# Macroeconomics II, Lecture I:
# The Real Business Cycle Model: Basics

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# This course part

- Business cycle models
  - 6 lectures
  - Frameworks for studying aggregate fluctuations
- Frictional labor markets
  - 4 lectures
  - Digging deeper into the determinants of household income
- Incomplete asset markets
  - 3 lectures
  - Digging deeper into the determinants of consumption-savings dynamics, taking the income process as given
- 1 Dynare tutorial session; 6 problem sets
- Grading:
  - Problem sets 30%, Exam 70%
  - Part I: 25%, Part II: 75%
  - To pass, you need 50% of the points, with at least 30% from the exam

# Learning outcomes

1. You should know a few key empirical facts about business cycles, the labor market and the distribution/dynamics of earnings-consumption-wealth

2. You should be able to construct, solve and analyze workhorse models within the business-cycle, macro-labor and incomplete-markets literatures
   - Within these models, you should know which assumptions are essential and which can be relaxed
   - You should acquire the technical skills needed to solve/analyze the presented models

3. You should know key predictions of the models presented and how the models can be used to interpret the data

# Hidden agenda

- Two guiding principles:
  1. You should acquire sufficient tools to continue studying on your own (especially for those that decide to specialize in macro)
  2. You should acquire an overview about how research in this field looks like, and how it relates to other areas of economic research (labor, public finance etc.)

- Repeated emphasis on how to use models for *quantitative interpretation* of the data

- Repeated emphasis on how to use micro data for informing macroeconomic research

# My teaching style

- In class, we go through most steps, but not all, when solving the models
  - I expect you to work (or know how to work) through the missing pieces at home
- The problem sets are the heart of the course
  - Primary benefit is that you learn economics
  - Secondary benefit is that you practice for the main exam
  - Third benefit is that you gain som points for the exam
- References: I use convention Name-Name-... (journal, year)
  - Example: Gabaix-Lasry-Lions-Moll (Ecmtra, 2016)
  - Abbreviation when reference is recurrently repeated
  - If not published, I do not write out journal
- I very much appreciate questions and you pointing out errors, inconsistencies or anything else that makes my slides/teaching unclear

# Part I: Business cycles

- Basic questions:
  1. What are business cycles?
  2. What causes business cycles?
  3. What consequences do they have?
  4. When, and if so, how, should government policy intervene?

- The facts and models that we introduce represent *an attempt* to start reasoning about these questions

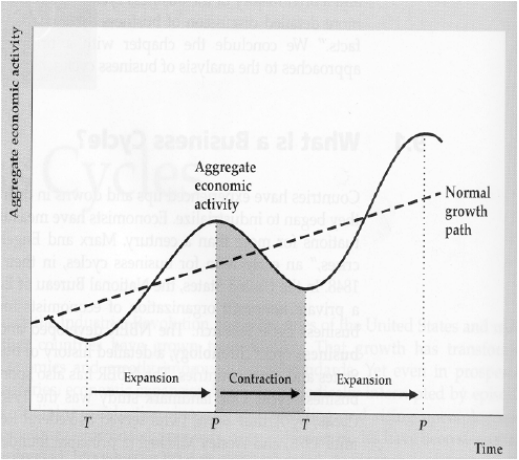
- As you will see, there are many questions raised by these facts and models that we still do not have great answers to

# Agenda

1. Business cycle facts
2. Math preliminaries
3. The Real Business Cycle model: Setup and solution
4. The Real Business Cycle model: Analysis

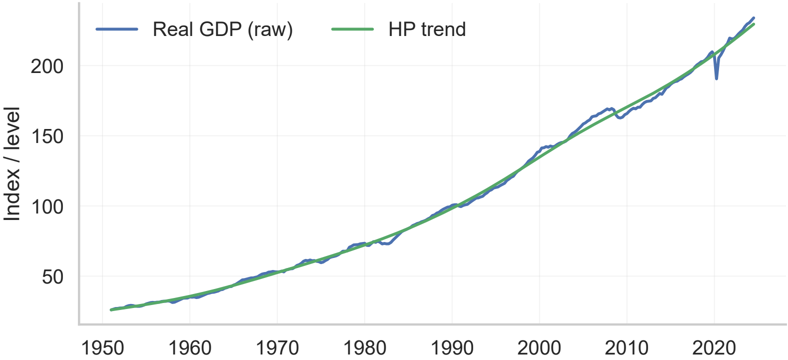
# Business cycle facts

# Two approaches to business cycle measurement




- NBER recession dating focus on periods of contraction
- In our course (and most academic literature): Periods where output is below trend

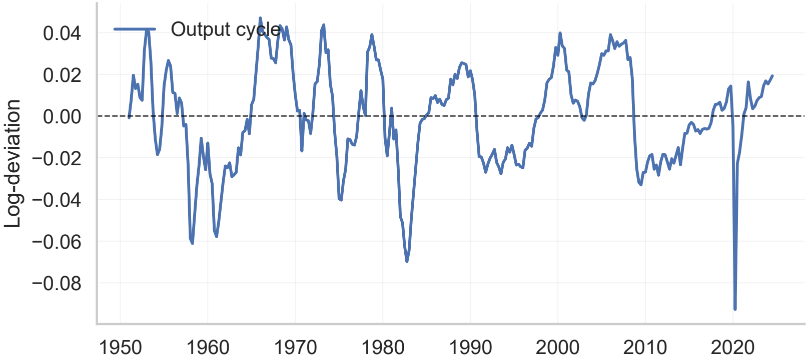
# US post-war real GDP: trend and cycle




Own calculations using FRED data

- Our focus: the deviations of blue from orange line

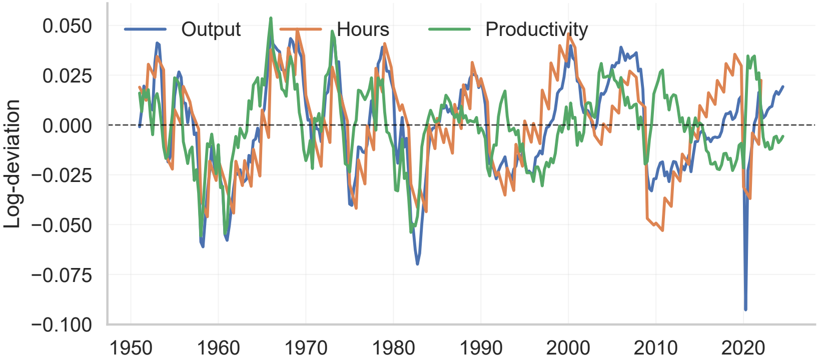
# US Cyclical Real GDP




Own calculations using FRED data

- Fact 1: considerable variations in GDP growth from year to year

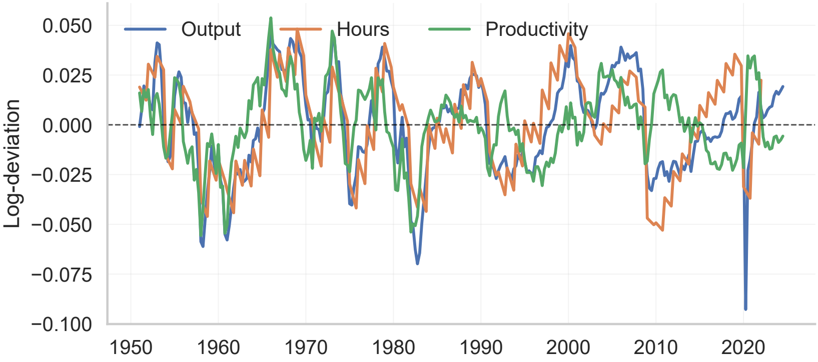
# US Cyclical Real GDP + Hours and productivity




Own calculations using FRED data

- Fact 2: Many key macroeconomic aggregates comove with GDP

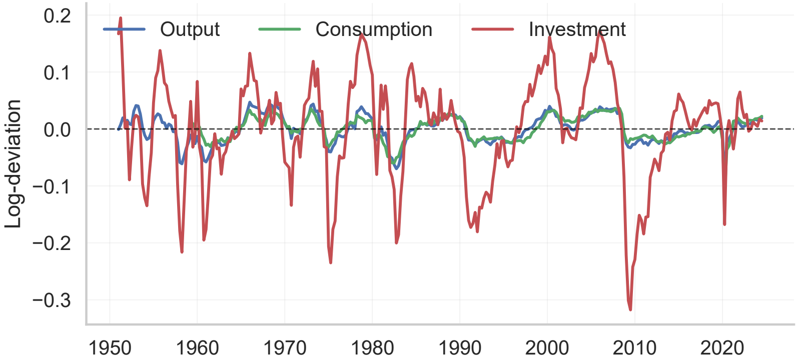
# US Cyclical Real GDP + Hours and productivity




Own calculations using FRED data

- Fact 3: Hours as volatile as GDP, Productivity less volatile than GDP

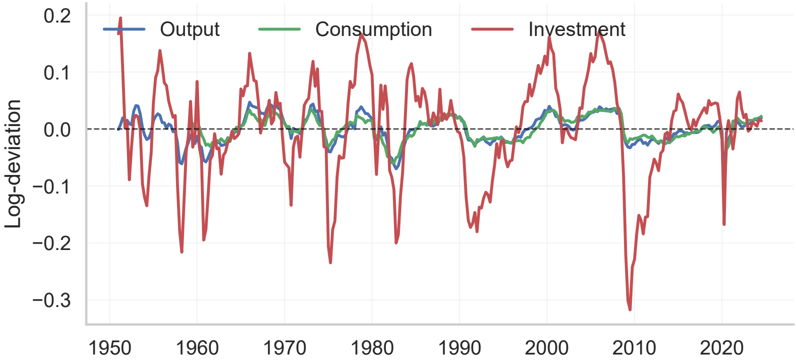
# US Cyclical Real GDP + Consumption and Investment




Own calculations using FRED data

- Fact 2 again: Many key macroeconomic aggregates comove with GDP

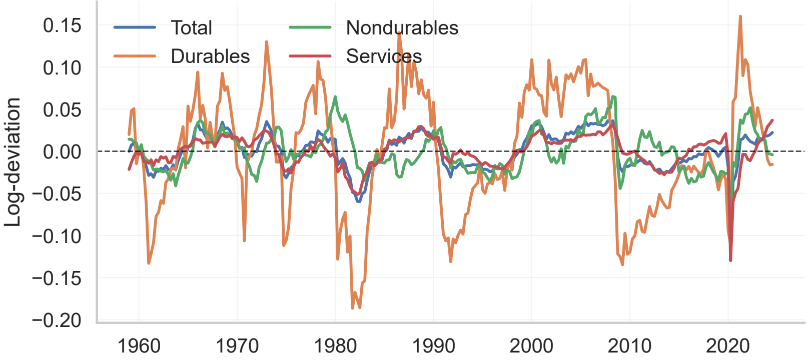
# US Cyclical Real GDP + Consumption and Investment




Own calculations using FRED data

- Fact 4: Investment more volatile, consumption less volatile than GDP

# US Cyclical Real consumption




Own calculations using FRED data

- Moreover: Investment-like consumption goods more volatile than other consumption goods...

# Summary of US business cycle moments

| Series | SD | Rel. SD | Corr $Y_t$ | Autocorr | Corr $Y_{t-4}$ | Corr $Y_{t+4}$ |
|---|---|---|---|---|---|---|
| $Y_t$ (Output) | 0.017 | 1.00 | 1.00 | 0.79 | 0.08 | 0.08 |
| $C_t$ (Consumption) | 0.011 | 0.66 | 0.76 | 0.67 | 0.15 | 0.02 |
| $I_t$ (Investment) | 0.044 | 2.67 | 0.76 | 0.86 | -0.05 | 0.23 |
| $N_t$ (Hours) | 0.021 | 1.27 | 0.87 | 0.82 | 0.28 | -0.06 |
| $A_t$ (TFP) | 0.013 | 0.76 | 0.78 | 0.76 | -0.30 | 0.31 |
| $W_t$ (Wage) | 0.012 | 0.69 | -0.01 | 0.71 | -0.11 | 0.17 |
| $R_t$ (Real rate) | 0.004 | 0.26 | 0.00 | 0.47 | 0.27 | -0.28 |
| $P_t$ (Price level) | 0.010 | 0.58 | -0.08 | 0.91 | 0.14 | -0.44 |

Eric Sims, RBC notes (Spring 2024), Table 1. Quarterly HP-filtered data 1947Q1–2022Q3 ($\lambda = 1600$).

On top of the facts already discussed, we see that

- All series display and considerable degree of persistence
- Wages and interest rates are not very correlated with output (especially not contempouraneously correlated)

# US Business cycle: key facts

1. Standard deviation of US quarterly Real GDP $\sim 2$ percent
2. Many macroeconomic variables comove with output
3. Productivity less volatile than output, Hours worked as volatile as output
4. Investment more volatile that output, consumption less volatile than output
5. All series display considerable degree of persistence
6. Wages and interest rates are not very correlated with output

# Math preliminaries

# Math preliminaries I: Natural Logarithms

- An appealing feature of the natural logarithm is that for small $x$, we have that

$$\log(1+x) \approx x$$

- As a result, we can interpret log differences as percentage growth rates

$$\begin{aligned}
\log(x_1) - \log(x_2) &= \log\left(\frac{x_1}{x_2}\right) \\
&= \log\left(1 + \frac{x_1 - x_2}{x_2}\right) \\
&\approx \frac{x_1 - x_2}{x_2}
\end{aligned}$$

# Math preliminaries II: Log-linearization

- An equilibrium characterization is set of $n$ equations in $n$ unknowns

$$F^1(\mathbf{X}) = 0, F^2(\mathbf{X}) = 0, ..., F^n(\mathbf{X}) = 0$$

where $\mathbf{X} = \{X_1, X_2, ..., X_n\}$ are endogenous variables

- For example, the Cobb-Douglas production function represent one such equation

$$Y_t \quad = \quad A_t K_t^{\alpha} N_t^{1-\alpha}$$

where the endogenous variables are $Y_t, A_t, K_t, N_t$.

- When analyzing the dynamic equilibrium response to shocks, we often consider a log-linear approximation of the equilibrium characterization $\{F^i(\mathbf{X})\}_{i=0}^{n}$ around its steady state

# Math preliminaries II: Log-linearization

- Let's focus on the two-variable case

- Taylor's theorem: the value of a differentiable function $F$ at the point $X_1, X_2$, can be approximated knowing its value at the point $X_1^*, X_2^*$, like

$$
\begin{aligned}
F(X_1, X_2) &\approx F(X_1^*, X_2^*) + \frac{\partial F(X_1^*, X_2^*)}{\partial X_1}(X_1 - X_1^*) + \frac{\partial F(X_1^*, X_2^*)}{\partial X_2}(X_2 - X_2^*) \\
\Leftrightarrow \Delta F(X_1, X_2) &\approx F_1(X_1^*, X_2^*)\Delta X_1 + F_2(X_1^*, X_2^*)\Delta X_2
\end{aligned}
$$

- If we take logs first, this becomes

$$
\begin{aligned}
\Delta \log F(X_1, X_2) &\approx \frac{\partial \log F(X_1^*, X_2^*)}{\partial X_1}\Delta X_1 + \frac{\partial \log F(X_1^*, X_2^*)}{\partial X_2}\Delta X_2 \\
&= \frac{F_1(X_1^*, X_2^*)}{F(X_1^*, X_2^*)}\Delta X_1 + \frac{F_2(X_1^*, X_2^*)}{F(X_1^*, X_2^*)}\Delta X_2 \\
&= \frac{F_1(X_1^*, X_2^*)X_1^*}{F(X_1^*, X_2^*)}\hat{x}_1 + \frac{F_2(X_1^*, X_2^*)X_2^*}{F(X_1^*, X_2^*)}\hat{x}_2
\end{aligned}
$$

where $\hat{x}_i = \frac{X_i - X_i^*}{X_i^*} \approx \Delta \log X_i$

# Math preliminaries II: Log-linearization

- Result: the percentage growth of the function value can be approximated with an appropriate linear combination of the percentage growths in the function variables

- Log-linearizing = applying this formula

- Lets consider a few examples from dynamic economic models

- For any variable $X_t$, denote
  - its steady state value with $X$
  - its log with $x_t$
  - its log steady state value $x$
  - its log difference to steady state with $\hat{x}_t$

## Math preliminaries II: Log-linearization (do on whiteboard)

- Example 1: Capital law of motion

$$K_{t+1} = (1-\delta)K_t + I_t$$

- Log-linearizing around the steady state gives

$$\hat{k}_{t+1} \approx (1-\delta)\hat{k}_t + \delta\hat{i}_t$$

- Example 2: Resource constraint

$$Y_t = C_t + I_t$$

- Log-linearizing around the steady state gives

$$\hat{y}_t \approx \frac{C}{Y}\hat{c}_t + \frac{I}{Y}\hat{i}_t$$

## Math preliminaries II: Log-linearization

- Multiplicative-exponential relationships are log-linear to start with, these need not to be approximated
- Example: Cobb-Douglas production function:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

- Taking logs

$$y_t = a_t + \alpha k_t + (1-\alpha) n_t$$

- Subtracting steady state

$$\begin{aligned} y_t - y &= a_t - a + \alpha(k_t - k) + (1-\alpha)(n_t - n) \\ \hat{y}_t &= \hat{a}_t + \alpha \hat{k}_t + (1-\alpha)\hat{n}_t \end{aligned}$$

# Math preliminaries III: Systems of linear difference equations

- After log-linearizing, the typical macro model can be written as a forward-looking auto-regressive system of linear difference equations:

$$\mathbf{A}_1 \mathbf{x}_t = \mathbf{A}_2 E_t \mathbf{x}_{t+1} + \mathbf{B_1} \epsilon_t$$

  where
  - $\mathbf{x}_t = [x_{1t}, ..., x_{nt}]'$ is a vector of endogenous variables
  - $\epsilon_t = [\epsilon_{1t}, ..., \epsilon_{kt}]'$ is a vector of exogenous shocks

# Math preliminaries III: Systems of linear difference equations

- We are typically interested in finding a bounded solution to this system, in response to the shocks $\epsilon_t$
- Question: Under what conditions does a unique bounded solution exist?
- Rewrite system as

$$\mathbf{x}_t = \mathbf{A} E_t \mathbf{x}_{t+1} + \mathbf{B}\epsilon_t$$

where $\mathbf{A} = \mathbf{A}_1^{-1}\mathbf{A}_2$ and $\mathbf{B} = \mathbf{A}_1^{-1}\mathbf{B}_1$

## Math preliminaries III: Systems of linear difference equations

- To gain intuition, consider a 1-equation system with one forward-looking variable

$$x_t = aE_t x_{t+1} + \epsilon_t$$

  with the shock sequence $\epsilon_t = \epsilon > 0$, $\epsilon_{t+s} = 0$ for all $s > 0$

- Any solution satisfies

$$\begin{aligned} x_t &= aE_t \left[ aE_{t+1} x_{t+2} \right] + \epsilon_t \\ &= .... \\ &= \lim_{T \to \infty} a^T E_t x_{t+T} + \epsilon_t \end{aligned}$$

- What is a solution? A: Any stochastic process for $x_t$ that satisifes this equation.

- What is a bounded solution? A: any stochastic process for $x_t$ such that $x_{t+s} \leq M$ for some $M < \infty$ and all $s \geq 0$

- Ergo, a bounded solution has $\lim_{T \to \infty} E_t x_{t+T} < \infty$

# Math preliminaries III: Systems of linear difference equations

- Our equation:

$$x_t = \lim_{T\to\infty} a^T E_t x_{t+T} + \epsilon_t$$

- Q: How many solutions to this equation has $\lim_{T\to\infty} E_t x_{t+T} < \infty$?

- Suposse $a < 1$, then
    - Given $\lim_{T\to\infty} E_t x_{t+T} < \infty$, we have $\lim_{T\to\infty} a^T E_t x_{t+T} = 0$
    - Hence, $x_t = \epsilon_t$ is the unique bounded solution

- Suppose $a \geq 1$, then
    - any stochastic process for $x_t$ with $\lim_{T\to\infty} E_t x_{t+T} = 0$ is a solution.
    - Example:

$$x_t = \epsilon_t + \nu_t, \quad \nu_t \sim F \text{ with } E_t \nu_t = 0$$

    - Infintely many bounded solutions!

# Math preliminaries III: Systems of linear difference equations

- Consider the general system of n equations:

$$\mathbf{x}_t = \mathbf{A} E_t \mathbf{x}_{t+1} + \mathbf{B}\epsilon_t \tag{1}$$

- The counterpart of the AR(1) scalar $a$ in our 1-equation system are the eigenvalues of $\mathbf{A}$

- The eigenvalues are the solution to the deterministic equation

$$\det(\mathbf{A} - \mathbf{I}\lambda) = 0$$

- Theorem (Blanchard-Kahn, Ecmtra 1981): There exist a unique bounded solution to the system (1) if and only if $\mathbf{A}$ has the same number of eigenvalues inside the unit circle as the number of forward-looking variables.

- Forward-looking variables = variables that are not pre-determined

# The Real Business Cycle Model: Setup and solution

# Overview

- Vanilla RBC model = Neoclassical growth model with stochastic TFP shocks

- No market frictions: dynamics caused by efficient response of production inputs to technology shocks

- Why use this as our starting point?
  1. To establish a minimal efficient benchmark
     - Modern business cycle models can be thought of as extensions to this framework
     - Policy analysis often boils down to the question: "how we can make the world behave more like an RBC model?"
  2. Use this model as an example for understanding commonly employed methods
     - Log-linear approximation of model dynamics
     - Calibration

- Origination: Kydland-Prescott (Ecmtra, 1982); King-Plosser (AER, 1984)
  - Important prior developments: Rational expectations paradigm (Lucas, JET 1972); structural econometrics (Sargent, JPE 1976)
  - These models and methods completely transformed economic research (not only macro!)

# Model structure

- A representative household chooses consumption $C$, labor supply $N$ and investment $I$, taking $W$ and $R$ as given
  - $\Rightarrow$ Supply curves of $N$ and $K$, Demand curve for goods $Y$
- A representative firm chooses capital and labor input, taking $W$ and $R$ and a stochastic process for TFP $A_t$ as given
  - $\Rightarrow$ Demand curves of $N$ and $K$, Supply curve for $Y$
- Markets are complete: every good can be traded at very point in time.
- Equilibrium concept: $W$ and $R$ has to be such that when agents optimize, we have that
  - Labor supply = Labor Demand
  - Capital supply = Capital Demand
  - Goods supply = Goods Demand
- Since markets are complete and there are no distortions, the decentralized equilibrium and the social planner solution yield the same allocation
- We proceed with using the decentralized setup

## Household problem

- Program of the representative household

$$\begin{aligned}
\max_{\{C_t, N_t^s, I_t, K_{t+1}^s\}} \quad & E_O \sum_{t=0}^{\infty} \beta^t \left[U(C_t) - V(N_t^s)\right] \\
\text{s.t} \quad & C_t + I_t \leq W_t N_t^s + R_t^r K_t^s, \\
& K_{t+1}^s \leq (1-\delta) K_t^s + I_t, \\
& C_t, N_t^s, I_t \geq 0,
\end{aligned}$$

with $U(C_t)$, $V(N_t^s)$ satisfying the usual regularity conditions

- Note:
  - Separable preferences for simplicity
  - The household owns the capital stock
  - $R_t^r$ = rental rate earned on capital stocked rented to firm in period $t$
  - $R_t^r \neq$ risk-free real interest rate $R_t$
  - Return on period $t$ investment, $R_{t+1}^r + (1-\delta)$, not known in period $t$
  - However, $E_t\left[R_{t+1}^r + (1-\delta)\right]$ intimately related to $R_t$
    - ⋆ In a first-order approximation, they are, in fact, the same (more on this in Lecture III)

# Firm problem

- The firm rents labor and capital from the household, can freely adjust in each period $\Rightarrow$ Static problem

- Program of the representative firm

$$\begin{aligned} \max_{\{N_t^d, K_t^d\}} \quad & A_t F(K_t^d, N_t^d) - R_t^r K_t^d - W_t N_t^d \\ \text{s.t} \quad & A_t = A_{t-1}^{\rho_a} exp(\epsilon_t) \end{aligned}$$

  with $F_t(\cdot)$ being homogeneous of degree 1

- Note:
  - Competitive markets ensures profits are zero
  - The process for $A_t$ is AR(1) in logs - parsimonious, and captures the fluctuations in measured TFP well

# Equilibrium

- A competitive equilibrium is a set of allocations $\{C_t, N_t^s, I_t, K_t^s, N_t^d, N_t^d\}$ and prices $\{W_t, R_t^r\}$ such that
  - Given $\{W_t, R_t^r\}$, $\{C_t, N_t^s, I_t, K_t^s\}$ solve the household problem
  - Given $\{W_t, R_t^r\}$, $\{N_t^d, K_t^d\}$ solve the firm problem
  - Markets clear:

$$\text{Goods Market: } \; C_t + I_t = A_t F(K_t^d, N_t^d) \;\text{ for all } t$$
$$\text{Labor Market: } \; N_t^s = N_t^d \;\text{ for all } t$$
$$\text{Capital Market: } \; K_t^s = K_t^d \;\text{ for all } t$$

- Going forward, I will skip supply-demand notation and simply use $N_t, K_t$ in both the household and firm problem

# Equilibrium (imposing some market clearing)

- A competitive equilibrium is a set of allocations $\{C_t, N_t, I_t, K_t\}$ and prices $\{W_t, R_t^r\}$ such that
  - Given $\{W_t, R_t^r\}$, $\{C_t, N_t, I_t, K_t\}$ solve the household problem
  - Given $\{W_t, R_t^r\}$, $\{N_t, K_t\}$ solve the firm problem
  - Markets clear:

$$C_t + I_t = A_t F(K_t, N_t) \text{ for all } t$$

- Comment: In fact, now the last equation is redudant. Since we have imposed that the capital and labor market clear, the goods market will clear by Walras law

# Equilibrium (imposing some market clearing and Walras' law)

- A competitive equilibrium is a set of allocations $\{C_t, N_t, I_t, K_t\}$ and prices $\{W_t, R_t^r\}$ such that
  - Given $\{W_t, R_t^r\}$, $\{C_t, N_t, I_t, K_t\}$ solve the household problem
  - Given $\{W_t, R_t^r\}$, $\{N_t, K_t\}$ solve the firm problem

# Model solution

- What is a model solution?
  - A set of policy functions that specifies the equilibrium response of the endogenous variables as a function of parameters and the realization of the exogenous shocks

- Two ways to solve the for the equilibrium:
  1. Global solution: solve for global policy functions, using, e.g., value function iteration (or a neural net)
  2. Log-linear approximation: Do a local approximation of the policy functions around some point of the equilibrium

- Here, we will explore option 2

- Why?
  - Because we know how to handle linear difference equations
  - Because log-differences have an appealing interpretation (percent growth)
  - Because, in practice, a large class of models that we use are, in fact, not very non-linear

- In practice, this means that we will log-linearize the model around the (non-stochastic) steady state

# Log-linearization cook book

1. Start with making an equilibrium characterization
   - List all equations that must hold true in equilibrium
   - Given our equilibrium defintion, they must consist of
     - Equations that must be satisfied in a solution to the household problem
     - Equations that must be satisfied in a solution to the firm problem
     - Market clearing conditions
2. Solve for the steady state
   - Typically a simple algebraic exercise once you have the equilibrium characterization
3. Log-linearize the equilibrium characterization around the steady state

# Step 1: Equilibrium characterization

- Lagrangian to the household problem:

$$\mathbf{L} = E_0 \left( \sum_{t=0}^{\infty} \beta^t \left[ U(C_t) - V(N_t) \right] - \sum_{t=0}^{\infty} \lambda_t \left[ C_t + K_{t+1} - W_t N_t - (R_t^r + (1-\delta)) K_t \right] \right)$$

  where I have substituted the capital accumulation equation into the budget constraint

- First order conditions

$$\begin{aligned} C_t: &\quad \beta^t U'(C_t) - \lambda_t = 0 \\ N_t: &\quad -\beta^t V'(N_t) + \lambda_t W_t = 0 \\ K_{t+1}: &\quad -\lambda_t + E_t \lambda_{t+1}(R_{t+1}^r + (1-\delta)) = 0 \end{aligned}$$

- In an interior solution, these equations + the constraints must be satisfied, and also the transversality condition:

$$\lim_{T\to\infty} \beta^T K_T U'(C_T) \leq 0$$

- In practice, we search for a candidate solution to the household problem, then check that this candidate also satisfies tranversality

# Equilibrium characterization II

- Necessary conditions for household optimality:

$$
\begin{aligned}
U'(C_t)W_t &= V'(N_t) \\
U'(C_t) &= \beta E_t(R^r_{t+1} + (1-\delta))U'(C_{t+1}) \\
C_t + I_t &= W_t N_t + R^r_t K_t \\
K_{t+1} &= (1-\delta)K_t + I_t
\end{aligned}
$$

- Necessary conditions for firm optimality:

$$
\begin{aligned}
R^r_t &= A_t F_k(K_t, N_t) \\
W_t &= A_t F_n(K_t, N_t) \\
A_t &= A^{\rho_a}_{t-1} exp(\epsilon_t)
\end{aligned}
$$

- This completes the equilibrium characterization

- Again: resource constraint $C_t + I_t = A_t F(K_t, N_t)$ is implied by Walras' law
  - Going forward, however, I will add the resource constraint, and drop the household budget constraint instead

# Equilibrium characterization III

- Summing up, the equilibrium is characterized by:

$$
\begin{aligned}
U'(C_t)W_t &= V'(N_t) \\
U'(C_t) &= \beta E_t\left[(R^r_{t+1} + (1-\delta))U'(C_{t+1})\right] \\
C_t + I_t &= A_t F(K_t, N_t) \\
\\
K_{t+1} &= (1-\delta)K_t + I_t \\
R^r_t &= A_t F_k(K_t, N_t) \\
W_t &= A_t F_n(K_t, N_t) \\
A_t &= A^{\rho_a}_{t-1} exp(\epsilon_t)
\end{aligned}
$$

- Seven variables $\{C_t, N_t, I_t, K_t, W_t, R^r_t, A_t\}$ and seven equations

# Equilibrium characterization III (it's nice with output as a separate variable)

- Summing up, the equilibrium is characterized by:

$$
\begin{aligned}
U'(C_t)W_t &= V'(N_t) \\
U'(C_t) &= \beta E_t \left[ (R^r_{t+1} + (1-\delta))U'(C_{t+1}) \right] \\
C_t + I_t &= Y_t \\
Y_t &= A_t F(K_t, N_t) \\
K_{t+1} &= (1-\delta)K_t + I_t \\
R^r_t &= A_t F_k(K_t, N_t) \\
W_t &= A_t F_n(K_t, N_t) \\
A_t &= A^{\rho_a}_{t-1} exp(\epsilon_t)
\end{aligned}
$$

- Eight variables $\{C_t, N_t, I_t, K_t, Y_t, W_t, R^r_t\}$ and eight equations

# Equilibrium characterization III (it's nice with output as a separate variable)

- Summing up, the equilibrium is characterized by:

$$
\begin{aligned}
U'(C_t)W_t &= V'(N_t) \\
U'(C_t) &= \beta E_t\left[(R^r_{t+1} + (1-\delta))U'(C_{t+1})\right] \\
C_t + I_t &= Y_t \\
Y_t &= A_t F(K_t, N_t) \\
K_{t+1} &= (1-\delta)K_t + I_t \\
R^r_t &= A_t F_k(K_t, N_t) \\
W_t &= A_t F_n(K_t, N_t) \\
A_t &= A_{t-1}^{\rho_a} exp(\epsilon_t)
\end{aligned}
$$

- Eight variables $\{C_t, N_t, I_t, K_t, Y_t, W_t, R^r_t\}$ and eight equations

- Question: Which are the state variables and which are the shocks?

## Functional forms

- To compute the steady state, we need to impose some functional forms
- Note: choice of functional forms of course restricts the quantitative properties of the model - it should be treated as part of the calibration
- Cobb-Douglas (AER, 1928) production function:

$$F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$$

- MacCurdy (JPE, 1981) consumption-leisure preferences:

$$U(C_t) - V(N_t) = \log C_t - \theta \frac{N_t^{1+\varphi}}{1+\varphi}$$

- Note:
    - MacCurdy is a special case of balance-growth path preferences (King-Plosser-Rebelo, JME 1988; Boppart-Krusell JPE 2019)
    - Generate constant hours if wage and non-wage (capital) income grow at the same rate
    - With MacCurdy, $\frac{1}{\varphi}$ measures the Frisch elasticity (more on this next class)

# Equilibrium characterization with assumed functional forms

$$
\begin{aligned}
\frac{1}{C_t} W_t &= \theta N_t^{\varphi} \\
\frac{1}{C_t} &= \beta E_t \left[ (R_{t+1}^r + (1-\delta)) \frac{1}{C_{t+1}} \right] \\
C_t + I_t &= Y_t \\
Y_t &= A_t K_t^{\alpha} N_t^{1-\alpha} \\
K_{t+1} &= (1-\delta) K_t + I_t \\
R_t^r &= \alpha A_t \left( \frac{K_t}{N_t} \right)^{\alpha - 1} \\
W_t &= (1-\alpha) A_t \left( \frac{K_t}{N_t} \right)^{\alpha} \\
A_t &= A_{t-1}^{\rho_a} exp(\epsilon_t)
\end{aligned}
$$

# Step 2: solve for steady state

- Set $A_t = 1$ and impose $X_t = X_{t+1}$ for all variables $X$, then work through the algebra
- Take-home exercise: show that the steady state is given by:

$$
\begin{aligned}
R^r &= \frac{1}{\beta} - (1-\delta) \\
W &= (1-\alpha)\left(\frac{R^r}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} \\
N &= \left[\frac{1}{\theta}\frac{W}{\frac{R^r}{\alpha}-\delta}\left(\frac{R^r}{\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{1+\varphi}} \\
K &= \left(\frac{R^r}{\alpha}\right)^{-\frac{1}{1-\alpha}} N \\
Y &= \frac{R^r K}{\alpha} \\
I &= \delta K \\
C &= \left(\frac{R^r}{\alpha} - \delta\right) K
\end{aligned}
$$

- (Trick: after solving for $R^r$ and $W$, write the intratemporal household optimality condition in terms of $\frac{K}{N}$)

# Step 3: Log-linearize

- From levels to log deviations: (Do an example on whiteboard)

$$
\begin{aligned}
\frac{1}{C_t} W_t = \theta N_t^{\varphi} \quad &\Rightarrow \quad \hat{w}_t = \hat{c}_t + \varphi \hat{n}_t \\
\frac{1}{C_t} = \beta E_t \left[ (R^r_{t+1} + (1-\delta)) \frac{1}{C_{t+1}} \right] \quad &\Rightarrow \quad \hat{c}_t = -\beta R^r E_t \hat{r}^r_{t+1} + E_t \hat{c}_{t+1} \\
C_t + I_t = Y_t \quad &\Rightarrow \quad \frac{C}{Y} \hat{c}_t + \frac{I}{Y} \hat{i}_t = \hat{y}_t \\
Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \quad &\Rightarrow \quad \hat{y}_t = \hat{a}_t + \alpha \hat{k}_t + (1-\alpha) \hat{n}_t \\
K_{t+1} = (1-\delta) K_t + I_t \quad &\Rightarrow \quad \hat{k}_{t+1} = (1-\delta) \hat{k}_t + \delta \hat{i}_t \\
R^r_t = \alpha A_t \left( \frac{K_t}{N_t} \right)^{\alpha - 1} \quad &\Rightarrow \quad \hat{r}^r_t = \hat{a}_t - (1-\alpha)(\hat{k}_t - \hat{n}_t) \\
W_t = (1-\alpha) A_t \left( \frac{K_t}{N_t} \right)^{\alpha} \quad &\Rightarrow \quad \hat{w}_t = \hat{a}_t + \alpha(\hat{k}_t - \hat{n}_t) \\
A_t = A_{t-1}^{\rho_a} exp(\epsilon_t) \quad &\Rightarrow \quad \hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_t
\end{aligned}
$$

- Note: we can interpret $\hat{r}^r_{t+1}$ as:
  - percent deviation in gross rental rate $R^r_{t+1}$ from steady state
  - percentage point deviation in net rental rate $(R^r_{t+1} - 1)$ from steady state

# Log-linear equilibrium system

- The log-linear system can be written as:

$$\mathbf{A}_1 \mathbf{x}_t = \mathbf{A}_2 E_t \mathbf{x}_{t+1} + \mathbf{B_1} \epsilon_t$$

where $\mathbf{x}_t = [\hat{r}_t, \hat{w}_t, \hat{c}_t, \hat{n}_t, \hat{i}_t, \hat{y}_t, \hat{k}_t, \hat{a}_{t-1}]'$, and

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & -1 & -\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{C}{Y} & 0 & \frac{I}{Y} & -1 & 0 & 0 \\ 0 & 0 & 0 & -(1-\alpha) & 0 & 1 & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\delta & 0 & -(1-\delta) & 0 \\ 1 & 0 & 0 & -(1-\alpha) & 0 & (1-\alpha) & 0 & \\ 0 & 1 & 0 & \alpha & 0 & -\alpha & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\rho_a \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta R^r & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Which variables in $\mathbf{x}_t$ are pre-determined?

# Log-linear equilibrium system

- We rewrite this as:

$$\begin{aligned} \mathbf{A}_1\mathbf{x}_t &= \mathbf{A}_2 E_t\mathbf{x}_{t+1} + \mathbf{B_1}\epsilon_t \\ \Rightarrow \qquad \mathbf{x}_t &= \mathbf{A} E_t\mathbf{x}_{t+1} + \mathbf{B}\epsilon_t \end{aligned}$$

  where $\mathbf{A} = \mathbf{A}_1^{-1}\mathbf{A}_2$, and $\mathbf{B} = \mathbf{A}_1^{-1}\mathbf{B}_1$

- Goal: simulate this system

- Simulate = Solve the path of endogenous variables $\{\mathbf{x}_t\}$ given some sequence of shocks $\{\epsilon_t\}$

- Given that the system has one unique bounded solution, solving this system amounts to some clever usage of matrix algebra
  - Older approach: Blanchard-Kahn (Ecmtra, 1981)
  - Modern approach: QZ-method (Klein, JEDC 2000)

- Nowadays, there exist ready-made routines that do the job for us, e.g., Dynare

# The Real Business Cycle Model: Analysis

# Quantitative analysis

- We have discussed how to solve and simulate the system
- So let's proceed and analyze it
- To do so, we need to pick parameter values

# Calibration

- Main idea:
  1. Estimate driving process for exogenous TFP shocks
  2. Pick the other parameters to a) be consistent with external estimates and/or b) that the model steady state matches long-run data moments

- The idea that you could calibrate a theoretical model to quantitatively analyze data was the second major contribution of Kydland-Prescott (Ecmtra, 1982)
  - Traditional method: use theory to generate hypotehses, and reduced-form econometrics for quantification
  - Prior development: structural econometrics, where theory is used to derive an estimating equation
  - Calibration was very controversial when introduced
  - Now: bread and butter in all of economics

# Calibration II

- The model has 6 parameters: $\varphi, \delta, \beta, \alpha, \rho_a, \sigma_\epsilon$
- Typical procedure:
  - Pick $\delta$ to match NIPA estimates of average yearly capital depreciation rate $\sim 10\%$
  - Pick $\beta$ to match average gross yearly real return on capital $\sim 1 + 0.04 + \delta_{yearly}$
    - Recall steady-state relationship $R^r = \frac{1}{\beta} - (1 - \delta)$
  - Pick $\varphi$ to match outside estimates of the Frish elasticity $\sim 1$ (to be discussed more!)
  - Pick $\alpha$ to match long-run labor share $\sim 2/3$
    - Recall steady-state relationship $\frac{R^r K}{Y} = \alpha$
- For TFP, one starting point is to assume that these shocks has to be consistent with the fluctuations of the Solow residuals
  - Suppose we have quarterly data on $Y_t, K_t, N_t$
  - Taking logs of the production function, we can estimate SR's as the residuals from the regression

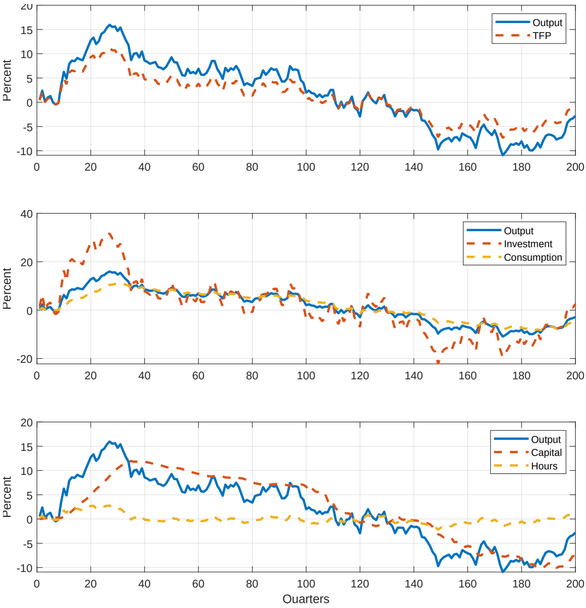
$$y_t - n_t = \alpha(k_t - n_t) + a_t$$

  - Having estimated $a_t$, we can estimate $\rho_a$ and $\sigma_\epsilon$ of

$$a_t = \rho_a a_{t-1} + \epsilon_t$$

  Sims (Mitman) reports $\rho_a = 0.979$ (0.95) and $\sigma_\epsilon = 0.009$ (0.007)

# Simulation results



# Simulation results (HP-filtered)

| | SD | | Rel. SD | | Corr $Y_t$ | | Autocorr | |
|---|---|---|---|---|---|---|---|---|
| | Data | Model | Data | Model | Data | Model | Data | Model |
| $Y_t$ | 0.017 | 0.015 | 1.00 | 1.00 | 1.00 | 1.00 | 0.79 | 0.72 |
| $C_t$ | 0.011 | 0.006 | 0.66 | 0.40 | 0.76 | 0.95 | 0.67 | 0.78 |
| $I_t$ | 0.044 | 0.041 | 2.67 | 2.73 | 0.76 | 0.99 | 0.86 | 0.72 |
| $N_t$ | 0.021 | 0.005 | 1.27 | 0.33 | 0.87 | 0.98 | 0.82 | 0.72 |
| $W_t$ | 0.012 | 0.010 | 0.69 | 0.66 | -0.01 | 1.00 | 0.71 | 0.74 |
| $R_t$ | 0.004 | 0.015 | 0.26 | 1.00 | 0.00 | 0.97 | 0.47 | 0.71 |
| $A_t$ | 0.013 | 0.012 | 0.76 | 0.80 | 0.78 | 1.00 | 0.76 | 0.72 |

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  - positive comovement of all GDP components
  - big swings in investment and small swings in consumption

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- **Consistent with the data, the RBC model has**
  - positive comovement of all GDP components
  - big swings in investment and small swings in consumption
- **In contrast to the data, the model has**
  - too little amplification
  - no persistence beyond that inherited by TFP process
  - way too little volatility in hours worked
  - too much volatility in prices

## Simulation results: what did we just look at?

- We computed moments from a time series of our endogenous variables $[\mathbf{x}_t]_{t=0}^{\infty}$
- This time series $[\mathbf{x}_t]$ solved

$$\mathbf{x}_t = \mathbf{A} E_t \mathbf{x}_{t+1} + \mathbf{B}\epsilon_t$$

  when feeding a sequence of shocks $\{\epsilon_0, \epsilon_1, ...\}$

- Since the system is linear, the solution is linear in the underlying shocks:

$$\mathbf{x}_t = \sum_{s=0}^{t} \mathbf{a}_{t-s}\epsilon_s,$$

  i.e.,

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{a}_0\epsilon_0, \\ \mathbf{x}_1 &= \mathbf{a}_1\epsilon_0 + \mathbf{a}_0\epsilon_1, \\ \mathbf{x}_2 &= \mathbf{a}_2\epsilon_0 + \mathbf{a}_1\epsilon_1 + \mathbf{a}_0\epsilon_2, ... \end{aligned}$$

# Simulation = superimposing IRFs

- Define the impulse-response function as

$$\begin{aligned} F(\epsilon) &= [a_0\epsilon, a_1\epsilon, a_2\epsilon, ...] \\ &= \epsilon[a_0, a_1, a_2, ...] \end{aligned}$$

  - IRF = vector of responses in period $t, t+1, t+2, ...$ to a singular shock in period $t$
  - Note: A linear IRF scales linearly with the size of the shock $\epsilon$

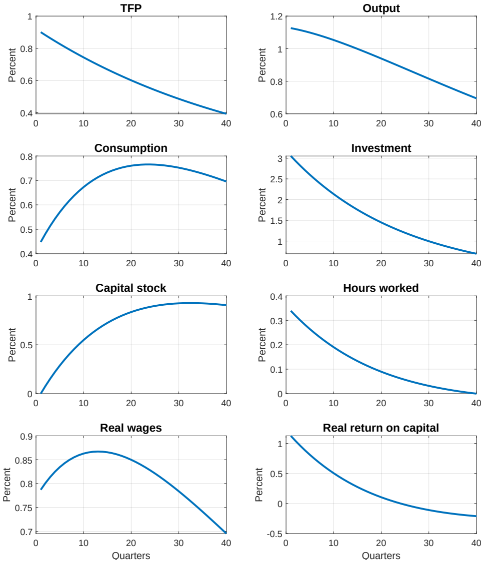
- Boppart-Krusell-Mitman (JEDC 2018): the simulation solution is a superimposition of IRFs:

$$\begin{aligned} [\mathbf{x}_t] &= [\mathbf{a}_0\epsilon_0, \mathbf{a}_1\epsilon_0 + \mathbf{a}_0\epsilon_1, \mathbf{a}_2\epsilon_0 + \mathbf{a}_1\epsilon_1 + \mathbf{a}_0\epsilon_2, ...] \\ &= F(\epsilon_0) + [0, F(\epsilon_1)] + [0, 0, F(\epsilon_2)] + ... \\ &= \epsilon_0 F(1) + \epsilon_1[0, F(1)] + \epsilon_2[0, 0, F(1)] + ... \end{aligned}$$

- $\Rightarrow F(1)$ is a sufficient statistic for the model simulation results

- Put differently, the mechanism of the model revealed by studying the impulse-response function to a unitary single-period shock

# IRFs to single persistent TFP shock



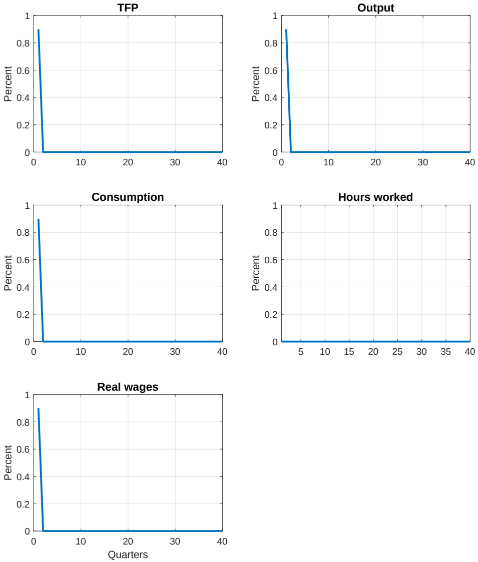
# How to unpack the responses?

- When staring at IRFs, it can be hard to discern the mechanism and to discern *impulse* from *propagation*

- How to unpack any model: simplify as much as you can, and then build the model gradually up again

- Two simplifications:
  1. $\alpha = 0$ (such that $I = K = 0$, and therefore $Y = F(N) = C$)
  2. $\rho_a = 0$ (no persistence in the impulse)

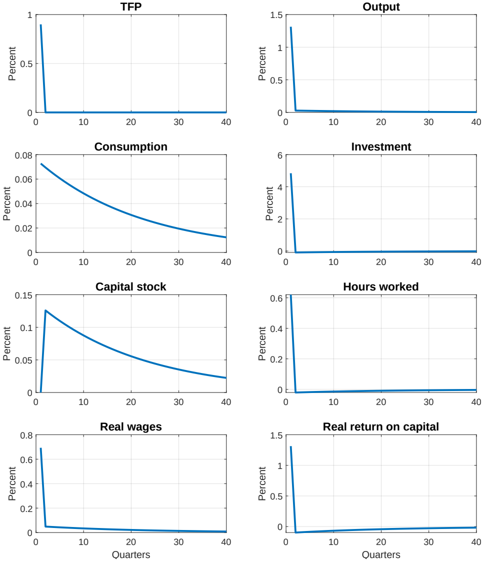
# IRFs using a blip shock in model without capital



# What's going on?

- When TFP increases; output, consumption and wages jump
  - Production function: $y_t = a_t + n_t$
  - Market clearing: $c_t = y_t$
  - Firm F.O.C.: $w_t = a_t$
- Why is hours worked flat?
- Household intratemporal F.O.C: $w_t = c_t + \varphi n_t$
- Holding marginal utility of consumption fixed, hours increase as wages increase (substitution effect)
- But in equilibrium, consumption increases, dampening hours (income effect)
- Balance-growth path preferences: income and substitution effect cancels
- This model has no internal propagation!

# IRFs using a blip shock with capital



# What's going on?

- With capital, household can now smooth consumption

$$\hat{c}_t = -\beta R^r E_t \hat{r}^r_{t+1} + E_t \hat{c}_{t+1}$$

- TFP up, households feel wealthier, consumption increases (but much less so compared to previous model)

- Consumption smoothing $\Rightarrow$ investment jumps

$$\frac{C}{Y}\hat{c}_t + \frac{I}{Y}\hat{i}_t = \hat{y}_t$$

- When TFP increases, wages and current interest rate jumps

$$\hat{r}^r_t = \hat{a}_t - (1-\alpha)(\hat{k}_t - \hat{n}_t)$$
$$\hat{w}_t = \hat{a}_t + \alpha(\hat{k}_t - \hat{n}_t)$$

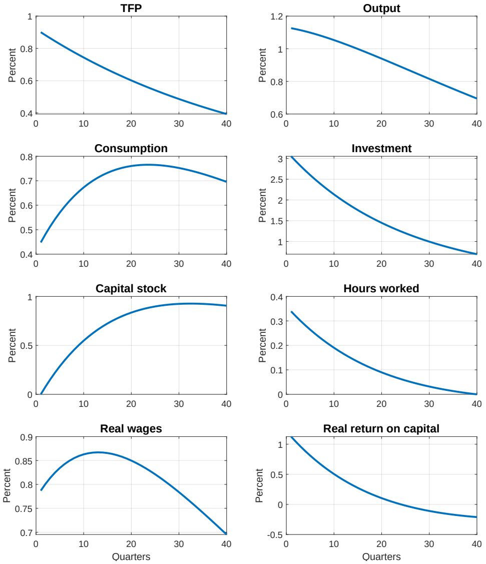
- Moreover, higher capital stock means that wages (rental rate) will be persistently higher (lower)

- Now, because of lower consumption response, hours worked increases

$$\hat{w}_t = \hat{c}_t + \varphi \hat{n}_t$$

- Now there is some propagation!

# IRFs in core model (persistent shock and capital)



# What's going on?

- With a persistent shock, all responses become more persistent
- Some, in particular household consumption, even hump-shaped
- Now there is an additional motive for investing besides consumption smoothing: future capital is unusually productive
- Households face trade-off when saving: smoothing consumption vs. maximizing lifetime consumption
- Turns out hump-shape is the optimal path

# Summing up

- RBC = minimal GE model to get started with business cycles analysis
  - Abstracts from a lot of things, but this was intentional
- Key features:
  - No distortions
  - TFP is the only driving process
  - Propagation happens through equilibrium responses of hours worked and investment
- Underlying philosophy:
  - Business cycles analyzed in the same framework as long-run growth
  - GE models can be calibrated and used to quantitatively interpret the data
- Results:
  - The model can seemingly explain a whole lot of business cycle moments
  - Fails in some key aspects
- Next up:
  - RBC as a diagnosis tool
  - Thinking deeper about mechanisms and model fit