Team Reference Document
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Contents

Graph Algorithms

SCC

```
struct edge{int e, nxt;};
int V, E;
edge e[MAXE], er[MAXE];
int sp[MAXV], spr[MAXV];
int group_cnt, group_num[MAXV];
bool v[MAXV];
int stk[MAXV];
void fill_forward(int x)
{
  int i;
  v[x]=true;
  for(i=sp[x];i;i=e[i].nxt) if(!v[e[i].e]) fill_forward(e[i].e);
  stk[++stk[0]]=x;
}
void fill_backward(int x)
{
  int i;
  v[x]=false;
  group_num[x]=group_cnt;
  for(i=spr[x];i;i=er[i].nxt) if(v[er[i].e]) fill_backward(er[i].e);
}
void add_edge(int v1, int v2) //add edge v1->v2
{
  e [++E].e=v2; e [E].nxt=sp [v1]; sp [v1]=E;
  er[  E].e=v1; er[E].nxt=spr[v2]; spr[v2]=E;
}
void SCC()
{
  int i;
  stk[0]=0;
  memset(v, false, sizeof(v));
  for(i=1;i<=V;i++) if(!v[i]) fill_forward(i);
  group_cnt=0;
  for(i=stk[0];i>=1;i--) if(v[stk[i]]){group_cnt++; fill_backward(stk[i
      ]);}
}
```

EulerianPath

```
struct Edge;
typedef list<Edge>::iterator iter;

struct Edge
{
        int next_vertex;
        iter reverse_edge;

        Edge(int next_vertex)
                :next_vertex(next_vertex)
                { }
};

const int max_vertices = ;
int num_vertices;
list<Edge> adj[max_vertices];           // adjacency list

vector<int> path;

void find_path(int v)
{
        while(adj[v].size() > 0)
        {
                int vn = adj[v].front().next_vertex;
                adj[vn].erase(adj[v].front().reverse_edge);
                adj[v].pop_front();
                find_path(vn);
        }
        path.push_back(v);
}

void add_edge(int a, int b)
{
        adj[a].push_front(Edge(b));
        iter ita = adj[a].begin();
        adj[b].push_front(Edge(a));
        iter itb = adj[b].begin();
        ita->reverse_edge = itb;
        itb->reverse_edge = ita;
}
```

ArticulationPoints

```
vector<int> adj[MAXN];
bool used[MAXN];
int timer, tin[MAXN], fup[MAXN];
```

```
void dfs (int v, int p = -1) {
  used[v] = true;
  tin[v] = fup[v] = timer++;
  int children = 0;

  for (int to:adj[v]){
    if (to == p)  continue;
    if (used[to])
      fup[v] = min (fup[v], tin[to]);
    else {
      dfs (to, v);
      fup[v] = min (fup[v], fup[to]);
      if (fup[to] >= tin[v] && p != -1){
        IS_CUTPOINT(v);
      }
      ++children;
    }
  }
  if (p == -1 && children > 1)
    IS_CUTPOINT(v);
}
```

MinCostMaxFlow

```
// Implementation of min cost max flow algorithm using adjacency
// matrix (Edmonds and Karp 1972).  This implementation keeps track of
// forward and reverse edges separately (so you can set cap[i][j] !=
// cap[j][i]).  For a regular max flow, set all edge costs to 0.
//
// Running time, O(|V|^2) cost per augmentation
//     max flow:           O(|V|^3) augmentations
//     min cost max flow:  O(|V|^4 * MAX_EDGE_COST) augmentations
//
// INPUT:
//     - graph, constructed using AddEdge()
//     - source
//     - sink
//
// OUTPUT:
//     - (maximum flow value, minimum cost value)
//     - To obtain the actual flow, look at positive values only.

typedef vector<int> VI;
typedef vector<VI> VVI;
typedef long long L;
typedef vector<L> VL;
typedef vector<VL> VVL;
typedef pair<int, int> PII;
typedef vector<PII> VPII;

const L INF = numeric_limits<L>::max() / 4;

struct MinCostMaxFlow {
  int N;
  VVL cap, flow, cost;
  VI found;
  VL dist, pi, width;
  VPII dad;

  MinCostMaxFlow(int N) :
    N(N), cap(N, VL(N)), flow(N, VL(N)), cost(N, VL(N)),
    found(N), dist(N), pi(N), width(N), dad(N) {}

  void AddEdge(int from, int to, L cap, L cost) {
    this->cap[from][to] = cap;
    this->cost[from][to] = cost;
  }

  void Relax(int s, int k, L cap, L cost, int dir) {
    L val = dist[s] + pi[s] - pi[k] + cost;
    if (cap && val < dist[k]) {
      dist[k] = val;
      dad[k] = make_pair(s, dir);
      width[k] = min(cap, width[s]);
    }
  }

  L Dijkstra(int s, int t) {
    fill(found.begin(), found.end(), false);
    fill(dist.begin(), dist.end(), INF);
    fill(width.begin(), width.end(), 0);
    dist[s] = 0;
    width[s] = INF;

    while (s != -1) {
      int best = -1;
      found[s] = true;
      for (int k = 0; k < N; k++) {
        if (found[k]) continue;
        Relax(s, k, cap[s][k] - flow[s][k], cost[s][k], 1);
        Relax(s, k, flow[k][s], -cost[k][s], -1);
        if (best == -1 || dist[k] < dist[best]) best = k;
      }
      s = best;
    }

    for (int k = 0; k < N; k++)
      pi[k] = min(pi[k] + dist[k], INF);
    return width[t];
  }

  pair<L, L> GetMaxFlow(int s, int t) {
    L totflow = 0, totcost = 0;
    while (L amt = Dijkstra(s, t)) {
      totflow += amt;
      for (int x = t; x != s; x = dad[x].first) {
        if (dad[x].second == 1) {
          flow[dad[x].first][x] += amt;
          totcost += amt * cost[dad[x].first][x];
        } else {
          flow[x][dad[x].first] -= amt;
          totcost -= amt * cost[x][dad[x].first];
        }
      }
```

```
    }
    return make_pair(totflow, totcost);
  }
};
```

MaxFlow: PushRelabel

```
// Adjacency list implementation of FIFO push relabel maximum flow
// with the gap relabeling heuristic.  This implementation is
// significantly faster than straight Ford-Fulkerson.  It solves
// random problems with 10000 vertices and 1000000 edges in a few
// seconds, though it is possible to construct test cases that
// achieve the worst-case.
//
// Running time:
//     O(|V|^3)
//
// INPUT:
//     - graph, constructed using AddEdge()
//     - source
//     - sink
//
// OUTPUT:
//     - maximum flow value
//     - To obtain the actual flow values, look at all edges with
//       capacity > 0 (zero capacity edges are residual edges).

typedef long long LL;

struct Edge {
  int from, to, cap, flow, index;
  Edge(int from, int to, int cap, int flow, int index) :
    from(from), to(to), cap(cap), flow(flow), index(index) {}
};

struct PushRelabel {
  int N;
  vector<vector<Edge> > G;
  vector<LL> excess;
  vector<int> dist, active, count;
  queue<int> Q;

  PushRelabel(int N) : N(N), G(N), excess(N), dist(N), active(N), count
      (2*N) {}

  void AddEdge(int from, int to, int cap) {
    G[from].push_back(Edge(from, to, cap, 0, G[to].size()));
    if (from == to) G[from].back().index++;
    G[to].push_back(Edge(to, from, 0, 0, G[from].size() - 1));
  }

  void Enqueue(int v) {
    if (!active[v] && excess[v] > 0) { active[v] = true; Q.push(v); }
  }

  void Push(Edge &e) {
    int amt = int(min(excess[e.from], LL(e.cap - e.flow)));
    if (dist[e.from] <= dist[e.to] || amt == 0) return;
    e.flow += amt;
    G[e.to][e.index].flow -= amt;
    excess[e.to] += amt;
    excess[e.from] -= amt;
    Enqueue(e.to);
  }

  void Gap(int k) {
    for (int v = 0; v < N; v++) {
      if (dist[v] < k) continue;
      count[dist[v]]--;
      dist[v] = max(dist[v], N+1);
      count[dist[v]]++;
      Enqueue(v);
    }
  }

  void Relabel(int v) {
    count[dist[v]]--;
    dist[v] = 2*N;
    for (int i = 0; i < G[v].size(); i++)
      if (G[v][i].cap - G[v][i].flow > 0)
        dist[v] = min(dist[v], dist[G[v][i].to] + 1);
    count[dist[v]]++;
    Enqueue(v);
  }

  void Discharge(int v) {
    for (int i = 0; excess[v] > 0 && i < G[v].size(); i++) Push(G[v][i])
        ;
    if (excess[v] > 0) {
      if (count[dist[v]] == 1)
        Gap(dist[v]);
      else
        Relabel(v);
    }
  }

  LL GetMaxFlow(int s, int t) {
    count[0] = N-1;
    count[N] = 1;
    dist[s] = N;
    active[s] = active[t] = true;
    for (int i = 0; i < G[s].size(); i++) {
      excess[s] += G[s][i].cap;
      Push(G[s][i]);
    }

    while (!Q.empty()) {
      int v = Q.front();
      Q.pop();
      active[v] = false;
      Discharge(v);
    }

    LL totflow = 0;
    for (int i = 0; i < G[s].size(); i++) totflow += G[s][i].flow;
```

```
    return totflow;
  }
};
```

MaxFlow: Dinic

```
#defineNN 105 // the maximum number of vertices
int cap[NN][NN], deg[NN], adj[NN][NN]; //cap[u][v] is the capacity of
    the edge u->v
int q[NN], prev[NN]; // BFS stuff
int dinic( int n, int s, int t ) {
  int flow = 0;
  while( true ){ // find an augmenting path
    memset( prev, -1, sizeof( prev ) );
  int qf = 0, qb = 0;  prev[q[qb++] = s] = -2;
  while( qb > qf && prev[t] == -1 )
    for( int u = q[qf++], i = 0, v; i < deg[u]; i++ )
      if( prev[v = adj[u][i]] == -1 && cap[u][v] )
        prev[q[qb++] = v] = u;
  if( prev[t] == -1 ) break; // we're done finding paths
    for( int z = 0; z <n; z++ )
      if( cap[z][t] && prev[z] != -1 ){
        int bot = cap[z][t];
        for( int v = z,u = prev[v]; u >= 0;v = u,u = prev[v] )
          bot = min(bot,cap[u][v]);
        if( !bot ) continue;
          cap[z][t] -= bot; cap[t][z] += bot;
        for( int v =z,u = prev[v]; u >= 0;v = u, u =prev[v] )
          cap[u][v] -= bot; cap[v][u] += bot;
        flow += bot;
      }
  }
  return flow;
}

int main() {
  memset( cap, 0, sizeof( cap ) );
  int n, s, t, m;
  scanf( " %d %d %d %d", &n, &s, &t, &m );
  while( m-- ) {
    int u, v, c; scanf( " %d %d %d", &u, &v, &c );
    cap[u][v] = c;
  }
  memset( deg, 0, sizeof( deg ) );
  for( int u = 0; u < n; u++ )
    for( int v = 0; v < n; v++ )
      if( cap[u][v] || cap[v][u] ) adj[u][deg[u]++] = v;
  printf( "%d\n", dinic( n, s, t ) );
}
```

MinCostMatching

```
//
    //////////////////////////////////////////////////////////////////////
// Min cost bipartite matching via shortest augmenting paths
//
// This is an O(n^3) implementation of a shortest augmenting path
// algorithm for finding min cost perfect matchings in dense
// graphs.  In practice, it solves 1000x1000 problems in around 1
// second.
//
//   cost[i][j] = cost for pairing left node i with right node j
//   Lmate[i] = index of right node that left node i pairs with
//   Rmate[j] = index of left node that right node j pairs with
//
// The values in cost[i][j] may be positive or negative.  To perform
// maximization, simply negate the cost[][] matrix.
//
    //////////////////////////////////////////////////////////////////////


typedef vector<double> VD;
typedef vector<VD> VVD;
typedef vector<int> VI;

double MinCostMatching(const VVD &cost, VI &Lmate, VI &Rmate) {
  int n = int(cost.size());

  // construct dual feasible solution
  VD u(n);
  VD v(n);
  for (int i = 0; i < n; i++) {
    u[i] = cost[i][0];
    for (int j = 1; j < n; j++) u[i] = min(u[i], cost[i][j]);
  }
  for (int j = 0; j < n; j++) {
    v[j] = cost[0][j] - u[0];
    for (int i = 1; i < n; i++) v[j] = min(v[j], cost[i][j] - u[i]);
  }

  // construct primal solution satisfying complementary slackness
  Lmate = VI(n, -1);
  Rmate = VI(n, -1);
  int mated = 0;
  for (int i = 0; i < n; i++) {
    for (int j = 0; j < n; j++) {
      if (Rmate[j] != -1) continue;
      if (fabs(cost[i][j] - u[i] - v[j]) < 1e-10) {
        Lmate[i] = j;
        Rmate[j] = i;
        mated++;
        break;
      }
    }
  }

  VD dist(n);
  VI dad(n);
  VI seen(n);

  // repeat until primal solution is feasible
  while (mated < n) {

    // find an unmatched left node
```

```
  int s = 0;
  while (Lmate[s] != -1) s++;

  // initialize Dijkstra
  fill(dad.begin(), dad.end(), -1);
  fill(seen.begin(), seen.end(), 0);
  for (int k = 0; k < n; k++)
    dist[k] = cost[s][k] - u[s] - v[k];

  int j = 0;
  while (true) {

    // find closest
    j = -1;
    for (int k = 0; k < n; k++) {
      if (seen[k]) continue;
      if (j == -1 || dist[k] < dist[j]) j = k;
    }
    seen[j] = 1;

    // termination condition
    if (Rmate[j] == -1) break;

    // relax neighbors
    const int i = Rmate[j];
    for (int k = 0; k < n; k++) {
      if (seen[k]) continue;
      const double new_dist = dist[j] + cost[i][k] - u[i] - v[k];
      if (dist[k] > new_dist) {
        dist[k] = new_dist;
        dad[k] = j;
      }
    }
  }

  // update dual variables
  for (int k = 0; k < n; k++) {
    if (k == j || !seen[k]) continue;
    const int i = Rmate[k];
    v[k] += dist[k] - dist[j];
    u[i] -= dist[k] - dist[j];
  }
  u[s] += dist[j];

  // augment along path
  while (dad[j] >= 0) {
    const int d = dad[j];
    Rmate[j] = Rmate[d];
    Lmate[Rmate[j]] = j;
    j = d;
  }
  Rmate[j] = s;
  Lmate[s] = j;

  mated++;
}

double value = 0;
  for (int i = 0; i < n; i++)
    value += cost[i][Lmate[i]];

  return value;
}
```

MaxBipartiteMatching

```
// This code performs maximum bipartite matching.
//
// Running time: O(|E| |V|) -- often much faster in practice
//
//   INPUT: w[i][j] = edge between row node i and column node j
//   OUTPUT: mr[i] = assignment for row node i, -1 if unassigned
//           mc[j] = assignment for column node j, -1 if unassigned
//           function returns number of matches made

typedef vector<int> VI;
typedef vector<VI> VVI;

bool FindMatch(int i, const VVI &w, VI &mr, VI &mc, VI &seen) {
  for (int j = 0; j < w[i].size(); j++) {
    if (w[i][j] && !seen[j]) {
      seen[j] = true;
      if (mc[j] < 0 || FindMatch(mc[j], w, mr, mc, seen)) {
        mr[i] = j;
        mc[j] = i;
        return true;
      }
    }
  }
  return false;
}

int BipartiteMatching(const VVI &w, VI &mr, VI &mc) {
  mr = VI(w.size(), -1);
  mc = VI(w[0].size(), -1);

  int ct = 0;
  for (int i = 0; i < w.size(); i++) {
    VI seen(w[0].size());
    if (FindMatch(i, w, mr, mc, seen)) ct++;
  }
  return ct;
}
```

MinCut

```
// Adjacency matrix implementation of Stoer-Wagner min cut algorithm.
//
// Running time:
//     O(|V|^3)
//
// INPUT:
//     - graph, constructed using AddEdge()
//
// OUTPUT:
```

```
//     - (min cut value, nodes in half of min cut)

typedef vector<int> VI;
typedef vector<VI> VVI;

const int INF = 1000000000;

pair<int, VI> GetMinCut(VVI &weights) {
  int N = weights.size();
  VI used(N), cut, best_cut;
  int best_weight = -1;

  for (int phase = N-1; phase >= 0; phase--) {
    VI w = weights[0];
    VI added = used;
    int prev, last = 0;
    for (int i = 0; i < phase; i++) {
      prev = last;
      last = -1;
      for (int j = 1; j < N; j++)
        if (!added[j] && (last == -1 || w[j] > w[last])) last = j;
      if (i == phase-1) {
        for (int j = 0; j < N; j++) weights[prev][j] += weights[last][j
            ];
        for (int j = 0; j < N; j++) weights[j][prev] = weights[prev][j];
        used[last] = true;
        cut.push_back(last);
        if (best_weight == -1 || w[last] < best_weight) {
          best_cut = cut;
          best_weight = w[last];
        }
      } else {
        for (int j = 0; j < N; j++)
          w[j] += weights[last][j];
        added[last] = true;
      }
    }
  }
  return make_pair(best_weight, best_cut);
}
```

Trees

Segment Tree

```
// TODO: Define num_elems (~N)
const int num_elems = 1 << 20;
const int seg_size = 2 * num_elems;
const int off = num_elems - 1;
int segtree[seg_size];

int left(int x) {return 2 * x + 1;}
int right(int x) {return 2 * x + 2;}
int parent(int x) {return (x - 1) / 2;};

// TOTO: Define Operator. Example: Sum.
int op (int a, int b) {return a + b; }
```

```
void update (int pos) {
  segtree[pos] = op(segtree[left(pos)], segtree[right(pos)]);
  if (parent (pos) != pos) update(parent(pos)); }

void set (int pos, int data) {
  segtree[pos + off] = data;
  update(parent(pos + off)); }

int query (int i, int j, int l, int r, int curr_node) {
  if (i <= l && j >= r) return segtree[curr_node];
  if (i > r || j < l) return 0; // Neutral Element
  int m = (l + r) / 2;
  return op(query(i, j, l, m, left(curr_node)),
    query(i, j, m + 1, r, right(curr_node))); }

int query(int i, int j) { // op[i, j];
  return query(i, j, 0, off, 0); }

int main() {
  // Initialize
  fill_n(segtree, seg_size, 0);

return 0; }
```

BIT

```
int bit[M],n;
void add(int i, int v){
  for(; i <= n ; i += i & -i)
    bit[i] += v;
}
int sum(int i){
  int ret=0;
  for(; i >= 1 ; i -= i & -i)
    ret += bit[i];
  return ret;
}
```

BIT2D

```
int bit[M][M], n;
int sum( int x, int y ){
  int ret = 0;
  while( x > 0 ){
    int yy = y;
    while( yy > 0 )
      ret += bit[x][yy], yy -= yy & -yy;
    x -= (x & -x);
  }
  return ret;
}
void update(int x , int y , int val){
  int y1;
  while (x <= n){
    y1 = y;
    while (y1 <= n){ bit[x][y1] += val; y1 += (y1 & -y1); }
    x += (x & -x);
```

```
  }
}
```

RMQDP

```
#define better(a,b) A[a]<A[b]?(a):(b)
int make_dp(int n) { // N log N
  REP(i,n) H[i][0]=i;
  for(int l=0,k; (k=1<<l) < n; l++)
    for(int i=0;i+k<n;i++)
      H[i][l+1] = better(H[i][l], H[i+k][l]);
}
int query_dp(int a, int b) {
  int l = __lg(b-a);
  return better(H[a][l], H[b-(1<<l)+1][l]);
}
```

OrderStatisticsTree

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;

typedef tree<int, null_type, less<int>, rb_tree_tag,
    tree_order_statistics_node_update> ordered_set;
//null_mapped_type instead of null_type in older versions

int main()
{
  ordered_set X;
  for(int i=1;i<=10;i++)
    X.insert(i); // insert element

  cout<< X.order_of_key(7)<<endl; // result: 6 (number of keys less than 7)
  cout<< *X.find_by_order(6)<<endl; // result: 7 (6th element (0-based))

  X.erase(8); // remove element
  return 0;
}
```

LCA

```
typedef vector<int> VI;
typedef int INFO; // information that is written into the nodes

INFO combine(const INFO &a, const INFO &b){ // combines two infos, e.g. returns min(a,b)
  return min(a,b);
}

// Nodes are numbered 1..n
// After all edges are added with addEdge(), build(root) has to be called
// query(a,b) is used to make query of path a, ..., b
struct LCA_Tree{
  int N;
  vector<VI> adj, p;
  vector<vector<INFO> > p_info;
  vector<INFO> node_info;
  VI depth, log;

  LCA_Tree(int n):N(n), adj(n+1), p(n+1, VI(20)), p_info(n+1, vector<INFO>(20)), node_info(n+1), depth(n+1), log(n+1){
    for(int k=1, l=0; k <= N; k++){
      if((1<<(l+1)) <= k)
        l++;
      log[k] = l;
    }
  }

  void set_node_info(int i, INFO info){
    node_info[i] = info;
  }

  void addEdge(int a, int b){
    adj[a].push_back(b);
    adj[b].push_back(a);
  }

  void build(int root=1){
    dfs(root, root);
    for(int l=1; l<=log[N]; l++)
      for(int k=1; k<=N; k++){
        p[k][l] = p[ p[k][l-1] ][ l-1 ];
        p_info[k][l] = combine(p_info[k][l-1], p_info[ p[k][l-1] ][l-1]);
      }
  }

  void dfs(int i, int prev, int d=0){
    depth[i] = d;
    p[i][0] = prev;
    p_info[i][0] = node_info[prev];
    for(int c:adj[i])
      if(c!=prev)
        dfs(c, i, d+1);
  }

  INFO query(int a, int b){
    if(depth[a]>depth[b])
      swap(a,b);

    INFO res = node_info[b];

    while(depth[a]!=depth[b]){
      res = combine(res, p_info[b][ log[ depth[b]-depth[a] ] ]);
      b = p[ b ][ log[ depth[b]-depth[a] ] ];
    }

    if(a==b) // a is LCA
      return res;

    res = combine(res, node_info[a]);
```

```
    for(int l=log[N]; l>=0; l--)
      if(p[a][l]!=p[b][l]){
        res = combine(res, combine(p_info[a][l], p_info[b][l]));
        a = p[a][l], b = p[b][l];
      }

    return combine(res, p_info[a][0]); // a is LCA
  }
};


int main(){
  LCA_Tree tree(5);
  tree.addEdge(1,2);  tree.addEdge(1,4);   tree.addEdge(3,2);   tree.
      addEdge(5,2);
  for(int i=1;i<=5;i++)
    tree.set_node_info(i, i);

  tree.build(1);
  cout<<tree.query(2,5)<<endl; // 2
  cout<<tree.query(5,3)<<endl; // 2
  cout<<tree.query(4,3)<<endl; // 1
  cout<<tree.query(3,3)<<endl; // 3
  return 0;
}
```

∞ DP in Tree

```
#define MAXN 100005

typedef int DATA; // DP Data
typedef pair<int,int> SDATA; // Sum DP Data

struct EDGE{
  int to;
  int back_idx;
  DATA dp;

  EDGE(int t):to(t),dp(-1){}
};

vector<EDGE> adj[MAXN];
char vis[MAXN];
SDATA sum[MAXN];
EDGE* missing_edge[MAXN];

void add_to_sum(SDATA &s, const DATA &d){
  if(d > s.first){
    swap(s.first, s.second);
    s.first = d;
  }
  else if(d > s.second)
    s.second = d;
}

DATA sub_from_sum(const SDATA &s, const DATA &d){
  return s.first==d ? s.second : s.first;
}

void add_edge(int a, int b){ // adds edge in both directions
  adj[a].push_back(EDGE(b));
  adj[b].push_back(EDGE(a));
  adj[b].back().back_idx = adj[a].size()-1;
  adj[a].back().back_idx = adj[b].size()-1;
}

void dfs(int n, int from, EDGE& from_e){
  if(from_e.dp != -1)
    return;
  if(adj[n].size()==1 && from!=-1){ // leaf
      from_e.dp = 1;
      return;
    }

  if(vis[n] >= 1){
    if(missing_edge[n] != NULL){
          dfs(missing_edge[n]->to, n, *missing_edge[n]);
          add_to_sum(sum[n], missing_edge[n]->dp);
          missing_edge[n] = NULL;
        }

      if(from==-1)
        from_e.dp = sub_from_sum(sum[n], 0); // subtract sth. neutral
      else
        from_e.dp = sub_from_sum(sum[n], adj[n][from_e.back_idx].dp);

      from_e.dp++;
      return;
    }


  missing_edge[n] = NULL;
  sum[n] = SDATA(0,0);

  for(auto &e:adj[n]){
    if(e.to == from){
      missing_edge[n] = &e;
      continue;
    }

    dfs(e.to, n, e);
    add_to_sum(sum[n], e.dp);
  }

  from_e.dp = sub_from_sum(sum[n], 0) + 1; // subtract sth. neutral
  vis[n] = 1;
}

DATA calculate(int n){
  EDGE e(n);
  dfs(n,-1,e);
  return e.dp-1;
}
```

```
void init(int n){
  for(int i=0;i<=n;i++){
    adj[i].clear();
    vis[i] = 0;
  }
}

int main(){
  int n;
  cin >> n;

  init(n);

  for(int i = 0; i < n - 1; i++){
    int a, b; cin >> a >> b;
    add_edge(a+1,b+1);
  }

  int best_ttl=n+1;
  for(int i = 1; i <= n; i++)
    best_ttl = min(best_ttl,calculate(i));

  cout<<(best_ttl)<<endl;
  return 0;
}
```

Geometry

ConvexHull

```
// Compute the 2D convex hull of a set of points using the monotone
    chain
// algorithm.  Eliminate redundant points from the hull if
    REMOVE_REDUNDANT is
// #defined.
//
// Running time: O(n log n)
//
//   INPUT:   a vector of input points, unordered.
//   OUTPUT:  a vector of points in the convex hull, counterclockwise,
    starting
//            with bottommost/leftmost point

#define REMOVE_REDUNDANT

typedef double T;
const T EPS = 1e-7;
struct PT {
  T x, y;
  PT() {}
  PT(T x, T y) : x(x), y(y) {}
  bool operator<(const PT &rhs) const { return make_pair(y,x) <
      make_pair(rhs.y,rhs.x); }
  bool operator==(const PT &rhs) const { return make_pair(y,x) ==
      make_pair(rhs.y,rhs.x); }
};

T cross(PT p, PT q) { return p.x*q.y-p.y*q.x; }
T area2(PT a, PT b, PT c) { return cross(a,b) + cross(b,c) + cross(c,a);
     }

#ifdef REMOVE_REDUNDANT
bool between(const PT &a, const PT &b, const PT &c) {
  return (fabs(area2(a,b,c)) < EPS && (a.x-b.x)*(c.x-b.x) <= 0 && (a.y-b
      .y)*(c.y-b.y) <= 0);
}
#endif

void ConvexHull(vector<PT> &pts) {
  sort(pts.begin(), pts.end());
  pts.erase(unique(pts.begin(), pts.end()), pts.end());
  vector<PT> up, dn;
  for (int i = 0; i < pts.size(); i++) {
    while (up.size() > 1 && area2(up[up.size()-2], up.back(), pts[i]) >=
         0) up.pop_back();
    while (dn.size() > 1 && area2(dn[dn.size()-2], dn.back(), pts[i]) <=
         0) dn.pop_back();
    up.push_back(pts[i]);
    dn.push_back(pts[i]);
  }
  pts = dn;
  for (int i = (int) up.size() - 2; i >= 1; i--) pts.push_back(up[i]);

#ifdef REMOVE_REDUNDANT
  if (pts.size() <= 2) return;
  dn.clear();
  dn.push_back(pts[0]);
  dn.push_back(pts[1]);
  for (int i = 2; i < pts.size(); i++) {
    if (between(dn[dn.size()-2], dn[dn.size()-1], pts[i])) dn.pop_back()
        ;
    dn.push_back(pts[i]);
  }
  if (dn.size() >= 3 && between(dn.back(), dn[0], dn[1])) {
    dn[0] = dn.back();
    dn.pop_back();
  }
  pts = dn;
#endif
}
```

Geometry

```
// C++ routines for computational geometry.

double INF = 1e100;
double EPS = 1e-12;

struct PT {
  double x, y;
  PT() {}
  PT(double x, double y) : x(x), y(y) {}
  PT(const PT &p) : x(p.x), y(p.y)    {}
  PT operator + (const PT &p)  const { return PT(x+p.x, y+p.y); }
  PT operator - (const PT &p)  const { return PT(x-p.x, y-p.y); }
```

```
  PT operator * (double c)     const { return PT(x*c,   y*c  ); }
  PT operator / (double c)     const { return PT(x/c,   y/c  ); }
};

double dot(PT p, PT q)     { return p.x*q.x+p.y*q.y; }
double dist2(PT p, PT q)   { return dot(p-q,p-q); }
double cross(PT p, PT q)   { return p.x*q.y-p.y*q.x; }
ostream &operator<<(ostream &os, const PT &p) {
  os << "(" << p.x << "," << p.y << ")";
}

// rotate a point CCW or CW around the origin
PT RotateCCW90(PT p)   { return PT(-p.y,p.x); }
PT RotateCW90(PT p)    { return PT(p.y,-p.x); }
PT RotateCCW(PT p, double t) {
  return PT(p.x*cos(t)-p.y*sin(t), p.x*sin(t)+p.y*cos(t));
}

// project point c onto line through a and b
// assuming a != b
PT ProjectPointLine(PT a, PT b, PT c) {
  return a + (b-a)*dot(c-a, b-a)/dot(b-a, b-a);
}

// project point c onto line segment through a and b
PT ProjectPointSegment(PT a, PT b, PT c) {
  double r = dot(b-a,b-a);
  if (fabs(r) < EPS) return a;
  r = dot(c-a, b-a)/r;
  if (r < 0) return a;
  if (r > 1) return b;
  return a + (b-a)*r;
}

// compute distance from c to segment between a and b
double DistancePointSegment(PT a, PT b, PT c) {
  return sqrt(dist2(c, ProjectPointSegment(a, b, c)));
}

// compute distance between point (x,y,z) and plane ax+by+cz=d
double DistancePointPlane(double x, double y, double z,
                          double a, double b, double c, double d)
{
  return fabs(a*x+b*y+c*z-d)/sqrt(a*a+b*b+c*c);
}

// determine if lines from a to b and c to d are parallel or collinear
bool LinesParallel(PT a, PT b, PT c, PT d) {
  return fabs(cross(b-a, c-d)) < EPS;
}

bool LinesCollinear(PT a, PT b, PT c, PT d) {
  return LinesParallel(a, b, c, d)
      && fabs(cross(a-b, a-c)) < EPS
      && fabs(cross(c-d, c-a)) < EPS;
}

// determine if line segment from a to b intersects with
// line segment from c to d
bool SegmentsIntersect(PT a, PT b, PT c, PT d) {
  if (LinesCollinear(a, b, c, d)) {
    if (dist2(a, c) < EPS || dist2(a, d) < EPS ||
      dist2(b, c) < EPS || dist2(b, d) < EPS) return true;
    if (dot(c-a, c-b) > 0 && dot(d-a, d-b) > 0 && dot(c-b, d-b) > 0)
      return false;
    return true;
  }
  if (cross(d-a, b-a) * cross(c-a, b-a) > 0) return false;
  if (cross(a-c, d-c) * cross(b-c, d-c) > 0) return false;
  return true;
}

// compute intersection of line passing through a and b
// with line passing through c and d, assuming that unique
// intersection exists; for segment intersection, check if
// segments intersect first
PT ComputeLineIntersection(PT a, PT b, PT c, PT d) {
  b=b-a; d=c-d; c=c-a;
  assert(dot(b, b) > EPS && dot(d, d) > EPS);
  return a + b*cross(c, d)/cross(b, d);
}

// compute center of circle given three points
PT ComputeCircleCenter(PT a, PT b, PT c) {
  b=(a+b)/2;
  c=(a+c)/2;
  return ComputeLineIntersection(b, b+RotateCW90(a-b), c, c+RotateCW90(a
      -c));
}

// determine if point is in a possibly non-convex polygon (by William
// Randolph Franklin); returns 1 for strictly interior points, 0 for
// strictly exterior points, and 0 or 1 for the remaining points.
// Note that it is possible to convert this into an *exact* test using
// integer arithmetic by taking care of the division appropriately
// (making sure to deal with signs properly) and then by writing exact
// tests for checking point on polygon boundary
bool PointInPolygon(const vector<PT> &p, PT q) {
  bool c = 0;
  for (int i = 0; i < p.size(); i++){
    int j = (i+1)%p.size();
    if ((p[i].y <= q.y && q.y < p[j].y ||
      p[j].y <= q.y && q.y < p[i].y) &&
      q.x < p[i].x + (p[j].x - p[i].x) * (q.y - p[i].y) / (p[j].y - p[i
          ].y))
      c = !c;
  }
  return c;
}

// determine if point is on the boundary of a polygon
bool PointOnPolygon(const vector<PT> &p, PT q) {
  for (int i = 0; i < p.size(); i++)
    if (dist2(ProjectPointSegment(p[i], p[(i+1)%p.size()], q), q) < EPS)
      return true;
    return false;
```

```
}

// compute intersection of line through points a and b with
// circle centered at c with radius r > 0
vector<PT> CircleLineIntersection(PT a, PT b, PT c, double r) {
  vector<PT> ret;
  b = b-a;
  a = a-c;
  double A = dot(b, b);
  double B = dot(a, b);
  double C = dot(a, a) - r*r;
  double D = B*B - A*C;
  if (D < -EPS) return ret;
  ret.push_back(c+a+b*(-B+sqrt(D+EPS))/A);
  if (D > EPS)
    ret.push_back(c+a+b*(-B-sqrt(D))/A);
  return ret;
}

// compute intersection of circle centered at a with radius r
// with circle centered at b with radius R
vector<PT> CircleCircleIntersection(PT a, PT b, double r, double R) {
  vector<PT> ret;
  double d = sqrt(dist2(a, b));
  if (d > r+R || d+min(r, R) < max(r, R)) return ret;
  double x = (d*d-R*R+r*r)/(2*d);
  double y = sqrt(r*r-x*x);
  PT v = (b-a)/d;
  ret.push_back(a+v*x + RotateCCW90(v)*y);
  if (y > 0)
    ret.push_back(a+v*x - RotateCCW90(v)*y);
  return ret;
}

// This code computes the area or centroid of a (possibly nonconvex)
// polygon, assuming that the coordinates are listed in a clockwise or
// counterclockwise fashion.  Note that the centroid is often known as
// the "center of gravity" or "center of mass".
double ComputeSignedArea(const vector<PT> &p) {
  double area = 0;
  for(int i = 0; i < p.size(); i++) {
    int j = (i+1) % p.size();
    area += p[i].x*p[j].y - p[j].x*p[i].y;
  }
  return area / 2.0;
}

double ComputeArea(const vector<PT> &p) {
  return fabs(ComputeSignedArea(p));
}

PT ComputeCentroid(const vector<PT> &p) {
  PT c(0,0);
  double scale = 6.0 * ComputeSignedArea(p);
  for (int i = 0; i < p.size(); i++){
    int j = (i+1) % p.size();
    c = c + (p[i]+p[j])*(p[i].x*p[j].y - p[j].x*p[i].y);
  }
  return c / scale;
}

// tests whether or not a given polygon (in CW or CCW order) is simple
bool IsSimple(const vector<PT> &p) {
  for (int i = 0; i < p.size(); i++) {
    for (int k = i+1; k < p.size(); k++) {
      int j = (i+1) % p.size();
      int l = (k+1) % p.size();
      if (i == l || j == k) continue;
      if (SegmentsIntersect(p[i], p[j], p[k], p[l]))
        return false;
    }
  }
  return true;
}

int main() {

  // expected: (-5,2)
  cerr << RotateCCW90(PT(2,5)) << endl;

  // expected: (5,-2)
  cerr << RotateCW90(PT(2,5)) << endl;

  // expected: (-5,2)
  cerr << RotateCCW(PT(2,5),M_PI/2) << endl;

  // expected: (5,2)
  cerr << ProjectPointLine(PT(-5,-2), PT(10,4), PT(3,7)) << endl;

  // expected: (5,2) (7.5,3) (2.5,1)
  cerr << ProjectPointSegment(PT(-5,-2), PT(10,4), PT(3,7)) << " "
       << ProjectPointSegment(PT(7.5,3), PT(10,4), PT(3,7)) << " "
       << ProjectPointSegment(PT(-5,-2), PT(2.5,1), PT(3,7)) << endl;

  // expected: 6.78903
  cerr << DistancePointPlane(4,-4,3,2,-2,5,-8) << endl;

  // expected: 1 0 1
  cerr << LinesParallel(PT(1,1), PT(3,5), PT(2,1), PT(4,5)) << " "
       << LinesParallel(PT(1,1), PT(3,5), PT(2,0), PT(4,5)) << " "
       << LinesParallel(PT(1,1), PT(3,5), PT(5,9), PT(7,13)) << endl;

  // expected: 0 0 1
  cerr << LinesCollinear(PT(1,1), PT(3,5), PT(2,1), PT(4,5)) << " "
       << LinesCollinear(PT(1,1), PT(3,5), PT(2,0), PT(4,5)) << " "
       << LinesCollinear(PT(1,1), PT(3,5), PT(5,9), PT(7,13)) << endl;

  // expected: 1 1 1 0
  cerr << SegmentsIntersect(PT(0,0), PT(2,4), PT(3,1), PT(-1,3)) << " "
       << SegmentsIntersect(PT(0,0), PT(2,4), PT(4,3), PT(0,5)) << " "
       << SegmentsIntersect(PT(0,0), PT(2,4), PT(2,-1), PT(-2,1)) << " "
       << SegmentsIntersect(PT(0,0), PT(2,4), PT(5,5), PT(1,7)) << endl;

  // expected: (1,2)
  cerr << ComputeLineIntersection(PT(0,0), PT(2,4), PT(3,1), PT(-1,3))
     << endl;
```

```
  // expected: (1,1)
  cerr << ComputeCircleCenter(PT(-3,4), PT(6,1), PT(4,5)) << endl;

  vector<PT> v;
  v.push_back(PT(0,0));
  v.push_back(PT(5,0));
  v.push_back(PT(5,5));
  v.push_back(PT(0,5));

  // expected: 1 1 1 0 0
  cerr << PointInPolygon(v, PT(2,2)) << " "
       << PointInPolygon(v, PT(2,0)) << " "
       << PointInPolygon(v, PT(0,2)) << " "
       << PointInPolygon(v, PT(5,2)) << " "
       << PointInPolygon(v, PT(2,5)) << endl;

  // expected: 0 1 1 1 1
  cerr << PointOnPolygon(v, PT(2,2)) << " "
       << PointOnPolygon(v, PT(2,0)) << " "
       << PointOnPolygon(v, PT(0,2)) << " "
       << PointOnPolygon(v, PT(5,2)) << " "
       << PointOnPolygon(v, PT(2,5)) << endl;

  // expected: (1,6)
  //           (5,4) (4,5)
  //           blank line
  //           (4,5) (5,4)
  //           blank line
  //           (4,5) (5,4)
  vector<PT> u = CircleLineIntersection(PT(0,6), PT(2,6), PT(1,1), 5);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleLineIntersection(PT(0,9), PT(9,0), PT(1,1), 5);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleCircleIntersection(PT(1,1), PT(10,10), 5, 5);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleCircleIntersection(PT(1,1), PT(8,8), 5, 5);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleCircleIntersection(PT(1,1), PT(4.5,4.5), 10, sqrt(2.0)/2.0);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
  u = CircleCircleIntersection(PT(1,1), PT(4.5,4.5), 5, sqrt(2.0)/2.0);
  for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;

  // area should be 5.0
  // centroid should be (1.1666666, 1.166666)
  PT pa[] = { PT(0,0), PT(5,0), PT(1,1), PT(0,5) };
  vector<PT> p(pa, pa+4);
  PT c = ComputeCentroid(p);
  cerr << "Area: " << ComputeArea(p) << endl;
  cerr << "Centroid: " << c << endl;

  return 0;
}
```

Delaunay Triangulation

```
// Slow but simple Delaunay triangulation. Does not handle
// degenerate cases (from O'Rourke, Computational Geometry in C)
//
// Running time: O(n^4)
//
// INPUT:    x[] = x-coordinates
//           y[] = y-coordinates
//
// OUTPUT:   triples = a vector containing m triples of indices
//                     corresponding to triangle vertices

typedef double T;

struct triple {
    int i, j, k;
    triple() {}
    triple(int i, int j, int k) : i(i), j(j), k(k) {}
};

vector<triple> delaunayTriangulation(vector<T>& x, vector<T>& y) {
        int n = x.size();
        vector<T> z(n);
        vector<triple> ret;

        for (int i = 0; i < n; i++)
            z[i] = x[i] * x[i] + y[i] * y[i];

        for (int i = 0; i < n-2; i++) {
            for (int j = i+1; j < n; j++) {
                for (int k = i+1; k < n; k++) {
                    if (j == k) continue;
                    double xn = (y[j]-y[i])*(z[k]-z[i]) - (y[k]-y[i])*(z
                        [j]-z[i]);
                    double yn = (x[k]-x[i])*(z[j]-z[i]) - (x[j]-x[i])*(z
                        [k]-z[i]);
                    double zn = (x[j]-x[i])*(y[k]-y[i]) - (x[k]-x[i])*(y
                        [j]-y[i]);
                    bool flag = zn < 0;
                    for (int m = 0; flag && m < n; m++)
                        flag = flag && ((x[m]-x[i])*xn +
                                        (y[m]-y[i])*yn +
                                        (z[m]-z[i])*zn <= 0);
                    if (flag) ret.push_back(triple(i, j, k));
                }
            }
        }
        return ret;
}

int main()
{
    T xs[]={0, 0, 1, 0.9};
    T ys[]={0, 1, 0, 0.9};
    vector<T> x(&xs[0], &xs[4]), y(&ys[0], &ys[4]);
    vector<triple> tri = delaunayTriangulation(x, y);

    //expected: 0 1 3
    //          0 3 2
```

```
    int i;
    for(i = 0; i < tri.size(); i++)
        printf("%d %d %d\n", tri[i].i, tri[i].j, tri[i].k);
    return 0;
}
```

Math

NumberTheory

```
// Primes less than 1000:
//      2     3     5     7    11    13    17    19    23    29    31    37
//     41    43    47    53    59    61    67    71    73    79    83    89
//     97   101   103   107   109   113   127   131   137   139   149   151
//    157   163   167   173   179   181   191   193   197   199   211   223
//    227   229   233   239   241   251   257   263   269   271   277   281
//    283   293   307   311   313   317   331   337   347   349   353   359
//    367   373   379   383   389   397   401   409   419   421   431   433
//    439   443   449   457   461   463   467   479   487   491   499   503
//    509   521   523   541   547   557   563   569   571   577   587   593
//    599   601   607   613   617   619   631   641   643   647   653   659
//    661   673   677   683   691   701   709   719   727   733   739   743
//    751   757   761   769   773   787   797   809   811   821   823   827
//    829   839   853   857   859   863   877   881   883   887   907   911
//    919   929   937   941   947   953   967   971   977   983   991   997

// Other primes:
//     The largest prime smaller than 10 is 7.
//     The largest prime smaller than 10^2 is 97.
//     The largest prime smaller than 10^3 is 997.
//     The largest prime smaller than 10^4 is 9973.
//     The largest prime smaller than 10^5 is 99991.
//     The largest prime smaller than 10^6 is 999983.
//     The largest prime smaller than 10^7 is 9999991.
//     The largest prime smaller than 10^8 is 99999989.
//     The largest prime smaller than 10^9 is 999999937.
//     The largest prime smaller than 10^10 is 9999999967.
//     The largest prime smaller than 10^11 is 99999999977.
//     The largest prime smaller than 10^12 is 999999999989.
//     The largest prime smaller than 10^13 is 9999999999971.
//     The largest prime smaller than 10^14 is 99999999999973.
//     The largest prime smaller than 10^15 is 999999999999989.
//     The largest prime smaller than 10^16 is 9999999999999937.
//     The largest prime smaller than 10^17 is 99999999999999997.
//     The largest prime smaller than 10^18 is 999999999999999989.
//     (1<<61)-1 is prime

// This is a collection of useful code for solving problems that
// involve modular linear equations.  Note that all of the
// algorithms described here work on nonnegative integers.

typedef vector<int> VI;
typedef pair<int,int> PII;

// return a % b (positive value)
int mod(int a, int b) {
  return ((a%b)+b)%b;
}

// computes gcd(a,b)
int gcd(int a, int b) {
  int tmp;
  while(b){a%=b; tmp=a; a=b; b=tmp;}
  return a;
}

// computes lcm(a,b)
int lcm(int a, int b) {
  return a/gcd(a,b)*b;
}

// returns d = gcd(a,b); finds x,y such that d = ax + by
int extended_euclid(int a, int b, int &x, int &y) {
  int xx = y = 0;
  int yy = x = 1;
  while (b) {
    int q = a/b;
    int t = b; b = a%b; a = t;
    t = xx; xx = x-q*xx; x = t;
    t = yy; yy = y-q*yy; y = t;
  }
  return a;
}

// finds all solutions to ax = b (mod n)
VI modular_linear_equation_solver(int a, int b, int n) {
  int x, y;
  VI solutions;
  int d = extended_euclid(a, n, x, y);
  if (!(b%d)) {
    x = mod (x*(b/d), n);
    for (int i = 0; i < d; i++)
      solutions.push_back(mod(x + i*(n/d), n));
  }
  return solutions;
}

// computes b such that ab = 1 (mod n), returns -1 on failure
int mod_inverse(int a, int n) {
  int x, y;
  int d = extended_euclid(a, n, x, y);
  if (d > 1) return -1;
  return mod(x,n);
}

// Chinese remainder theorem (special case): find z such that
// z % x = a, z % y = b.  Here, z is unique modulo M = lcm(x,y).
// Return (z,M).  On failure, M = -1.
PII chinese_remainder_theorem(int x, int a, int y, int b) {
  int s, t;
  int d = extended_euclid(x, y, s, t);
  if (a%d != b%d) return make_pair(0, -1);
  return make_pair(mod(s*b*x+t*a*y,x*y)/d, x*y/d);
}

// Chinese remainder theorem: find z such that
```

```
// z % x[i] = a[i] for all i.  Note that the solution is
// unique modulo M = lcm_i (x[i]).  Return (z,M).  On
// failure, M = -1.  Note that we do not require the a[i]'s
// to be relatively prime.
PII chinese_remainder_theorem(const VI &x, const VI &a) {
  PII ret = make_pair(a[0], x[0]);
  for (int i = 1; i < x.size(); i++) {
    ret = chinese_remainder_theorem(ret.second, ret.first, x[i], a[i]);
    if (ret.second == -1) break;
  }
  return ret;
}

// computes x and y such that ax + by = c; on failure, x = y =-1
void linear_diophantine(int a, int b, int c, int &x, int &y) {
  int d = gcd(a,b);
  if (c%d) {
    x = y = -1;
  } else {
    x = c/d * mod_inverse(a/d, b/d);
    y = (c-a*x)/b;
  }
}

int main() {

  // expected: 2
  cout << gcd(14, 30) << endl;

  // expected: 2 -2 1
  int x, y;
  int d = extended_euclid(14, 30, x, y);
  cout << d << " " << x << " " << y << endl;

  // expected: 95 45
  VI sols = modular_linear_equation_solver(14, 30, 100);
  for (int i = 0; i < (int) sols.size(); i++) cout << sols[i] << " ";
  cout << endl;

  // expected: 8
  cout << mod_inverse(8, 9) << endl;

  // expected: 23 56
  //           11 12
  int xs[] = {3, 5, 7, 4, 6};
  int as[] = {2, 3, 2, 3, 5};
  PII ret = chinese_remainder_theorem(VI (xs, xs+3), VI(as, as+3));
  cout << ret.first << " " << ret.second << endl;
  ret = chinese_remainder_theorem (VI(xs+3, xs+5), VI(as+3, as+5));
  cout << ret.first << " " << ret.second << endl;

  // expected: 5 -15
  linear_diophantine(7, 2, 5, x, y);
  cout << x << " " << y << endl;

}
```

RabinMiller

```
bool Miller(LL p, LL s, int a){
    if(p==a) return 1;
    LL mod=expmod(a,s,p); // a^s
    for(;s-p+1 && mod-1 && mod-p+1;s*=2) mod=mulmod(mod,mod,p);// mod^2
    return mod==p-1 || s%2;
}
bool isprime(LL n) {
    if(n<2)return 0; if(n%2==0) return n==2;
    LL s=n-1;
    while(s%2==0) s/=2;
    return Miller(n,s,2) && Miller(n,s,7) && Miller(n,s,61);
} // for 341*10^12  primes <= 17
```

GaussJordan

```
// Gauss-Jordan elimination with full pivoting.
//
// Uses:
//   (1) solving systems of linear equations (AX=B)
//   (2) inverting matrices (AX=I)
//   (3) computing determinants of square matrices
//
// Running time: O(n^3)
//
// INPUT:    a[][] = an nxn matrix
//           b[][] = an nxm matrix
//
// OUTPUT:   X      = an nxm matrix (stored in b[][])
//           A^{-1} = an nxn matrix (stored in a[][])
//           returns determinant of a[][]

const double EPS = 1e-10;

typedef vector<int> VI;
typedef double T;
typedef vector<T> VT;
typedef vector<VT> VVT;

T GaussJordan(VVT &a, VVT &b) {
  const int n = a.size();
  const int m = b[0].size();
  VI irow(n), icol(n), ipiv(n);
  T det = 1;

  for (int i = 0; i < n; i++) {
    int pj = -1, pk = -1;
    for (int j = 0; j < n; j++) if (!ipiv[j])
      for (int k = 0; k < n; k++) if (!ipiv[k])
        if (pj == -1 || fabs(a[j][k]) > fabs(a[pj][pk])) { pj = j; pk =
            k; }
    if (fabs(a[pj][pk]) < EPS) { cerr << "Matrix is singular." << endl;
        exit(0); }
    ipiv[pk]++;
    swap(a[pj], a[pk]);
    swap(b[pj], b[pk]);
```


14

```
    if (pj != pk) det *= -1;
    irow[i] = pj;
    icol[i] = pk;

    T c = 1.0 / a[pk][pk];
    det *= a[pk][pk];
    a[pk][pk] = 1.0;
    for (int p = 0; p < n; p++) a[pk][p] *= c;
    for (int p = 0; p < m; p++) b[pk][p] *= c;
    for (int p = 0; p < n; p++) if (p != pk) {
      c = a[p][pk];
      a[p][pk] = 0;
      for (int q = 0; q < n; q++) a[p][q] -= a[pk][q] * c;
      for (int q = 0; q < m; q++) b[p][q] -= b[pk][q] * c;
    }
  }

  for (int p = n-1; p >= 0; p--) if (irow[p] != icol[p]) {
    for (int k = 0; k < n; k++) swap(a[k][irow[p]], a[k][icol[p]]);
  }

  return det;
}

int main() {
  const int n = 4;
  const int m = 2;
  double A[n][n] = { {1,2,3,4},{1,0,1,0},{5,3,2,4},{6,1,4,6} };
  double B[n][m] = { {1,2},{4,3},{5,6},{8,7} };
  VVT a(n), b(n);
  for (int i = 0; i < n; i++) {
    a[i] = VT(A[i], A[i] + n);
    b[i] = VT(B[i], B[i] + m);
  }

  double det = GaussJordan(a, b);

  // expected: 60
  cout << "Determinant: " << det << endl;

  // expected: -0.233333 0.166667 0.133333 0.0666667
  //           0.166667 0.166667 0.333333 -0.333333
  //           0.233333 0.833333 -0.133333 -0.0666667
  //           0.05 -0.75 -0.1 0.2
  cout << "Inverse: " << endl;
  for (int i = 0; i < n; i++) {
    for (int j = 0; j < n; j++)
      cout << a[i][j] << ' ';
    cout << endl;
  }

  // expected: 1.63333 1.3
  //           -0.166667 0.5
  //           2.36667 1.7
  //           -1.85 -1.35
  cout << "Solution: " << endl;
  for (int i = 0; i < n; i++) {
    for (int j = 0; j < m; j++)
      cout << b[i][j] << ' ';
    cout << endl;
  }
}
```

FFT

```
struct cpx
{
  cpx(){}
  cpx(double aa):a(aa){}
  cpx(double aa, double bb):a(aa),b(bb){}
  double a;
  double b;
  double modsq(void) const
  {
    return a * a + b * b;
  }
  cpx bar(void) const
  {
    return cpx(a, -b);
  }
};

cpx operator +(cpx a, cpx b)
{
  return cpx(a.a + b.a, a.b + b.b);
}

cpx operator *(cpx a, cpx b)
{
  return cpx(a.a * b.a - a.b * b.b, a.a * b.b + a.b * b.a);
}

cpx operator /(cpx a, cpx b)
{
  cpx r = a * b.bar();
  return cpx(r.a / b.modsq(), r.b / b.modsq());
}

cpx EXP(double theta)
{
  return cpx(cos(theta),sin(theta));
}

const double two_pi = 4 * acos(0);

// in:     input array
// out:    output array
// step:   {SET TO 1} (used internally)
// size:   length of the input/output {MUST BE A POWER OF 2}
// dir:    either plus or minus one (direction of the FFT)
// RESULT: out[k] = \sum_{j=0}^{size - 1} in[j] * exp(dir * 2pi * i * j
    * k / size)
void FFT(cpx *in, cpx *out, int step, int size, int dir)
{
  if(size < 1) return;
```

```
  if(size == 1)
  {
    out[0] = in[0];
    return;
  }
  FFT(in, out, step * 2, size / 2, dir);
  FFT(in + step, out + size / 2, step * 2, size / 2, dir);
  for(int i = 0 ; i < size / 2 ; i++)
  {
    cpx even = out[i];
    cpx odd = out[i + size / 2];
    out[i] = even + EXP(dir * two_pi * i / size) * odd;
    out[i + size / 2] = even + EXP(dir * two_pi * (i + size / 2) / size)
          * odd;
  }
}

// Usage:
// f[0...N-1] and g[0..N-1] are numbers
// Want to compute the convolution h, defined by
// h[n] = sum of f[k]g[n-k] (k = 0, ..., N-1).
// Here, the index is cyclic; f[-1] = f[N-1], f[-2] = f[N-2], etc.
// Let F[0...N-1] be FFT(f), and similarly, define G and H.
// The convolution theorem says H[n] = F[n]G[n] (element-wise product).
// To compute h[] in O(N log N) time, do the following:
//   1. Compute F and G (pass dir = 1 as the argument).
//   2. Get H by element-wise multiplying F and G.
//   3. Get h by taking the inverse FFT (use dir = -1 as the argument)
//      and *dividing by N*. DO NOT FORGET THIS SCALING FACTOR.

int main(void)
{
  printf("If rows come in identical pairs, then everything works.\n");

  cpx a[8] = {0, 1, cpx(1,3), cpx(0,5), 1, 0, 2, 0};
  cpx b[8] = {1, cpx(0,-2), cpx(0,1), 3, -1, -3, 1, -2};
  cpx A[8];
  cpx B[8];
  FFT(a, A, 1, 8, 1);
  FFT(b, B, 1, 8, 1);

  for(int i = 0 ; i < 8 ; i++)
  {
    printf("%7.2lf%7.2lf", A[i].a, A[i].b);
  }
  printf("\n");
  for(int i = 0 ; i < 8 ; i++)
  {
    cpx Ai(0,0);
    for(int j = 0 ; j < 8 ; j++)
    {
      Ai = Ai + a[j] * EXP(j * i * two_pi / 8);
    }
    printf("%7.2lf%7.2lf", Ai.a, Ai.b);
  }
  printf("\n");

  cpx AB[8];
  for(int i = 0 ; i < 8 ; i++)
    AB[i] = A[i] * B[i];
  cpx aconvb[8];
  FFT(AB, aconvb, 1, 8, -1);
  for(int i = 0 ; i < 8 ; i++)
    aconvb[i] = aconvb[i] / 8;
  for(int i = 0 ; i < 8 ; i++)
  {
    printf("%7.2lf%7.2lf", aconvb[i].a, aconvb[i].b);
  }
  printf("\n");
  for(int i = 0 ; i < 8 ; i++)
  {
    cpx aconvbi(0,0);
    for(int j = 0 ; j < 8 ; j++)
    {
      aconvbi = aconvbi + a[j] * b[(8 + i - j) % 8];
    }
    printf("%7.2lf%7.2lf", aconvbi.a, aconvbi.b);
  }
  printf("\n");

  return 0;
}
```

Strings

SuffixArray

```
// Suffix array construction in O(L log^2 L) time.  Routine for
// computing the length of the longest common prefix of any two
// suffixes in O(log L) time.
//
// INPUT:   string s
//
// OUTPUT:  array suffix[] such that suffix[i] = index (from 0 to L-1)
//          of substring s[i...L-1] in the list of sorted suffixes.
//          That is, if we take the inverse of the permutation suffix[],
//          we get the actual suffix array.

struct SuffixArray {
  const int L;
  string s;
  vector<vector<int> > P;
  vector<pair<pair<int,int>,int> > M;

  SuffixArray(const string &s) : L(s.length()), s(s), P(1, vector<int>(L
      , 0)), M(L) {
    for (int i = 0; i < L; i++) P[0][i] = int(s[i]);
    for (int skip = 1, level = 1; skip < L; skip *= 2, level++) {
      P.push_back(vector<int>(L, 0));
      for (int i = 0; i < L; i++)
        M[i] = make_pair(make_pair(P[level-1][i], i + skip < L ? P[level
            -1][i + skip] : -1000), i);
      sort(M.begin(), M.end());
      for (int i = 0; i < L; i++)
        P[level][M[i].second] = (i > 0 && M[i].first == M[i-1].first) ?
            P[level][M[i-1].second] : i;
```


16

```
    }
  }

  vector<int> GetSuffixArray() { return P.back(); }

  // returns the length of the longest common prefix of s[i...L-1] and s
      [j...L-1]
  int LongestCommonPrefix(int i, int j) {
    int len = 0;
    if (i == j) return L - i;
    for (int k = P.size() - 1; k >= 0 && i < L && j < L; k--) {
      if (P[k][i] == P[k][j]) {
        i += 1 << k;
        j += 1 << k;
        len += 1 << k;
      }
    }
    return len;
  }
};

int main() {

  // bobocel is the 0'th suffix
  //  obocel is the 5'th suffix
  //   bocel is the 1'st suffix
  //    ocel is the 6'th suffix
  //     cel is the 2'nd suffix
  //      el is the 3'rd suffix
  //       l is the 4'th suffix
  SuffixArray suffix("bobocel");
  vector<int> v = suffix.GetSuffixArray();

  // Expected output: 0 5 1 6 2 3 4
  //                  2
  for (int i = 0; i < v.size(); i++) cout << v[i] << " ";
  cout << endl;
  cout << suffix.LongestCommonPrefix(0, 2) << endl;
}
```

Z-Algorithm

```
vector<int> z_function(string s) {
    int n = (int) s.length();
    vector<int> z(n);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r)
            z[i] = min (r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]])
            ++z[i];
        if (i + z[i] - 1 > r)
            l = i, r = i + z[i] - 1;
    }
    return z;
}
```

KMP

```
/*
Searches for the string w in the string s (of length k). Returns the
0-based index of the first match (k if no match is found). Algorithm
runs in O(k) time.
*/

typedef vector<int> VI;

void buildTable(string& w, VI& t)
{
  t = VI(w.length());
  int i = 2, j = 0;
  t[0] = -1; t[1] = 0;

  while(i < w.length())
  {
    if(w[i-1] == w[j]) { t[i] = j+1; i++; j++; }
    else if(j > 0) j = t[j];
    else { t[i] = 0; i++; }
  }
}

int KMP(string& s, string& w)
{
  int m = 0, i = 0;
  VI t;

  buildTable(w, t);
  while(m+i < s.length())
  {
    if(w[i] == s[m+i])
    {
      i++;
      if(i == w.length()) return m;
    }
    else
    {
      m += i-t[i];
      if(i > 0) i = t[i];
    }
  }
  return s.length();
}

int main()
{
  string a = (string) "The example above illustrates the general
      technique for assembling "+
    "the table with a minimum of fuss. The principle is that of the
      overall search: "+
    "most of the work was already done in getting to the current
      position, so very "+
    "little needs to be done in leaving it. The only minor complication
      is that the "+
    "logic which is correct late in the string erroneously gives non-
      proper "+
```

```
    "substrings at the beginning. This necessitates some initialization
        code.";

  string b = "table";

  int p = KMP(a, b);
  cout << p << ": " << a.substr(p, b.length()) << " " << b << endl;
}
```

IO

```
#include <bits/stdc++.h>
using namespace std;

int main()
{
    ios_base::sync_with_stdio(0); // fast cin,cout

    // Ouput a specific number of digits past the decimal point,
    // in this case 5
    cout.setf(ios::fixed); cout << setprecision(5);
    cout << 100.0/7.0 << endl;
    cout.unsetf(ios::fixed);

    // Output the decimal point and trailing zeros
    cout.setf(ios::showpoint);
    cout << 100.0 << endl;
    cout.unsetf(ios::showpoint);

    // Output a '+' before positive values
    cout.setf(ios::showpos);
    cout << 100 << " " << -100 << endl;
    cout.unsetf(ios::showpos);

    // Output numerical values in hexadecimal
    cout << hex << 100 << " " << 1000 << " " << 10000 << dec << endl;
}
```

Miscellaneous

Divide and Conquer Optimization

```
#include <bits/stdc++.h>
using namespace std;

// Story: - You have a queue of passengers p1, p2... p_n waiting for
    gondolas
//              - You should pack them in k gondolas, in the order
    listed
//              - Packing passengers i to j into one gondola costs C[i
    ][j]
//              - minimize the cost
//              - Sufficient constraint on cost function:
//                      C[a][c] + C[b][d] <= C[a][d] + C[b][c], a < b <
    c < d (typically satisfied)
//                      means longer intervals cost more than smaller
    ones
// Refinement: in our story (not necessary) in every gondola needs to be
     at least one person

int LARGE = 100000000;

int U[4009][4009];
int C(int i, int j)
{
        return U[j][j] - U[i - 1][j] - U[j][i - 1] + U[i - 1][i - 1];
}

// -> to compute
int f(int i, int j)
{
        if(i == 0) return j == 0 ? 0 : LARGE;
        if(j == 0) return LARGE;

        int result = LARGE;
        for(int k = 0; k < j; k++)
                result = min(result, f(i - 1, k) + C(k + 1, j));
        return result;
}

// Trick:
//                      Define opt[i][j] := smallest k such that f(i, j)
     = f(i - 1, k) + C(k + 1, j)
//                      i.e. "position" of the optimal solution where
    you perfere to take the most
//                      notice that opt[i][1] <= opt[i][2] <= opt[i][3]
    ...
//                      thus if we want to calculate all values [f(i, a)
    , f(i, a + 1) .. f(i, b)]
//                      we have a restriction on the possible search
    range, as we do not have to search
//                      for an optimal solution at k < opt[i][b + 1]
//                      The following has runtime O(k*n*log(n))


int dp[4009][4009];

void divConq(int i, int a, int b, int optL, int optR)
{
        if(a > b) return;

        int m = (a + b) / 2, opt;

        // calculate value for the middle element - as in f
        dp[i][m] = LARGE;
        for(int k = optL; k <= optR; k++)
                if(dp[i - 1][k] + C(k + 1, m) <= dp[i][m])
                {
                        dp[i][m] = dp[i - 1][k] + C(k + 1, m);
                        opt = k;
                }

        divConq(i, a, m - 1, optL, opt);
        divConq(i, m + 1, b, opt, optR);
}
```


18

```
int dpOpt(int k, int n)
{
        for(int i = 0; i <= k; i++)
        {
                if(i == 0)
                {
                        for(int j = 0; j <= n; j++)
                                dp[i][j] = j == 0 ? 0 : LARGE;
                                continue;
                }
                dp[i][0] = LARGE;

                divConq(i, 1, n, 0, n);
        }

        return dp[k][n];
}
```

C++ IO (Gregor)

```
#include <bits/stdc++.h>

struct cmp {
        bool operator()(int a, int b) {
                return a < b; }};

int main() {
        set<int, cmp> set;
        map<int, int, cmp> map;
        priority_queue<int, vector<int>, cmp> pq;

        for (int i = 1; i < 10; i++) set.insert(i);

        auto itlow = set.lower_bound(3); // -> 3
        auto itup = upper_bound(set.begin(), set.end(), 6, cmp()); // ->
            7

        for (int i = 0; i < 10; ++i) map[i] = i * 10;

        auto it = map.find(5);
        if (it != map.end())
                printf("(%d %d)\n", it->first, it->second);

        int cmb[] = { 1, 2, 3 };
        do { // (1 2 3), (1 3 2), (2 1 3), (2 3 1), (3 1 2), (3 2 1)
                printf("(%d %d %d)\n", cmb[0], cmb[1], cmb[2]);
        } while (next_permutation(cmb, cmb + 3));

        // massively improve cout and cin performance for large streams
        ios::sync_with_stdio(false);
        cin.tie(0);

        // Ouput a specific number of digits past the decimal point, in
            this case 5
        cout.setf(ios::fixed); cout << setprecision(5);
        cout << 100.0/7.0 << endl;
        cout.unsetf(ios::fixed);

        // Output the decimal point and trailing zeros
        cout.setf(ios::showpoint);
        cout << 100.0 << endl;

        // Output a '+' before positive values
        cout.setf(ios::showpos);
        cout << 100 << " " << -100 << endl;

        // Output numerical values in hexadecimal
        cout << hex << 100 << " " << 1000 << " " << 10000 << dec << endl
            ;
}
```

GCC Builtin Functions

```
int __builtin_clz (unsigned int x);
// Returns the number of leading 0-bits in x, starting at the most
    significant bit position. If x is 0, the result is undefined.

int __builtin_ctz (unsigned int x)
// Returns the number of trailing 0-bits in x, starting at the least
    significant bit position. If x is 0, the result is undefined.

int __builtin_popcount (unsigned int x)
// Returns the number of 1-bits in x.

int __builtin_ffs (int x)
// Returns one plus the index of the least significant 1-bit of x, or if
     x is zero, returns zero.

// For long long arguments add ll to the function name, e.g.
    __builtin_clzll(long long x)
```

Longest Increasing Subsequence

```
\subsection{Longest Increasing Subsequence}
\begin{lstlisting}[language=C++]
// Given a list of numbers of length n, this routine extracts a
// longest increasing subsequence.
//
// Running time: O(n log n)
//
//   INPUT: a vector of integers
//   OUTPUT: a vector containing the longest increasing subsequence
typedef vector<int> VI;
typedef pair<int,int> PII;
typedef vector<PII> VPII;

#define STRICTLY_INCREASNG

VI LongestIncreasingSubsequence(VI v) {
  VPII best;
  VI dad(v.size(), -1);

  for (int i = 0; i < v.size(); i++) {
```

```
#ifdef STRICTLY_INCREASNG
    PII item = make_pair(v[i], 0);
    VPII::iterator it = lower_bound(best.begin(), best.end(), item);
    item.second = i;
#else
    PII item = make_pair(v[i], i);
    VPII::iterator it = upper_bound(best.begin(), best.end(), item);
#endif
    if (it == best.end()) {
      dad[i] = (best.size() == 0 ? -1 : best.back().second);
      best.push_back(item);
    } else {
      dad[i] = dad[it->second];
      *it = item;
    }
  }

  VI ret;
  for (int i = best.back().second; i >= 0; i = dad[i])
    ret.push_back(v[i]);
  reverse(ret.begin(), ret.end());
  return ret;
}
\end{lstlisting}
```

Longest Palindrome

```
int longest_palindrome (char *text, intn) {
  int rad [2*n], i, j , k;
  for (i = 0, j = 0; i < 2*n; i += k , j = max (j-k, 0)) {
    while (i-j >= 0 && i+j+1 < 2*n&&text [(i-j) /2] == text [(i+j+1)
        /2]) ++j;
    rad [i] = j;
    for (k = 1; i-k >= 0 && rad [i] - k >= 0 && rad [i-k] != rad [i] - k
        ; ++k)
      rad [i+k] = min (rad [i-k], rad [i] - k);
  }
  return *max_element (rad, rad+2*n); // ret. centre of the longest
      palindrome
}
```

Theorems

Euler's theorem. For any planar graph, $V - E + F = 1 + C$, where V is the number of graph's vertices, E is the number of edges, F is the number of faces in graph's planar drawing, and C is the number of connected components. Corollary: $V - E + F = 2$ for a 3D polyhedron.

Vertex covers and independent sets. Let M, C, I be a max matching, a min vertex cover, and a max independent set. Then $|M| \leq |C| = N - |I|$, with equality for bipartite graphs. Complement of an MVC is always a MIS, and vice versa. Given a bipartite graph with partitions (A, B), build a network: connect source to A, and B to sink with edges of capacities, equal to the corresponding nodes' weights, or 1 in the unweighted case. Set capacities of the original graph's edges to the infinity. Let (S, T) be a minimum $s - t$ cut. Then a maximum(-weighted) independent set is $I = (A \cap S) \cup (B \cap T)$, and a minimum(-weighted) vertex cover is $C = (A - T) \cup (B \cap S)$.

2-SAT. Build an implication graph with 2 vertices for each variable - for the variable and its inverse; for each clause $x \vee y$ add edges $(\neg x, y)$ and $(\neg y, x)$. The formula is satisfiable if x and $\neg x$ are in distinct SCCs, for all x. To find a satisfiable assignment, consider the graph's SCCs in topological order from sinks to sources (i.e. Kosaraju's last step), assigning 'true' to all variables of the current SCC (if it hasn't been previously assigned 'false'), and 'false' to all inverses.

Pick's theorem. $I = A - B/2 + 1$, where A is the area of a lattice polygon, I is number of lattice points inside it, and B is number of lattice points on the boundary. Number of lattice points minus one on a line segment from $(0, 0)$ and (x, y) is $gcd(x, y)$.

Combinatorics

Mathematical Sums

$\sum_{k=0}^{n} k = n(n+1)/2$	$\sum_{k=a}^{b} k = (a+b)(b-a+1)/2$
$\sum_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$	$\sum_{k=0}^{n} k^3 = n^2(n+1)^2/4$
$\sum_{k=0}^{n} k^4 = (6n^5 + 15n^4 + 10n^3 - n)/30$	$\sum_{k=0}^{n} k^5 = (2n^6 + 6n^5 + 5n^4 - n^2)/12$
$\sum_{k=0}^{n} x^k = (x^{n+1} - 1)/(x - 1)$	$\sum_{k=0}^{n} kx^k = (x - (n+1)x^{n+1} + nx^{n+2})/(x-1)^2$

Burnsides Lemma. The number of orbits under Group G's action on set X:
$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X_g|$, where $X_g = \{x \in X : g(x) = x\}$ ("Average number of fixed points.")
Let $w(x)$ be weight of x's orbit. Sum of all orbit's weights:
$\sum_{o \in X/G} w(o) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X_g} w(x)$.

Simpson Formula. $\int_a^b f(x)dx = \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$.
Error: $|E(f)| \leq \frac{(b-a)^5}{2880} \max_{a \leq x \leq b} |f^{(4)}(x)|$.

20

Combinatorics

Mathematical Sums

$\sum_{k=0}^{n} k = n(n+1)/2$ $\quad\sum_{k=a}^{b} k = (a+b)(b-a+1)/2$
$\sum_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$ $\quad\sum_{k=0}^{n} k^3 = n^2(n+1)^2/4$
$\sum_{k=0}^{n} k^4 = (6n^5+15n^4+10n^3-n)/30$ $\quad\sum_{k=0}^{n} k^5 = (2n^6+6n^5+5n^4-n^2)/12$
$\sum_{k=0}^{n} x^k = (x^{n+1}-1)/(x-1)$ $\quad\sum_{k=0}^{n} kx^k = (x-(n+1)x^{n+1}+nx^{n+2})/(x-1)^2$

Binomial coefficients

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1
	0	1	2	3	4	5	6	7	8	9	10	11	12

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$
$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$
$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$
$\binom{n+1}{k} = \frac{n+1}{n-k+1}\binom{n}{k}$
$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$

$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$
$\sum_{k=1}^{n} k^2\binom{n}{k} = (n+n^2)2^{n-2}$

$\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$
$\binom{n}{k} = \prod_{i=1}^{k}\frac{n-k+i}{i}$

Number of ways to pick a multiset of size k from n elements: $\binom{n+k-1}{k}$
Number of n-tuples of non-negative integers with sum s: $\binom{s+n-1}{n-1}$, at most s: $\binom{s+n}{n}$
Number of n-tuples of positive integers with sum s: $\binom{s-1}{n-1}$
Number of lattice paths from $(0,0)$ to (a,b), restricted to east and north steps: $\binom{a+b}{a}$

Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. $\quad C_0 = 1$, $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$. $\quad C_{n+1} = C_n \frac{4n+2}{n+2}$.
$C_0, C_1, \ldots = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, \ldots$
C_n is the number of: properly nested sequences of n pairs of parentheses; rooted ordered binary trees with $n+1$ leaves; triangulations of a convex $(n+2)$-gon.

Derangements . Number of permutations of $n = 0, 1, 2, \ldots$ elements without fixed points is $1, 0, 1, 2, 9, 44, 265, 1854, 14833, \ldots$ Recurrence: $D_n = (n-1)(D_{n-1} + D_{n-2}) = nD_{n-1} + (-1)^n$. Corollary: number of permutations with exactly k fixed points is $\binom{n}{k}D_{n-k}$.

Stirling numbers of 1^{st} kind . $s_{n,k}$ is $(-1)^{n-k}$ times the number of permutations of n elements with exactly k permutation cycles. $\left[{n \atop k}\right] = |s_{n,k}| = |s_{n-1,k-1}| + (n-1)|s_{n-1,k}|$ $\quad s(0,0) = 1$ and $s(n,0) = s(0,n) = 0$.

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	−1	1							
3	0	2	−3	1						
4	0	−6	11	−6	1					
5	0	24	−50	35	−10	1				
6	0	−120	274	−225	85	−15	1			
7	0	720	−1764	1624	−735	175	−21	1		
8	0	−5040	13068	−13132	6769	−1960	322	−28	1	
9	0	40320	−109584	118124	−67284	22449	−4536	546	−36	1
	0	1	2	3	4	5	6	7	8	9

Stirling numbers of 2^{nd} kind . $S_{n,k} = \left\{ {n \atop k} \right\}$ is the number of ways to partition a set of n elements into exactly k non-empty subsets. $\left\{ {n \atop k} \right\} = S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$; $S_{n,1} = S_{n,n} = 1$.
$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$.

n/k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	1	1								
3	0	1	3	1							
4	0	1	7	6	1						
5	0	1	15	25	10	1					
6	0	1	31	90	65	15	1				
7	0	1	63	301	350	140	21	1			
8	0	1	127	966	1701	1050	266	28	1		
9	0	1	255	3025	7770	6951	2646	462	36	1	
10	0	1	511	9330	34105	42525	22827	5880	750	45	1
	0	1	2	3	4	5	6	7	8	9	10

Bell numbers . B_n is the number of partitions of n elements. $B_0, \ldots = 1, 1, 2, 5, 15, 52, 203, 877, \ldots$
$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k = \sum_{k=1}^{n+1} S_{n,k}$. Bell triangle: $B_r = a_{r,1} = a_{r-1,r-1}$, $a_{r,c} = a_{r-1,c-1} + a_{r,c-1}$.

Eulerian numbers . $E(n,k) = \left\langle {n \atop k} \right\rangle$ is the number of permutations with exactly k descents ($i : \pi_i < \pi_{i+1}$) / ascents ($\pi_i > \pi_{i+1}$) / excedances ($\pi_i > i$) / $k+1$ weak excedances ($\pi_i \geq i$).
Formula: $E(n,m) = (m+1)E(n-1,m) + (n-m)E(n-1,m-1)$. $E(n,0) = E(n,n-1) = 1$. $E(n,m) = \sum_{k=0}^{m} (-1)^k \binom{n+1}{k} (m+1-k)^n$.

Double factorial . Permutations of the multiset $\{1, 1, 2, 3, \ldots n, n\}$ such that for each k, all the numbers between two occurrences of k in the permutation are greater than k. $(2n-1)!! = \prod_{k=1}^{n} (2k-1)$.

Eulerian numbers of 2^{nd} kind . Related to Double factorial, number of all such permutations that have exactly m ascents. $\left\langle\!\left\langle {n \atop m} \right\rangle\!\right\rangle = (2n-m-1) \left\langle\!\left\langle {n-1 \atop m-1} \right\rangle\!\right\rangle + (m+1) \left\langle\!\left\langle {n-1 \atop m} \right\rangle\!\right\rangle$. $\left\langle\!\left\langle {n \atop 0} \right\rangle\!\right\rangle = 1$

Multinomial theorem . $(a_1 + \cdots + a_k)^n = \sum \binom{n}{n_1, \ldots, n_k} a_1^{n_1} \ldots a_k^{n_k}$, where $n_i \geq 0$ and $\sum n_i = n$.
$\binom{n}{n_1, \ldots, n_k} = M(n_1, \ldots, n_k) = \frac{n!}{n_1! \ldots n_k!}$. $M(a, \ldots, b, c, \ldots) = M(a + \cdots + b, c, \ldots) M(a, \ldots, b)$

Necklaces and bracelets Necklace of length n is an equivalence class of n-character strings over an alphabet of size k, taking all rotations as equivalent. Number of necklases: $N_k(n) = \frac{1}{n} \sum_{d|n} \varphi(d) k^{n/d}$
Bracelet is a necklace such that strings may also be equivalent under reflection. Number of bracelets:

$$B_k(n) = \begin{cases} \frac{1}{2} N_k(n) + \frac{1}{4}(k+1)k^{n/2} & \text{if } n \text{ is even} \\ \frac{1}{2} N_k(n) + \frac{1}{2} k^{(n+1)/2} & \text{if } n \text{ is odd} \end{cases}$$

An aperiodic necklace of length n is an equivalence class of size n, i.e., no two distinct rotations of a necklace from such class are equal. Number of aperiodic necklaces: $M_k(n) = \frac{1}{n} \sum_{d|n} \mu(d) k^{n/d}$.
Each aperiodic necklace contains a single Lyndon word. Lyndon word is n-character string over an alphabet of size k, and which is the minimum element in the lexicographical ordering of all its rotations. $w = uv$ is a Lyndon word if and only if u and v are Lyndon words and $u < v$.

Number theory (*gcd, totient, sieve, modexp, rabinmiller)

Linear diophantine equation . $ax + by = c$. Let $d = \gcd(a, b)$. A solution exists iff $d|c$. If (x_0, y_0) is any solution, then all solutions are given by $(x, y) = (x_0 + \frac{b}{d}t, y_0 - \frac{a}{d}t)$, $t \in \mathbb{Z}$. To find some solution (x_0, y_0), use extended GCD to solve $ax_0 + by_0 = d = \gcd(a, b)$, and multiply its solutions by $\frac{c}{d}$.
Linear diophantine equation in n variables: $a_1x_1 + \cdots + a_nx_n = c$ has solutions iff $\gcd(a_1, \ldots, a_n)|c$. To find some solution, let $b = \gcd(a_2, \ldots, a_n)$, solve $a_1x_1 + by = c$, and iterate with $a_2x_2 + \cdots = y$.
Multiplicative inverse of a modulo m: x in $ax + my = 1$, or $a^{\varphi(m)-1} \pmod m$.

Chinese Remainder Theorem . System $x \equiv a_i \pmod{m_i}$ for $i = 1, \ldots, n$, with pairwise relatively-prime m_i has a unique solution modulo $M = m_1m_2 \ldots m_n$: $x = a_1b_1\frac{M}{m_1} + \cdots + a_nb_n\frac{M}{m_n} \pmod M$, where b_i is modular inverse of $\frac{M}{m_i}$ modulo m_i.
System $x \equiv a \pmod m$, $x \equiv b \pmod n$ has solutions iff $a \equiv b \pmod g$, where $g = \gcd(m, n)$. The solution is unique modulo $L = \frac{mn}{g}$, and equals: $x \equiv a + T(b - a)m/g \equiv b + S(a - b)n/g \pmod L$, where S and T are integer solutions of $mT + nS = \gcd(m, n)$.

Prime-counting function . $\pi(n) = |\{p \le n : p \text{isprime}\}|$. $n/\ln(n) < \pi(n) < 1.3n/\ln(n)$. $\pi(1000) = 168$, $\pi(10^6) = 78498$, $\pi(10^9) = 50\ 847\ 534$. n-th prime $\approx n \ln n$.
List of primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71...

Miller-Rabin's primality test . Given $n = 2^rs + 1$ with odd s, and a random integer $1 < a < n$. If $a^s \equiv 1 \pmod n$ or $a^{2^js} \equiv -1 \pmod n$ for some $0 \le j \le r - 1$, then n is a probable prime. With bases 2, 7 and 61, the test indentifies all composites below 2^{32}. Probability of failure for a random a is at most $1/4$.

Pollard-ρ . Choose random x_1, and let $x_{i+1} = x_i^2 - 1 \pmod n$. Test $\gcd(n, x_{2^k+i} - x_{2^k})$ as possible n's factors for $k = 0, 1, \ldots$ Expected time to find a factor: $O(\sqrt{m})$, where m is smallest prime power in n's factorization. That's $O(n^{1/4})$ if you check $n = p^k$ as a special case before factorization.

Fermat primes . A Fermat prime is a prime of form $2^{2^n} + 1$. The only known Fermat primes are 3, 5, 17, 257, 65537. A number of form $2^n + 1$ is prime only if it is a Fermat prime.

Perfect numbers . $n > 1$ is called perfect if it equals sum of its proper divisors and 1. Even n is perfect iff $n = 2^{p-1}(2^p - 1)$ and $2^p - 1$ is prime (Mersenne's). No odd perfect numbers are yet found.

Carmichael numbers . A positive composite n is a Carmichael number ($a^{n-1} \equiv 1 \pmod n$ for all a: $\gcd(a, n) = 1$), iff n is square-free, and for all prime divisors p of n, $p - 1$ divides $n - 1$.

Number/sum of divisors . $\tau(p_1^{a_1} \ldots p_k^{a_k}) = \prod_{j=1}^k (a_j + 1)$. $\sigma(p_1^{a_1} \ldots p_k^{a_k}) = \prod_{j=1}^k \frac{p_j^{a_j+1}-1}{p_j-1}$. $\sigma_x(n) = \prod_{i=1}^r \frac{p_i^{(a_i+1)x}-1}{p_i^x-1}$

Euler's phi function . $\varphi(n) = |\{m \in \mathbb{N}, m \le n, \gcd(m, n) = 1\}|$. $\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)$
$\varphi(mn) = \frac{\varphi(m)\varphi(n)\gcd(m,n)}{\varphi(\gcd(m,n))}$. $\varphi(p^a) = p^{a-1}(p - 1)$. $\sum_{d|n} \varphi(d) = \sum_{d|n} \varphi(\frac{n}{d}) = n$.

Euler's theorem . $a^{\varphi(n)} \equiv 1 \pmod n$, if $\gcd(a, n) = 1$.
Wilson's theorem . p is prime iff $(p - 1)! \equiv -1 \pmod p$.

Mobius function . $\mu(1) = 1$. $\mu(n) = 0$, if n is not squarefree. $\mu(n) = (-1)^s$, if n is the product of s distinct primes. Let f, F be functions on positive integers. If for all $n \in N$, $F(n) = \sum_{d|n} f(d)$,

then $f(n) = \sum_{d|n} \mu(d)F(\frac{n}{d})$, and vice versa. $\varphi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$. $\sum_{d|n} \mu(d) = 1$.
If f is multiplicative, then $\sum_{d|n} \mu(d)f(d) = \prod_{p|n}(1 - f(p))$, $\sum_{d|n} \mu(d)^2 f(d) = \prod_{p|n}(1 + f(p))$.

Legendre symbol . If p is an odd prime, $a \in \mathbb{Z}$, then $\left(\frac{a}{p}\right)$ equals 0, if $p|a$; 1 if a is a quadratic residue modulo p; and -1 otherwise. Euler's criterion: $\left(\frac{a}{p}\right) = a^{\left(\frac{p-1}{2}\right)} \pmod p$.

Jacobi symbol . If $n = p_1^{a_1} \cdots p_k^{a_k}$ is odd, then $\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{k_i}$.

Primitive roots . If the order of g modulo m (min $n > 0$: $g^n \equiv 1 \pmod m$) is $\varphi(m)$, then g is called a primitive root. If Z_m has a primitive root, then it has $\varphi(\varphi(m))$ distinct primitive roots. Z_m has a primitive root iff m is one of 2, 4, p^k, $2p^k$, where p is an odd prime. If Z_m has a primitive root g, then for all a coprime to m, there exists unique integer $i = \mathrm{ind}_g(a)$ modulo $\varphi(m)$, such that $g^i \equiv a \pmod m$. $\mathrm{ind}_g(a)$ has logarithm-like properties: $\mathrm{ind}(1) = 0$, $\mathrm{ind}(ab) = \mathrm{ind}(a) + \mathrm{ind}(b)$.
If p is prime and a is not divisible by p, then congruence $x^n \equiv a \pmod p$ has $\gcd(n, p-1)$ solutions if $a^{(p-1)/\gcd(n,p-1)} \equiv 1 \pmod p$, and no solutions otherwise. (Proof sketch: let g be a primitive root, and $g^i \equiv a \pmod p$, $g^u \equiv x \pmod p$. $x^n \equiv a \pmod p$ iff $g^{nu} \equiv g^i \pmod p$ iff $nu \equiv i \pmod p$.)

Discrete logarithm problem . Find x from $a^x \equiv b \pmod m$. Can be solved in $O(\sqrt{m})$ time and space with a meet-in-the-middle trick. Let $n = \lceil\sqrt{m}\rceil$, and $x = ny - z$. Equation becomes $a^{ny} \equiv ba^z \pmod m$. Precompute all values that the RHS can take for $z = 0, 1, \ldots, n-1$, and brute force y on the LHS, each time checking whether there's a corresponding value for RHS.

Pythagorean triples . Integer solutions of $x^2 + y^2 = z^2$ All relatively prime triples are given by: $x = 2mn, y = m^2 - n^2, z = m^2 + n^2$ where $m > n, \gcd(m, n) = 1$ and $m \not\equiv n \pmod 2$. All other triples are multiples of these. Equation $x^2 + y^2 = 2z^2$ is equivalent to $(\frac{x+y}{2})^2 + (\frac{x-y}{2})^2 = z^2$.

Postage stamps/McNuggets problem . Let a, b be relatively-prime integers. There are exactly $\frac{1}{2}(a-1)(b-1)$ numbers *not* of form $ax+by$ $(x, y \geq 0)$, and the largest is $(a-1)(b-1)-1 = ab-a-b$.

Fermat's two-squares theorem . Odd prime p can be represented as a sum of two squares iff $p \equiv 1 \pmod 4$. A product of two sums of two squares is a sum of two squares. Thus, n is a sum of two squares iff every prime of form $p = 4k + 3$ occurs an even number of times in n's factorization.

Congruence $ax \equiv b \pmod n$

```
int congruence( int a, int b, int n ) { // finds ax = b(mod n)
    int d = gcd( a, n );
    if( b % d != 0 ) return 1<<30; // no solution
    pii ans = egcd( a, n );
    int ret = ans.x * ( b/d + 0LL ), mul = n/d;
    ret %= mul;
    if( ret < 0 ) ret += mul;
    return ret;
}
```

Extended GCD

```
pii egcd( LL a, LL b ) {  // returns x,y | ax + by = gcd(a,b)
    if( b == 0 ) return pii( 1, 0 );
    else {
        pii d = egcd( b, a % b );
        return pii( d.y, d.x - d.y * ( a / b ) );
    }
}
```

GCD

```
LL gcd( LL a, LL b ) { return !b ? a : gcd( b, a%b ); }
```

Euler's theorem . For any planar graph, $V - E + F = 1 + C$, where V is the number of graph's vertices, E is the number of edges, F is the number of faces in graph's planar drawing, and C is the number of connected components. Corollary: $V - E + F = 2$ for a 3D polyhedron.

Vertex covers and independent sets . Let M, C, I be a max matching, a min vertex cover, and a max independent set. Then $|M| \le |C| = N - |I|$, with equality for bipartite graphs. Complement of an MVC is always a MIS, and vice versa. Given a bipartite graph with partitions (A, B), build a network: connect source to A, and B to sink with edges of capacities, equal to the corresponding nodes' weights, or 1 in the unweighted case. Set capacities of the original graph's edges to the infinity. Let (S, T) be a minimum s-t cut. Then a maximum(-weighted) independent set is $I = (A \cap S) \cup (B \cap T)$, and a minimum(-weighted) vertex cover is $C = (A \cap T) \cup (B \cap S)$.

Matrix-tree theorem . Let matrix $T = [t_{ij}]$, where t_{ij} is the number of multiedges between i and j, for $i \neq j$, and $t_{ii} = -\deg_i$. Number of spanning trees of a graph is equal to the determinant of a matrix obtained by deleting any k-th row and k-th column from T.

Euler tours . Euler tour in an undirected graph exists iff the graph is connected and each vertex has an even degree. Euler tour in a directed graph exists iff in-degree of each vertex equals its out-degree, and underlying undirected graph is connected.

Stable marriages problem . While there is a free man m: let w be the most-preferred woman to whom he has not yet proposed, and propose m to w. If w is free, or is engaged to someone whom she prefers less than m, match m with w, else deny proposal.

Stoer-Wagner's min-cut algorithm . Start from a set A containing an arbitrary vertex. While $A \neq V$, add to A the most tightly connected vertex ($z \notin A$ such that $\sum_{x \in A} w(x, z)$ is maximized.) Store cut-of-the-phase (the cut between the last added vertex and rest of the graph), and merge the two vertices added last. Repeat until the graph is contracted to a single vertex. Minimum cut is one of the cuts-of-the-phase.

2-SAT . Build an implication graph with 2 vertices for each variable – for the variable and its inverse; for each clause $x \vee y$ add edges $(\overline{x}, y)$ and $(\overline{y}, x)$. The formula is satisfiable iff x and $\overline{x}$ are in distinct SCCs, for all x. To find a satisfiable assignment, consider the graph's SCCs in topological order from sinks to sources (i.e. Kosaraju's last step), assigning 'true' to all variables of the current SCC (if it hasn't been previously assigned 'false'), and 'false' to all inverses.

Randomized algorithm for non-bipartite matching . Let G be a simple undirected graph with even $|V(G)|$. Build a matrix A, which for each edge $(u, v) \in E(G)$ has $A_{i,j} = x_{i,j}$, $A_{j,i} = -x_{i,j}$, and is zero elsewhere. Tutte's theorem: G has a perfect matching iff $\det G$ (a multivariate polynomial) is identically zero. Testing the latter can be done by computing the determinant for a few random values of $x_{i,j}$'s over some field. (e.g. Z_p for a sufficiently large prime p)

Prufer code of a tree . Label vertices with integers 1 to n. Repeatedly remove the leaf with the smallest label, and output its only neighbor's label, until only one edge remains. The sequence has length $n - 2$. Two isomorphic trees have the same sequence, and every sequence of integers from 1 and n corresponds to a tree. Corollary: the number of labelled trees with n vertices is n^{n-2}.