Continuous Logistic Equation with Time Delay

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Contents

1 Abstract

In this project we looked at the continuous logistic equation with time delay. We found that altering the values of λ completely altered the behavior of our solutions. Analytically we found that the perturbations from the equilibrium solutions either grow or shrink infinitely, depending on whether λ is larger or smaller than $\frac{\pi}{2}$, respectively. This means that for λ values smaller than $\frac{\pi}{2}$, the equilibrium solution is stable, given that small perturbations away from the solution move toward the solution .For λ values larger than $\frac{\pi}{2}$, the equilibrium solution is unstable, given that small perturbations from the solution grow away from the solution.

2 Introduction

The continuous logistic equation with delay is given by

$$\frac{dN}{dt} = rN(1 - \frac{N(t-\tau)}{k}) \tag{1}$$

This project included looking at the equations and trying to find both numerical and analytic solutions and observe how they changed as we altered certain parameters. The original logistic equation had 3 parameters, r,k and τ . By non dimensionalizing the equation so that it had only one parameter, λ, it became easier to analyze.

3 Discussion

3.1

Consider the transformations

$$y(x) = \frac{N(t)}{k}$$

$$x = \frac{t}{\tau}$$

this implies

$$x - 1 = \frac{t}{\tau} - 1$$

$$x - 1 = \frac{t}{\tau} - \frac{\tau}{\tau}$$

$$x - 1 = \frac{t - \tau}{\tau}$$

so

$$\frac{N(\frac{t-\tau}{\tau})}{k} = y(\frac{t - \tau}{\tau}) = y(x - 1)$$

$$\frac{dN}{dt} = \frac{d(ky)}{dt} = k\frac{d(y)}{dt} * \frac{dx}{dx} = k\frac{dy}{dx}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{\tau}$$

$$\frac{dN}{dt} = \frac{k}{\tau}\frac{dy}{dx}$$

plugging into equation 1 we get

$$\frac{k}{\tau}\frac{dy}{dx} = rky(x)(1 - y(x - 1))$$

$$\frac{1}{\tau}\frac{dy}{dx} = ry(x)(1 - y(x - 1))$$

$$\frac{dy}{dx} = r\tau \cdot y(x)(1 - y(x - 1))$$

let λ=$r\tau$, we get

$$\frac{dy}{dx} = \lambda y(x)(1 - y(x - 1)) \tag{2}$$

3.2

Plotting equation 2 using ddde23 in MATLAB, which gives numerical approximations for the delay differential equation over a given time span , with history function y=0.5 for -1 $\leq x<0$, using λ values of 1 and 1.8 gives

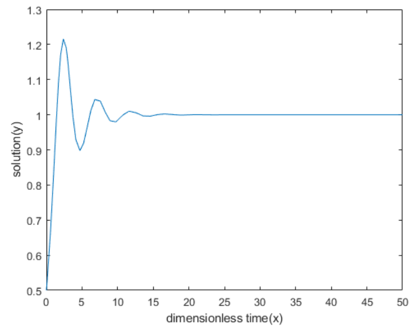

Figure 1: $\lambda = 1$

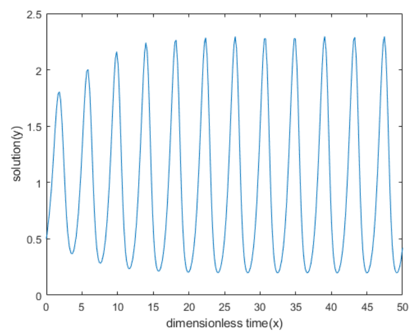

Figure 2: $\lambda = 1.8$

Plotting on the same graph we get

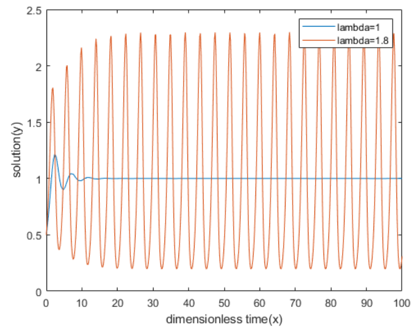

Figure 3: $\lambda = 1$ vs. $\lambda = 1.8$

It appears that in the case of $\lambda = 1$, values of y oscillate but grow smaller in amplitude and converge to y=1. In the case of $\lambda = 1.8$, values of y oscillate and grow a bit larger in amplitude, and never converge. Doing a parameter sweep for different values of λ, we can see a change in behavior for solutions.

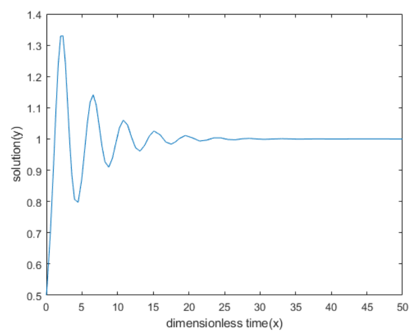

Figure 4: $\lambda = 1.2$

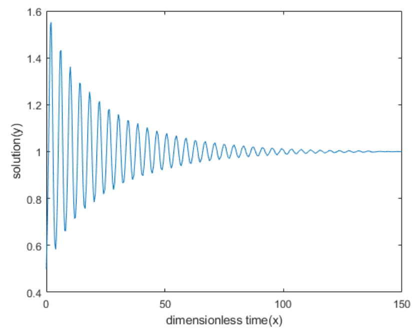

Figure 5: $\lambda = 1.5$

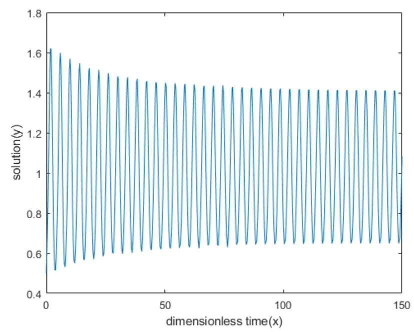

Figure 6: $\lambda = 1.6$

It appears that the solutions change from converging to non converging between λ values of 1.5 and 1.6. We will explore this more looking at the stability of equilibrium solutions.

3.3

Next we want to classify equilibrium solutions of equation 2 by stability, setting $\frac{dy}{dx} = 0$ we get

$$\lambda y(1 - y(x-1)) = 0$$

from here our solutions are y=0 or y=1. To determine the stability of y=1, we will examine small perturbations about the solution.
Let u(x) = y(x)-1
If u(x) grows large as x $\rightarrow$ ∞, this means values of y move away from the equilibrium solution, unstable
If u(x) becomes small as x $\rightarrow$ ∞, this means values of y move toward the equilibrium solution, stable
u=y-1 $\rightarrow$ y=u+1

$$\frac{du}{dx} = \frac{dy}{dx}$$

plugging into equation 2 we get

$$\frac{du}{dx} = \lambda(u(x) + 1)(1 - [u(x-1) + 1]$$

$$\frac{du}{dx} = \lambda(u(x) + 1)(1 - u(x-1) - 1]$$

$$\frac{du}{dx} = \lambda(u(x) + 1)(-u(x-1)$$

$$\frac{du}{dx} = -\lambda(u(x))(u(x-1)) - \lambda u(x-1)$$

For small values of u, the first term

$$-\lambda(u(x))(u(x-1))$$

will be a quadratic function of u , meaning it is negligibly small. So

$$\frac{du}{dx} \approx -\lambda u(x-1) \tag{3}$$

because this is a linear function we can assume u=$e^{\alpha x}$
Plugging into equation 3, we get

$$\alpha e^{\alpha x} = -\lambda e^{\alpha(x-1)}$$

$$\alpha e^{\alpha x} + \lambda e^{\alpha(x-1)} = 0$$

$$\alpha e^{\alpha x} + \lambda e^{\alpha x} \cdot e^{-\alpha} = 0$$

$$e^{\alpha x}(\alpha + \lambda \cdot e^{-\alpha}) = 0$$

because $e^{\alpha x} \neq 0$, this gives

$$\alpha + \lambda \cdot e^{-\alpha} = 0$$

$$\alpha = -\lambda \cdot e^{-\alpha}$$

$$\alpha \cdot e^{\alpha} = -\lambda$$

assume α is complex, α= a+bi

$$(a + bi) \cdot e^{a+bi} = -\lambda$$

$$(a + bi) \cdot e^{a} \cdot e^{bi} = -\lambda$$

$$(a + bi) \cdot e^{a} \cdot [\cos(b) \, + \, i \, \sin(b)] = -\lambda$$

$$-\lambda = e^{a}[a \cdot \cos(\mathrm{b}) - b \cdot \sin(\mathrm{b}) + i(a \cdot \sin(b) + b \cdot \cos(b))]$$

equating real and imaginary parts we get

$$-\lambda = e^{\alpha}[a \cdot \cos(\mathrm{b}) - b \cdot \sin(\mathrm{b})]$$

$$0 = e^{\alpha}[a \cdot \sin(b) + b \cdot \cos(b)]$$

Solutions have the form $u = e^{\alpha x} = e^{(a+bi)x} = e^{ax} \cdot e^{bxi} = e^{ax} \cdot [cos(bx) + i \cdot sin(bx)]$
The second term, $cos(bx) + i \cdot sin(bx)$, is sinusoidal and will oscillate. The first term e^{ax} will determine if u(x) will grow or shrink as x $\rightarrow \infty$.
If a $>$ 0, u(x) will grow as x $\rightarrow \infty$
If a $<$ 0, u(x) will shrink as x $\rightarrow \infty$
By setting a = 0 , we can examine what will change the stability of the solution y=1. This gives

$$\lambda = b \cdot \sin(b)$$

$$0 = b \cdot \cos(b)$$

Solutions to this system are
b = 0, $\lambda = 0$
b = $\frac{\pi}{2} + 2\pi k$, $\lambda = \frac{\pi}{2} + 2\pi k$ $(k \in Z)$
b = $\frac{3\pi}{2} + 2\pi k$, $\lambda = \frac{\pi}{2} + 2\pi k$ $(k \in Z)$
So a bifurcation likely occurs when $\lambda = \frac{\pi}{2}$ This makes sense given that we saw that the solutions of y changed in behavior as λ changed from 1.5 to 1.6 ($\frac{\pi}{2}$ $\approx$ 1.5708).

Graphing numerical solutions for equation 3, $\frac{du}{dx} \approx -\lambda u(x-1)$, to see how the perturbations behave given different λ values we observe

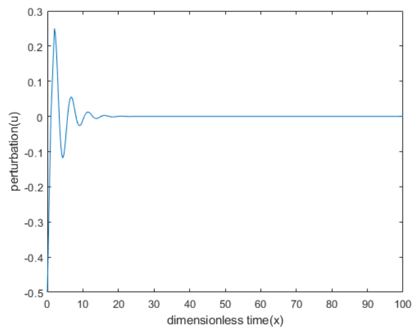

Figure 7: $\lambda = 1$

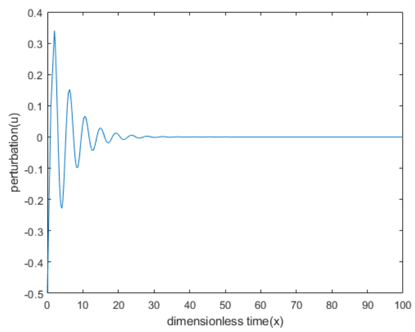

Figure 8: $\lambda = 1.2$

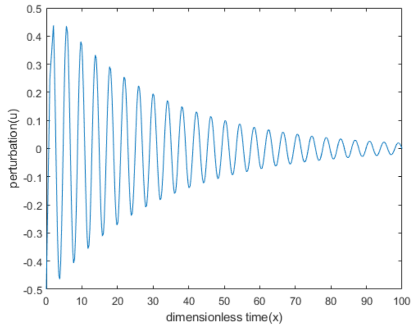

Figure 9: $\lambda = 1.5$

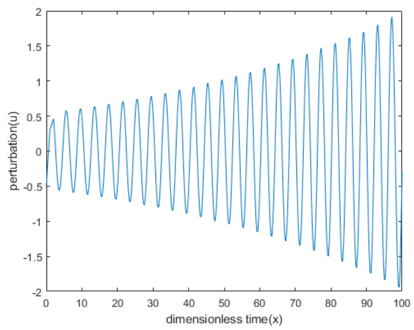

Figure 10: $\lambda = 1.6$

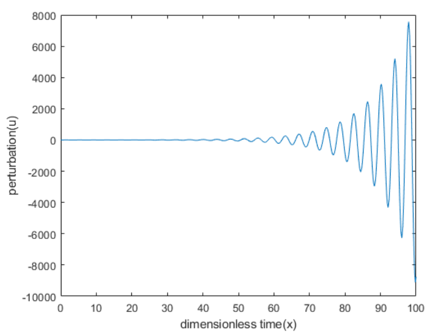

Figure 11: $\lambda = 1.8$

This supports the assumption that the solutions of y change behavior as we alter the parameter λ. We see that for values of $\lambda<\frac{\pi}{2}$, u(x) becomes smaller in amplitude as x $\to$ ∞. For values of $\lambda>\frac{\pi}{2}$, u(x) grows infinitely large in amplitude as x $\to$ ∞. This means that for values of $\lambda<\frac{\pi}{2}$ the equilibrium solution of equation 2, y=1, is stable. For values of $\lambda>\frac{\pi}{2}$, the equilibrium solution is unstable.

4 Appendix

```
%% This code will use dde23 to compute a numerical approximation to solve
%% the dimensionless logistic equation
lags=[1]
tspan = [0 150];
history=[0.5]
sol = dde23(@ddefun, lags, history, tspan)
plot(sol.x,sol.y)
xlabel('dimensionless time(x)')
ylabel('solution(y)')




function dydx = ddefun(x,y,Z)
  ylag1 = Z(:,1)


  dydx = lambda*y(1)*(1-ylag1)
end
```

```
%% This code will use dde23 to compute a numerical approximation to solve
%% the perturbation equation
lags=[1]
tspan = [0 100];
history=[-0.5]
sol = dde23(@ddefun, lags, history, tspan)
plot(sol.x,sol.y)
xlabel('dimensionless time(x)')
ylabel('perturbation(u)')


function dydx = ddefun(x,y,Z)
  ylag1 = Z(:,1)


  dydx = -lambda*ylag1
end
```