IMPORTANCE SAMPLING

- IS THERE A WAY TO USE <u>ALL</u> SAMPLES THAT ARE GENERATED FROM A PROPOSAL DISTRIBUTION?

- SAY THAT WE ARE INTERESTED IN ESTIMATING $E_\pi(f(\theta))$

  USUAL APPROACH → MC INTEGRATION

PROBLEM

NOT EASY TO SAMPLE $\pi(\theta|x)$

$$\hat{E}_\pi(f(\theta)) = \frac{1}{n}\sum_i^n f(\theta^i) \quad \text{WHERE } \theta^i \text{ ARE FROM } \pi(\theta|x)$$

- WE WOULD LIKE TO USE A PROPOSAL $g(\theta)$ AND SAMPLE $\theta_g^1 \ldots \theta_g^n \sim g(\theta)$

⇒ **NOTE**

$$E_\pi(f(\theta)) = \int f(\theta)\pi(\theta|x)\,d\theta = \int \frac{f(\theta)\pi(\theta|x)}{g(\theta)} g(\theta)\,d\theta = E_g\left[\frac{f(\theta)\pi(\theta|x)}{g(\theta)}\right]$$

$$\to \hat{E}_\pi(f(\theta)) = \hat{E}_g\left[\frac{f(\theta)\pi(\theta|x)}{g(\theta)}\right] = \frac{1}{n}\sum_i^n \frac{\pi(\theta_g^i|x)}{g(\theta_g^i)} f(\theta_g^i) = \frac{1}{n}\sum_i^n w(\theta_g^i) f(\theta_g^i)$$

$$w_i = \frac{\pi(\theta_g^i \mid x)}{g(\theta_g^i)} = \text{"IMPORTANCE WEIGHTS"}$$

- MORE EFFICIENT THAN REJECTION SAMPLING! WE USE ALL THE SAMPLES FROM THE PROPOSAL $g(\theta)$
- IT WORKS ALSO WITH UNNORMALISED POSTERIORS (TARGETS)

UNNORMALISED $p(\theta \mid x) = Z \cdot \tilde{p}(\theta \mid x)$ WHERE $\begin{cases} \tilde{p}(\theta \mid x) = \text{PROPER DISTRIBUTION} \\ Z = \text{UNKNOWN CONSTANT} \end{cases}$

$$\rightarrow Z = \int p(\theta \mid x)\, d\theta = \int \frac{p(\theta \mid x)}{g(\theta)} g(\theta)\, d\theta$$

$$= E_g\left[\frac{p(\theta \mid y)}{g(\theta)}\right] \approx \frac{1}{n}\sum_i^n w_i$$

$$\Rightarrow E[p(\theta \mid x)] \approx \frac{\frac{1}{n}\sum_i^n w_i \theta_g^i}{\frac{1}{n}\sum_i^n w_i}$$

EXAMPLE 1

$\pi(\theta) \sim BETA(3,4)$

$g(\theta) \sim U(0,1)$

- 1) SAMPLE $\theta_g^1 \sim \theta_g^n \sim U(0,1)$
- 2) WEIGHTS $w_i = \dfrac{\pi(\theta_g^i)}{g(\theta_g^i)} = \dfrac{\Gamma(7)}{\Gamma(4)\Gamma(3)} \theta^{i\,2}(1-\theta^i)^3$

$\Rightarrow \hat{E}_\pi(\theta) = \frac{1}{n}\sum_i^n w_i \theta_g^i$

$\Rightarrow \hat{E}_\pi\left[(\theta-\mu_\theta)^2\right] = \frac{1}{n}\sum^n w_i \theta^{i\,2} - \left[\frac{1}{n}\sum_i^n w_i \theta_i\right]^2$

SAMPLING IMPORTANCE RESAMPLING (SIR)

- $\theta^1 \ldots \theta^n$ FROM $g(\theta)$
- $w_i = p(\theta^i | x) / g(\theta^i)$ AND $w_i^* = \dfrac{w_i}{\sum^n w_i}$
- SAMPLE WITH REPLACEMENT $\theta^1 \ldots \theta^m$ FROM $\theta^1 \ldots \theta^n$ WHERE EACH θ^i IS CHOSEN WITH PROBABILITY w_i^*

$\Rightarrow$ THE RESULTING $\theta^1 \ldots \theta^m$ ARE SAMPLES FROM $p(\theta | x)$

POSSIBLE PROBLEM

"PARTICLE DEPLETION" $\rightarrow$

$$
\begin{array}{c|ccccc}
g(\theta) & \theta^{(1)} & \theta^{(2)} & \ldots & \theta^{(n)} \\
w_i^* & 0.5 & 0.49 & \ldots & -
\end{array}
$$

$$\sum_i^2 w_i^* = 0.99 \; !$$