LECTURE 16 – HIERARCHICAL MODELS

EARTHQUAKES EXAMPLE

$$\begin{cases} Y_{ik} \sim \text{Exp}(\lambda_k) \;;\; i = 1 \dots n_k,\; k = 1 \dots K \\ \lambda_k \sim \Gamma(\alpha, \beta) \\ \alpha \sim U[0,\infty] \;;\; \beta \sim U[0,\infty] \end{cases}$$

$$\pi(\lambda_1 \dots \lambda_K, \alpha, \beta \mid x) \propto f(x \mid \lambda_1 \dots \lambda_K, \alpha, \beta)\, p(\lambda_1 \dots \lambda_K \mid \alpha, \beta)\, p(\alpha)\, p(\beta) =$$

$$= \prod_{k=1}^{K} \lambda_k^{n_k} \exp\left\{-\lambda_k \sum^{n_k} Y_{ik}\right\} \cdot \prod_{k=1}^{K} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda_k^{\alpha-1} \exp\left\{-\beta \lambda_k\right\} \cdot 1 \cdot 1 \quad \text{with } n_k = \sum_i \mathbb{I}(Y_{ik} = k)$$

$$\Rightarrow \pi(\lambda_j \mid \lambda_{(j)}, \alpha, \beta, Y) \propto \lambda_j^{n_j+\alpha-1} \exp\left\{-\lambda_j \left(\sum^{n_j} Y_{ij} + \beta\right)\right\}$$

$$\Rightarrow \pi(\alpha, \beta \mid \lambda_1 \dots \lambda_K, Y) \propto \left(\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right)^{K} \exp\left\{-\beta \sum^{K} \lambda_k\right\} \prod_{k=1}^{K} \lambda_k^{\alpha-1}$$

LECTURE 17 – BAYES FACTORS

HYPOTHESIS TESTING

$H_0 : \theta \in \Theta_0$

$H_1 : \theta \in \Theta_1$

e.g. $H_0 : \theta < \theta_0$

$H_1 : \theta \geq \theta_1$

FREQUENTIST APPROACH

- 2 POSSIBLE OUTCOMES:
  a) REJECT H_0 ; b) FAIL TO REJECT H_0
- $P(T \geq t \mid H_0) \leq \alpha \Rightarrow$ a

!!

$P(H_0 \text{ IS TRUE} \mid t)$

$P(H_1 \text{ IS TRUE} \mid t)$

WE WANT TO EVALUATE THESE PROBABILITIES INSTEAD!

BAYESIAN HYPOTHESIS TESTING

1. PRIOR BELIEF

$$P(H_i) = P(\theta \in \Theta_i) = p_i$$

2. COLLECT $\underline{x}$

3. FORM OUR POSTERIOR BELIEFS

→ BAYES THEOREM $\quad P(H_i \mid \underline{x})$

A SIMPLE HYPOTHESIS : $H_i : \theta = \theta_i$

$$P(\theta = \theta_i \mid x) = \frac{f(\underline{x} \mid \theta_i)\, p_i}{f(x)} \propto f(x \mid \theta_i)\, p_i$$

$$\left[\begin{array}{l} f(x) = \sum_i f(\underline{x} \mid \theta_i)\, p_i \\ \sum_i p_i = 1 \end{array}\right]$$

B COMPOSITE HYPOTHESIS : $H_i : \theta \in \Theta_i$

$$P(\theta \in \Theta_i \mid \underline{x}) = \int_{\theta \in \Theta_i} \pi(\theta \mid \underline{x})\, d\theta = \frac{1}{f(x)} \int f(x \mid \theta)\, p(\theta)\, d\theta$$

BAYES FACTORS

$$\frac{\text{POSTERIOR ODDS}}{\text{PRIOR ODDS}} \to \frac{P(\theta \in \Theta_0 \mid x)}{P(\theta \in \Theta_1 \mid x)} \cdot \frac{p_1}{p_0} = B_{01}$$

- $B_{01} > 3$; POSITIVE EVIDENCE FOR H_0
- $B_{01} > 20$; STRONG EVIDENCE
- $B_{10} = 1/B_{01}$
- $B_{01} < 1/3 \to$ POSITIVE EVIDENCE FOR H_1
- $B_{01} < 1/20 \to$ STRONG EVIDENCE H_1

BF FOR SIMPLE HYPOTHESIS

$$B_{01} = \frac{P(\theta = \theta_0 \mid \underline{x})}{P(\theta = \theta_1 \mid \underline{x})} \cdot \frac{p_1}{p_0} = \frac{f(x \mid \theta_0)}{f(x \mid \theta_1)} = \text{LIKELIHOOD RATIO}$$

BF FOR COMPOSITE HYPOTHESIS

$$B_{01} = \frac{\int_{\Theta_0} f(x \mid \theta)\, p_0(\theta)\, d\theta}{\int_{\Theta_1} f(x \mid \theta)\, p_1(\theta)\, d\theta} \xrightarrow{\text{CONSEQUENCE}}$$

IMPROPER PRIORS

$$\begin{cases} p_0(\theta) = c_0\, h_0(\theta) \\ p_1(\theta) = c_1\, h_1(\theta) \end{cases}$$

where $h_i(\theta)$ is a FUNCTION s.t. $\int_{\Theta_i} h_i(\theta) = \infty$ AND c_i IS CONSTANT

- $$\pi_i(\theta \mid x) = \frac{f(x \mid \theta)\, p_i(\theta)}{\int f(x \mid \theta)\, p_i(\theta)\, d\theta} = \frac{f(x \mid \theta)\, \cancel{c_i}\, h_i(\theta)}{\cancel{c_i} \int f(x \mid \theta)\, h_i(\theta)\, d\theta}$$

$$\to B_{01} = \frac{c_0 \int_{\Theta_0} f(x \mid \theta)\, h_0(\theta)\, d\theta}{c_1 \int_{\Theta_1} f(x \mid \theta)\, h_1(\theta)\, d\theta} \to$$

c_0/c_1 WILL IMPACT FINAL BF ESTIMATE! $\to$ AVOID IMPROPER PRIORS WHEN CALCULATING BFs

Ex. 1

$X_{i=1..n} \sim \text{Poisson}(\lambda)$; $H_0: \lambda = 100$; $H_1: \lambda = 110$

(a) COMPUTE BF FOR H_0 VS H_1 GIVEN $\underline{x}$

$$B_{01} = \frac{f(\underline{x} \mid \lambda = 100)}{f(\underline{x} \mid \lambda = 110)} = \frac{\lambda_0^{\Sigma x_i} \exp\{-\lambda_0 n\}}{\lambda_1^{\Sigma x_i} \exp\{-\lambda_1 n\}} = \left(\frac{\lambda_0}{\lambda_1}\right)^{\Sigma x_i} \exp\{-n(\lambda_0 - \lambda_1)\}$$

(b) SUPPOSE $n = 10$; $\bar{x} = 102.7$ → EVALUATE AND INTERPRET BF

$$B_{01} = \frac{10}{11}^{(10 \cdot 102.7)} \exp\{100\} = 8.302$$ → POSITIVE EVIDENCE IN FAVOUR OF H_0

(c) ASSUME EQUAL PROB ON EACH HYPOTHESIS, CALCULATE THE POSTERIOR PROBABILITIES FOR EACH HYPOTHESIS.

$B_{01} = 8.302$; $P_0 = P_1 = 1/2$; $B_{01} = \dfrac{P(H_0 \mid \underline{x})}{P(H_1 \mid \underline{x})} \cdot \dfrac{P_1}{P_0}$ $\xrightarrow{\text{SOLUTION}}$ .... $P(H_0 \mid x) = 0.892$

TRY THIS AT HOME!