Sequential convex optimization for quantum process tomography

There are occasions when we want to identify or approximate a physical process that transforms an initial quantum state ρ to a final quantum state ρ'.

Figure 1: Can we identify a quantum process given initial and final states?

This procedure, called *quantum state tomography*, generally requires an exponentially large number of initial-final state pairs to fully specify the process. A system of N qubits has dimension $d = 2^N$, and complete tomography requires $d^2 = 2^{2N}$ initial-final state pairs.

This project aims to develop an algorithm to approximate a quantum channel $\mathcal{E}$ given a single initial-final state pair represented by PSD density matrices ρ, ρ'. A general quantum channel $\mathcal{E}$ can be described by a set of matrices $K_i \in \mathbb{C}^{d\times d}$ for $i = 1, \ldots, d^2$ called *Kraus operators* [1], where,

$$\rho' = \mathcal{E}(\rho) = \sum_{i=1}^{d^2} K_i \rho K_i^\dagger \tag{1}$$

$$\sum_{i}^{d^2} K_i^\dagger K_i \leq I. \tag{2}$$

With this representation, finding a quantum channel is an optimization problem,

$$\min_{\{K_i\}} \quad \left\| \sum_{i}^{M} K_i \rho K_i^\dagger - \rho' \right\|_F \tag{3}$$

$$\text{s.t.} \quad \sum_{i}^{M} K_i^\dagger K_i \leq I. \tag{4}$$

We will implement a sequential optimization algorithm to solve this problem. For each term in the sum, we will fix one of the K_i matrices and optimize over the other. We can add a regularization term $\|K_i' - K_i\|_F$ if the two operators in each term do not converge to be the equal.

If successful, we can explore extending this algorithm to a set of initial-final state pairs $\{(\rho^{(i)}, \rho'^{(i)})\}$ and efficiently representing the quantum states and Kraus operators with tensor networks.

All left/right operators

There are multiple ways to solve the optimization problem sequentially. One option is to fix the Kraus operators on the left of the density matrix, and optimize the Kraus operators on the right of the density matrix. Then, the Kraus operators on the left can be fixed, and the Kraus operators on the right can be optimized. This process can be repeated until there is convergence. This process can be expressed as follows,

$$\min_{\{K_i^{(i)}\}} \quad \left\| \sum_i^M K_i^{(i-1)} \rho K_i^{(i)\dagger} - \rho' \right\|_F \tag{5}$$

$$\text{s.t.} \quad \sum_i^M K_i^{(i-1)\dagger} K_i^{(i)} \leq I. \tag{6}$$

Since $K_i^{(i-1)}$ is fixed, this now becomes a convex problem over $K_i^{(i)}$. This approach was implemented and tested with with a bit-flip channel with probability p of flipping the bit,

$$K_1 = \sqrt{1-p}I, \; K_2 = \sqrt{p}X$$

$$\rho_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho_1 = \sum_{i=1}^{2} K_i \rho_0 K_i^\dagger = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$$

This is a simple and purely classical initial state and channel. Also, the Kraus operators were assumed to be real and were initialized as,

$$K_1^{(0)} = \frac{1}{\sqrt{2}}I, \; K_2^{(0)} = \frac{1}{\sqrt{2}}X$$

This approach did not converge,

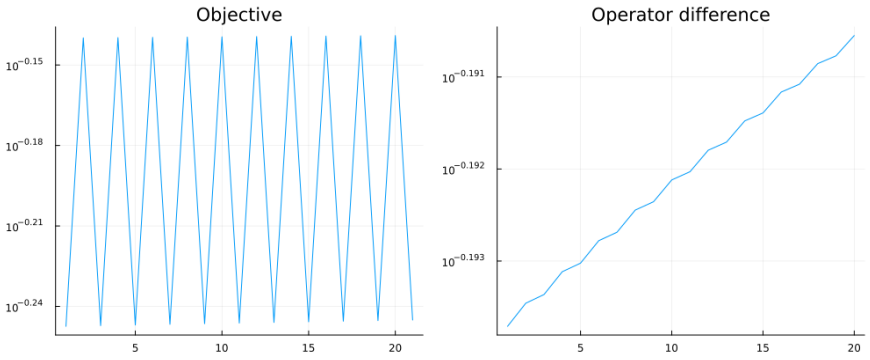

Figure 2: Results without regularization term

To combat this issue, a regularization term was added that would keep $K_i^{(i-1)}$ and $K_i^{(i)}$ from diverging too much. The regularized objective function is formulated as follows,

$$\min_{\{K_i^{(i)}\}} \left\| \sum_i^M K_i^{(i-1)} \rho K_i^{(i)\dagger} - \rho' \right\|_F + \frac{\lambda}{M} \sum_i^M \left\| K_i^{(i)} - K_i^{(i-1)} \right\|_F^2 \tag{7}$$

For some values of λ, this approach converged nicely.

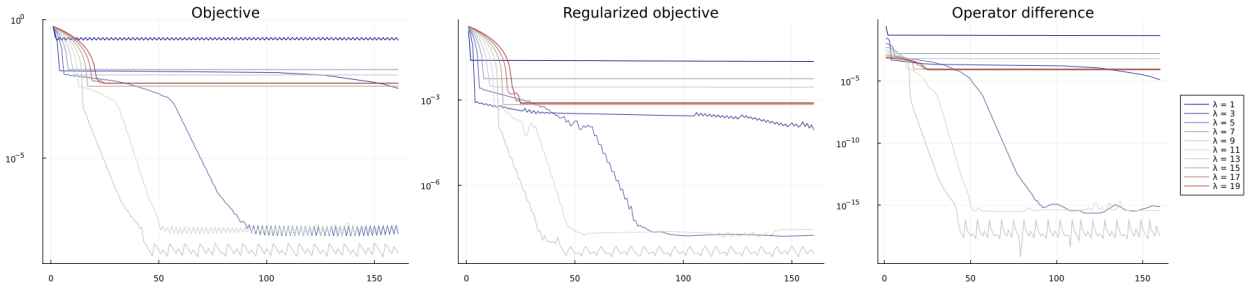

Figure 3: Sweep of regularization weight

When λ is too small, the optimizer got stuck in an oscillation. When λ is too big, the optimizer seems to not be able to search adequately.

Next, random initialization of the Kraus operators was examined,

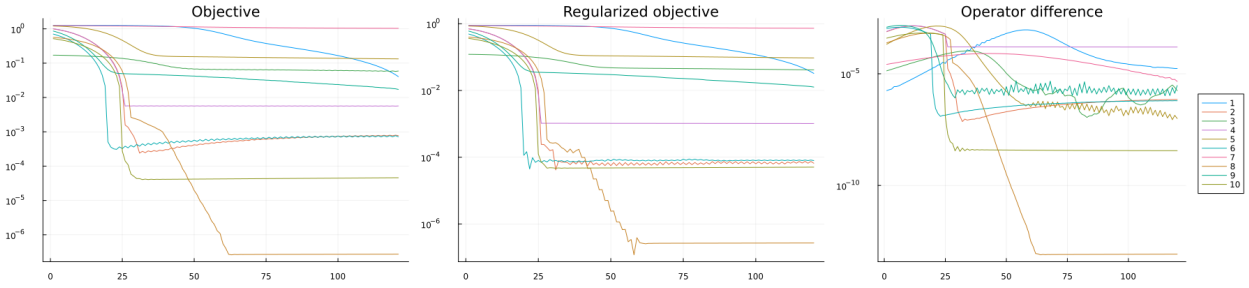

Figure 4: Random Kraus operator initialization

Not all instances were able to converge.

Single left/right operators

Instead of optimizing all M Kraus operators in each step, we can optimize one operator K_j at a time. This can be expressed as follows,

$$\min_{K_j^{(i)}} \left\| \left(\sum_{i \neq j}^{M} K_i^{(i-1)} \rho K_i^{(i-1)\dagger} \right) + K_j^{(i-1)} \rho K_j^{(i)\dagger} - \rho' \right\|_F + \lambda \left\| K_j^{(i)} - K_j^{(i-1)} \right\|_F^2 \tag{8}$$

This approach would allow us to parallelize the optimization over the M Kraus operators. Since the dimension of the Kraus operators grows as $D = 2^N$ in the number of qubits, and the number of Kraus operators is $M = D^2$, this parallelizability may be crucial for the problem to be tractable.

Again we perform a sweep of λ with the bit-flip channel and densities presented earlier.

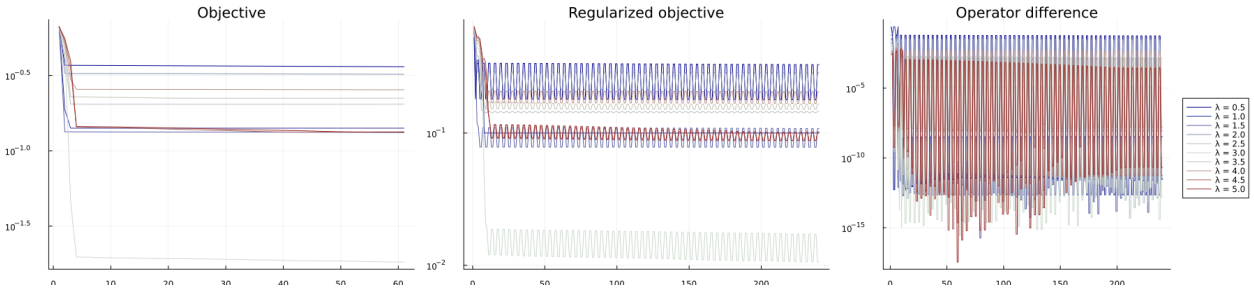

Figure 5: Optimizing one Kraus operator at a time

This method does not seem to converge as readily as the first method. However, it may be possible that this method can be further optimized.

Next steps

1. Refine the sequential optimization approach(es) with input from peers and course staff.
2. Examine other canonical channels, like depolarization, dephasing, and amplitude dampening.
3. Extend to complex Kraus operators. A different normalization constraint will be needed as $\sum_i^M K_i^{(i-1)\dagger} K_i^{(i)}$ may no longer be real-valued.
4. Optimize over multiple input-output density pairs.
5. Once single-qubit optimization works well, we can extend this to many-body channel learning.

References

[1] Michael A. Nielsen & Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.