INVERSION SAMPLING

- PROBABILITY INTEGRAL TRANSFORM THEOREM

CONSIDER A CONTINUOUS R.V. $X \sim f$ WITH $F_X = \int_{-\infty}^{x} f_X(u)\,du$ I.E. THE CDF

DEFINE $Y = F_X(X)$, THEN

$$F_Y = P(Y \leq y) = P(F_X \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

$Y \sim U(0,1)$

$\Rightarrow$ IF $Y = F_X(X)$ THEN $X = F_X^{-1}(Y)$ WHERE $Y \sim U(0,1)$

INVERSION ALGORITHM

1 - GENERATE y FROM $U(0,1)$

2 - SET $x = F_X^{-1}(y)$. THIS IS AN INDEPENDENT SAMPLE FROM $X \sim f$

Ex. 1

$X \sim \mathrm{Exp}(\lambda), \lambda > 0$

$f(x) = \lambda e^{-\lambda x}, x > 0$

- FIND $F_X(x) = \int_{-\infty}^{x} \lambda e^{-\lambda u} du = \int_{0}^{x} \lambda e^{-\lambda u} du = \left[-e^{-\lambda u}\right]_0^x = -e^{-\lambda x} + 1$

$= 1 - e^{-\lambda x}$

- FIND $F_X^{-1}(x)$

$1 - e^{-\lambda x} = y \iff -e^{-\lambda x} = y - 1 \iff 1 - y = e^{-\lambda x} \iff$

$\log(1-y) = -\lambda x \iff x = -\frac{1}{\lambda}\log(1-y) = F_X^{-1}(y)$

INVERSION ALGORITHM

1 - SAMPLE y FROM $U(0,1)$

2 - SET $x = -\frac{1}{\lambda}\log(1-y)$

REPEAT N TIMES TO GET N INDEPENDENT SAMPLES FROM $X \sim \mathrm{Exp}(\lambda)$

Ex. 2

$X \sim \text{PARETO}(x_m, \alpha)$, $x_m > 0$, $\alpha > 0$

$$f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}} \quad \text{if } x \geq x_m$$

- $F(x) = \int_{-\infty}^{x} \frac{\alpha x_m^\alpha}{u^{\alpha+1}} du = \int_{x_m}^{x} \frac{\alpha x_m^\alpha}{u^{\alpha+1}} du = \alpha x_m^\alpha \left[-\frac{1}{\alpha} u^{-\alpha} \right]_{x_m}^{x}$

$= \alpha x_m^\alpha \left[-\frac{1}{\alpha} x^{-\alpha} + \frac{1}{\alpha} x_m^{-\alpha} \right] = -\frac{x_m^\alpha}{x^\alpha} + 1 = 1 - \left(\frac{x_m}{x} \right)^\alpha$

- $F_X^{-1}(x) \longrightarrow 1 - \left(\frac{x_m}{x} \right)^\alpha = y \iff 1 - y = \left(\frac{x_m}{x} \right)^\alpha \iff (1-y)^{1/\alpha} = \frac{x_m}{x} \iff$

$$x = \frac{x_m}{(1-y)^{1/\alpha}}$$

ALGORITHM

1 SAMPLE y FROM $U(0,1)$

2 SET $x = x_m / (1-y)^{1/\alpha}$

- HOW DO WE SIMULATE FROM DISCRETE DISTRIBUTIONS?

SUPPOSE X TAKES DISCRETE VALUES $x_1 \dots x_n$ with $x_{i-1} < x_i$

THEN $\boxed{1}$ GENERATE y FROM $U(0,1)$

$\boxed{2}$ SET $X = x_i$ IF $F_X(x_{i-1}) < y \leq F_X(x_i)$

PROOF

$$P(F_X(x_{i-1}) < y \leq F_X(x_i)) = F_X(x_i) - F_X(x_{i-1}) = P(X \leq x_i) - P(X \leq x_{i-1})$$

$$= P(X = x_i)$$