GIBBS SAMPLER

IDEA SAMPLE FROM $\pi(\underline{\theta} \mid \underline{x})$ USING FULL CONDITIONAL DISTRIBUTIONS $\pi(\theta_j \mid \underline{\theta}_{(-j)}, \underline{x})$

HOW DOES IT WORK?

$$\underline{\theta} = (\theta_1, \theta_2, \theta_3)$$

→ TO CONSTRUCT A MC WE FIRST SET INITIAL VALUES $\theta_1^0, \theta_2^0, \theta_3^0$

- EXTRACT NEW VALUES $\theta_1^1, \theta_2^1, \theta_3^1$ IN THIS WAY:

$$\theta_1^1 \sim p(\theta_1^0 \mid \theta_2^0, \theta_3^0, \underline{x})$$

$$\theta_2^1 \sim p(\theta_2^0 \mid \theta_1^1, \theta_3^0, \underline{x})$$

$$\theta_3^1 \sim p(\theta_3^0 \mid \theta_1^1, \theta_2^1, \underline{x})$$

$$\underline{\theta}^0 = (\theta_1^0, \theta_2^0, \theta_3^0) \qquad \underline{\theta}^1 = (\theta_1^1, \theta_2^1, \theta_3^1)$$

THIS COMPLETES ONE CYCLE FROM ITERATION 0 TO ITERATION 1

IF WE APPLY THE ALGORITHM FOR T ITERATIONS

$\rightarrow \underline{\theta}^0, \ldots, \underline{\theta}^T$ WHERE

- $\underline{\theta}^0 - \underline{\theta}^B$ ARE DISCARDED (BURN-IN)
- $\underline{\theta}^{B+1} \ldots \underline{\theta}^T$ USED TO OBTAIN OUR MONTE CARLO ESTIMATES OF INTEREST

MARKOV CHAIN DETAILS

- THE TRANSITION KERNEL $K_G(\theta^t, \theta^{t+1}) = \prod_{i=1}^{3} p(\theta_i^{t+1} \mid \theta_j^{t+1}, \theta_{j^*}^{t}, \underline{x})$

  WHERE $j < i$ AND $j^* > i$

- STATIONARY DISTRIBUTION IS THE POSTERIOR DISTRIBUTION

TO PROVE IT $\rightarrow$ $$E_{\theta^t}\left[K_G(\theta^t, \theta^{t+1})\right] = p(\theta^{t+1} \mid \underline{x})$$

TO BUILD A GIBBS SAMPLER

1) CONSIDER $\pi(\underline{\theta} \mid x) \propto f(x \mid \underline{\theta})\, p(\underline{\theta})$

2) DERIVE THE FULL CONDITIONALS $\pi(\theta_i \mid \underline{\theta}_{(-i)}, \underline{x})$

$$\pi(\theta_i \mid \underline{\theta}_{(-i)}, x) = \frac{\pi(\underline{\theta} \mid x)}{\pi(\underline{\theta}_{(-i)} \mid x)}$$

- LOOK AT $\pi(\underline{\theta} \mid x)$
- TREAT ALL FACTORS THAT ONLY DEPEND ON $\theta_{(-i)}$ AND x AS CONSTANTS ABSORBED IN THE PROPORTIONALITY
- WHATEVER IS LEFT, GIVES YOU THE FULL CONDITIONAL DISTRIBUTION OF θ_i UP TO PROPORTIONALITY

3) WE NEED A WAY TO SAMPLE FROM THE THE FULL CONDITIONALS

- IF THEY HAVE STANDARD FORMS $\rightarrow$ SIMPLE! rbeta, rnorm ...
- IF THEY ARE NOT STANDARD? $\rightarrow$ NEXT LECTURE!

EXAMPLE

WE OBSERVE $Y_1 - Y_n$ WHERE $Y_i = \begin{cases} \text{Exp}(\lambda_1) & \text{if } i \le \tau \\ \text{Exp}(\lambda_2) & \text{if } i > \tau \end{cases}$

" CHANGE POINT MODEL "

$\underline{\theta} = (\lambda_1, \lambda_2, \tau)$

PRIORS

$$\begin{cases} p(\lambda_1) = p(\lambda_2) = \text{GAMMA}(\alpha, \beta) \\ p(\tau) = \dfrac{1}{n-1}, \quad 1 \le \tau < n \end{cases}$$

ONLY $n-1$ POTENTIAL LOCATIONS FOR τ, CHANGE POINT CANNOT OCCUR AT n

- JOINT POSTERIOR $\pi(\lambda_1, \lambda_2, \tau \mid Y) \propto p(Y \mid \lambda_1, \lambda_2, \tau)\, p(\lambda_1)\, p(\lambda_2)\, p(\tau)$

- LIKELIHOOD $\prod_{i=1}^{\tau} p(Y \mid \lambda_1) \cdot \prod_{i=\tau+1}^{n} p(Y \mid \lambda_2) = \lambda_1^{\tau} e^{-\lambda_1 \sum_1^{\tau} Y_i} \cdot \lambda_2^{n-\tau} e^{-\lambda_2 \sum_{\tau+1}^{n} Y_i}$

$$\pi(\lambda_1,\lambda_2,\tau \mid Y) \propto \lambda_1^{\tau} e^{-\lambda_1 \sum^{\tau} Y_i} \lambda_2^{n-\tau} e^{-\lambda_2 \sum_{\tau+1} Y_i} \left[\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda_1^{\alpha-1} e^{-\beta\lambda_1}\right]\left[\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda_2^{\alpha-1} e^{-\beta\lambda_2}\right]\left[\frac{1}{n-1}\right]$$

$$= \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^2 \left(\frac{1}{n-1}\right) \lambda_1^{\alpha-1+\tau} \lambda_2^{\alpha-1+n-\tau} e^{-\lambda_1\left(\beta + \sum^{\tau} Y_i\right)} e^{-\lambda_2\left(\beta + \sum_{\tau+1}^{n} Y_i\right)}$$

FULL CONDITIONALS

$$P(\lambda_1 \mid \lambda_2, \tau, Y) \propto \lambda_1^{\alpha-1+\tau} e^{-\lambda_1\left(\beta+\sum^{\tau} Y_i\right)} \to \text{GAMMA}\left(\alpha+\tau, \beta + \sum^{\tau} Y_i\right)$$

$$P(\lambda_2 \mid \lambda_1, \tau, Y) \propto \lambda_2^{\alpha-1+n-\tau} e^{-\lambda_2\left(\beta+\sum_{\tau+1}^{n} Y_i\right)} \to \text{GAMMA}\left(\alpha+n-\tau, \beta + \sum_{\tau+1}^{n} Y_i\right)$$

$$P(\tau \mid \lambda_1, \lambda_2, Y) \propto \lambda_1^{\alpha-1+\tau} \lambda_2^{\alpha-1+n-\tau} e^{-\lambda_1\left(\beta+\sum^{\tau} Y_i\right)} e^{-\lambda_2\left(\beta+\sum_{\tau+1}^{n} Y_i\right)}$$

→ SINCE τ IS DISCRETE, $\tau \in \{1 \dots n-1\}$

JUST EVALUATE $\tilde{w}_k = P(\tau = k \mid \lambda_1, \lambda_2, Y)$ FOR $k = 1 \dots n-1$

AND NORMALISE THE $\tilde{w}_k$ SO THAT $\dfrac{\tilde{w}_k}{\sum \tilde{w}_k} = w_k$ $\Rightarrow$ DRAW A VALUE FOR τ USING A CATEGORICAL DISTRIBUTION WITH $p = w_1 \dots w_k$