Density Matrix Calculations Using Master Equations In Light-Cavity Interactions

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Bachelor's project

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Abstract

The purpose of this project is to give the reader an understanding of a new approach to light matter Interaction developed by Alexander Kiilerich and Klaus Mølmer. The theory is applied to an optical parametric amplifier through numerical simulations in Python using QuTip. Based on the simulations, it was concluded that the amplifier increases the photon number and that the selected modes for the outgoing wave were not optimal. The system and the theory are also analysed in the interaction picture, using techniques from the squeezing of quantum states. The simulations of the Interaction picture were unsuccessful, this is argued to be a fault of the code itself and the handling of the Hamiltonian.

Resumé

Formålet med dette projekt er at give læseren en forståelse af en ny tilgang til lysstofinteraktion udviklet af Alexander Kiilerich og Klaus Mølmer. Teorien anvendes på et system med en optisk parametrisk forstærker via numeriske simuleringer i Python ved bruge af QuTip. Det blev konkluderet at forstærkeren øger fotonantallet og at de valgte bølgetilstande for den udgående bølge ikke var optimale. Systemet og teorien analyseres også i vekselvirkningsbilledet ved hjælp af teknikker fra klemte kvantetilstande. Vekselvirkings simuleringerne var ikke succesfulde og der argumenteres for at dette var grundet koden selv og dens evne til at håndtere Hamiltonen.

Colophon

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Bachelor's project by Erik Holm Steenberg.

The project was supervised by Professor Klaus Mølmer.

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Preface

This document is a 10 ECTS bachelor project about introducing and using the theoretical approach of [6] to simulate light matter interactions in a cavity system experiencing the effect of an Optical Parametric Amplifier. Besides the obligatory courses of a bachelor in Physics at Aarhus University I also took the Masters course quantum Mechanics II, which I found very useful during this project.

Before we begin I would like to acknowledge and thank the people who helped me make this project possible.
I would like to acknowledge my supervisor professor Klaus Mølmer, who was always ready to explain and advise when needed.
I would also like to thank Aleksander Brøndum Bille, Michel Hardenberg, Nikolaj Roager Christensen and Kristoffer Grimstrup Ragn for proof-reading and giving feedback. I would also like to give an extra thanks to Kristoffer for patiently listening to my endless complaining about the simulations not working.
In addition I would like to thank all the people who were willing to let me try to explain the theory behind this project and all those who gave small tips and feedback.

Contents

Chapter 1

Introduction

What happens when a light wave enters an object? Said simply, a light wave hits a target and the target emits a new wave. Of course it is not as easy to calculate that as it is to say it. The problem is, that if one were to actually describe the interaction between light and matter, it would be necessary to include the system that described the light before it enters the system, what happens inside the material (a cavity system in our case) and all the possible ways that the light could exit the material. Because there are an infinite amount of ways for the wave to exit the cavity system, it would be necessary to combine an infinite amount of systems, which makes any serious attempts at solving the system impossible. The goal of this projects is thus to introduce, apply and discuss one approach of this question proposed by Kiilerich and Mølmer in [6], where the infinite amount of combined systems is avoided.

The project is organized as follows, in chapter 2 background knowledge and the main theory are introduced. In chapter 3 the theory is applied to an optical parametric amplifier and then numerically simulated with a discussion of the results. In chapter 4 the theory is applied in the Interaction picture followed by a discussion of the unsuccessful simulations of this. A conclusion is given in chapter 5.

Chapter 2

Theory

In this chapter, the main theory, referred to as Pulse theory and necessary background knowledge is introduced, starting with notation.

2.1 Notation

The following summary of notation is based upon [5, section 1.2]. We will begin by briefly going through the notation that we will be using in this project. As the reader knows, quantum mechanics builds on the idea of quantization. A particle is described as being in either a specific state or a superpostion of multiple states. We follow the standard, that a state can be represented as a vector and use the standard ket-bra notation to represent these.

When we say that a particle is in a superposition, mathematically we mean that the vector representing the wavefunction is a linear combination of several other vectors or kets. The ket representation of the wavefunction Ψ (the function that represents the probability distribution of the particle) can then be written as

$$|\Psi\rangle = \sum_n c_n |n\rangle . \tag{2.1}$$

Where $|c_n|^2$ is the probability of the particle being in n'th state. $|\Psi\rangle$ and c_n are also normalized to

$$\langle\Psi|\Psi\rangle = 1 \qquad \sum_n |c_n^2| = 1,$$

where $\langle\Psi|$ is the bra representation of Ψ. Additionally, we use a hat $\hat{A}$ to denote an operator. The standard of using matrix notation to represent states and operators is also used. Finally, with the exception of introducing eq. (2.2) and the theory behind the quantum pictures in section 4, we assume Plancks constant $\hbar = 1$.

2.2 Closed and open Systems

As the main theory deals with time evolution of open systems, we need to explain what that is. This explanation is based upon [4]. In the world of quantum mechanics, there are fundamentally two types of systems, closed and open systems. A closed system, is a system where the rest of the universe is ignored. There is no interaction with the outside world. The open system on the other hand does interact with the world outside of the system. An example of this, would be decay or the effects of outside measurements. One consequence of the difference between these types of systems is that the time evolution of states is calculated differently in these systems.

2.2.1 Time evolution in the closed system

The following explanation is based upon [5, section 2.1].
In the case of closed systems, time evolution is determined by the Hamiltonian (the operator that represents the energy of the system) and is given by the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{dt}}|\Psi\rangle = \hat{H}\,|\Psi\rangle\,. \tag{2.2}$$

With this differential equation the time evolution of any given state in a closed system can then be solved. One approach to this is to represent time evolution as an operator, although we won't use this for now, it will become relevant when we discuss the different pictures of quantum mechanics in section 4.1.

2.2.2 Density Matrix, the Master equation & open system time evolution

The introduction of the density operator in this section is based upon [5, section 3.4]. The time evolution in an open system is slightly more complex than that of a closed system, to explain it, we need to introduce the density operator. The density operator is essentially an operator representation of states, defines as

$$\rho = \sum_n p_n \left|\psi_n\right\rangle \left\langle \psi_n\right|, \tag{2.3}$$

where $|\psi_n\rangle$ is a set of states in a given system and the p_n is the relative population of that state with normalization

$$\sum_n p_n = 1. \tag{2.4}$$

The reason why eq. (2.3) is useful, is because it is a way of calculating the expectation value. If you know ρ at a certain time t you can calculate $\langle A(t)\rangle$. Specifically

$$\langle \hat{A}(t)\rangle = \operatorname{Tr}\left(\rho(t)\hat{A}\right), \tag{2.5}$$

where Tr is the trace (the sum of the diagonal elements). As for the time evolution of ρ, we are interested in evolution of the matrix representation of ρ and it is calculated using the Master equation. Note here from now on whenever we write ρ we refer to the matrix representation of the density operator.

The Master equation controls the time evolution of the ρ. We will use what is called the Lindblad Master equation[4, p. 2103](as it is the one that the main theory uses) and it is given as

$$\frac{\mathrm{d}}{\mathrm{dt}}\rho = -i[\hat{H}, \rho] - \sum_i \hat{L}_i \rho \hat{L}_i - \frac{1}{2}\left\{\hat{L}_i^\dagger \hat{L}_i, \rho\right\}, \tag{2.6}$$

where $[,]$ is the commutator and $\{,\}$ is the anticommutator and the $\hat{L}_i$ operators are representations of the interaction (decay/measurement) between the system and the outside world, they are called Lindblad operators. So the sum represents the systems interaction with the outside world. Solving this equation and then using the ρ to find the expectation value, to describe the system, is what was done in the simulations.

We are now ready to introduce the theory that this project is based upon.

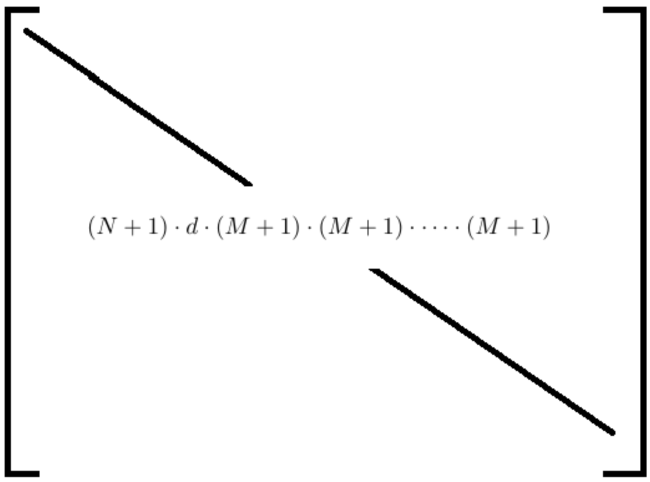


Figure 2.1: The purpose of this matrix is to illustrate the size of the matrix needed to represent ρ. The product represents the dimension of this square matrix with N and M being the maximum number of excitations in the incoming wave and all the possible outgoing waves respectively and d being the dimension of the system absorbing and emitting the waves. The +1 in the product is there to include the ground states. The M's may differ in value as the different outgoing wave may have different levels of possible excitations. This figure was made by Erik Steenberg.

2.3 Pulse Theory

When describing the interaction between a system and wave packets of radiation, like a light beam, it can be extremely difficult to solve eq. (2.6). This comes from the fact that the Hamiltonian has to describe how the shape of the incoming wave package effects the system, the system itself and all the possible outgoing wave packages. This means that we have to combine an seemingly infinite amount of systems, because of the virtually unlimited amount of possible outgoing waves. The result is that the matrix needed to represent ρ becomes enormous. This makes it numerically very challenging to properly solve eq. (2.6). Please see fig. 2.1 for a visualization of the size of the needed matrix. In order to solve this, we introduce a new approach to this problem, by Alexander Holm Kiilerich and Klaus Mølmer. The approach of Kiilerich and Mølmer[6], which we will henceforth refer to as "Pulse

Theory", is to limit our calculations to only one outgoing wave mode (we thus only look at one of the possible ways that the wave may leave the system). This means that instead of having to take care of all the possible outgoing waves and the resulting matrix, we only need to combine 3 systems. This puts serious limits on the size of the matrix representation of ρ.

In pulse theory, we limit ourselves to only focusing on specific modes by doing the following. We describe the incoming wave packages as having been ejected from a previous cavity system (a cavity is a system where it is possible to create standing waves), with certain specific coupling conditions, dependent on the shape of the wave. The same is done for the outgoing wave packages, being described as being absorbed by a third cavity, that also have coupling conditions dependent on its shape, see fig. 2.2 for an illustration of the theory. We denote these pre- and post-cavity conditions as $g_u(t)$ and $g_v(t)$ respectively and they are used to calculate how the shape of the incoming and outgoing wave effects the system. They are defined as

$$g_u(t) = \frac{u^*(t)}{\sqrt{1 - \int_o^t dt'|u(t')|^2}} \quad g_v(t) = \frac{v^*(t)}{\sqrt{\int_o^t dt'|v(t')|^2}}, \tag{2.7}$$

where $u(t)$ and $v(t)$ represents the incoming and outgoing waves and are normalized. Using this approach in [6], Kiilerich and Mølmer were able to construct a Hamiltonian capable of describing the entire setup,

$$\hat{H}(t) = \hat{H}_s(t) + \frac{i}{2}[\sqrt{\gamma}g_u(t)\hat{a}_u^\dagger\hat{c} + \sqrt{\gamma}g_u^*(t)\hat{c}^\dagger\hat{a}_v + g_u(t)g_v^*(t)\hat{a}_u^\dagger\hat{a}_v + H^c]. \tag{2.8}$$

Here $\hat{H}_s$ is the Hamiltonian of the cavity-system, $\hat{a}_{u/v}$ is the annihilation operator of the respective wave packages and $\hat{c}$ is the annihilation operator of the cavity system with a decay of excitation γ. H^c represents the conjugated and transposed versions of the other terms inside the bracket. The second part of eq. (2.8), i.e. the non $\hat{H}_s(t)$ part, describes the energy exchange of the incoming wave, the system and the outgoing wave. Although eq. (2.8) might at first not appear like much of an improvement, we have to remember that we are comparing a matrix representation of that equation with a matrix of the size of fig. 2.1. That is why it is an improvement and it is all thanks

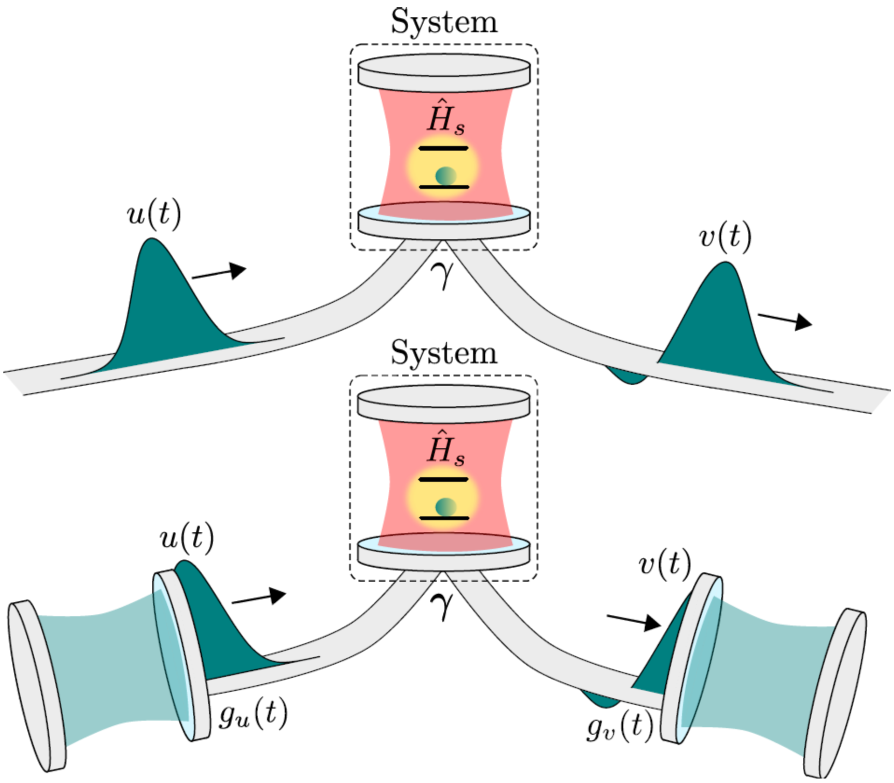


Figure 2.2: This figure visualizes the technique used in the Pulse Theory. It depicts how the $u(t)$ and $v(t)$ are ejected and absorbed by the other systems with the coupling conditions $g_v(t)$ and $g_u(t)$ on to describe the appropriate wave. γ represents the decay of the system being targeted by the incoming wave. $\hat{H}_s$ is the system Hamiltonian. The figure was taken from[6].

to the conditions $g_u(t)$ and $g_v(t)$. The limitation of the size of the matrix representations also applies to ρ. The Lindblad operators are also altered in this approach, in [6] Kiilerich and Mølmer showed that the Lindblad operator representing loss of the wavepackets (for instance, parts of the waves that aren't picked up by the g_v cavity) can be expressed as

$$\hat{L}_o(t) = \sqrt{\gamma}\hat{c} + g_u^*(t)\hat{a}_u + g_v^*(t)\hat{a}_v. \tag{2.9}$$

With $\hat{H}$ and $\hat{L}_0$ having been replaced the obvious question of how are $v(t)$ and $u(t)$ calculated, arises. This is what is so useful about this theory. Since we are the ones who decide what goes into the system, we can pick $u(t)$ to be whatever we want, as long as it is normalized. From the shape of $u(t)$ the shape of $v(t)$ is determined. Our goal is thus to find the most populated modes for $v(t)$. Another really useful thing is that the theory works for unpopulated as well as populated modes, regardless of what shape for $v(t)$ we pick, what will differ is how much of the outgoing wave is picked up in the selected $v(t)$ mode. So instead of calculating a superposition of 25 modes to describe $v(t)$, we instead ask "If the wave starts in this mode (u), then how populated is this mode? What about this mode? Or what about that one?" One mode might pick up 5 another might pickup 95. This also means that if we pick a mode that does not represent any of the possible outgoing waves it won't pick up anything. It is also important to note that we aren't limited to guessing, we can actually calculated the most populated $v(t)$ modes. This is done explicitly in [7] by Kiilerich and Mølmer using eigenmode decomposition of the autocorrelated function and quantum regression theory. The formalism of [6] can be altered to work for other types of setups although we shall stick to the wave cavity interactions, derived in [6]. With the theory in place we are now ready for the simulations and their discussions.

Chapter 3

Simulation Results & Discussion

Having explained the Pulse theory in the previous chapter, we are now ready to introduce, explain and discuss the simulations. The different wave modes and the setup behind the Hamiltonian will also be explained.

3.1 Explanation of The Simulated System

The system that was simulated, was a system with a Hamiltonian

$$\hat{H}_{sys} = \alpha[\hat{c}^2 + (\hat{c}^\dagger)^2], \tag{3.1}$$

where α represents the strength of the optical parametric amplifier. As we are not including $\hbar$ due to our trick of setting $\hbar = 1$, the units is then in Hz, which is the unit of α. An optical parametric amplifier is a non linear component that is cable of creating or annihilating photons in pairs, in our simulated system it is pointing at the cavity system that the $u(t)$ and $v(t)$ modes are entering and leaving. This means that the system that we will simulate is a system where a wave ($u(t)$) enters a cavity, meanwhile the cavity is being affected by an optical parametric amplifier that alters the photons inside it. The strength and how long the amplifier is active will then affect the wave that exits the cavity ($v(t)$). But there is of course several ways that we can look at or slightly alter this scenario. Suppose that we first apply the parametric amplifier on the cavity for a while before the $u(t)$ wave enters or we could alter the strength of α to see how the strength of the amplifier eq. (3.1) effects the outgoing wave.

3.2 The incoming and outgoing wave modes used

In the simulations a gaussian wave was used to represent our $u(t)$

$$u(t) = \frac{1}{\pi^{1/4}\sqrt{\tau}} e^{\frac{-(t-t_P)^2}{2\tau^2}}, \tag{3.2}$$

t_p represented when $u(t)$ peaked, which for our interest told us when most of the wavepackage hit the cavity. $\tau = 1/\gamma$ where γ is the γ of eq. (2.8). In the simulations $\gamma = 1$ was used. For $v(t)$ we tested each simulation twice, using a gaussian and a filtered gaussian (these will be explained in a moment). The reason for why we stuck to these two which where also used in [6], is because actually calculating the optimal $v(t)$–mode was not the goal of this project. The goal was to illustrate the theory and apply it to the optical parametric amplifier, hence the filtered gaussian was selected because it served as a good enough potential mode. We thus limited ourselves to two modes to illustrate the differences in the mode populations. Now for explaining the possible $v(t)$. The gaussian $v(t)$ is just that, a gaussian, so when $v(t)$ is gaussian we will essentially ask how much of the original $u(t)$ remains. As for the filtered gaussian, it is a wavepackage that is altered/distorted through the cavity interaction, it is numerically calculated and how to do so is explained in one of the examples in [6]. The calculations of this filtered gaussian makes certain assumptions that aren't really valid for systems that involve $\hat{c}^2$. The filtered gausssian was the optimal $v(t)$–mode for a system with $\hat{H}_s = \omega_c \hat{c}^\dagger \hat{c}$, where ω_c is a resonance frequency, see [6] for further details. This means that for the filtered gaussian we are thus essentially asking if this optimal mode for another system is also useful for other systems.

3.3 The simulations

The simulations were done using QuTip, a Quantum Toolbox in the programming language Python[3]. The actual code was based upon the code used for the simulations in [6], with extensive modifications to account for the amplifier as well as the Interaction picture version of the simulations, which will be discussed in section 4.3. The simulations could vary in the following ways:

1. Changing α which affects the overall strength of $\hat{H}$.
2. The time for which the Hamiltonian is turned on or off. It was for example, possible to start with $\alpha = 0$ and then at a later point t_0 change α to some other value $\alpha \neq 0$.
3. Whether or not $u(t)$ or the cavity contains any initial photons.
4. Although this wasn't done, we could also vary the amount of states that the photons in $u(t)$, $v(t)$ and the cavity could be in.
5. changing t_p which affected when the $u(t)$ pulse hit the cavity was also an option, but not one used in the given examples.

Because the whole point of the simulations was to find ρ, but writing out the matrix for each time step wouldn't give a good idea about what happens in the system, expectation values using eq. (2.5) were used as a way of visualization. The expectation values of the number operators of the photons $\hat{a}^\dagger \hat{a}$ (where $\hat{a}$ is the annihilation operator) were chosen to help visualize the time evolution of the system. The simulation graphs (not fig. 3.1 which is decay rate over time) depicts the number of photons in the different parts along the vertical axis with time (in units of γ^{-1}) on the horizontal axis. For each simulations there two graphs with different $v(t)$–modes, but the same photon number for the cavity and $u(t)$. All figures are denoted with a vShape="$v(t)$–mode" and whether or not there is an active system Hamiltonian present.
Before we discuss the simulations of the photon numbers, it could be useful to look at how $g_v(t)$ and $g_u(t)$ behaved.

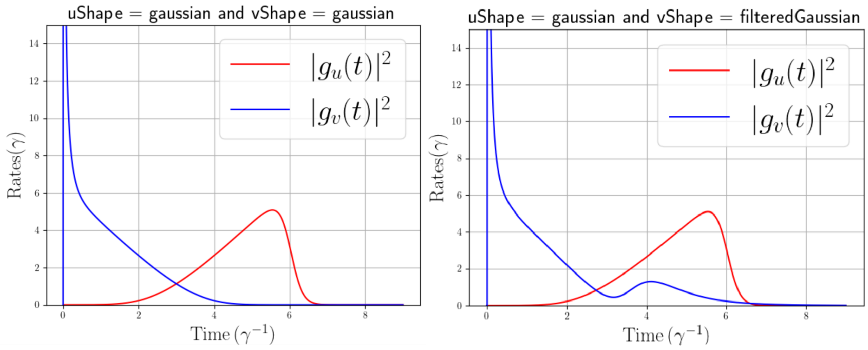


Figure 3.1: This depicts the differences in the $g_v(t)$ and $g_u(t)$ factors for the setup of a wave entering the cavity at $\gamma t_p = 3$. On the left we look at the gaussian $v(t)$–mode and on the right we look at the filtered gaussian $v(t)$–mode. The reason why this is not depicted for each simulation is that the factors do not depend on changes in $\hat{H}$.

The $|g_v|^2$ and $|g_u|^2$ plots in fig. 3.1 illustrates, how large the effect of the different parameters in the Hamiltonian and the Lindblad operator, as seen in eq. (2.8) and eq. (2.9), were. For example, for the simulation, at around $\gamma t = 7$ the non system Hamiltonian part of eq. (2.8) was effectively dead. We can also see on the graph that terms with the factor $g_v(t)$ had more of an effect for the filtered gaussian as it had an extra peak at $\gamma t = 4$. One of the more numerical challenging parts of the simulations was to insure that $g_v(t)$ did not go to infinity, given how it is defined, so certain limitations were put in place, specifically that it was manually set to zero at the start. This may have had a small effect on the system, but it seems unlikely, as it was only set to zero from 0 to either 10^{-6} or 10^{-8} time units.
To get a sense of the layout, we start out by running the system without the amplifier turned on, $\alpha = 0$. Before discussing fig. 3.2 let us briefly discuss how to understand these figures. In fig. 3.2 the red line starts at 1 and then goes to 0, this represents how at $\gamma t = 0$ non of the $u(t)$ photons have entered the cavity, that it then begins to decrease represents how the wave is starting to be absorbed by the cavity and after around $\gamma t = 5$ the entire wave has been absorbed by the cavity. The green graph depicts how there are initial no photons but then for a period of time there are photons, until again there are no photons in the cavity system. That the blue graph starts out in zero and then starts to increase, depicts how initially no photons are

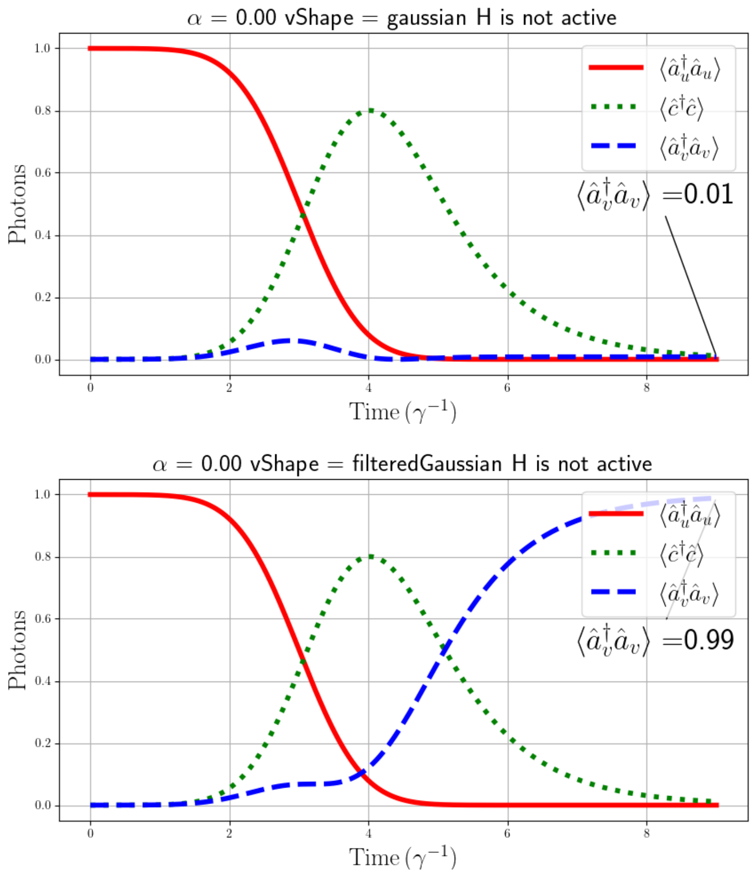
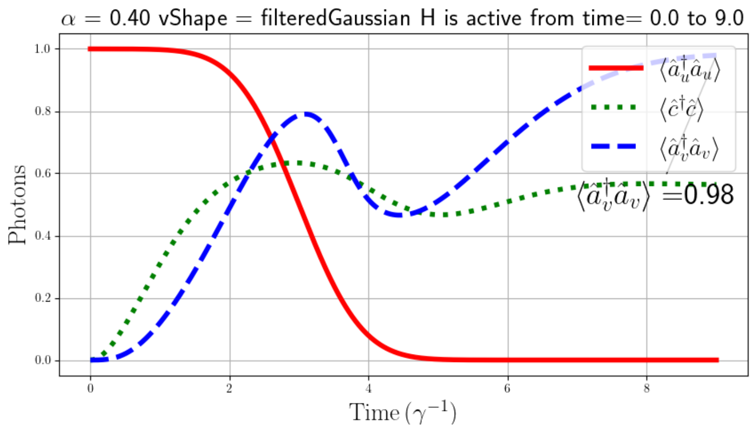


Figure 3.2: With no active $\hat{H}$, the $u(t)$ wave packages enters the cavity, thus increasing the number of photons inside the cavity. As the majority of the $u(t)$ photons enters the cavity the photons begin to leave the cavity. But only a negligible amount of the exiting photons is picked up in the gaussian v–mode, Unlike the filtered mode takes in almost all of the photons.

leaving the cavity system, but then as time goes on photons begin to leave. Now back to discussing the figures. For the fig. 3.2, where the amplifier is not active, it makes sense that the filtered gaussian would work as well as it does, as there are no $\hat{c}^2$ terms present, the wavepackage enters the cavity and then leaves it. So the assumptions for the filtered gaussian aren't violated. But the gaussian $v(t)$ mode cannot be completely ignored as from $\gamma t = 2$ to 4 there is a small jump peak. The filtered $v(t)$ mode also starts here, it just doesn't end. As for the cavity photons, their amount peaks at $\gamma t = 4$ after the primary part of $u(t)$ has entered the cavity. The cavity number is gaussian-like to the left of $\gamma t = 4$ and it is a gaussian to the right of $\gamma t = 4$ it is a gaussian with a smaller width than to the left. So the number rises quickly and then decays slower than it arose.
Having illustrated the system with no active amplifier present, let us turn it on to see what happens as depicted in fig. 3.3.



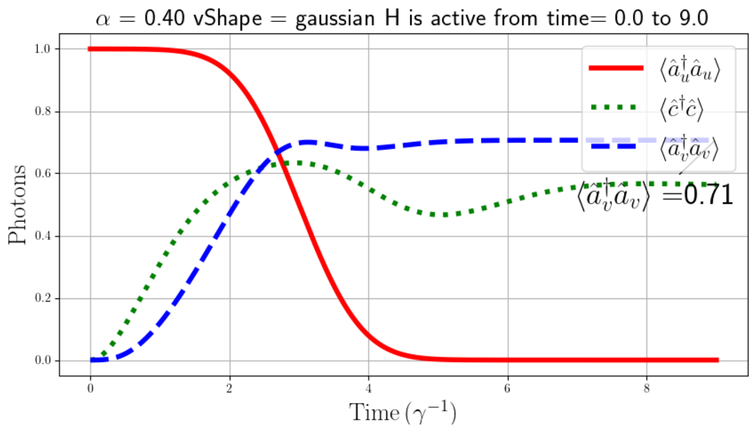


Figure 3.3: With an active amplifier on with no breaks, the cavity photons reach their maximum as $u(t)$ hits the cavity and then goes into a steady state. The $v(t)$ modes are far more evenly split than in fig. 3.2, but we do see that unlike the gaussian that is fairly even after $\gamma t = 3$ while the filtered gaussian varies a lot more after $\gamma t = 3$.

fig. 3.3 depicts an active Hamiltonian from $\gamma t = 0$ to 9, The first thing worth noting is that the cavity number begins to grow immediately, peaking as the peak of $u(t)$ enters the cavity. This makes sense as this is where the cavity receives the majority of the $u(t)$ wave, so its peak is at the point where it receives the most $u(t)$ wave while also receiving external energy. The cavity photons then declines a bit until it reaches steady state. It is to be expected that if the Hamiltonian is never turned off the number of photons will never decrease to zero, as there is a constantly affecting Hamiltonian. The $v(t)$-modes photon numbers tell us that the wave that leaves the cavity is more populated

than the one that entered. Thus confirming that applying an amplifier, amplifies the wavepackage. Noteworthy is the fact that unlike when the amplifier was turned off, The gauss-mode is now quite populated and although still less populated than the filtered mode, it is still quite large. The population of the gaussian v-mode also seems not to be determined by the $u(t)$ wave as it goes into its peak population as $u(t)$ hits the cavity but then it remains in that position, more or less unaltered. The filtered gaussian is again more populated than the gaussian $v(t)$-mode but it no longer contains the vast amount of the population of photons in $v(t)$. It is also interesting to note that the filtered gaussian actually drops as the last of the $u(t)$ is absorbed by the cavity. This might be explained by the fact that the filtered gaussian is starting to become a less and less optimal $v(t)$-mode. Unlike the gaussian $v(t)$-mode, the filtered $v(t)$–mode appears to depend far more on the $u(t)$ wavepackage than the gaussian mode and just as the $u(t)$ wave finishes entering it actually drops to less then the gaussian mode before picking up again. To summarize, fig. 3.3 confirms that the amplifier increases the overall wave and that an active system Hamiltonian effects the $v(t)$-modes.
For the next simulation, the question that inspired it was, what if there was already a photon in the cavity, how would this affect the $v(t)$–modes compared to fig. 3.3.

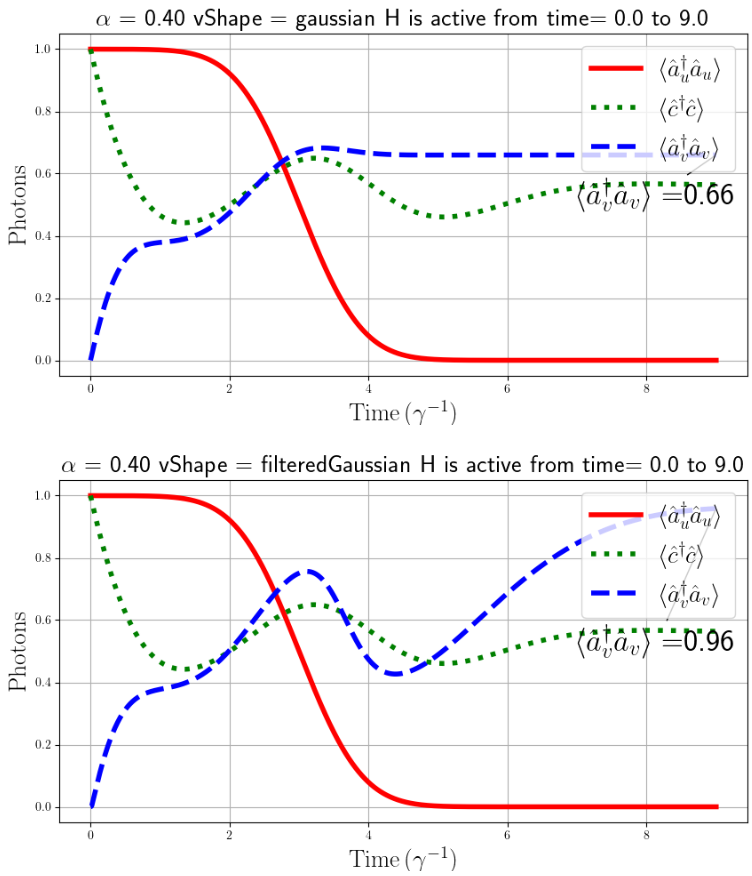


Figure 3.4: We start with a photon in the cavity the starts to decay into the $v(t)$–modes. The cavity photons along with the $v(t)$–modes reach a peak as the majority of the $u(t)$ photons enters the cavity. the cavity then go into steady state and the $v(t)$–modes leaves the cavity with a population split between them in a similar way as to fig. 3.3.

Looking at fig. 3.4 the interesting thing is that the only real difference at the end compared to the previous situation, where there was no initial cavity photon, is that the two $v(t)$-modes are slightly less populated than in the previous setup. Also the $v(t)$–modes are populated quicker than the previous example with slightly lower peaks at $\gamma t = 3$. This leads to the obvious question, why would an initial photon in the cavity have a decreasing effect on the population in our two $v(t)$–modes. One explanation is that other $v(t)$–modes are beginning to become more populated. It is not really a surprise that the filtered gaussian is not affected by the extra cavity photon as this photon most likely decayed into an entirely different mode. So new $v(t)$–modes that are potentially more populated than the gaussian and the filtered gaussian are beginning to make an impact. The cavity goes into a steady state, this is not a surprise when we remember that the amplifier is constantly active.

The next simulation fig. 3.5 was done to explore what would happen, if we started with a photon in the cavity, turn on the $\hat{H}$ then turn it off just as $u(t)$ entered the cavity. For fig. 3.5 where we have the same setup as fig. 3.4, but now we turn off the Hamiltonian at $\gamma t = 3$. In this one, the most eye catching graph is without a doubt the cavity photon number. Starting out with one initial cavity photon, the cavity photon number start to decline, then it takes a turn and starts to increase and then experiences two small peaks, before fading away. We also see that the population of the two $v(t)$–modes combined are only some what larger than the initial wave with $1.35 > 1$. This is quite noteworthy, when we remember that there was also an initial photon in the cavity and there was an amplifier present for a period of time. The gaussian $v(t)$–mode goes into its final state with a slightly higher value than the system where the amplifier wasn't turned off in fig. 3.4. Meanwhile the filtered gaussian $v(t)$–mode reaches its final state with a much lower value than in fig. 3.4. The cavity also does not go into a steady state. The filtered gaussian $v(t)$–mode is in fact less populated than the gaussian $v(t)$–mode. From this we can determine the following. It seems more and more likely that if we want to get an idea of which modes are the most populated, we should probably try alternative modes or try to calculate the optimal $v(t)$–mode. Also not surprisingly the cavity number is quite dependent on the amplifier.

In the previous simulations, the strength of the amplifier α was never larger than 0.4, so an obvious idea for a simulation, was to use a much stronger α, specifically $\alpha = 2$.

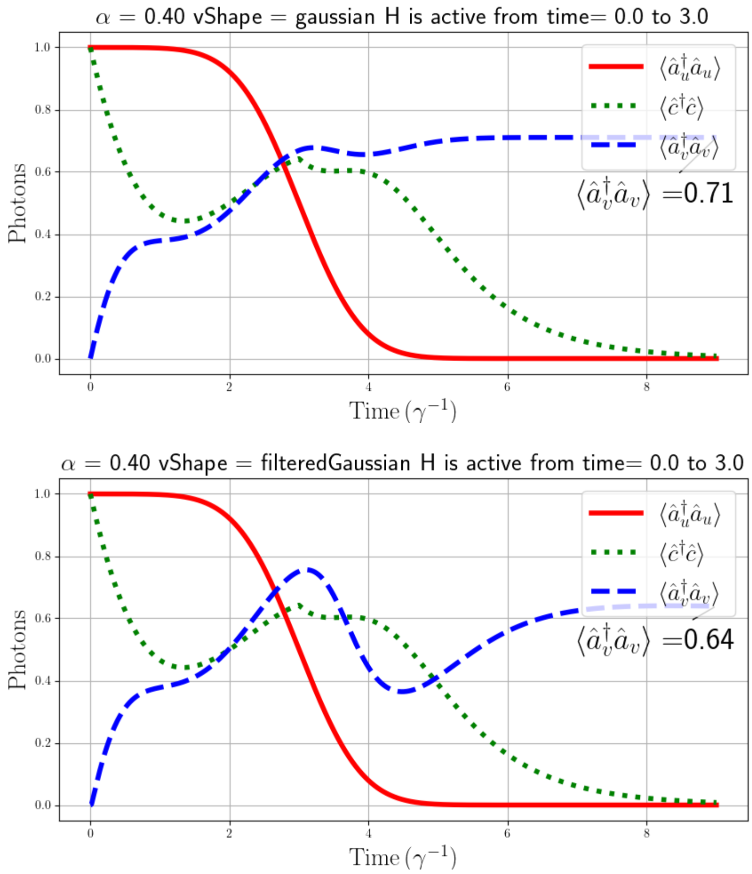


Figure 3.5: We start out with a photon in the cavity that immediately starts to decay, then as the cavity photons starts to increase again the Hamiltonian is turned off and the number of cavity photon drops off again. Then the $u(t)$ enters the cavity and we see another increase in the cavity photons followed by their decrease to zero. As with fig. 3.3 and fig. 3.4 the gaussian $v(t)$–mode is fairly constant after $\gamma t = 3$, while the filtered $v(t)$–mode peaks at $\gamma t = 3$, then it begins to decrease and then it start to increase again.

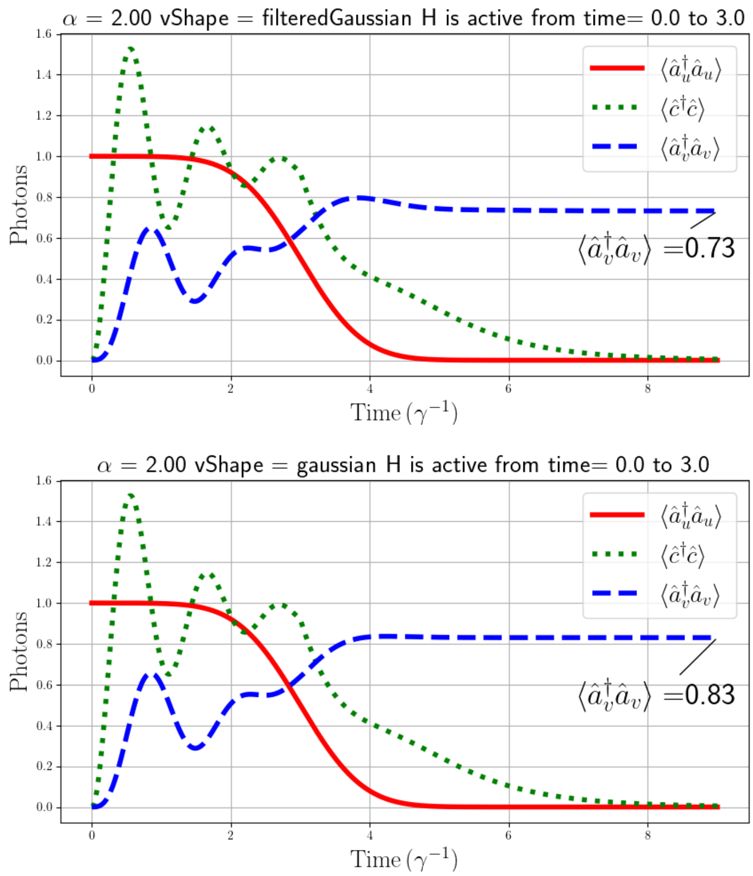


Figure 3.6: A strong Hamiltonian is applied with a stop. We see a massive increase in the cavity photon followed by its oscillating decay. The two $v(t)$–modes are more or less equally filled until around γt = 3.8, after the main part of the $u(t)$ has hit the cavity. The behaviour of the $v(t)$–modes are also very similar.

The simulation depicted in fig. 3.6 uses a Hamiltonian that is at least 5 times as strong as any of the previous ones. In this final mode comparison, something quite interestingly happens. The cavity reaches a higher steady state than the two v-modes. In this system, it appears that the cavity number is not noticeably affected by the $u(t)$ wave, as it's damped oscillating behaviour doesn't change as the $u(t)$ hits the cavity. Given that the cavity photons reaches heights of around 1.5 and we send in 1 photon, it would appear that there are other more populated modes than the ones we used.

In these simulations that we have discussed one idea keeps repeating itself. The chosen modes cannot possibly be the only modes that are populated and it is in fact not even guaranteed that they are the most populated ones either. The decay of a photon initially in the cavity does not appear to be optimally modeled by the gaussian and the filtered gaussian modes, but this is again not surprising as the filtered gaussian was never meant to do that. It was mentioned that the amount of possible states for the photons to be in, was something that could but wasn't altered. To insure that the amplifier would have a noticeable effect, the simulations were set up in such a way, that the photons could in three states for $u(t)$, $v(t)$ and the cavity. Increasing these numbers did not affect simulation results (although it did increase the time required to run the simulation), but decreasing them did. An obvious follow up to this project would be to try and calculate the optimal $v(t)$–mode and then repeat the simulations.
With the discussion of the simulations over, it is now to time to talk about the second part of this project, the translation of the setup and the theory into the interaction picture.

Chapter 4

Interaction picture

This chapter introduces and explains the pictures in quantum mechanics and how the Interaction picture was applied to the Pulse theory. The application and lack of results of the Interaction picture in the simulations is also discussed.

4.1 The Schrödinger and Heisenberg pictures

This section is based upon [5, section 2.2]. In section 2.2.1 it was mentioned, that time evolution can be represented as an operator. If we have a state at time equal zero, $|\Psi(0)\rangle$, then we can represent the state at a later point in time as $|\Psi(t)\rangle = \hat{U}(t)\,|\Psi(0)\rangle$. This means that if we wished to calculate the expectation value of an operator $\hat{A}$ then we could write it as

$$\langle \hat{A}(t) \rangle = \langle \psi(0)|\, \hat{U}^{\dagger}(t)\hat{A}\hat{U}(t)\,|\psi(0)\rangle\,. \tag{4.1}$$

This equation is an excellent way of introducing the pictures in quantum mechanics. The quantum pictures are different ways of viewing / understanding what is effected by the time evolution operator in eq. (4.1). This is because the time evolution operators can be applied to either the states or the operators, but not both at the same time. So the one way of looking at it is that $\hat{U}(t)$ works on the states

$$[\langle \psi(0)| \hat{U}^{\dagger}(t)] \hat{A} [\hat{U}(t) |\psi(0)\rangle] \tag{4.2}$$

$$= \langle \psi(t)| \hat{A} |\psi(t)\rangle . \tag{4.3}$$

This is what is called the Shrödinger picture, the states evolve through time while the operators are constant.
Of course it is equally legitimate to look at it the other way around. That the states are constant and the operators are the ones that change over time,

$$\langle \psi(0)| [\hat{\mathrm{U}}^{\dagger}(\mathrm{t})\hat{\mathrm{A}}\hat{\mathrm{U}}(\mathrm{t})] |\psi(0)\rangle \tag{4.4}$$

$$= \langle \psi(0)| \hat{\mathrm{A}}(\mathrm{t}) |\psi(0\rangle \tag{4.5}$$

$$= \langle \hat{A}(t)\rangle. \tag{4.6}$$

This is called the Heisenberg picture. Now it is an individual choice whether to use the Schrödinger picture or the Heisenberg picture. They are useful for different things. For example, if you already have the expectation value for an operator, but now you want to find the derivative of the expectation value, it can be very useful to work in the Heisenberg picture. In the simulations and the Pulse theory, we use the Schrödinger picture. This can be seen by the fact that it is the density operator, the representation of the states, that evolves in time. From the density operator we then calculate the expectation value of the operators, we never ask about the time evolution of the operators themselves. Besides the Heisenberg and Schrödinger pictures, there is also the Interaction picture, which is a third quantum picture. that we will now explain.

4.2 Interaction picture

This section is based upon [5, section 5.5]. A problem that both the Schrödinger and the Heisenberg pictures have, is that the complexity and time dependence of the Hamiltonian, can often make it quite difficult to calculate the time operator $\hat{U}(t)$. So when dealing with the time dependent Hamiltonian one may use the Interaction picture. The Interaction picture is essentially a middle ground between the

Heisenberg and Schrödinger pictures. What is done in the Interaction picture is that the Hamiltonian is split into a time independent and an time dependent part

$$\hat{H} = \hat{H}_0 + \hat{V}(t), \tag{4.7}$$

where $\hat{V}(t)$ is the time dependent part and $\hat{H}_0$ the time independent part. The Hamiltonian is split up and then the two terms affect different things. H_0 is what affects the operators

$$\hat{A} \longrightarrow \hat{A}_I = e^{itH_0/\hbar}\hat{A}e^{-itH_0/\hbar}, \tag{4.8}$$

where the I in $\hat{A}_I$ denote that the operator is represented in the Interaction picture the same goes for the states with $|\rangle_I$. $\hat{V}_I$ is the time dependent part of the Hamiltonian in eq. (4.7) represented in the Interaction picture and is what drives the evolution of the states. Interestingly the time evolution of the states is described by a very Schrödinger-like equation

$$i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle_I = \hat{V}_I\left|\Psi(t)\right\rangle_I. \tag{4.9}$$

Now it can be complicated setting up all the terms, so like the other pictures, it is only sometimes that it makes sense to use the Interaction picture.

4.3 Input-Output in the Interaction picture

This project had two goals. The first was to take older pieces of code to create a python script that could simulate the system of eq. (3.1) in the standard approach where eq. (2.6) is used. The second goal was to see if it was possible to take the equations of (2.6), (2.8) and (2.9) and translate them into the Interaction picture, to then see if our simulations would give the same results as those in the Schrödinger picture (which is what the simulations in chapter 3 was referred to as throughout the project). The following equations are thus not sourced from anywhere, they were derived as a part of the project itself using the Interaction picture and the equations of [6].

The approach was as follows, eq. (2.8) was treated as eq. (4.7) where $\hat{H}_s$ was the $\hat{H}_0$. The equations

$$\rho = e^{-iH_0 t}\rho_I e^{iH_0 t} \quad \rho_I = e^{iH_0 t}\rho e^{-iH_0 t} \tag{4.10}$$

$$V = e^{-iH_0 t}V_I e^{iH_0 t} \quad V_I = e^{iH_0 t}V e^{-iH_0 t} \tag{4.11}$$

$$L_i = e^{-iH_0 t}L_{iI} e^{iH_0 t} \quad L_{iI} = e^{iH_0 t}L_i e^{-iH_0 t}, \tag{4.12}$$

were then used to derive the Interaction versions of the equations (2.6), (2.8) and (2.9), with eq. (2.6) becoming

$$\frac{\mathrm{d}\rho_I}{\mathrm{d}t} = -i[V_I, \rho_I] + \sum_{i=0} L_{iI}\rho_I L_{iI}^\dagger - \frac{1}{2}\{L_{iI}^\dagger L_{iI}, \rho_I\}. \tag{4.13}$$

A brief comment on this equation, ρ and the Lindblad operators were of course replaced by their Interaction counterparts, but $\hat{H}$ was replaced by $\hat{V}_I$ the Interaction version of the non $\hat{H}_0$ part of the hamiltonian. From an Interaction picture in the closed system point of view this makes sense as eq. (4.9) tells us that the time evolution of the states is driven by $\hat{V}_I$ so it is not surprising that the same applies for ρ in the open system. Now if we look at how $\hat{V}_I$ actually looks

$$V_I = e^{i\hat{H}_0 t}\frac{i}{2}[\sqrt{\gamma}g_u(t)\hat{a}_u^\dagger\hat{c} + \sqrt{\gamma}g_u^*(t)\hat{c}^\dagger\hat{a}_v + g_u(t)g_v^*(t)\hat{a}_u^\dagger\hat{a}_v + H^c]e^{-i\hat{H}_0 t}, \tag{4.14}$$

we might notice that this could get really difficult to calculate given the $e^{\pm iH_0 t}$ terms. Fortunately the Hamiltonian is on a form that allows us to use a neat trick from squeezed states. A squeezed state is a state where the variance for either momentum or space is below the vacuum state, that is to say that the variance is ultra low in one of those parameters[1]. If an operator can then be written as a squeezing operator[2] on the form

$$\hat{S}(z) = \exp\left(\frac{1}{2}(z^*\hat{c}^2 - z\hat{c}^{\dagger 2})\right), \qquad z = re^{i\theta}, \tag{4.15}$$

where $\hat{c}$ is an annihilation operator, then the following relations are valid

$$\hat{S}^\dagger(z)\hat{c}\hat{S}(z) = \hat{c}\cosh r - e^{i\theta}\hat{c}^\dagger\sinh r \tag{4.16}$$

$$\hat{S}^\dagger(z)\hat{c}^\dagger\hat{S}(z) = \hat{c}^\dagger\cosh r - e^{-i\theta}\hat{c}\sinh r. \tag{4.17}$$

$e^{\pm i\hat{H}_0 t}$ can then be expressed as eq. (4.15) if $z = z_0 = 2\alpha t e^{i\frac{\pi}{2}}$, where $e^{i\hat{H}_0 t} = \hat{S}^\dagger(z_0)$ and $\hat{S}(z_0)$ for $e^{-i\hat{H}_0 t}$. Using the fact that H_0 has no effect on $\hat{a}_u$ or $\hat{a}_v$, we can let it slip past them without incident and $\hat{V}_I$ becomes

$$\hat{V}_I = \frac{i}{2}[\sqrt{\gamma} g_u(t)\hat{a}_u^\dagger e^{iH_0 t}\hat{c}e^{-iH_0 t} + \sqrt{\gamma} g_u^*(t) e^{iH_0 t}\hat{c}^\dagger e^{-iH_0 t}\hat{a}_v \tag{4.18}$$
$$+ g_u(t)g_v^*(t)\hat{a}_u^\dagger \hat{a}_v + e^{iH_0 t}H^c e^{-iH_0 t}],$$
$$\hat{V}_I = \frac{i}{2}[\sqrt{\gamma} g_u(t)\hat{a}_u^\dagger \hat{S}^\dagger(z_0)\hat{c}\hat{S}(z_0) + \sqrt{\gamma} g_u^*(t)\hat{S}^\dagger(z_0)\hat{c}^\dagger \hat{S}(z_0)\hat{a}_v \tag{4.19}$$
$$+ g_u(t)g_v^*(t)\hat{a}_u^\dagger \hat{a}_v + \hat{S}^\dagger(z_0)H^c\hat{S}(z_0)],$$

using the identities and writing out V_I completely without using $\hat{H}^c$, we get

$$\begin{aligned} V_I = {} & \frac{i}{2}[\sqrt{\gamma} g_u(t)\hat{a}_u^\dagger \left[\hat{c}\cosh 2\alpha t - i\hat{c}^\dagger \sinh 2\alpha t\right] \\ & + \sqrt{\gamma} g_v^*(t)\left[\hat{c}^\dagger \cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right]\hat{a}_v + g_u(t)g_v^*(t)\hat{a}_u^\dagger \hat{a}_v] \\ & - \frac{i}{2}[\sqrt{\gamma} g_u^*(t)\left[\hat{c}^\dagger \cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right]\hat{a}_u \\ & + \sqrt{\gamma} g_v(t)\hat{a}_v^\dagger \left[\hat{c}\cosh 2\alpha t - i\hat{c}^\dagger \sinh 2\alpha t\right] + g_u^*(t)g_v(t)\hat{a}_v^\dagger \hat{a}_u]. \end{aligned} \tag{4.20}$$

And we now have an expression for V_I, large and complicated as it might be. Using the same trick as for V_I, we can also derive an expression for $\hat{L}_{0I}$

$$\hat{L}_{0I} = \hat{S}^\dagger(2i\alpha t)\hat{L}_0\hat{S}(2i\alpha t)$$
$$\hat{L}_{0I} = \sqrt{\gamma}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^\dagger \sinh 2\alpha t\right] + g_u^*(t)\hat{a}_u + g_v^*(t)\hat{a}_v. \tag{4.21}$$

Having derived the Interaction expression for the equations that were used to simulate the system, several attempts were made to simulate the system using these new equations. For a more in depth explanation of how these equations were derived see chapter 6. As the Interaction picture is simply another way of describing a system, it was believed that a simulation of these new Interaction equations would give the same results. This was unfortunately not the case and the simulation of the Interaction equations were not successful. A discussion on the errors and possible cause of the failed simulations will now follow.

4.4 Interaction simulation discussion

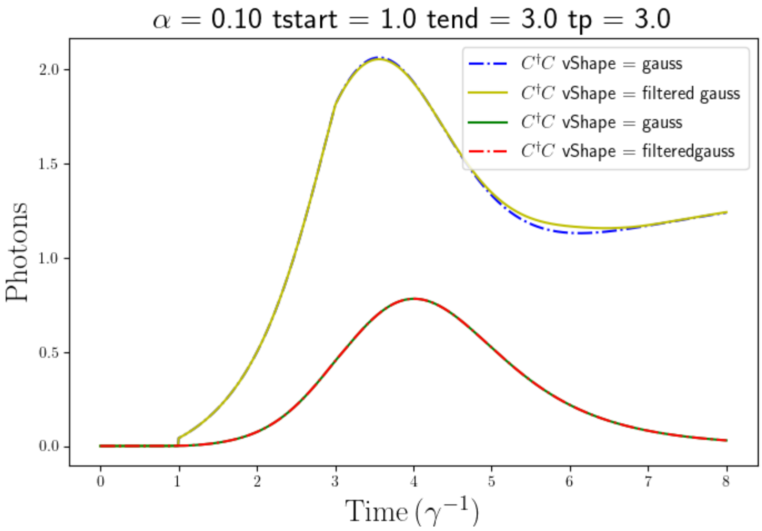


Figure 4.1: On the figure we have photon number over time, with the expectation value of the photon number operators of the cavity system with different $v(t)$–modes in different pictures plotted. The lines in the different picture should follow exactly, they don't. The different modes do not agree within the Interaction picture. That the photon number for the Interaction picture jumps straight up as system Hamiltonian is turned on at $\gamma t = 1$ is also unfortunate.

The main explanation for why the simulations of the Interaction picture failed, is that the code was not able to properly handle the system Hamiltonian. An indicator of this was that eq. (4.18) had several hyperbolic terms of $2\alpha t$. This meant that simulations were only realistically possible for very low values of α (below 0.3) and even these would eventually grow very very large. Another example is seen on fig. 4.1, although it is not as clear as it was for large values α, the mean value of the cavity photon number would just jump when the amplifier was activated as is seen at $\gamma t = 1$ it just flat out jumps, which seems quite unrealistic. But the even worse part, which is also commented on in the caption of the graph, is that different cavity photon numbers were reached, depending on which $v(t)$–mode was used. Small as the deviation is, it is a clear sign of a flaw in the code

as the $v(t)$–modes themselves cannot have any effect on the cavity system that it is leaving. It is also worth noting that the Interaction simulations were actually in agreement with the other simulations if the amplifier was not active, again suggesting a problem with how the Hamiltonian was handled in the code.
The limiting effect on the size of α in the Interaction picture is why only one of the Schrödinger simulations had a $\alpha > 1$. Wanting to compare the different pictures, the same value for α was selected for both pictures. The Interaction simulations where $\alpha > 0.3$ required an extensive time to run and for $\alpha = 1$ they were simply not possible, as an example of how big the values got let us look at the $\cosh(2\alpha t)$, for $\alpha = 1$ after just 2 time units $\gamma t = 2$ $cosh(2) \approx 27.3$, then $\cosh(6) \approx 201$ and then $\cosh(8) \approx 1490$. Future attempts at simulating the Interaction picture, must take the effects of the Hamiltonian in the code seriously, not that it wasn't here, it was simply realized too late and several attempts were made to fix it, but all ultimately failed. It also did not appear to be a question of processor capacity and several retreading of the derivation of the Interaction equations didn't reveal any analytical mistakes. It thus appears that the error is a mistake in the code itself.

Chapter 5

Conclusion

To conclude. We have successfully presented and simulated the theory laid out in [6] and [7]. From the simulation of a system experiencing an optical parametric amplifier using this theory, we can conclude that the gaussian and the filtered gaussian cannot possibly be the only outgoing modes for $v(t)$ for our system. This is probably not a surprise as the optimal modes were never calculated, but still, it is a noteworthy fact. The simulations show how applying an optical parametric amplifier does indeed increase the photon number, but exactly how large this increase is cannot be estimated, as the optimal mode was not calculated. We also introduced the theoretical framework for simulating the Interaction picture version of the chosen system, even though the simulations themselves were unsuccessful. Recommendations for how to simulate the Interaction picture were given with a primary focus on the effects of the Hamiltonian. Although the project was limited to calculating the expectation values of the number operators we must stress that because we can calculate the density matrix we can find the expectation values of any operator that we want. It would be very interesting to reattempt a simulation of the Interaction equations. If not specifically with the ones derived in this project, then some other system, with a different Hamiltonian. Perhaps a system where the Interaction expression does not lead to hyperbolic terms. It would also be interesting to test if the outgoing states of the used system were actually squeezed states. This would require that the uncertainties of position and the momentum were calculated and then compared. Finally, it would be interesting to look at a genuine experiment, simulate the setup, calculate the optimal $v(t)$–mode and then compare the experimental data with the optimal $v(t)$–mode as well as with the ones used in this project.

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Chapter 6

Appendix

This appendix exists to give a more detailed walkthrough of the equations derived in chapter 4

$$\frac{\mathrm{d}\rho_I}{\mathrm{d}t} = -i[V_I, \rho_I] + \sum_{i=0} L_{iI}\rho_I L_{iI}^{\dagger} - \frac{1}{2}\{L_{iI}^{\dagger}L_{iI}, \rho_I\}. \tag{6.1}$$

This is the Master equation in the Interaction picture, what follows is its derivation.
The following techniques and terms were used, for the operators $\hat{A}$ and $\hat{B}$ we had

$$(AB)^{\dagger} = B^{\dagger}C^{\dagger}, \tag{6.2}$$

and

$$\rho = e^{-iH_0 t}\rho_I e^{iH_0 t} \tag{6.3}$$
$$V_I = e^{iH_0 t} V e^{-iH_0 t} \tag{6.4}$$
$$\rho_I = e^{iH_0 t}\rho e^{-iH_0 t}, \tag{6.5}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[\hat{H}, \rho] + \sum_{i=0} \hat{L}_i\rho\hat{L}_i^{\dagger} - \frac{1}{2}\{\hat{L}_i^{\dagger}\hat{L}_i, \rho\}. \tag{6.6}$$

We insert eq. (6.3) i eq. (6.6), first on the left side and then on the right side

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-iH_0 t}\rho_I e^{iH_0 t}\right) = -iH_0 e^{-iH_0 t}\rho_I e^{iH_0 t} + e^{-iH_0 t}\frac{\mathrm{d}\rho_I}{\mathrm{d}t}e^{iH_0 t} + e^{-iH_0 t}\rho_I iH_0 e^{iH_0 t} \tag{6.7}$$

$$
\begin{aligned}
[\hat{H}, \rho] &= -i[H_0 + V, e^{-iH_0 t}\rho_I e^{iH_0 t}] && (6.8)\\
&= -iH_0 e^{-iH_0 t}\rho_I e^{iH_0 t} - iVe^{-iH_0 t}\rho_I e^{iH_0 t} + e^{-iH_0 t}\rho_I e^{iH_0 t} iH_0 \\
&+ e^{-iH_0 t}\rho_I e^{iH_0 t} iV. && (6.9)
\end{aligned}
$$

We see that the terms cancel and we get

$$
\begin{aligned}
e^{-iH_0 t}\frac{\mathrm{d}\rho_I}{\mathrm{d}t}e^{iH_0 t} &= -iVe^{-iH_0 t}\rho_I e^{iH_0 t} + e^{-iH_0 t}\rho_I e^{iH_0 t} iV \\
&+ \sum_{i=0} \hat{L}_i \rho \hat{L}_i^\dagger - \frac{1}{2}\{\hat{L}_i^\dagger \hat{L}_i, \rho\}. && (6.10)
\end{aligned}
$$

We will wait with the sum for a moment. We now multiply the exponents on the sides away and thus we get the time dependent ρ term on its own

$$
\begin{aligned}
\frac{\mathrm{d}\rho_I}{\mathrm{d}t} &= -ie^{iH_0 t}Ve^{-iH_0 t}\rho_I e^{iH_0 t}e^{-iH_0 t} + e^{iH_0 t}e^{-iH_0 t}\rho_I e^{iH_0 t} iVe^{-iH_0 t} + \\
&+ e^{iH_0 t}\left(\sum_{i=0} \hat{L}_i \rho \hat{L}_i^\dagger - \frac{1}{2}\{\hat{L}_i^\dagger \hat{L}_i, \rho\}\right)e^{-iH_0 t} && (6.11)\\
&= -iV_I\rho_I + i\rho_I V_I + e^{iH_0 t}\left(\sum_{i=0} \hat{L}_i \rho \hat{L}_i^\dagger - \frac{1}{2}\{\hat{L}_i^\dagger \hat{L}_i, \rho\}\right)e^{-iH_0 t}. && (6.12)
\end{aligned}
$$

Having used the fact that $1 = e^{iH_0 t}e^{-iH_0 t}$, we reintroduce the commutator relationship

$$
\begin{aligned}
\frac{\mathrm{d}\rho_I}{\mathrm{d}t} &= -i[V_I, \rho_I] + \sum_{i=0} e^{iH_0 t}\hat{L}_i e^{-iH_0 t}e^{iH_0 t}\rho e^{-iH_0 t}e^{iH_0 t}\hat{L}_i^\dagger e^{-iH_0 t} && (6.13)\\
&- \frac{1}{2}\Big(e^{iH_0 t}\hat{L}_i^\dagger e^{-iH_0 t}e^{iH_0 t}\hat{L}_i e^{-iH_0 t}e^{iH_0 t}\rho e^{-iH_0 t} \\
&+ e^{iH_0 t}\rho e^{-iH_0 t}e^{iH_0 t}\hat{L}_i^\dagger e^{-iH_0 t}e^{iH_0 t}\hat{L}_i e^{-iH_0 t}\Big). && (6.14)
\end{aligned}
$$

We now use the following

$$
\begin{aligned}
L_{iI} &= e^{iH_0 t}L_i e^{-iH_0 t} && (6.15)\\
L_{iI}^\dagger &= e^{iH_0 t}L_i^\dagger e^{-iH_0 t} && (6.16)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\rho_I}{\mathrm{d}t} &= -i[V_I, \rho_I] + \sum_{i=0} L_{iI}\rho_I L_{iI}^\dagger - \frac{1}{2}\left(L_{iI}^\dagger L_{iI}\rho_I + \rho_I L_{iI}^\dagger L_{iI}\right) && (6.17)\\
\frac{\mathrm{d}\rho_I}{\mathrm{d}t} &= -i[V_I, \rho_I] + \sum_{i=0} L_{iI}\rho_I L_{iI}^\dagger - \frac{1}{2}\{L_{iI}^\dagger L_{iI}, \rho_I\}. && (6.18)
\end{aligned}
$$

And thus the Interaction representation of the Master equation is achieved.

The Hamiltonian and the implementation of L_I and V_I

We now show how the used cases of the Hamiltonian, V_I and L_I were derived. The equations are given as

$$\hat{H}(t) = \hat{H}_0(t) + \frac{i}{2}\Bigg(\sqrt{\gamma} g_u(t)\hat{a}_u^\dagger \hat{c} + \sqrt{\gamma} g_v^*(t)\hat{c}^\dagger \hat{a}_v + g_u(t)g_v^*(t)\hat{a}_u^\dagger \hat{a}_v - \sqrt{\gamma} g_u^*(t)\hat{c}^\dagger \hat{a}_u - \sqrt{\gamma} g_v(t)\hat{a}_v^\dagger \hat{c} - g_u^*(t)g_v(t)\hat{a}_v^\dagger \hat{a}_u \Bigg) \tag{6.19}$$

$$= \hat{H}_0(t) + \hat{V}(t). \tag{6.20}$$

Here $\hat{H}_0 = \hat{H}_s$, we then set $i = 0$ and introduce $\hat{H}_0$ and L_i and calculate V_I and L_{iI}

$$\hat{H}_0 = \alpha \left((\hat{c}^\dagger)^2 + (\hat{c})^2 \right) \tag{6.21}$$

$$\hat{L}_0 = \sqrt{\gamma}\hat{c} + g_u^*(t)\hat{a}_u + g_v^*(t)\hat{a}_v. \tag{6.22}$$

To calculate V_I and L_{iI} we use the squeezing operator

$$\hat{S}(z) = \exp\left(\frac{1}{2}(z^*\hat{c}^2 - z\hat{c}^{\dagger 2}) \right), \qquad z = r\, e^{i\theta} \tag{6.23}$$

$$\hat{S}^\dagger(z)\hat{c}\hat{S}(z) = \hat{c}\cosh r - e^{i\theta}\hat{c}^\dagger \sinh r \tag{6.24}$$

$$\hat{S}^\dagger(z)\hat{c}^\dagger\hat{S}(z) = \hat{c}^\dagger \cosh r - e^{-i\theta}\hat{c}\sinh r \tag{6.25}$$

$$\tag{6.26}$$

Then we get

$$\hat{S}(2i\alpha t) = \exp\left(\frac{1}{2}((2i\alpha t)^*\hat{c}^2 - 2i\alpha t\hat{c}^{\dagger 2}) \right) \qquad z = z_0 = 2\alpha t\, e^{i\frac{\pi}{2}} \tag{6.27}$$

$$\hat{S}^\dagger(z_0)\hat{c}\hat{S}(2i\alpha t) = \hat{c}\cosh 2\alpha t - i\hat{c}^\dagger \sinh 2\alpha t \tag{6.28}$$

$$\hat{S}^\dagger(z_0)\hat{c}^\dagger\hat{S}(z_0) = \hat{c}^\dagger \cosh 2\alpha t + i\hat{c}\sinh 2\alpha t \tag{6.29}$$

$$\hat{L}_{0I} = \hat{S}^\dagger(z_0)\hat{L}_0\hat{S}(z_0) = \sqrt{\gamma}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^\dagger \sinh 2\alpha t\right] + \hat{H}_L \tag{6.30}$$

$$\hat{L}_{0I}^\dagger = \hat{S}^\dagger(z_0)\hat{L}_0^\dagger\hat{S}(z_0) = \sqrt{\gamma}\left[\hat{c}^\dagger \cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right] + \hat{H}_L^\dagger \tag{6.31}$$

$$\text{where} \qquad \hat{H}_L = g_u^*(t)\hat{a}_u + g_v^*(t)\hat{a}_v. \tag{6.32}$$

With that we have our expression for $\hat{L}_{0I}$. We now calculate V_I

$$
\begin{aligned}
V_I &= e^{iH_0 t}\hat{V}e^{-iH_0 t} = \hat{S}^{\dagger}(2i\alpha t)\hat{V}\hat{S}(2i\alpha t) \\
&= \frac{i}{2}\Bigg[\sqrt{\gamma}g_u(t)\hat{a}_u^{\dagger}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^{\dagger}\sinh 2\alpha t\right] \\
&+ \sqrt{\gamma}g_v^{*}(t)\left[\hat{c}^{\dagger}\cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right]\hat{a}_v + g_u(t)g_v^{*}(t)\hat{a}_u^{\dagger}\hat{a}_v\Bigg] \\
&- \frac{i}{2}\Bigg[\sqrt{\gamma}g_u^{*}(t)\left[\hat{c}^{\dagger}\cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right]\hat{a}_u \\
&+ \sqrt{\gamma}g_v(t)\hat{a}_v^{\dagger}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^{\dagger}\sinh 2\alpha t\right] + g_u^{*}(t)g_v(t)\hat{a}_v^{\dagger}\hat{a}_u\Bigg].
\end{aligned}
\tag{6.33}
$$

With this we have now derived the used expressions in the project.

Additional expressions

As is stated the simulations of the Interaction picture were unsuccessful, hence there aren't given any simulation results. However lots of interesting expression could be derived by applying the approach just used. What follows are three of such expressions derived but not used.

The expressions for V_I and L_{0I} are fairly large so why not set things to zero and see what happens

$$g_u(t) = g_v(t) = g_u^{*}(t) = g_v^{*}(t) = 0 \tag{6.34}$$

$$\hat{H}_L = 0 \tag{6.35}$$

$$\hat{L}_{0I} = \sqrt{\gamma}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^{\dagger}\sinh 2\alpha t\right] \tag{6.36}$$

$$\hat{L}_{0I}^{\dagger} = \sqrt{\gamma}\left[\hat{c}^{\dagger}\cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right] \tag{6.37}$$

$$V_I = 0 \tag{6.38}$$

$$\frac{\mathrm{d}\rho_I}{\mathrm{d}t} = L_{0I}\rho_I L_{0I}^{\dagger} - \frac{1}{2}\{L_{0I}^{\dagger}L_{0I}, \rho_I\} \tag{6.39}$$

or

$$g_u(t) = g_u^*(t) = 0 \tag{6.40}$$

$$\hat{H}_L = g_v(t)\hat{a}_v \qquad \hat{H}_L^{\dagger} = g_v^*(t)\hat{a}_v^{\dagger} \tag{6.41}$$

$$\hat{L}_{0I} = \sqrt{\gamma}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^{\dagger}\sinh 2\alpha t\right] + g_v(t)\hat{a}_v \tag{6.42}$$

$$\hat{L}_{0I}^{\dagger} = \sqrt{\gamma}\left[\hat{c}^{\dagger}\cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right] + g_v^*(t)\hat{a}_v^{\dagger} \tag{6.43}$$

$$V_I = \frac{i}{2}\Bigg[\sqrt{\gamma}g_v^*(t)\left[\hat{c}^{\dagger}\cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right]\hat{a}_v \tag{6.44}$$

$$-\sqrt{\gamma}g_v(t)\hat{a}_v^{\dagger}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^{\dagger}\sinh 2\alpha t\right]\Bigg]$$

$$\frac{\mathrm{d}\rho_I}{\mathrm{d}t} = -i[V_I, \rho_I] + L_{0I}\rho_I L_{0I}^{\dagger} - \frac{1}{2}\{L_{0I}^{\dagger}L_{0I}, \rho_I\} \tag{6.45}$$

or

$$g_v(t) = g_v^*(t) = 0 \tag{6.46}$$

$$\hat{H}_L = g_u^*(t)\hat{a}_u \qquad \hat{H}_L^{\dagger} = g_u(t)\hat{a}_u^{\dagger} \tag{6.47}$$

$$\hat{L}_{0I} = \sqrt{\gamma}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^{\dagger}\sinh 2\alpha t\right] + g_u^*(t)\hat{a}_u \tag{6.48}$$

$$\hat{L}_{0I}^{\dagger} = \sqrt{\gamma}\left[\hat{c}^{\dagger}\cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right] + g_u(t)\hat{a}_u^{\dagger} \tag{6.49}$$

$$V_I = \frac{i}{2}\Bigg\{\sqrt{\gamma}g_u(t)\hat{a}_u^{\dagger}\left[\hat{c}\cosh 2\alpha t - i\hat{c}^{\dagger}\sinh 2\alpha t\right] \tag{6.50}$$

$$-\sqrt{\gamma}g_u^*(t)\left[\hat{c}^{\dagger}\cosh 2\alpha t + i\hat{c}\sinh 2\alpha t\right]\hat{a}_u\Bigg\}$$

$$\frac{\mathrm{d}\rho_I}{\mathrm{d}t} = -i[V_I, \rho_I] + L_{0I}\rho_I L_{0I}^{\dagger} - \frac{1}{2}\{L_{0I}^{\dagger}L_{0I}, \rho_I\}. \tag{6.51}$$