八省联考导数压轴解答

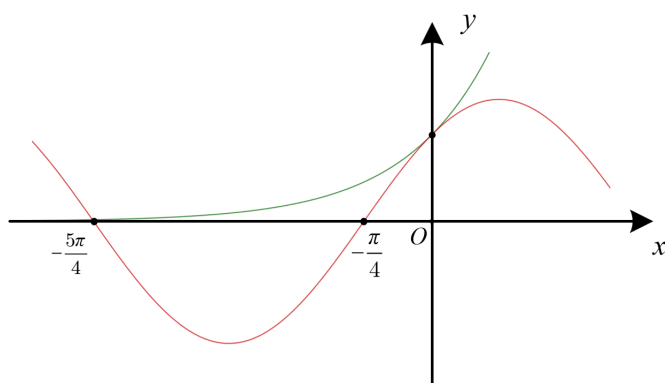## 潘世维

Friday, October 1, 2021

已知函数 $f(x) = e^x - \sin x - \cos x, g(x) = e^x + \sin x + \cos x$

(1) 证明: 当 $x > -\frac{5\pi}{4}$时$,f(x) \geq 0$

(2)若 $g(x) \geq 2 + ax,$ 求 a

(1)证法1:

证明$e^x \geq \sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$

①$x \in \left(-\frac{5\pi}{4}, -\frac{\pi}{4}\right], e^x > 0, \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \leq 0 \Rightarrow f(x) > 0$

②$x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), f'(x) = e^x - \cos x + \sin x, f''(x) = e^x + \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) > 0$

$f'(0) = 0 \Rightarrow f(x) \geq f(0) = 0$

③$x \in \left[\frac{\pi}{4}, +\infty\right), e^x \geq e^{\frac{\pi}{4}}, \sin x + \cos x \leq \sqrt{2}, \left(e^{\frac{\pi}{2}}\right)^{\frac{1}{2}} > 2^{\frac{1}{2}} \Rightarrow f(x) > 0$

证法2:

证法2：$e^x \geq \sin x + \cos x \Leftrightarrow F(x) = \frac{\sin x + \cos x}{e^x} \leq 1, F'(x) = -\frac{2\sin x}{e^x}$

①$x \in \left(-\frac{5\pi}{4}, \pi\right], F\left(-\frac{5\pi}{4}\right) = 0, F(0) = 1, F(x) \leq 0$

②$x \in (\pi, +\infty), f(x) = e^x - \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) > e^\pi - \sqrt{2} > 0$

(2)证法1:(同除e^x变形):

$e^x + \sin x + \cos x \geq ax + 2 \Leftrightarrow h(x) = \frac{ax+2-\sin x-\cos x}{e^x} \leq 1$

$h\left(-\frac{\pi}{2}\right) \leq 1 \Rightarrow \frac{\pi}{2}a \geq 3 - e^{-\frac{\pi}{2}} > 2 \Rightarrow a > \frac{4}{\pi} > 1$

$h'(x) = \frac{2\sin x-ax+a-2}{e^x}$,记$\varphi(x) = 2\sin x + a(1-x) - 2, \varphi'(x) = 2\cos x - a$

①$a > 2, \varphi'(x) < 0 \Rightarrow \varphi(x) \downarrow$，又$\varphi(0) = a - 2 > 0, \varphi(\pi) = (1-\pi)a - 2 < 0$

$\exists x_1 \in (0,\pi)$，使$h(x)$在$(0,x_1) \uparrow$，在$(x_1,\pi) \downarrow$，在$x \in (0,x_1)$上

有$h(x) > h(0) = 1$

②$a = 2, \varphi'(x) = 2\cos x - 2 \leq 0, \varphi(x) \downarrow, \varphi(0) = 0$

$h(x)$在$(-\infty,0) \uparrow$,在$(0,+\infty) \downarrow$

$h(x) \geq h(0) = 1$

③$a \in (1,2), \varphi'(x)$在$\left(-\frac{\pi}{2},0\right) \uparrow, \varphi'\left(-\frac{\pi}{2}\right) = -a < 0, \varphi'(0) = 2 - a > 0$

$\exists x_2 \in \left(-\frac{\pi}{2},0\right)$,使$\varphi(x)$在$\left(-\frac{\pi}{2},x_2\right) \downarrow$,在$(x_2,0) \uparrow$,在$x \in (x_2,0)$上

有$\varphi(x) < \varphi(0) = a - 2 < 0 \Rightarrow h(x)$在$(x_2,0) > h(0) = 1$

证法2(放缩)：

由(1),当$x > -\frac{5\pi}{4}$时,有$e^x \geq \sin x + \cos x$

$e^x \geq 2 + ax - e^x \Rightarrow e^x \geq \frac{a}{2}a + 1 \Rightarrow a = 2$

下面证明充分性

$a = 2, \varphi'(x) = 2\cos x - 2 \leq 0, \varphi(x) \downarrow, \varphi(0) = 0$

$h(x)$在$(-\infty,0) \uparrow$,在$(0,+\infty) \downarrow$ $h(x) \geq h(0) = 1$

证法3(必要性探路)：

$g(x) \geq 0 \Leftrightarrow F(x) = e^x + \sin x + \cos x - ax - 2 \geq 0$

$F(0) = 0 \Rightarrow F'(0) = e^0 + \cos 0 - \sin 0 - a = 0, a = 2$

下面证明充分性

$a = 2, \varphi'(x) = 2\cos x - 2 \leq 0, \varphi(x) \downarrow, \varphi(0) = 0$

$h(x)$在$(-\infty,0) \uparrow$,在$(0,+\infty) \downarrow$

$h(x) \geq h(0) = 1$