Network Dynamics II

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1 Traffic Tolls in Los Angeles

1.1

The shortest path between node 1 and 17 is [1, 2, 3, 9, 13, 17] with a travel time of 0.6490. The path has been plotted in Figure 1.

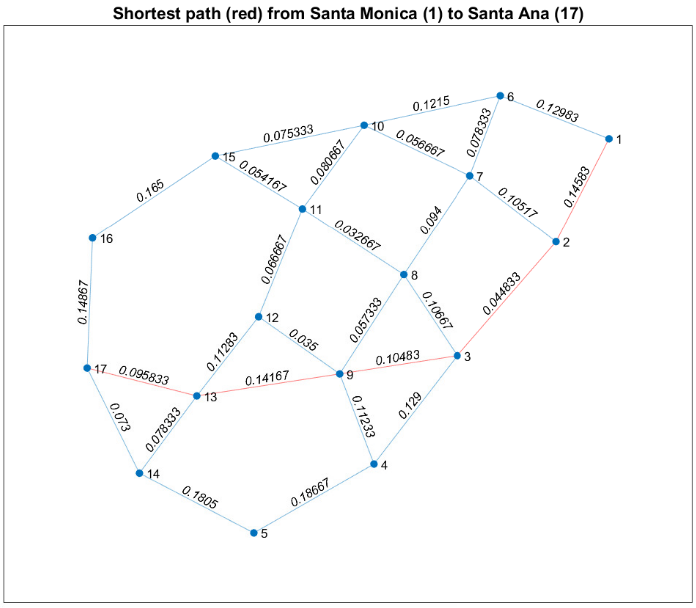

Figure 1: Graph of road traffic in Los Angeles. Shortest path from Santa Monica (node 1) to Santa Ana (node 17) has been plotted in red. Edges are labeled with the travel time for each edge.

1.2

The maximum flow between node 1 and 17 is 22448.

1.3

The external in/out-flow vector (at each node) ν can simply be calculated with the equation

$$Bf = \nu, \tag{1}$$

where B is the node-link incidence matrix and f is the flow vector for each edge.

For the given flow vector in assignment,

$$\nu = \begin{bmatrix} 16806 \\ 8570 \\ 19448 \\ 4957 \\ -746 \\ 4768 \\ 413 \\ -2 \\ -5671 \\ 1169 \\ -5 \\ -7131 \\ -380 \\ -7412 \\ -7810 \\ -3430 \\ -23544 \end{bmatrix}.$$

1.4

Finding the social optimum flow f^* for a flow-network means I need to solve the optimization problem

$$f^* = \underset{f \geq 0}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} f_e d_e(f_e), \tag{2}$$
$$Bf = \nu,$$

where the delay function is

$$d_e(f_e) = \frac{l_e}{1 - f_e/C_e}, 0 \leq f_e < C_e, \tag{3}$$

where l_e and C_e is the travel time and capacity for each edge, respectively. For $f_e \geq C_e$ the value of $d_e(f_e)$ is considered as $+\infty$.

It's easy to check that this is a convex problem $(f_e d_e(f_e))'' \geq 0$ and so I can solve this problem using cvx in MATLAB. I get the answer

$$f^* = \begin{bmatrix} 6642 \\ 6059 \\ 3132 \\ 3132 \\ 10164 \\ 4638 \\ 3006 \\ 2543 \\ 3132 \\ 583 \\ 0 \\ 2927 \\ 0 \\ 3132 \\ 5525 \\ 2854 \\ 4886 \\ 2215 \\ 464 \\ 2338 \\ 3318 \\ 5656 \\ 2373 \\ 0 \\ 6414 \\ 5505 \\ 4886 \\ 4886 \end{bmatrix},$$

with a total (optimal) cost $C_{\text{opt}} = 25943.6$.

1.5

Finding the Wardrop equilibrium $f^{(0)}$ for a flow-network means I need to solve the optimization problem

$$f^{(0)} = \underset{f \geq 0}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(s) ds, \quad (4)$$
$$Bf = \nu.$$

One can interpret this as every person in a road-network acting in their own self-interest to find their own fastest travel time. The Wardrop equilibrium is found when each player can no longer improve their own situation, and is thus a form of a Nash Equilibrium. This usually means that the cost of the whole network suffers and is measured in Price of Anarchy (PoA) defined as

$$\text{PoA}^{(0)} = \frac{\text{Total cost at } f^0}{\text{Total cost at } f^*}. \quad (5)$$

Note that $\text{PoA}^{(0)} \geq 1$ with equality when the Wardrop equlibrium coincides with the social optimum.

The cost-function

$$c_e(f_e) = \int_0^{f_e} d_e(s) ds = \int_0^{f_e} \frac{l_e}{1 - f_e/C_e} = -C_e l_e \ln(1 - \frac{f_e}{C_e}), \quad (6)$$

is convex since $c_e(f_e)'' \geq 0$ and one can also see that $\lim_{f_e \to C_e^+} c_e(f_e) = +\infty$.

I solve this problem and get

$$
f^{(0)} = \begin{bmatrix} 6716 \\ 6716 \\ 2367 \\ 2367 \\ 10090 \\ 4645 \\ 2804 \\ 2284 \\ 3418 \\ 0 \\ 177 \\ 4171 \\ 0 \\ 2367 \\ 5445 \\ 2353 \\ 4933 \\ 1842 \\ 697 \\ 3036 \\ 3050 \\ 6087 \\ 2587 \\ 0 \\ 6919 \\ 4954 \\ 4933 \\ 4933 \end{bmatrix},
$$

with a total cost $C_{\text{Wardrop}} = 26293$ and $\text{PoA}^{(0)} = 1.0135$.

1.6

I introduce tolls and calculate the new Wardrop equilibrium $f^{(\omega)}$. This means essentially that we add punishment to each actor so that we can force the flow to reach a social optimum. The problem can be formulated as

$$f^{(\omega)} = \underset{f \geq 0}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(s)ds + \omega_e f_e, \tag{7}$$
$$Bf = \nu.$$

where $\omega_e = f_e^* d_e'(f_e^*)$ where f_e^* is flow at social optimum and

$$d_e'(f_e) = \frac{l_e}{(1 - f_e/C_e)^2}. \tag{8}$$

Now technically the linear function is both convex and concave, but since $f_e \geq = 0$ it will work out in the end.

Solving this problem yields

$$f^{(\omega)} = \begin{bmatrix} 6642 \\ 6059 \\ 3132 \\ 3132 \\ 10164 \\ 4638 \\ 3006 \\ 2543 \\ 3132 \\ 583 \\ 0 \\ 2927 \\ 0 \\ 3132 \\ 5525 \\ 2854 \\ 4886 \\ 2215 \\ 464 \\ 2338 \\ 3318 \\ 5656 \\ 2373 \\ 0 \\ 6414 \\ 5505 \\ 4886 \\ 4886 \end{bmatrix},$$

with a total cost $C_{\text{Tolls}} = 25943.6$ and $\text{PoA}^{(0)} = 1.0000$. We have succesfully reached a social optimum using tolls!

1.7

Now we will compute the social optimum f^* and Wardrop equilibrium with tolls $f^{(\omega^*)}$ with a new cost function $c_e(f_e) = f_e(d_e(f_e) - l_e)$. The problems become

$$\begin{aligned} f^* = &\operatorname*{argmin}_{f\geq 0} \sum_{e\in\mathcal{E}} f_e(d_e(f_e) - l_e), \\ &Bf = \nu, \end{aligned} \tag{9}$$

and

$$\begin{aligned} f^{(\omega^*)} = &\operatorname*{argmin}_{f\geq 0} \sum_{e\in\mathcal{E}} \int_0^{f_e} d_e(s)ds + \omega_e f_e - l_e f_e, \\ &Bf = \nu. \end{aligned} \tag{10}$$

Again $\omega_e = f_e^* d_e'(f_e^*)$ where f_e^* is the flow for the new social optimum. Now $-l_e f_e$ has minimum for $f_e \to +\infty$ but the other terms involving $d_e(f_e)$ and integrals of it will go towards ∞ as well before that so it cancels out.

I do the calculations and get indeed that $C_{\text{Opt}} = C_{\text{Tolls}} = 15095.5$ and $\text{PoA}^{(\omega^*)} = 1.0000$ so we have successfully forced a social optimum using tolls.