Lecture 7: Buffer-Stock Consumption Model

Dynamic Programming

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Three generations of models

- **1st:** *Permanent income hypothesis* (Friedman, 1957) or *life-cycle model* (Modigliani and Brumberg, 1954)
- **2nd:** *Buffer-stock consumption model* (Deaton, 1991, 1992; Carroll 1992, 1997, 2012)
- **3nd:** *Multiple-asset buffer-stock consumption models* (e.g. Kaplan and Violante, 2014)

Model

- **Objective:**

$$\max_{C_t, C_{t+1}, \dots} \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta^t u(C_t) \right]$$

subject to

$$\begin{aligned}
A_t &= M_t - C_t \\
M_{t+1} &= RA_t + Y_{t+1} \\
Y_{t+1} &= \xi_{t+1} P_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
P_{t+1} &= G P_t \psi_{t+1} \\
A_t &\geq -\lambda P_t \\
A_T &\geq 0
\end{aligned}$$

- **Mean-one shocks:** $\mathbb{E}_t[\epsilon_{t+1}] = \mathbb{E}_t[\psi_{t+1}] = 1 \Rightarrow$

$$\mathbb{E}_t[Y_{t+1}] = \mathbb{E}_t[P_{t+1}] = GP_t$$

Vocabulary

1. M_t: Cash-on-hand
2. P_t: Permanent income
3. Y_t: Income
4. C_t: Consumption
5. A_t: End-of-period assets
6. ξ_t: Transitory shock ($\log \epsilon_t \sim \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2)$)
7. ψ_t: Permanent shock ($\log \psi_t \sim \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2)$)
8. $u(C_t) = C_t^{1-\rho}/(1-\rho), \rho > 1$: CRRA utility

Bellman equation

$$
\begin{aligned}
V(M_t, P_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V(M_{t+1}, P_{t+1}) \right] \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= R A_t + Y_{t+1} \\
Y_{t+1} &= \xi_{t+1} P_{t+1} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
P_{t+1} &= G P_t \psi_{t+1} \\
A_t &\geq -\lambda P_t \\
A_T &\geq 0
\end{aligned}
$$

Bellman equation - normalized

- **Define**

$$c_t \equiv C_t/P_t,\ m_t \equiv M_t/P_t, \ldots$$
$$v(m_t) \equiv V(M_t, P_t)/P_t^{1-\rho}$$

- The problem can be written in **ratio-form**

$$\begin{aligned}
v(m_t) &= \max_{C_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t\left[(G\psi_{t+1})^{1-\rho} v(m_{t+1})\right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{1}{G\psi_{t+1}} R a_t + \xi_{t+1} \\
a_t &\geq -\lambda \\
a_T &\geq 0
\end{aligned}$$

Euler-equation

- **Optimal choice:** $c(m_t)$
- All optimal **interior choices** must satisfy

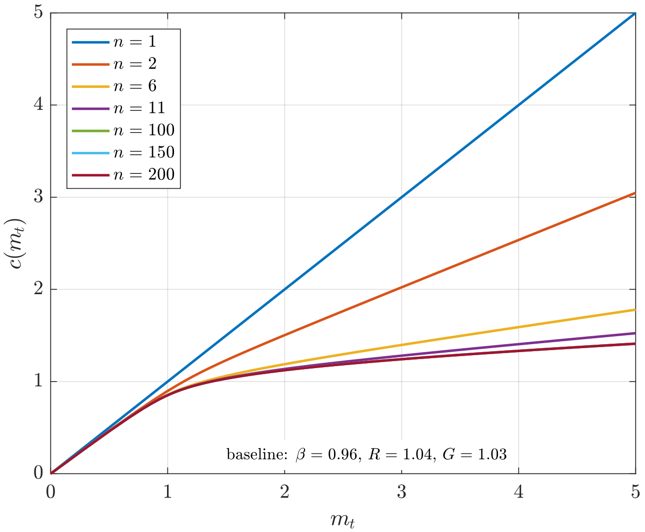
$$\begin{aligned} C_t^{-\rho} &= \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right] \Leftrightarrow \\ c_t^{-\rho} &= \beta R \mathbb{E}_t \left[(G\psi_{t+1} c_{t+1})^{-\rho} \right] \end{aligned}$$

- Else optimal choice is **constrained**

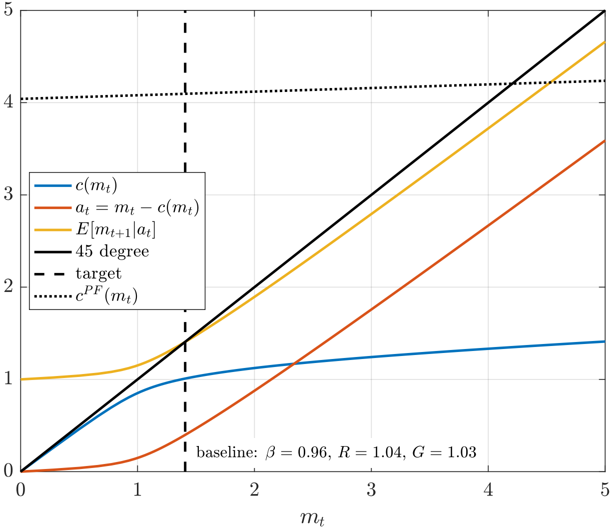
$$\begin{aligned} C_t^{-\rho} &\geq \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right] \Leftrightarrow \\ C_t &= M_t + \lambda P_t \Leftrightarrow \\ c_t &= m_t + \lambda \end{aligned}$$

Model
EGM
Perfect foresight
Buffer-stock
Preferences
Life-cycle
Until next
Appendix

Convergence of $c(m_t)$



Buffer-stock target



Endogenous grid method (EGM) ($\lambda = 0$)

- **Prerequisites** ($\lambda = 0 \Rightarrow \underline{m}_t = 0 \Rightarrow \underline{a}_t = 0$)
  1. **Inverted Euler-equation:** $c_t = \left[\beta R \mathbb{E}_t\left[(G\psi_{t+1}c_{t+1})^{-\rho}\right]\right]^{-\frac{1}{\rho}}$
  2. **Next-period consummption function**: $c_{t+1}(m_{t+1})$
  3. **Asset grid:** $\mathcal{G}_a = \{a_1, a_2, \dots, a_\#\}$ with $a_1 = \underline{a}_t + 10^{-6}$
- **Algorithm:** For each $a_i \in \mathcal{G}_a$
  1. Find consumption

$$c_i = \left[\beta R \mathbb{E}_t\left[(G\psi_{t+1}c_{t+1}(\frac{R}{G\psi_{t+1}}a_i + \xi_{t+1}))^{-\rho}\right]\right]^{-\frac{1}{\rho}}$$

  2. Find endogenous state

$$a_i = m_i - c_i \Leftrightarrow m_i = a_i + c_i$$

- The **consumption function**, $c_t(m_t)$, is given by

$$\{0, c_1, c_2, \dots, c_\#\} \text{ for } \{\underline{a}_t, m_1, m_2, \dots, m_\#\}$$

- *We can find all consumption functions in this way!*

Perfect foresight (PF)

- **No uncertainty**

$$\sigma_\xi = \sigma_\psi = \pi = 0 \Rightarrow Y_t = P_t = G^t P_0$$

- **No borrowing constriant**

$$\lambda = \infty$$

- **Euler-equation** is simpler

$$\begin{aligned} C_t^{-\rho} &= \beta R C_{t+1}^{-\rho} \Leftrightarrow \\ C_{t+1} &= (\beta R)^{1/\rho} C_t \Leftrightarrow \\ c_{t+1} &= (\beta R)^{1/\rho} / G c_t \end{aligned}$$

PF - Taxonomy of patience

1. **Utility impatience (UI)**

$$\beta < 1$$

2. **Return impatience (RI)**

$$(\beta R)^{1/\rho}/R < 1 \Rightarrow \mathbf{PDV}(C_t, C_{t+1}, \ldots) < \infty \Rightarrow C_t > 0$$

3. **Growth impatience (GI)**

$$(\beta R)^{1/\rho}/G < 1 \Rightarrow c_{t+1} < c_t$$

4. **Absolute impatience (AI)**

$$(\beta R)^{1/\rho} < 1 \Rightarrow C_{t+1} < C_t$$

PF - Solution

- **Return impatience (RI):** $(\beta R)^{1/\rho}/R < 1$
- **Finite human wealth (FHW):** $G/R < 1$
- **Solution** if RI + FHW then

$$\begin{aligned}
\text{PDV}(C_0, C_1, \dots) &= M_0 + \text{PDV}(P_1, P_2, \dots) \\
\sum_{t=0}^{\infty}((\beta R)^{1/\rho})^t C_0 R^{-t} &= M_0 + \sum_{t=0}^{\infty} G^t P_0 R^{-t} - P_0 \\
(1-\text{RI})^{-1} C_0 &= M_0 + (1-G/R)^{-1} P_0 - P_0 \Leftrightarrow \\
c_t &= (1-\text{RI})[m_t + (1-G/R)^{-1} - 1]
\end{aligned}$$

- **Value function**

$$\begin{aligned}
V_0 &= \sum_{t=0}^{\infty} \beta^t u(C_0((\beta R)^{1/\rho})^t) = \sum_{t=0}^{\infty}(\beta(\beta R)^{(1-\rho)/\rho})^t u(C_0) \\
&= \sum_{t=0}^{\infty}((\beta R)^{1/\rho}/R)^t u(C_0) = \frac{1}{1-\text{RI}} u(C_0)
\end{aligned}$$

PF-con - Adding constraint

- We now **assume** $\lambda = 0$.
- **Solution:** RI + FHW is still *sufficient*, but not necessary
- **Standard solutions:** RI + FHW
  1. **GI** $\Rightarrow$ *constraint will eventually be binding*

     $$c(m_t) \text{ converge to } c^{PF}(m_t) \text{ from below as } m_t \to \infty$$

  2. **Not GI** $\Rightarrow$ *constraint is never reached*

     $$c(m_t) = c^{PF}(m_t) \text{ for } m_t \geq 1$$

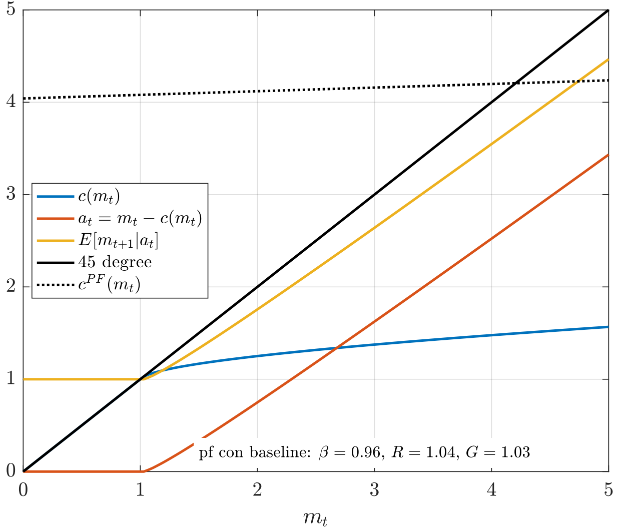
- **Exotic solutions without FHW** (GI necessary)
  1. GI with RI

     $$\lim_{m \to \infty} c'(m_t) = c^{PF\prime}(m_t) = 1 - \text{RI}$$

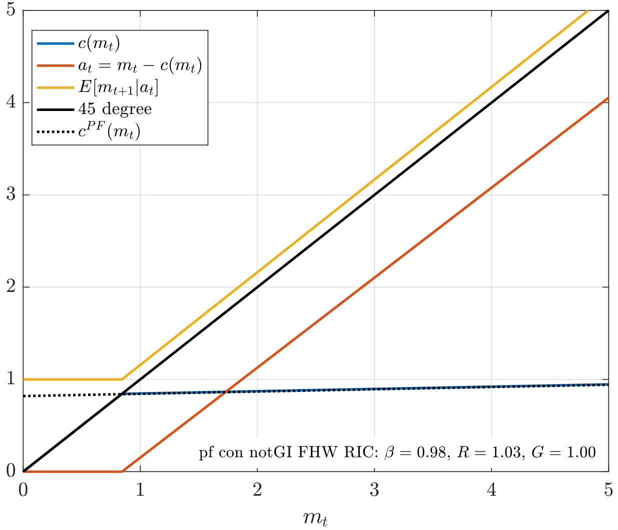
  2. GI without RI

     $$\lim_{m \to \infty} c'(m_t) = 0$$

PF-con with GI



PF-con without GI



Taxonomy of uncertainty adjusted patience

1. **Utility impatience (UI):**
$$\beta < 1$$
2. **Return impatience (RI):**
$$(\beta R)^{1/\rho}/R < 1$$
3. **Weak return impatience (WRI):**
$$\pi^{1/\rho}(\beta R)^{1/\rho}/R < 1$$
4. **Growth impatience (GI)** $(\mathbb{E}_t \psi_{t+1}^{-1} > 1)$**:**
$$(\beta R)^{1/\rho}\mathbb{E}_t \psi_{t+1}^{-1}/G < 1$$
5. **Absolute impatience (AI):**
$$(\beta R)^{1/\rho} < 1$$
6. **Finite value of autarky (FVA)** $(\mathbb{E}_t \psi_{t+1}^{1-\rho} < 1)$**:**
$$\beta \mathbb{E}_t (G\psi_{t+1})^{1-\rho} < 1$$

Analytical results

- **Zero-income risk:** $\mu = 0, \pi > 0$
- **Natural borrowing constraint**

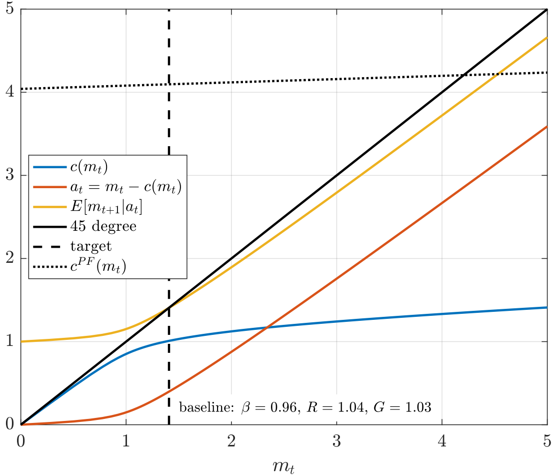
$$\lim_{c_t \to 0} \frac{c_t^{1-\rho}}{1-\rho} = -\infty \Rightarrow c(m_t) < m_t \Rightarrow \lambda \text{ does not matter}$$

- **Liquidity constrained model reached in the limit:**

$$\lim_{\pi \to 0} c(m_t; \pi) = c(m_t; \pi = 0, \lambda = 0)$$

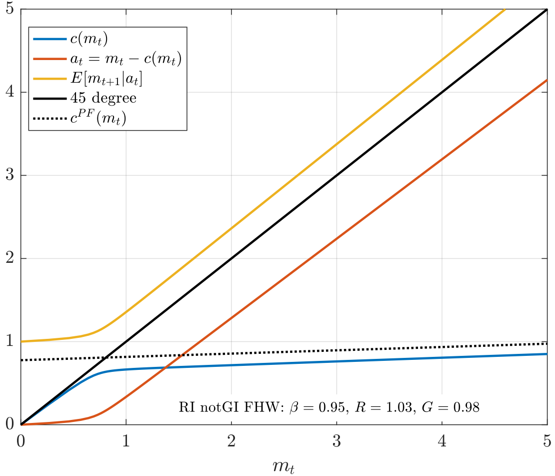
- **Solution:** WRIC + FVA
  - **Proof:** Use *Boyds weighted contraction mapping theorem*
  - **Standard assumptions:** FHW, RI, GI
- The **consumption function** is twice continuously differentiable**, increasing** and **concave**

Buffer-stock target



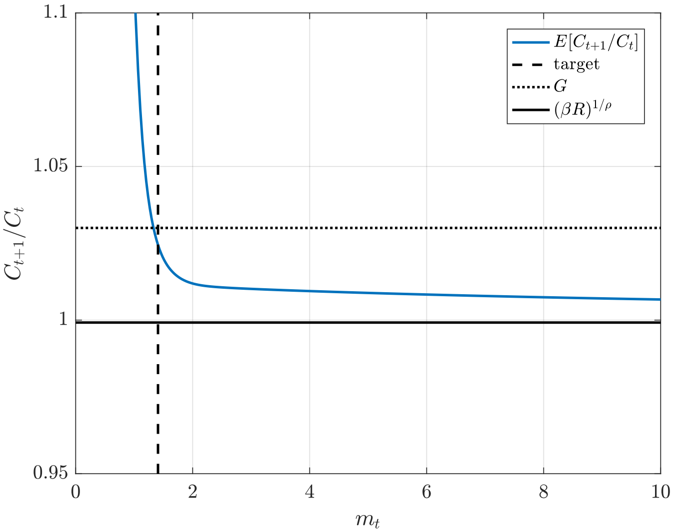
- **GI for finite target:** $(\beta R)^{1/\rho}\mathbb{E}_t\psi_{t+1}^{-1}/G < 1$

Not GI: $(\beta R)^{1/\rho}\mathbb{E}_t\psi_{t+1}^{-1}/G \geq 1$



- No buffer-stock target

Consumption growth I



Consumption growth II

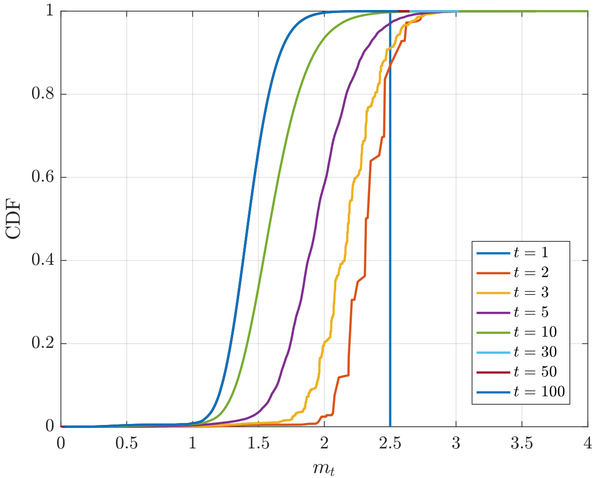
- Remember **Euler-equation**

$$C_t^{-\rho} \quad = \quad \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right]$$

- **Results**
  1. C_{t+1}/C_t is declining in m_t
  2. $\lim_{m_t \to \infty} C_{t+1}/C_t = (\beta R)^{1/\rho} = \text{RI}$
  3. $\lim_{m_t \to 0} C_{t+1}/C_t = \infty$
  4. $C_{t+1}/C_t < G$ at target
- **Intuition** for $C_{t+1}/C_t > (\beta R)^{1/\rho}$
  1. Uncertainty $\uparrow \Rightarrow$ expected marginal utility $\uparrow$ (because $c_{t+1}^{-\rho}$ is convex function)
  2. Consumer must be lowered today, $C_t \downarrow$
  3. Consumption growth will increase, $C_{t+1}/C_t \uparrow$

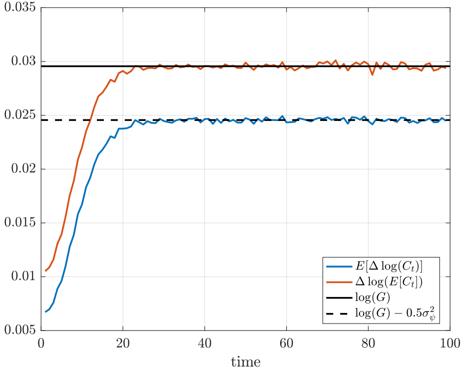
Further: *The above arguments are stronger for low cash-on-hand relative to permanent income*

Simulated distribution of cash-on-hand



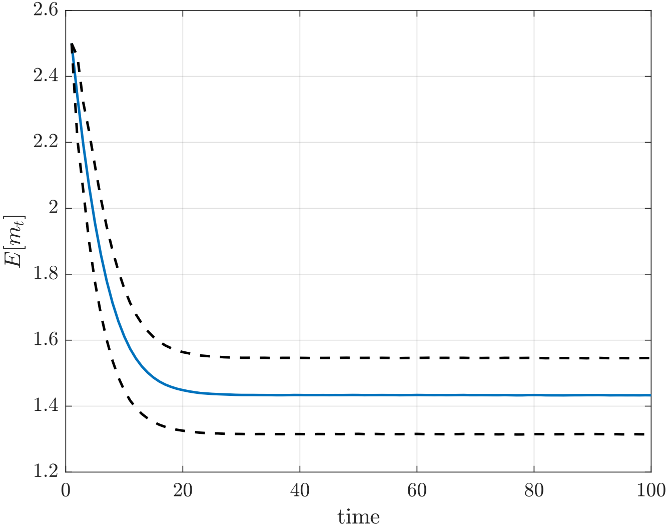
- **Proof of convergence:** Szeidl (2006)

Simulated $\mathbb{E}_t\left[\Delta \log(C_t)\right]$ vs. $\Delta\mathbb{E}_t\left[\log(C_t)\right]$



$$
\begin{aligned}
\mathbb{E}_t\left[\Delta \log(C_t)\right] &= \mathbb{E}_t\left[\log(c_t P_t) - \log(c_{t-1}P_{t-1})\right] \\
&= \mathbb{E}_t\left[\log c_t - \log c_{t-1} + \log G - 0.5\sigma_\psi^2 + \log P_{t-1} - \log P_{t-1}\right] \\
&= \log G - 0.5\sigma_\psi^2 \\
\Delta\mathbb{E}_t\left[\log(C_t)\right] &= \log G \text{ (only proven for } \sigma_\psi = 0)
\end{aligned}
$$

Simulated convegence of avg. cash-on-hand



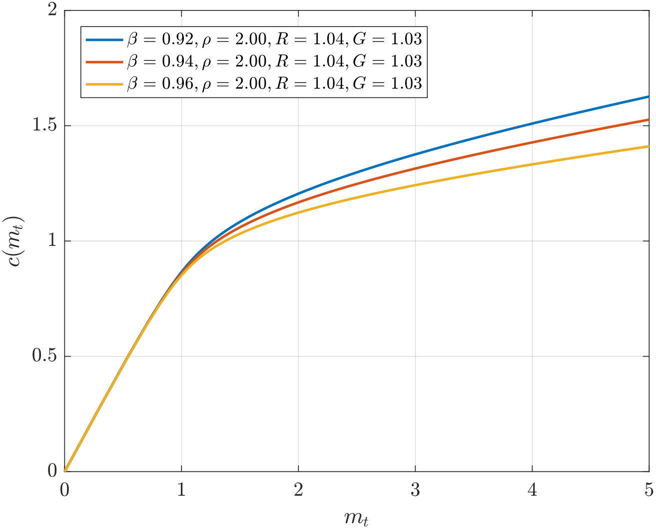
Preferences and saving

1. $\beta \uparrow \Rightarrow$ higher target, more savings
2. $G \uparrow \Rightarrow$ lower target, less savings
3. $R \uparrow \Rightarrow$ lower target, less savings
4. $\rho \uparrow \Rightarrow$ more complicated
   - *Risk-aversion*: $\rho \uparrow \Rightarrow$ higher target, more savings
   - *Intertemporal-subsitution elasticy:* $\rho \uparrow \Rightarrow$ flatter consumption profile $\Rightarrow$

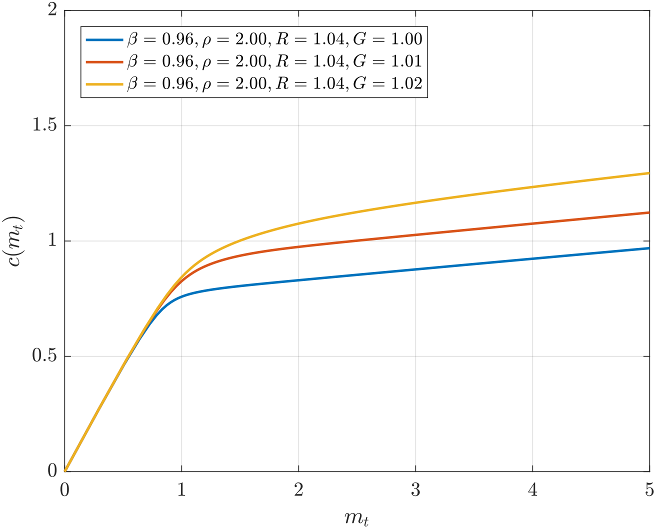
$$C_{t+1} > C_t : \rho \uparrow \Rightarrow C_t \uparrow \Rightarrow \text{ less savings}$$

$$C_{t+1} < C_t : \rho \uparrow \Rightarrow C_t \downarrow \Rightarrow \text{ more savings}$$

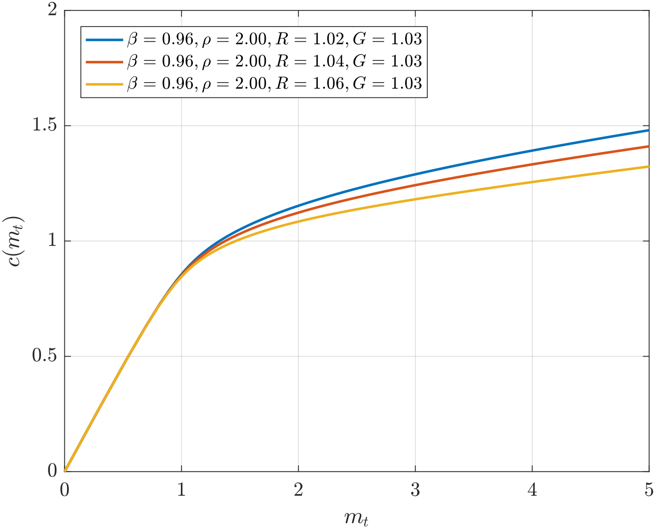
Varying β



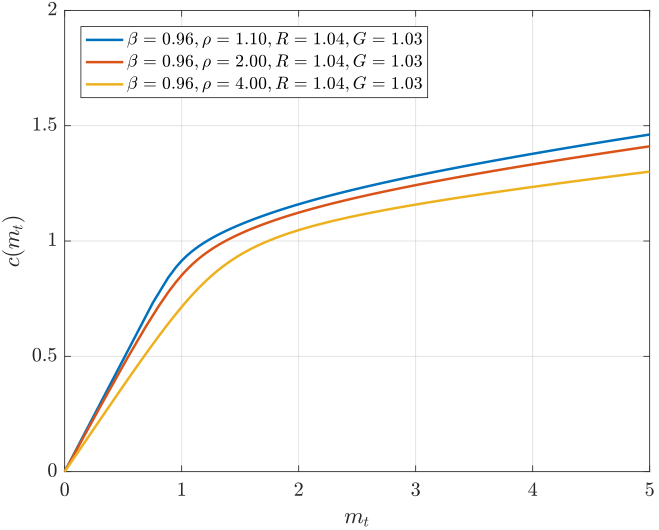
Varying G


$\beta = 0.96, \rho = 2.00, R = 1.04, G = 1.00$
$\beta = 0.96, \rho = 2.00, R = 1.04, G = 1.01$
$\beta = 0.96, \rho = 2.00, R = 1.04, G = 1.02$
$c(m_t)$
m_t
0
0.5
1
1.5
2
0
1
2
3
4
5

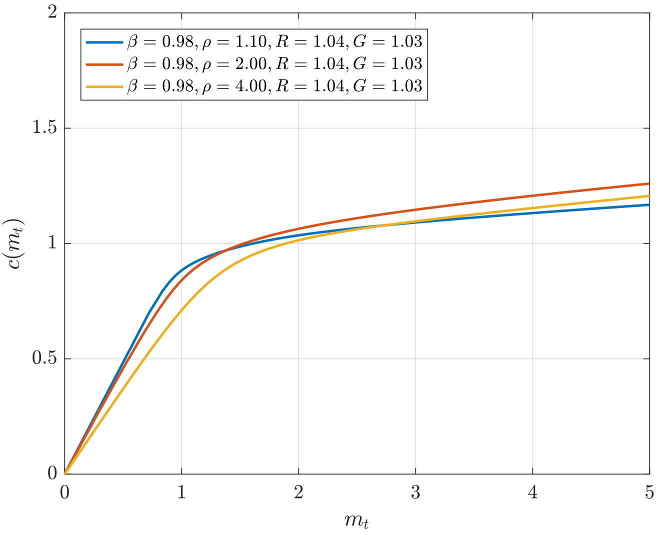
Varying R


$\beta = 0.96, \rho = 2.00, R = 1.02, G = 1.03$
$\beta = 0.96, \rho = 2.00, R = 1.04, G = 1.03$
$\beta = 0.96, \rho = 2.00, R = 1.06, G = 1.03$
$c(m_t)$
m_t
0
0.5
1
1.5
2
0
1
2
3
4
5

Varying ρ


$\beta = 0.96, \rho = 1.10, R = 1.04, G = 1.03$
$\beta = 0.96, \rho = 2.00, R = 1.04, G = 1.03$
$\beta = 0.96, \rho = 4.00, R = 1.04, G = 1.03$
$c(m_t)$
m_t
0
0.5
1
1.5
2
0
1
2
3
4
5

Varying ρ (with higher β)



Adding life-cycle

$$
\begin{aligned}
v_t(m_t) &= \max_{C_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1}) \right] \\
& \text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \begin{cases} \frac{1}{GL_t \psi_{t+1}} R a_t + \xi_{t+1} & \text{if } t+1 \leq T_R \\ \frac{1}{GL_t} R a_t + 1 & \text{else} \end{cases} \\
\xi_{t+1} &= \begin{cases} \mu & \text{with prob } \pi \\ (\epsilon_{t+1} - \pi\mu)/(1-\pi) & \text{else} \end{cases} \\
a_t &\geq \lambda_t = \begin{cases} -\lambda & \text{if } t < T_R \\ 0 & \text{if } t \geq T_R \end{cases}
\end{aligned}
$$

- **Retirement age:** $T_R \leq T$ (end-of-period)
- **Income profile:** $P_{t+1} = GL_t P_t \psi_{t+1}$ (with a drop in L_t at T_R)
- *No uncertainty or borrowing in retirement*

Consumption functions

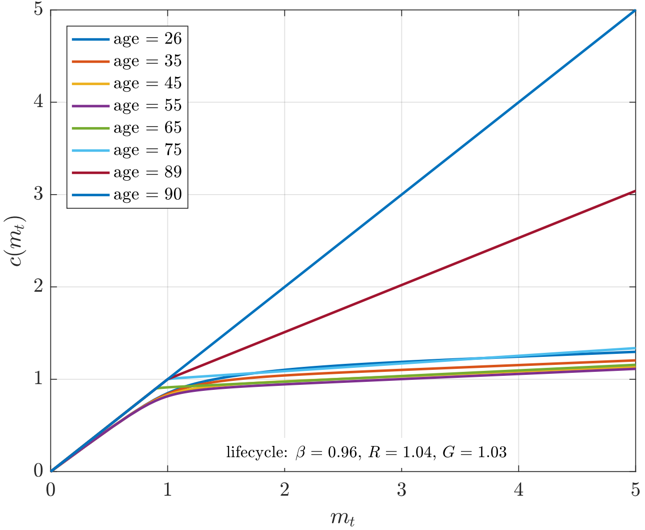
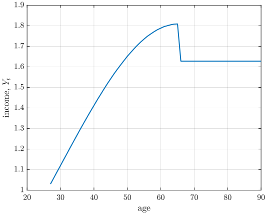
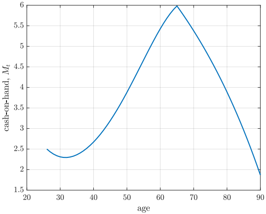


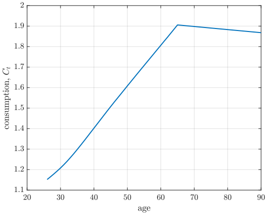
Figure: Life-cycle profiles



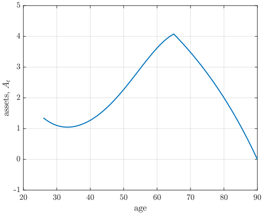
(a) Income, Y_t



(b) Cash-on-hand, M_t



(c) Consumption, C_t



(d) End-of-period assets, A_t

The natural borrowing constraint

- The optimal end-of-period asset choice satisfies

$$A_t \geq \underline{A}_t = \begin{cases} 0 & \text{if } t \geq T_R \\ -\min\{\Lambda_t, \lambda_t\} GL_t \underline{\psi} & \text{if } t < T_R \end{cases}$$

where

$$\Lambda_t \equiv \begin{cases} R^{-1} GL_t \underline{\psi}\underline{\xi} & \text{if } t = T_R - 1 \\ R^{-1} \left[\min\{\Lambda_{T-1}, \lambda_t\} + \underline{\xi}\right] GL_t \underline{\psi} & \text{if } t < T - 1 \end{cases}$$

and $\underline{\psi}$ and $\underline{\xi}$ are the minimum realizations of ψ_{t+1} and ξ_{t+1}

- **Proof:** Can be shown as a consequence of the household wanting to avoid $C_t = 0$ at *any cost*

Until next

- **Ensure that you understand:**
  1. The normalization of the Bellman equation
  2. The Euler-equation
  3. *The endogenous grid point method*
  4. Conditions for existence of solution
  5. The life-cycle dynamics
- Go to **PadLet** and ask or answer a question (https://padlet.com/jeppe_druedahl/dynamic_programming)
- **Think about:** How can we estimate the buffer-stock conumption model if we have panel data on income?

Theoretical bounds

- **Lower bound on MPC**

$$\begin{aligned}\underline{\kappa}^{-1} &= 1 + \text{RI}^1 + \text{RI}^2 + \cdots \Rightarrow \\ \underline{\kappa} &= \begin{cases} 0 & \text{if not RI} \\ 1 - \text{RI} & \text{if RI} \end{cases}\end{aligned}$$

Intuition: High MPC if impatience is high relative to return

- **Upper bound on MPC**

$$\begin{aligned}\overline{\kappa}^{-1} &= 1 + \text{WRI}^1 + \text{WRI}^2 + \cdots \Rightarrow \\ \overline{\kappa} &= 1 - \text{WRI}\end{aligned}$$

Intuition: ....

- **Theoretical bounds are given by**

$$\overline{\kappa} m_t \leq c(m_t) \leq \underline{\kappa} m_t$$

Exotic solutions

1. **Not FHW,** but **RI**

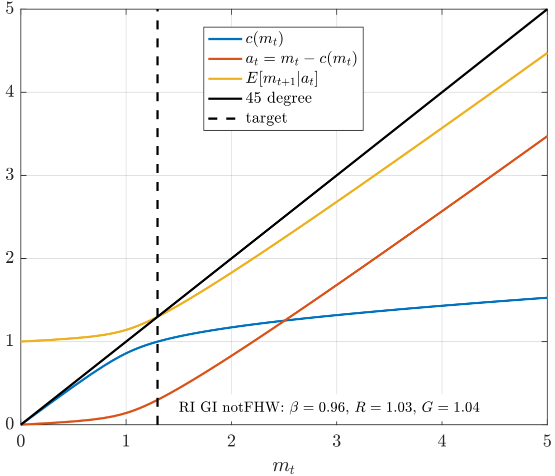
$$\lim_{m \to \infty} c'(m_t) = \underline{\kappa} = 1 - \text{RI}$$

2. **Not RI** $\Rightarrow$ **not FHW**

$$\lim_{m \to \infty} c'(m_t) = \underline{\kappa} = 0$$

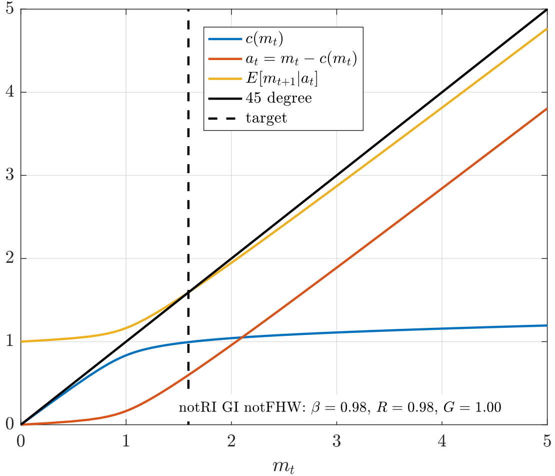
3. **No GI** $\Rightarrow$ no buffer-stock target

Not FHW, but RI: $G \geq R$, $(\beta R)^{1/\rho}/R < 1$



$$\lim_{m \to \infty} c'(m_t) = \underline{\kappa} = 1 - \text{RI}$$

Not FHW, not RI: $G \geq R$, $(\beta R)^{1/\rho}/R \geq 1$



$$\lim_{m\to\infty} c'(m_t) = \underline{\kappa} = 0$$

Approximating the Euler-equation I

- **True:**

$$C_t^{-\rho} = \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho} \right]$$

- **First order approximation:**

$$\mathbb{E}_t \left[\Delta \log(C_t) \right] \approx \rho^{-1} \log(\beta R)$$

- **Second order approximation:**

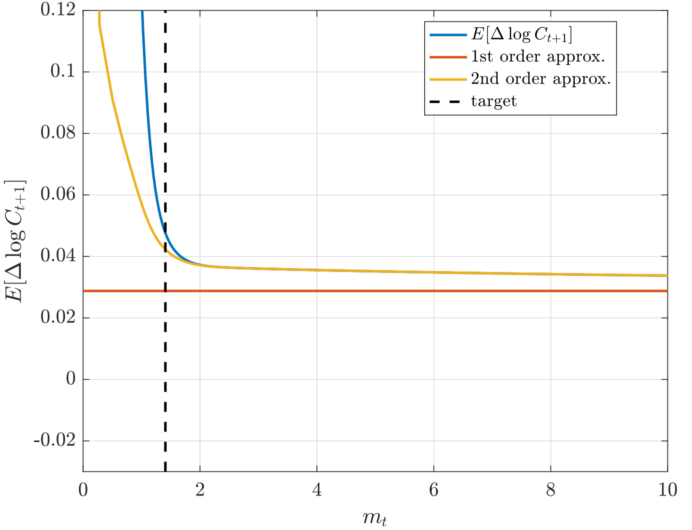
$$\mathbb{E}_t \left[\Delta \log(C_t) \right] \approx \rho^{-1} \log(\beta R) + \frac{\rho}{2} \mathrm{var}_t \left[(\Delta \log C_{t+1}) \right]$$

- **Reconciliation with previous evidence:**

$$\begin{aligned} \mathbb{E}_t \left[\Delta \log(C_t) \right] &= \log G - 0.5\sigma_\psi^2 \Leftrightarrow \\ \mathrm{var}_t \left[(\Delta \log C_{t+1}) \right] &\approx \frac{2}{\rho} (\log G - 0.5\sigma_\psi^2 - \rho^{-1} \log(\beta R)) \end{aligned}$$

i.e. the second order term is *endogenous*

Approximating the Euler-equation II



Approximating the Euler-equation III

- **General approximation** arround $\xi_{t+1} = \psi_{t+1} = 1$:

$$\mathbb{E}_t\left[\Delta C_{t+1}\right] = \rho^{-1}\log(\beta R) + \varphi_t$$

- **Properties of** $\varphi_t = \varphi(m_t) = \varphi(m_0, \xi_0, \ldots, \xi_t, \psi_0, \ldots, \psi_t)$
  1. Always positive: $\varphi_t > 0$ for $m_t < \infty$
  2. Converge to zero: $\varphi_t \to 0$ for $m_t \to \infty$
  3. Declining in cash-on-hand: $\varphi_t \downarrow$ for $m_t \uparrow$
  4. Declining in *past* transitory income shocks: $\varphi_t \downarrow$ for $\xi_{t-k} \uparrow \Rightarrow m_t \uparrow$
  5. Declining in *past* permanent income shocks? (we don't know... seems so in simulations)
- **See also:** Commault (2017)