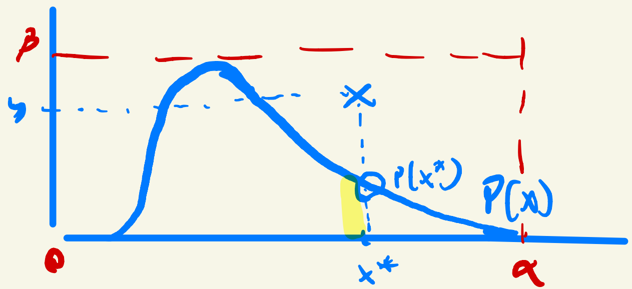
REJECTION SAMPLING

IDEA 1 RECTANGLE ENVELOPE



1) SELECT $x^* \sim U[0, \alpha]$
2) GENERATE $y \sim U(0, \beta]$
3) ACCEPT x^* IF $y \leq P(x^*)$
   ELSE REJECT x^* AND GO BACK TO (1)

NOTE

→ POINT (2) AND (3) CAN BE REPLACED BY (2') SIMULATE y FROM $U(0,1)$ AND ACCEPT x^* IF $y \leq \dfrac{P(x^*)}{\beta}$

PROBLEMS

1. WHAT IF $P(x)$ DEFINED ON INFINITE SUPPORT?
2. PROB. OF REJECTION CAN BECOME LARGE!

IDEA 2

① CHOOSE $g(x)$:

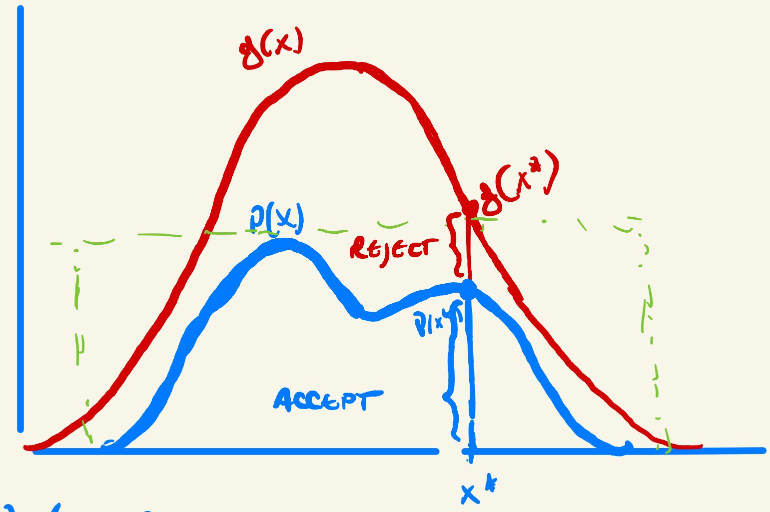
$g(x) \geq P(x) \quad \forall x$

$P(x) / g(x) < \infty$

② SIMULATE x^* FROM $g(x)$

③ EVALUATE $P(x^*) / g(x^*)$

④ ACCEPT x^* IF $y \leq \frac{P(x^*)}{g(x^*)}$; with $y \sim U(0,1)$



$\rightarrow$ NOT EASY TO FIND A $g(x)$ WITH THESE PROPERTIES,
WE COULD MULTIPLY A KNOWN $g(x)$ BY A POSITIVE CONSTANT $M \geq 1$
AND REQUIRE $M g(x) \geq P(x) \quad \forall x$

STANDARD REJECTION SAMPLING ALGORITHM

1) SAMPLE x^* FROM A PROPOSAL $g(x)$
2) SAMPLE y FROM $U(0,1)$
3) IF $y \leq \dfrac{P(x^*)}{M g(x^*)}$ THEN SET $x^{t+1} = x^*$, OTHERWISE REJECT x^* AND GO BACK TO (1)

HOW MANY?

→ REPEAT N-TIMES TO GET K INDEPENDENT SAMPLES FROM $P(x)$

$$P(x^* \text{ IS ACCEPTED}) = P\left(y \leq \frac{P(x^*)}{M g(x^*)}\right) = \int_{-\infty}^{+\infty} P\left(y \leq \frac{P(x^*)}{M g(x^*)} \,\middle|\, x^* = x\right) g(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \frac{P(x)}{M \cancel{g(x)}} \cdot \cancel{g(x)}\, dx = \frac{1}{M} \int_{-\infty}^{+\infty} P(x)\, dx = \frac{1}{M}$$

→ ON AVERAGE WE NEED M-SAMPLES BEFORE ONE IS ACCEPTED!

⇒ WE WANT M TO BE AS SMALL AS POSSIBLE!

OPTIMAL M
$$\begin{cases} \cdot \text{ CLOSE TO } 1 \quad (M \geq 1) \\ \cdot \text{ SUBJECT TO } M g(x) \geq p(x) \ \forall x \end{cases}$$

$$M \geq \frac{p(x)}{g(x)} \ \forall x \quad \text{i.e.} \quad M \geq \max_x \left(\frac{p(x)}{g(x)} \right) \rightarrow M^{\text{OPTIM}} = \max_x \left(\frac{p(x)}{g(x)} \right)$$

EXAMPLE 1

- TARGET $p(x) \propto \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$ with $\mu = 0$, $\sigma^2 = 1$
- PROPOSAL $g(x) = U[-5, 5]$ $\quad g(x) = \begin{cases} 1/10 & \text{if } -5 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$
- SELECT M: $\quad M g(x) \geq p(x) \ \forall x \rightarrow M \geq \max_x \left(\frac{p(x)}{g(x)} \right)$

  SINCE $g(x)$ CONSTANT OVER $[-5, 5]$ AND
  $\max(p(x))$ IS 1 (WHEN $x = 0$) THEN $\rightarrow M = \frac{1}{0.1} = 10$

(1) SAMPLE x^* FROM $U[-5, 5]$

(2) SAMPLE u FROM $U(0, 1)$

(3) IF $u \leq \frac{1}{10} \cdot \frac{p(x^*)}{g(x^*)}$ THEN ACCEPT x^*

WHERE $\frac{p(x^*)}{10\, g(x^*)} = \exp\left\{ -\frac{1}{2} {x^*}^2 \right\}$