

八省联考导数压轴解答

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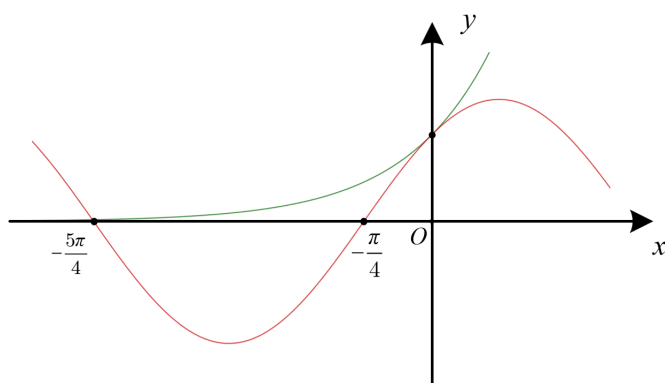
已知函数 $f(x) = e^x - \sin x - \cos x$, $g(x) = e^x + \sin x + \cos x$

(1) 证明: 当 $x > -\frac{5\pi}{4}$ 时, $f(x) \geq 0$

(2) 若 $g(x) \geq 2 + ax$, 求 a

(1) 证法1:

证明 $e^x \geq \sin x + \cos x = \sqrt{2} \sin(x + \frac{\pi}{4})$



① $x \in (-\frac{5\pi}{4}, -\frac{\pi}{4}]$, $e^x > 0$, $\sqrt{2} \sin(x + \frac{\pi}{4}) \leq 0 \Rightarrow f(x) > 0$

② $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$, $f'(x) = e^x - \cos x + \sin x$, $f''(x) = e^x + \sqrt{2} \sin(x + \frac{\pi}{4}) > 0$

$f'(0) = 0 \Rightarrow f(x) \geq f(0) = 0$

③ $x \in [\frac{\pi}{4}, +\infty)$, $e^x \geq e^{\frac{\pi}{4}}$, $\sin x + \cos x \leq \sqrt{2}$, $(e^{\frac{\pi}{2}})^{\frac{1}{2}} > 2^{\frac{1}{2}} \Rightarrow f(x) > 0$

证法2:

证法2: $e^x \geq \sin x + \cos x \Leftrightarrow F(x) = \frac{\sin x + \cos x}{e^x} \leq 1$, $F'(x) = -\frac{2 \sin x}{e^x}$

① $x \in (-\frac{5\pi}{4}, \pi]$, $F(-\frac{5\pi}{4}) = 0$, $F(0) = 1$, $F(x) \leq 0$

② $x \in (\pi, +\infty)$, $f(x) = e^x - \sqrt{2} \sin(x + \frac{\pi}{4}) > e^\pi - \sqrt{2} > 0$

(2)证法1:(同除 e^x 变形):

$$e^x + \sin x + \cos x \geq ax + 2 \Leftrightarrow h(x) = \frac{ax+2-\sin x-\cos x}{e^x} \leq 1$$

$$h\left(-\frac{\pi}{2}\right) \leq 1 \Rightarrow \frac{\pi}{2}a \geq 3 - e^{-\frac{\pi}{2}} > 2 \Rightarrow a > \frac{4}{\pi} > 1$$

$$h'(x) = \frac{2\sin x - ax + a - 2}{e^x}, \text{记 } \varphi(x) = 2\sin x + a(1-x) - 2, \varphi'(x) = 2\cos x - a$$

$$\textcircled{1} a > 2, \varphi'(x) < 0 \Rightarrow \varphi(x) \downarrow, \text{又 } \varphi(0) = a - 2 > 0, \varphi(\pi) = (1 - \pi)a - 2 < 0$$

$$\exists x_1 \in (0, \pi), \text{使 } h(x) \text{在 } (0, x_1) \uparrow, \text{在 } (x_1, \pi) \downarrow, \text{在 } x \in (0, x_1) \text{上}$$

$$\text{有 } h(x) > h(0) = 1$$

$$\textcircled{2} a = 2, \varphi'(x) = 2\cos x - 2 \leq 0, \varphi(x) \downarrow, \varphi(0) = 0$$

$$h(x) \text{在 } (-\infty, 0) \uparrow, \text{在 } (0, +\infty) \downarrow$$

$$h(x) \geq h(0) = 1$$

$$\textcircled{3} a \in (1, 2), \varphi'(x) \text{在 } \left(-\frac{\pi}{2}, 0\right) \uparrow, \varphi'\left(-\frac{\pi}{2}\right) = -a < 0, \varphi'(0) = 2 - a > 0$$

$$\exists x_2 \in \left(-\frac{\pi}{2}, 0\right), \text{使 } \varphi(x) \text{在 } \left(-\frac{\pi}{2}, x_2\right) \downarrow, \text{在 } (x_2, 0) \uparrow, \text{在 } x \in (x_2, 0) \text{上}$$

$$\text{有 } \varphi(x) < \varphi(0) = a - 2 < 0 \Rightarrow h(x) \text{在 } (x_2, 0) > h(0) = 1$$

证法2(放缩):

$$\text{由(1), 当 } x > -\frac{5\pi}{4} \text{ 时, 有 } e^x \geq \sin x + \cos x$$

$$e^x \geq 2 + ax - e^x \Rightarrow e^x \geq \frac{a}{2}a + 1 \Rightarrow a = 2$$

下面证明充分性

$$a = 2, \varphi'(x) = 2\cos x - 2 \leq 0, \varphi(x) \downarrow, \varphi(0) = 0$$

$$h(x) \text{在 } (-\infty, 0) \uparrow, \text{在 } (0, +\infty) \downarrow, h(x) \geq h(0) = 1$$

证法3(必要性探路):

$$g(x) \geq 0 \Leftrightarrow F(x) = e^x + \sin x + \cos x - ax - 2 \geq 0$$

$$F(0) = 0 \Rightarrow F'(0) = e^0 + \cos 0 - \sin 0 - a = 0, a = 2$$

下面证明充分性

$$a = 2, \varphi'(x) = 2\cos x - 2 \leq 0, \varphi(x) \downarrow, \varphi(0) = 0$$

$$h(x) \text{在 } (-\infty, 0) \uparrow, \text{在 } (0, +\infty) \downarrow$$

$$h(x) \geq h(0) = 1$$

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made by Erikpsw

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