Analytical Neutrino structure in Einstein DeSitter

Analytisk Neutrino struktur i Einstein DeSitter

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Master's Thesis in Physics
June 2023

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Colophon

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Typeset by the author with LaTeX and the memoir document class, using Linux Libertine and Linux Biolinum 12.0/14.5pt.

Printed at Aarhus University

Abstract (English)

We …

Resumé (Dansk)

Vi …

Preface

This thesis concludes my Master's degree in/at .....

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Introduction

Note to self. Det her er ikke en introduktion, det er et resume.

This thesis is titled "A semi Analytical Neutrino Perutbation Equation in the Einstein deSitter Model". That was not the original name. Originally the working title was "The neutrino bispectrum in second-order perturbation theory". The original goal of this thesis was to to use the ideas of a describing Neutrinos as particles with a sound speed to see if it might be possible to get a semi analytical expression for the Neutrino Bispectrum. We did not succeed in this, what we did succeed in was using the sound speed to derive an semi analytical equation that describes the Neutrino perturbation in A Einstein deSitter Model (A matter dominated universe).

The timeline was as follows, to start time was spend rederiving the equations in [1] concerning the Newtonian perturbation theory that was used to derive the second order CDM perturbation and is Newtonian kernel. Then time was spend first getting familiar with the cosmic simulation code CLASS and the numerically solving the Newtonian equations of [1] which was then compared with CLASS. Then time as spend introducing the sound speed into these numerical equation resulting a successfully replication of the CDM and Neutrino power spectrum fitting very well with the power spectra of CLASS. Then focus was put on trying to work with the second order Neutrino perturbation using the sound speed method. This took quit time but is also where we found the analytical equation for the first order Neutrino perturbations. In the end however, even with several assumptions we concluded that we would not solve if it even is possible to solve, in the time that we had left. Because of this and the lack of time, we also choose not to work with simulating bispectra using the code CONCEPT. And so we choose to focus on the the neutrino equation, specifically we tried to see if we could reduce its complexity without losing its accuracy. Although we did find an interesting expansion that works for very late time, we didn't reduce the overall complexity.

Chapter 1

The universe

The universe is expanding. This will become relevant.

1.1 The components of the universe

[1] Through analysis of observations astronomers have come up with a model of the universe where its components distribution as depicted in fig. 1.1. Radiation is light, an example of this would be the cosmic microwave background.

As can be seen on fig. 1.1 there two types of matter baryonic and dark Matter. Baryonic matter is the matter of the everyday, for life on Earth it is what we humans can see with our eyes. The term baryonic is a particle physics term that refers to the particle with an odd number of quarks.

Although called dark a more proper term would be non-baryonic matter (Do note that there are so theories that suggest that there is baryonic dark matter[3]). Unlike baryonic matter, Dark matter does not interact with the electromagnetic spectrum. This means that we can not observe it with light. Explaining what dark matter isn't, does not truly explain what dark matter truly is. The short answer is that we don't really know. Like normal matter, dark matter does interact with gravity meaning that it has mass. So we can say that dark matter is a non-baryonic, non-light interacting particle with mass. Surprisingly we do know of such a particle, the neutrino[2].

Neutrinos are an electrically neutral type of particle called leptons (which are spin half particles that do not interact with the strong force but do interact with the weak force another example of leptons are electrons). They were originally proposed by Pauli so account for energy and momentum

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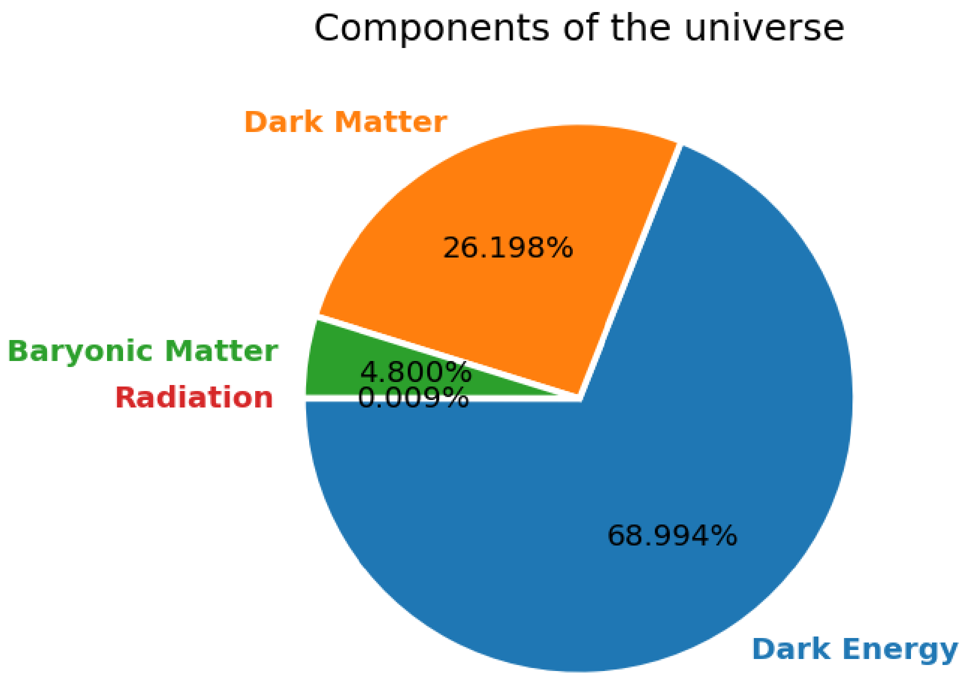


FIGURE 1.1: Contents of the universe today. As can be seen Radiation is effectively zero while "dark" components make up the vast majority of the universe. Based on data from [2]. .

conservation of beta decay. As a result it has always been a requirement that Neutrinos must be very light[4].

However there is a fundamental problem here and that is the different between hot and cold. In cosmology the term hot and cold refers essentially to how relativistic the particles are. If something is hot than it is highly energetic and relativistic, but if it is cold then it means that it is not very energetic and not relativistic and it thus more likely to clump together. While Neutrinos are hot, observations generally support the notion that dark matter is cold. Thus dark matter cannot simply be massive Neutrinos. In this thesis we will denote cold dark matter with "CDM".

By structure formation refers to the clustering togetherness of stuff in the universe. An example of structure would be galaxies. So by Neutrino structure we refer to how potential effect on cosmic structure that Neutrinos may have. An in deep on the effects of Neutrino structure is beyond the scope of this thesis.

1.2 Epochs of the universe

As what type of particle is dominant in the universe is quite important in this thesis, we will quickly go through the different domination eras of the universe. By domination eras we refer to the fact that although fig. 1.1 might depict the universe of today it does not show how the universe has always been. Different components have dominated throughout different time periods of the universe. Whichever particle species was dominant at a given time effected how the universe expanded in that time period. The reason for this changing behaviour comes about as a consequence of how different particle types are effected differently by the expansion of the universe[2]. Because of this it is also possible to estimate when which particle was dominant.

In cosmology we speak of 4 different eras. The inflation era, a very brief period of time where the universe experienced massive expansion in a very short time, which explains the flatness and uniformly of the universe[2]. Since the inflation era, the universe has had three other eras, where each era has marked the domination of a certain component of the universe. The first era was the radiation domination, lasting from the Big Bang until Radiation-Matter equality at 50000 yrs (at $a = 2.9 \cdot 10^{-4}$[2]), it is the era where subatomic particles and nuclear synthesis took place and the universe was dominated by hot particles ("a" is scala factor which represents how much smaller the universe was at a given as compared with today). Then came the matter domination era, where the gravitational effects of matter and dark matter became the dominating force in the universe. This era ended when Matter-dark energy equality was reached at 10.2 Gyr (at $a = 0.77$[2]). We are current in the dark energy era as one can see in fig. 1.1. We don't know much about dark energy and it is currently an active field of study, what we can say is that it is something that is active throughout the universe and is causing the expansion of the universe to accelerate[2].

1.3 Expansion of the universe

In 1929[2] Edwin Hubble discovered that the universe was expanding. This was quite a big deal. What is relevant for us, is that how this expansion and its time evolution are explained mathematically and that is through the Hubble parameter, the logarithmic time derivative of a,

$$\frac{\dot{a}}{a} = H, \tag{1.1}$$

Where we use a as a scale of the size of the universe throughout time and a time derivative of a thus denotes by how much the universe was expanding.

The Hubble parameter H has several equations related to it, but for now the important one is

$$H^2 = H_0^2(\Omega_m/a^3 + \Omega_R/a^4 + \Omega_\Lambda + \frac{1-\Omega_0}{a^2}) \tag{1.2}$$

where H_0 is the Hubble factor, Ω_m is the proportion of matter (baryonic and dark) in the universe today, Ω_R is the proportion of the universe that is radiation and Ω_Λ represents dark energy[2]. The final term represents effects from curvature, with scientists generally agreeing that $\Omega_0 = 1$. This equations tells us how the Hubble parameter and thus the expansion rate and its time evolution is dependent on the components of the universe. As eq. (1.2) is a differential equation one might wish to solve it to get a clear relationship between time and expansion. Unfortunately there is no simple analytical solution to eq. (1.2)[2]. But if you choose to focus on a specific time period and thus a specific domination era, you con get away with neglecting the other components in eq. (1.2) . As an example and relevant later, if we focus on matter domination, we can let $\Omega_M \to 1$ which reduces eq. (1.2) to

$$H^2 = H_0^2(1/a^3) \tag{1.3}$$

Letting $\Omega_M \to 1$ is a specific type of approximation that we call the Einstein deSitter Universe, which we will use extensively in this thesis.

The Hubble factor is currently one of the most focused on factors in cosmology as no one can quite agree on its value. The two primary candidates for the value comes from [5], which is based on measurements of the comic microwave background, with a value of $H_0 = 67.27 \pm 0.6 \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$ and from [6] with a value of $H_0 = 73.2 \pm 1.3 \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$. We choose to work with $H_0 = 68$ km/s/Mpc .

Chapter 2

Theory

2.1 On Notation

To avoid confusing we start off by summarizing the notation used. In some parts of this thesis, the "Einstein summation" notation was used, where indices that repeat, represents an implicit summation. For instance $x_\mu x^\mu$ represents

$$x_\mu x^\mu = \sum_{\mu=0}^{3} x_\mu x^\mu = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3 \tag{2.1}$$

when Greek letters are used we summarized the time and spacial components while Latin indices only represents spacial components, so

$$x_\mu x^\mu = x_0 x^0 + x_i x^i. \tag{2.2}$$

We will also use two forms of shorthand for differentiation, the first is

$$\partial_i = \frac{\partial}{\partial^i} \tag{2.3}$$

and the second is

$$A_{,\alpha} = \frac{\partial A}{\partial x^\alpha}. \tag{2.4}$$

Unless specifically stated we will use natural units

$$\hbar = c = k_B = 1. \tag{2.5}$$

We also need to discuss our time variables, in cosmology we like to switch around our time variables once in a while. There is the standard physical time that is use in most of physics, then there is conformal time where we take into account the expansion of the universe using the transformation

$$dt = a d\tau. \tag{2.6}$$

Having explained the difference in physical and conformal time it is relevant to bring up their different representations of the Hubble parameter,

$$\mathcal{H} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau}, \quad H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{2.7}$$

Finally we may also use the scala factor as a time variable.

2.2 General relativity

We will start off by introduce general relativity.
The main difference between classical Newtonian physics and GR is that time and space stops being completely separate components and our coordinates systems begin to experience curvature. Like with special relativity there is also a huge emphasis how things are observed in different reference frames. An important part of GR is the metric $g_{\mu\nu}$ which is used to describe the curvature of a given system. There is also a big focus on invariant terms, chief among these being the space-time interval squared[7]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.8}$$

with $dx = (dt, dx, dy, dz)$ representing infinitesimal changes in space time. The behaviour/curvature of a given system may then be described using the metric, as an example in flat Euclidean space $g_{\mu\nu} = diag(-1, 1, 1, 1)$, called the Minkowski metric[7],

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 \tag{2.9}$$

where we collect the spacial coordinates into dr.
Another difference between GR and Newton is the equation of motion for a free object. In Newtonian mechanics,

$$\frac{d^2x^i}{dt^2} = 0 \tag{2.10}$$

in GR the motion of a free object is described by the geodesic equation[7]

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{2.11}$$

where the gammas are Christoffel symbols which describe the curvature of a space [7]

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\nu\beta,\mu} + g_{\beta\mu,\nu} - g_{\mu\nu,\beta}). \tag{2.12}$$

We then move on to the governing equation of GR, the Einstein equation[7]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.13}$$

where G is the gravitational constant, $T_{\mu\nu}$ is the energy-momentum tensor and $R_{\mu\nu}$ and $\mathcal{R}$ are the Ricci tensor and Ricci scalar ($\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$) respectively. The Ricci tensor is given in terms of Christoffel symbols [7]

$$R_{\alpha\nu} = \partial_\mu \Gamma^\mu_{\nu\alpha} - \partial_\nu \Gamma^\mu_{\mu\alpha} + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\mu\alpha} - \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\mu\alpha}, \tag{2.14}$$

and is meant to give a measure of the general curvature of a given space. Is energy-momentum tensor as its name suggest describes the energy and momentum of a system and so the point of the Einstein equation is to describe the relationship between the curvature of space and its energy and momentum.

2.3 The Friedmann equations

eq. (2.13) is a long list of differential equations which we can use to describe the curvature and behaviour of space. Simplifications and generalizations are thus be expected. If we assume that the universe is filled up with an isotropic, perfect and homogeneous gas, then the metric needed to describe the universe can be written as

$$ds^2 = -dt^2 + a(t)^2\left[dr^2 + S_k(r)^2 d\Omega^2\right] \tag{2.15}$$

where $d\Omega^2$ represent spherical coordinates[2] and S_k is a curvature dependent function given as

$$S_k = \begin{cases} R_0 \sin(r/R_0) & (\kappa = +1) \\ r & (\kappa = 0) \\ > R_0 \sinh(r/R_0) & (\kappa = -1) \end{cases}, \tag{2.16}$$

where κ represents the curvature and R_0 is the radius of curvature[2]. eq. (2.15) is called the Friedmann-Lemaître-Robertson-Walker metric[2]. From it, the Friedmann equation can be derived[2],

$$H^2 = \frac{8\pi G}{3}\varepsilon(t) - \frac{\kappa}{R_0^2 a(t)^2} \tag{2.17}$$

where ε is the energy density[2]. As the universe is generally accepted to be flat[5] we set $\kappa = 0$, additional equation can then be derived from the Friedmann metric to describe the universe, most importantly the equation of acceleration

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon + 3P) \tag{2.18}$$

where P is the pressure of the components of the universe. This equation tells us that the behaviour of expansion comes down to a relation between the energy and pressure of the components of the universe. Very fortunately, because of the assumptions about the universe, we may write the pressure as [2] $P = w\varepsilon$ where w is a dimensionless number. w varies for different components of the universe. For radiation $w = 1/3$ for matter $w = 0$ and fascinatingly for dark energy $w = -1$[2].

2.4 Gauges

As discussed in the previous section on GR one of the main difference between GR and Newtonian mechanics is that you must take into account the curvature of the space that you are working with. And although the laws of physics are the same everywhere, that cannot be assumed about the curvature. The way that GR gets around this problem is gauge fixing. Gauge fixing, like that of electrodynamics, is where you set a general metric so that you can assume that the physics of the system behaves the same in different places[7]. Examples of gauges would be the Newtonian and synchronous gauges[7]. Normally neutrinos are see as the more energetic and relativistic particles, but as we were interested in the case were we could use Newtonian mechanics to describe the behaviour of the system, we needed a gauge that allowed for that.
Now it would be wrong to say that we actively sat down and wrote up a gauge that could do that. Instead we wanted to use/modify

$$\dot{\delta}^{(1)} = -\theta^{(1)} \tag{2.19}$$

$$\dot{\theta}^{(1)} = -\mathcal{H}\theta^{(1)} - \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta^{(1)}, \tag{2.20}$$

From [1] to describe the neutrinos (where δϽ is the density contrast and θ is the divergence of the velocity flow), where we would limit ourselves to Newtonian mechanics and those requirements are fulfilled by the Newtonian motion gauges[8]. Newtonian motion gauges are gauges where space time behaves in such a way that they allow for the use of Newtonian equations of motion. So essentially we did not pick a gauges as much as we say that these equations work for Newtonian motion gauges. To be a Newtonian motion gauge one must observe the requirement that the equations of motion for density contrast δ and the divergence of the velocity flow θ follow eq. (2.19) and eq. (2.20)[8].

2.5 Particles in cosmology

As we are interested in the structure formation of different particles species, we wish to understand their density perturbations/contraction and velocity.

To derive equations that describe these things we turn to the Boltzmann equation, as it provides a framework for describing the statistical behavior of and interactions between particles[9]

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t} = C[f] \tag{2.21}$$

which governs the behaviour of the f, the distribution function in phase space, where C represents all possible collision terms,x represents position, p momentum and $\hat{p}^i$ is the momentum unit vector.

2.5.1 Dark matter

As an important example we have dark matter. When working with eq. (2.21) a problem that often comes up is that the collision term adds a great deal of complicity, see for example [9]. What is fascinating about Dark matter is that because of how we understand the collision term must be zero (no interactions). Now we will refer to [1] for a a step by step guide, but briefly said, using that the collision term must be zero and standard cosmology assumptions, we get that the behaviour of the density contrast δ and the divergence of the velocity flow θ are govern by[1] ,

$$\dot{\delta} = -\partial_j\left[(1+\delta)\partial_j\nabla^{-2}\theta\right] \tag{2.22}$$

$$\dot{\theta} = \mathcal{H}\theta - \partial_i\partial_j\nabla^{-2}\theta\partial_j\partial_i\nabla^{-2}\theta - \partial_j\nabla^{-2}\theta\partial_j\theta - \frac{3}{2}\frac{H_0^2\Omega_m}{a}\delta, \tag{2.23}$$

where ∇^{-2} is the inverse laplace operator.

2.5.2 Massive neutrinos

What separates the description of Neutrinos from that of dark matter is that for the neutrinos a hierarchy of momentum variables are needed. We start with our distribution function which we write as

$$f(x,\boldsymbol{p},\tau) = f_0(p)\left[1+\Psi(x,\boldsymbol{p},\tau)\right] \tag{2.24}$$

where $f_0(p)$ is the zeroth-order phases space distribution Ψ the perturbed component[10] . Because of the complexity of Ψ in f we then choose to switch to fourier space and then expand in a Legendre series

$$\Psi(\boldsymbol{k},p,\tau) = \sum_{l=0}^{\infty}(-i)^l(2l+1)\Psi_l(\boldsymbol{k},p,\tau)P_l(\hat{k}\cdot\hat{n}), \tag{2.25}$$

where P_l is the Legendre polynomials. Since eq. (2.25) is an infinite series any numerical solutions must truncate at some value l_{max}. Unfortunately the Ψ_l cause a problem is they dependent on one another, for instance the time evolution of Ψ_1 is dependent on both Ψ_0 and Ψ_2 so simply setting all Ψ_l to zero after a certain l_{max} will propagate errors[10].

2.5.3 A neutrino sound speed

A way around using the neutrino hierarchy problems is to use a sound speed. which is where you re interpret variation in the Pressure over density relation as a sound speed. Specifically

$$c_s^2 = \frac{\delta P}{\delta \rho} \tag{2.26}$$

The actual value for this is given as [11]

$$c_s^2 \equiv \frac{\delta P}{\delta \rho} = \frac{1}{3} \frac{\int q^2 dq \frac{q^2}{\varepsilon(q)} f_0(q) \Psi_0}{\int q^2 dq \varepsilon(q) f_0(q) \Psi_0} \tag{2.27}$$

where q is momentum. This equation is not easily solvable so we follow [11] and state that $c_s \simeq \frac{\sqrt{5}}{3} \sigma_\nu$ where σ_ν is the velocity dispersion given as

$$\sigma_\nu^2 = \frac{15\zeta(5)}{\zeta(3)} \left(\frac{4}{11}\right)^{2/3} \frac{T_\gamma^2 (1+z)^2}{m_\nu} \tag{2.28}$$

where T_γ is the current photon temperature of the universe. We insert the sound speed into the continuity and Euler equation [11]

$$\dot{\delta}^{(1)} + \theta^{(1)} = 0 \tag{2.29}$$

$$\dot{\theta}^{(1)} = -\mathcal{H}\theta^{(1)} - \frac{3}{2} \frac{H_0^2 \Omega_M}{a} \delta_{\mathrm{cdm}}^{(1)} + k^2 c_s^2 \delta^{(1)}, \tag{2.30}$$

Do note though that unlike the equation in [11] we only source gravitationally from matter not neutrinos.

2.6 Correlation Functions

Although it is a common and true statement the the universe is homogeneous, as mentioned in chapter 1, it isn't the whole truth. The universe is homogeneous on very large scales, but not on smaller scales. But there are inhomogeneities on smaller scales, like hte density contrast

$$\delta(\boldsymbol{x}) = (n(\boldsymbol{x}) - \bar{n})/\bar{n} \tag{2.31}$$

where $\bar{n}$ is the mean density of particles (galaxies) in the universe and $n(x)$ is the number density at position x. As mentioned it is quite important to consider which scale we are working with. An easy way to distinguish between scales is to switch to Fourier space, then $\delta(\boldsymbol{x})$ becomes $\tilde{\delta}(\boldsymbol{k})$. Now because of the definition of $\delta(\boldsymbol{x})$ taking a average (doing a volume integral) would yield $\langle \delta(x) \rangle = 0$. But that is not the case if you look at correlation functions,

$$\xi_2 = \langle \delta(x_1)\delta(x_2) \rangle \tag{2.32}$$
$$\xi_3 = \langle \delta(x_1)\delta(x_2)\delta(x_3) \rangle \tag{2.33}$$
$$\xi_N = \langle \delta(x_1)\delta(x_2)...\delta(x_N) \rangle \tag{2.34}$$

The correlation functions are defined as the joint density average at N different locations[12]. In other words what is the expectation value of the product of n perturbations in different positions[13] . As mentioned it can be useful to work in Fourier space. As an example this turns the two point correlation function into

$$\langle \tilde{\delta}(\boldsymbol{k})\tilde{\delta}(\boldsymbol{k}') \rangle = \int d^3x d^3x' \xi_2(x') e^{-i(k+k')x - ik'x'} \tag{2.35}$$
$$(2\pi)^3 \delta^3(k+k') \int d^3x' \xi_2(x') e^{ikx'} = (2\pi)^3 \delta^3(\boldsymbol{k}+\boldsymbol{k}') P(k) \tag{2.36}$$

where $\delta^3()$ is the delta Dirac function and $P(k)$ is the power spectrum[12]. We may thus think of the power spectrum as a statistical variations between two random points.

Now we have seen how one can express the fourier transformations of the two-points correlation function, but this is not restricted to 2 points and generally be done for all correlation functions as [12]

$$\langle \delta(\boldsymbol{k_1})..\delta(\boldsymbol{k_N}) \rangle = (2\pi)^3 \delta^3(\boldsymbol{k_1} + ... + \boldsymbol{k_N}) P_N(\boldsymbol{k_1}, ..., \boldsymbol{k_N}). \tag{2.37}$$

The Bispectrum is the name the 3 point correlation function, which is then an indicator of the variance between 3 random points. To numerically calculate the power spectrum we used [14]

$$P(k) = 2\pi^2 Y^2(\tau, k) k^{-3} \mathcal{P}(k) \tag{2.38}$$

where Y the quantity that we make the power spectrum of (in our case δ for CDM and neutrinos) $\mathcal{P}(k)$ is the primordial background.

Chapter 3

Analytical Results

the structure of this chapter is as follows. First we derive the first order cold dark matter perturbation, then the second order perturbation in Λcdm. Then we show the effects of switching to Eisten deSitter.
Then we will do it for neutrinos and show what we manage to accomplish there.

To begin, we note that we closely follow the derivation for first and second order cold dark matter perturbation in [1]. We repeat it here for clarity.

A

s a reminder the idea in perturbation theory is that we can solve otherwise difficult equations by by "expanding" the functions to a certain order, like

$$\delta = \delta^{(1)} + \frac{1}{2}\delta^{(2)} \tag{3.1}$$

$$\theta = \theta^{(1)} + \frac{1}{2}\theta^{(2)} \tag{3.2}$$

The we insert these expansion into the equations and ignore everything that is above our order of expansion (for a 2nd order case $\delta^{(1)}\delta^{(2)}$ is a 3rd order term and would thus be ignored).

3.1 First order dark matter

We start off by taking the first order expansion of δ and θ in eq. (2.22) and eq. (2.23), this gives us

$$\dot{\delta}^{(1)} = -\theta^{(1)} \tag{3.3}$$

$$\dot{\theta}^{(1)} = -\mathcal{H}\theta^{(1)} - \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta^{(1)}, \tag{3.4}$$

these equations of motion combined gives us

$$\ddot{\delta}^{(1)} + \mathcal{H}\dot{\delta}^{(1)} = \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta^{(1)}. \tag{3.5}$$

δ and θ are time and space dependent, but as there is no explicit space dependent terms in eq. (3.5) we split the dependencies up as $\delta^{(1)}(\tau, x) = D(\tau, x)\tilde{\delta}(x)$, we can then insert this into eq. (3.5) and we gain a purely time dependent equation,

$$\ddot{D} + \mathcal{H}\dot{D} - \frac{3}{2}\frac{H_0^2\Omega_M}{a}D = 0. \tag{3.6}$$

So we see that we have a differential equation for time component of $\delta^{(1)}$. As for $\theta^{(1)}$ we can reexpress it as $\theta^{(1)} = -\frac{\dot{D}}{D}\delta^{(1)}$. We then go to the second order.

3.2 Second order Dark Matter

To find the second order eqaution for dark Matter, we start off by taking the second order expansion of δ and θ

$$\delta = \delta^{(1)} + \frac{1}{2}\delta^{(2)}$$
$$\theta = \theta^{(1)} + \frac{1}{2}\theta^{(2)}.$$

combining these expansions with eq. (2.22), eq. (2.23), eq. (2.19) and eq. (2.20) we get two 2nd order equations of motion

$$\dot{\delta}^{(2)} = -2\partial_j\nabla^{-2}\theta^{(1)}\partial_j\delta^{(1)} - 2\delta^{(1)}\theta^{(1)} - \theta^{(2)} \tag{3.7}$$
$$\dot{\theta}^{(2)} = -\mathcal{H}\theta^{(2)} - 2\Big(\partial_j\partial_i\nabla^{-2}\theta^{(1)}\partial_i\partial_j\nabla^{-2}\theta^{(1)}\Big) - 2\partial_j\nabla^{-2}\theta^{(1)}\partial_j\theta - \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta^{(2)}. \tag{3.8}$$

We then combine eq. (3.7) and eq. (3.8) into a single second order differential equation,

$$\begin{aligned}\ddot{\delta}^{(2)} + \mathcal{H}\dot{\delta}^{(2)} =& 2\Big(\mathcal{H}\dot{D}D + \ddot{D}D + \dot{D}^2\Big)\Big[\partial_j\nabla^{-2}\tilde{\delta}\partial_j\tilde{\delta} + \tilde{\delta}^2\Big] + \frac{3}{2}\frac{H_0^2\Omega_m}{a}\delta^{(2)} \\ &+ 2\dot{D}^2\Big[\partial_i\partial_j\nabla^{-2}\tilde{\delta}\partial_i\partial_j\nabla^{-2}\tilde{\delta} + \partial_j\nabla^{-2}\tilde{\delta}\partial_j\tilde{\delta}\Big].\end{aligned} \tag{3.9}$$

Now this is a rather complex equation with no clear solution. While we will be fairly brief on the solution to this equation, We refer to [1] for the step by step approach. To introduce the solution to eq. (3.9) we introduce the growth function F which is governed by

$$\ddot{F} + \mathcal{H}\dot{F} = \frac{3}{2}\frac{H_0^2\Omega_M}{a}\Big(F + D^2\Big). \tag{3.10}$$

With that we can introduce the solution to the cold dark matter density perturbation as [1],

$$\begin{aligned}\delta^{(2)} =& 2\partial_j\nabla^{-2}\delta^{(1)}\partial_j\delta^{(1)} + \left(1+\frac{F}{D^2}\right)\delta^{(1)}\delta^{(1)} \\ &+ \left(1-\frac{F}{D^2}\right)\partial_i\partial_j\nabla^{-2}\delta^{(1)}\partial_i\partial_j\nabla^{-2}\delta^{(1)}\end{aligned} \tag{3.11}$$

Although it is a solution eq. (3.11) is still rather complex. To simplify we switch to Fourier space, this turns eq. (3.11) into

$$\frac{1}{2}\delta^{(2)}(k) = C_k\left\{\mathcal{K}_N(k_1,k_2,k)\delta^{(1)}(k_1)\delta^{(1)}(k_2)\right\}, \tag{3.12}$$

with

$$\mathcal{K}_N(k_1,k_2,k) \equiv (\beta_N - \alpha_N) + \frac{\beta_N}{2}\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2\left(\frac{k_2}{k_1}+\frac{k_1}{k_2}\right) + \alpha_N\left(\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2\right)^2 \tag{3.13}$$

and

$$\alpha_N = \frac{7-3v}{14}, \quad \beta_N = 1, \quad v = 7F/3D^2, \tag{3.14}$$

where C_k is the convolution integral,

$$C_k\{f(\boldsymbol{k}_1,\boldsymbol{k}_2)\} \equiv \int \frac{d\boldsymbol{k}_1\boldsymbol{k}_2}{(2\pi)^3} f(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta^D(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) \tag{3.15}$$

We now have the equation for the second order equation in Λ_{cdm} and a Kernel which we could use to generate a cdm Bispectrum. With this summary of newtonian section of [1] we now swithc to Einstein deSitter Universe.

3.3 Dark Matter using the Einstein deSitter Model

We will go in the reversed order here and briefly talk about the second order first. This comes from the fact that for second order CDM the only difference between the Λ_{cdm} and EdS is that the kernel eq. (3.12) is altered so that in eq. (3.14) $v \to 1$ and thus $\alpha_N \to 2/7$[1]. With that done let us turn to the first order.

As discussed previously, the main principle of Einstein deSitter is the assumption that there is only matter in the universe, e.g. $\Omega_M \simeq 1$. This has the effect of reducing the Hubble parameter to (in conformal time)

$$\mathcal{H} = \frac{\dot{a}}{a} = H_0\sqrt{\Omega_M/a} = \frac{H_0}{\sqrt{a}}, \tag{3.16}$$

which lets us analytically solve the scala factor as

$$\dot{a} = H_0\sqrt{a} \rightarrow a = \frac{1}{4}H_0^2\tau^2, \tag{3.17}$$

This means that we can rewrite eq. (3.6) as

$$\ddot{D} + \mathcal{H}\dot{D} - \frac{3}{2}\frac{H_0^2\Omega_M}{a}D = 0.$$

as

$$\ddot{D} + \frac{H_0}{\sqrt{a}}\dot{D} = \frac{3}{2}\frac{H_0^2}{a}D. \tag{3.18}$$

$$\ddot{D} + 2\frac{H_0}{H_0\tau}\dot{D} = 4\frac{3}{2}\frac{H_0^2}{H_0^2\tau^2}D. \tag{3.19}$$

$$\ddot{D} + \frac{2}{\tau}\dot{D} = \frac{6}{\tau^2}D. \tag{3.20}$$

This is a solvable equation but instead of solving it in conformal time we choose to switch to scala time.
eq. (3.20) is rewritten in scala time becomes[1]

$$a^2\ddot{D}_a + \frac{3}{2}a\dot{D}_a = \frac{3}{2}D_a \tag{3.21}$$

where the index a, represent that we are working with scala. eq. (3.21) has the simple solution of

$$D \propto a \tag{3.22}$$

$$D = D_0 a, \tag{3.23}$$

where D_0 is a currently unknown value. This is interpreted as cdm perturbations scaling with a. This fits with standard cosmology about the growth of cdm perturbations in the matter dominated era in Λ_{cdm} . We will come back to eq. (3.23) but for now we move unto the neutrinos. This marks the end of the [1] walkthrough.

3.4 First order neutrinos

As explained in section 2.5.3 may represent the equation for density contrast and velocity flow as the CDM equation.
One important note is that we are still dependent on the CDM because of the gravitational sourcing. Although the more stringent way of writing these equation would be to include a sourcing of the neutrinos as well like in [15] we make the assumption like in [11] that the neutrino sourcing may be neglected as it is minute compared with the CDM sourcing.

1: See appendix for derivation.

As before we take the first order, this gives us[2]

$$\dot{\delta}_\nu^{(1)} = -\theta_\nu^{(1)} \tag{3.24}$$

$$\dot{\theta}_\nu^{(1)} = -\mathcal{H}\theta_\nu^{(1)} - \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta_{cdm}^{(1)} - c_s(a)^2\nabla^2\delta_\nu^{(1)} \tag{3.25}$$

$$\ddot{\delta}_\nu^{(1)} + \mathcal{H}\dot{\delta}_\nu^{(1)} = \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta_{\text{cdm}}^{(1)} + c_s(a)^2\nabla^2\delta_\nu^{(1)}. \tag{3.26}$$

This is where we run into the main challenge of the neutrinos as compared with the cdm, in that we also need to account for the cdm.

Firstly we choose to make the switch to Fourier/phase space now instead of at the end like section 3.2, this has the effect of removing the gradient on the sound speed term

$$\ddot{\delta}_{\nu,F}^{(1)} + \mathcal{H}\dot{\delta}_{\nu,F}^{(1)} = \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta_{cdm,F}^{(1)} - k^2c_s(a)^2\delta_{\nu,F}^{(1)}, \tag{3.27}$$

where the $_F$ denote the Fourier transform of the perturbation. For readability we will note include the fourier denotation bu the reader may assume that we are working with the fourier transformed components unless otherwise stated, giving us,

$$\ddot{\delta}_\nu^{(1)} + \mathcal{H}\dot{\delta}_\nu^{(1)} = \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta_{\text{cdm}}^{(1)} - k^2c_s(a)^2\delta_\nu^{(1)}. \tag{3.28}$$

With that done we then switch to conformal to scala time [3]

$$\left(\frac{3}{2}\Omega_M + 3\Omega_\Lambda a^3\right)\dot{\delta}_\nu^{(1)} + \left(a\Omega_m + a^4\Omega_\Lambda\right)\ddot{\delta}_\nu^{(1)} = \frac{3}{2}\frac{\Omega_m}{a}\delta_{\text{cdm}}^{(1)} - \frac{k^2c_s(a)^2}{H_0^2}\delta_\nu^{(1)}. \tag{3.29}$$

To simplify this we then switch to the EdS Model

$$\frac{3}{2}\dot{\delta}_\nu^{(1)} + a\ddot{\delta}_\nu^{(1)} = \frac{3}{2}\frac{1}{a}\delta_{\text{cdm}}^{(1)} - \frac{k^2c_s(a)^2}{H_0^2}\delta_\nu^{(1)}. \tag{3.30}$$

Beside simplifying eq. (3.29) the reason why we use the EdS is so that we may use the solution to the $\delta_{\text{cdm}}^{(1)}$ from eq. (3.23). Using that we may rewrite eq. (3.30) as

$$\frac{3}{2}\dot{\delta}_\nu^{(1)} + a\ddot{\delta}_\nu^{(1)} = \frac{3}{2}\frac{1}{a}aD_0 - \frac{k^2c_s(a)^2}{H_0^2}\delta_\nu^{(1)}. \tag{3.31}$$

$$\frac{3}{2}\dot{\delta}_\nu^{(1)} + a\ddot{\delta}_\nu^{(1)} = \frac{3}{2}D_0 - \frac{k^2c_s(a)^2}{H_0^2}\delta_\nu^{(1)}. \tag{3.32}$$

2: See Apprendix
3: see appendix

Which we rearrange

$$\frac{3}{2}\dot{\delta}_\nu^{(1)} + a\ddot{\delta}_\nu^{(1)} + \frac{k^2 c_s(a)^2}{H_0^2}\delta_\nu^{(1)} = \frac{3}{2}D_0. \tag{3.33}$$

This differential equation has a solution. The solution to that equation is [4]

$$\delta_\nu^{(1)} = aD_0 + c_1 \cos\left(\frac{2kc_s(a)\sqrt{a}}{H_0}\right) - c_2 \sin\left(\frac{2kc_s(a)\sqrt{a}}{H_0}\right) \tag{3.34}$$

$$+ \left(\frac{2kc_s(a)a}{H_0}\right)^2 D_0\Bigg(\cos\left(\frac{2kc_s(a)\sqrt{a}}{H_0}\right)\mathrm{Ci}\left(\frac{2kc_s(a)\sqrt{a}}{H_0}\right) \tag{3.35}$$

$$+ \sin\left(\frac{kc_s(a)\sqrt{a}}{H_0}\right)\mathrm{Si}\left(\frac{kc_s(a)\sqrt{a}}{H_0}\right)\Bigg) \tag{3.36}$$

Where Ci and Si are the cosine and sine integrals respectively. To make this equation a bit more readable we introduce the variable

$$M(a) = \left(\frac{2kc_s(a)\sqrt{a}}{H_0}\right)$$

$$\delta_\nu^{(1)} = aD_0 + c_1 \cos\Big(M(a)\Big) - c_2 \sin\Big(M(a)\Big) \tag{3.37}$$

$$+ aM(a)^2 D_0\Big(\cos\Big(M(a)\Big)\mathrm{Ci}(M(a)) + \sin(M(a))\mathrm{Si}(M(a))\Big) \tag{3.38}$$

We are not done yet.

We need to specify the factors c_1 and c_2 . It was not possible for us to use Mathmatica so find c_1 and c_2 using initial conditions so we found another way. To do this let us think a bit about the equation. This equation is supposed to represent the neutrinos in a matter dominated universe. That means that they will start out oscillating until a certain point where the effects of gravity will start to dominate the Neutrinos and so they will start to follow the CDM. we can thus interpret terms with D_0 as representations of the dark matter and the interplay between the effects of gravity and the Neutrino sound speed. The terms with c_1 and c_2 are the cos and sin terms, e.g. the oscillating terms, representing the oscillating behaviour of the Neutrinos before they start to experience the effect of the CDM. This is then our justification for doing an expansion around $a = 0$.
So we expanded around $t = 0$ giving

$$\delta_\nu = c_1 \cos\Big(M(a)\Big) - (2\pi P - c_2)\sin\Big(M(a)\Big), \tag{3.39}$$

4: Solved using Mathmatica

with us defining $P \equiv D_0\left(\frac{kc_s(a)a}{H_0}\right)^2$. It is worth noting that because the sound speed is proportional to $1/a$, P is actually a constant. We then claim that we can rewrite eq. (3.39) as simply a cosine,

$$\delta_\nu = A\cos(M(a) + \Phi) \tag{3.40}$$

wiht Φ being a phase using trigonometry we can write that as

$$\delta_\nu = A\cos(M(a))\cos(\Phi) + \sin(M(a))\sin(\Phi) \tag{3.41}$$

if we then set $\Phi = 0$ then eq. (3.40) and eq. (3.41) are equivalent to eq. (3.39) if

$$c_1 = A\ c_2 = 2\pi P \tag{3.42}$$

With that we have a value of c_2 but what about c_1 what is A ? . Another reason that we rewrote the expansion as eq. (3.40) is that we can then do the following,

$$\dot{\delta}_\nu = -A\dot{M}(a)\sin(M(a)) \tag{3.43}$$

which means that

$$\delta_\nu^2 + (\dot{\delta}_\nu/\dot{M}(a))^2 = A^2(\cos^2(M(a)) + \sin^2(M(a))) = A^2 \tag{3.44}$$

meaning

$$A = \sqrt{\delta_\nu^2 + (\dot{\delta}_\nu/\dot{M}(a))^2} \tag{3.45}$$

With that, all we need is some initial value of $\delta_\nu^{(1)}$ and $\dot{\delta}_\nu^{(1)}$ at some value of a , $a_{initial}$ and we have an equation for the Neutrino perturbations in a Matter dominated universe given as,

$$\begin{aligned} M(a) =&\left(\frac{2kc_s(a)\sqrt{a}}{H_0}\right) \quad A = \sqrt{\delta(a_0)_\nu^2 + (\dot{\delta}(a_0)_\nu/\dot{M}(a_0))^2} \quad P \equiv D_0\left(\frac{kc_s(a)a}{H_0}\right)^2 \\ \delta_\nu^{(1)} =& aD_0 + A\cos\Big(M(a)\Big) - 2\pi P\sin\Big(M(a)\Big) \\ &+ aM(a)^2D_0\Big(\cos\Big(M(a)\Big)\mathrm{Ci}(M(a)) + D_0\sin(M(a))\mathrm{Si}(M(a))\Big) \end{aligned} \tag{3.46}$$

This is the main result of this thesis.

3.5 2nd order Neutrinos

Så jeg skal lige fikse hvad der sker her.

We also tried to derive a second order equation for the Neutrinos although we did not truely success. As previous we started out with following along the method for the δ_{CDM} where we inserted a sound speed term giving eq. (3.26) , now we have

$$\dot{\delta}_\nu^{(1)} = -\theta_\nu^{(1)} \quad \dot{\theta}_\nu^{(1)} = -\mathcal{H}\theta_\nu^{(1)} - \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta_M^{(1)} - c_s^2\nabla^2\delta_\nu^{(1)} \tag{3.47}$$

$$\dot{\delta}_\nu^{(2)} = -2\partial_j\nabla^{-2}\theta_\nu^{(1)}\partial_j\delta_\nu^{(1)} - 2\delta_\nu^{(1)}\theta_\nu^{(1)} - \theta_\nu^{(2)} \tag{3.48}$$

$$\begin{aligned}\dot{\theta}_\nu^{(2)} =& -\mathcal{H}\theta_\nu^{(2)} - 2\Big(\partial_i\partial_j\nabla^{-2}\theta_\nu^{(1)}\partial_i\partial_j\theta_\nu^{(1)}\Big) - 2\partial_j\nabla^{-2}\theta_\nu^{(1)}\partial_j\theta_\nu^{(1)} \\ &- \frac{3}{2}\frac{H_0^2\Omega_M}{a}\delta_M^{(2)} - c_s^2\nabla^2\delta_\nu^{(2)}\end{aligned} \tag{3.49}$$

This can be reduced to the following [5]

$$\begin{aligned}\ddot{\delta}_\nu^{(2)} + \mathcal{H}\dot{\delta}_\nu^{(2)} =& \frac{3}{2}\frac{H_0^2}{a}\Big(2\partial_j\nabla^{-2}\delta_M^{(1)}\partial_j\delta_\nu^{(1)} + 2\delta_\nu^{(1)}\delta_M^{(1)} + \delta_M^{(2)}\Big) \\ &+c_s^2\Big(2\partial_j\delta_\nu^{(1)}\partial_j\delta_\nu^{(1)} + 2\delta_\nu^{(1)}\nabla^2\delta_\nu^{(1)} + \nabla^2\delta_\nu^{(2)}\Big) \\ +4\partial_j\nabla^{-2}\dot{\delta}_\nu^{(1)}\partial_j\dot{\delta}_\nu^{(1)} &+ 2\dot{\delta}_\nu^{(1)}\dot{\delta}_\nu^{(1)} + 2\partial_i\partial_j\nabla^{-2}\dot{\delta}_\nu^{(1)}\partial_i\partial_j\nabla^{-2}\dot{\delta}_\nu^{(1)}\end{aligned} \tag{3.50}$$

Now we were unable to more beyond this and we thus did not success in in repeat the method for the 2nd order cdm kernel.

3.6 Expanding on the neutrino equation

After having successfully derived and implemented eq. (3.46) we made attempts at simplifying it. Simple put, what we did was to expand around the sound speed. Our reasoning for this was what we were under the belief that once the neutrino started to follow the dark matter, e.g. that the gravitational sourcing term in eq. (3.26) became dominate, the sound speed term would be become negligible. With the reasoning we then expand the sound speed around zero to second order, which gave[6]

$$\begin{aligned}\hat{\delta}_0 =& aD_0 - \frac{2c_s(a)\sqrt{a}}{H_0}c_2 \\ &+ \frac{2ac_s(a)^2k^2}{H_0^2}\Big(2aD_0\gamma_{Euler} - c_1 + 2aD_0\Big(\log(c_s(a)a) + \log(\frac{2k}{\sqrt{a}H_0})\Big)\Big)\end{aligned} \tag{3.51}$$

where the $_0$ index with a hat denotes an expansion around zero and γ_{Euler} is Eulers γ which is a constant. We also tried to around infinity, here the logic

5: see Appendix
6: Done in mathmatica

was to expanding around the sound speed when it is the dominating term. This gave[7]

$$\hat{\delta}_\infty = \cos(\frac{2kc_s(a)\sqrt{a}}{H_0})\big(c_1 + 2D_0\frac{k^2c_s(a)^2}{H_0^2}a^2\big(2\big(\log(\frac{kc_s(a)\sqrt{a}}{H_0})\big) - \log(\frac{k^2c_s^2(a)a}{H_0^2})\big)\big) \tag{3.52}$$
$$+ \frac{3D_0H_0^2}{2c_s(a)^2k^2} + \big(\frac{2c_s(a)^2a^2D_0k^2}{H_0^2}\pi - c_2\big)\sin(\frac{2kc_s(a)\sqrt{a}}{H_0})\big)$$

where where the $_\infty$ index with a hat denotes an expansion around infinity.

7: Again done in mathmatica

Chapter 4

Numerical Results

Before we begin let us first give a brief mention to CLASS, the code we used to compare and get initial values from.

4.1 CLASS

The "Cosmic Linear Anisotropy Solving System"[16] is a code designed to, amongst other tings, simulate the evolution for linear perturbations in cosmology. The code has many uses but the way we used CLASS was to

1. get initial values for our numerical solutions.
2. compare with our own numerical and analytical solutions.

We use the Python jupyterNotebook edition available online.

4.2 Explanation

As explained in the intro a lot of time was spent on numerically solutions of cold dark matter and neutrinos. This chapter will contain a walkthrough of everything we did. All numerical solutions were done using the numerical differential equation solver integrate in the python package scipy, using Explicit Runge-Kutta method of order 5(4). As a comparison and source of initial value we used CLASS[16]. To run any cosmic simulation one needs to choose Cosmological parameters. For the Λcdmruns we choose $H_0 = 68$ 1/Mpc , $\Omega_{\text{cdm}} = 0.31$, $\Omega_b = 0.01$ and $\Omega_\Lambda = 0.68$ [2] and unless otherwise stated this were the values we used. As mentioned previously we also work alot with the Einsten deSitter Model, here we sat $\Omega_{\text{cdm}} = 0.98$. In addition we also included a non cdm particle to represent the Neutrinos, where we varied the mass to check the flexibility of our numerical solutions.

Given that our goal was the Neutrino density contrast δ_ν we did not focus on the velocity flow θ. One thing of note here is that the edition of CLASS that we used required us to either work in newtonian or Synchronous gauges. WE worked with Synchronous which meant that all the θ values were zero. The way around this was to use the combined 2nd order differential equations so that initial values for θ would not be neccesary.

4.3 Dark matter without Neutrinos.

We began with numerically solving eq. (2.19) and eq. (2.20) in for different k values in Λcdm and the Einstein deSitter Model. As can be seen in fig. 4.1 the solutions fit very well to those of CLASS. The best solutions were those in the Einstein deSitter model, while those in Λcdmmodel were slightly ahead for higher k-values and slightly behind for the very small k-values. But the behaviour does fit very well. I

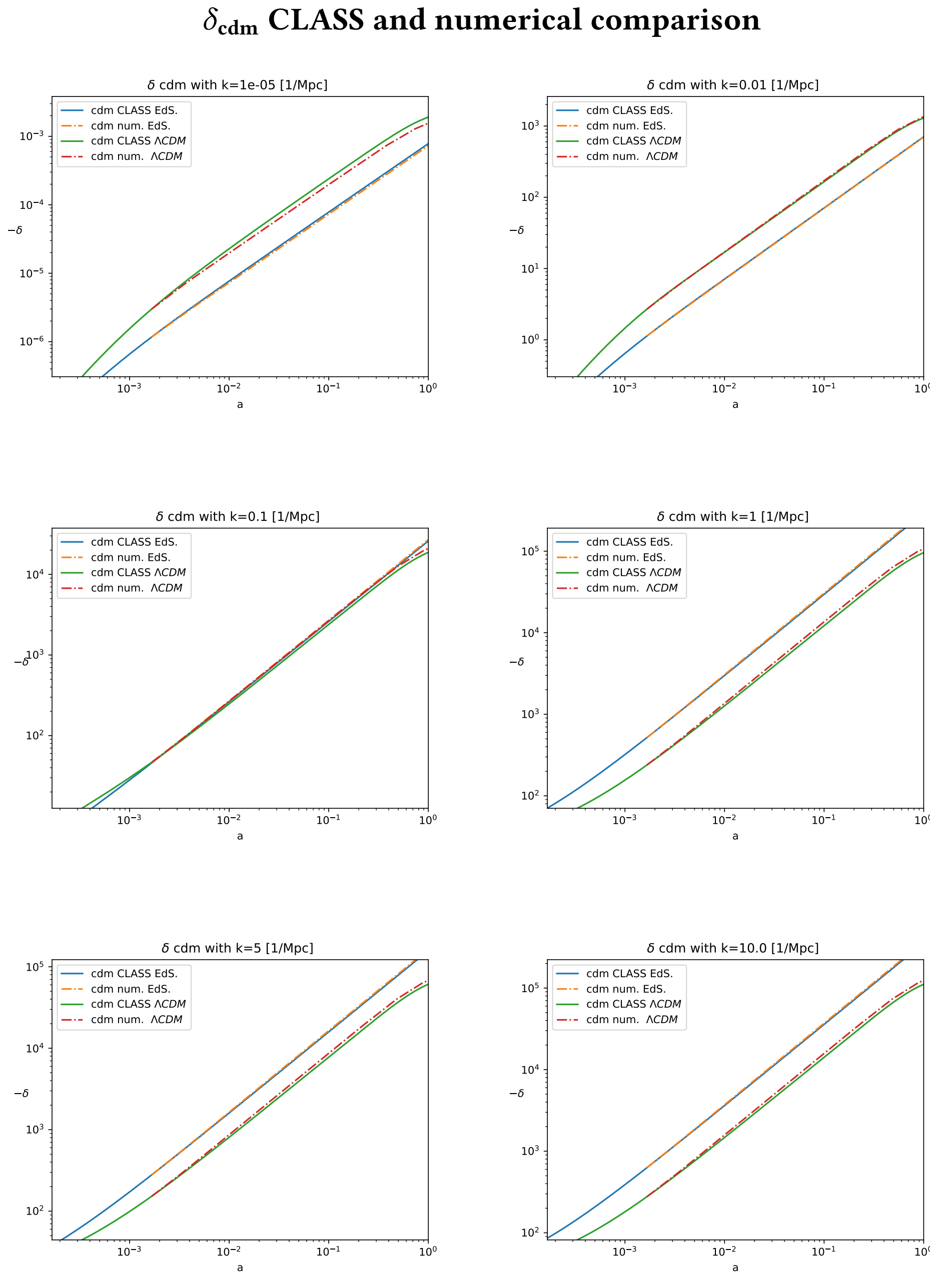
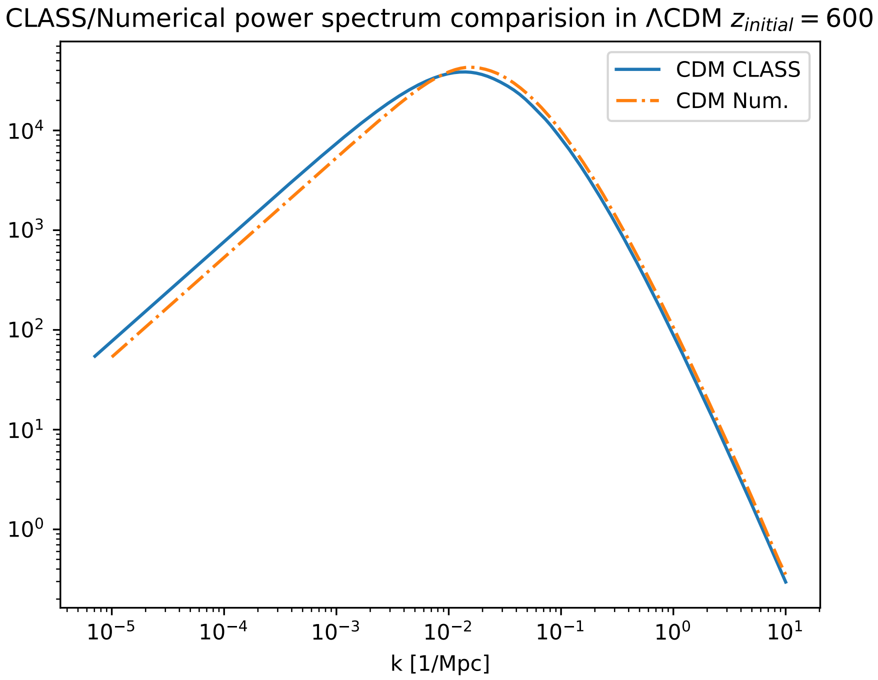


FIGURE 4.1: This figure shows the comparison between the numerical solution of eq. (3.6) and the solutions from CLASS in both Λcdmand the Einsten deSitter Model with varying k. The numerical solution were initialized at $z = 600$..

As a way to check how well the numerical solutions worked compared to CLASS as well as a step towards the Bispectrum, we then created power spectres in both Λcdmand in EdS. This time we also varied the initial z value To illustrate its effect. As can we see on fig. 4.2 and fig. 4.3, our numerical power spectrum matches the behaviour and scale of the Power spectrum form CLASS. The Λcdmz_{ini} = 600, fig. 4.2, is the one that is the most off. The numerical solutions worked the best in the Einstein deSitter model and at z_{ini} $z = 100$(fig. 4.3), where is is effectively a $1 : 1$ overlap. This is not that surprising since the EdS. model is the simpler model and and with a lower z there is less time for the solution to go off.



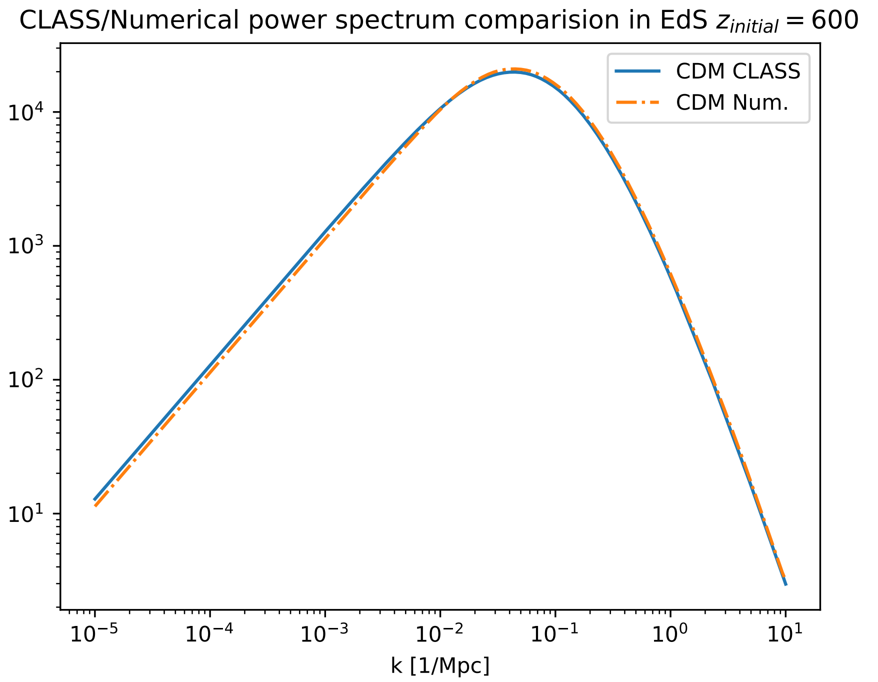
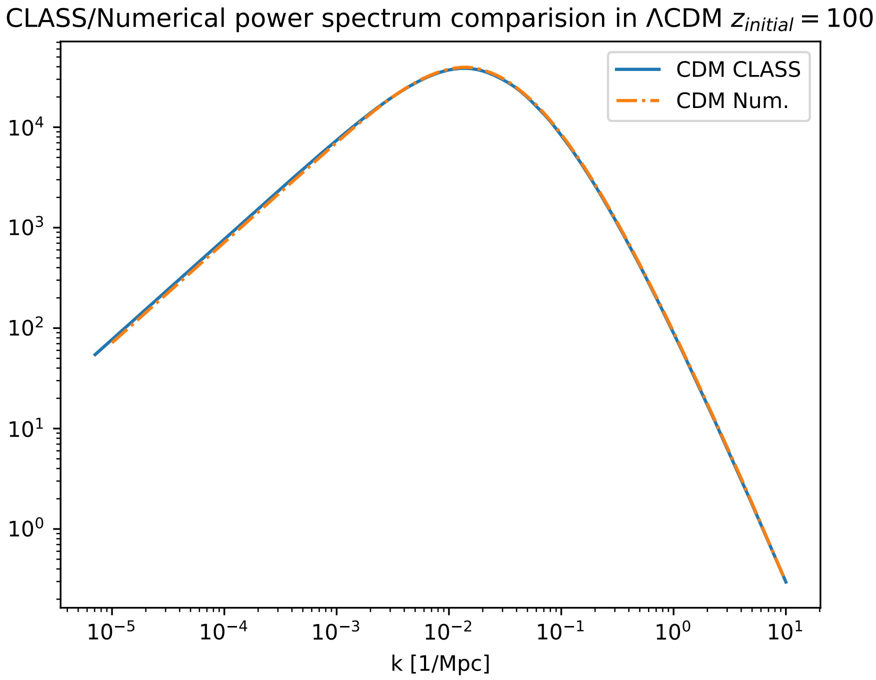


FIGURE 4.2: Comparison of the CLASS power spectrum and the power spectrum created by the eq. (2.19) and eq. (2.20) in the Λcdm. .



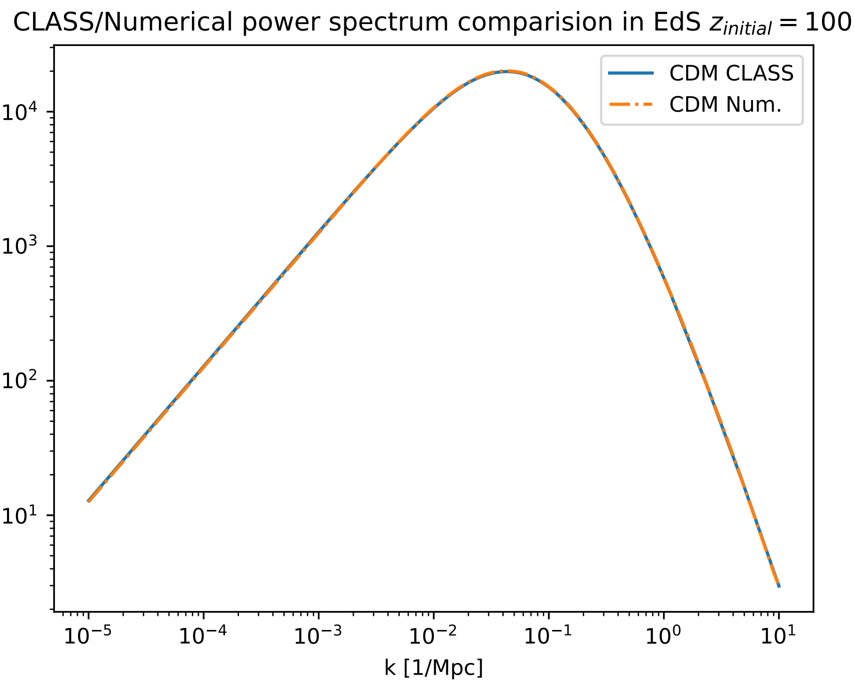


FIGURE 4.3: Comparison of the CLASS power spectrum and the power spectrum created by the eq. (2.19) and eq. (2.20) in the Λcdm. .

4.4 Neutrinos

After working with the CDM power spectra we then began work on solving the neutrino equations, as has been explained earlier, to describe the behaviour of the neutrinos, we only needed to implement a sound speed in the CDM equations and then take initial values for neutrinos although we did also need initial CDM values as we source from the CDM. The fascinating thing about the sound is its strength. In [11] a value for the sound speed is given as

$$\sigma_s(z)^2 = \frac{5}{9}\sigma_\nu^2 = \frac{5}{9}\frac{15\zeta(5)}{\zeta(3)}\left(\frac{4}{11}\right)^{2/3}\frac{T_{\gamma,0}(1+z)^2}{m_\nu^2} \tag{4.1}$$

where $T_{\gamma,0}$ is the initial temperature of the photons and m_ν^2 the mass of the neutrino. Now replacing $\left(\frac{4}{11}\right)^{2/3} T_{\gamma,0}$ with $T_{\nu,0}$ σ_s and so we get that

$$\sigma_s(a) \simeq 2.68\frac{T_{\nu,0}}{m_\nu a} \tag{4.2}$$

This is our "theoretical" value for the sound speed but we did try other values to seen if there was anything that might work better. We give the cases of using $\frac{2.68}{2}$ and 2 as comparisons to 2.68. We will discuss the these when we get there.

Starting things off, beside the strength of the sound speed we could also vary the mass of the Neutrinos. The numerical solution for the k-values $0.1, 1$ and 10 for Neutrinos with mass' 0.1 and 0.4 eV in both Λcdmand in the Einstein Model, using eq. (4.2) are shown in fig. 4.4 fig. 4.5. In these two figures, as a rule, the numerical solutions lack behind. They do follow the correct behaviour of neutrinos, which is good. But in the cases of Neutrinos with mass $m_\nu = 0.1$eV where the sound speed is thus a factor of $1/0.4$ strong than in the $m_\nu = 0.4$eV cases, they simple couple to the dark matter later than the Neutrinos of CLASS. This is seen the clearest in the case of fig. 4.4 in the lower left corner where $k = 10$. IN general the lower the k-value and the higher the mass the better the numerical solution fitted to CLASS. This may be seen in the $k = 0.1$ and 1 cases in fig. 4.4 and fig. 4.5.

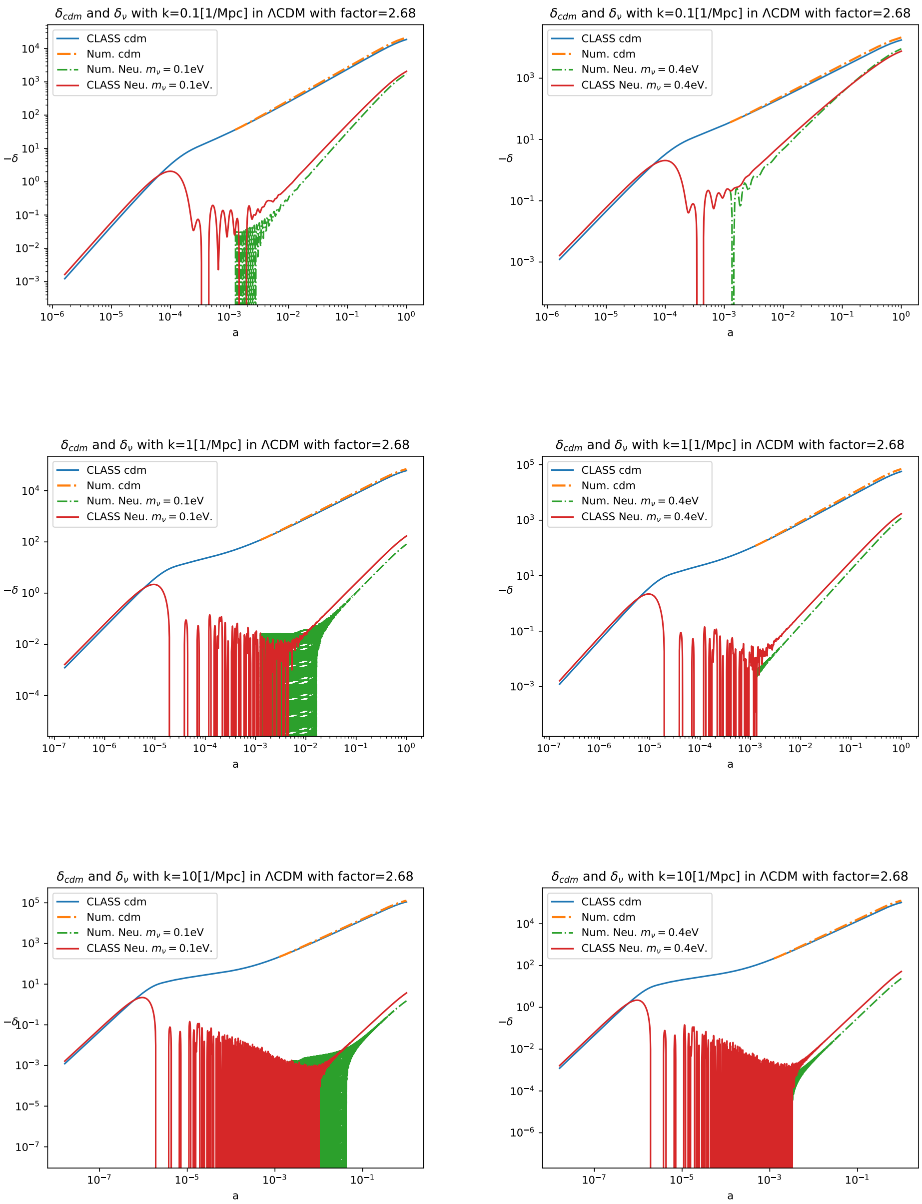


FIGURE 4.4: This figure shows the comparison between the numerical solution of eq. (3.6) and the solutions from CLASS with different neutrino mass' and k value. Here we are in Λcdm. Initialized at $z = 600$ using 2.68 as the strength of our neutrino sound speed..

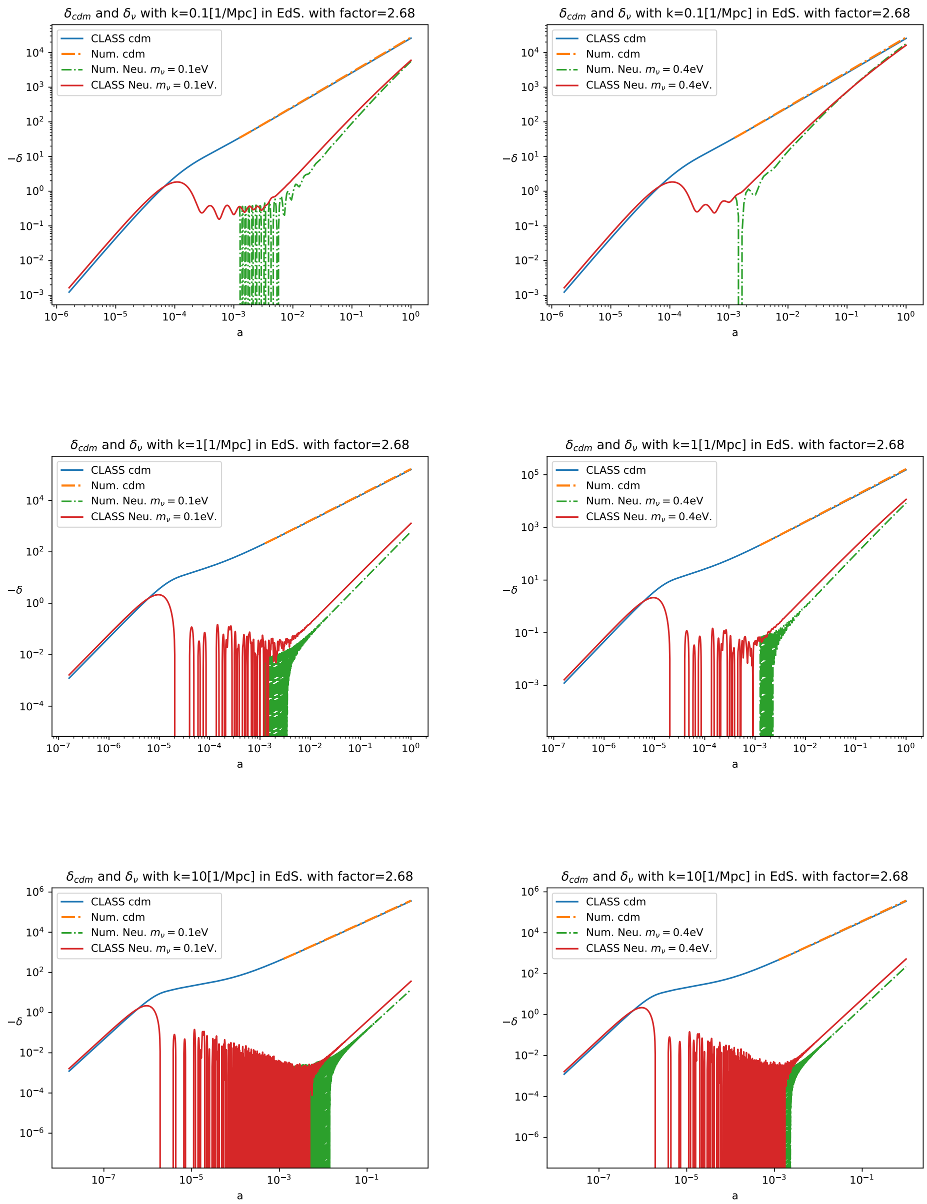


FIGURE 4.5: This figure shows the comparison between the numerical solution of eq. (3.6) and the solutions from CLASS with different neutrino mass' and k value. Here we are in Einstein deSitter. Initialized at $z = 600$ using 2.68 as the strength of our neutrino sound speed..

Many of the statements about fig. 4.5 and fig. 4.4 also apply for fig. 4.6 . The purpose of fig. 4.6 was not so much to see the effectiveness of the sound speed as the effectiveness of using eq. (3.23) to act as a source for the Neutrinos. So comparing it fig. 4.6 with fig. 4.4 we see that the figures are very very similar. The CDM approximation as plotted in fig. 4.6 is slightly off as compared with the numerical solution but it is not by any means a big offset.

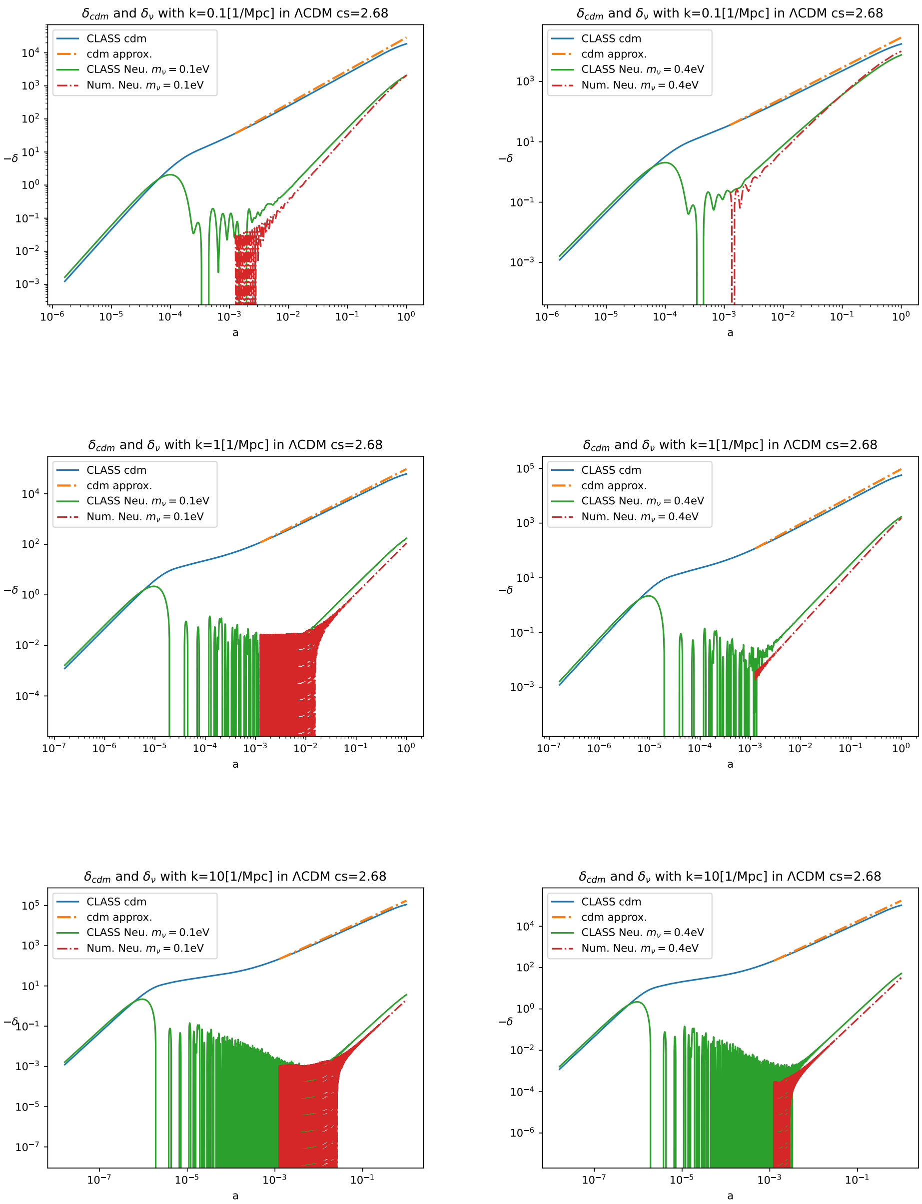


FIGURE 4.6: This figure shows the comparison between the numerical solution of eq. (3.6) and eq. (3.23) and the solutions from CLASS with different neutrino mass' and k value. Here we are in Λcdm. Unlike the previous figures the Neutrinos no source from an approximation of the CDM eq. (3.18). Initialized at $z = 600$.

fig. 4.7 is one of the more important figures as it shows the effects of the strength of the sound speed. In fig. 4.7 we can see that while the Neutrinos with a sound speed strength of $\frac{2.68}{2}$ essentially have the opposite problem that those using eq. (4.2) (2.68) But the Neutrinos who have a sound speed factor of 2 actually fits to Neutrinos from CLASS better that the once using the "theoretical" value. Especially the case of $k = 1$ in fig. 4.7 fits extremely well with the CLASS solution. As one may expect we also observe that the weaker the sound speed the earlier do the numerical neutrinos couple to dark matter. The effect is not huge but it is there.

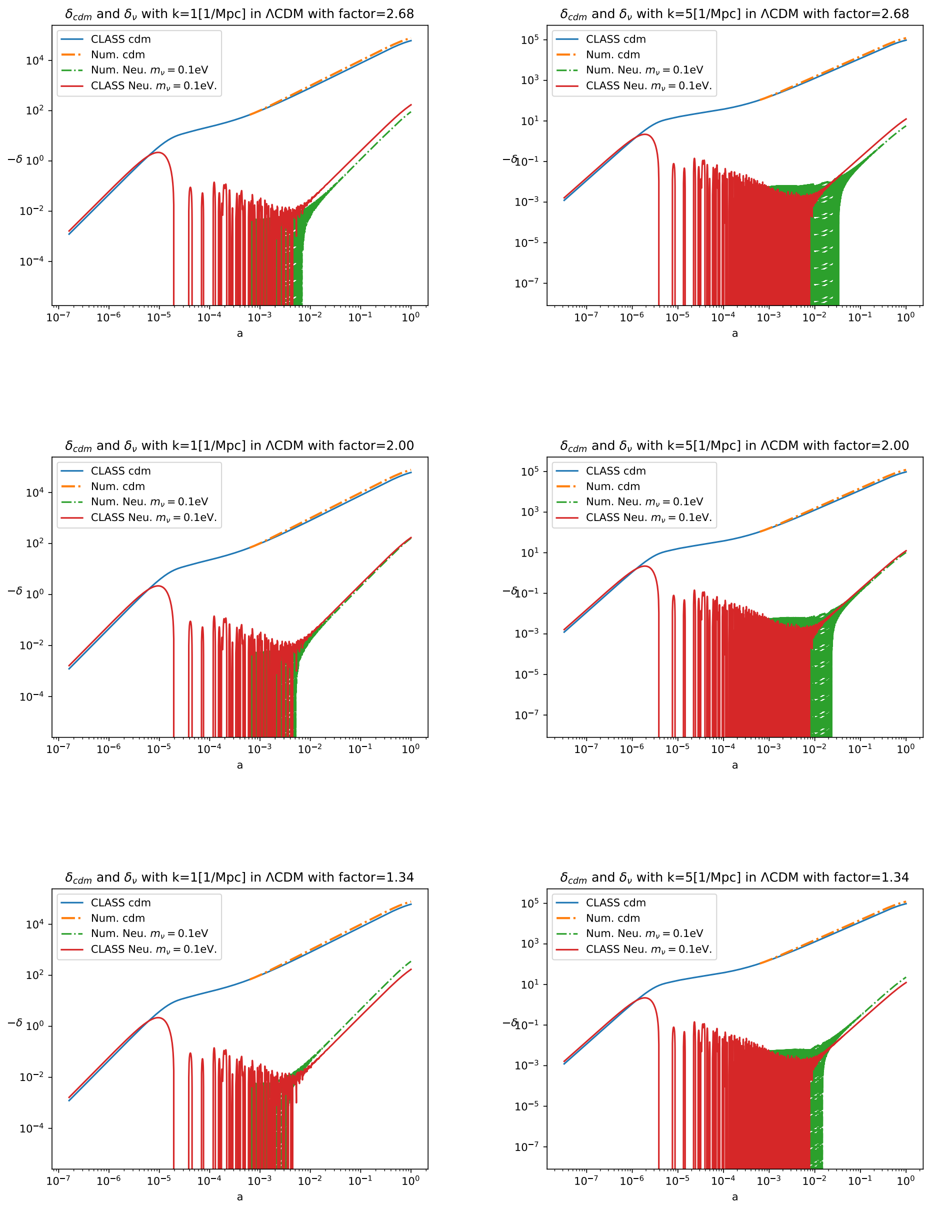


FIGURE 4.7: In this figure the effect of using different strengths of the neutrino sound speed are shown for two different neutrinos. .

4.5 Neutrino power spectrum

Building on fig. 4.7 and repeating the steps of the CDM-only solutions, we generated power spectre for the neutrinos in fig. 4.8, where we varied the sound speed strengths. In general we see that the numerical Neutrinos do follow the behaviour of CLASS however we also note a clear difference between the power spectre that used 1.34 as a factor than the other two. It is simply off and while the power spectre are virtually identical for k-value below 10^{-2} 1.34 is consistently off for k-values above 10^{-2}. mean while for the 2 and the 2.68 and interesting pattern emerges. the neutrinos using 2 are better for higher values of k while 2.68 is fits better to the neutrinos of CLASS in the $10^{-1} - 10^{-2}$ range. Finally we may also note the interesting fact that for k values below 10^{-2} fig. 4.7 show a very similar behaviour to that of Λcdmfigure in fig. 4.2 where the numerical solutions are consistently off

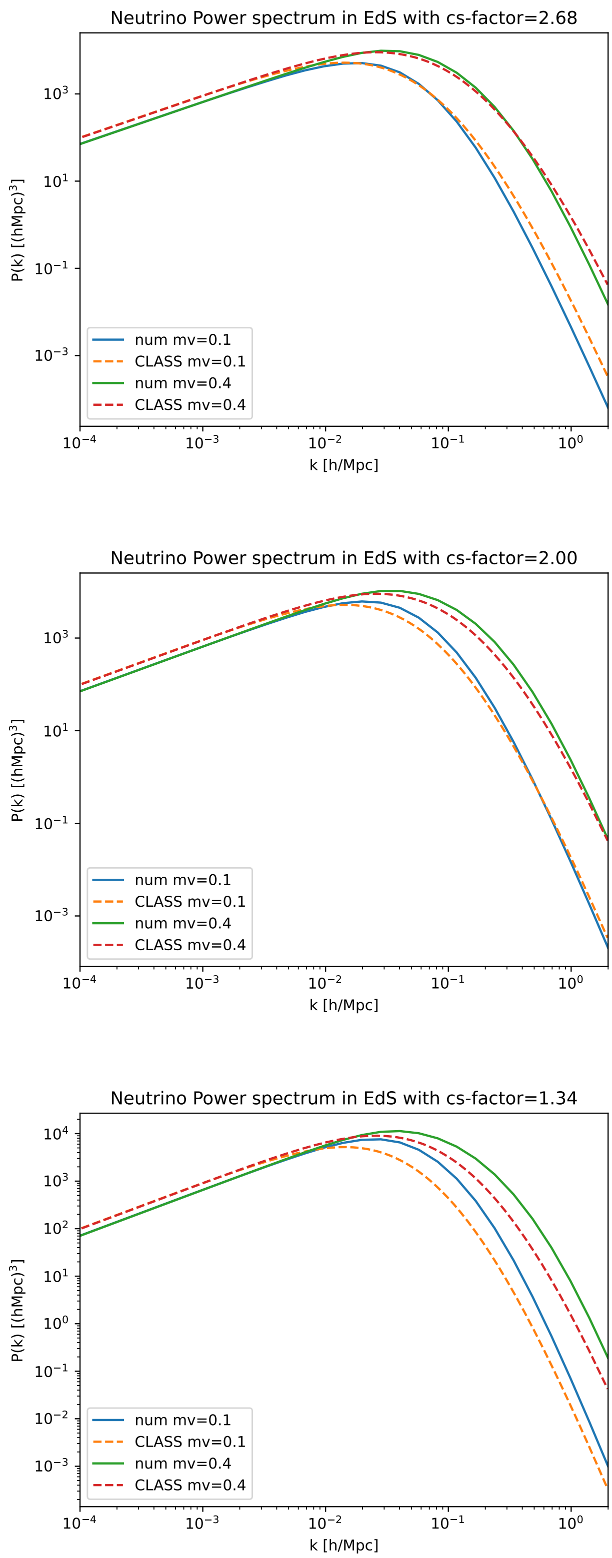


FIGURE 4.8: Comparison of the effects of the strength of the neutrino sound speed using power spectra.

This marked the end of us working in the Λcdmmodel. This is a result of us now using the equation which only counted in EdS.

4.6 Neutrino equation

Having shown the numerical part of this thesis we now turn to the Neutrino equation of eq. (3.46). In fig. 4.9 and fig. 4.10 we see how well the equation works as compared to both CLASS and the numerical solutions of the previous figures, where we vary the mass and k values. We note first off how well the equation actually fits the numerical solution. In fact the equation fits so well that there are several figures in fig. 4.9 and fig. 4.10 where we can't see the numerical solution because it is covered over by the equation. We can see in that for the lower $k$value 0.1 the equation does appear to be slightly out of phase with the numerical solutions. This out-of-phaseiness does not seem to have a serious effect that the equations as they still follow one another closely. Like in the precious figures were we looked at using different sound speed 8fig. 4.7) strengths we again see that 2 does appear to be a slightly better strength than 2.68.

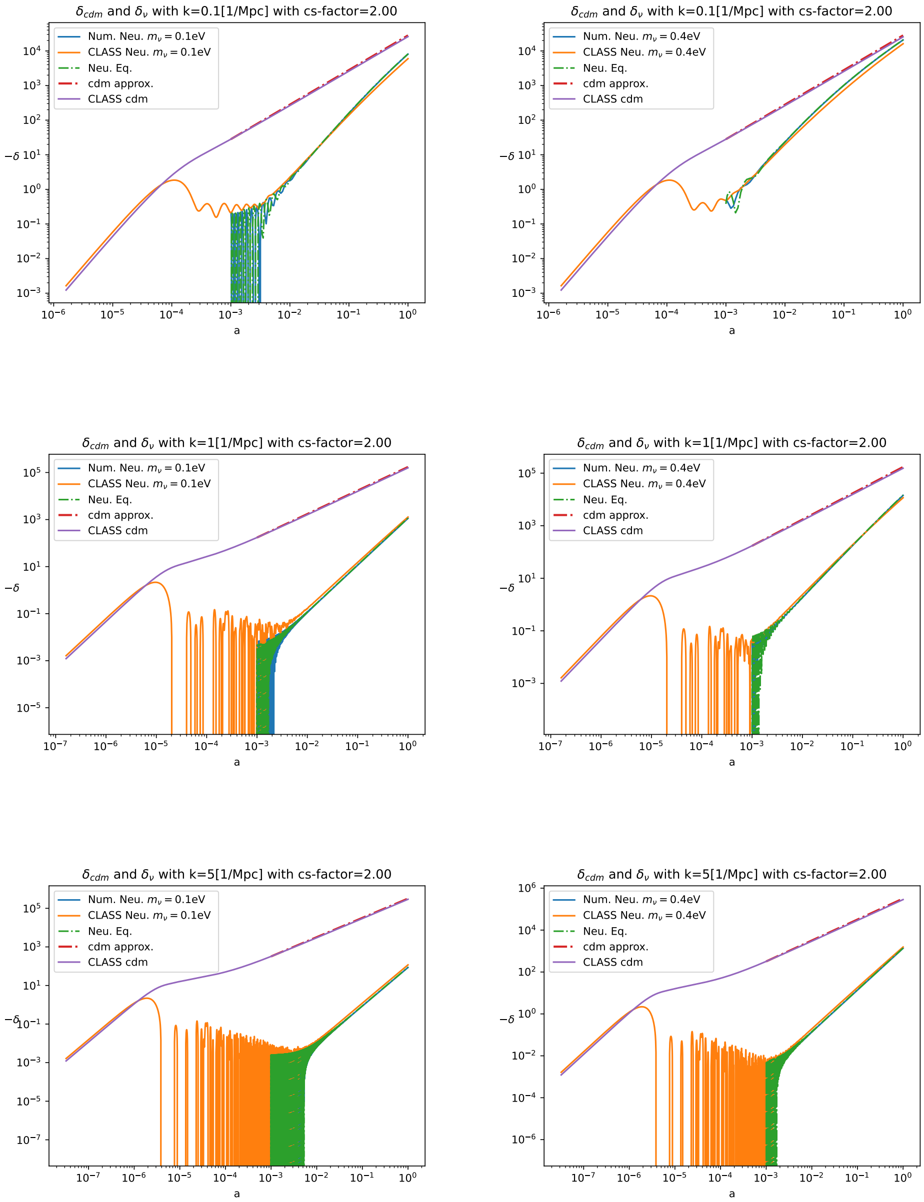


FIGURE 4.9: This figure shows the comparison between the neutrino solutions of CLASS, the numerical solutions and the Neutrino equation. Where we vary the mass and the $k$values and use a sound speed factor of 2. This figure is best compared with fig. 4.10 where we have set the sound speed strength to that of eq. (4.2) .

4.6.1 Power spectra

Testing the effectiveness of eq. (3.46) by making power spectra. we have fig. 4.11 where we have again varied the strength of the neutrino sound speed. fig. 4.11 is almost identical to that of fig. 4.8, this is not surprising as eq. (3.46) a solution to the equations that were solved so create fig. 4.8 . There is not a

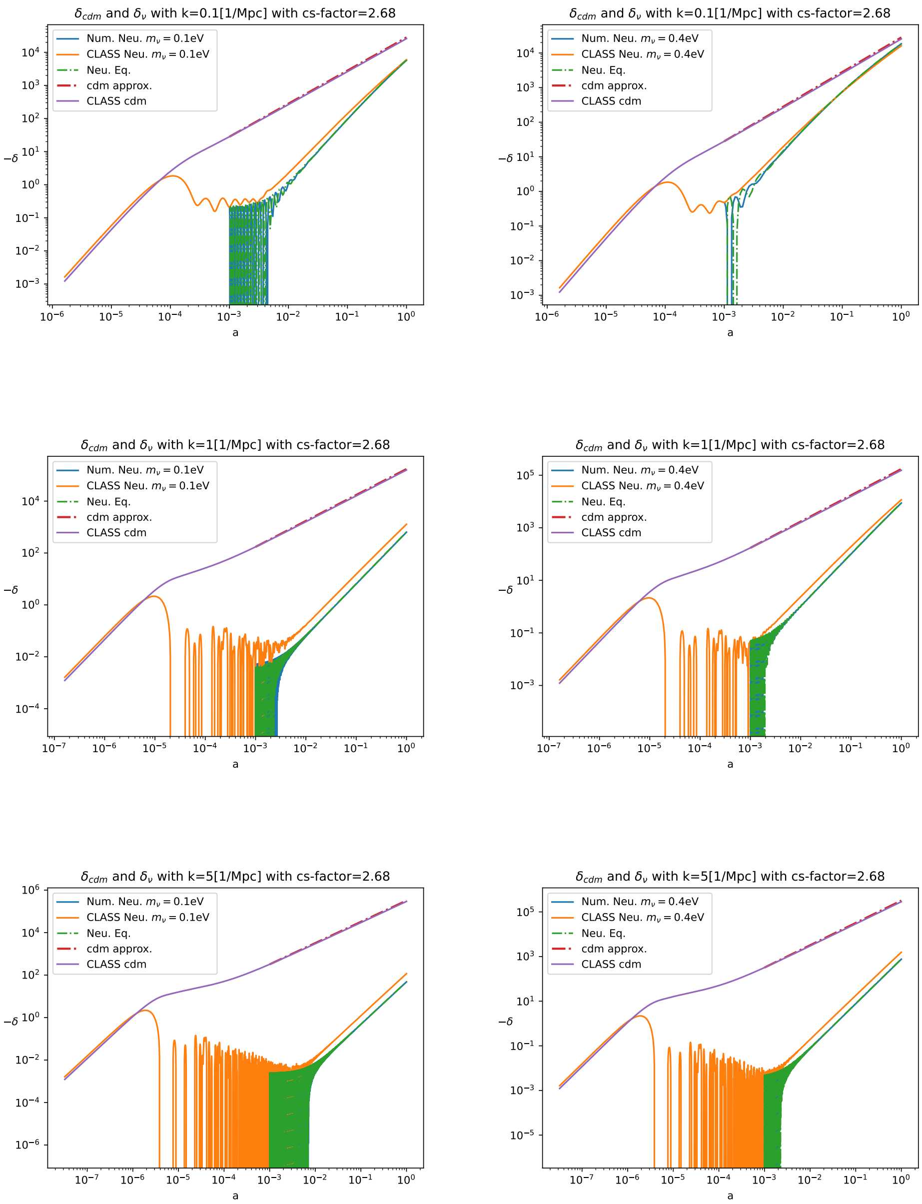
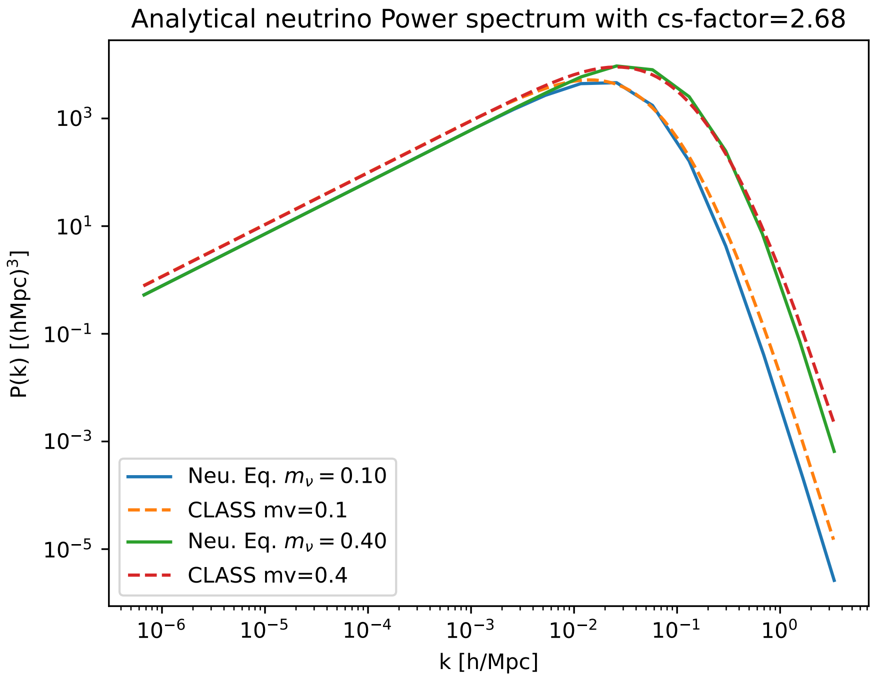


FIGURE 4.10: This figure shows the comparison between the neutrino solutions of CLASS, the numerical solutions and the Neutrino equation. Where we vary the mass and the $k$values and use a sound speed factor of 2.68. This figure is best compared with fig. 4.9 ..

lot to say that was not said in the fig. 4.8 section, except to note again how well the equation fits.



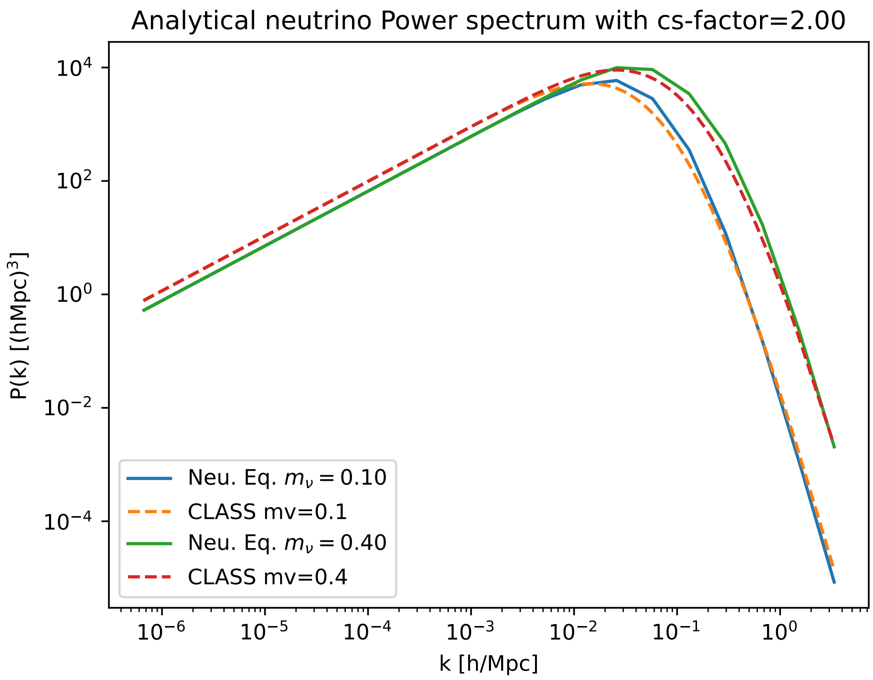


FIGURE 4.11: A comparison of Neutrino power spectra using different neutrino mass' and sound speed strengths, where we use the neutrino equation of eq. (3.46).

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4.7 Bispectrum

As stated before, we did not succeed on the Bispectrum front. However we did attempt, out of curiosity to replicate the Neutrino equilateral Bispectrum using the CDM kernel. The argument for that attempt was essentially that sense bispectra is a higher order phenomena and the neutrinos to couple to CDM for late time. It didn't work. Although we did implement an equilateral version of the kernel from eq. (3.12)

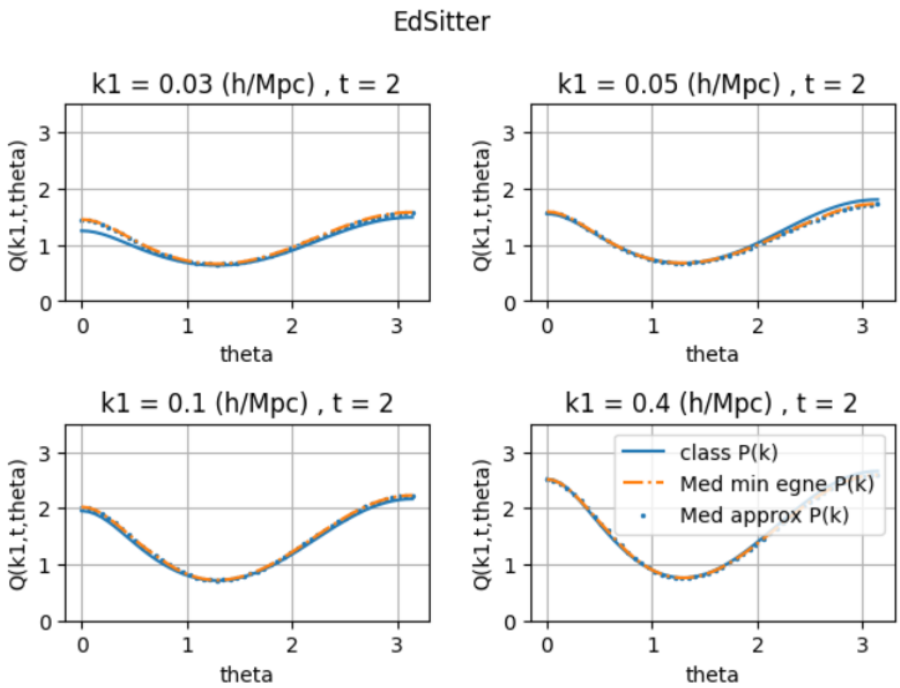

FIGURE 4.12: The CDM Equilateral Bispectrum in Einstein deSitter..

Which did work fairly fine

Find noget som du kan brug esom argument for at det virker. f.eks. Johans speciale.

As mentioned in the section on expanding the neutrino equation, that since neutrino will eventual start to follow the dark matter, we tried to see what would happen if we used the CDM kernel for the Neutrinos, so we would use combinations of $\delta_\nu^{(1)}$ but the kernel from eq. (3.12). This gave us
For those not well versed in the art of neutrino bispectra that is not what the neutrino power spectra are supposed to look like. Fortunately we can compare with the numerical bispectra generate in [17] and although fig. 4.12 does match well, but for fig. 4.13 the neutrino bispctra simply do not match.

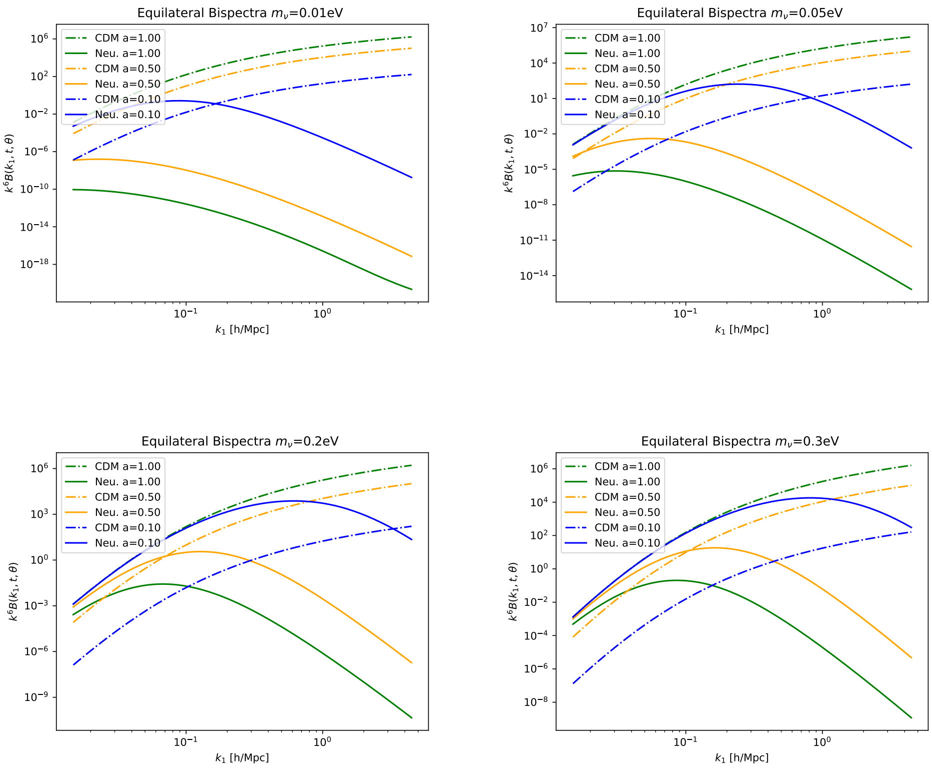

FIGURE 4.13: NEutrino/CDM bispectra using the CDM approximation from eq. (3.23) with a sound speed factor of 2.68 in Einstein deSitter. Depicted are the Bispectra for CDM and the bispectrum where we use the CDM kernel from eq. (3.12) on the neutrino $\delta^{(1)}$.

4.8 Approximations to the Neutrino equation

We also made some attempts to approximate the neutrino equation eq. (3.46), where we expanded around the sound speed. The initial reasoning behind this was that we believed that as the neutrinos started to follow to CDM, the effects of the sound speed would be negligible.
Finally we also make a figure to illustrate the effectively of these expansions as an extension to fig. 4.14 which is fig. 4.15-. As can we can see both orders of expansion work well from $a = 10^{-4}$ to around $a = 10^{-2}$ interestingly enough the 4th order expansion is actually worse that the second order expansion.

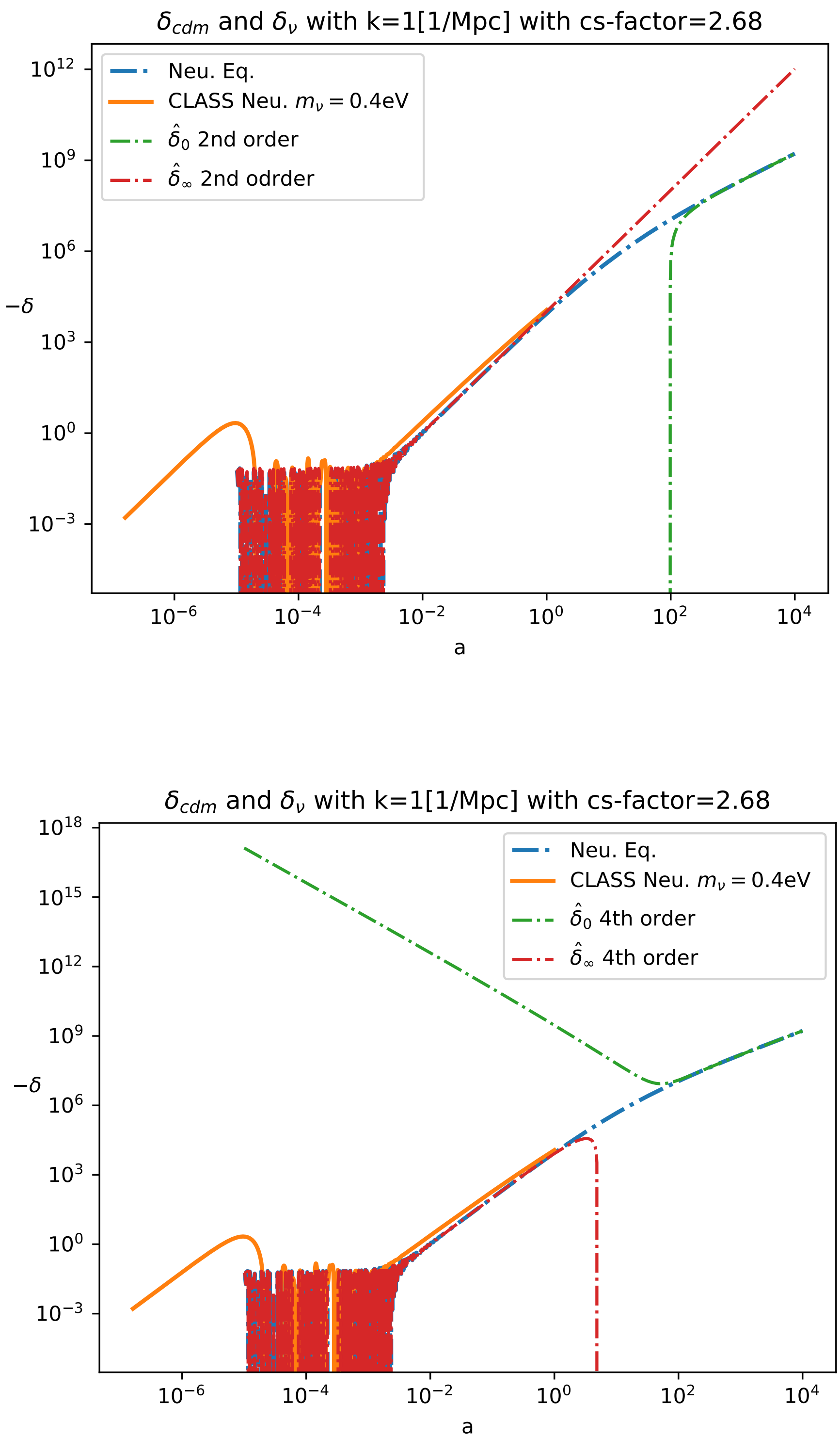

FIGURE 4.14: This figure shows the expansion of the sound speed around zero and infinity, compared with the neutrino contrast equation and from CLASS. The neutrino equation was initializaed at z=800. Unlike the previous figures we ahve here choosen to go far beyond $a = 1$..

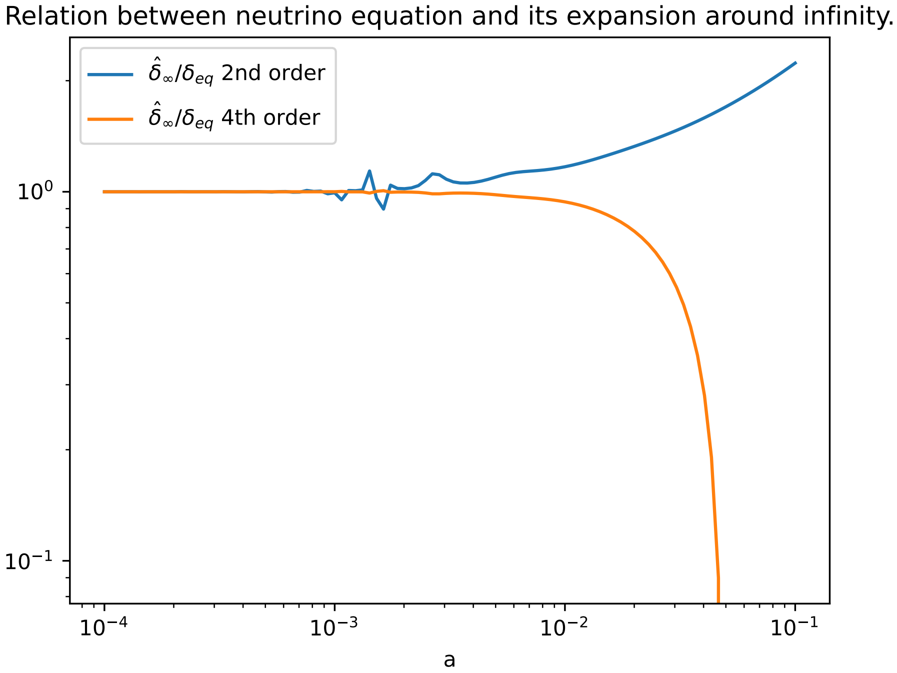

FIGURE 4.15: This figure depicts the ration between the neutrino equation and the two expansion that we made around zero and infinity. k=0.10.cs-factor=2.68 $m_\nu = 0.40$eV initialized at $z = 800$.

Chapter 5

Discussion

So we can see from the figures that our numerical solutions worked. Were able to numerically solve the equations for the CDM and neutrinos in both Λcdmand Einstein deSitter. Although the Λcdmsolutions do work well, he Einstein deSitter solution were in general better. This is not surprising when one considers that the Einstein deSitter model is a simpler model than Λcdmand we should thus not be surprised when the numerical solutions work better. As stated previously because we used a version of CLASS that did not have Newtonian motion gauge we had to use another gauge, we used the Synchronous gauge. We were thus required to work around the problem that θ was constantly zero. we did this by using second order equations where θ had been removed. Instead we approximated the derivative of δ by using an interpolated version of δ using "scipy.interpolate.interp1d" so we could make

$$\dot{\delta}_{approx} = \frac{\delta(a_0 + h/2) - \delta(a_0 - h/2)}{h} \tag{5.1}$$

where h was set to a low value and a_0 the starting point of the numerical solution. This appears to have worked. It is worth noting that even without this method and instead using θ from CLASS as we did in the beginning so $\dot{\delta}(a_0) = -\theta(a_0) = 0$, the numerical solution would still work although it would lack behind CLASS' solutions by a small offset[1] Unlike the numerical solutions, this derivative-problem was not much of a an issue for the Neutrino equation eq. (3.46) . In eq. (3.46) the is a $\dot{\delta}_\nu(a_0)$ dependent term, however we found no issue in setting this to zero. Our rational for this was that since for all our z_{ini} value (usually 600, but a few time we used 800) the neutrinos were still oscillating very intensively, the derivative of a specific z_{ini} should be set to be at a peak of an oscillation as these oscillations were so rapid. AS stated

1: Note til migselv, jeg skal lige dobbelttjekke om det ikke primært var et CDM problem, for jeg tror det var.

it worked, with no issue, we did also test with eq. (5.1) but these numerical values were consistently very small. Initializing at a much lower $z-$value of around 80 was also checked using both $\delta_\nu = 0$ and using eq. (5.1) and this did not alter the effectiveness of the equation.

The power spectra were a very useful way for us to check the success of the numerical and analytical solutions. Unsurprisingly the best power spectra fig. 4.3 as also the most simplistic, begin initialized only at $z_{init} = 100$ and using the Einstein deSitter model. So simple equations and with less time for errors to propagate. meanwhile the neutrino power spectrum using a sound speed of 1.34 was the most off from the power spectrum of CLASS as compared with all other power spectra, but it did still follow the behaviour of CLASS. But the more interesting ones were those of fig. 4.8 where we varied the the strength of the neutrinos sound speed. As stated the power spectrum where the sound speed factor was set to 1.34 was the one that was the most off as compared with CLASS. But interestingly it is not clear whether the values 2 or 2.68 is the better. Based on fig. 4.7 one might assume that that 2.68 would not be the best as 2.00 line up better with CLASS solution. it would appear that the equation slightly under performs for higher values of k when using 2.00 while the equation slightly over performs when using 2.68. This might suggest that neither 2.68 or 2.00 are optimal values and we should note that we simply pick 2 as an example. Finding a numerically optimal sound speed factor would thus be a clear next step forward.

The equation works. It is in fact quite remarkable how well it works. We would not initially have assume that using the sound speed method for the neutrinos to account for their oscillation, to replicate CLASS' solution as well as it did in fig. 4.7 where the sound speed fact was set to 2 and and using 2.68 did also work quite well as in fig. 4.10. Deriving the equation was of cause only possible because of the δ_{CDM} in Einstein deSitter eq. (3.23) which we called the approximation for simplicity, we also checked it in fig. 4.6. It fits quite well with the solutions of CLASS.

Of high interest is the expansions. Initially we believed that the expansion around zero would be the logical choice as we believed the sound speed to be negligible after the Neutrinos stopped oscillating, but that is apparently not the case. The figures in fig. 4.14 illustrates that the expansions around zero only work for high values of a ($a >> 1$). Meanwhile the expansion around infinite worked surprisingly well although they do start to break at around $a = 1$. Rigorously testing the limits of these expansions was not something that we did it would would be a clear next step to check just how well these expansions work.

Finally we have the Bispectrum. On this front we did not succeed. Our initial idea of testing whether one could use the bispectrum kernel for cold dark matter was in part motivated by our wrong assessment that the sound speed could be neglected after the oscillations. This would explain how fig. 4.13 did not match the numerical Neutrino bispectra from [17].

Chapter 6

Conclusion

We have successfully derived an analytical equation to describe the density contrast of neutrinos in Einstein deSitter along with an expansion for early time and investigations into the limits of it would be a logical next step. Although we did not make progress on the Bispectrum front, we have made progress with the sound speed and it would be of interest to see if it is possible to generate a neutrino Bispectrum using the Neutrino sound speed. We also conclude that the sound speed plays a more import role after the neutrinos start to follow the matter density contractions that we initially realized.
The value for the sound speed from [11] would appear to be a bit too strong and an investigation into a numerical value of the Neutrino factor would be a clear next step.

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