The Complexity of Determining Critical Sets

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Given an NP search problem, we have a set of instances I, and a set of solutions S. We let a clue be a subset of a solution.

The fewest clues problem asks for the smallest clue with a unique solution for a given instance.

1 Reductions

General Clue Reduction We have a function mapping instances of A to instances of B, $f : I_A \to I_B$ such that $x \in A \iff f(x) \in B$. We also want a function $g : C_B \to C_A$ such that g is surjective, g is a bijection on solutions, and subsets are preserved. So $c \subset c' \iff g(c) \subset g(c')$. We also require that if c is a clue set for $f(x)$, then $g(c)$ is a clue set for x.

So on input x, we find the minimum clue problem for $f(x)$, $c_{f(x)}$ and then use g to get a clue c_x for x.

Theorem 1. *Suppose A reduces to B with this reduction and* $\mathsf{FCP}A$ *is* FCP*-hard, then* $\mathsf{FCP}B$ *is* FCP*-hard.*

Proof. We want to show that if $c_{f(x)}$ is a minimum clue set, then $g(c_{f(x)})$ is a minimum clue set as well.

First, we show that $g(c_{f(x)})$ is a clue. This is true because $c_{f(x)}$ is a clue. That is, there exists some solution $S_{f(x)}$ such that $c_{f(x)} \subset S_{f(x)}$, which means that $g(c_{f(x)}) \subset g(S_{f(x)})$, which is a solution.

Second, we need to show that $g(c_{f(x)})$ has a unique superset solution. Suppose there were two different solutions, $S_x^{(1)}$ and $S_x^{(2)}$, then $g(c_{f(x)} \subset S_x^{(1)}, S_x^{(2)}$, and so $c_{f(x)} \subset S_{f(x)}^{(1)}, S_{f(x)}^{(2)}$ because g is surjective, and because solutions have inverse solutions. Therfore, $c_{f(x)}$ is not unique.

Third, we need to show that $g(c_{f(x)})$ is minimal. Suppose there was some other $c' \subset g(c_{f(x)})$, since the g is surjective, there exists some other $c'_{f(x)} \subset c_{f(x)}$ which is smaller. So by assumption, this must have two distinct solutions. But then mapping them again gives us two distinct solutions for c'.

Therefore, $g(c_{f(x)})$ is a minimal clue set with a unique superset solution for x. □

Sometimes, the requirement that g be a bijection on solutions is too restrictive, so we can give a more specialized sort of reduction that reduces the clue set.

Special Subset Clue Reduction Suppose we have a problem A, with instance set I_A, solution set S_A, we let the set of clues be C_A. Note that $S_A \subset C_A$. We also have a problem B with instance I_B, solution set S_B and clue set C_B.

We will first need a function $f : I_A \to I_B$ such that $x \in I_A$ has a solution if and only if $f(x) \in I_B$ has a solution. Note this is the normal concept of reduction.

We additionally specify $C'_B \subset C_B$ and an algorithm M that maps any minimal clue set in C_B into a minimal clue set in C'_B of the same size which runs in polynomial time.

We will now let $g : C'_B \to C_A$ which satisfies the following properties:

1. g is surjective.
2. g is a bijection on solutions, and S_x is a solution for x, if and only if $g^{-1}(S_x)$ is a solution to $f(x)$.
3. $c \subset c' \iff g(c) \subset g(c')$.

Theorem 2. *If we can find f, algorithm M, and function g satisfying the above properties for A and B, then if $\mathsf{FCP}A$ is FCP-hard, then $\mathsf{FCP}B$ is FCP-hard.*

Proof. Suppose $x \in I_a$. We compute $y = f(x)$. If x has no solution, we can identify it by noting that y will have no solution.

Suppose c_y is a fewest clues solution for y. Then $c'_y = M(c_y)$ will be another fewest clues solution to y. So for now on, suppose c_y is a fewest clues solution for y and $c_y \in C'_B$.

Then let $c_x = g(c_y)$.

Claim. c_x is a fewest clues solution to x.

We know $c_y \subset S_y$ since c_y is a clue. Which means that $g(c_y) \subset g(S_y)$. Now we know that $c_x \subset g(S_y)$ and since S_y is a clue of maximal size, then $c_x \subset S_x$, and by the property 2 of g, S_x is a solution. So c_x is a clue.

Suppose c_x does not have a unique solution superset. So $c_x \subset S_x^{(1)}$ and $c_x \subset S_x^{(2)}$. Then since g is bijective on solutions, let $S_y^{(1)} = g^{-1}(S_x^{(1)})$ and $S_y^{(2)} = g^{-1}(S_x^{(2)})$. So $S_y^{(1)}$ and $S_y^{(2)}$ are solutions to y, and so by property 3 of g, $c_y \subset S_y^{(1)}$ and $c_y \subset S_y^{(2)}$, which means c_y does not have a unique superset solution. A contradiction.

Suppose c_x is not minimal. Then let c'_x be the smallest subset of c_x with S_x be the unique superset solution for c'_x. Then the smallest of $g^{-1}(c'_x) = c'_y$ is a proper subset of c_y, and if there exists two superset solutions for c'_y, $S_y^{(1)}$ and $S_y^{(2)}$, then c'_x does not have unique superset solutions.

This means that c_x is the minimal set. □