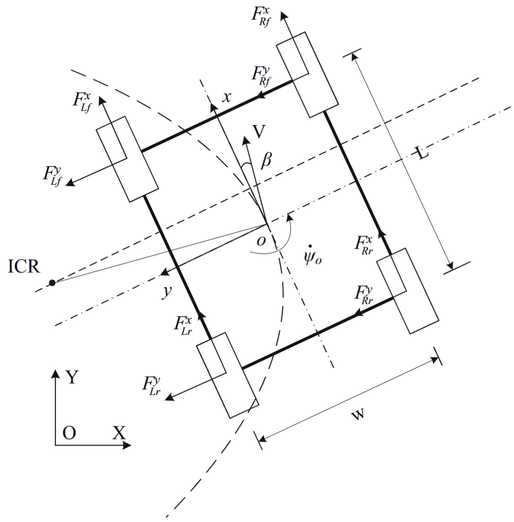
For rigid body, the local coordinate



Using a similar triangle, we have

$$v_{ox} = \frac{v_L + v_R}{2}$$
$$\dot{\psi}_o = \frac{v_L - v_R}{W}$$

for velocity in y direction, using ICR (instantaneous centers of rotation)

$$ICR_x = -\frac{v_{oy}}{\dot{\psi}_o}$$
$$ICR_y = \frac{v_{ox}}{\dot{\psi}_o}$$

Combine these we get

$$\begin{pmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{\psi}_o \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{ICR_x}{W} & -\frac{ICR_x}{W} \\ \frac{1}{W} & -\frac{1}{W} \end{pmatrix} \begin{pmatrix} v_L \\ v_R \end{pmatrix}$$

Since $ICR_x = 0$, we can simplify our model to

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{pmatrix} \begin{pmatrix} v_L \\ v_R \end{pmatrix}$$

Using inverse

$$\begin{pmatrix} v_L \\ v_R \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{pmatrix}^{-1} \begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} 1 & \frac{W}{2} \\ 1 & -\frac{W}{2} \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix}$$

For state vector

$$\xi = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}, u = \begin{pmatrix} v \\ \omega \end{pmatrix}$$
$$\dot{\xi} = f(\xi, u) \Leftrightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} v \cdot \cos\theta \\ v \cdot \sin\theta \\ \omega \end{bmatrix}$$

Taylor linearization

$$\dot{\xi} = f(\xi_r, u_r) + \frac{\partial f}{\partial \xi}_{|\xi=\xi_r, u=u_r} (\xi - \xi_r) + \frac{\partial f}{\partial u}_{|\xi=\xi_r, u=u_r} (u - u_r)$$

We define

$$A = \begin{bmatrix} 0 & 0 & -v_r \cdot \sin\theta_r \\ 0 & 0 & v_r \cdot \cos\theta_r \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} \cos\theta_r & 0 \\ \sin\theta_r & 0 \\ 0 & 1 \end{bmatrix} \quad O = -A \begin{bmatrix} x_r \\ y_r \\ \theta_r \end{bmatrix}$$
$$\dot{\xi} = A\xi + Bu + O$$

Discretization

$$\frac{\xi_{k+1} - \xi_k}{T} = A_k \xi_k + B_k u_k + O_k$$
$$\xi_{k+1} = (I + TA_k)\xi_k + TB_k u_k + TO_k = \hat{A}_k \xi_k + \hat{B}_k u_k + \hat{O}_k$$

Plug in we get

$$\hat{A}_k = \begin{bmatrix} 1 & 0 & -Tv_r \sin\theta_r \\ 0 & 1 & Tv_r \cos\theta_r \\ 0 & 0 & 1 \end{bmatrix} \quad \hat{B}_k = \begin{bmatrix} T\cos\theta_r & 0 \\ T\sin\theta_r & 0 \\ 0 & T \end{bmatrix}$$
$$\hat{O}_k = \begin{bmatrix} T\theta_r v_r \sin\theta_r \\ -T\theta_r v_r \cos\theta_r \\ 0 \end{bmatrix}$$

For future state

$$X = \begin{bmatrix} \xi_{k+1} \\ \xi_{k+2} \\ \cdots \\ \xi_{k+N} \end{bmatrix} \qquad U = \begin{bmatrix} U_k \\ U_{k+1} \\ \cdots \\ U_{k+N-1} \end{bmatrix}$$
$$X = M\xi_k + CU + NO$$
$$M = \begin{bmatrix} \hat{A}_k & \hat{A}_k \hat{A}_{k+1} & \ldots & \prod_{i=0}^{N-1} \hat{A}_{k+i} \end{bmatrix}^T_{3N\times 3}$$
$$C = \begin{bmatrix} \hat{B}_k & 0 & \cdots & 0 \\ \hat{A}_{k+1}\hat{B}_k & \hat{B}_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{i=1}^{N-1} \hat{A}_{k+i}\hat{B}_k & \prod_{i=2}^{N-1} \hat{A}_{k+i}\hat{B}_{k+1} & \ldots & \hat{B}_{k+N-1} \end{bmatrix}_{3N\times 2N}$$
$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \hat{A}_{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{i=1}^{N-1} \hat{A}_{k+i} & \prod_{i=2}^{N-1} \hat{A}_{k+i} & \cdots & 1 \end{bmatrix}_{3N\times 3N}$$
$$O = \begin{bmatrix} \hat{O}_k \\ \hat{O}_{k+1} \\ \vdots \\ \hat{O}_{k+N-1} \end{bmatrix}_{3N\times 1}$$

Error

$$\tilde{X} = X - X_r = MX_k + CU + DO - X_r$$

So the objective function is

$$
\begin{aligned}
\tilde{X}^T Q \tilde{X} &= (MX_k + DO - X_r + CU)^T Q (MX_k + DO - X_r + CU) \\
&= (E + CU)^T Q (E + CU) \\
&= E^T QE + E^T QCU + U^T C^T QCU + U^T C^T QE \\
&= U^T \left(C^T QC\right) U + \left(2E^T QC\right) U + E^T QE
\end{aligned}
$$

Optimization

$$
\begin{aligned}
\min_{U}\ J = \frac{1}{2} U^T \left(2C^T QC + 2W\right) U + \left(2E^T QC\right) U \\
\text{s.t.}\ \ U_{\min} \leq U \leq U_{\max}
\end{aligned}
$$