

# 1 Precautionary Savings: Prudence and Borrowing Constraints

In this section we study conditions under which savings react to changes in income uncertainty. Recall that in the PIH, when you abstract from borrowing constraints, certainty equivalence implies that “mean preserving spreads” of the income distribution do not impact on saving. But quadratic utility is very special, what happens with more general utility functions?

## 1.1 Prudence: A two-period model

Consider the simple two-period consumption-saving problem studied by Leland (1968):

$$\begin{aligned} & \max_{\{c_0, c_1, a_1\}} u(c_0) + \beta E[u(c_1)] \\ & \text{s.t.} \end{aligned}$$

$$\begin{aligned} c_0 + a_1 &= y_0 \\ c_1 &= Ra_1 + \tilde{y}_1 \end{aligned}$$

where  $y_0$  is given, and income next period  $\tilde{y}_1$  is also exogenous but stochastic.<sup>1</sup> If we retain the assumption  $\beta R = 1$  to simplify the algebra, the Euler equation gives

$$u'(y_0 - a_1) = E[u'(Ra_1 + \tilde{y}_1)],$$

which is one equation in one unknown,  $a_1$ . The LHS is increasing in  $a_1$  since  $u'' < 0$ , and the RHS is decreasing for the same reason, hence  $a_1^*$  is uniquely determined.

### FIGURE HERE

Note that current consumption  $c_0$  is determined by the period-zero budget constraint

$$c_0^* = y_0 - a_1^*,$$

hence a rise in savings leads to a fall in current consumption.

---

<sup>1</sup>Note that the timing of this problem is slightly different from the one adopted in the description of the PIH. There, we assumed that individuals receive income and consume at the beginning of the period and the payments of interests occurs at the end of the period. Here, we assume the payment of interests occurs at the beginning of the period, income is paid at the end of the period and individuals consume at the end of the period. In general, results are robust to this timing, it is a matter of convenience which one to choose.

**Mean-preserving spread**— What happens to optimal consumption at  $t = 0$  if the uncertainty over income next period  $\tilde{y}_1$  rises, i.e. as future income becomes more risky? Consider a mean-preserving spread of  $\tilde{y}_1$ . Define

$$\tilde{y}_1 = \bar{y}_1 + \varepsilon_1,$$

where  $\varepsilon_1$  is the stochastic component and  $\bar{y}_1$  is the mean. Assume that  $E(\varepsilon_1) = 0$  and  $\text{var}(\varepsilon_t) = \sigma_\varepsilon$ . The Euler equation becomes

$$u'(y_0 - a_1) = E[u'(Ra_1 + \bar{y}_1 + \varepsilon_1)],$$

which shows that *if  $u'$  is convex*, then by Jensen's inequality, a mean-preserving spread of  $\varepsilon_1$  will increase the value of the RHS, which then shifts upward, inducing a rise in  $a_1^*$  and a fall in  $c_0^*$ . This is an application of the famous result by Rotschild and Stiglitz (1970).

**Prudence**— The convexity of the marginal utility (or  $u''' > 0$ ) is called “prudence” and is a property of preferences, like risk aversion: risk-aversion refers to the curvature of the utility function, whereas prudence refers to the curvature of the marginal utility function.<sup>2</sup>

*Result 2.3: If the marginal utility is convex ( $u''' > 0$ ), then the individual is “prudent” and a rise in future income uncertainty leads to a rise in current savings and a decline in current consumption.*

It can be easily seen that any utility function with decreasing absolute risk aversion, i.e. in the DARA class (which includes CRRA), displays positive third derivative. Let  $\alpha(c)$  be the coefficient of absolute risk aversion. Then:

$$\alpha(c) = \frac{-u''(c)}{u'(c)} \Rightarrow \alpha'(c) = \frac{-u'''(c)u'(c) + [u''(c)]^2}{[u'(c)]^2}.$$

Since with DARA  $\alpha'(c) < 0$ , then we have that

$$-u'''(c)u'(c) + [u''(c)]^2 < 0 \Rightarrow u'''(c) > \frac{[u''(c)]^2}{u'(c)} > 0.$$

Intuitively, a rise in uncertainty reduces the certainty-equivalent income next period and with DARA effectively increases the degree of risk-aversion of the agent, inducing him to save more.

---

<sup>2</sup>Precisely, Kimball (1990) defines the index of absolute prudence as the ratio  $-u'''(c)/u''(c)$ , so in a similar vein to the Arrow-Pratt index of absolute risk-aversion  $-u''(c)/u'(c)$ .

Prudence is a motive for additional savings in order to take precaution against possible negative realizations of the income shock next period. In this sense, savings induced by prudence are called “precautionary savings” or “self-insurance”. In this simple, two-period partial equilibrium model one can define precautionary wealth due to income uncertainty  $\sigma_\varepsilon$  as the difference between the optimal asset choice under uncertainty  $a_1^*(\sigma_\varepsilon)$  and the optimal asset choice under certainty over next period income, i.e.  $a_1^*(0)$ .

**Saving motives**— This is a good time to make a short remark about “saving motives”. The saving motive associated to  $\beta R > 1$  which pushes the individual to postpone consumption because of patience and/or returns to savings is called *intertemporal motive*. The saving motive of the pure PIH where utility is quadratic (hence uncertainty has no role) and  $\beta R = 1$  (hence intertemporal motives are inactive) is called *smoothing motive*. The individual wants to smooth consumption through income shocks. Finally, as explained above, the saving motive associated to future income uncertainty is called *precautionary or self-insurance motive*. We add that in a life-cycle model where the individual faces a retirement period, during the working stage of the life-cycle the individual would have a *life-cycle motive* for saving associated to the desire of smoothing consumption between working life and retirement. In presence of altruism towards their offsprings, we have an additional saving motive aimed at leaving behind some assets as *bequest*.

## 1.2 Prudence: Multi-period case

Let’s generalize the two-period model to a multiperiod model with *iid* income shocks and finite-horizon. In the multi-period case (with time horizon  $T$ ), the problem of the household can be written, in recursive form, as

$$\begin{aligned} V_t(a_t, y_t) &= \max_{\{c_t, a_{t+1}\}} u(c_t) + \beta E[V_{t+1}(a_{t+1}, y_{t+1})] \\ &\text{s.t.} \\ c_t + a_{t+1} &= Ra_t + y_t \end{aligned}$$

Note that when the income shocks  $\{y_t\}$  are *iid*, we can define a unique state variable which is a sufficient statistics for the household choice, “cash in hand”  $x_t \equiv Ra_t + y_t$  since  $(a_t, y_t)$  always enter additively and current levels of  $y_t$  do not provide any information about the future realizations of income shocks. Note that

$$x_{t+1} = Ra_{t+1} + y_{t+1} = R(x_t - c_t) + y_{t+1},$$

which is the law of motion for the new individual state variable. See LS, 17.5.1 for a presentation of cash-in-hand as state variable. This leads to the simpler formulation

$$\begin{aligned} V_t(x_t) &= \max_{\{c_t, x_{t+1}\}} u(c_t) + \beta E[V_{t+1}(x_{t+1})] \\ &\text{s.t.} \\ x_{t+1} &= R(x_t - c_t) + y_{t+1} \end{aligned}$$

From the FOC's with respect to  $c_t$  and the constraint, we obtain

$$u'\left(x_t + \frac{y_{t+1} - x_{t+1}}{R}\right) = \beta RE[V'_{t+1}(x_{t+1})]. \quad (1)$$

Applying the same logic as in the two-period model, it is easy to conclude that precautionary saving arises as long as the derivative of the value function ( $V'_{t+1}$ ) is convex, i.e.  $V''_{t+1} > 0$ . Sibley (1975) showed that when the time-horizon  $T$  is finite, it can be proved that *if*  $u''' > 0$ , *then*  $V'''_t > 0$  for all  $t = 1, \dots, T$ . The proof is based on backward induction: in the last period  $V'_T = u'$  which is convex by assumption. Then, one can show that  $V'_{T-1}$  is also convex, and so on.

### 1.3 Borrowing Constraints

To isolate the role of borrowing constraints for precautionary saving, we abstract from prudence altogether and focus on the quadratic utility case. To account for the possibility that the borrowing constraint is binding, the Euler equation needs to be modified. Suppose households face a *no-borrowing constraint*  $a_{t+1} \geq 0$ . Then, (??) becomes

$$c_t = \begin{cases} E_t c_{t+1} & \text{if } a_{t+1} > 0 \\ y_t + a_t & \text{if } a_{t+1} = 0 \end{cases}$$

where the first line is just the FOC of the agent when the constraint is not binding, while the second line descends directly from the budget constraint  $a_{t+1} = R(y_t + a_t - c_t)$  when the constraint is binding ( $a_{t+1} = 0$ ). The constrained household would like to borrow to finance consumption, but it is not allowed, so it consumes all its resources.

In which scenarios is the borrowing constraint binding? For example, imagine that  $a_t = 0$  and that income  $y_t = \bar{y} + \varepsilon_t$ , where  $\varepsilon_t$  follows an iid process with mean zero. In this case, we know that optimal unconstrained consumption will be  $c_t^* = \bar{y} + \frac{r}{1+r}\varepsilon_t$  while

her total resources are  $y_t = \bar{y} + \varepsilon_t$ . Therefore, if  $\varepsilon_t$  is negative, the agent, to smooth income, would like to consume  $c_t > y_t$  (which she would if she could borrow) but she is constrained at  $c_t = y_t$ . In general, the constraint is likely to bind whenever  $y_t$  follows a mean-reverting process. The above pair of conditions can be written in compound form as

$$c_t = \min \{y_t + a_t, E_t c_{t+1}\} = \min \{y_t + a_t, E_t [\min \{y_{t+1} + a_{t+1}, E_{t+1} c_{t+2}\}]\}.$$

Now, suppose that the uncertainty about income  $y_{t+1}$  increases. Very low realizations of income  $y_{t+1}$  become more likely, which makes the borrowing constraint more likely to bind in the future and reduces the value of  $E_t [\min \{y_{t+1} + a_{t+1}, E_{t+1} c_{t+2}\}]$ . This, in turn, reduces the value of  $E_t c_{t+1}$ . Thus, if the borrowing constraint is not already binding at time  $t$  but it may be binding in the future, then agents consume less today.

Intuitively, when agents face borrowing constraints, they fear getting several consecutive bad income realizations which would push them towards the constraint and force them to consume their income *without the ability of smoothing consumption*. To prevent this situation, they save for self-insurance (precautionary motive). Thus, we have an important result: prudence is not strictly necessary for precautionary saving behavior, or:

*Result 2.4: Even in absence of prudence (e.g. with quadratic preferences), in presence of borrowing constraints a rise in future income uncertainty leads to a rise in current savings for precautionary reasons and to a decline in current consumption.*

Even though we showed this result for quadratic utility, it is a general results that holds for concave utility.

### 1.3.1 A Natural Debt Limit

This is a good place to discuss how to model debt limits. We started by imposing an exogenous borrowing constraint like  $a_{t+1} \geq -\phi$ , where  $\phi$  is a parameter (in our previous case  $\phi = 0$ ). However, one may wonder if there is a “natural” borrowing limit that the household faces.

Suppose the income process  $\{y_t\}_{t=0}^\infty$  is deterministic. Impose non-negativity of consumption throughout the life of the household, i.e.  $c_t \geq 0$  for all  $t$  and iterate forward on

the budget constraint

$$\begin{aligned} c_t &= a_t + y_t - \frac{a_{t+1}}{1+r} \geq 0 \Rightarrow a_t \geq -y_t + \frac{a_{t+1}}{1+r} \\ a_t &\geq -y_t + \frac{a_{t+1}}{1+r} \geq -y_t + \frac{1}{1+r} \left[ -y_{t+1} + \frac{a_{t+2}}{1+r} \right] \geq \dots \\ a_t &\geq -\sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j y_{t+j}. \end{aligned}$$

In other words, by imposing this constraint, the household is not allowed to accumulate more debt than what she will ever be able to repay by consuming just zero every period.

If the income process is stochastic, then how can we be sure that whatever the household borrows she will repay *almost surely* (*i.e.* *with probability 1*)? Then, we need to substitute  $y_t$  at each  $t$  with the lowest possible realization of the income shock, call it  $y_{\min}$ , and we have the *natural debt limit*

$$a_t \geq - \left( \frac{1+r}{r} \right) y_{\min}. \quad (2)$$

This is the loosest possible debt limit. No exogenous borrowing constraint can ever be looser than the natural debt limit. Note however that if  $y_{\min} = 0$ , then the natural debt limit is zero!

**Inada conditions and natural borrowing limit**— Keep in mind an important property: if the utility function satisfies the Inada condition  $u'(0) = -\infty$ , then the consumer will never want to borrow up to the natural debt limit.<sup>3</sup> Suppose she does borrow up to  $a_t = -\left(\frac{1+r}{r}\right)y_{\min}$  and suppose the income realization  $y_t$  is precisely  $y_{\min}$  which has positive probability. From the budget constraint at date  $t$ :

$$\begin{aligned} c_t &= a_t + y_{\min} - \frac{a_{t+1}}{1+r} = - \left( \frac{1+r}{r} \right) y_{\min} + y_{\min} - \frac{a_{t+1}}{1+r} \\ &= -\frac{1}{r} y_{\min} - \frac{a_{t+1}}{1+r} \leq -\frac{1}{r} y_{\min} - \underbrace{\frac{1}{1+r} \left[ - \left( \frac{1+r}{r} \right) y_{\min} \right]}_{\text{max that can be borrowed}} \\ &= 0 \end{aligned}$$

which shows that, with positive probability next period the consumer will have to consume zero. However that would lead to an infinitely negative utility, and so the consumer will never reach that state.

---

<sup>3</sup>For example, CRRA utility  $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$  with  $\gamma \geq 1$  satisfies the Inada condition  $u'(0) = -\infty$ .

The preferences alone will insure that the *natural* borrowing limit will never bind. In other words, in solving for the optimal consumption you can safely assume interior solutions for the Euler equation. This is not true for ad-hoc debt limits!

## 2 The Income Fluctuation Problem

We are now ready to formally examine the general problem

$$\begin{aligned} & \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & \text{s.t.} \\ & c_t + a_{t+1} = Ra_t + y_t \\ & a_{t+1} \geq 0 \end{aligned}$$

where we combine general preferences (i.e., we just impose  $u' > 0, u'' < 0$ ) with a no-borrowing constraint. The value for the interest rate is always *exogenously given* and constant. Note the slight difference in timing of the budget constraint with respect to the problem stated when we studied the PIH. We explained that it only depends on the timing of the consumption choice.

The first order necessary condition for optimality is

$$u'(c_t) = \beta R E_t [u'(c_{t+1})] + \lambda_t, \quad (3)$$

where  $\lambda_t > 0$  is the multiplier on the no-borrowing constraint. Condition (3) implies the Euler equation

$$u'(c_t) \geq \beta R E_t [u'(c_{t+1})]. \quad (4)$$

We want to understand whether the optimal consumption sequence  $\{c_t\}$  is bounded above or whether it will be diverging as  $t \rightarrow \infty$ . This characterization is important because, if the consumption sequence is bounded, since  $c'(a) > 0$  then the endogenous state space for assets  $[0, \bar{a}]$  is compact, i.e. there exists an upper bound  $\bar{a}$  which is finite. This is a crucial requirement for proving the existence of an equilibrium in economies populated by many agents who chooses their optimal consumption by solving income fluctuations problems. If the consumption sequence diverges, then  $\bar{a} \rightarrow \infty$ . This means that, in such an economy, there will be an infinite supply of assets which prevents existence of an equilibrium.

The convergence properties of the consumption sequence will depend on the value of the term  $\beta R$ . We always examine three separate cases:  $\beta R$  above, equal to or below one.

We start from a problem where income fluctuations are deterministic, i.e., perfectly foreseen. Next we move to stochastic income fluctuations.

## 2.1 Deterministic Income Fluctuations

**Case  $\beta R > 1$ :** Without uncertainty, since  $u$  is strictly concave, the Euler equation (4) implies

$$u'(c_t) \geq \beta R u'(c_{t+1}) > u'(c_{t+1}) \Rightarrow c_{t+1} > c_t,$$

thus consumption grows indefinitely. Since borrowing is limited at zero, assets must grow to finance consumption, so also assets diverge. The individual is “too patient” or/and the rate of return on savings is too high. Both forces push her to accumulate too much wealth.

**Case  $\beta R = 1$ :** From the Euler Equation,  $u'(c_t) \geq u'(c_{t+1})$ . Ideally, households want perfectly smooth consumption ( $c_{t+1} = c_t$ ) when the liquidity constraint is not binding. If it is binding, then  $c_{t+1} > c_t$ . Overall, consumption is a *nondecreasing* sequence. One can prove that the existence of the borrowing constraint affects the problem only until a given time  $\tau$  and vanishes thereafter. Until  $t = \tau$ , consumption will grow and then it remains constant thereafter. How is  $\tau$  determined? Define

$$h_t = \frac{r}{1+r} H_t = \frac{r}{1+r} \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j y_{t+j}$$

as the annuity value of the of human wealth (i.e., the discounted present value of future income). Then  $\tau$  satisfies

$$\tau = \arg \sup_t h_t.$$

So, consumption and assets converge to a finite value. See LS (16.3.1) for a formal proof of this result.

To gain some intuition, consider a sequence for  $y_t$  that increases until time  $t^*$  and then is decreasing from there on. It is easy to see that  $\tau < t^*$ . Households would like to borrow initially against future higher income, but they cannot so the constraint will bind for a while and they will keep increasing consumption until  $\tau$  and thereafter they

will follow a constant consumption path where they will save at the beginning to finance future consumption when income is low.

Clearly, there may be some paths for  $\{y_t\}$  such that  $h_t$  is always increasing, in which case the individual will always be constrained (or,  $\tau \rightarrow \infty$ ). However, assuming that the sequence  $\{y_t\}$  is bounded above (reasonable, since  $y_t$  is individual income) is enough to guarantee that  $h_t$  has a maximum for finite  $\tau$ .

**Case  $\beta R < 1$ :** Consider the simple case where the endowment sequence is constant at  $y$ . Let's write the above problem in DP form with cash-in-hand as a state variable, i.e.,  $x \equiv Ra + y$ . Then:

$$\begin{aligned} V(x) &= \max_c \{u(c) + \beta V(x')\} \\ &\text{s.t.} \\ x' &= R(x - c) + y \\ x' &\geq y \end{aligned}$$

If at time  $t$  the liquidity constraint is binding, then next period  $a_{t+1} = 0$ , and  $c_{t+1} = y$ . Since  $y$  is constant, this situation perpetuates and  $c$  is constant forever: the individual is always constrained. In this case the consumption sequence is bounded.

The interesting case is when the liquidity constraint is not binding and wealth is positive. From the envelope condition of the above problem,

$$u_c(c) = V_x(x),$$

which differentiated wrt to  $x$  gives

$$u_{cc}(c) \frac{dc}{dx} = V_{xx}(x) \implies \frac{dc}{dx} = \frac{V_{xx}(x)}{u_{cc}(c)} > 0, \quad (5)$$

thus, consumption is increasing in cash-in-hand, under concavity of  $V$ .<sup>4</sup>

Next, we want to show that, as long as the borrowing constraint is not binding (i.e. if  $x' > y$ ), then  $x' < x$ . When  $x' > y$ , the Euler Equation holds with equality, and:

$$\begin{aligned} u_c(c) &= \beta RV_x(x') \\ V_x(x) &= \beta RV_x(x') < V_x(x') \implies x' < x, \end{aligned}$$

---

<sup>4</sup>Here we are assuming the concavity of  $V$  but we know that under certain conditions the Bellman operator preserves strict concavity of  $u$ .

where the second line follows from the envelope condition, from  $\beta R < 1$  and from the concavity of  $V$ . Therefore, if we start from a positive level of assets  $a_0$ , assets will be optimally decumulated over time and consumption will decline because of (5). Because  $dc/dx > 0$ , since  $x$  declines over time, the consumption sequence is bounded above.

Finally, we can also show that when  $x$  reaches  $y$  from above, then  $x' = y$  and  $c = y$ , i.e. in the limit as assets get depleted and reach zero (and cash in hand equals  $y$ ), the consumption sequence converges to the constant endowment stream.<sup>5</sup> We prove it *by contradiction*. Suppose that at the point where  $x = y$ ,  $c < x$ , hence  $x' > y$ . Then, the FOC holds with equality and

$$\begin{aligned} u_c(c) &= \beta RV_x(x') \\ V_x(y) &= \beta RV_x(R(x - c) + y) < V_x(R(x - c) + y) < V_x(y), \end{aligned}$$

where the second line uses the envelope condition, the fact that  $\beta R < 1$  and the strict concavity of the value function. The second line contains the contradiction. Thus, when  $\beta R < 1$  consumption converges to a finite value.

It is not difficult to generalize this result, and show that in this case the consumption sequence will converge even if the  $y_t$  sequence is time-varying.

**Taking stock-** How can we take stock of these results? In the deterministic case, the desire to save is increasing in patience ( $\beta$ ) and the interest rate ( $R$ ). When  $\beta R$  is too large, households' desire to accumulate is "too strong" and both assets and consumption diverge to infinity. To prevent divergence, we need  $\beta R \leq 1$ .

## 2.2 Stochastic Income Fluctuations

We now turn to the stochastic case. Here there is an additional motive for saving, the *precautionary motive*, due to 1) prudence (for some utility functions) and 2) the interaction between risk-aversion (i.e., aversion to consumption fluctuations) and the borrowing constraint. Given the extra saving motive that pushes consumption upward over time, we should expect that the condition under which  $\{c_t\}$  converges will be more stringent than in the deterministic case. It turns out that we will need  $\beta R < 1$ .

**A useful supermartingale—** Multiply both sides of

$$u'(c_t) \geq \beta RE_t[u'(c_{t+1})]$$

---

<sup>5</sup>You should try to prove that the strictly decreasing cash in hand sequence will reach  $y$ .

by  $(\beta R)^t$  and define  $M_t \equiv (\beta R)^t u'(c_t) > 0$ . Then equation (4) can be written as

$$M_t \geq E_t M_{t+1}$$

which asserts that  $M_t$  follows a *supermartingale*. Since  $M_t$  is non-negative, by the supermartingale convergence theorem (Doob, 1995), this stochastic process converges almost surely to a non-negative finite limit  $\bar{M}$ , i.e.,

$$\lim_{t \rightarrow \infty} M_t = \bar{M} < \infty. \quad (6)$$

**Case  $\beta R > 1$ :** According to the convergence theorem above,  $(\beta R)^t u'(c_t)$  has a finite limit. Since  $(\beta R)^t \rightarrow \infty$ , then marginal utility  $u'(c_t)$  can only converge to  $u'(\bar{c}) = 0$  or, given the Inada condition,  $c_t \rightarrow \infty$ . Since debt is limited (at zero, in our benchmark, but could be any finite limit), this large consumption cannot be financed by debt but must be financed by saving. And hence divergence of consumption means  $a_t \rightarrow \infty$ , which in turn implies that there is no upper bound in the asset space. This is the same result we found for the certainty case.

**Case  $\beta R = 1$ :** We can give a simple proof of this result for general income process if we assume that  $u''' > 0$ . From the Euler Equation

$$u'(c_t) \geq E_t [u'(c_{t+1})].$$

From convexity of the marginal utility, by Jensen's inequality

$$u'(c_t) \geq E_t [u'(c_{t+1})] > u'(E_t(c_{t+1})),$$

and by concavity of  $u$ , we have that  $E_t(c_{t+1}) > c_t$ , so consumption will always tend to ratchet upward over time which prevents the consumption sequence from converging almost surely to a finite number.

Another simple argument is available for the case where  $y$  is iid. The proof is by contradiction. Suppose there exists an upper bound for cash-in-hand  $\bar{x}$ . Let  $\bar{y}$  be the highest possible realization of  $y$ , and  $\bar{x} = \bar{y} + Ra'(\bar{x})$  be the maximum amount of cash in hand. Using the envelope condition into the Euler equation

$$V_x(\bar{x}) \geq E_t [V_x(y_{t+1} + Ra_{t+1}(\bar{x}))] > V_x(\bar{y} + Ra_{t+1}(\bar{x})) = V_x(\bar{x})$$

where the second inequality descends from the strict concavity of the value function, and the last equality, which states the contradiction, comes from the definition of cash in hand. See LS (16.5) for a more complete formal proof of this result.

Finally, note that the convergence result with  $\beta R = 1$  of the deterministic case does not hold in the stochastic case.

**Case  $\beta R < 1$ :** Consider the case of *iid* income shocks. Let  $x$  be cash in hand. From the Euler Equation:

$$u_c(c(x)) = \beta R E[u_c(c(x'))] = \beta R \frac{E[u_c(c(x'))]}{u_c(c(\bar{x}'))}, \quad (7)$$

where  $x' = Ra'(x) + y'$  is cash in hand next period given that today's cash in hand is  $x$  and given the current income realization is  $y$ . Let  $\bar{x}' = Ra'(x) + \bar{y}$  be the cash in hand associated to the maximum realization of income next period, given that today's cash in hand is  $x$ . Suppose that the limit

$$\lim_{x \rightarrow \infty} \frac{E[u_c(c(x'))]}{u_c(c(\bar{x}'))} = 1. \quad (8)$$

Then, for  $x$  large enough, since  $\beta R < 1$ , the Euler equation (7) yields

$$u_c(c(x)) = \beta R u_c(c(\bar{x}')) < u_c(c(\bar{x}')).$$

Concavity of  $u$  and monotonicity of  $c$  wrt  $x$  (proved in the deterministic case earlier) implies that

$$c(\bar{x}') < c(x) \Rightarrow \bar{x}'(x) < x$$

thanks to the fact that  $c_x(x) > 0$ . And we would be done, because we have demonstrated that cash in hand does not increase forever: when  $x$  is large enough  $x' < x$  for sure.

Therefore, we only need to establish conditions under which the limit in (8) holds. Consider  $u_c(c(x'))$  and compute a first-order Taylor approximation around  $x' = \bar{x}'$ :

$$u_c(c(x')) \simeq u_c(c(\bar{x}')) + u_{cc}(c(\bar{x}')) c_x(\bar{x}')(x' - \bar{x}').$$

Taking expectations of both sides

$$\begin{aligned} E[u_c(c(x'))] &\simeq u_c(c(\bar{x}')) - u_{cc}(c(\bar{x}')) E[\bar{x}' - x'] c_x(\bar{x}') \\ &= u_c(c(\bar{x}')) - u_{cc}(c(\bar{x}')) E[\bar{y} - y'] c_x(\bar{x}') \end{aligned} \quad (9)$$

where in the first line we use the fact that  $\bar{x}'$  is deterministic since it is implied by the specific income realization  $\bar{y}$ . And in the second line we use the fact that  $x' \equiv Ra' + y'$  which implies  $\bar{x}' - x' \equiv \bar{y} - y'$ .

Dividing equation (9) by  $u_c(c(\bar{x}'))$  we obtain:

$$\frac{E[u_c(c(x'))]}{u_c(c(\bar{x}'))} \simeq 1 + \alpha(c(\bar{x}')) [\bar{y} - E(y')] c_x(\bar{x}'),$$

where  $\alpha(c)$  is the coefficient of absolute risk aversion at consumption level  $c(\bar{x}')$ . Since both  $[\bar{y} - E(y')]$  and  $c_x(\bar{x}')$  are positive and finite, a sufficient condition for the limit in (8) to hold is

$$\lim_{x \rightarrow \infty} \alpha(c(x)) = 0. \quad (10)$$

If, for example, *absolute risk aversion is monotonically decreasing with asset holdings* then condition (10) holds. The faster  $\alpha$  decreases, the smaller the upper bound on the asset space. The intuition is clear: DARA means that the agent is less worried about income uncertainty as she gets rich because she becomes less risk averse, so she will consume more and accumulate less. This force offsets, and eventually overcomes, precautionary accumulation as wealth increases.

Note that CRRA utility has DARA, so it satisfies condition (10). Remember that DARA is a sufficient condition and that this result holds for *iid* shocks. Huggett (1993) generalizes this result to a 2-state Markov chain for the income process (but only for the CRRA utility case).

We conclude by summarizing our findings in:

*Result 2.5: In presence of borrowing constraints and uncertain income, the condition  $\beta R < 1$  is necessary for the optimal consumption sequence and for the asset space to be bounded. Moreover, when  $\beta R < 1$ , if income shocks are iid and absolute risk-aversion is decreasing (DARA utility), then the asset space is bounded. More in general, even with Markov shocks, as long as absolute risk aversion decreases fast enough with  $c$ , the state space will remain bounded.*

### 3 Notes

Leland (1968) and Sandmo (1970) showed in a two-period setting that a positive third derivative is needed to obtain precautionary savings. Sibley (1975) extended the result to

a multi-period setting. Kimball (1990) further generalized these results. Schechtman and Escudero (1977) characterized the income fluctuation problem with *iid* shocks. Huggett (1993) generalizes their proofs to the case of a 2-state Markov chain for the income process and CRRA utility. Chamberlain and Wilson (2000) extend it to any arbitrary endowment sequence that is sufficiently stochastic.

## References

- [1] Chamberlain, Gary and Charles Wilson (2000); Optimal Intertemporal Consumption under Uncertainty, *Review of Economic Dynamics*
- [2] Huggett, Mark (1993); The risk-free rate in heterogeneous-agent incomplete-insurance economies, *Journal of Economic Dynamics and Control*
- [3] Kimball, Miles (1990); Precautionary Saving in the Small and in the Large, *Econometrica*
- [4] Leland, H. (1968); Savings and Uncertainty: The Precautionary Demand for Saving, *Quarterly Journal of Economics*
- [5] Sandmo Agnar (1970); The Effect of Uncertainty on Saving Decisions, *Review of Economic Studies*.
- [6] Schechtman J. and V. Escudero (1977); Some Results on An Income Fluctuation Problem, *Journal of Economic Theory*
- [7] Sibley David (1975); Permanent and Transitory Income Effects in a Model of Optimal Consumption with Wage Income Uncertainty, *Journal of Economic Theory*