Network Dynamics III

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1 Single-particle random walk

In continuous time the movement of a particle in a graph is defined through the transition state rate matrix Λ. Shortly summarized the transition rate matrix corresponds to probabilities of a random particle moving to different node, and the weight $w_i = \sum_j \Lambda_{ij}$ corresponds to the rate at which a particle i moves to a different node. Rate here means refer to that the time a particle stays in a node is w_i-rate exponentially distributed. An important aspect of exponential distributions is that they are memory less: $\mathbb{P}(X \geq t+s | X \geq t) = \mathbb{P}(X \geq s)$, i.e. the probability of switching state is independent of the total time already passed.

These properties means that the process can be modeled as a Markov chain $X(t)$, i.e. the probability of the current state depends only on the previous.

Now consider

$$\omega_* = \max_i \omega_i, \tag{1}$$

the Markov chain can be interpreted as the entire graph having a common rate-w_* Poisson Clock with transition probabilities:

$$\bar{P} = \frac{\Lambda_{ij}}{w_*}, \quad i \neq j, \quad \bar{P} = 1 - \sum_{j \neq i} \bar{P}_{ij}. \tag{2}$$

In technical terms the Markov chain has an associated jump-chain U, but I find the intuition to be simpler: Imagine a large clock striking for everyone which should tell each particle when to move. However the particle with the highest rate w_* will of course switch states more often than the other particles, so in order to model this we must add the transition probability of remaining in that state for the other particles, which is just 1 - (the other transition probabilities).

1.1

The return time for a node i in a graph $\mathcal{G}$ is defined as: $T_i^+ = \inf\{t \geq 0 : X(t) = i$ and $X(s) = i$ for some $s \in (0; t)\}$, where $X(t)$ is the position of the particle at time t. The interpretation is simple: T_i^+ is the time it takes for a particle X to leave node i and then return back to it. With 10^5 iterations, the average return time for node a was 6.7380.

1.2

If Λ is irreducible ($\mathcal{G}_\Lambda$ is connected) then the Markov X chain will always converge to a probability distribution $\bar{\pi}$ and the expected return time is given by

$$\mathbb{E}_i[T_i^+] = \frac{1}{w_i \bar{\pi}}. \tag{3}$$

The expected return time for node a was $\mathbb{E}_a[T_a^+] = 6.7500$, which is reasonable close to the simulated value.

1.3

The expected time it takes for a particle beginning in node i to reach a state $\mathcal{S}$ is given by the hitting time $\mathbb{E}_i[T_\mathcal{S}]$. Setting $S = \{d\}$, the simulated average hitting time for node o was $\mathbb{E}_o[T_\mathcal{S}] = 8.7826$.

1.4

If $\mathcal{S}$ is globally reachable in $\mathcal{G}_\Lambda$ then the expected hitting time is given by the solution to the system of equations

$$\begin{aligned} \mathbb{E}_i[T_\mathcal{S}] &= 0, && i \in \mathcal{S}, \\ \mathbb{E}_i[T_\mathcal{S}] &= \frac{1}{w_i} + \sum_j P_{ij}\mathbb{E}_j[T_\mathcal{S}], && i \notin \mathcal{S}, \end{aligned} \tag{4}$$

where $P = \mathrm{diag}(\omega)^{-1}\Lambda$. If one removes all the rows and columns corresponding to $\mathcal{S}$ from P then this can be represented as a matrix equation $x = (w)^{-1} + Px =>$ $x = (I - P)^{-1}(w)^{-1}$. We know that $(I - P)$ is invertible from the spectral radius discussed in the beginning of the course.

For node o the expected hitting time for $S = \{d\}$ is $\mathbb{E}_o[T_\mathcal{S}] = 8.7857$ which is reasonable close to the simulated average hitting time.

2 Graph coloring and network games

The graph $\mathcal{G}$ with weight matrix W is undirected, and the game is a network anti-coordination game, which we know is a symmetric two-player potential game with potential function $\phi(a,b) = \phi(b,a)$. Thus the potential for this network game is given by

$$\Phi(\mathbf{x}) = \frac{1}{2}\sum_{i,j} W_{ij}\phi(x_i, x_j). \tag{5}$$

We can model this game with noisy best response dynamics. Every player has an independent rate-1 Poisson clock and after every tick particle i in the Markov chain $X(t)$ is updated to a different color with probability

$$\mathbb{P}(X_i(t) = a | X(t^-)) \propto \exp(\eta u_i(a, X_{-i}(t^-)), \tag{6}$$

where a is the possible color and η is the inverse noise. A lower η will introduce more randomness in the model. For this problem the utility u_i is given by

$$u_i(a, X_{-i}(t^-)) = -\sum_j W_{ij} c(a, X_j(t)), \tag{7}$$

where c is a cost function. This makes sense since there are no self-loops and cost = -utility.

2.1

For the line graph and cost function given the potential function for $\eta = t/100$ has been plotted in Figure 1. As you can see it is quite random in the beginning but is eventually minimized and as expected $X(500) = [1,2,1,2,1,2,1,2,1,2]$ which is a Nash Equilibrium for this network game.

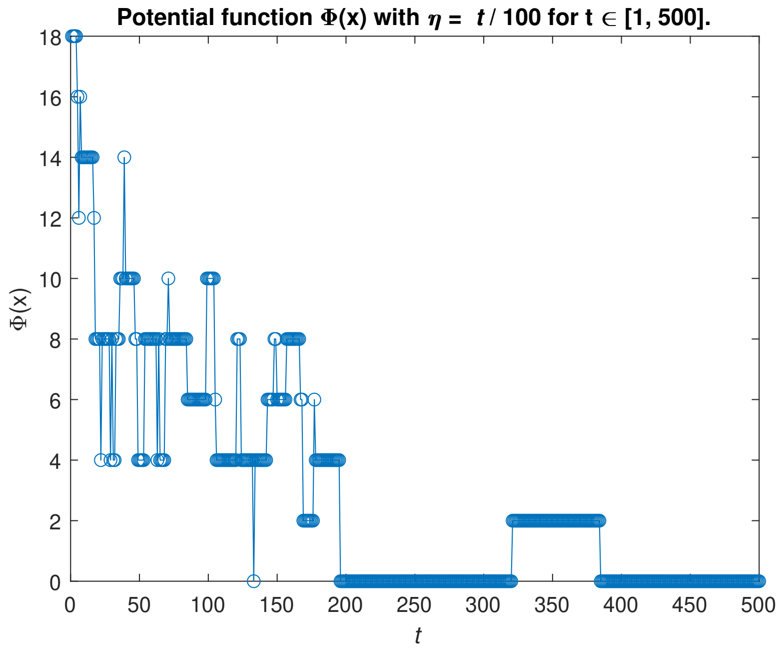

Figure 1: Plot of potential function $\Phi(\mathbf{x})$ for noisy best response dynamics for anti-coordination game for line graph with $\eta = t/100$ for $t \in [1, 500]$.

2.2

The figures for the potential functions for different η have been plotted in Figures 2-4 below but it was hard to see the best response from them. I looked at the node-graph instead and found that there was a trade-off for noise: More noise makes it more random and thus less likely to find best response, but without randomness you're not exploring the action space enough and also don't get the best reponse. I settled for the baseline $\eta = t/100$ which has been plotted in Figure 5 below.

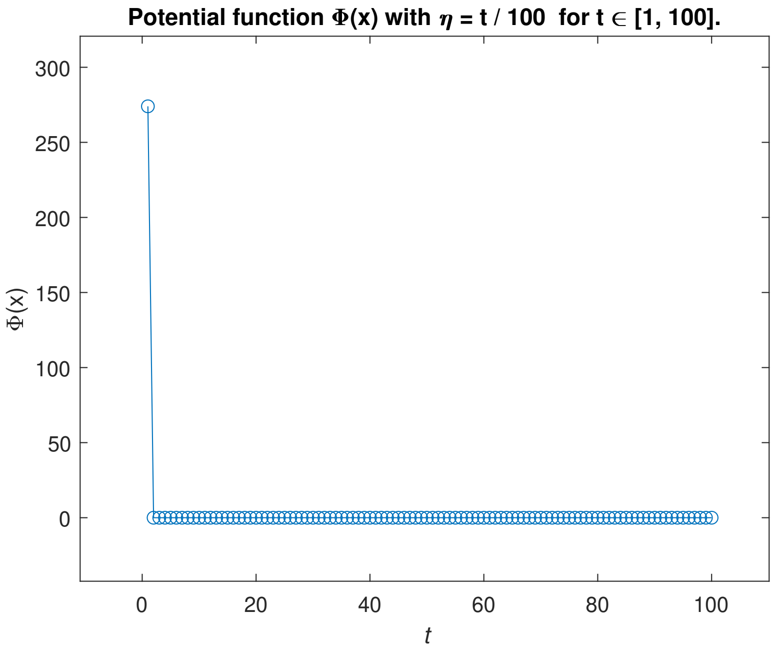

Figure 2: Plot of potential function with $\eta = t/100$.

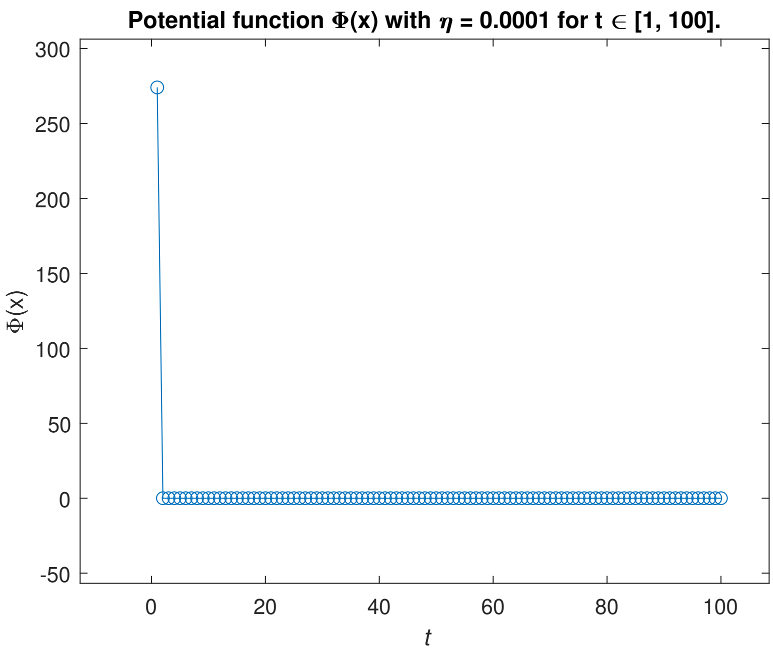

Figure 3: Plot of potential function with $\eta = 0.0001$.

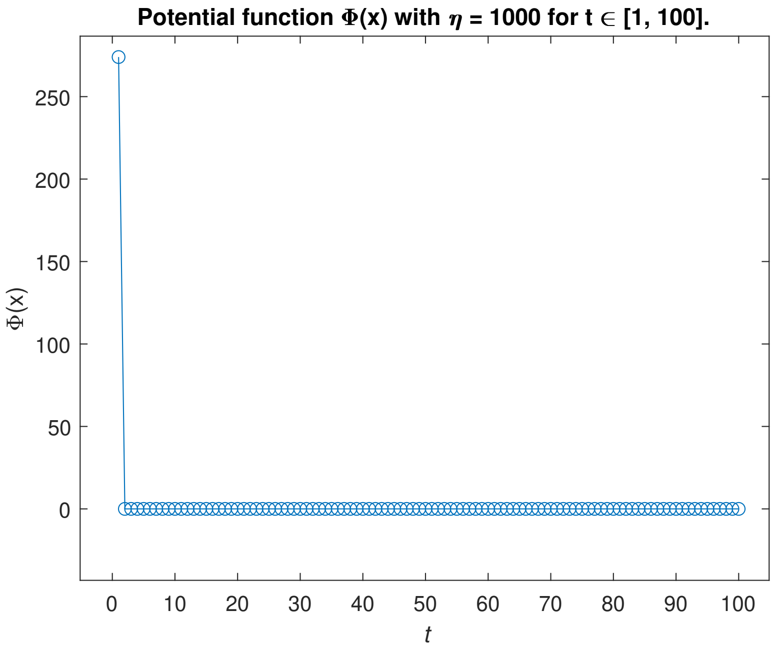

Figure 4: Plot of potential function with $\eta = 1000$.

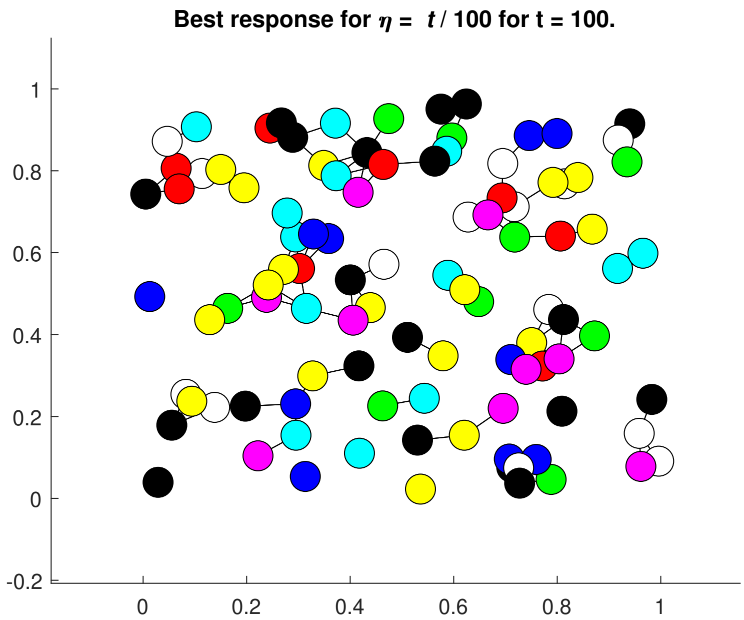

Figure 5: Plot of graph with $\eta = t/100$.