MT4531/5731: (Advanced) Bayesian Inference

Inversion Sampling

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Outline

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Sampling from Univariate distributions

- One generally applicable method for sampling from a distribution is the Metropolis-Hastings algorithm.
- (The Gibbs sampler is a method for how to *combine* samples from univariate distributions to generate samples from a multivariate one.)
- We know of course that we can use commands such as 'rnorm()' or 'rbeta()' to directly sample independent samples from standard distributions in R. But what methods does R use to sample from such distributions?
- Various methods have been developed for sampling directly from some distribution.
- This is typically the case for standard or univariate/bivariate distributions.

Outline

The Inversion method

- This is the simplest of all procedures, as long as one can calculate the inverse cumulative distribution function for the target distribution.
- If $X \sim f$, with F the corresponding cumulative distribution function, then $F(X) \sim U[0,1]$.
- Suppose that we wish to simulate a continuous random variable X with cumulative distribution function

$$F(x) = \mathbb{P}(X \leq x).$$

- Suppose also that the inverse function $F^{-1}(u)$ is well defined for $0 \leq u \leq 1$. Then we can use the following algorithm to sample from f.

The Inversion algorithm

- The algorithm for the Inversion method is

  STEP 1. GENERATE u AS A REALISATION FROM $U \sim U[0,1]$.
  STEP 2. SET $x = F^{-1}(u)$. THIS IS A SAMPLE FROM $X \sim f$

- For the proof that this algorithm samples from the f distribution, see the next slide.

Proof of the validity of the Inversion method

- Proof: If $X = F^{-1}(U)$, then what is the distribution of X?

$$\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x).$$

- Since F is the cumulative distribution function of a continuous random variable, F is a strictly monotonic, increasing and continuous function of x. Hence,

$$\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)).$$

- But, as U is a $U[0,1]$ random variable,

$$\mathbb{P}(U \leq F(x)) = F(x),$$

i.e.,

$$\mathbb{P}(X \leq x) = F(x).$$

- Therefore, $X \sim f(x)$.

Outline

Example

- Let $X \sim Exp(\lambda)$, $(\lambda > 0)$. How can we sample from this distribution? Remember that,

$$f(x) = \lambda e^{-\lambda x} \qquad \text{for } x \geq 0,$$

$$F(x) = \int_0^x \lambda e^{-\lambda u} du = 1 - e^{-\lambda x}, \qquad x \geq 0.$$

- To obtain F^{-1}, let $F(x) = u = 1 - e^{-\lambda x}$.
- Then $x = -\frac{1}{\lambda} \ln(1-u)$.
- So, to sample from the exponential distribution, set

$$x = F^{-1}(u) = -\frac{1}{\lambda} \ln(1-u), \text{ where } U \sim U[0,1].$$

where u is a realisation (sample) from $U \sim U[0,1]$.
- See other examples discussed in class.