Let $\epsilon > 0$ Be Given

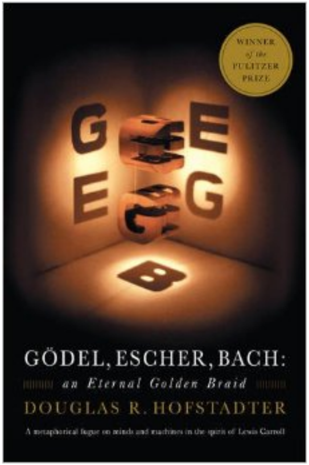
Erik Taubeneck

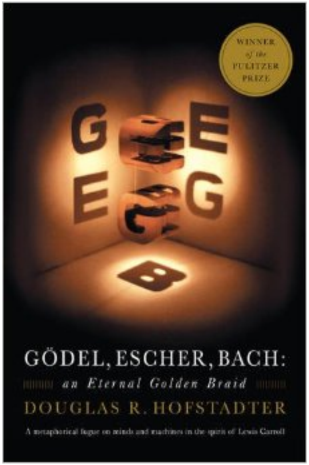
GameChanger

November 16th, 2016

Welcome Erik and Maura!

The Beginning of the Story





Propositional Calculus

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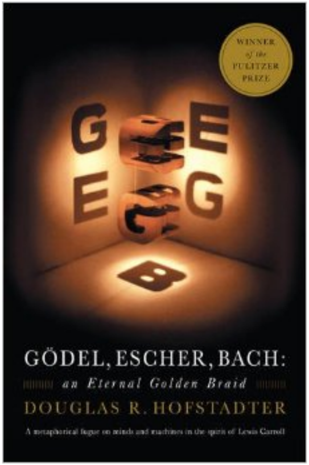
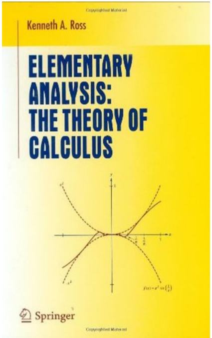


~~Propositional~~ Calculus

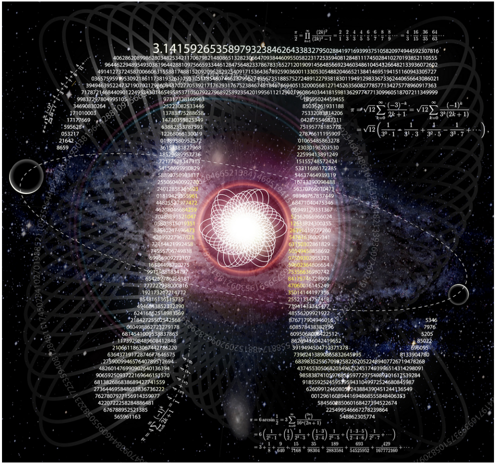
The *Real* Beginning of the Story



Outline:

- $\mathbb{N}$: The *natural numbers* (i.e. counting numbers)
- $\mathbb{Z}$: The *integers* (i.e. $\mathbb{N} \cup 0 \cup$ negative $\mathbb{N}$)
- $\mathbb{Q}$: The *rational numbers* (i.e. fractions)
- $\mathbb{R}$: The *real numbers* (i.e. $\mathbb{Q} \cup$ the crazy numbers like e and π)
- Sequence: a sequence of numbers, (e.g. $\{1, 1, 2, 3, 5, 8, ...\}$)
- ∞ (infinity)
- Hyper-dimensional balls in crazy high dimensions

Math - The Universal Language?



Nope!

Math is rooted in Definitions and Axioms

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Zermelo–Fraenkel set theory

From Wikipedia, the free encyclopedia

"ZFC" redirects here. For other uses, see ZFC (disambiguation).

In mathematics, **Zermelo–Fraenkel set theory**, named after mathematicians Ernst Zermelo and Abraham Fraenkel, is one of several axiomatic systems that were proposed in the early twentieth century to formulate a theory of sets free of paradoxes such as Russell's paradox. Zermelo–Fraenkel set theory with the historically controversial axiom of choice included is commonly abbreviated **ZFC**, where C stands for choice.[1] Many authors use **ZF** to refer to the axioms of Zermelo–Fraenkel set theory with the axiom of choice excluded. Today ZFC is the standard form of axiomatic set theory and as such is the most common foundation of mathematics.

ZFC is intended to formalize a single primitive notion, that of a hereditary well-founded set, so that all entities in the universe of discourse are such sets. Thus the axioms of ZFC refer only to pure sets and prevent its models from containing urelements (elements of sets that are not themselves sets). Furthermore, proper classes (collections of mathematical objects defined by a property shared by their members which are too big to be sets) can only be treated indirectly. Specifically, ZFC does not allow for the existence of a universal set (a set containing all sets) nor for unrestricted comprehension, thereby avoiding Russell's paradox. Von Neumann–Bernays–Gödel set theory (NBG) is a commonly used conservative extension of ZFC that does allow explicit treatment of proper classes.

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- N4. If n and m have the same successors, then $n = m$.
- N5. A subset of $\mathbb{N}$ which contains 1, and which contains $n + 1$ whenever it contains n, must equal $\mathbb{N}$.

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- $\mathbb{R}$ (talk to me afterwards...)

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- $\left(2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, ...\right) : s_n = \left(1 + \frac{1}{n}\right)^n$

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A *sequence* can do one of three things as $n \to \infty$:

- Converge to $m \in \mathbb{R}$
- Diverge to ∞ or $-\infty$
- Not Converge or Diverge

Each of these has their own definition.

Convergent Sequences

A *sequence* is said to *converge* to $s \in \mathbb{R}$ if for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|s_n - s| > \epsilon$ for all $n > N$ $(n \in \mathbb{N})$.

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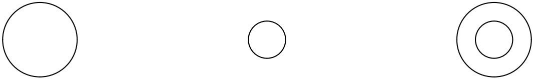
A Circle



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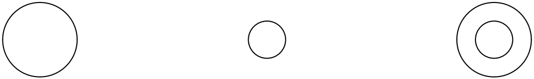

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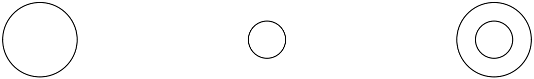

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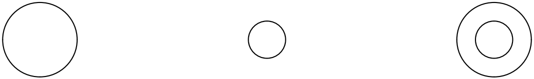

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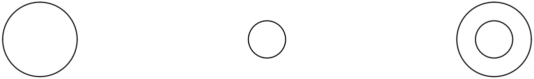
Now, let $A(x) = \pi r^2$ (the area function) and note that
$A(c_1) = \pi \quad A(c_{0.5}) = 0.25 * \pi \quad \mathrm{A(d)} = A(c_1) - A(c_{0.5}) = 0.75\pi$

A Circle

Now, note the proportion of area contained within the "donut" d

$$\frac{A(d)}{A(c_1)} = \frac{A(c_1) - A(c_{0.5})}{A(c_1)} = \frac{0.75\pi}{\pi} = 0.75$$

A Sphere



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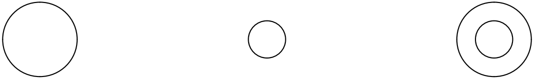

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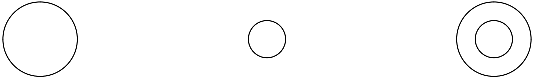

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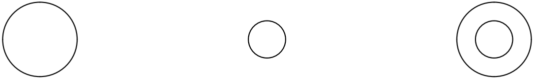

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Then let d^3 be the "donut" left over when we remove $c_{0.5}$ from c_1.

Now, let $V_3(x) = \frac{4}{3}\pi r^3$ (the volume function) and note that

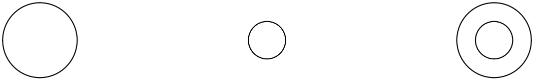
$$V_3(c_1^3) = \frac{4}{3}\pi \quad V_3(c_{0.5}^3) = \frac{4}{3}\frac{1}{2^3} * \pi = \frac{1}{6}\pi \quad V_3(d^3) = V_3(c_1^3) - V_3(c_{0.5}^3) = \frac{7}{6}\pi$$

A Sphere

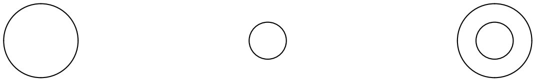
Again, note the proportion of volume contained within the "donut" d^3

$$\frac{V_3(d^3)}{V_3(c_1^3)} = \frac{V_3(c_1^3) - V_3(c_{0.5}^3)}{V_3(c_1^3)} = \frac{\frac{7}{6}\pi}{\frac{4}{3}\pi} = \frac{7}{8}$$

We Can Go Higher!



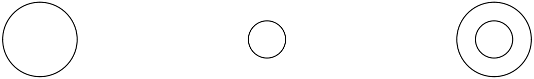
We Can Go Higher!



In general, the hyper-volume of a hyper-sphere in n dimensions is

$$V_n(r) = \frac{r^n \pi^{n/2}}{\Gamma(n/2+1)}.$$

We Can Go Higher!



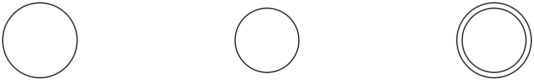
In general, the hyper-volume of a hyper-sphere in n dimensions is

$$V_n(r) = \frac{r^n \pi^{n/2}}{\Gamma(n/2+1)}.$$

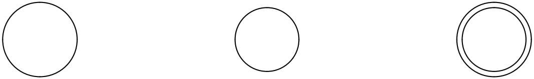
Then, the proportion of volume contained within the hyper "donut" is

$$\frac{V_n(1) - V_n(0.5)}{V_n(1)} = \frac{\frac{1^n \pi^{n/2}}{\Gamma(n/2+1)} - \frac{(0.5)^n \pi^{n/2}}{\Gamma(n/2+1)}}{\frac{1^n \pi^{n/2}}{\Gamma(n/2+1)}} = \frac{\frac{(1-(0.5)^n)\pi^{n/2}}{\Gamma(n/2+1)}}{\frac{\pi^{n/2}}{\Gamma(n/2+1)}} = 1 - (0.5)^n$$

We Can Go Higher!



We Can Go Higher!



The proportion of volume contained within the hyper "donut" that is within $\delta > 0$, $\delta < 1$ of the surface is

$$\frac{V_n(1) - V_n(1-\delta)}{V_n(1)} = \frac{\frac{1^n \pi^{n/2}}{\Gamma(n/2+1)} - \frac{(1-\delta)^n \pi^{n/2}}{\Gamma(n/2+1)}}{\frac{1^n \pi^{n/2}}{\Gamma(n/2+1)}} = \frac{\frac{(1-(1-\delta)^n)\pi^{n/2}}{\Gamma(n/2+1)}}{\frac{\pi^{n/2}}{\Gamma(n/2+1)}} = 1-(1-\delta)^n$$

We Can Go Higher!

Let $s_n = 1 - (1 - \delta)^n$.

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Claim: s_n converges to $s = 1$ for all $\delta > 0, \delta < 1$.

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Let $N = \left\lceil \frac{\log(\epsilon)}{\log(1-\delta)} \right\rceil$.

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Then, for all $n > N$

We Can Go Higher!

$$n > N \geq \frac{\log(\epsilon)}{\log(1-\delta)}$$

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$$(1-\delta)^n < \epsilon$$

$$1 - (1 - (1-\delta)^n) < \epsilon$$

$$|1 - (1 - (1-\delta)^n)| < \epsilon$$

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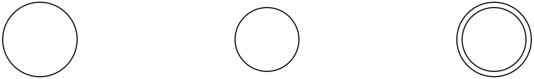
QED

Woah!



In crazy high enough dimensions, an arbitrarily high proportion of the outer hyper volume of any hyper sphere is concentrated arbitrarily close the boundary!