LECTURE 8 - PREDICTION

- WE WISH TO MAKE INFERENCE ABOUT A FUTURE RANDOM OBS. Y WITH pdf $= f(y|\theta)$ AND UNKNOWN PARAMETER $\theta \in \Theta$

→ PREDICTIVE DISTRIBUTION FOR Y

$$f(y) = \int_{\theta \in \Theta} f(y|\theta)\, g(\theta)\, d\theta$$

WHERE $g(\theta)$ = OUR BELIEFS ABOUT θ

IF

$\boxed{A}$ $g(\theta) = p(\theta)$ i.e. PRIOR DISTRIBUTION $\Rightarrow$ $f(y)$ IS THE PRIOR PREDICTIVE DISTRIBUTION

$\boxed{B}$ $g(\theta) = \pi(\theta|x)$ i.e. POSTERIOR DISTRIBUTION $\Rightarrow$ $f(y)$ IS THE POSTERIOR PREDICTIVE DISTRIBUTION

$\boxed{A}$ PRIOR PD

$$f(Y) = \int_{\theta \in \Theta} f(Y|\theta)\, p(\theta)\, d\theta \qquad \text{IF } f(Y = y) \Rightarrow \text{MARGINAL LIKELIHOOD}$$

EX. 1

$X \sim Exp(\lambda)$; $\lambda > 0$

$\lambda \sim \Gamma(\alpha, \beta)$; $\alpha > 0$; $\beta > 0$

FIND PRIOR PD

$$f(x) = \int_0^\infty f(x|\lambda)\, p(\lambda)\, d\lambda = \int_0^\infty \lambda e^{-\lambda x} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \underbrace{\lambda^\alpha e^{-\lambda[\beta+x]}}_{\propto\, \Gamma(\alpha+1,\, \beta+x)} d\lambda$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{(\beta+x)^{\alpha+1}}$$

IS THIS A VALID PD??

$$\begin{cases} \cdot\ f(x) \geq 0 \\ \cdot\ \int f(x) = 1 \end{cases} \rightarrow \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \Gamma(\alpha+1) \cdot \int_0^\infty (\beta+x)^{-\alpha-1} dx$$

$$\frac{\cancel{\beta^\alpha}}{\cancel{\Gamma(\alpha)}} \cdot \cancel{\alpha}\,\cancel{\Gamma(\alpha)} \cdot \left[-\frac{1}{\alpha} [\beta+x]^{-\alpha} \right]_0^\infty = \frac{1}{\cancel{\alpha}} \cdot \cancel{\beta^{-\alpha}} = 1$$

$\boxed{B}$ POSTERIOR PD

$f(a,b) = f(a|b)\, f(b)$

- $f(Y|x) = \int_\theta f(Y,\theta|x)\, d\theta = \int_\theta f(Y|x,\theta)\, \pi(\theta|x)\, d\theta$

$$f(Y|x) = \int_\theta f(Y|\theta)\, \pi(\theta|x)\, d\theta$$

ASSUMING $Y \perp\!\!\!\perp X$ GIVEN θ

i.e. $P(Y|x,\theta) = P(Y|\theta)$

EX. 2 DREW 5 RED AND 2 WHITE BALLS FROM URN

$P(\theta) \sim \text{BETA}(1,1)$

$\Rightarrow \theta | x \sim \text{BETA}(5+1, 2+1)$

→ SUPPOSE WE EXTRACT ~~OTH~~ 3 OTHER BALLS, WHAT IS PROB AT LEAST ONE RED BALL?

$$f(y|x) = \int_\theta f(y|\theta)\,\pi(\theta|x)\,d\theta = \int_\theta \binom{3}{y} \theta (1-\theta)^{3-y} \cdot \frac{\Gamma(6+3)}{\Gamma(6)\Gamma(3)} \cdot \theta^5 (1-\theta)^2 \, d\theta$$

$$= \frac{3!}{x!(3-x)!} \cdot \frac{\Gamma(9)}{\Gamma(6)\Gamma(3)} \cdot \int_\theta \theta^{y+5} (1-\theta)^{3-x+2} \, d\theta$$

$$= \frac{3!}{x!(3-x)!} \cdot \frac{\Gamma(9)}{\Gamma(6)\Gamma(3)} \cdot \frac{\Gamma(x+6)\Gamma(6-x)}{\Gamma(12)}$$

← BETA-BINOMIAL DISTRIBUTION

$$P(y \geq 1) = 1 - P(y=0) = 1 - \frac{3}{6} \cdot \frac{\Gamma(9) \cdot \Gamma(6)}{\Gamma(12)} \cong 0.94$$