# LECTURE 6 – INFORMATIVE PRIORS

- WE HAVE PRIOR KNOWLEDGE ON $\theta$

OUR JOB TRANSLATE PRIOR INFO INTO $P(\theta)$ ["PRIOR ELICITATION"]

ELICITATION ERRORS

1. FITTING ERROR
2. SPECIFICATION ERROR

$P(\theta)$ IS OFTEN DERIVED VIA AN ITERATIVE PROCESS

1. INITIAL DISCUSSION
2. DERIVATION OF A $P(\theta)$
3. PRESENT PRIOR TO EXPERT(S)
4. IF EXPERT(S) HAPPY THEN STOP; OTHERWISE START DISCUSSION AGAIN
5. SENSITIVITY ANALYSIS

TRY DIFFERENT PRIORS AND REPORT EFFECT ON POSTERIOR

EXAMPLE : "HIDDEN FIGURES" – ESTIMATING # STUDENTS CHEATING IN EXAMS IN SCOTLAND DURING 2020

TASK CREATE A PRIOR $P(\theta)$

PRIOR INFO
- $N^o$ OF CHEATERS CAUGHT 480
- DISCOVERY RATE ESTIMATE 0.535 [0.293, 0.979] 95% CI

→ ESTIMATED # OF CHEATERS $\hat{\theta} = 897$ [493, 1638] 95% CI

=> POSSIBLE WAYS TO FIND SUITABLE $P(\theta)$ :

A • SET $P(\theta)$ ASYMMETRIC $\xrightarrow[\text{e.g.}]{}$ GAMMA$(a, b)$

$\frac{a}{b} = 897 \quad a = 897b$

FIND $a, b$ SUCH THAT LOWER & UPPER 2.5% QUANTILES ARE [493, 1638]

B • CONSIDER A TRANSFORMATION $\phi = \log \theta$

$\phi = 6.8\ [6.2, 7.4]$ 95% CI

$6.8 \pm 1.96\,\sigma = [6.2, 7.4]$

$\sigma = 0.306$

$\phi \sim N(6.8, 0.306^2)$

$\Rightarrow \theta \sim \log N(6.8, 0.306^2) \rightarrow \underset{\substack{= \\ 897}}{\text{Exp}(6.8)}$

→ WE CAN FORMALLY JUSTIFY DIFFERENT AVERAGES WITHIN A DECISION THEORY APPROACH

- $\theta \in \Theta$ = PARAMETER
- $L(\theta, a)$ = ASSOCIATED LOSS FOR ESTIMATE $a$ WHEN TRUE VALUE IS $\theta$

- $L(\theta, a) = |\theta - a|$

  $\hat{\theta} = M_\pi(\theta)$

- $L(\theta, a) \begin{cases} 1 & \text{if } a \neq \theta \\ 0 & \text{if } a = \theta \end{cases}$

  $\hat{\theta} = Mo_\pi(\theta)$

⇒ <u>BAYES ESTIMATE</u> $\hat{\theta}$ = VALUE $a$ THAT MINIMISES THE POSTERIOR EXPECTED LOSS

$$\hat{\theta} = \min_{a \in \Theta} E_\pi\big(L(\theta, a)\big) = \min_{a \in \Theta} \left[\int_\theta L(\theta, a)\, \pi(\theta \mid x)\, d\theta\right]$$

(1) $L(\theta, a) = (\theta - a)^2 \Rightarrow \hat{\theta} = E_\pi(\theta)$ = POSTERIOR MEAN

$$E_\pi\left[(\theta - a)^2\right] = E_\pi\left[\left(\theta - E_\pi(\theta) + E_\pi(\theta) - a\right)^2\right] = E_\pi\left[\left(\theta - E_\pi(\theta)\right)^2\right] + \left(E_\pi(\theta) - a\right)^2 + 2\left(E_\pi(\theta) - a\right) \underbrace{E_\pi\left[\theta - E_\pi(\theta)\right]}_{=0}$$

$$\min_a \left[\mathrm{VAR}_\pi(\theta) + \left(E_\pi(\theta) - a\right)^2\right]$$

→ $a = E_\pi(\theta)$ ✱