 + y_2) + \beta w_2 \geq (1-\beta)u(t_1 + y_2) + \beta w_1$$

where π is the probability of low income y_1. Let us proceed under the assumption that V is differentiable (which is not straightforward to prove). The the first order and envelope conditions read as

$$\pi(1-\beta) - \lambda\pi(1-\beta)u'(t_1 + y_1) + \mu(1-\beta)u'(t_1 + y_2) = 0$$
$$(9.11)$$

$$(1-\pi)(1-\beta) - \lambda(1-\pi)(1-\beta)u'(t_2 + y_2) - \mu(1-\beta)u'(t_2 + y_2) = 0$$
$$(9.12)$$

$$\pi\beta V'(w_1) - \lambda\pi\beta + \mu\beta = 0$$
$$(9.13)$$

$$(1-\pi)\beta V'(w_2) - \lambda(1-\pi)\beta - \mu\beta = 0$$
$$(9.14)$$

$$V'(w) = \lambda$$
$$(9.15)$$

Rewriting equations (9.13) and (9.14) and using (9.15) yields

$$\lambda = V'(w) = V'(w_1) + \frac{\mu}{\pi} = V'(w_2) - \frac{\mu}{1-\pi}$$

Thus, since $\mu > 0$, we have (by strict convexity of the cost function V) that $w_1 < w < w_2$, that is, utility promises spread out over time. Rewriting (9.11) and (9.12) yields

$$
\begin{aligned}
1 &= \lambda u'(t_1 + y_1) - \frac{\mu}{\pi} u'(t_1 + y_2) \\
&= \left(\lambda - \frac{\mu}{\pi}\right) u'(t_1 + y_1) + \frac{\mu}{\pi} \left[u'(t_1 + y_1) - u'(t_1 + y_2)\right] > \left(\lambda - \frac{\mu}{\pi}\right) u'(t_1 + y_1) \\
1 &= \left(\lambda + \frac{\mu}{1 - \pi}\right) u'(t_2 + y_2)
\end{aligned}
$$

and thus

$$
\begin{aligned}
1 &= V'(w_2)u'(c_2) = V'(w_1)u'(c_1) + \frac{\mu}{\pi} \left[u'(c_1) - u'(t_1 + y_2)\right] \\
&> V'(w_1)u'(c_1)
\end{aligned}
$$

The first equality (and again using strict convexity of the cost function) show that if future promises w_2 are increased (in response to, say, an increase in w), then it is necessarily optimal to also increase c_2. That is, the principal should spread costs over time, increasing both current costs and well as future costs. [To be completed]

9.2 Endogenous Interest Rates in General Equilibrium

See the papers by Phelan and Townsend (1991) and Atkeson and Lucas 1992, 1995 should be discussed

9.3 Applications

9.3.1 New Dynamic Public Finance

Paper that makes the closest connection to the traditional Ramsey tax literature is Werning (2007). Key paper starting this literature is Golosov, Kocherlakota and Tsyvinski (2003) and accessible summaries are Golosov, Tsyvinski and Werning (2006) as well as Kocherlakota (20xx).

Part IV

Conclusions

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