Assignment 9 Proofs

a) $HH_R = H_R$ and $HX_1 = H_RX_1 = X_1$

First, want to show $HX_1 = X_1$

Know $X = (X_1 | X_2)$

$HX = X$ by HW5, so $H(X_1|X_2) = HX = X$

and if $H(X_1|X_2) = X$

then $H(X_1|X_2) = (HX_1 | HX_2) = X$

$= (HX_1 | HX_2) = (X_1|X_2)$

Therefore, $\underline{HX_1 = X_1}$

Now, want to show $HH_R = H_R$

$H_R = X_1(X_1^TX_1)^{-1}X_1^T$

So, $\underbrace{HX_1}_{\text{by above } = X_1}(X_1^TX_1)^{-1}X_1^T \Rightarrow X_1(X_1^TX_1)^{-1}X_1^T = H_R$

So $\underline{HH_R = H_R}$

Now, $H_RX_1 = X_1$ because H_RX_1 is essentially the same as HX_1 as proved above.

Also, $H_RX_1 = (X_1(X_1^TX_1)^{-1}X_1^T)X_1 \to X_1\underbrace{(X_1^TX_1)^{-1}(X_1^TX_1)}_{I,\ b/c\ A^{-1}A = I} \to X_1I \to X_1$

b) $H - H_R$ is Symmetric and Idempotent

Symmetric

$(H - H_R)^T = H - H_R$

$H^T - H_R^T = H - H_R$

$(X(X^TX)^{-1}X^T)^T - (X_1(X_1^TX_1)^{-1}X_1^T)^T$

$X((X^TX)^{-1})^TX^T - X_1((X_1^TX_1)^{-1})^TX_1^T$

$X(X^{-1}(X^T)^{-1})X^T - X_1(X_1^{-1}(X_1^T)^{-1})X_1^T$

$X(X^TX)^{-1}X^T - X_1(X_1^TX_1)^{-1}X_1^T = \boxed{H - H_R}$

Idempotent

$(H - H_R)(H - H_R) = H - H_R$

$H^2 - HH_R - H_RH + H_R^2$ by Part a, and class notes $H_RH = H_R$

$H^2 - H_R - H_R + H_R^2$

$H^2 - H_R = \boxed{H - H_R}$

by HW5 $\downarrow$

$X\underbrace{(X^TX)^{-1}X^TX}_{I_p}(X^TX)^{-1}X^T$

$XI_p(X^TX)^{-1}X^T \to X(X^TX)^{-1}X^T \to \underline{H}$

c) Under H_0, $\frac{1}{\sigma^2} SSreg \sim \chi^2_q$ with $SSreg = RSS_R - RSS_F$

$RSS_R = y^T(I - H_R)y$ $\quad RSS_F = y^T(I-H)y$

$SSreg = y^T(I-H_R)y - y^T(I-H)y$

$= (y^T I - y^T H_R)y - (y^T I - y^T H)y$

$= (y^T - y^T H_R)y - (y^T - y^T H)y$

$SSreg = y^T((I-H_R)-(I-H))y = y^T(I - H_R - I + H)y = \underline{y^T(H-H_R)y}$

Now, by theorem 2: if $y \sim N(\emptyset_p, V)$, $q = y^T A y$ then $q \sim \chi^2_r$ with $rank(A) = r$, if AV is idempotent.

So, need to now prove $Y = (y - X\beta)$ because we know $E(Y - X\beta) = \emptyset$

Given here that $\beta_2 = \emptyset$:

$E(Y|X) = X\beta$ and now $X\beta = X_1\beta_1$

Want to show that $(H - H_R)(Y - X_1\beta_1) = (H - H_R)Y$ from SSreg above.

$\downarrow$

$(H - H_R)Y - (\underline{H} X_1\beta_1) - (\underline{H_R} X_1\beta_1)$

both $= X_1$ from part a.

$(H - H_R)Y - X_1\beta_1 - X_1\beta_1$

So, $\underline{Y = (Y - X_1\beta_1)}$. Then, by a similar argument with the transpose, $\underline{Y^T = (Y - X_1\beta_1)^T}$

So, can say $SSreg = \underbrace{(Y - X_1\beta_1)^T}_{Y^T}\underbrace{(H - H_R)}_{A}\underbrace{(Y - X_1\beta_1)}_{Y} = q$ by theorem 2

also by theorem 2: $y = (y - X_1\beta_1) \sim N(\emptyset, \sigma^2 I)$

- Now, to show AV is idempotent, need $A = \frac{(H - H_R)}{\sigma^2}$ to cancel σ^2 in V.

$(AV)(AV) = AV$

$\left(\frac{1}{\sigma^2}(H - H_R)(\sigma^2 I)\right)^2 = \frac{1}{\sigma^2}(H - H_R)(\sigma^2 I)$

$(HI - H_R I)^2 = HI - H_R I$

by part b $\downarrow$

$H - H_R = H - H_R$ ✓

Now, rank(A):

$rank\left(\frac{H - H_R}{\sigma^2}\right) = rank(H - H_R)$ and from notes and lemma $H - H_R$ is symmetric and idempotent

$\downarrow$

$trace(H - H_R) = trace(X(X^TX)^{-1}X^T - X_1(X_1^TX_1)^{-1}X_1^T)$

$= tr(X^TX(X^TX)^{-1} - X_1^TX_1(X_1^TX_1)^{-1})$

$= trace(I_{p'} - I_{(p'-q)}) = p' - p' + q$

$= q$ ✓

So, by theorem 2: $q = y^T \frac{(H - H_R)}{\sigma^2} y = \frac{1}{\sigma^2} SSreg \sim \chi^2_q$ ■

d) Show SSreg and $\hat{\sigma}^2$ are independent

$SSreg = Y^T(H-H_R)Y \quad \hat{\sigma}^2 = Y^T\frac{(I-H)}{n-p'}Y$

Using theorem 3': if $y \sim N(\mu, V)$, $q_1 = y^TA_1y$ and $q_2 = y^TA_2y$ then q_1 and q_2 are independent if $A_1VA_2 = \emptyset$

$q_1 = SSreg = y^T\underbrace{(H-H_R)}_{A_1}y \quad q_2 = y^T\underbrace{\frac{(I-H)}{n-p'}}_{A_2}y$ and $y \sim N(XB, \underbrace{\sigma^2 I}_{V})$

So, need to show $A_1VA_2 = \emptyset$.

$$(H-H_R)\sigma^2 I\left(\frac{I-H}{n-p'}\right) \rightarrow \frac{\sigma^2}{n-p'}\left((H-H_R)I(I-H)\right) \rightarrow \frac{\sigma^2}{n-p'}\left((HI-H_RI)(I-H)\right)$$

$$\downarrow$$

$$\frac{\sigma^2}{n-p'}\left(HI^2 - H^2I - H_RI^2 + HH_RI\right)$$

$$\downarrow$$

$$\frac{\sigma^2}{n-p'}\left(H - H^2 - H_R + HH_R\right)$$

$$\downarrow \text{by part a and HW5}$$

$$\frac{\sigma^2}{n-p'}\left(H - H - H_R + H_R\right) = \emptyset \checkmark$$

So, SSreg and $\hat{\sigma}^2$ are independent $\blacksquare$

e) Show $\dfrac{\frac{SSreg}{q}}{\frac{RSS}{n-p'}} \sim F_{q,n-p'}$

by theorem 5: if $W_1 \sim \chi^2_i$, $W_2 \sim \chi^2_k$ and W_1 and W_2 are independent, then $\frac{W_1/i}{W_2/k} \sim F_{i,k}$

here $W_1 = SSreg$ and $W_2 = RSS$, need to show they are independent

Already know $\frac{1}{\sigma^2}SSreg \sim \chi^2_q$ from part c.

Need to show now that $\frac{1}{\sigma^2}RSS \sim \chi^2_{n-p'}$ so $\dfrac{\frac{1}{\sigma^2}\frac{SSreg}{q}}{\frac{1}{\sigma^2}\frac{RSS}{n-p'}} \sim F_{q,n-p'}$

$\frac{1}{\sigma^2}$ terms cancel.

First: $\frac{1}{\sigma^2}RSS = \frac{1}{\sigma^2}Y^T(I-H)Y$

need to show $(I-H)y = (I-H)(y - X_1B_1)$ like before

$= (I-H)y - (I-H)(X_1B_1)$

$= (I-H)y - X_1B_1 + X_1B_1$

$= (I-H)y$, and like before can show same for y^T via transpose.

So, $\frac{1}{\sigma^2}(y - X_1B_1)^T(I-H)(y-X_1B_1)$

and using LM theorem 2. $A = \frac{1}{\sigma^2}(I-H)$, $V = \sigma^2 I$ $\quad AV = I-H$ which is idempotent by HW5.

Now, $r = rank(A) = rank(I-H) = trace(I-H) = trace(I) - trace(H) = \underline{n-p'}$

So, $\frac{1}{\sigma^2}RSS \sim \chi^2_{n-p'}$ and know that $\frac{1}{\sigma^2}SSreg \sim \chi^2_q$

- Now, need to show W_1 and W_2 are independent.

Know SSreg and $\hat{\sigma}^2 = \frac{RSS}{n-p'}$ are independent by part d.

And, because $\hat{\sigma}^2$ and RSS are only different by a constant $(\frac{1}{n-p'})$, SSreg is also independent from RSS.

Therefore: W_1 and W_2 are independent, and

$$\frac{W_1/J}{W_2/K} = \frac{SSres/q}{RSS/n-p'} \sim F_{q,n-p'}$$