Hacker School Analysis Seminar - Sequences

Attribution

Much of the content is closely adapted from Elementary Analysis: The Theory of Calculus by Kenneth Ross. This book, or others on mathematical analysis, should be consulted for further inquiry.

Useful Properties

Triangle Inequality

Claim:

$|a+b| \leq |a| + |b|$ *for all* $a, b \in \mathbb{R}$.

Proof: See Ross text.

Archimedean Property

Claim: *If* $a > 0$ *and* $b > 0$*, then there exists* $n \in \mathbb{N}$ *such that* $na > b$.

In essence, this states that no matter how small of an a and how large of a b we choose, we can always find an integer multiple of a that will exceed b.

Proof: Suppose, by way of contradiction, that the Archimedean property fails. Then there exists $a > 0$ and $b > 0$ such that $na \leq b$ for all $nin\mathbb{N}$. By definition, b is then an upper bound on the set $S = \{na : n \in \mathbb{N}\}$.

Using the completeness axiom, let $s_0 = \sup S$. Since $a > 0$, we have $s_0 < s_0 + a$, so $s_0 - a < s_0$. Since s_0 is the least upper bound of S, $s_0 - a$ cannot be an upper bound of S. This implies that there exists an $s = n_0 a \in S$ such that $s_0 - a < n_0 a$ for some $n_0 \in \mathbb{N}$. Adding a to each side of the inequality, we get $s_0 < (n_0 + 1)a$. Since $n_0 + 1 \in \mathbb{N}$, $(n_0 + 1)a \in S$, and so s_0 is not an upper bound for S. Contradiction. $\mathbb{QED}$.

Denseness of $\mathbb{Q}$

Claim: *If* $a, b \in \mathbb{R}$ *and* $a < b$*, then there exists a rational number* $r \in \mathbb{Q}$ *such that* $a < r < b$.

Pseudo-Proof: We need to show that $a < \frac{m}{n} < b$ for some $m, n \in \mathbb{N}$, where $n > 0$. Thus, we need

$$an < m < bn$$

Since $b - a > 0$, the Archimedean property shows that there exists an $n \in \mathbb{N}$ such that $n(b-a) > 1$. Thus, $bn - an > 1$, and so it is *fairly evident* that there exists an m between an and bn, so our claim holds. (Actually *proving* that this m exists is a bit more delicate, see page 24/25 in Ross for more detail.)

Sequences

Formally, a *sequence* is a function from $\{n \in \mathbb{Z} : n \geq m\}$, where m is usually 0 or 1, into $\mathbb{R}$. However, it is customary to write s_n rather than $s(n)$, and is also convenient to write $(s_n)_{n=m}^{\infty}$, (s_m, s_{m+1}, s_{m+2}), or, when $m = 1$, $(s_n)_{n\in\mathbb{N}}$.

Examples (1)

(a). Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{1}{n^2}$. This is the sequence $(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ...)$. Formally this is the function with domain $\mathbb{N}$ with value $\frac{1}{n^2}$ for each n. The *set* of values is $\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ...\}$.

(b). Consider the sequence $(b_n)_{n=0}^{\infty}$ where $b_n = (-1)^n$. The sequence is $(1, -1, 1, -1, ...)$, however its *set* of values is $\{-1, 1\}$.

It's important to distinguish between a sequence and its set of values, and we will always use parentheses ( ) to signify a sequence and braces { } to signify a set.

(c). Consider the sequence $(c_n)_{n\in\mathbb{N}}$ where $c_n = (1 + \frac{1}{n})^n$. This is the sequence $(2, (\frac{3}{2})^2, (\frac{4}{3})^3, (\frac{5}{4})^4, ...)$, or approximately

$$(2, 2.25, 2.3704, 2.4414, 2.4883, 2.5216, 2.5465, 2.5658, ...).$$

c_{100} is approximately 2.7048, and c_{1000} is approximately 2.7169.

(d.) Consider the sequence $(d_n)_{n\in\mathbb{N}}$ where $d_n = \pi^{n-1}$. This is the sequence $(\pi^0, \pi^1, \pi^2, \pi^3, ...)$, or approximately

$$(0, 3.1416, 9.8696, 31.0063, ...).$$

d_{100} is approximately 5.1878×10^{49} and d_{1000} causes an OverflowError in Python.

Limits

The *limit* of a sequence (s_n) is a real number which the values s_n are "close" to for large values of n. In example (a), the values are "close" to 0 for large n, and in example (c), the values are close to Euler's constant, e, for large n. However, example (b) doesn't seem to get close to any number, but instead jumps between -1 and 1. As we'll see in the following definition, a *limit* will require the sequence values to be close to the limit value for *all* large n, so neither 1 or -1 will be limits of (c_n).

Definition (1): A sequence (s_n) of real numbers is said to *converge* to the real number s if and only if for every $\epsilon > 0$ there exists a number N such that $|s_n - s| < \epsilon$ for all $n > N$.

Definition (2): A sequence (s_n) of real numbers is said to *diverge towards* ∞ if and only if for every $M \in \mathbb{R}$ there exists a number N such that $s_n > M$ for all $n > N$.

Definition (3): A sequence (s_n) of real numbers is said to *diverge towards* $-\infty$ if and only if for every $M \in \mathbb{R}$ there exists a number N such that $s_n < M$ for all $n > N$.

Examples (2)

We will now prove a couple of the above examples (1).

(a.) We aim to prove that $\lim_{n\to\infty} a_n = 0$, so we must show that there exists $N in \mathbb{N}$ such that for all $n > N$, $|a_n - 0| < \epsilon$ for all $\epsilon > 0$.

Let $\epsilon > 0$ be given. Now, define $N = \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil$. Then for all $n > N$,

$$|a_n - 0| = a_n = \frac{1}{n^2} < \frac{1}{N^2} = \frac{1}{\left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil^2} \leq \frac{1}{\left(\frac{1}{\sqrt{\epsilon}}\right)^2} = \frac{1}{\left(\frac{1^2}{\sqrt{\epsilon}^2}\right)} = \left(\sqrt{\epsilon}\right)^2 = \epsilon.$$

$\mathbb{QED}$.

The relevant function (in python) would be:

```
from math import sqrt, ceil
def example_a(epsilon):
    N = int(ceil(sqrt(epsilon)))
    return N
```

(d.) We aim to prove that $\lim_{n\to\infty} d_n$ diverges towards ∞, so we must show that for all $M \in \mathbb{R}$ their exists $N \in \mathbb{N}$ such that $d_n > M$ for all $n > N$.

Let $M \in \mathbb{R}$ be given, and note that if $M \leq 0$ and we show that $d_n > 0$ for all $n \in \mathbb{N}$, then $d_n > M$ as well. So let $\hat{M} = \max(M, 1)$, and define $N = \lceil \log_\pi \hat{M} \rceil$. Then for all $n > N$,

$$d_n = \pi^n > \pi^N = \pi^{\lceil \log_\pi \hat{M} \rceil} \geq \pi^{\log_\pi \hat{M}} = \hat{M} = \max(M, 1) \geq M.$$

$\mathbb{QED}$.

The relevant function (in python) would be:

```
from math import pi, log, ceil
def example_d(M):
    M = max(1,M)
    N = int(ceil(log(M,pi)))
    return N
```

Exercises

(1) Write out the first five terms of the following sequences.

(a). $a_n = \frac{n}{n+1}$

(b). $b_n = 2^{-n}$

(c). $c_n = n!$

(d). $d_n = 1 + \frac{2}{n}$

(e). $e_n = \frac{6n^2+7}{4n^2-9}$

(f). $f_n = (-1)^n n$

(g). $g_n = \frac{n^3+4}{7n^2-13}$

(h). $h_n = \sin(n\pi)$

(i). $i_n = \frac{1}{n}\cos(n)$

(j). $j_n = \cos(n)$

(2) For each sequence above, determine whether it converges, diverges to $\pm\infty$, or doesn't converge/diverge, and if so give it's limit. Proofs are not required.

(3) For each sequence above that converges (or diverges to $\pm\infty$), find the mathematical function for N given M or ϵ.

(4) For each sequence above that converges (or diverges to $\pm\infty$), write a function in a programming language of your choice for N given M or ϵ.

(5) For each sequence above that converges (or diverges to $\pm\infty$), prove that it converges to the limit (or diverges to $\pm\infty$).