Erin Parker
S632 - HW Ø

1. We know $\text{bias}(\hat{y}|x_i) = (x_{1i}^T(X_1^TX_1)^{-1}X_1^TX_2 - x_{2i}^T)\beta_2$

Show $\sum_{i=1}^{n}(\text{bias}(\hat{y}|x_i))^2 = (X_2\beta_2)^T(I-H_1)(X_2\beta_2)$ and recall $\underset{\sim}{a}^T\underset{\sim}{a} = \sum a_i^2$

$$\sum\left((x_{1i}^T(X_1^TX_1)^{-1}X_1^TX_2 - x_{2i}^T)\beta_2\right)^2 \rightarrow \left((X_1(X_1^TX_1)^{-1}X_1^TX_2 - X_2)\beta_2\right)^T(X_1(X_1^TX_1)^{-1}X_1^TX_2 - X_2)\beta_2$$

$$\beta_2^T(X_2^T\underbrace{X_1((X_1^TX_1)^{-1})^TX_1^T}_{H_1} - X_2^T)(\underbrace{X_1(X_1^TX_1)^{-1}X_1^T}_{H_1}X_2 - X_2)\beta_2$$

$$(\beta_2^TX_2^TH_1 - X_2^T\beta_2^T)(H_1X_2\beta_2 - X_2\beta_2)$$

$$(\beta_2^TX_2^T)(H_1 - I)(H_1 - I)(X_2\beta_2)$$

$$(X_2\beta_2)^T(H_1^2 - H_1I - IH + I^2)(X_2\beta_2) \rightarrow (X_2\beta_2)^T(-H_1 + I)(X_2\beta_2)$$

$$(X_2\beta_2)^T(I - H_1)(X_2\beta_2) \checkmark$$

2a. Show $H - H_1$ is Positive Semidefinite i.e., for any non-zero vector $\underset{\sim}{z}$,

$$\underset{\sim}{z}^T(H - H_1)\underset{\sim}{z} \geq 0 \quad ; \text{ recall again that } \underset{\sim}{a}^T\underset{\sim}{a} = \sum a_i^2$$

- We know from last term that $H - H_1$ is both symmetric and idempotent, and that $HH_1 = H_1$.

So first, $\underset{\sim}{z}^T(H - H_1)(H - H_1)\underset{\sim}{z} \geq 0$

then $\rightarrow \underbrace{((H-H_1)\underset{\sim}{z})^T}_{\underset{\sim}{a}^T}\underbrace{(H-H_1)\underset{\sim}{z}}_{\underset{\sim}{a}} \geq 0 \xrightarrow{\text{so}} \sum_{i=1}^{n}\sum_{j=1}\left((h_{ij} - h_{1,ij})z_j\right)^2 \geq 0$ - Which is always true, because of the squared term.

2b. Find the $n \times 1$ vector $\underset{\sim}{z}$, such that using $\underset{\sim}{z}^T(H - H_1)\underset{\sim}{z}$ we can show that $h_{ii}^F \geq h_{ii}^R$ for all $i = 1, \ldots, n$ where h_{ii}^F and h_{ii}^R are the ith diagonal element of H and H_1.

From 2a:
$$\underset{\sim}{z}^T(H - H_1)\underset{\sim}{z} \geq 0$$
$$\underset{\sim}{z}^TH\underset{\sim}{z} - \underset{\sim}{z}^TH_1\underset{\sim}{z} \geq 0$$
$$\underset{\sim}{z}^TH\underset{\sim}{z} \geq \underset{\sim}{z}^TH_1\underset{\sim}{z} \rightarrow \text{then if we set } \underset{\sim}{z} = \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix} \leftarrow i\text{th element for } i = 1, \ldots, n$$

and $\underset{\sim}{z}^T = [0\,0\,0\,1\,0\,0]$

Thus, $h_{ii}^F \geq h_{ii}^R$ for any $i = 1, \ldots, n$.
When z is a vector of Ø's except for a 1 in the ith place.

We see that

$$\underset{\sim}{z}^TH\underset{\sim}{z} = [0\,0\,0\,1\,0\,0]\begin{bmatrix}h_{11} & & & & \\ & h_{22} & & & \\ & & h_{33} & & \\ & & & h_{44} & \\ & & & & h_{nn}\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}$$

$$= h_{44}$$