Inverted Pendulum Control using a BeagleBone Blue

Erik Sandström

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Introduction

The objective of this project is design and implement a complete control system for an inverted pendulum. Control will be applied to the angle of the robotic body and to the position relative to the ground. Two nested control loops will be used in order to achieve this. MatLab is used as a tool to design the concept and C is used to implement the design on the eduMIP platform provided by the UCSD Coordinated Robotics Lab.

Inner loop design

The first task is to design an inner loop controller to stabilize the body angle denoted θ around a given setpoint θ_{ref}. This process begins with finding the system transfer function G1 from the motor input duty cycle u to the body angle θ. The transfer function is given by,

$$G1 = \frac{\theta}{u} = \frac{-882.7s}{s^3 + 44.15s^2 - 192.8s - 2299} \tag{1}$$

The Bode plot of G1 is given by figure 1.

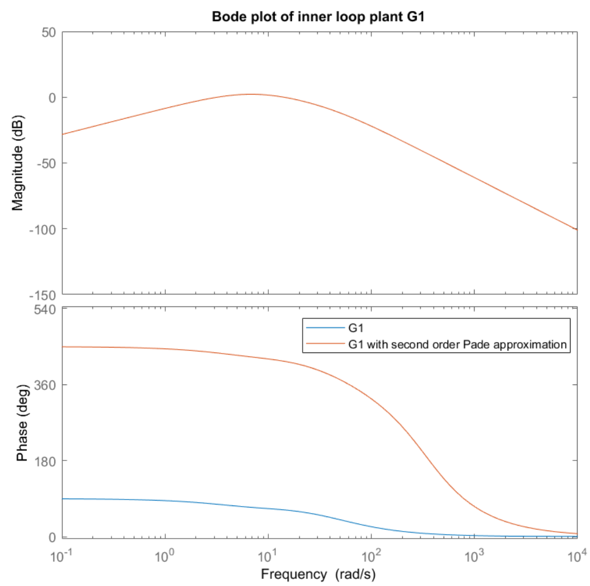

Figure 1: Bode plot of the inner loop plant with and without second order Pade approximation for delay coming from zero order hold of the discrete inner loop controller.

An analysis of the pole and zero placement of G1 reveals that it has poles located at $-47.2061,\ 8.6696,\ -5.6165$ and one zero at the origin. To get this system stable, a lag compensator that cancels the zero at the origin is necessary to bring the right hand pole to the left half plane. Cancelling a zero at the origin can be tricky business, but since it is known at this stage, that an outer loop will be implemented, we can assure that stability is kept, even though we might miss by a small amount when cancelling. The lag controller $D1_{lag}$ is of the form

$$D1_{lag} = \frac{s + 5.6165}{s}, \tag{2}$$

where a cancellation of the left half plane pole of the plant G1 is a safe cancellation since it contributes in the event of a missed cancellation to an exponentially decaying term in the expression of the output in the time domain. When the controller $D1_{lag}$ is tuned with some proportional gain, stability can be achieved, but the system is not robust enough as can be seen by the low phase margin and oscillatory behavior of figure 2.

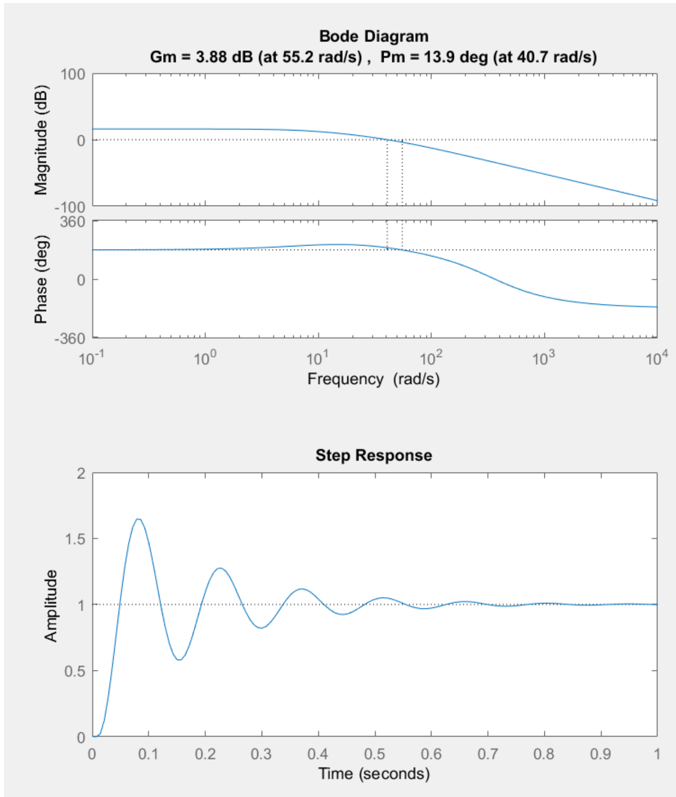

Figure 2: Bode plot of the open loop system $D1_{lag}G1$ tuned with some proportional gain to achieve stability.

To get a more stable system, a lead compensator $D1_{lead}$ is added. The lead compensator will be centered to have its maximum phase gain at the desired cross-over frequency ω_{c1} of the open loop system $L1 = D1G1$ to increase robustness. A good starting point for the the cross-over frequency ω_{c1} is to set it to a tenth of the Nyquist frequency where the Nyquist frequency is half of the sampling frequency of the IMU, which is 100 Hz. This yields a cross-over frequency

$$\omega_{c1} = 10\pi \ \ rad/sec. \tag{3}$$

One can also evaluate the chosen cross-over frequency according to the tuning guideline ($\omega_n = 1.8/t_r$) that specifies the cross-over frequency when the rise time to a step input is given. A rise time of around 0.05 seconds yields the specified cross-over frequency. This should reflect the natural time period of the pendulum when it is not inverted, but a simply experiment shows that the real world period time is closer to 0.5 seconds. This means that the inputs u will likely vary around this critical frequency during operation and it is thus important that stability is preserved for these frequencies. Using a cross-over frequency higher than the natural frequency means that we are on the safe side of operation since the closed loop system will not reject the important inputs frequencies.

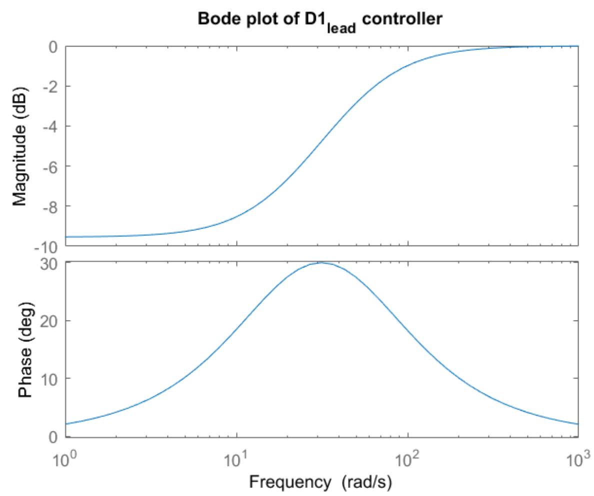

Figure 3: Bode plot of the controller $D1_{lead}$ centered at ω_{c1}.

$$D1_{lead} = \frac{s + 18.14}{s + 54.41} \tag{4}$$

with a Bode plot given by figure 3. Using $D1_{lead}$ together with $D1_{lag}$ in a cascade formation together with an appropriate gain constant yields the complete inner loop controller determined by

$$D1 = K_{D1} D1_{lead} D1_{lag} = \frac{-2.94s^2 - 69.84s - 299.5}{s^2 + 54.41} \tag{5}$$

The root locus of the system displaying the placement of the closed loop system poles is plotted together with the step response of the closed loop system, open loop $L1$ Bode plot and inner loop controllers in figure 4

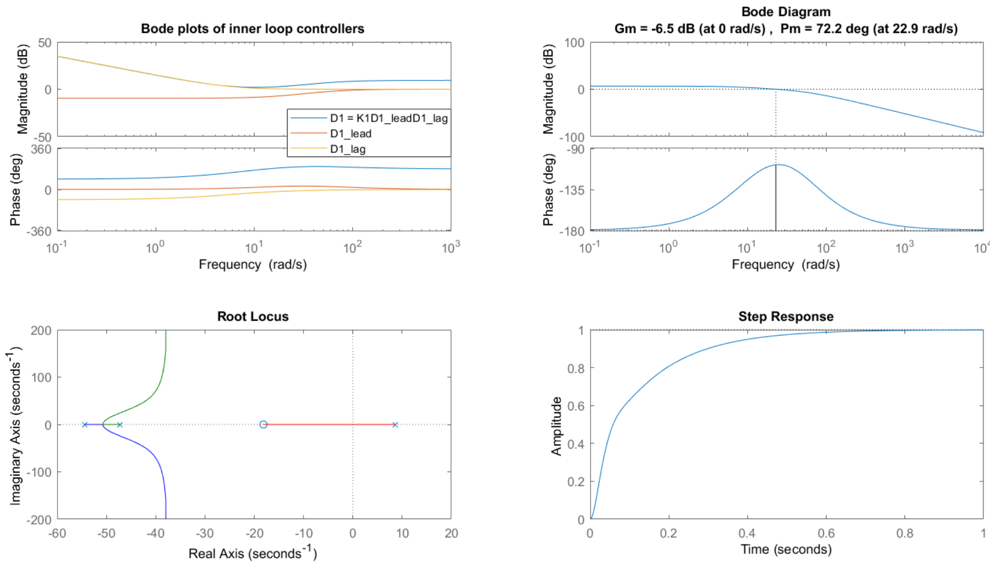

Figure 4: The root locus of the system displaying the placement of the closed loop system poles is plotted together with the step response of the closed loop system, open loop $L1$ Bode plot and inner loop controllers.

The discrete equivalent controller to $D1(s)$ is found by transforming the controller with Tustin's approximation with prewarping around the crossover frequency. The result is

$$D1(z) = \frac{u}{\theta_e} = \frac{-2.589z^2 + 4.602z - 2.037}{z^2 - 1.569z + 0.5695}, \tag{6}$$

where $\theta_e = \theta_{ref} - \theta$, the body angle error and u is the duty cycle which is the input to the PWM signal going to the motors.

As a final check of the inner loop controller, a second order Pade approximation of a delay can be applied to the plant G1. This is tested due to the fact that there will inherently be a delay of the form e^{-ds} from the DAC (Digital to Analog Converter) when the discrete controller delivers it output to the continuous plant G1. The sampling period time of the IMU is h = 0.01 s and the delay from the ZOH is $d = h/2$. The plant using a second order approxiamation is given by figure 1 and a corresponding analysis as given in figure 4 can be found in figure 5. As can be seen, the phase margin has droppen by around 7 degrees and is therefore still far from instability.

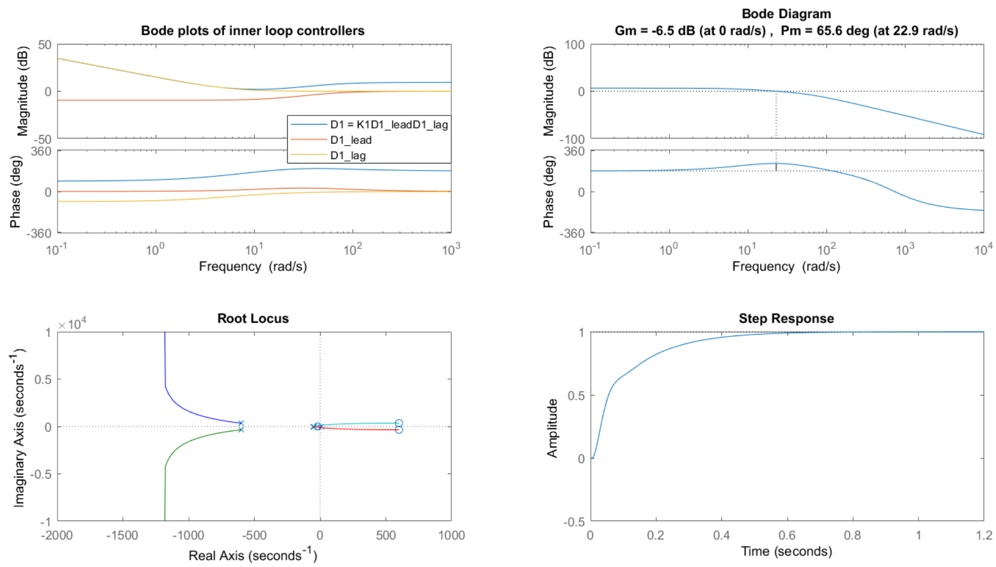

Figure 5: Bode plot of the controller $D1_{lead}$ centered at ω_{c1}.

Outer loop design

First assuming that the outer loop has much slower dynamics compared to the inner loop, the outer loop can be designed on the assumption that the closed inner loop will be very close to unity for all frequencies of interest for the outer loop. Therefore, a crossover frequency ten times smaller than the crossover frequency of the inner loop is used and the design of the outer loop controller is first made with the inner closed loop assumed to be one. From Newtons equations of motion the relationship between the body angle θ and the position ϕ can be calculated in the Laplace domain given by a transfer function

$$G2 = \frac{\phi}{\theta} = \frac{-1.476s^2 + 2.622 \times 10^{-15} + 128.9}{s^2}. \tag{7}$$

The Bode plot of the outer loop system is given by figure 6.

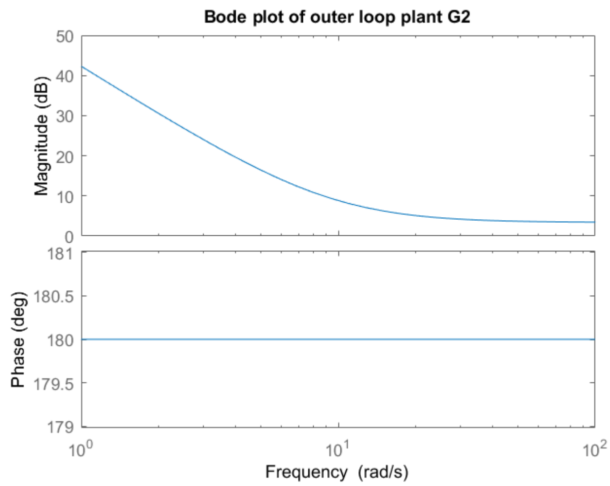

Figure 6: Bode plot of the outer loop plant G2.

Looking at the phase response which is -180 degrees over all frequencies, it is obvious that a lead compensator is needed. Designing a lead compensator $D2_{lead}$ centered at $\omega_{c2} = \omega_{c1}/10$ yields

$$D2_{lead} = \frac{s + 0.5736}{s + 17.21} \tag{8}$$

A lag compensator $D2_{lag}$ was also added to bump up the gain at lower frequencies. The robot works absolutely fine without this compensation, but the robot behaves more smoothly with it than without.

$$D2_{lag} = \frac{s + 0.06283}{s + 0.01571} \tag{9}$$

With an appropriate gain factor the complete outer loop controller then becomes

$$D2 = K2 D2_{lead} D2_{lag} = \frac{0.1738s^2 + 0.1106s + 0.006263}{s^2 + 17.22s + 0.2703} \tag{10}$$

The corresponding plot to figure 4 but for the outer loop is given by 7

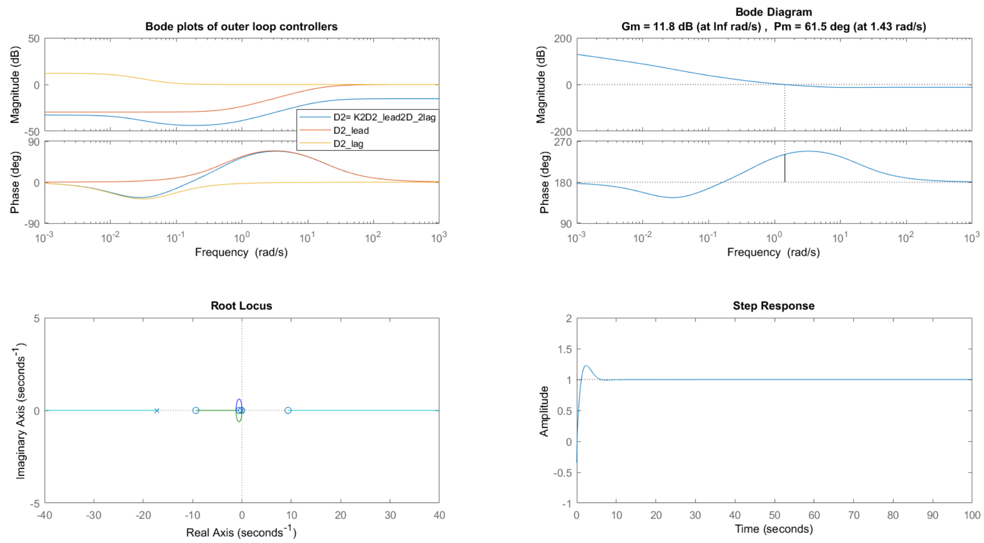

Figure 7: Bode plot of outer loop controllers, Bode plot of open loop $L2 = D2G2$ system, root locus and step response of the closed loop system of $L2$.

Adding a Pade approximated delay of e^{-ds} of fourth order with $d = h/2$ with $h = 0.05\ \ s$ yields the following result given by figure 8

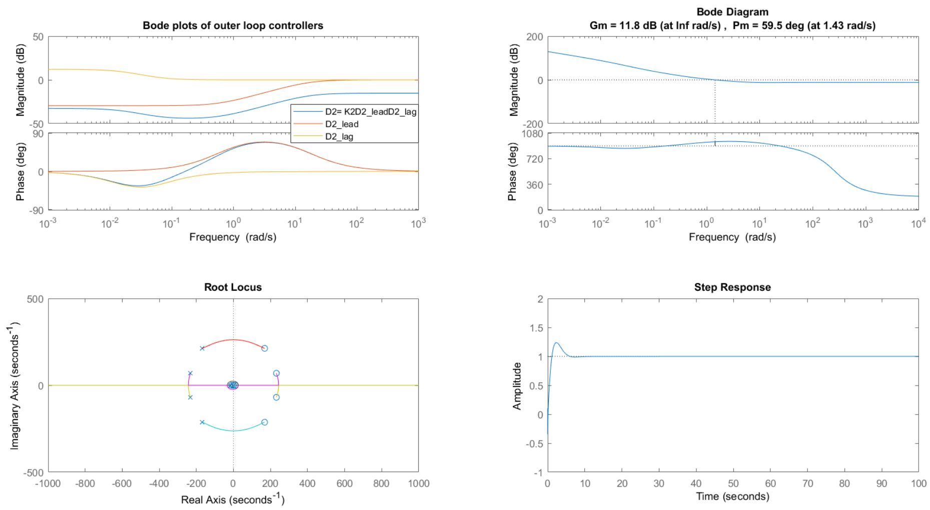

Figure 8: Bode plot of outer loop controllers, Bode plot of open loop $L2 = D2G2$ system, root locus and step response of the closed loop system of $L2$. The only difference to figure 7 is the addition of a the delay approximation to G2.

The delay occurs due to the zero order hold of the DAC in D2. A phase loss of only 2 degrees can be seen in figure 8 and thus the system is stable.

The final step is to include the closed loop transfer function $T1 = \frac{L1}{1+L1}$, where $L1 = D1G1$ to the outer loop system. The results, using a second order Pade approximation in G1 and a fourth order Pade approximation in G2 yields the final result as seen in figure 9

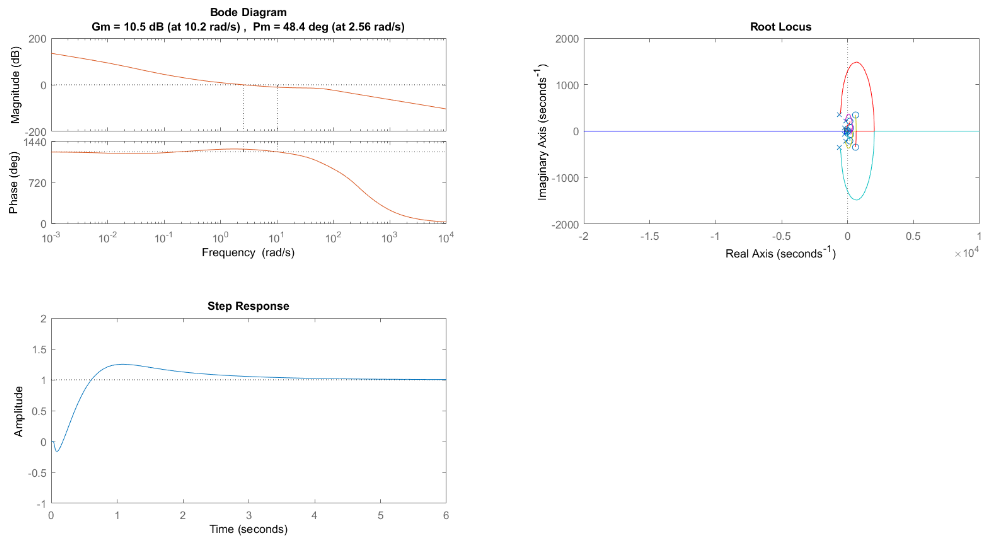

Figure 9: System analysis with the characteristics of the inner loop included. Root locus, step response and Bode plot all show stable behaviors.

Note the nonminimum phase behavior as can be seen in the step response of figure 9. This is expected and is a natural behavior of a MiP.

Finally the discrete version of D2 is found by utilizing Tustin's approximation around the crossover frequency ω_{c2} and the result is

$$D2(z) = \frac{0.1233z^2 - 0.2428z + 0.1195}{z^2 - 0.397 + 0.3972} \tag{11}$$