DFT Leakage Phenomena

Erik Sandström

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Introduction

The objective of this project is to study the phenomena usually referred to spectral leakage or DFT leakage. The phenomena is important to understand in order to analyze discrete time signals correctly. MatLab is used for implementation.

Sampling Continuous Signals

Three signals of frequencies 2.3 kHz, 23 kHz and 36 kHz were first plotted in the time domain according to figure 1, 2 and 3. In addition, using a sampling frequency of 48 kHz, 100 sampled data points were plotted on top of the continuous signal to specify what points in time the samples were taken from.

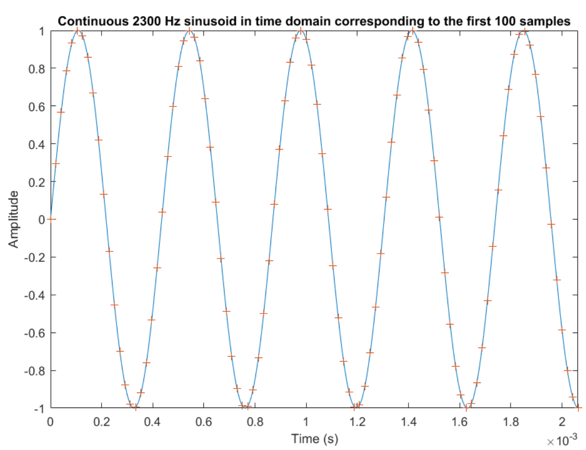

Figure 1: 2.3 kHz continuous sinusoid in time domain with samples marked with red crosses sampled at 48 kHz.

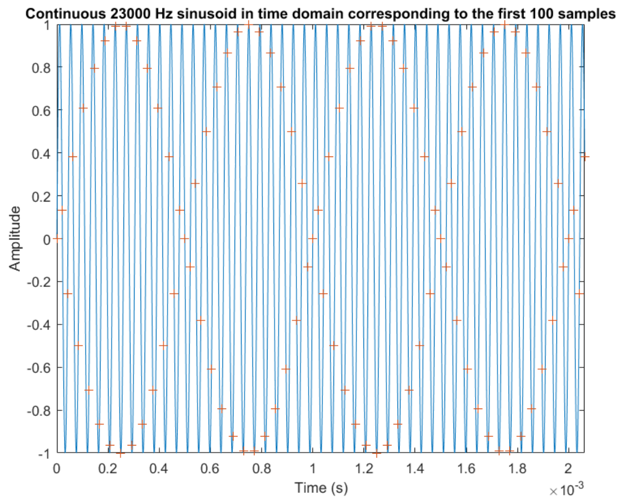

Figure 2: 23 kHz continuous sinusoid in time domain with samples marked with red crosses sampled at 48 kHz.

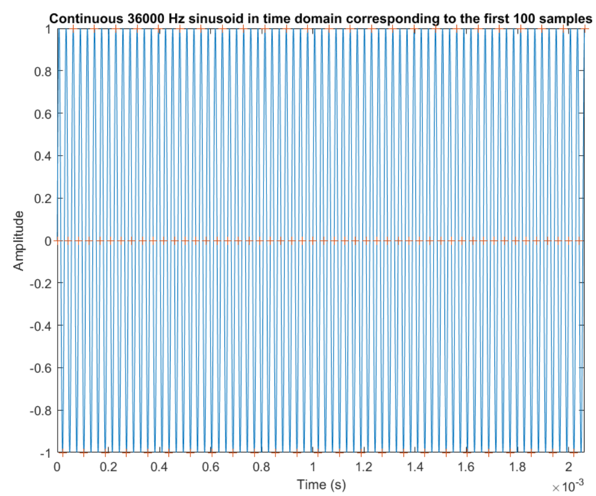

Figure 3: 36 kHz continuous sinusoid in time domain with samples marked with red crosses sampled at 48 kHz.

The resulting plots of the sampled sequences are displayed in figure 4 and a plot connecting the samples can be seen in figure 5.

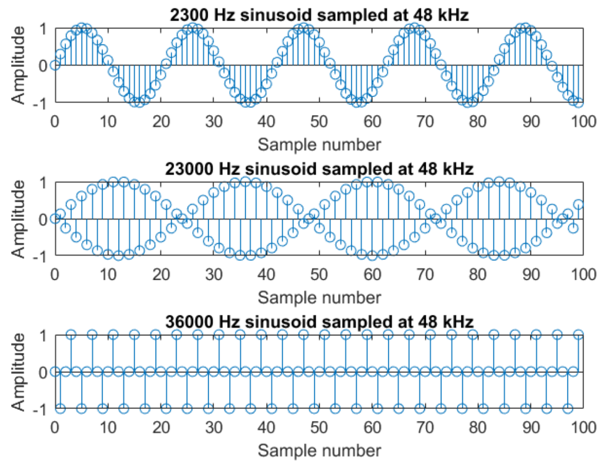

Figure 4: Resulting sampled signals in units of sample number.

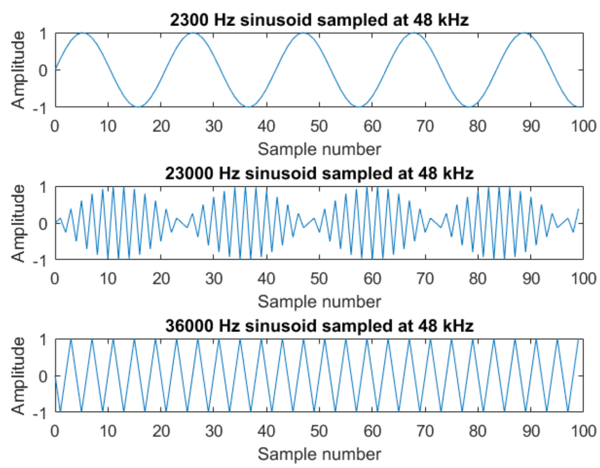

Figure 5: Resulting sampled signals in units of sample number.

DFT Representation of the Sampled Signals

A 64 point DFT was used in order to transform the sampled signals of length 100 to the frequency domain. The resulting DFT can be seen in figure 6. A 1000 point DFT visualizing the DTFT is overlapped in red.

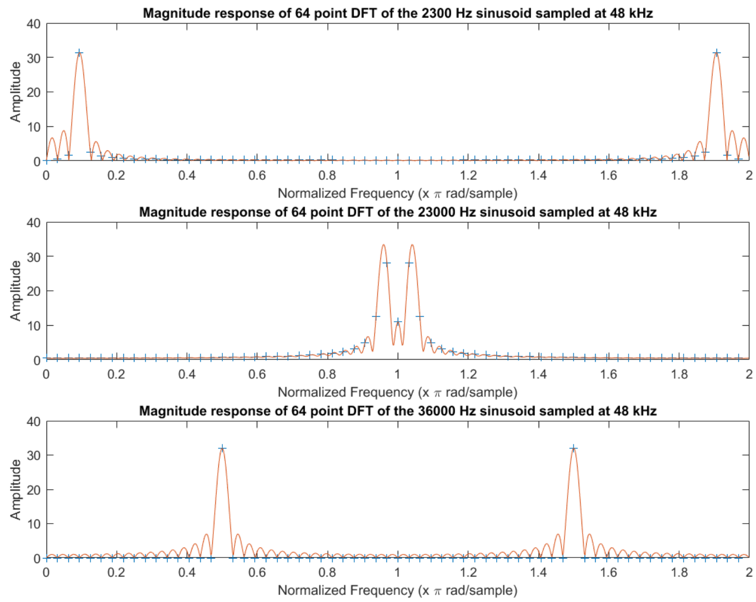

Figure 6: 64 point DFT of the sampled signals in figure 5. An approximate version of the DTFT is also displayed in red.

DFT Leakage Theory

The discrete Fourier transform of $x[n]$ is given by

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-i2\pi kn/N} \tag{1}$$

As can be seen by equation 3, an infinite sequence is truncated when taking the DFT. This means that only the first N points of the sequence will have an effect on the output of the DFT. When cutting out a segment of N samples from a sinusoid, information about the frequency content is lost. To see this, assume a sampled sinusoid $x[n] = sin(\omega_0 n)$. This can also be written, using Eulers formula as

$$sin(w_0 n) = \frac{e^{j\omega_0} - e^{-j\omega_0}}{2j} \tag{2}$$

The input to the DFT will be $x_{sample}[n]$ which is a windowed version of $x[n]$ given by

$$x_{sample}[n] = x[n] \times w[n], \tag{3}$$

where $\omega[n]$ is a rectangular window that is 1 over the duration of the duration of the sequence $x_{sample}[n]$ and 0 otherwise. Note that the symbol $\times$ is multiplication. Lets analyze equation 3 in the frequency domain by taking the DTFT on both sides. The result is

$$X_{sample}(e^{j\omega}) = X(e^{j\omega}) * W(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \qquad (4)$$

It is rather difficult to calculate this, but if we write the right hand side of equation 3 using equation 2, we get

$$x_{sample}[n] = \frac{e^{j\omega_0} - e^{-j\omega_0}}{2j} \times w[n] = \frac{1}{2j}(e^{j\omega_0}\omega[n] - e^{-j\omega_0}\omega[n]). \qquad (5)$$

This result can easily be transformed to the Fourier domain using the DTFT. The result is

$$X_{sample}(e^{j\omega}) = \frac{1}{2j}(W(e^{j(\omega-\omega_0)}) - W(e^{j(\omega+\omega_0)})), \qquad (6)$$

where

$$W(e^{j\omega}) = e^{-j\frac{\omega}{2}(N-1)}\frac{sin(N\frac{\omega}{2})}{sin(\frac{\omega}{2})}, \qquad (7)$$

where N is the duration of the sequence $x_{sample}[n]$. In our case the input sequence is 100 samples long, but we are taking a 64 point DFT, which means only the first 64 of our 100 points in $x_{sample}[n]$ will be accounted for. This is not perticularly useful to do, but for the sake of this paper, it has no effect on the underlying theory. We therefore just assume that the input sequence is of length 64 instead of 100. The function given by equation 7 is a sinc-function illustrated by figure 7, for the real-valued case.

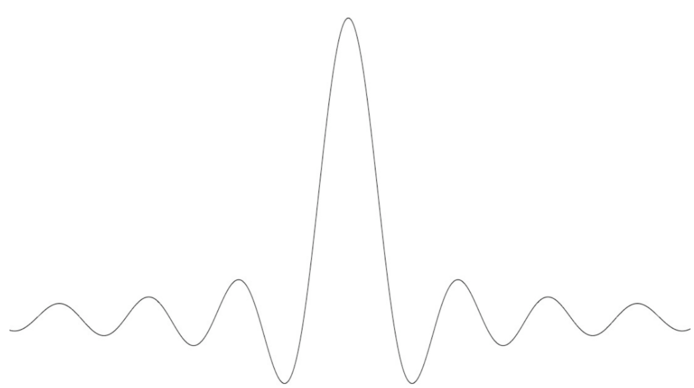

Figure 7: Real valued sinc-function.

Consider the case when the normalized frequency is $\omega_0 = 2\pi 2300/48000$ in equation 6, which is the case for the frequency 2.3 kHz. This means that the

DTFT of the sequence $x_{sample}[n]$ is the sum of two shifted sinc-functions given by equation 6 when N = 64. Taking 64 samples of the DTFT, given in red by the top graph in figure 6, yields the blue dots. They are evenly spaced with a distance $2\pi/64$ apart. It is apparent that leakage is present since more than two instances of the blue dots are non zero. Taking a look at the middle red graph in figure 6 yields looking at a DTFT with $\omega_0 = 2\pi 23000/48000$. The samples are equally spaced with a distance $2\pi/64$ and are illustrated by the blue dotted samples. Yet again leakage is confirmed. The bottom graph is, however, different. Here the DFT samples exactly hit the points in the DTFT where the value is identically zero. Consider a single non-shifted sinc-function as given by equation 7. The function is zero at multiple values of $\omega = 2\pi/64$. When shifting the sinc-function by ω_0, zeros will appear spaced at multiples of $\omega = 2\pi/64$ but centered around ω_0 instead. If $\omega_0 = \frac{2\pi}{64}k$ where k is an integer, the sinc-function will have zeros that align with the non-shifted sinc-function (except at ω_0 where the sinc-function has a global maxima). This means that the sum of the sinc-functions given by equation 6 will match up such that the result has zeros at multiples of $\omega = \frac{2\pi}{64}$. When the DTFT is sampled at every multiple of $\omega = \frac{2\pi}{64}$, due to the usage of a 64 point DFT, only zeros will show up in the DFT except where the DTFT peaks.

Aliasing effects

The sampling theorem says that aliasing will occur if the sampling frequency is less than twice the bandwidth of the signal. For the cases where the frequency is 2.3 kHz and 23 kHz, no aliasing will occur. This means that these frequencies displayed in figure 6 are correctly visualized. For the sinusoid of frequency 36 kHz, aliasing will occur. This is apparent from the fact that the normalized frequency displayed for the 36 kHz signal in figure 6 translates to a real frequency of

$$f = \frac{0.5\pi}{2\pi} F_s = \frac{1}{4} \times 48000 = 12kHz. \tag{8}$$