M-H : BLOCK AND SINGLE UPDATING

α^t, β^t WE WANT NEW VALUES FOR $\theta^{t+1} \sim q(\alpha^{t+1}, \beta^{t+1} \mid \theta^t)$

SIMPLE SOLUTION : 2 RANDOM WALKS

$$\alpha^{t+1} = \alpha^t + \varepsilon_\alpha \,, \quad \varepsilon_\alpha \sim N(0, \sigma^2_\alpha)$$
$$\beta^{t+1} = \beta^t + \varepsilon_\beta \,, \quad \varepsilon_\beta \sim N(0, \sigma^2_\beta)$$

$\boxed{A}$ BLOCK UPDATING

$\to$ PROPOSE $\theta^{t+1} \sim q(\alpha^{t+1}, \beta^{t+1} \mid \theta^t) = q(\alpha^{t+1} \mid \alpha^t)\, q(\beta^{t+1} \mid \beta^t)$

$$\alpha(\theta^t, \theta^{t+1}) = \min[1, A] \text{ WHERE } A = \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} \cdot \cancel{\frac{q(\alpha^t \mid \alpha^{t+1})}{\underbrace{q(\alpha^{t+1} \mid \alpha^t)}_{=1}}} \cdot \cancel{\frac{q(\beta^t \mid \beta^{t+1})}{\underbrace{q(\beta^{t+1} \mid \beta^t)}_{=1}}}$$

$$A = \frac{\pi(\alpha^{t+1}, \beta^{t+1})}{\pi(\alpha^t, \beta^t)}$$

$\Rightarrow$ BOTH $\alpha^{t+1}, \beta^{t+1}$ NEED TO BE GOOD MOVES i.e. LOWER ACCEPTANCE!

BUT IF ACCEPTED, IT LEADS TO LARGER JUMPS = LESS AUTOCORRELATION!

$\boxed{B}$ SINGLE UPDATES

CONSIDER $\theta^t = (\alpha^{t+1}, \beta^t)$ WE PROPOSE $\beta^{t+1} \sim q(\beta^{t+1} \mid \beta^t)$

ACCEPT β^{t+1} WITH PROB $= \min[1, A]$

$$A = \frac{\pi(\alpha^{t+1}, \beta^{t+1})}{\pi(\alpha^{t+1}, \beta^t)} \cdot \underbrace{\cancel{\frac{q(\beta^t \mid \beta^{t+1})}{q(\beta^{t+1} \mid \beta^t)}}}_{=1} = \frac{\pi(\beta^{t+1} \mid \alpha^{t+1})\, \cancel{\pi(\alpha^{t+1})}}{\pi(\beta^t \mid \alpha^{t+1})\, \cancel{\pi(\alpha^{t+1})}}$$

$\Rightarrow$ if $q(\phi \mid \theta) = \pi(\phi_j \mid \theta_{(j)})$ $\Rightarrow$ GIBBS SAMPLER

POSTERIOR CONDITIONAL DISTRIBUTION OF ϕ_j GIVEN ALL THE OTHER PARAMETERS

PROPOSAL $\beta^{t+1} \sim \pi(\beta^{t+1} \mid \alpha^{t+1})$

ACCEPT β^{t+1} WITH PROB $= \min[1, A]$

$$A = \frac{\pi(\alpha^{t+1}, \beta^{t+1})}{\pi(\alpha^{t+1}, \beta^t)} \cdot \frac{\pi(\beta^t \mid \alpha^{t+1})}{\pi(\beta^{t+1} \mid \alpha^{t+1})} = \frac{\cancel{\pi(\beta^{t+1} \mid \alpha^{t+1})}\, \cancel{\pi(\alpha^{t+1})}}{\cancel{\pi(\beta^t \mid \alpha^{t+1})}\, \cancel{\pi(\alpha^{t+1})}} \cdot \frac{\cancel{\pi(\beta^t \mid \alpha^{t+1})}}{\cancel{\pi(\beta^{t+1} \mid \alpha^{t+1})}} = 1$$

PROPOSALS ALWAYS ACCEPTED WITH GIBBS!

— LECTURE 15 —

- DATA AUGMENTATION
- MIXTURE EXAMPLE

$$\begin{cases} x_1 \ldots x_n \sim N(\mu_{z_i = j+1}, s^{-1}) \\ z_1 \ldots z_n \sim \text{Bernoulli}(\pi) \\ \mu_j \sim N(\emptyset, \lambda^{-1}) \; ; \; j = 1, 2 \\ \pi \sim \text{Beta}(a, b) \end{cases}$$

$$\pi(\pi, \underline{\mu}, \underline{z} \mid x) \propto p(x \mid \pi, \underline{\mu}, \underline{z})\, p(\pi, \underline{\mu}, \underline{z})$$

$$= p(x \mid \underline{z}, \underline{\mu}, \pi)\, p(\underline{z} \mid \underline{\mu}, \pi)\, p(\pi) \cdot p(\underline{\mu})$$

$$\propto \prod_{i=1}^{n} \exp\left\{-\frac{s}{2}\left(x_i - \mu_{z_i = j+1}\right)^2\right\} \cdot \pi^{z_i}(1-\pi)^{1-z_i} \cdot \exp\left\{\frac{\lambda}{2}(\mu_1 - \emptyset)^2\right\} \exp\left\{\frac{\lambda}{2}(\mu_2 - \emptyset)^2\right\} \cdot \pi^{a-1}(1-\pi)^{b-1}$$

FULL CONDITIONALS:

- $$\pi(\pi \mid \underline{\mu}, \underline{z}, x) \propto \prod_{i=1}^{n}\left[\pi^{z_i}(1-\pi)^{1-z_i}\right]\pi^{a-1}(1-\pi)^{b-1} = \pi^{n_1}(1-\pi)^{n_0}\pi^{a-1}(1-\pi)^{b-1}$$

$$= \pi^{n_1 + a - 1}(1-\pi)^{n_0 + b - 1}$$

where $n_j = \sum_i \mathbb{1}(z_i = j)$

$$\pi \mid \underline{\mu}, \underline{z}, x = \pi \mid z, x \sim \text{BETA}(n_1 + a, n_0 + b)$$

$$\bullet\ \pi(\mu_j \mid \pi, z, x) \propto \prod_{i=1}^{n} \exp\left\{-\frac{s}{2}\left(x_i - \mu_{z_i=j+1}\right)^2\right\} \exp\left\{-\frac{\lambda}{2}\left(\mu_{z_i=j+1} - \phi\right)^2\right\}$$

$\to$ <u>CONJUGACY</u>

$$\mu_j \mid \underline{z}, x \sim N\left(\frac{s \sum_{i: z_i = j} x_i + \lambda\phi}{n_j s + \lambda} ; [n_j s + \lambda]^{-1}\right)$$

$$\bullet\ \pi(\underline{z} \mid \underline{\mu}, \pi, x) \propto \prod_{i=1}^{n} \left[\exp\left\{-\frac{s}{2}\left(x_i - \mu_{z_i=j+1}\right)^2\right\} \pi^{z_i} (1-\pi)^{1-z_i}\right]$$

$$\prod_{i=1}^{n} \left[\exp\left\{-\frac{s}{2}\left(x_i - \mu_2\right)^2\right\} \pi\right]^{z_i} \cdot \left[\exp\left\{-\frac{s}{2}\left(x_i - \mu_1\right)^2\right\} (1-\pi)\right]^{1-z_i}$$

$$\prod_{i=1}^{n} \text{BERNOULLI}\left(\frac{\theta_{i2}}{\theta_{i2} + \theta_{i1}}\right)$$

where $\theta_{i2} = N(\mu_2, s^{-1})\pi$ and $\theta_{i1} = N(\mu_1, s^{-1})(1-\pi)$