八省联考导数压轴解答

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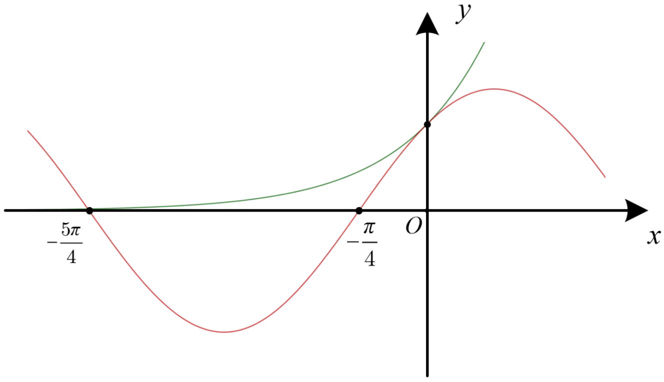
已知函数 $f(x)=e^x-\sin x-\cos x, g(x)=e^x+\sin x+\cos x$

(1) 证明: 当 $x>-\frac{5\pi}{4}$时,$f(x)\geq 0$

(2)若 $g(x)\geq 2+ax$, 求 a

(1)证法1:

证明$e^x\geq \sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$



①$x\in\left(-\frac{5\pi}{4},-\frac{\pi}{4}\right], e^x>0, \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\leq 0\Rightarrow f(x)>0$

②$x\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right), f'(x)=e^x-\cos x+\sin x, f''(x)=e^x+\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)>0$

$f'(0)=0\Rightarrow f(x)\geq f(0)=0$

③$x\in\left[\frac{\pi}{4},+\infty\right), e^x\geq e^{\frac{\pi}{4}}, \sin x+\cos x\leq\sqrt{2}, \left(e^{\frac{\pi}{2}}\right)^{\frac{1}{2}}>2^{\frac{1}{2}}\Rightarrow f(x)>0$

证法2:

证法2: $e^x\geq\sin x+\cos x\Leftrightarrow F(x)=\frac{\sin x+\cos x}{e^x}\leq 1, F'(x)=-\frac{2\sin x}{e^x}$

①$x\in\left(-\frac{5\pi}{4},\pi\right], F(-\frac{5\pi}{4})=0, F(0)=1, F(x)\leq 0$

②$x\in(\pi,+\infty), f(x)=e^x-\sqrt{2}\sin(x+\frac{\pi}{4})>e^{\pi}-\sqrt{2}>0$

(2)证法1:(同除e^x变形):

$e^x+\sin x+\cos x \geq ax+2 \Leftrightarrow h(x)=\frac{ax+2-\sin x-\cos x}{e^x} \leq 1$

$h\left(-\frac{\pi}{2}\right) \leq 1 \Rightarrow \frac{\pi}{2}a \geq 3-e^{-\frac{\pi}{2}} > 2 \Rightarrow a > \frac{4}{\pi} > 1$

$h'(x)=\frac{2\sin x-ax+a-2}{e^x}$,记$\varphi(x)=2\sin x+a(1-x)-2, \varphi'(x)=2\cos x-a$

①$a>2, \varphi'(x)<0 \Rightarrow \varphi(x)\downarrow$，又$\varphi(0)=a-2>0, \varphi(\pi)=(1-\pi)a-2<0$

$\exists x_1 \in (0,\pi)$，使$h(x)$在$(0,x_1)\uparrow$，在$(x_1,\pi)\downarrow$，在$x\in(0,x_1)$上

有$h(x)>h(0)=1$

②$a=2, \varphi'(x)=2\cos x-2 \leq 0, \varphi(x)\downarrow, \varphi(0)=0$

$h(x)$在$(-\infty,0)\uparrow$,在$(0,+\infty)\downarrow$

$h(x) \geq h(0)=1$

③$a\in(1,2), \varphi'(x)$在$\left(-\frac{\pi}{2},0\right)\uparrow, \varphi'\left(-\frac{\pi}{2}\right)=-a<0, \varphi'(0)=2-a>0$

$\exists x_2 \in \left(-\frac{\pi}{2},0\right)$,使$\varphi(x)$在$\left(-\frac{\pi}{2},x_2\right)\downarrow$,在$(x_2,0)\uparrow$,在$x\in(x_2,0)$上

有$\varphi(x)<\varphi(0)=a-2<0 \Rightarrow h(x)$在$(x_2,0)>h(0)=1$

证法2(放缩)：

由(1),当$x>-\frac{5\pi}{4}$时,有$e^x \geq \sin x+\cos x$

$e^x \geq 2+ax-e^x \Rightarrow e^x \geq \frac{a}{2}a+1 \Rightarrow a=2$

下面证明必要性

$a=2, \varphi'(x)=2\cos x-2 \leq 0, \varphi(x)\downarrow, \varphi(0)=0$

$h(x)$在$(-\infty,0)\uparrow$,在$(0,+\infty)\downarrow$ $h(x)\geq h(0)=1$

证法3(必要性探路)：

$g(x)\geq 0 \Leftrightarrow F(x)=e^x+\sin x+\cos x-ax-2 \geq 0$

$F(0)=0 \Rightarrow F'(0)=e^0+\cos 0-\sin 0-a=0, a=2$

下面证明必要性

$a=2, \varphi'(x)=2\cos x-2 \leq 0, \varphi(x)\downarrow, \varphi(0)=0$

$h(x)$在$(-\infty,0)\uparrow$,在$(0,+\infty)\downarrow$

$h(x) \geq h(0)=1$

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