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· Momentum & Force;

$\Rightarrow \bar{F}_{net}(i) = \frac{d}{dt}\bar{p}(i)$ } Newton's 2^{nd} Law.

$\Rightarrow \bar{p}_i = m_i \times \frac{d}{dt}\bar{r}_i(t)$

——x——

Deterministic Chaos;

ex; $y_{n+1} = \alpha y_n(1-y_n)$ → This is deterministic since previous state is a dependency for current state

$y_n \in [0,1]$

$\alpha \in [0,4], \alpha > 3.8$

$y_1 = 3.8 \times y_0 \times (1-y_0)$

⇒ Randomness that occurs even in well defined deterministic systems like above.

——x——

* Scientific Model;

- We need to frame equations for each of our variables.

ex; $\frac{dx}{dt} = \alpha x - \beta xy - \varepsilon x$

$\frac{dy}{dt} = \gamma xy - Ky$

} where x, y are variables and $\alpha, \beta, \gamma, K, \varepsilon$ are parameters (constants)

Not a unique matrix

$\Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha-\varepsilon & -\beta x \\ \gamma y & -K \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\boxed{\frac{d}{dt}(\bar{r}) = \bar{M}(\bar{r}) \cdot \bar{r}}$

where $\bar{r}$ is the vector $\begin{pmatrix} x \\ y \end{pmatrix}$

We need to get this form of any model

This can be any order derivative

⇒ Steady state of the system is when

$$\frac{d}{dt}\bar{r} = 0$$ //

∴ $$\alpha x - \beta xy - \varepsilon x = 0 = \frac{dx}{dt}$$

$$\gamma xy - Ky = 0 = \frac{dy}{dt}$$

$$\Rightarrow \gamma x\cancel{y} = K\cancel{y}$$

$$x = K/\gamma$$

$$\Rightarrow \frac{\alpha K}{\cancel{\gamma}} - \frac{\beta K}{\cancel{\gamma}} y - \frac{\varepsilon K}{\cancel{\gamma}} = 0$$

$$\therefore y = \frac{\alpha K - \varepsilon K}{\cancel{\gamma}\beta K} = \frac{\alpha - \varepsilon}{\beta}$$ //

(For steady state)

$$\cancel{\bar{r} = \left(0, \frac{\alpha-\varepsilon}{\beta}\right)} \text{ or } \cancel{\bar{r} = (K/\gamma, 0)}$$

$$\bar{r} = (0,0) \text{ or } \bar{r} = \left(K/\gamma, \frac{\alpha-\varepsilon}{\beta}\right)$$

are the steady states.

(We can replace $\alpha - \varepsilon$ with α (New α) to make it simpler)

$$\Rightarrow \bar{r} = \cancel{\left(0, \frac{\alpha}{\beta}\right)}, (0,0), \cancel{(K/\gamma, 0)} \text{ and } (K/\gamma, \alpha/\beta)$$

<u>Note</u>; Stable fixed points are where the system returns here when disturbed. Unstable fixed points are the opposite.

(Both of these are steady states)

⇒ Suppose we generalized our case,

$$\frac{d\bar{r}}{dt} = \bar{f}(\bar{r})$$

$$\frac{dx}{dt} = f_1(x,y)$$

$$\frac{dy}{dt} = f_2(x,y)$$

⇒ At steady state,

$$\frac{d\bar{r}}{dt} = 0 \Rightarrow \underbrace{\bar{f}(\bar{r})}_{\text{Use Taylor expansion}} = 0$$

$$\Rightarrow \bar{f}(\bar{r}) = \bar{f}(\bar{r}_0) + (\bar{r} - \bar{r}_0)\left.\frac{\partial \bar{f}}{\partial \bar{r}}\right|_{\bar{r}=\bar{r}_0} + \ldots$$

$$|\bar{r} - \bar{r}_0| \to 0$$

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*Note: ⇒

$$\frac{\partial f_1}{\partial \bar{r}} = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y}\right) \Big\} \text{ Gradient of function } f_1 \ (\bar{\nabla} f_1)$$

$\bar{r} = (x,y)$

Note; Parabolic analysis and tensors.

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*Newtons Laws;

1) Inertia (Inertial & Non-Inertial frames)
2) $\frac{d}{dt}\bar{p}_i = \bar{F}_i$, $\bar{p}_i = m_i \bar{v}_i = m_i \frac{d}{dt}\bar{r}_i$
3) Action-Reaction (not true for all situations)

** Note; $\frac{df(t)}{dt} = cf(t)$, its solutions depend on c

$\Rightarrow f(t) = \alpha e^{ct}$ (even for complex)

(They are all oscillating function)

Continuing, Scientific Model;

$\Rightarrow f_1 = \alpha x - \beta xy$ (Ignore Σx) $\quad$ ($f_1 \to dx/dt$)

$f_2 = \delta xy - \gamma y \quad , \bar{r} = (x, y)^T$ $\quad$ ($f_2 \to dy/dt$)

$$\frac{d\bar{r}}{dt} = F(\bar{r}) = \begin{pmatrix} f_1(\bar{r}) \\ \vdots \\ f_n(\bar{r}) \end{pmatrix}$$

From Taylor expansion, (1^{st} order terms)

$$\Rightarrow f_1(x, y) = f_1(x_0, y_0) + \frac{\partial f_1}{\partial x}\Big|_{x_0}(x - x_0) + \frac{\partial f_1}{\partial y}\Big|_{y_0}(y - y_0)$$

$\Rightarrow$ Now here $n = 2$

$$\therefore \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1(\bar{r}_0) \\ f_2(\bar{r}_0) \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x}\Big|_0 & \frac{\partial f_1}{\partial y}\Big|_0 \\ \frac{\partial f_2}{\partial x}\Big|_0 & \frac{\partial f_2}{\partial y}\Big|_0 \end{pmatrix}}_{\text{Jacobian matrix}} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

*
$$\Rightarrow \boxed{\frac{d}{dt}(\bar{r} - \bar{r}_0) = F(\bar{r}_0) + J(\bar{r}_0) \cdot (\bar{r} - \bar{r}_0)}$$
*

$$\frac{d\bar{z}(t)}{dt} = J(\bar{z} = 0) \cdot \bar{z}(t)$$

Given, Lets say

$\Rightarrow \bar{z} = \bar{r} - \bar{r}_0$

$|\bar{z}| \sim 0$

$$\frac{d\bar{z}(t)}{dt} = J(\bar{z}=0)\cdot\bar{z}(t)$$

$\bar{J}\cdot\bar{V} = \lambda\bar{V}$ → For each eigen value λ_i, we have an eigen vector $\bar{V}_i$

(Assign 2 Q 4)

⇒ Now let us assume eigen vectors form a basis & are linearly independent

$$\Rightarrow \bar{z}(t) = \sum_{i=1} c_i(t) \times \bar{V}_i$$

we know solution is of form $e^{\lambda t}$

prove this is eigen vector → $\bar{V}_i$, eigen value c_i

$$\therefore \text{LHS} \Rightarrow \sum_{i=1} \bar{V}_i \cdot \frac{d}{dt} c_i = \bar{J}(r_0)\sum_{i=1} c_i \bar{V}_i$$

$$= \sum_{i=1} c_i\, \bar{J}\cdot\bar{V}_i$$

$$= \sum_{i=1} c_i \lambda_i \bar{V}_i$$

$$\Rightarrow \frac{dc_i}{dt} = c_i\lambda_i$$

$\therefore c_i = e^{\lambda_i t}$ } ∴ Solution exists for C.

$\cancel{\Rightarrow \sum_{i=1} c_i\lambda_i V_i = \sum_{i=1} J\cdot(c_i \times \bar{V}_i)}$

$\cancel{\Rightarrow \sum_{i=1} c_i(\bar{V}_i\lambda_i - J\cdot\bar{V}_i) = 0}$

$\cancel{\Rightarrow \sum_{i=1} (\bar{V}_i\lambda_i - J\cdot\bar{V}_i) = 0 \ \forall i \geq 1}$

~~∴ λ_i is the eigen values & $\bar{V}_i$ are eigen vectors~~

∴ We can show $\bar{z}(t)$ as a linear combination

∴ A linear combination of eigen vectors is the answer to our $\bar{z}(t)$

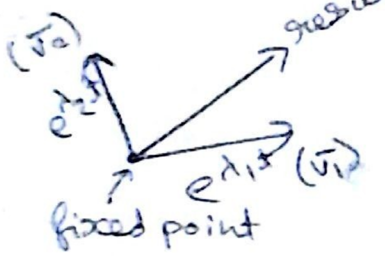
Characteristics of System (Analysis); $\left(\frac{d\bar{z}(t)}{dt} = \bar{M}\cdot \right)$ Given

$\bar{z}(t) = c_1(t)\bar{v}_1 + c_2(t)\bar{v}_2$ (Assume)

$\underbrace{\frac{dc_1}{dt}(t) = \lambda_1 c_1(t)}_{\text{We are given earlier.}} \rightarrow c_1(t) = e^{\lambda_1 t}$ → eigen values

$\frac{dc_2}{dt}(t) = \lambda_2 c_2(t) \rightarrow c_2(t) = e^{\lambda_2 t}$ → eigen value

→ ∴ (Assuming eigen vectors are orthogonal) → they form a basis



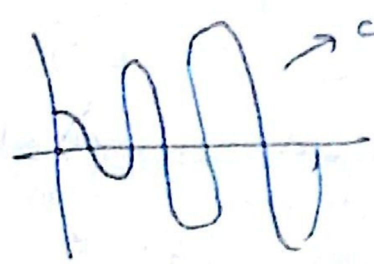
⇒ So our resultant vector $\bar{z}(t)$ depends on components

→ If $\lambda_1, \lambda_2 > 0$, then resultant goes away from fixed point.

⇒ If $\lambda_1, \lambda_2 < 0$, then goes towards fixed point.

(If λ_1, λ_2 are complex numbers)

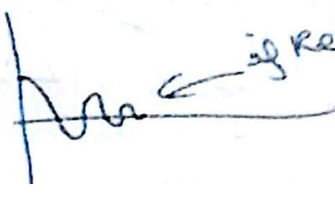
$e^{\lambda t} \Rightarrow$



→ $\cos\theta \times e^{Re(\lambda)\times t}$, $Im(\lambda)\times t$

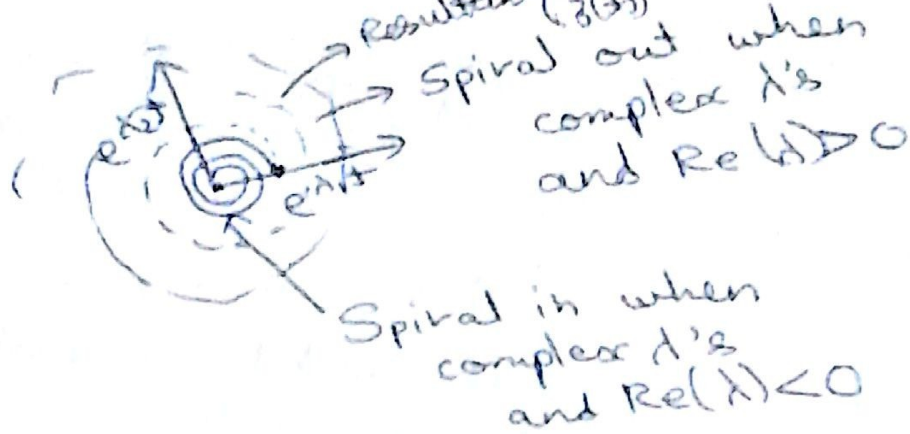
← if $Re(\lambda) > 0$ (Ignoring Imaginary part of $e^{i\,Im(\lambda)t}$) $(i\sin\theta)$

$e^{\lambda t} \Rightarrow$



← if $Re(\lambda) < 0$

⇒ So



— x —

Continuing our scientific models

$$\frac{dx}{dt} = \alpha x - \beta xy$$ (Assuming our first fixed point = (0,0))

$$\frac{dy}{dt} = \delta xy - \gamma y$$

Stable point 1:

1) ⇒ $J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$ → Jacobian matrix

$$z = \bar{x} - \bar{x}_0 = \bar{x}$$

$$\therefore \frac{d\bar{x}}{dt} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}(\bar{x})$$

$\left.\begin{array}{l}\lambda_1 = \alpha \\ \lambda_2 = -\gamma\end{array}\right\}$ Eigen values

Eigen vectors $\left\{\begin{array}{l}\bar{v}_1 = (1,0) \\ \bar{v}_2 = (0,1)\end{array}\right.$

So $\frac{d\bar{x}}{dt}$ ⇒ Our 2 eigen values are +ve & -ve.

So goes away on one axis & comes close on other axis. ($\bar{x}(t)$)

Stable Point 2:

⇒ $J(\gamma/\delta, \alpha/\beta) = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix}$

$\lambda^2 = -\alpha\gamma$, $\lambda_{1,2} = \pm i\sqrt{\alpha\gamma}$ ← eigen values

So here $\frac{d\bar{r}}{dt} \Rightarrow$ Both our eigen values are purely imaginary.

∴ We will move in a circle (Sinusoidal on both ⊥'lar eigen vectors)

$e^{\lambda t}$

↓

λ = Purely imaginary

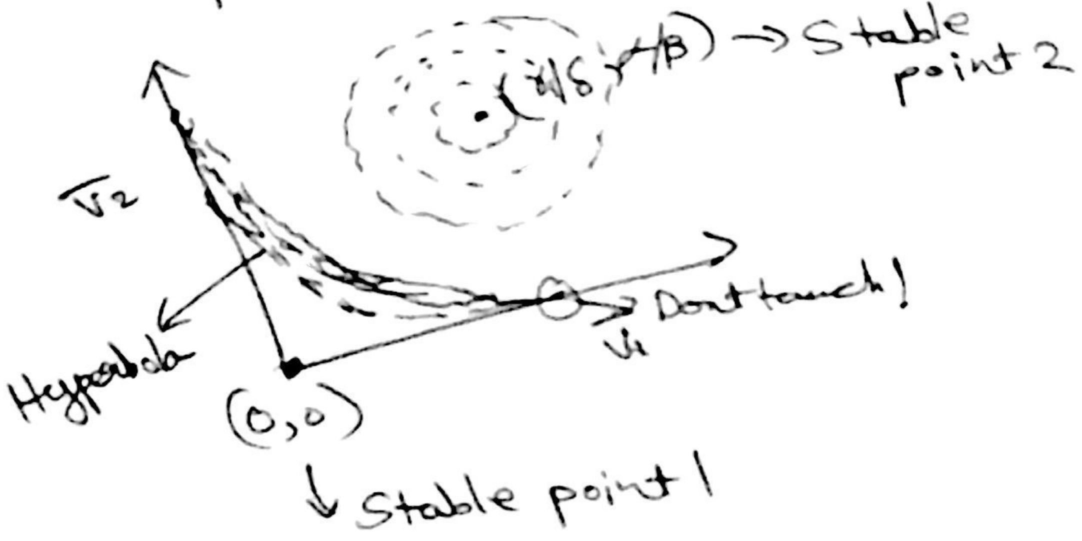
$e^{ibt} \Rightarrow \underbrace{\cos bt}_{\text{cos wave}} + \underbrace{i \sin bt}_{\text{Ignore}}$

∴ Cos on $\bar{V}_1$ & cos on $\bar{V}_2$

$(\bar{V}_1 \perp \bar{V}_2)$

∴ We go in a circle

∴ If we are near stable point 1, there is one type of behaviour, and near stable point 2 its a different behav



—— x ——

*Classical Mechanics;

- $\frac{d}{dt}\overline{p_i} = \overline{F_i}$ → Force → ①

  $\overline{p_i}$ ↓ Momentum

$* \Rightarrow \overline{F_i}\cdot\overline{s} = 0$ → ②

$\Rightarrow \frac{d}{dt}(\overline{p_i}\cdot\overline{s}) = 0$ } Putting ② in ①

$\Rightarrow \overline{p_i}\cdot\overline{s}$ = Constant

—— x ——

Work

- $d\overline{W} = \overline{F}_{ext}\cdot d\overline{r}$

—— x ——

Conservation of
4) Angular Momentum; (Total for a system is sum of individuals)

- $\overline{L} = \overline{r}\times\overline{p}$ → Momentum

($\overline{L}$ is conserved if $\overline{\tau} = 0$)
or $\overline{N}$ (Torque)

$$\frac{d\overline{L}}{dt} = \frac{d\overline{r}}{dt}\times\overline{p} + \overline{r}\times\frac{d\overline{p}}{dt}$$

$$= \overline{v}\times\overline{p} + \overline{r}\times\overline{F}$$

$$= \underbrace{\overline{v}\times\overline{p}}_{\text{Collinear vectors}} + \overline{N} \text{ } \}\text{Torque}$$

$$\frac{d\overline{L}}{dt} = 0 + \overline{N}$$

, ∴ Only $\overline{N}$ effects angular momentum.

—— x ——

2) Law of Conservation of energy;

$$dW = \bar{F}(\bar{r}) \cdot \underbrace{d\bar{r}}_{\text{Small distance}} \rightarrow \bar{r}_2 - \bar{r}_1 \text{ when } |\bar{r}_2 - \bar{r}_1| \sim 0$$

a) $W = \int_{\bar{r}_i}^{\bar{r}_f} \bar{F}(\bar{r}) \cdot d\bar{r}$ } Work Done on System (Spacial) → Leads to potential energy

b) $W = \int_{t_i}^{t_f} \bar{F}(t) \cdot \underbrace{\bar{v}(t) \cdot dt}_{d\bar{r}}$ } Temporal } Leads to Kinetic energy

i) K.E: (Conservative forces)

$$\bar{F}(t) \cdot \bar{v}(t) \cdot dt = \left| m \frac{d\bar{v}}{dt} \right| \cdot \bar{v} \cdot dt = m\bar{v} \cdot d\bar{v}$$

$$= d\left(\frac{1}{2} m \bar{v} \cdot \bar{v}\right)$$

derivative of dot product

$$\therefore dW = d\left(\frac{1}{2} m\bar{v} \cdot \bar{v}\right) = d(KE)$$

$$d(\bar{v} \cdot \bar{v}) = \bar{v} d\bar{v} + d\bar{v} \cdot \bar{v} = 2\bar{v} \cdot d\bar{v}$$

$$\therefore W = KE(t_2) - KE(t_1)$$

$$W = KE_2 - KE_1$$

ii) P.E;

For conservative forces

$$\bar{F}(\bar{r}) = -\bar{\nabla} u(\bar{r}), \quad u(\bar{r}) = mgz \rightarrow z \text{ axis}$$

We just need some $u(\bar{r})$

Gradient

$$\bar{\nabla}(z) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(z) = (0,0,1)$$

$$\bar{F} = -mg(0,0,1)$$

$$\therefore dW = \bar{F}(\bar{r}) \cdot d\bar{r} = -\bar{\nabla} u(\bar{r}) \cdot d\bar{r}$$

$$\therefore W = -(u_2 - u_1)$$

$\Rightarrow$ Now, $W = K_2 - K_1$

Since we are dealing with conservative forces,

$W = -(U_2 - U_1)$

$\therefore K_2 - K_1 = -(U_2 - U_1)$

$\Rightarrow K_2 + U_2 = U_1 + K_1$

$\therefore$ Conservation //

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3) <u>Frictional Forces;</u>

- $\bar{F} \propto -\bar{v}$ //

$F = -c\bar{v} \neq -\bar{\nabla} u(\bar{r})$

We cannot find any function $u(\bar{r})$ where $-\bar{\nabla} u(\bar{r}) = -c\bar{v}$

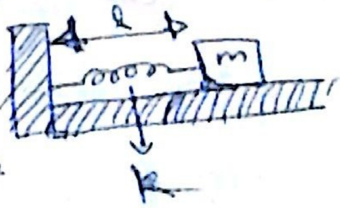
$\therefore$ Its a non-conservative force. (Energy is not conserved)

Oscillators ; (Phase Diagram)
(1D Oscillator) (Harmonic)

1) $F(x) = -kx$, x is the extension of spring
$x = l(t) - l_0$

$m\ddot{x} = F(x)$

$\therefore m\ddot{x} = -kx \rightarrow (1)$
Conservative



Total Energy = PE(Spring) + KE(mass)

$\frac{1}{2} m \times \left[\frac{d\,x(t)}{dt}\right]^2$

$\Rightarrow$ Now, $\frac{-dU}{dx} = -kx$, $U(x) = \frac{1}{2} kx^2 + \text{Const}$

From (i),

$\ddot{x} = -cx$, $c = k/m$

Linear, 2nd Order, Homogeneous differential eqns.

no constant terms

Note:

→ For linear, homogeneous eqns, try $x(t) = e^{\lambda t}$.

$\Rightarrow x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t}$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\therefore \lambda^2 e^{\lambda t} + c e^{\lambda t} = 0$

$e^{\lambda t}(c + \lambda^2) = 0$

$\Rightarrow \lambda^2 + c = 0$

$\therefore \lambda = \pm i\sqrt{c}$ } 2 roots.

$\therefore \cancel{x(t) = e^{\pm i\sqrt{c} \times t}}$

$\therefore x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$x(t) = \underbrace{c_1 e^{+i\sqrt{c} \times t} + c_2 e^{-i\sqrt{c} \times t}}$

This sum should be real

Considering both roots

$\therefore x(t) = c_1 \cos(\sqrt{c}t) + i c_1 \sin(\sqrt{c}t)$

$+ c_2 \cos(\sqrt{c}t) - i c_2 \sin(\sqrt{c}t)$

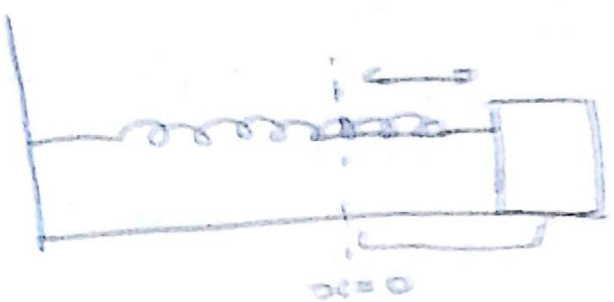
$= (c_1 + c_2)\cos(\sqrt{c}t) + i(c_1 - c_2) \times \sin(\sqrt{c}t)$

$\Rightarrow$ To make this real, $c_1 = c_2$

$\therefore x(t) = (c_1 + c_2)\cos(\sqrt{c}t)$

Now, consider

$x(t) = \alpha \cos(\sqrt{c}\,t) + i\beta \sin(\sqrt{c}\,t)$



At $t=0$,
$x(t) = x_0$
$\frac{dx}{dt} = 0$
At starting point

$\therefore x(0) = \alpha, \quad \therefore x_0 = \alpha$

$\frac{dx}{dt} = \alpha\sqrt{c}\sin(\sqrt{c}\,t) + i\cos(\sqrt{c}\,t) \times \beta\sqrt{c}$

at $t=0$,

$\frac{dx}{dt} = \alpha \times 0 + i\beta\sqrt{c} \times 1$, But this is 0

$\therefore \beta = 0$

$\therefore x(t) = \overset{\text{at } t=0}{x_0} \cos(\sqrt{c}\,t)$

$\therefore$ Total Energy $= \frac{1}{2}kx^2(t) + \frac{1}{2}m\left[\frac{dx}{dt}\right]^2$

$= \frac{1}{2}kx_0^2(\cos^2\sqrt{c}\,t + \sin^2\sqrt{c}\,t)$

$\boxed{T.E = \frac{1}{2}kx_0^2}$ → At any time is conserved

Initial at $t=0$

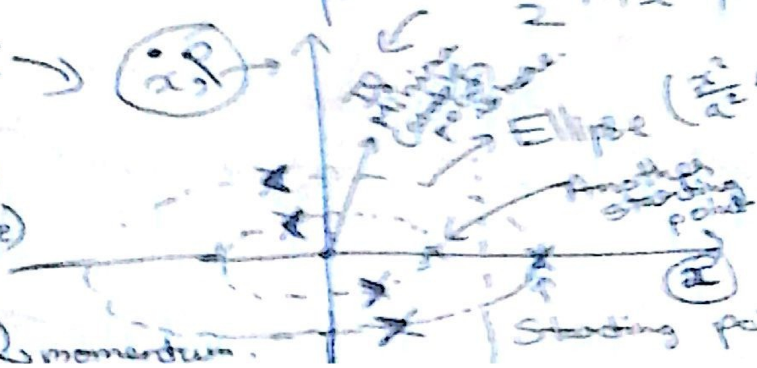
*Now, at some (t);

$x, \dot{x} \to$ velocity

$T.E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ } At some instant

Phase Plots :→ (x, P)

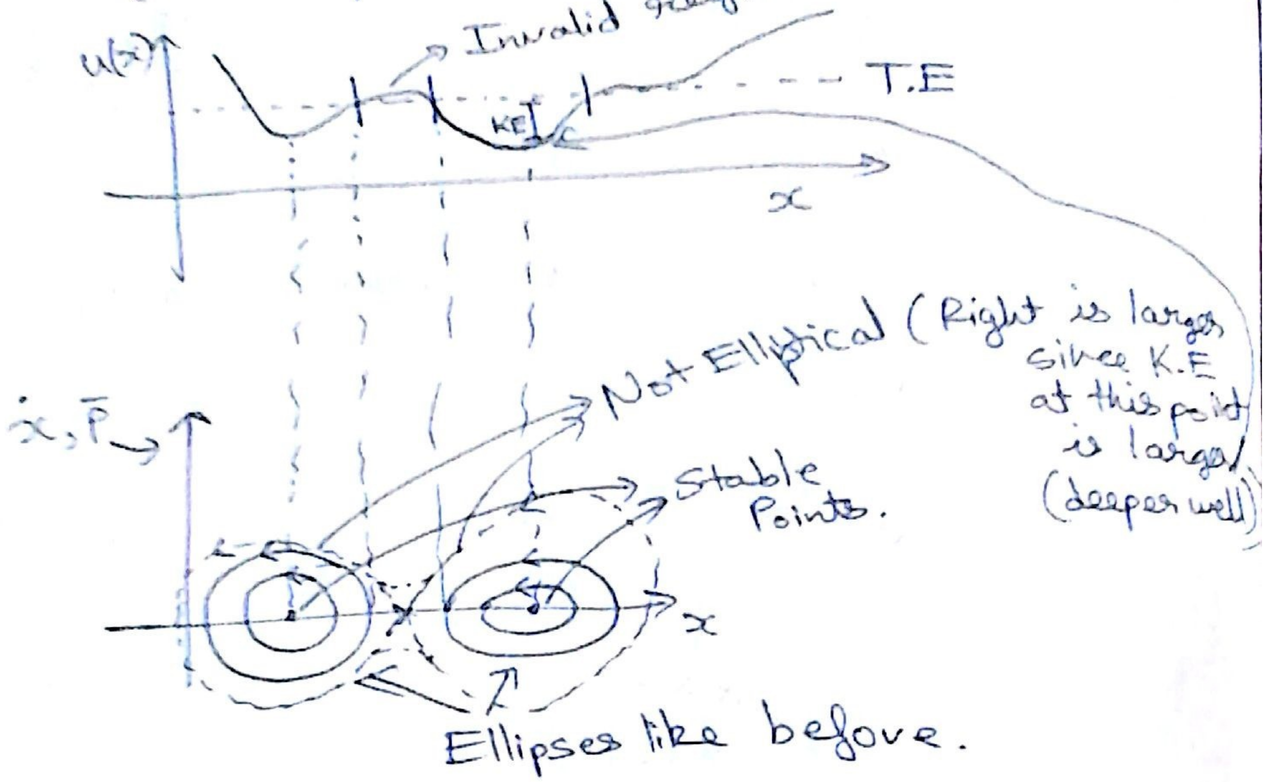
(Closed loops)
(clockwise)
(x vs $\dot{x}$ or x vs P → momentum.)



(Negative momentum/velocity when we start are extended)

⇒ If we go too far from extremum for start points, we don't get ellipses.

⇒ So if there are multiple minima (stable points), then



** 2D Oscillators;

* $\vec{F} = -k\vec{r}$ } Simple Harmonic 2D Oscillation

$\Rightarrow F_x = -kx = m\ddot{x}$

$F_y = -ky = m\ddot{y}$

Solving we get, $(x, y) = (x_0 \cos(\omega_x t - \delta_x), y_0 \cos(\omega_y t - \delta_y))$ (6 constants)

* $\Rightarrow \omega_x = \omega_y = \sqrt{\frac{k}{m}}$

$x_0 = x(t=0)$ given $\frac{dx}{dt}(t=0) = 0$

$y_0 = y(t=0)$ given $\frac{dy}{dt}(t=0) = 0$

* $\left.\begin{matrix}\delta x \\ \delta y\end{matrix}\right\} = 0$

→ These might differ in some situations

$$\Rightarrow x = x_0 \cos(\omega_x t - \delta_x)$$
$$y = y_0 \cos(\omega_y t - \delta_y)$$

$$\Rightarrow m\ddot{x} = -m x_0 \omega_x^2 \cos(\omega_x t - \delta_x) = -m\omega_x^2 x$$
$$m\ddot{y} = -m\omega_y^2 y$$

$$\Rightarrow \bar{r} = (x, y)^T \Rightarrow$$
$$\ddot{\bar{r}} = -\begin{pmatrix} \omega_x^2 & 0 \\ 0 & \omega_y^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{vmatrix} \omega_x^2 & 0 \\ 0 & \omega_y^2 \end{vmatrix} \bar{r}$$

$$\therefore \ddot{\bar{r}} = -\bar{M} \cdot \bar{r}$$

Matrix $\longrightarrow$ This matrix has to have a positive determinant (positive definite matrix)

↑ all eigen values of the matrix are positive

We have seen earlier,

$$\dot{\bar{r}} = \bar{J}_0 \cdot \bar{r}$$ (Scientific Model)

$\bar{J}_0 \rightarrow$ Jacobian matrix. (At stationary point)

—x—

Note; Positive definite matrix (eigen values of matrix are ≥ 0)

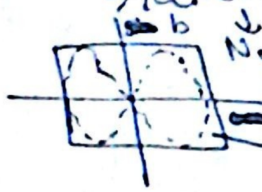
$\Rightarrow$ Determinant is +ve.

—x—

<u>Note; H.W</u>

$\Rightarrow$ Plot $\cos(\omega_x t)$ vs $\sin(\omega_y t)$ where $\frac{\omega_x}{\omega_y}$ is rational & ω_x/ω_y is irrational.

↓ visit entire square (area) (almost)



b ↓ Not visit entire square

$\rightarrow$ When $\omega_x/\omega_y = 2$

—x—

Note; $K + U = T$ } Kinetic & Potential energies

<u>Lagrangian</u> $= K - U$

$\boxed{L = K - U}$ } Lagrangian function

$= \sum_{i=1}^{?} \frac{1}{2} m \dot{x}_i^2 - U(\{x_i\})$ } Function of coords & velocities

$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{-\partial L}{\partial x_i}}$ } Lagrange stated that his function satisfies this

Need not be cartesian coordinates

$\forall i \in [1, f]$ → f degrees of freedom

ex; Testing Lagrangian;

$U(x_1, x_2, x_3)$ (Single particle moving in a potential)

a) Newtons eqn of motion;

$\Rightarrow m\ddot{x}_1 = F_1 = -\frac{\partial U}{\partial x_1}$

$m\ddot{x}_i = F_i = -\frac{\partial U}{\partial x_i}$ } Along all 3 directions

b) Lagrangian;

$L = K - U = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - U(x_1, x_2, x_3)$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{-\partial L}{\partial x_i} \quad i \in \{1, 2, 3\}$

$\frac{\partial L}{\partial \dot{x}_i} = \frac{1}{2} m \cdot 2\dot{x}_i$ (Treat velocities independant to position)

LHS $\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d}{dt}(m\dot{x}_i) = m\ddot{x}_i$

RHS $\Rightarrow -\frac{\partial L}{\partial x_i} = 0 - \frac{\partial U}{\partial x_i}$

$\Rightarrow$ So, we have retrieved the Newtons eqn of motion from Lagrangian

Hamiltonian Principle; (Optimization principle)

Note: Snells Law

$$\sigma = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\, dt \rightarrow \text{Minimize this (action)}$$

⇒ Path followed by system is where the above integral is minimum.

⟹ $\delta\sigma = 0$ } Variation of trajectory

Derive Lagrangian; (see video)

(1) ex; $J = \int_{x_1}^{x_2} f(y, y'; x)\, dx$ where $y' = dy/dx$

⇒ Find $y(x)$ such that J is stationary

ie $\delta J = 0$.

⇒ Consider, $y(\alpha, x) = y(0, x) + \alpha \eta(x)$, with solution being $y(0, x)$

∴ We are looking for $y(0, x)$ such that

$$\left(\frac{\delta J}{\delta \alpha}\right)_{\alpha=0} = 0$$

$$\Rightarrow \frac{\partial y(\alpha, x)}{\partial \alpha} = 0 + \eta(x)$$

$$\Rightarrow \frac{\partial y'(\alpha, x)}{\partial \alpha} = \frac{\partial}{\partial \alpha} \alpha \eta' = \frac{\partial \eta}{\partial x}$$

$$\Rightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \times \frac{\partial y'}{\partial \alpha} \right)$$

$$= \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \frac{\partial \eta}{\partial x} \right)$$

⇒ Now we can use uv rule for

$$\Rightarrow \cancel{\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\left(\frac{\partial \eta}{\partial x}dx\right)}$$

$$= \frac{\partial f}{\partial y'}\eta \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$\Rightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y}\eta(x) - \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)$$

$$\Rightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx\, \eta(x)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)$$

true for all η

$\therefore$ We get $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)}_{\text{Euler Lagrangian Equation.}}$$

Euler Lagrangian Equation. (Obeys the optimization principle)

—— x ——

* Note; We can also derive euler lagrangian from Newton's IInd law

—— x ——

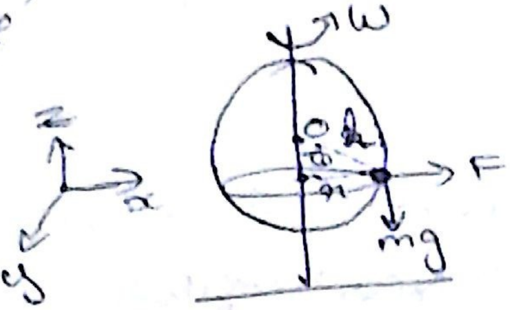
Note; Make sure constraints are accounted for in Lagrangian equation before solving it (ex: pendulum $\Rightarrow x^2+z^2=l^2$)

—— x ——

—— x ——

Scanned by CamScanner

*(Q) Solve using Lagrangian (Bead on Rotating Ring)
**



Rotating ring with bead.

KE ⇒ ?
PE ⇒ ?

$\Rightarrow KE = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - (mgh\cos\theta)$

Considering the rotational frame as reference, $\underbrace{x = l\sin\theta \;,\; z = -l\cos\theta \;, y=0}_{\text{For the bead.}}$

* ⇒ Converting to actual frame, our constraints are

$$r \leftarrow \left.\begin{aligned} x &= l\sin\theta \times \cos\omega t \\ y &= l\sin\theta \times \sin\omega t \\ z &= -l\cos\theta \end{aligned}\right\}$$

Considering → circle inside with radius $l\sin\theta$

⇒ Now we can just plug this into lagrangian equation.

(Note; Answer: $\ddot{\theta} = -\frac{g}{r}\sin\theta + \frac{\omega^2}{r^2}\sin\theta\cos\theta$)

—— x ——

**

*Central Force; (ex: Gravitational Forces) (2 body system) (Prove Kepler's 2nd Law)

**
- Force of interaction directed along line connecting centers of 2-bodies (m_1 at location r_1 & m_2 at r_2) vectors

⇒ Generalized Lagrangian:

Function of r (Proved fact)

$$L = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - U(r)$$

$r = |r_1 - r_2|$

$\Rightarrow m_1\ddot{r}_1 = -\nabla_1 U(r)$?

$m_2\ddot{r}_2 = -\nabla_2 U(r)$? $\;\therefore\; m_1\ddot{r}_1 + m_2\ddot{r}_2 = 0$?

$\Rightarrow R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$

R = C.O.M (Center of Mass) (r_1 & r_2 variable)

$\frac{dR}{dt}$ = Constant

⇒ Now $r = r_1 - r_2$ } Relative distance

$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$ } Center of Mass

$\therefore L(\vec{r}, R, \dot{r}, \dot{R}) = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 - U(|r|)$

Vector where $M = m_1 + m_2$ & $\mu = \frac{m_1 m_2}{m_1 + m_2}$

⇒ Let us consider an inertial frame, with velocity of $\dot{R}$ (Velocity of C.O.M), so we get, (r is independent of which frame we take)

$$\boxed{L(\bar{r}, \dot{\bar{r}}) = \frac{1}{2} \mu \dot{\bar{r}}^2 - U(|\bar{r}|)}$$

vector

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

$r = r_1 - r_2$

We can say that the 2-body problem is no longer a 2-body problem but a (2-1) body problem.
(With center of this body at R from origin & mass μ).

So our degrees of freedom drops ~~...~~ $2 \times 3 - 3$ degrees of freedom.

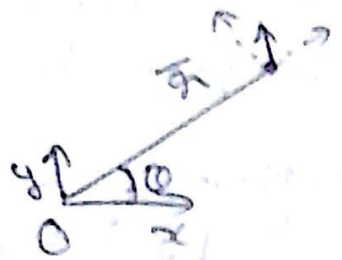
Initially we have 6 degrees of freedom $x_1, x_2, y_1, y_2, z_1, z_2$, but now, in this inertial frame of reference, we are fixing one object & only varying the other, ∴ 3 degrees = x_r, y_r, z_r

⇒ The combined mass system does not move in our chosen inertial frame.

⇒ However the two individual ~~masses~~ masses can move but in a fixed plane (the plane in which they were present at start).

———x———

So, we can say let this plane be the x,y plane



$\bar{r} \times \bar{v}_t = $ Const ?

$\bar{r} = (r\cos\theta, r\sin\theta) = (x, y)$

↓ distance

$$L = \frac{1}{2}\mu\dot{x}^2 + \frac{1}{2}\mu\dot{y}^2 - U(r)$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta}$$

$$\Rightarrow \boxed{L = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(\dot{r}^2 + r^2\dot{\theta}^2) - U(\bar{r})}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \quad \& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

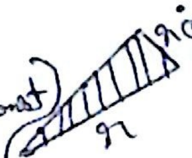
Note; $p_i = \frac{\partial L}{\partial \dot{q}_i}$, ex $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}\mu r^2 \times 2\dot{\theta}$

Generalized Momentum

$P_\theta = \mu r^2\dot{\theta}$

= Constant since $\frac{d}{dt}(P_\theta) = 0$

$$\Rightarrow r^2\dot{\theta} = \underbrace{r \times r\dot{\theta}}_{\text{Area}} = \text{Const}$$



Area covered in dt interval

∴ We can see that area covered in a certain time interval is constant

(So we have proved Kepler's 2nd Law)

Note (We can also prove Kepler's 3rd Law with Lagrangian)

---x---

---x---

* Note;

* a) Time Homogenity; Symmetry which tells us we get same results whenever we perform experiment.

b) Space Homogenity; Symmetry tells us we get same results wherever we perform experiment.

(On earth, only x, y axis homogenity is seen)

(Since z axis is effected by gravity)

(Only z-axis isotropy is homogeneous on earth)

→ Every location → (Isotropy = Every angle)

Homogenity — homogenity

---x---

---x---

Begin Of Mid 2

Time Homogenity (leads to (Conservation of energy))

~~Proof~~

$$\frac{d}{dt}L(q,\dot{q},t) = \sum_{i=1}^{s}\left(\frac{\partial L}{\partial q_i}\times\frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i}\times\frac{d\dot{q}_i}{dt}\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}L = \sum_{i=1}^{s}\left(\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{\text{From Lagrangian formula}}\cdot \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\times\frac{d\dot{q}_i}{dt}\right) + 0$$

$$\Rightarrow \frac{d}{dt}L = \underbrace{\sum \frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right)}_{\text{From (uv) rule}} \Rightarrow \frac{d}{dt}\left(L - \sum_i \dot{q}_i p_i\right) = 0$$

$$\left(p_i \equiv \frac{\partial L}{\partial \dot{q}_i}\right)$$

$$\Rightarrow H = \sum_i \dot{q}_i p_i - L$$ } We know

$$\therefore \frac{d}{dt}(H) = -\frac{\partial L}{\partial t}$$ } When we assume $\frac{\partial L}{\partial t} \neq 0$ in the above proof

$$\Rightarrow U \equiv U(q)$$

$$\frac{\partial U}{\partial \dot{q}_i} = 0,\ q_i \equiv q_i(x(t))$$

$$\Rightarrow \frac{\partial q_i}{\partial t} = 0,$$

$$\Rightarrow \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$ } Prove this!

Kinetic energy

$$\therefore H = \sum_i \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L = \sum_i \dot{q}_i\frac{\partial T}{\partial \dot{q}_i} - L$$

$L = T - U$ and $\frac{\partial U}{\partial \dot{q}_i} = 0$

$$\therefore H = 2T - L = T + U$$

Not a function of velocity

$$\therefore \boxed{H = T + U}$$

—x—

* Translational Symmetry ; (Spacial Homogeneity) (Conservation of linear momentum)

* If we move system by δr in the new frame,

$$\delta L = 0 \Rightarrow \sum_i \left(\frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i \right) = 0$$

But $\delta \dot{x}_i = \delta \frac{d}{dt} x_i = \frac{d}{dt} \delta x_i = 0$ } Velocity remains same

$$\therefore \delta L = 0 = \sum_i \frac{\partial L}{\partial x_i} \delta x_i$$

$$\therefore \delta L = 0 \Rightarrow \frac{\partial L}{\partial x_i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i} \right) = 0$$

$$\Rightarrow p_i \equiv \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i = \text{const}$$

$\therefore$ Conservation of linear momentum is a consequence of space homogenity.

—— x ——

* Space Isotropy ; (Rotation) (Conservation of angular momentum)

* Let us assume rotation around z axis,

$$\delta r = \delta\omega \, \hat{n} \times r \quad \text{axis of rotation}$$

$$\Rightarrow \delta \dot{r} = \delta\omega \, \hat{n} \times \dot{r}$$

$$\delta L = \sum_i \left(\frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i \right) = \sum_i \left(\dot{p}_i \delta x_i + p_i \delta \dot{x}_i \right) ??$$

$$= \dot{p}\,\delta x + p \cdot \delta \dot{x}$$

$$\Rightarrow \delta L = \delta\omega \left(\dot{p} \cdot \hat{n} \times r + p \cdot \hat{n} \times \dot{r} \right) = \delta\omega \, \hat{n} \left(r \times \dot{p} + \dot{r} \times p \right)$$

Cyclic permutation

$$\delta L = \delta\theta\, \hat{n} \frac{d}{dt} L_m = \delta\theta \frac{d}{dt} \hat{n} \cdot L_m$$

where $L_m = r \times p$ } Angular momentum

$$\delta L = 0, \therefore \hat{n} L_m = \text{Constant}$$

$\therefore L_m = \text{Constant}$ } For ~~every~~ every axis of rotation $\hat{n}$

— x —

* Hamiltonian Mechanics;

$$H = \sum_{i=1}^{s} \dot{q}_i p_i - L(q, \dot{q}, t)$$

$$\Rightarrow dH = \sum_{i=1}^{s} (p_i\, d\dot{q}_i + \dot{q}_i\, dp_i) - \left[\sum_{i=1}^{s} (\dot{p}_i dq_i + p_i d\dot{q}_i) + \frac{\partial L}{\partial t} \times dt\right]$$

$$\Rightarrow dH = \sum_{i=1}^{s} (\dot{q}_i\, dp_i - \dot{p}_i\, dq_i) - \frac{\partial L}{\partial t} \times dt \Big\} \underbrace{p_i, q_i, t}$$

$\longrightarrow$ (1)

Variables (coordinates, momentum and time) (No velocity!)

Now $H(\{q_i\}, \{p_i\}, t)$

$$\Rightarrow dH = \sum_{i=1}^{s} \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i\right) + \frac{\partial H}{\partial t} \times dt$$

$\longrightarrow$ (2)

p_i, q_i conjugates $\Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i}, \; \dot{p}_i = \frac{\partial H}{-\partial q_i}$

Practically useful since they depend on conjugate derivatives

*Equations of motion in Hamiltonian mechanics

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

} Comparing (1) & (2)

From time homogenity proof.

Note: $\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$

— x —

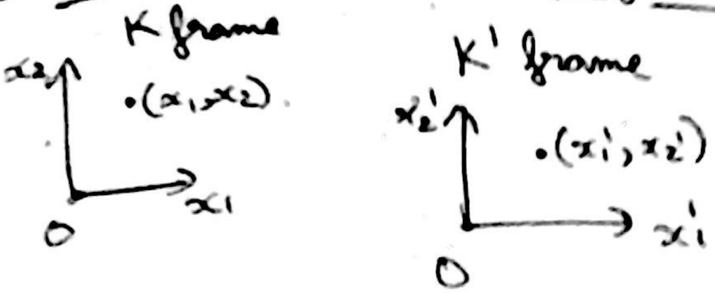
Note: In Hamiltonian mechanics, if q_i's or p_i's are unused, then our number of equations of motion are reduced unlike in Lagrangians. (See slides for example) (Central forces)

——— × ———

Special Theory of Relativity

(Chapter 14)

⇒ <u>Inertial frames of reference</u>; (Example)



* • If we consider K' frame moving along x_1 axis with velocity v.

$$\rightarrow \quad x_1' = x_1 - vt$$
$$x_2' = x_2$$

$$\Rightarrow \quad \frac{dx_1'}{dt} = \frac{dx_1}{dt} - v \quad , \quad u_1' = u_1 - v$$

$$\Rightarrow \quad \frac{dx_2'}{dt} = \frac{dx_2}{dt} \quad , \quad u_2' = u_2$$

— x —

~~Galilean Variance~~

— x —

** <u>Galilean Relativity</u>;

$$\left.\begin{aligned} \cdot\ u_1' &= u_1 - v_{x_1} \\ \cdot\ u_2' &= u_2 - v_{x_2} \end{aligned}\right\}$$ When K' frame moves on x_1 axis with velocity v_{x_1} & x_2 axis with velocity v_{x_2}

— x —

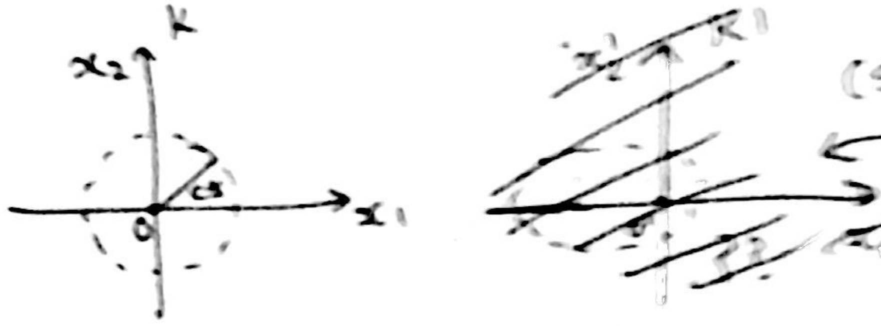
*<u>Note</u>: This galilean relativity <u>fails</u> for light, since experiments done to calculate velocity of light at different points in earth's orbit, but light velocity is always constant, so, the frame's (earth's) velocity

— x —

Note; The relativity also fails with the help of lamp experiment (it turned on & off instantaniously at $t=0$)

So in K frame $v = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$

$\Rightarrow$ In K' frame $v = \sqrt{\dot{x}_1'^2 + \dot{x}_2'^2}$ } Speed of light



(Shape ~~[illegible]~~ ... speed changes)

But this ~~change~~ does not obey galilean relativity

$\Rightarrow$ Consider x_1 axis ~~[illegible]~~

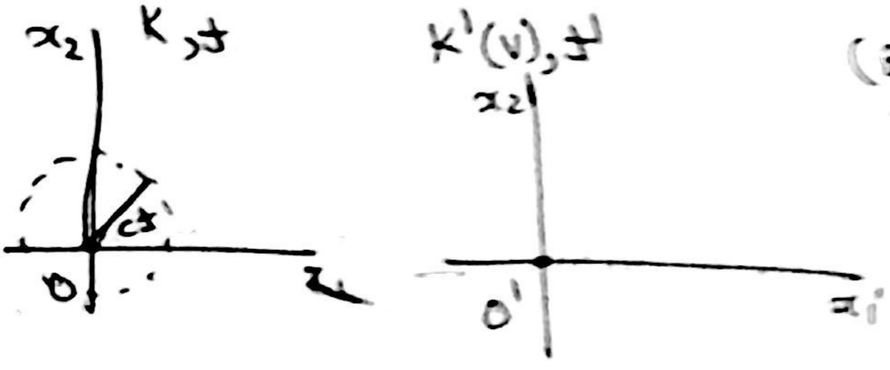
$\Delta x_1' = \Delta x_1 - v\Delta t$ } From galilean

- But experimentally this is not true

So we drop the theory for light

Lamp Experiment, (K' frame moving in x axis with velocity v)

- This is how galilean's theory was disproved for light



(But we know that it could not be false so

- Let us consider the wavefront on x_1 axis
- ~~Newton~~ Einstein proposed that time varies different in these cases.

$\Rightarrow$ at $x = 0, t = 0$ (So we use t & t')
$x' = 0, t' = 0$

$\rightarrow$ K wavefront, $x_1 = ct \rightarrow \frac{dx_1}{dt} = c$
K' wavefront, $x_1' = ct' \rightarrow \frac{dx_1'}{dt'} = c$

$\Rightarrow$ K wavefront, $x_1^2 + x_2^2 = (ct)^2$
K' wavefront, $x_1'^2 + x_2'^2 = (ct')^2$ } If we are trying to show wavefront moves similarly & galilean principle doesn't fail completely

$\Rightarrow$ Our only way is to say $t \neq t'$, so that galilean relativity is satisfied & velocity does not differ either.

Not very feasible either $\longrightarrow$ (Now, lets take $t' = t$ (Standard))

- $x_1(O') = vt$
  $x_1'(O') = 0$

Fails for light if $t' = t$
$x_1' \neq x_1 - vt$ X

$\Rightarrow$ Einstein proposed, $\boxed{x_1' = \gamma(x_1 - vt)}$ when the time is taken as same everywhere. So we get x_1' entirely from K frame.

$\Rightarrow$ γ is a function of velocity of the frame.

$\therefore x_1' = \gamma(v)(x_1 - vt)$

$\Rightarrow x_1 = \gamma(-v)(x_1' + vt')$ } Considering the frame K from frame K'.
(Make K' original frame)
(So K moves with -ve velocity).

$$\Rightarrow x_1' = \gamma(v)(x_1 - vt)$$
$$x_1 = \gamma(-v)(x_1' + vt')$$
$$x_2' = x_2$$
$$x_3' = x_3$$

Einstein said $\gamma(-v) = \gamma(v)$

- He said if this is not true the the theory fail[s]

$\therefore$ let $\gamma = \gamma(v) = \gamma(-v)$

$$\therefore \quad x_1' = \gamma(x_1 - vt) \longrightarrow (1)$$
$$x_1 = \gamma(x_1' + vt') \longrightarrow (2)$$

Putting (1) in (2),

$$x_1 = \gamma\left[\gamma(x_1 - vt) + vt'\right]$$

$$\Rightarrow vt' = \frac{x_1}{\gamma} - \gamma(x_1 - vt)$$

$$\Rightarrow \frac{x_1}{\gamma v}\left[1 - \gamma^2\right] + t\gamma = t'$$

∴ ~~We have shown that t≠~~

$$* \Rightarrow \boxed{t' = \gamma\left[t + \frac{x_1(1-\gamma^2)}{v\gamma^2}\right]}$$

∴ We have shown that $t' = t = 0$ initially but later on $t' \neq t$.
(ie. time varies in both frames).

→ Now coming back to the wavefronts,

- $x_1 = ct$, $x_1' = ct'$ $\longrightarrow (1)$, $\quad x_1 = \gamma(x_1' + vt')$, $x_1' = \gamma(x_1 - vt) \longrightarrow (2)$

Putting (1) in (2),

$$\Rightarrow ct = \gamma(ct' + vt') = \gamma t'(c + v)$$
$$ct' = \gamma(ct - vt) = \gamma t(c - v)$$

$$\Rightarrow ct = \gamma t'(c + v) \longrightarrow (1)$$
$$\gamma t(c - v) = ct' \longrightarrow (2)$$

$\Rightarrow$ Dividing the two,

$$\frac{c\cancel{t}}{\gamma \cancel{t'}(c-v)} = \frac{\gamma \cancel{t}'(c+v)}{c\cancel{t'}}$$

$$* \Rightarrow c^2 = \gamma^2(c^2 - v^2)$$

$$\gamma^2 = \frac{c^2}{c^2 - v^2}, \quad \therefore \boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$\therefore$ We can see $\gamma(-x) = \gamma(x)$

- Putting this in our equations,

$$x_1' = \gamma(x - vt), \quad x_2' = x_2, \quad x_3' = x_3$$

$$\cancel{t' = \gamma\left(t + \frac{x_1}{v} \times c^2\right)}$$

$$t' = \gamma\left(t - \frac{x_1 v}{c^2}\right)$$

Considering K' to be the original frame. (Like we did before)

$$\Rightarrow t = \gamma\left(t' + \frac{x_1' v}{c^2}\right)$$

* Finally,

*

$$x_1' = \gamma(x_1 - vt)$$
$$x_2' = x_2$$
$$x_3' = x_3$$
$$t' = \gamma\left(t - \frac{x_1 v}{c^2}\right)$$
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

* Lorentz Transformation

* (K' frame from K frame)

$$x_1 = \gamma(x_1' + vt')$$
$$x_2 = x_2'$$
$$x_3 = x_3'$$
$$t = \gamma\left(t' + \frac{x_1' v}{c^2}\right)$$
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Inverse Lorentz Transformations

(K frame from K' frame)

* Now lets calculate the velocities,

$\overline{u} = (u_1, u_2, u_3)$, $u_1 = \frac{dx_1}{dt}$, $u_2 = \frac{dx_2}{dt}$.

$\overline{u}' = ?$, $u_1' = \frac{dx_1'}{dt'}$, $u_2' = \frac{dx_2'}{dt'}$, $u_3' = \frac{dx_3'}{dt'}$

$\Rightarrow$ Now, $u_1' = \frac{dx_1/dt}{dt'/dt} = \frac{\cancel{\gamma}\left(\frac{dx_1}{dt} - v\right)}{\cancel{\gamma}\left(1 - \frac{dx_1 \times v}{dt \times c^2}\right)}$

$$u_1' = \frac{(u_1 - v)}{\left(1 - \frac{u_1 v}{c^2}\right)}$$

* (According to galilean,

$u_1' = u_1 - v$, but its not that here)

$\therefore \boxed{u_1' = \frac{\gamma(u_1 - v)}{\gamma\left(1 - \frac{u_1 v}{c^2}\right)}}$, $u_2' = \frac{u_2}{\gamma\left(1 - \frac{u_1 v}{c^2}\right)}$, $u_3' = \frac{u_3}{\gamma\left(1 - \frac{u_1 v}{c^2}\right)}$

*

Same denominators since dt'/dt is same

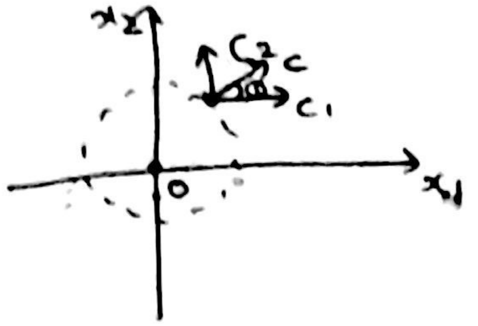
$\therefore u_1' = \frac{u_1 - v}{1 - \frac{u_1 v}{c^2}}$ | Wavefront in K frame

$u_1 = c$ } For wavefront

$\therefore u_1' = \frac{c - v}{1 - \frac{cv}{c^2}} \Rightarrow u_1' = \frac{\cancel{(c-v)}c}{\cancel{c-v}} = c$

$\therefore$ Only Light has same velocity in any frame according to this formula.

⇒ We can show this for any point on wavefront



$c_1 \longrightarrow c_1'$
$c_2 \longrightarrow c_2'$ } From these we can check & show that

$\sqrt{c_1'^2 + c_2'^2} = c$

↓

$\underbrace{u' \cdot u' = c^2}$ In K' frame (Check & Prove this)

• So light always moves with velocity c in any frame.

* Fitzgerald Lorentz;

(Length ~~Contradiction~~)
Contraction

• Frame K, rod of length L_0, $x_L = 0$, $x_R = L$

• Frame K' rod length = ?

• L = Rod Length

$$\Rightarrow \quad x_L(T) = 0 = \gamma\left(x_L'(T') + vT'\right)$$

$$\Rightarrow \quad x_R(T) = L_0 = \gamma\left(x_R'(T') + vT'\right)$$

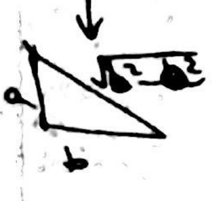
• $\Delta x = L_0 - 0 = \gamma\left(x_R'(T') - x_L'(T')\right) = \gamma L$

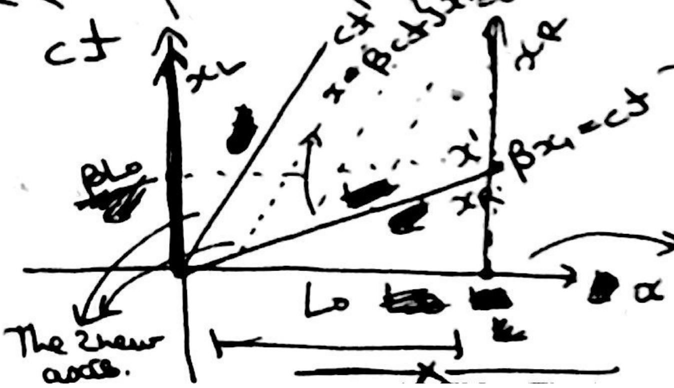
$$\therefore \boxed{L = L_0\sqrt{1 - v^2/c^2}}$$

Length in moving frame is contracted.

Look at Slides for graph

*
Hyperbolic coordinate system.





Time Dilation;

K Frame	K' Frame
• T_0 = time between 2 ticks	• $T'_0 = ?$

$\Rightarrow$ 2-ticks; $x_1(e_1) = 0$, $t(e_1) = 0$
$x_1(e_2) = 0$, $t(e_2) = T_0$

$$\Rightarrow t'(e_1) = \gamma\left(t(e_1) - x_1(e_1)\frac{v}{c^2}\right) = \gamma t(e_1)$$
$$t'(e_2) = \gamma\left(t(e_2) - x_1(e_2)\frac{v}{c^2}\right) = \gamma t(e_2)$$

• Hence $T' = t'(e_2) - t'(e_1) = \gamma T_0$

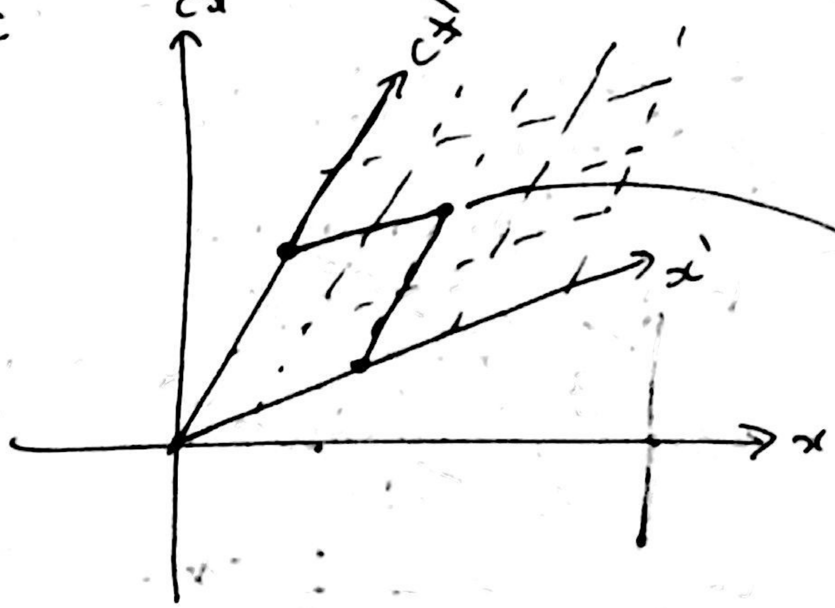
$$\therefore \boxed{T' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \times T_0}$$

• Time difference is larger $\rightarrow$ Dilation

$\Rightarrow$ Muon decay experiment proved this

(~~Length Contradiction~~ continuation);

Note:



From Length contradiction (Graph from prev page)

Given a point and its x' or t', we can get other corresponding coordinates

- In K frame, $x_L(t)=0$
  $x_R(t)=L_0$
- In K' frame,

$$x=\gamma(x'+\beta ct')$$

$$\Rightarrow x'=\frac{1}{\gamma}x - \beta ct'$$

$$\Rightarrow x_L'(ct')=-\beta ct' \;?$$

$$x_R'(ct')=\frac{1}{\gamma}L_0-\beta ct'$$

$$\therefore x_R'(ct')-x_L'(ct')=\frac{1}{\gamma}L_0$$

$$\therefore L'=L_0\sqrt{1-\beta^2} \quad ,\ \beta=v/c$$

→ Hyperbolic coord system

Hypotenuse, Sides = L_0 & βL_0

* Note; Similarly Time Dilation can be proved graphically (Look at slides)

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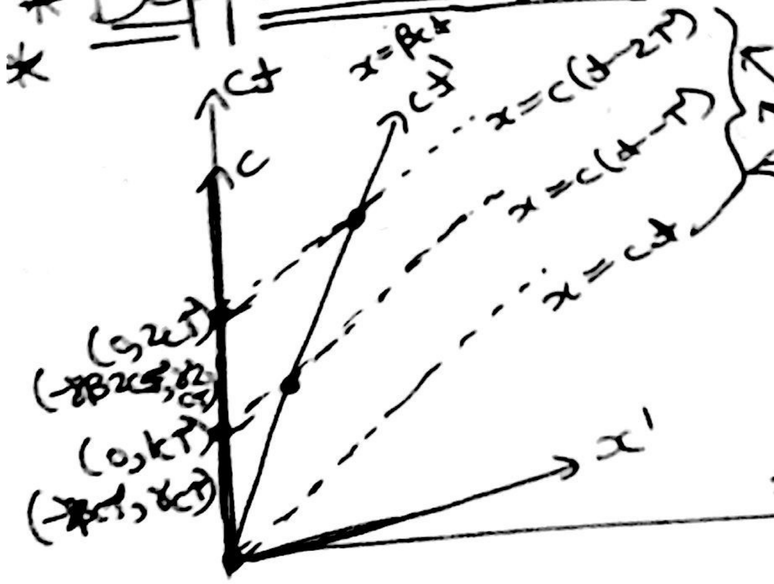
* Doppler Effect,



- Assume light is turned on & off at τ time intervals. (Flashes)

$$\Rightarrow x=\gamma(x'+\beta ct') \quad ,\ ct=\gamma(ct'+\beta x')$$

$$\Rightarrow \cancel{x=c(t+\beta ct)} \qquad x=c(t-n\tau)$$

$$\Rightarrow \gamma(x'+\beta ct')=\gamma(ct'+\beta x')-nc\tau$$

$$\Rightarrow (1-\beta)x' = ct'(1-\beta) - \frac{ncT}{\gamma}$$

$$\Rightarrow x' = 0,\ ct' = ncT\sqrt{\frac{1+\beta}{1-\beta}}$$

Now, K' frame frequency

$$\boxed{\nu' = \nu\sqrt{\frac{1-\beta}{1+\beta}}}, \quad \beta = v/c$$

When observer moves away from source, with velocity v.

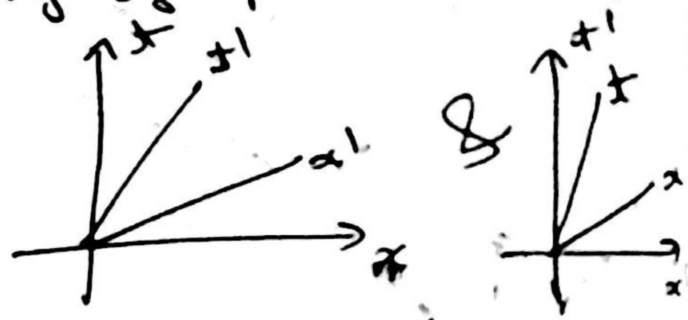
Smaller frequency in K' frame.

— x —

Note; When source moves away from stationary observer we get same result as above, but when source moves towards observer, what happens? , $\nu' = \nu\sqrt{\frac{1+\beta}{1-\beta}}$

— x —

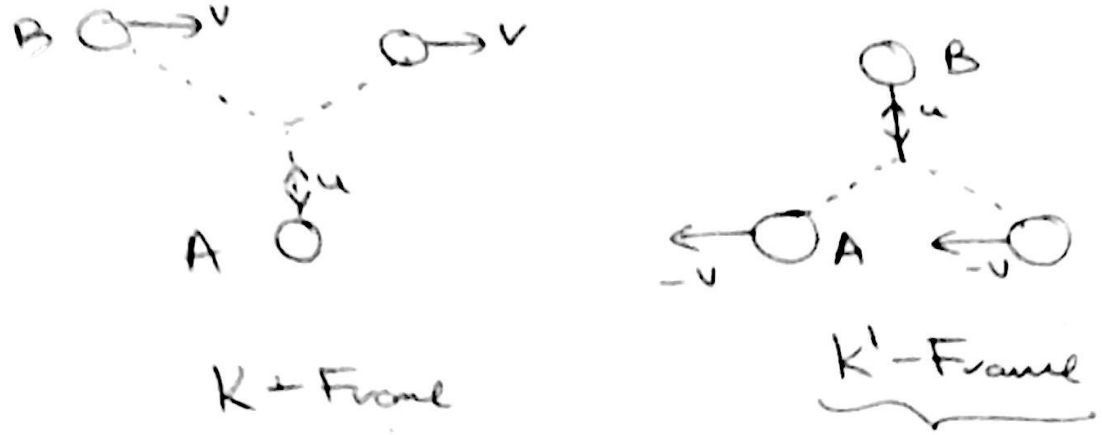
Note; From symmetry of special theory of relativity,



— x —

Momentum ; (Slides for proof on why $p = mv$ fails)

- Consider 2 frames,



Consider moving with velocity v in x-axis.

$\Rightarrow$ Here if we consider $p = mv$ } Then even though no forces are acting, momentum is not conserved

$\therefore$ We redefine $\boxed{p = mu/\sqrt{1-u^2/c^2}} = \gamma m u$

u: Velocity of object. $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

Energy ;

$\Rightarrow W_{12} = \int_1^2 F\,dr = \int_{t_1}^{t_2} F.u.dt = T_2 - T_1$ } Kinetic energies.

$\Rightarrow F = \frac{d}{dt} p = \frac{d}{dt}[\gamma m u]$

$\therefore W_{12} = m\int_0^u u\,d(\gamma u) = m\left[[u\gamma u]_0^u - \int_0^u \gamma u\,du\right]$

$\Rightarrow \int_0^u \frac{u\,du}{\sqrt{1-u^2/c^2}} = \int_0^u \frac{\frac{1}{2}d(u^2)}{\sqrt{1-u^2/c^2}} = c^2\left(\sqrt{1-\frac{u^2}{c^2}} - 1\right)$

$$\Rightarrow W_{12} = T_2 - T_1 = \cancel{\ } + mc^2 \sqrt{1-\frac{u^2}{c^2}} - \ldots$$

$$\therefore \quad T = \gamma mc^2 - mc^2$$

In absence of potential

T: Kinetic energy

γmc^2: Total energy

mc^2: Rest mass energy → $E_0 = mc^2$

- $E = T + E_0 = (\gamma - 1)mc^2 + mc^2 = \underbrace{\gamma mc^2}_{\text{Total energy}}$

Now, $p = \gamma mu$

$$\Rightarrow p^2 = \gamma^2 m^2 u^2 = \gamma^2 m^2 c^2 \left(\frac{1}{\gamma^2} - 1\right)$$

$$= m^2c^2 - \gamma^2 m^2 c^2$$

$\Rightarrow$ We know $\boxed{E = \gamma mc^2}$

$$p^2 = m^2c^2 - \gamma^2 m^2 c^2$$

$$\therefore \boxed{E^2 = p^2c^2 + (mc^2)^2}$$

When there is no potential so no potential energy

From K.E part

where $p = \gamma mu$, $\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$

From rest energy part.

$= = = \times = = =$

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* Space-Time; (Four Vectors 1) (S) Bottom
(Like Intervals)

* • In any frame,

$$(\Delta \bar{x}')^2 - (c\Delta t')^2 = \underbrace{(\Delta \bar{x})^2 - (c\Delta t)^2}_{(\Delta s)^2}$$

• Four vectors: $s = (x_1, x_2, x_3, ict)$

* ⇒ The length of this four vector is an invariant across any frame.

i) ⇒ If $(\Delta s)^2 < 0$ $(\Delta s = \underbrace{s_2 - s_1}_{\text{Two events}})$

• Then we can find a frame where $\Delta \bar{x}' = 0$ } Space Like intervals } K' frame

ii) ⇒ If $(\Delta s)^2 > 0$

• Then we can find a frame where $\Delta t' = 0$ } Time like intervals } K' frame

• $-p^2c^2 + E^2$ is also an invariant and is always equal to m^2c^4. //

⇒ Momentum Four vector ⇒ $(p_1, p_2, p_3, \frac{iE}{c})$

* Four velocity vector = (u_1, u_2, u_3, ic)

(video for proof of $(\Delta s)^2$ is constant)

———x———

———x———

Thermodynamics
(Statistical Mechanics)

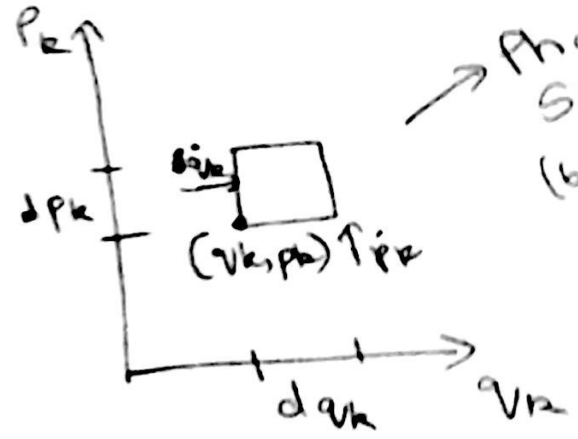
- When a particle exists at a constant hamiltonian = constant energy.

- The phase plot of p vs x

The particle can be anywhere on this curve.

$\ast \Rightarrow$ Boltzman stated that particle has equal probability of a particle to be anywhere on the curve.

** Liouville Theorem; (Slides)



$\nearrow$ Phase Space (6 dimensional) $\Rightarrow$ At time t, density is $\rho(\bar{q}, \bar{p}, t)$.

- What is density at $t + dt$?

$\Rightarrow$ So since every particle moves to the right as time passes,

$\Rightarrow$ at $t + dt$, number entering $= \rho \dot{q}_k \times dp_k$

$\Rightarrow$ At right side, number leaving $= ?$

leaving $= \left[\rho \dot{q}_k + \frac{\partial}{\partial q_k}(\rho \dot{q}_k)\, dq_k\right] \times dp_k$

$[\rho \dot{q}_k\, dp_k] \rightarrow$ Evaluated at $(q_k + dq_k, p_k)$

$\therefore$ Right $-$ Left $= \frac{\partial(\rho \dot{q}_k)}{\partial q_k}\, dp_k\, dq_k$

$\Rightarrow$ Similarly Top $-$ Bottom $= \frac{\partial(\rho \dot{p}_k)}{\partial p_k}\, dq_k\, dp_k$

$$\therefore \underbrace{\frac{\partial \rho}{\partial t}\, dV_{\text{phase-space}}}_{\text{Total particles exiting}} = -\sum_k \left(\frac{\partial(\rho \dot{q}_k)}{\partial q_k} + \frac{\partial(\rho \dot{p}_k)}{\partial p_k} \right) \times dV_{\text{phase-space}}$$

$dV \rightarrow$ Volume in phase space

$\rightarrow$ From right - left

$\rightarrow$ From top bottom

$$\therefore \Rightarrow \frac{\partial \rho}{\partial t} + \sum_k \left(\frac{\partial \rho}{\partial q_k} \dot{q}_k + \frac{\partial \rho}{\partial p_k} \dot{p}_k \right) + \rho \sum_k \left(\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \dot{p}_k}{\partial p_k} \right) = 0$$

Now, $\dot{q}_i = \frac{\partial H}{\partial p_i}$ & $\dot{p}_i = \frac{-\partial H}{\partial q_i}$ } From Hamiltonian

$$\therefore \frac{\partial \dot{q}_k}{\partial q_k} = \frac{\partial^2 H}{\partial q_k \partial p_k} \quad \& \quad \frac{\partial \dot{p}_k}{\partial p_k} = \frac{-\partial^2 H}{\partial p_k \partial q_k}$$

$$\therefore \frac{\partial \rho}{\partial t} + \sum_k \left(\frac{\partial \rho}{\partial q_k} \dot{q}_k + \frac{\partial \rho}{\partial p_k} \dot{p}_k \right) = 0$$

$$\therefore \boxed{\frac{d\rho}{dt} = 0}$$

$\therefore$ The density does not change!

If we follow the flow,

* Note; If we start with particles in a square □, then after ∞ time, if the particles assume a path ◯ in the phase plane then the areas of both the square & the path are the same since the density does not change.

---x---

** ⇒ If the system reaches Equilibriation, i.e. ∞ time, then $\frac{\partial}{\partial t}\rho(\bar{q},\bar{p},t) = 0$.

→ So every state $i \equiv x_i \equiv (\bar{q},\bar{p})_i$ has an associated probability density $P_i = P(x_i)$.

- Points with $H_i = H_j$, then $P(x_i) = P(x_j)$

---x---

---x---

*

Entropy → (Missing Information Function) for all states of a system.

⇒ We try create a mathematical function.

- This should be continuous, i.e. if state 1 probability $p_1 \to p_1 - \Delta$ & state 2 probability $p_2 \to p_2 + \Delta$, then entropy should be continuous
- If we make $p_1 \to p_2$ & $p_2 \to p_1$, then entropy should be the same.

⇒ We consider there are W states in the system consider regions of the line in phase space (earlier)(prev page).

⇒ Regrouping W states into n new groups should satisfy another constraint (slides)

∴ We derive entropy to be,

$$\boxed{S(\{p\}_n) = -k \underbrace{\sum_{i=1}^{n}}_{} p_i \ln p_i}$$

n states are present in system.

(ω = n)

---x---

Note, If we have ω states with $p_i = \frac{1}{\omega}$ (For maximum state) the $S(\{p\}_\omega)$ will be maximum

$\boxed{S = k\ln(\omega)}$ since our constraint is $\sum_i^{\omega} p_i = 1$

Boltzman Law.

---x---

Note:

Average of $\langle B \rangle = \int d\bar{q} \int d\bar{p}\ \rho_{eq}(\bar{p},\bar{q})\, B(\bar{p},\bar{q})$

---x---

I) Microcanonical Ensemble; (Slides)

- Constant number of particles, hamiltonian.

$\Rightarrow \therefore S = k\ln\Omega \rightarrow$ states.

$p_i = \frac{1}{\Omega}$

---x---

Note: **Information theory:** (Maximize missing info function)

$g = S - \lambda\left(\sum_i p_i - 1\right)$, when we change p_i of some state, then g no longer is maximum, so we will find a λ such that this is true.

constraint

(g should be maximum) (new S must be max)

$\Rightarrow \frac{\partial g}{\partial p_i} = \frac{\partial S}{\partial p_i} - \lambda \rightarrow (1)$

For all states i

---x---

Note. Lagrange Multiplier technique

From (v).

$\therefore$ We have $\dfrac{\partial g}{\partial p} = -K[\ln p_i + 1] - \lambda \equiv 0$

$\Downarrow$ For max

$\therefore \lambda = -K[\ln p_i + 1]$

$\downarrow$

$p_i =$ constant (Hamiltonian)

$\therefore \lambda$ is a constant.

where $\sum_i p_i = 1$ } Constraint.

—— x ——

- In this ensamble, we assume our system is isolated.
- In information theory method, our function is $g = S - \lambda_1 c_1 - \lambda_2 c_2 - \dots$ ← constraints.

Here $\underbrace{\sum_i p_i - 1 = 0}_{\text{constraint } c_1}$

—— x ——

—— x ——

II) Canonical Ensamble;

- Surroundings are at a temperature T. (we consider this as bath)

$\therefore \underset{\substack{\uparrow \\ \text{System}}}{E_S} + \underset{\substack{\uparrow \\ \text{Bath}}}{E_B} = E_0$.

- A system with energy E can have a max of $\Omega(E)$ states.

$\therefore$ System $\rightarrow \Omega_S(E) \rightarrow$ Assume

Bath $\rightarrow \Omega_B(E_0 - E)$

- Since system & bath are uncorrelated,

$\Rightarrow$ Total states available $= \Omega_S(E) \times \Omega_B(E_0 - E)$

$\Rightarrow$ Here all states are equally probable

→ Now if we consider a system state is

$p_i \propto \Omega_B(E_0 - E_i)$ } For rest of universe — (1)

⇒ Considering $E_0 - E_i \approx E_0$ (Large universe)

From (1),

$\ln(p_i) = c_1 + \ln \Omega_B(E_0 - E_i)$ } Taylor expand

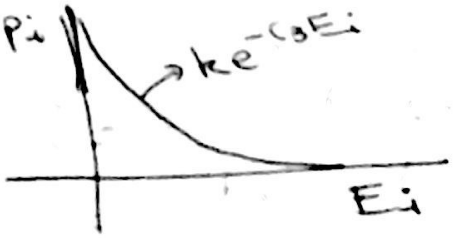
$\Rightarrow \ln(p_i) = c_1 + \ln \Omega_B(E_0) - E_i \left[\frac{\partial \ln \Omega_B(E)}{\partial E}\right]_{E_0 = E_i}$

$\Rightarrow \ln(p_i) = c_1 + \underbrace{c_2}_{\ln \Omega_B(E_0)} - E_i c_3$

$\therefore \boxed{p_i \propto e^{-c_3 E_i}}$ → Boltzman Law

$c_3 = 1/(k_B T)$ } Proof will be shown later.

$\because c_3 > 0$, As $E_i \uparrow$, $p_i \downarrow$



— x —

⇒ Using information theory:

$$g = \left[S - \lambda_E \sum_j p_j E_j - \lambda_1 \sum_j p_j\right]$$

Constraints

• We get the constraint $\underbrace{\sum p_i E_i}_{\text{Must be Constant}} = \bar{E}$ ← Average energy of distri

• Since $\bar{E}$ should only be dependant of T (surrounding) which is constant.

$$\therefore \frac{\partial}{\partial p_i}\left[S - \lambda_E \sum_j p_j E_j - \lambda_1 \sum p_j\right] = 0 \quad \forall \longrightarrow (1)$$

$$\Rightarrow S = -k \sum p_i \ln(p_i) \quad \Big\} \text{From Gibbs.}$$

$\therefore$ (1) becomes,

$$-k\left[\frac{p_i}{p_i} + \ln(p_i)\right] - \lambda_E E_i - \lambda_1 = 0$$

$$\Rightarrow 1 + \ln(p_i) = \frac{-\lambda_E E_i - \lambda_1}{k}$$

$$\therefore p_i = e^{-\frac{\lambda_E}{k} E_i} \Big/ e^{\lambda_1/k + 1}$$

$\therefore$ We can see $\boxed{p_i \propto e^{-CE_i}}$

Here too $C = 1/(k_B T)$

$\therefore$ We can also prove boltzman law from information theory.

x

** ~~Note~~ System with Constant energy;

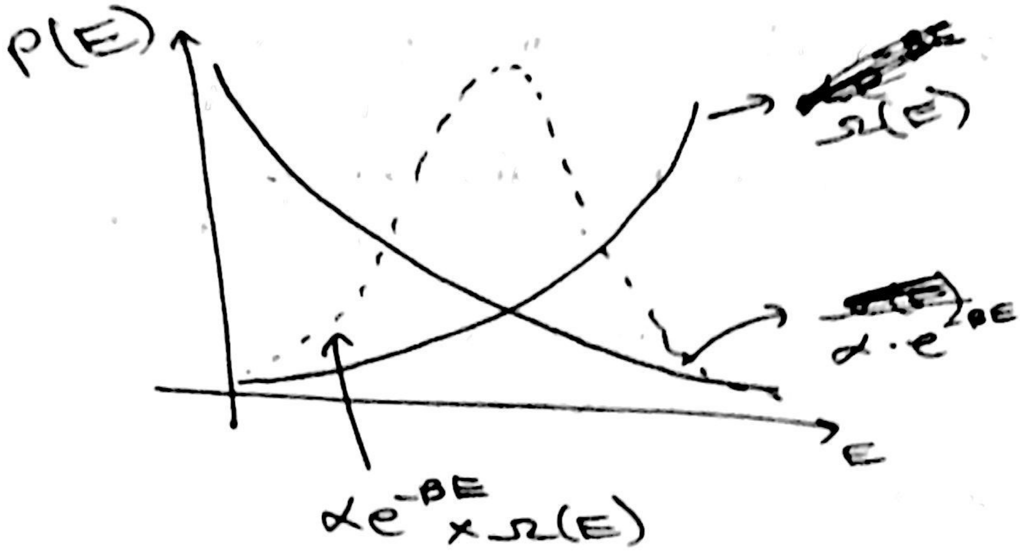
* $\Omega(E)$ = Number of states $\rightarrow$ Degeneracy.

* • Consider when each state has same energy E.

$$\Rightarrow P_i \propto e^{-\beta E} \quad , \therefore P(E) = \propto e^{-\beta E} \times \Omega(E)$$

• (Generally as $E\uparrow$, $\Omega(E)\uparrow$)

$\Rightarrow$ So since $\Omega(E)$ is an increasing function for E. & $\propto e^{-\beta E}$ is a decreasing function,



* • $P(E)$ is the possibility of system having energy E (All as constant. ~~particles~~ states in system with energy E)

⇒ The width & height of our hill (from above) depends on our temperature (equilibrium) since $\beta = 1/kT$.

- As $T \uparrow$, we get a higher peak & the width is smaller (since area needs to be the same).

———x———

———x———

*Types of Systems: (Thermodynamics)

1) Closed System → No material transfer, Heat & Work transfer

2) Open System → Material, heat, volume transfer

3) Isolated System → No material, No volume, No heat transfer.

———x———

*Equation of State: (For Thermodynamics)

- $PV = nRT$
- $\left(P + \frac{a}{V^2}\right)(V-b) = RT$ } V = Molar volume.

Vanderwaal } Real gases.

———x———

* Note: Intensive properties → Does not depend on quantity present.

Extensive properties → Depends on quantity present (mass).

ex: Energy

—x—

* Zeroth Law;

- If A is in equilibrium with B & B is in equilibrium with C, then A must be in equilibrium with C.

- Equilibrium ⇒ Thermal, Pressure, Chemical Potential } All these equilibriums must be satisfied.

—x—

* First Law;

- $\Delta E = \Delta q + \Delta w$

⇒ E = Internal Energy
q = Heat Energy
w = Work.

⇒ $\Delta w = -P_{ext} \times \Delta V$ → $\because dW = F_{ext} \cdot d\vec{r}$

⇒ $\Delta q = n C \Delta T$

C: Molar heat capacity.

- E is extensive.

⇒ $dE = E_B - E_A$ } Path Independent

⇒ Δq & Δw depend on path taken.

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Note: Adiabatic process $\rightarrow \Delta q = 0$

— x —

Reversible vs Non-Reversible processes;

- Reversible: At every stage of process, system is in equilibrium with external imposed constraints.
  (Quasi Static $\rightarrow$ Theoretically ∞ time)

- Non-Reversible: We cannot reverse the process. (Reversing our path gives us new system states).

— x —

* Work Done;

- $\Delta W = -P_{ext}\, dV$

$\Rightarrow P_{ext} = P_{int} = nRT/V \rightarrow$ For a reversible process.

$$\therefore \Delta W = -\int_i^f \frac{nRT}{V} dV = \underbrace{-nRT \log \frac{V_f}{V_i}}_{\text{Isothermal}}$$

$\Rightarrow$ When not done reversibly $\Delta W \geq \Delta W_{rev}$

$\Rightarrow$ Expansion against const pressure,

$$W = -P_{ext} \Delta V$$

Magnitude $|\Delta W| \leq |\Delta W_{rev}|$ but due to -ve sign.

— x —

* Heat Process;

- $\Delta q = nC_{v,m} \Delta T$ or $\Delta q = nC_{p,m} \Delta T$.

$$C_x = \left(\frac{\partial q}{\partial T}\right)_x, \quad x = V \text{ or } P.$$

$\Rightarrow$ Constant volume process,

$$\Delta W = 0$$

$$dE = (dq)_V = C_V dT \Rightarrow C_V = \left(\frac{dE}{dT}\right)_V$$

$\Rightarrow$ Constant pressure process,

$$H = E + pV \quad \} \text{Enthalpy}$$

$$dH = dE + pdV + Vdp$$

$$dH = \Delta q + \underset{-pdV}{\underset{\uparrow}{\Delta W}} + d(pV) = \Delta q + Vdp$$

$$\therefore \quad dH = (\Delta q)_p$$

$$\therefore \quad dH = C_p dT \quad , \quad C_p = \left(\frac{\partial H}{\partial T}\right)_p$$

—— x ——

After Mid 2

Note: C2-S11 Atkins (Joule thompson)

* Internal Energy Changes; (For real gases)

$E = E(T,V) \rightarrow$ Temp & Volume.

$$dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV = C_V dT + \pi_T dV$$

1) ⇒ Joule experiment: $\pi_T \approx 0$ (For real gases)

* • For ideal gas $\pi_T = 0$, $dE = C_V dT$ * $\boxed{\pi_T = \left(\frac{\partial E}{\partial V}\right)_T}$

(Expansion at ~~const temp~~ constant temp does not change internal energy)

2) Joule-Thompson effect;

- Expansion of gas at fixed external pressure adiabatically.

⇒ For const pressure processes. we consider enthalpy.

$H(p,T)$, $dH = \left(\frac{\partial H}{\partial T}\right)_p dT + \left(\frac{\partial H}{\partial P}\right)_T dP = 0$

* $\boxed{\mu_T = \left(\frac{\partial H}{\partial P}\right)_T}$, $\mu_T = -C_p\mu = -C_p\left(\frac{\partial T}{\partial P}\right)_H$

$\therefore dH = C_p\, dT$

⇒ For ideal gas $\mu_T = 0$, $\pi_T = 0$

⇒ For real gases $\mu_T \neq 0$, $\pi_T \neq 0$, since real gases mainly interact via attractive interactions.

---x---

* Note; For all const pressure, we consider ΔH, and const volume, we consider ΔE.

⇒ $\Delta H = 0$, for expansion at const pressure.

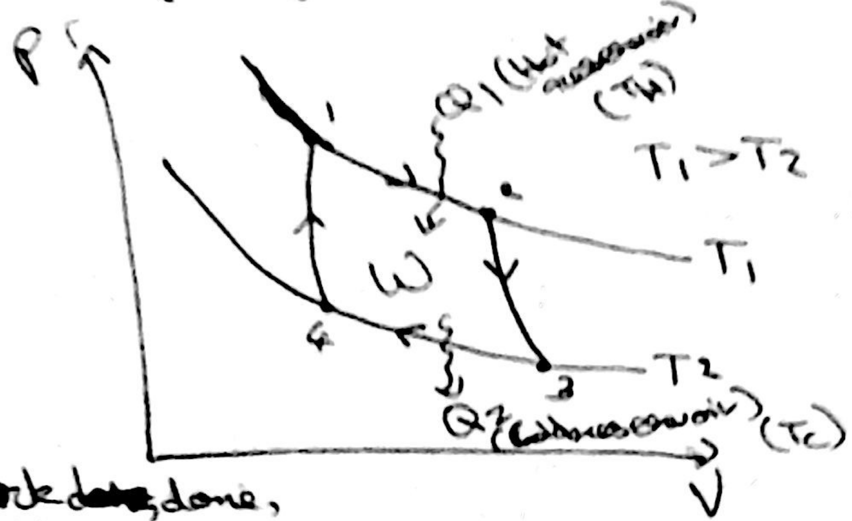
⇒ $\Delta E = 0$, for expansion at const ~~temp~~ volume.

Note: μ_T is -ve for certain temp,
** & μ_T is +ve for rest.

⟹ This is used to condense gases, since as $\partial P\uparrow$, $\partial T\downarrow$ if μ_T = +ve.
(We put pressure to cool gases).

Carnot Cycle (Carnot Engines)

- Every engine must be a cycle



Work done,

BY ① → ② is isothermal expansion, work done by system. (Heat from bath to system)

BY ② → ③ Adiabatic expansion, temp drops and does not take heat from bath. ($\Delta Q = 0$)

ON ③ → ④ Isothermal compression, work done on the system (Heat from system to bath)

ON system, ④ → ① Adiabatic compression, temp rises and does not give out heat to bath. ($\Delta Q = 0$)

⟹ Work extracted is area within 4 curves.

Note: μ_T is -ve for certain temperatures
** & μ_T is +ve for rest.

⇒ This is used to condense gases, since as $\partial P\uparrow, \partial T\downarrow$ if $\mu_T = +ve$.
(We put pressure to cool gases)

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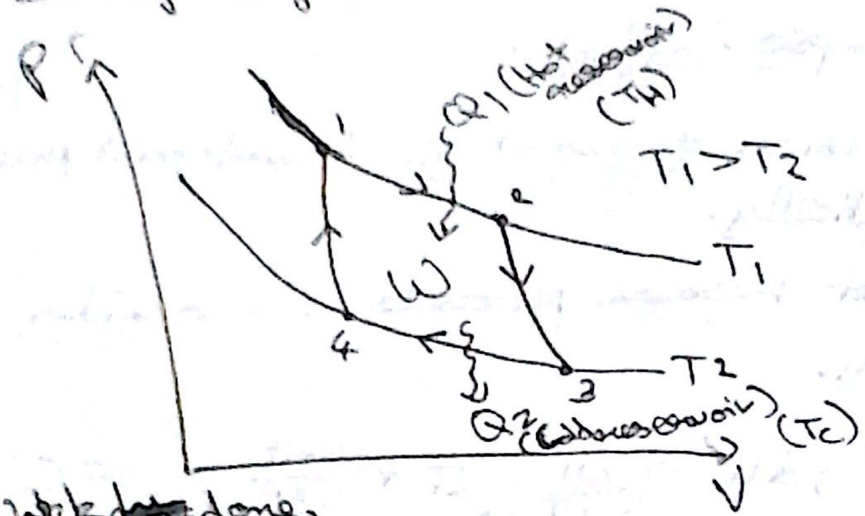

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ON system, ④ → ① Adiabatic compression, temp rises and does not give out heat to bath. ($\Delta Q = 0$)

⇒ Work extracted is area within 4 curves.

* $\boxed{\varepsilon = \left|\frac{W}{Q_1}\right|}$ } Efficiency

~~⇒~~

$\Rightarrow W_{1\to 2}^{rev} = -nRT\ln\left(\frac{V_2}{V_1}\right)$, $q_{1\to 2} = nRT_h\ln\left(\frac{V_2}{V_1}\right)$

$\because$ ~~E(T) = const~~

$\because E(T) = \text{const}$

$\Rightarrow W_{3\to 4}^{rev} = -nRT\ln\left(\frac{V_4}{V_3}\right)$, $q_{3\to 4} = nRT_c\ln\left(\frac{V_4}{V_3}\right)$

$\because E(T) = \text{const}$

- Efficiency $= \left|\frac{W}{q_{1\to 2}}\right| = \left|\frac{q_{1\to 2} + q_{3\to 4}}{q_{1\to 2}}\right|$

$$\Rightarrow \varepsilon = \left|1 + \frac{T_c \ln\frac{V_4}{V_3}}{T_h \ln\frac{V_2}{V_1}}\right| = \left|1 - \frac{T_c}{T_H}\right|$$

$$\therefore \boxed{\varepsilon = 1 - \frac{T_c}{T_h}} \rightarrow \text{Kelvin}$$

((Prove $\frac{\ln\frac{V_4}{V_3}}{\ln\frac{V_2}{V_1}} = -1$)

($\Rightarrow$ Adiabat $pV^\gamma = \text{const}$

Isotherm $pV = nRT$

(Efficiency, depends on the temperatures of outside system (Mainly the cold ~~reservoir~~ reservoir)

—— x ——

Entropy;

Also, $q_{1\to 2\to 3\to 4} = q_{1\to 2} + q_{3\to 4}$ } In above process

$$\Rightarrow \oint_{q_{1\to 2\to 3\to 4}} \frac{\delta_{rev} q}{T} = 0$$

based on this, clausius defined entropy S.

$$\boxed{\Delta S = \int_{init}^{final} \frac{d_{rev} q}{T}}$$

1) $\Rightarrow$ For adiabatic (reversible), $\Delta S = \int_i^f \frac{d_{rev} q}{T} = 0$

$\because dq = 0$

$\boxed{\Delta S = 0}$

2) $\Rightarrow$ For isothermal (reversible), $dE = 0 = \Delta q + dW$.

$\Rightarrow \Delta S = \int_i^f \frac{d_{rev} q}{T} = -\int_i^f \frac{d_{rev} W}{T}$

$\therefore \boxed{\Delta S = nR \ln\left(\frac{V_f}{V_i}\right)}$

———x———

** Note; $dS = \frac{d_{rev} q}{T} \geq \frac{dq}{T} \Rightarrow \boxed{dq \leq TdS}$

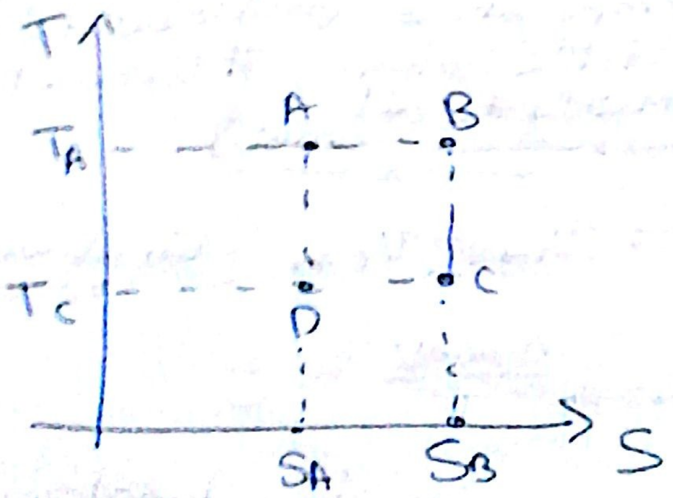
This is classius inequality

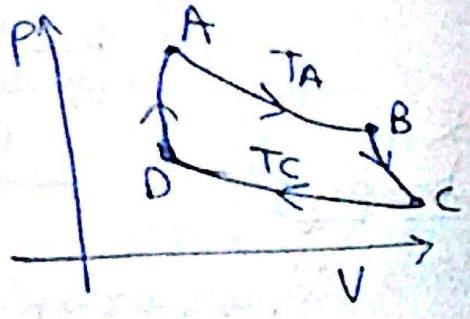
———x———

Note: $\oint dS = 0$ (Any cycle)

———x———

* Entropy in ~~[illegible]~~ carnot cycle;





• $TdS = d_{rev} q \longrightarrow S_B = S_C$ & $S_A = S_D$

$dE = dq + dW$.

⇒ $dE_{rev} = dQ + dW = TdS + dW$ (For infinitesimal small changes)

⇒ $\oint dE = \oint TdS + W$

↓

0, since integral of state function.

∴ $\oint TdS = -W$

⇒ From our plot $\oint TdS$ = area of rectangle

∴ $\oint TdS = (T_A - T_C)\times(S_B - S_A)$

⇒ $Q_{in} = T_A \times (S_B - S_A)$

$Q_{out} = T_C \times (S_B - S_A)$

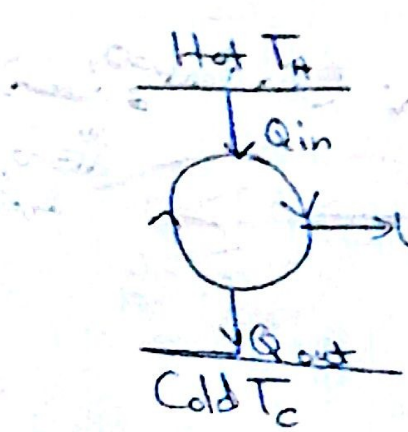
• $$\varepsilon = \left|\frac{W}{Q_{in}}\right| = \left|\frac{-(T_A - T_C)\times\cancel{(S_B - S_A)}}{T_A\cancel{(S_B - S_A)}}\right|$$

$$\boxed{\varepsilon = 1 - \frac{T_C}{T_A}}$$

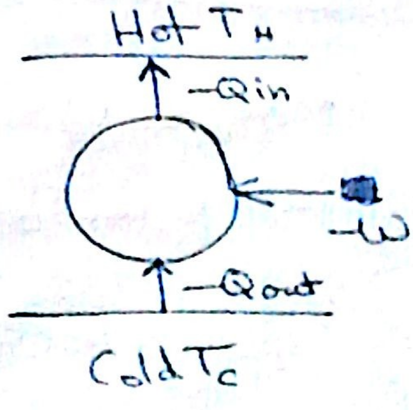
⇒ So we have shown efficiency again but this time with entropy.

—x—

Carnot Engine vs Refrigerator



$$\varepsilon = 1 - \frac{T_C}{T_H} = \left|\frac{W}{Q_{in}}\right|$$



$$e = \left|\frac{Q_{out}}{W}\right| = \frac{T_C}{T_H - T_C}$$

↑ Can be >1

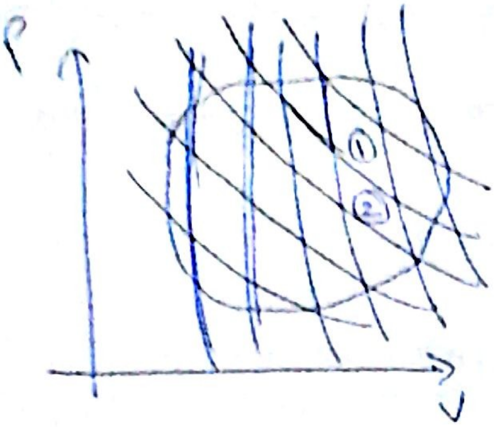
Note: Current day refrigerators are about $e \approx 7$

—x—

Note: $T_c = T_h(1-\varepsilon)$ (This was used to calculate surroundings temp, keeping T_h = Known hot bath).

—x—

* Generalizing any PV cycle;



- Consider infinitesimally small carnot cycles.

$\Rightarrow T_{h_1}, T_{c_1} = T_{h_2}, T_{c_2}$

$\therefore$ We can merge and consider these two cycles together as T_{h_1} to T_{c_2}

$\Rightarrow$ So any PV cycles can be broken down into carnot cycles.

—x—

** Clausius Equation;

- $dq \leq TdS$

At const volume $dE = dq$ (In $P_{ext}=0$, Vacuum) $dE=0$,

$\Rightarrow$ When $dq=0$, $TdS \geq 0 \longrightarrow dS \geq 0$ $\because T>0$

$\therefore \boxed{dS \geq 0}$

- (Wherever work (either expansion) or [illegible]tation etc is done, $dS \geq 0$).

* 2^{nd} Law of Thermodynamics;

a) ⟹ <u>Kelvin</u>: No process is possible in which the sole result is the absorption of heat from hot reservoir and in <u>complete</u> conversion to work.
(We can't have 100% efficiency).

b) ⟹ <u>Clausius</u>: We cannot transfer heat from cold reservoir to hot without doing any work.

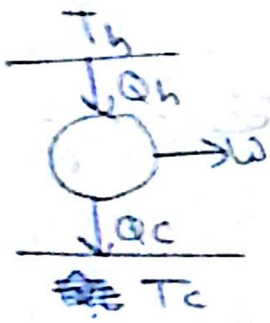
c) ⟹ <u>Entropy is additive</u>: Entropy of universe tends to a maximum $\Delta S \geq 0$.

$$\left(\Delta S = \int_i^f \frac{dQ_{rev}}{T}\right)$$

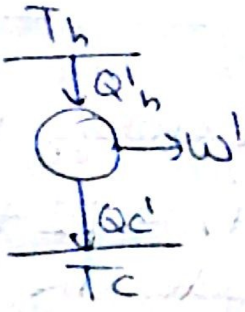
———x———

* <u>Real Systems</u> (Using 2^{nd} Law)

Q)



(I) → Ideal engine



(R) → Real system

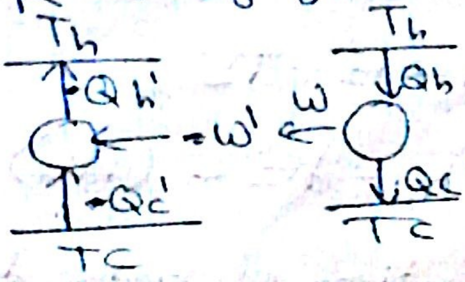
~~Disprove~~, Prove, → Ideal

$$\boxed{\left|\frac{W'}{Q'_h}\right| \not> \left|\frac{W}{Q_h}\right|}$$

Real ~~Not possible~~
Always (From 2^{nd} law)

⟹ Let us consider $R' \Rightarrow$ Refrigerator (Real)

(Scale W' & Q'_h & Q'_c, so $Q'_c = Q_c$)



$W = \left(W \times \frac{Q_c}{Q'_c}\right)$

$Q_h = \left(Q'_h \times \frac{Q_c}{Q'_c}\right)$

~~$\frac{W - W'}{Q_h - Q'_h}$~~

⟹ ~~$Q_h - Q'_h$ = Input~~, ~~$Q_c - Q'_c = 0$~~

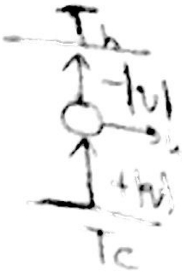
~~$W_{out} = W - W'$~~,

b) (System centric) (Prove we can't send heat from cold to hot without work)

$\Rightarrow \Delta S_{universe} = \Delta S_{hot} + \Delta S_{cold} + \underbrace{\Delta S_{system}}_{0}$ (Cycle Carnot)

$\therefore \Delta S_{universe} = \Delta S_{hot} + \Delta S_{cold}$

$= \frac{+|q|}{T_h} + \frac{-|q|}{T_c}$



$\Delta S_{univ} = |q|\left(\frac{1}{T_h} - \frac{1}{T_c}\right)$

$\Rightarrow \Delta S_{univ} < \underline{\underline{0}}$ } This should be false

$\Rightarrow$ This is against 2^{nd} Law, so such a system $W=0$ & heat from cold to hot bath is not possible!

—— x ——

* **Note**; Helmholtz free energy $(A) = E - TS$;

Enthalpy $(H) = E + PV$

$\Rightarrow$ Both are extensive

—— x ——

1) Helmholtz free energy;

$dA = dE - (TdS + SdT)$

$A = E - TS$,

$(dA)_{T,V} = dE - (TdS) = dq - TdS \underbrace{\leq 0}_{\text{From clausius inequality}}$

$\rightarrow$ Direction of natural process is such that it reduces helmholtz free energy.

#2) Gibbs free energy;

$$G = H - TS$$

$$dG = dH - TdS - SdT$$

$\Rightarrow$ For constant pressure,

$$(dG)_P = dq - TdS - SdT$$

$$(dG)_{P,T} = dq - \underbrace{TdS \leq 0}_{\text{From clausius inequality.}}$$

Natural

$\Rightarrow$ Direction of process is such that gibbs free energy is reduced.

---

*Extractable Work;

* a) Maximum work extractable $= \Delta A$ //
* b) Maximum non-expansion work $= \Delta G$ // (Battery etc)

a) $\Rightarrow dE = dq + dW \quad ; dq \leq TdS$

$$dE \leq TdS + dW$$

$dW \geq dE - TdS$ } In any real process

$\therefore$ Max work extracted $= dE - TdS = (dA)_T$

b) $dE = dq + dW = dq + \underbrace{\Delta W_{exp}}_{-P_{ext} \times dV} + \Delta W_{non\text{-}expansion}$

$dq_{rev} = TdS$

$$\Delta W_{non\text{-}exp} = dE - dq + P_{ext} \times dV$$

$\Delta W_{non\text{-}exp} = dE - dq + pdV$ } $dq \leq TdS$

$$= d(E + PV) - TdS - VdP$$

$$= dH - (TdS + SdT) - VdP + SdT$$

$$= dH - d(TS) - VdP + SdT$$

$$\Delta W_{non\text{-}exp} = dG - VdP + SdT //$$

$\rightarrow$ Const P, T, Reversible: $\Delta W_{non\text{-}exp} = dG$ //

Extensive vs Intensive, (Including material transfer) (Euler's Theorem)

- For extensive properties

$$E(T, P, \lambda V, \lambda n) = \lambda E(T, P, V, n)$$

→ Multiplying all extensive factors by λ, gives us λ times original value.

∴ Intensive = T, P
Extensive = V, n

* ⇒ In general,

$$f(\underbrace{i_1, i_2, \ldots, i_n}_{\text{Intensive}}, \underbrace{x_1, x_2, \ldots, x_m}_{\text{Extensive}}) = \frac{1}{\lambda} f(i_1, i_2, \ldots, i_n, \lambda x_1, \lambda x_2, \ldots, \lambda x_m)$$

⇒ *

$$\boxed{f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\right)_{i_1, \ldots, i_n, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m} \cdot x_j}$$

Keeping these constant

Now,

$$\because dE = dq + dW$$

$$\Rightarrow dE = TdS - pdV$$

$$\Rightarrow dG = dE + d(pV) - d(TS)$$

$$= TdS - pdV + (pdV + Vdp) - TdS - SdT$$

$$\therefore dG = VdP - SdT$$

$$* \Rightarrow \boxed{dG = VdP - SdT + \mu dn}$$

} So that homogenity is maintained even when mass is added. (Material Change)

$$dG = VdP - SdT + \sum_i \mu_i dn_i$$

* (μ = Chemical Potential)

$\Rightarrow$ Similarly, $dE = TdS - pdV + \sum_i \mu_i dn_i$

$$\boxed{dE = TdS - pdV + \mu n}$$ $\rightarrow$ Material I/O.

(we can go backwards from dG to dE)

— x —

* **Note;** $G = \mu n = \sum_i \mu_i n_i$

*

$dG = d\mu n + dn \cdot \mu$

$dG = VdP - SdT + \mu dn$

* $\therefore$ $$\boxed{d\mu \times n = VdP - SdT}$$ } Relation b/w T, P & μ changes.

— x —

— x —

* **Equilibriums** (Thermodynamic)

$\Rightarrow dE = TdS - pdV + \mu dn$

$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dn$

$\Rightarrow$ Consider initially, (2 systems are seperated)

(Adiababic initiall)

E_1 P_1 V_1 μ_1 T_1	E_2 P_2 V_2 μ_2 T_2

I) * * Now remove the seperation (But still make sure no material passes)

* $dE_1 = -dE_2$ (rest can change)

$dV_1 = -dV_2$ (n_1, n_2 are constant)

$S_{univ} = S_1 + S_2$

$dS_{univ} = dS_1 + dS_2$

$dS_{univ} = \left(\frac{1}{T_1}dE_1 + \frac{P_1}{T_1}dV_1 - \frac{\mu_1}{T_1}\times 0\right)$

$+ \left(\frac{1}{T_2}dE_2 + \frac{P_2}{T_2}dV_2 - \frac{\mu_2}{T_2}\times 0\right)$

$\Rightarrow$ From 2^{nd} law, $dS_{univ} \geq 0$ (Natural processes)

* $\Rightarrow dS_{univ} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)dE_1 + \left(\frac{P_1}{T_1} - \frac{P_2}{T_2}\right)dV_1 \geq 0$

a) $\Rightarrow \left(\frac{1}{T_1} - \frac{1}{T_2}\right) dE_1 \geq 0$ for every natural change

i) $dE_1 > 0$

$\Rightarrow \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \geq 0$

$\Rightarrow T_2 > T_1$

(Energy flows from 2 (system) to 1 (system))

ii) $dE_1 < 0$

$\Rightarrow \left(\frac{1}{T_1} - \frac{1}{T_2}\right) < 0$

$\Rightarrow T_2 < T_1$

(Energy flows from system 1 to system 2)

$\Rightarrow$ This talks about directionality of flow of heat.

Now,

b) $\Rightarrow \left(\frac{P_1}{T_1} - \frac{P_2}{T_2}\right) dV_1 \geq 0$ for every natural change

i) $dV_1 > 0$

$\Rightarrow \left(\frac{P_1}{T_1} - \frac{P_2}{T_2}\right) > 0$

$\Rightarrow P_1 T_2 > P_2 T_1$

ii) $dV_1 < 0$

$\Rightarrow \left(\frac{P_1}{T_1} - \frac{P_2}{T_2}\right) < 0$

$\Rightarrow P_1 T_2 < P_2 T_1$

$\Rightarrow$ This talks about how volume changes takes place.

$\Rightarrow$ If $\frac{P_1}{T_1} > \frac{P_2}{T_2}$, then $V_1 \uparrow$, and vice versa

c) Similarly we can show direction of flow for chemical potential ~~concentrations~~. (Using a & b)

——— x ———

* Maxwell Relations;

- Now, let us consider $E \equiv E(S, V, \{n\})$

$$dE = TdS - pdV + \sum_{i=1}^{n} \mu_i dn_i \rightarrow ①$$

$$* \quad T = \left(\frac{\partial E}{\partial S}\right)_{V, \{n\}}$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V, \{n\}}$$

$$\Rightarrow \Delta S_V = \int_1^2 \frac{dE}{T} = \int_1^2 \frac{dq}{T}$$

From ①,

$$\Rightarrow -p = \left(\frac{\partial E}{\partial V}\right)_{S, \{n\}}, \quad \mu_i = \left(\frac{\partial E}{\partial x_i}\right)_{\{V, S\}, \{n\}_{j \neq i}}$$

$\Rightarrow$ Considering $\{n\}$ as constant,

$$E(S, V) \equiv E = TdS - pdV$$

- $$\frac{\partial^2 E}{\partial S \partial V} = \left(\frac{\partial \left(\frac{\partial E}{\partial V}\right)_S}{\partial S}\right)_V = \left(\frac{\partial}{\partial S}\left(\frac{\partial E}{\partial V}\right)_S\right)_V \rightarrow ①$$

- $$\frac{\partial^2 E}{\partial V \partial S} = \left(\frac{\partial}{\partial V}\left(\frac{\partial E}{\partial S}\right)_V\right)_S \rightarrow ②$$

$\Rightarrow$ From maxwell relation, ① & ② are equal.

$$\Rightarrow \frac{\partial^2 E}{\partial S \partial V} = \left(\frac{\partial}{\partial S}(-p)\right)_V \rightarrow ③$$

$$\Rightarrow \frac{\partial^2 E}{\partial V \partial S} = \left(\frac{\partial T}{\partial V}\right)_S \rightarrow ④$$

$\Rightarrow$ So ③ = ④, $\therefore$ $\boxed{-\left(\frac{\partial S}{\partial P}\right)_V = \left(\frac{\partial V}{\partial T}\right)_S}$

One changes P, other changes T

So we can see that we can relate 2 process from same starting point.

Now, consider,

$$\Rightarrow H = E + PV$$

$$dH = dE + d(PV)$$

$$= TdS - \cancel{PdV} + \sum_{i=1}^{n} \mu_i n_i + \cancel{PdV} + Vd\text{P}$$

$$dH = TdS + VdP + \sum_{i=1}^{n} \mu_i n_i$$

$$\Rightarrow A = E - TS$$

$$dA = TdS - PdV + \sum_{i=0}^{n} \mu_i n_i - TdS - SdT$$

$$\Rightarrow dA = -SdT - PdV + \sum_{i=1}^{n} \mu_i n_i$$

$\therefore$ $E(S, V, \{n\})$, $H(S, P, \{n\})$, $A(T, V, \{n\})$ — These 3 can be used interchangeably as and when we need th

** Legendre Transformations

— x —

* Comparing Statistical Mech to Therm

1) $S = -k_B \sum_i p_i \ln(p_i)$ (Minimize Information)

missing info function

Const N, V, T

$$p_i = \frac{e^{-\beta E_i}}{Z(T,V)}$$

$$, S = +k_B \sum_i p_i [+\beta E_i + \ln Z]$$

$$= +k_B \beta \sum_i p_i E_i + k_B \ln Z \sum p_i$$

2) ~~$dS = \frac{1}{T} dE$~~

$$\therefore S = k_B \beta \bar{E} + k_B \ln Z$$

$\Rightarrow d\mathcal{S} = k_B \beta \, d\bar{E} + 0$ $\quad$ ($\because Z(T,V)$ here $T, V = $ const)

P

2) $dS = \frac{1}{T} dE + \frac{P}{T} dV = \frac{1}{T} dE \quad \because dV = 0$

$\Rightarrow$ So we can see that

$d\mathcal{S}$ & dS are both related to $d\bar{E}$,..

so we can assume, (Comparing $\mathcal{S}$ & S)

$\Rightarrow \mathcal{S} = S + k_1 \quad , k_B \beta = \frac{1}{T}$

$$\boxed{\beta = \frac{1}{k_B T}}$$

$\therefore$ We have proved that $\beta = \frac{1}{k_B T}$ & that missing info function is nothing but the entropy.

———x———

Quantum Mechanics

*Standing Waves; (2D)

For a certain wavelen / frequency, there's only one possible (one m

$$\boxed{\frac{n\lambda}{2} = L}, \quad \lambda = \text{Wavelength}$$

$$\Rightarrow A(x,t) = A\sin(kx)$$

$$k = 2\pi/\lambda. \rightarrow (1)$$

$$\Rightarrow A(x,t) = A_x(t) \times A_t(x) = A(t)\cdot \sin(kx)$$

$$\because \sin(kL) = 0, \quad kL = n\pi$$

$$k = n\pi/L \longrightarrow (2)$$

- We can see any arbitary wavelength is not possible.

—x—

Note: Maxwell ~~Law~~ Law $\rightarrow$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^3}\right)E = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E$$

E $\uparrow$ Electric field

$$\Rightarrow E(\bar{x},t) = E_0(t)\cdot E_1(x)\cdot E_2(y)\cdot E_3(z)$$

$$E_1(x) = c_1 \sin(k_1 x)$$

$$E_2(x) = c_2 \sin k_2(x)$$

$$E_3(x) = c_3 \sin k_3(x)$$

—x—

Standing Waves: (3D) (Planck's Theory)

• Consider a box, (Black body radiating)

$L = L_x = L_y = L_z$

$E_1(x) = 0$, $E_2(y) = 0$, $E_3(z) = 0$

when $x = L_x$, $y = L_y$, $z = L_z$.

• Using maxwells law,

$$\bar{E} = E_t \cdot E_x \cdot E_y \cdot E_z$$

$$\Rightarrow \frac{\partial E}{\partial x} = E_t \cdot \frac{\partial E_x}{\partial x} \cdot E_y \cdot E_z$$

$$\frac{\partial^2 E}{\partial x^2} = E_t \frac{\partial^2 E_x}{\partial x^2} \cdot E_y \cdot E_z \rightarrow (1)$$

$\Rightarrow$ Putting these (1)'s in Maxwell's, eqn,

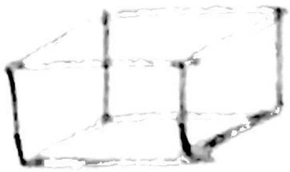
$$\left(\frac{1}{E_1}\left(\frac{\partial^2 E_1}{\partial x^2}\right) + \frac{1}{E_2}\left(\frac{\partial^2 E_2}{\partial y^2}\right) + \frac{1}{E_3}\left(\frac{\partial^2 E_3}{\partial z^2}\right)\right) \cdot E_0 E_1 E_2 E_3 = \frac{1}{c^2} \frac{\cancel{E_0 E_1 E_2 E_3}}{E_0} \frac{\partial^2 E_0}{\partial t^2}$$

$$\Rightarrow \underbrace{\frac{1}{E_1}\frac{\partial^2}{\partial x^2}E_1}_{f(x) \downarrow c_1} + \underbrace{\frac{1}{E_2}\frac{\partial^2 E_2}{\partial y^2}}_{g(y) \downarrow c_2} + \underbrace{\frac{1}{E_3}\frac{\partial^2 E_3}{\partial z^2}}_{h(z) \downarrow c_3} = \underbrace{\frac{1}{c^2} \times \frac{1}{E_0} \times \frac{\partial^2 E_0}{\partial t^2}}_{l(t) \downarrow c_4}$$

• This equality is satisfied only if all are constant. (Since if we change only 1 value, still the equation must hold).

* Standing Waves: (3D) (Planck's Theory)

* Consider a box, (Black body radiation)



$L = L_x = L_y = L_z$

$E_1(x) = 0,\ E_2(y) = 0,\ E_3(z)$

When $x = L_x$
$y = L_y$
$z = L_z$

- Using maxwells law,

$$E = E_t \cdot E_x \cdot E_y \cdot E_z$$

$$\Rightarrow \frac{\partial E}{\partial x} = E_t \cdot \frac{\partial E_x}{\partial x} \cdot E_y \cdot E_z$$

$$\frac{\partial^2 E}{\partial x^2} = E_t \frac{\partial^2 E_x}{\partial x^2} \cdot E_y \cdot E_z \rightarrow ①$$

$\rightarrow$ Putting these ①'s in Maxwells eqn,

$$\left(\frac{1}{E_1}\left(\frac{\partial^2 E_1}{\partial x^2}\right) + \frac{1}{E_2}\left(\frac{\partial^2 E_2}{\partial y^2}\right) + \frac{1}{E_3}\left(\frac{\partial^2 E_3}{\partial z^2}\right)\right) \cdot \cancel{E_0 E_1 E_2 E_3}$$

$$= \frac{1}{c^2} \frac{\cancel{E_0 E_1 E_2 E_3}}{E_0} \frac{\partial^2 E_0}{\partial t^2}$$

$$\Rightarrow \underbrace{\frac{1}{E_1}\frac{\partial^2}{\partial x^2}E_1}_{f(x)} + \underbrace{\frac{1}{E_2}\frac{\partial^2 E_2}{\partial y^2}}_{g(y)} + \underbrace{\frac{1}{E_3}\frac{\partial^2 E_3}{\partial z^2}}_{h(z)} = \underbrace{\frac{1}{c^2}\times\frac{1}{E_0}\frac{\partial^2 E_0}{\partial t^2}}_{l(t)}$$

$f(x) \downarrow c_1$, $g(y) \downarrow c_2$, $h(z) \downarrow c_3$, $l(t) \downarrow c_4$

- This equality is satisfied only if all are constant. (Since if we change only 1 value, still the equation must hold).

$\Rightarrow \frac{1}{E_1}\frac{\partial^2 E_1}{\partial x^2} = c_1$

- We see that sinusoids satisfy (experimentally)

$\Rightarrow E_1 = A_1 \sin(k_1 x)$, $c_1 = -k_1^2$

$E_2 = A_2 \sin(k_2 y)$, $c_2 = -k_2^2$

$E_3 = A_3 \sin(k_3 z)$, $c_3 = -k_3^2$

$E_4 = A_4 \sin(k_4 t)$, $c_4 = -c^2 k_4^2$

$\therefore E = A_1 \sin(k_1 x) \cdot A_2 \sin(k_2 x) \cdot \ldots\ldots$

$\Rightarrow$ Putting c_1, c_2, c_3, c_4 in maxwell's eqn,

$$(k_1^2 + k_2^2 + k_3^2) = k_4^2 \longrightarrow (3)$$

- Now, $\sin(k_1 x) = 0$ when $x = 0$ or $x = L_x$ (Boundaries) (Energy = 0)

$\Rightarrow \sin(k_1 L) = 0$

$\therefore k_1 L = n\pi$, $k_1 = \frac{n_1 \pi}{L}$

Similarly, $k_2 = \frac{n_2 \pi}{L}$, $k_3 = \frac{n_3 \pi}{L}$

Putting these in (3),

$$(n_1^2 + n_2^2 + n_3^2)\frac{\pi^2}{L^2} = k_4^2 = (2\pi\nu)^2 \rightarrow (4)$$

(n_1, n_2, n_3 are integers)

Prove yourself

$\nu\lambda = c$

$\Rightarrow$ If we have a standing wave of frequency ν, it has many possible pairs of (n_1, n_2, n_3) which satisfy (4).

$\therefore$ Many modes satisfy a certain freq.

$$n_1^2 + n_2^2 + n_3^2 = \left(\frac{L^2 (2\pi)^2}{\pi^2}\right) \gamma^2$$

$$\boxed{R = 2L\gamma} \rightarrow \text{radius}$$

$\Rightarrow$ Volume of this shell $= \frac{1}{8} \times 4\pi R^2 dR$

$$V = \frac{1}{8} \times 4\pi \times 4L^2\gamma^2 \times 2L d\gamma$$

$$V = 2\pi L^3 \gamma^2 d\gamma \rightarrow 5$$

** • The number of modes between γ & $\gamma + d\gamma$ $\propto L^3$

$\Rightarrow$ Boltzman said, $\boxed{P_i \propto e^{-E_i/kT}}$

He proposed,

$\boxed{E_i = nh\gamma}$

and he said E_i is not a continuous variable.

Since you get infinite energy in this way.

$$* \langle E \rangle = \frac{\sum_i \left(e^{-\beta E_i} \cdot E_i\right)}{\sum_i e^{-\beta E_i}}$$

$$= \frac{\sum_{n=0}^{\infty} \left(e^{-\beta n h\gamma} \cdot nh\gamma\right)}{\sum_{n=0}^{\infty} e^{-\beta n h\gamma}} = \frac{\sum_{n=0}^{\infty} \left(e^{-\beta n h\gamma} \cdot nh\gamma\right)}{\frac{1}{1 - e^{-\beta h\gamma}}}$$

• On solving it,

Average of a single ~~all~~ modes energy.

$$\boxed{\langle E \rangle = \frac{h\gamma}{e^{h\gamma/kt} - 1}} \rightarrow 6$$

Number of modes from 5

$$\therefore E_{\text{Black-body}} \propto (\gamma^2 d\gamma) \times \frac{h\gamma}{e^{h\gamma/kt} - 1} \propto \frac{\gamma^3 d\gamma}{e^{h\gamma/kt} - 1}$$

Energy of all modes combined

~~$E = nh\gamma$ is true~~

$$\therefore E_{\text{black-body}} \propto \frac{\nu^3 d\nu}{e^{\beta h\nu} - 1}$$

- This result satisfied experimental data.

* So $\boxed{E = nh\nu}$ is true.

—— x ——

* Planck Radiation Formula:

$$u(\nu)d\nu = \frac{8\pi \nu^3 d\nu}{c^3(e^{\beta h\nu} - 1)}$$

?

—— x ——

* X-Ray diffraction:

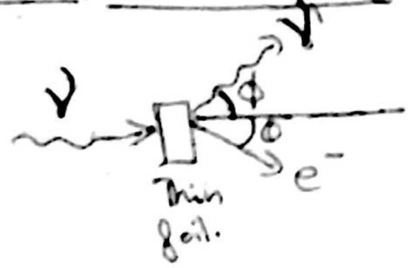
- Similar to YDSE but with ∞ holes, used to get info on the arrangement of atoms in a crystal.

$\Rightarrow$ Diffraction pattern depends on structure of crystal.

—— x ——

* Compton effect:

$\Rightarrow$



- Energy is conserved.
- $E^2 = p^2c^2 + (mc^2)^2$

$$\therefore h\nu = h\nu' + \underbrace{K.E}_{\text{Relativistic}} \rightarrow (4)$$

$(\gamma - 1)mc^2 \rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Photon: $|p| = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \rightarrow (1)$

→ Now momentum horizontal.

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cdot \cos\phi + |P_e| \cdot \cos\theta \rightarrow ②$$

$\Rightarrow |P_e| = \gamma m_e |V_e|$

→ Now conservation of momentum in vertical,

$$0 = \frac{h}{\lambda'} \sin\phi - |P_e| \sin\theta \rightarrow ③$$

** $\Rightarrow \therefore \boxed{\lambda' - \lambda = \frac{h}{mc}(1-\cos\phi)}$ from ①, ②, ③, ④

— x —

* De-Broglie;

⇒ Matter has waves associated with it & its wavelength $= \lambda = h/p$, $E = h\nu$

— x —

*** Born Postulate;

- Suppose we have a travelling wave, Height $\Psi(x,t)$ at a point x & at time t.
- He said $\boxed{P(x,t) = |\Psi(x,t)|^2}$

→ Ψ is a complex function

⇒ Suppose we have ~~the~~ these 2 ~~waves~~ travelling wave,

Wave1 $= A \cos(kx - \omega t)$

Wave 2 $= A \cos(k'x - \omega' t)$

⇒ If they ~~interfere~~ superimpose,

$$A_{tot} = A\cos(kx - \omega t) + A\cos(k'x - \omega' t)$$

$$\Rightarrow A_{TOT} = A\cos\left[\frac{1}{2}(k+k')x - \frac{1}{2}(\omega+\omega')t\right]$$

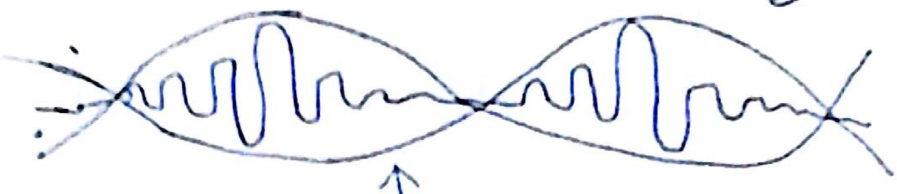
$(\omega' \sim \omega \;\&\; k' \sim k)$

$(k - k' = \Delta k$, $\omega - \omega' = \Delta\omega)$

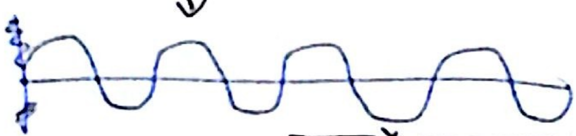
$$\times \cos\left[\frac{1}{2}\Delta k x - \frac{1}{2}\Delta\omega t\right]$$

$$A_{TOT} = A\cos(kx - \omega t) \times \underbrace{\cos\left(\frac{1}{2}\Delta k x - \frac{1}{2}\Delta\omega t\right)}_{\text{envelope}}$$

$\therefore$ Envelope is as such:



$\therefore \cos(kx - \omega t)$:

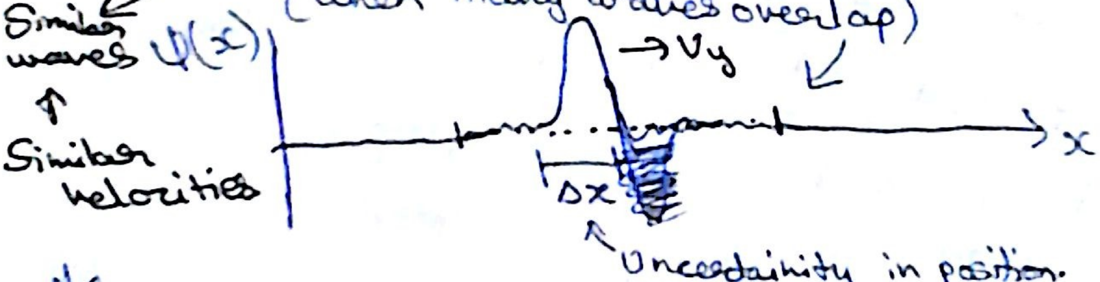


—x—

* Now, when many waves overlap we get a signal.

$\therefore$ When they overlap, (Highly localized wave)

(when many waves overlap)



$$V_g = \frac{\Delta\omega/2}{\Delta k/2}$$

$$V_g = \frac{\Delta\omega}{\Delta k}$$

** $\therefore$ Superposition of many waves generate a highly localized wave.

—x—

(Arthur Beizer Chapter 3)
(Modern Physics)

—x—

** ** $\Rightarrow \boxed{\Delta x \cdot \Delta k \geq 1/2}$

** $\boxed{\hbar = h/2\pi}$

—x—

De-Broglie Postulates: **

1) $\lambda = \frac{h}{p}$, $E = h\nu$

2) $k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = \frac{p}{\hbar}$, $\hbar = \frac{h}{2\pi}$

3) $\omega = 2\pi\nu = E/\hbar$ } Angular frequency

— x —

**Einstein: ** **

→ Now $E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$; $p = \gamma mv$

$$\omega = \frac{1}{\hbar}\gamma mc^2 \rightarrow \text{From } 3$$

$$\Rightarrow k = \frac{p}{\hbar} = \frac{\gamma mv}{\hbar}$$

— x —

∴ We got k & ω for relativistic particles using debroglie's postulates.

— x —

Now, $\boxed{V_p = \frac{\omega}{k} = \frac{c^2}{V}}$ → We will show that v = group velocity of waves (group of waves represent the particle).

** ⇒ ∵ $\boxed{V_g = \frac{d\omega}{dk}}$ } Group velocity

$$\Rightarrow \omega = \frac{1}{\hbar}\gamma mc^2 \ \& \ k = \frac{\gamma mv}{\hbar}$$

$$\Rightarrow V_g = \frac{d\omega/dv}{dk/dv} = \frac{c^2\, d\gamma/dv}{\frac{d}{dv}(\gamma v)} \quad , \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow V_g = c^2 \times \frac{\cancel{2}v}{c^2} \times \frac{1}{(1-v^2/c^2)^{3/2}} \Big/ \cancel{\frac{2}{2}}\left(\frac{c^2 \times}{} \times \frac{1}{(1-v^2/c^2)}\right.$$

$\therefore V_g = V$, So proved.

---

* Uncertainity;

$\Rightarrow \because \Delta k \cdot \Delta x \geq 1/2$

$$\Rightarrow \frac{\Delta p}{\hbar} \cdot \Delta x \geq 1/2$$

$$* \Rightarrow \boxed{\Delta p \cdot \Delta x \geq \frac{1}{2}\hbar}$$, $\hbar$ is very small $= h/2\pi$

- Since $\hbar$ is very small, in classical mechanics we dont really care much about this uncertainity.

$\Rightarrow$ But in quantum mech it matters quite a bit.

---

* Bohr's Model;

- He proposed electrons move around the nucleus as waves & they constructively interfere.

$\Rightarrow \underbrace{L = n\lambda}_{\text{Closed waves}}$ (Unlike $L = \frac{n\lambda}{2}$ in normal waves with 2 end points

$\Rightarrow 2\pi R = n\lambda$ } Radius of electron orbit.

- Using this & centrifugal forces, he showed $\boxed{E \propto \frac{-1}{n^2}} \rightarrow$ Energy in n^{th} orbit (electron)

$$\boxed{E = -13.6\,eV \times \frac{1}{n^2}}$$

*** Schrodinger (Slides for detailed proof).

* Pauli's Arguments;

$$\Rightarrow \nabla^2 \Psi = \frac{1}{V^2}\frac{\partial^2 \Psi}{\partial t^2}$$ } For any wave with velocity v.

1) $V = \lambda \nu = h/p \cdot \frac{E}{h} = E/p$ $\quad (\because \lambda = h/p)$

2) $\nabla^2 \Psi = \frac{p^2}{E^2}\frac{\partial^2}{\partial t^2}\Psi$

3) $E^2 = p^2c^2 + m^2c^4 \longrightarrow$ If moving in a potential, $E = \underbrace{\sqrt{p^2c^2+m^2c^4}}_{K.E} + \underbrace{\text{Potential }(V)}_{\underbrace{V(x)}_{\text{Potential}}}$

$\downarrow$

$(E-V)^2 = p^2c^2 + m^2c^4$

4) $\nabla^2\Psi = \left(\frac{E^2 - m^2c^4}{c^2E^2}\right)\frac{\partial^2 \Psi}{\partial t^2}$

$\downarrow$

$$\boxed{\nabla^2\psi = \left(\frac{(E-V)^2 - m^2c^4}{c^2E^2}\right)\cdot\frac{\partial^2\psi}{\partial t^2}} \rightarrow 1$$

**

5) $\Rightarrow \boxed{\psi(\bar{r}, t) = \psi(\bar{r}) \cdot e^{i\left(\frac{E}{\hbar}\right)t}}$ } $\rightarrow$ 2 For any stationary wave.

* Putting 2 in 1,

$$\therefore e^{i\frac{Et}{\hbar}}\nabla^2\psi = \left(\frac{(E-V)^2 - m^2c^4}{c^2E^2}\right)\psi(r)\cdot\left(\frac{iE}{\hbar}\right)^2 \cdot e^{i\frac{Et}{\hbar}}$$

$$\Rightarrow \boxed{\nabla^2\psi = \frac{-1}{\hbar^2}\left[\frac{(E-V)^2 - m^2c^4}{c^2}\right]\psi(r)}$$

$\Rightarrow E = \sqrt{p^2c^2 + m^2c^4} + V$, On considering $\frac{v}{c} \approx 0$ & taylor expand

$\Rightarrow E = \frac{1}{2}\frac{p^2}{m} + mc^2 + V$

$$\therefore \nabla^2 \psi = -\frac{1}{\hbar^2}\left[(2m(E-V))\psi(\bar{r})\right]$$

$$\text{***} \therefore \boxed{-\frac{\hbar^2}{2m}\nabla^2\psi = (E - V(x))\cdot\psi(\bar{r})}$$

Schrodinger's Equation (For non relativistic case).

—— x ——

*

Note: Time independent schrodinger eqn:
***

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(\bar{r}) + V(\bar{r})\cdot\psi(\bar{r}) = E\psi(\bar{r})$$

Can explain all sciences! (chem & bio)

(For phys we use time dependent schr eqn)

operators

An eigen value problem.

$\hat{H}\psi(r) = E\psi(r)$

—— x ——

$$\Rightarrow \int d\bar{r}\cdot|\psi(\bar{r})| = \text{finite.}$$

—— x ——

$\Rightarrow$ Prob electron is present at position $\bar{r}$ at time t,

$$P(\bar{r}, t) = |\psi(r,t)|^2 = \left|e^{\frac{iEt}{\hbar}}\psi(\bar{r})\right|^2 = |\psi(\bar{r})|^2$$

$$\text{***} \boxed{P(\bar{r}, t) = |\psi(\bar{r})|^2}$$

*Note: $\psi(x=0)=0$
$\psi(x=L)=0$ } Since we want continuity (And potential at $x<0$ & $x>L$ is ∞)

$\underbrace{\qquad\qquad}$
Old quantum theory.

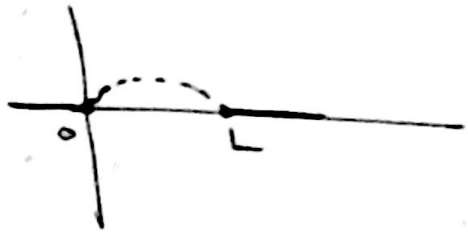
Note:
$\Rightarrow E=\frac{n^2h^2}{8mL^2}$ } Oscillating waves

---

*I) Old Quantum Theory: (Basic waves) (1D)

* • $\psi(x=0)$ & $\psi(x=L)=0$

$$E=\frac{n^2h^2}{8mL^2}$$



$\Rightarrow$ Potential $=0$ in $0\leq x\leq L$,

$$\therefore \frac{\partial^2}{\partial x^2}\psi = -\frac{2mE}{\hbar^2}\psi(x) \quad , \longrightarrow (1)$$

$\Rightarrow \psi(x) = A\sin kx + B\cos kx$ } General solutions are of this form

$\therefore$ LHS $\Rightarrow -Ak^2\sin kx - Bk^2\cos kx$

RHS $\Rightarrow -\frac{2mE}{\hbar^2}\left[A\sin kx + B\cos kx\right]$

$\rightarrow \psi(x=0)=0 \Rightarrow A\sin 0 + B\cos 0 = 0$
$\Rightarrow B=0$ //

$\psi(x=L)=0 \Rightarrow A\sin kL + B\cos kL = 0$
$\Rightarrow A\sin kL = 0 \quad (\because B=0)$

$\therefore kL = n\pi$, $n=0,1,2,\ldots$
(This is true)

$\therefore$ We have showed, $\boxed{\psi(x) = A\sin\left(\frac{n\pi}{L}\right)x}$

∴ The boundary conditions ruled out cos functions in $\psi(x)$, so only those sine functions at nodes $x=0$ & $x=L$ are allowed.

∴ From ①, LHS = RHS

$$-A k^2 \sin kx = \frac{-2mE}{\hbar^2} A \sin kx \quad , 0 \leq x \leq L$$

$$\therefore k^2 = \frac{2mE}{\hbar^2}$$

$$\rightarrow k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\therefore \psi(x) = A \sin(kx) \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}}$$

or $k = \frac{n\pi}{L}$

$$\therefore \frac{n\pi}{L} = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\Rightarrow E = \frac{\hbar^2}{2m} \cdot \frac{n^2\pi^2}{L^2} = \frac{n^2 h^2}{8mL^2}$$

$$** \therefore \boxed{E = \frac{n^2h^2}{8mL^2}}$$

⟹ Now we know total probability = 1.
(For a 1 dimensional problem)

$$\rightarrow \int_{-\infty}^{\infty} dx \, |\psi(x)|^2 = 1$$

$$\rightarrow \psi(x) = \begin{cases} 0 & , x<0 \\ A \sin kx & , 0 \leq x \leq L \\ 0 & , x>L \end{cases}$$

$$\therefore \int_0^L dx \, (A^2 \sin^2 kx) = 1$$

$$\Rightarrow A^2 \int_0^L \sin^2 kx \cdot dx = 1$$

$$\Rightarrow A^2 \int_0^L \frac{(1-\cos 2kx)}{2}\, dx$$

$$= A^2 \left(\frac{x}{2} - \frac{\sin 2kx}{4k} \right]_0^L$$

$$= A^2 \left(\frac{L}{2} - \frac{\sin 2kL}{4k} \right) = 1$$

$$\Rightarrow \frac{A^2 L}{2} - \frac{A^2 \sin(2n\pi)}{4k} = 1$$

$$\Rightarrow \frac{A^2 L}{2} - 0 = 1$$

$$\Rightarrow A^2 L = 2$$

$$A = \sqrt{2/L}$$

---x---

** Note: In classical mech, particles cannot cross potential barriers if their T.E < Barrier but in quantum mech they can with some probability.

---x---

* Time dependent S.E ;

- Note: T.I.S.E $\Rightarrow \hat{H}\psi(\bar{r}) = E\psi(r)$

$$\hat{H} = \left[\frac{-\hbar^2}{2m} \nabla^2 + V(\bar{r}) \right]$$

- Time-dependent;

$$\boxed{\frac{\hbar}{j} \frac{\partial}{\partial t} \psi(r,t) = \hat{H}\psi(\bar{r},t)}$$

(Linear homogeneous differential equation).

$\Rightarrow$ Suppose solutions for this are, ψ_1 & ψ_2, then any linear combination is also a solution

$$\therefore \psi = a\psi_1 + b\psi_2$$

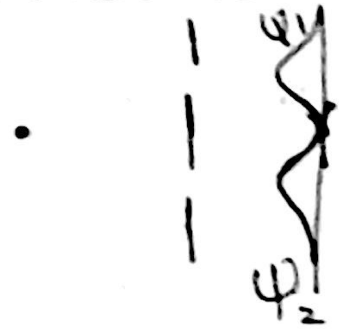
We know, $\dfrac{\hbar}{i}\dfrac{\partial \psi_1}{\partial t} = \hat{H}\psi_1$, $\dfrac{\hbar}{i}\dfrac{\partial \psi_2}{\partial t} = \hat{H}\psi_2$

Lets show,

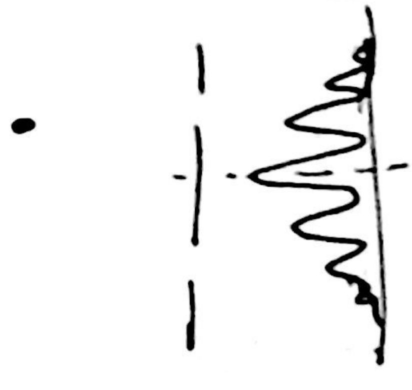
$$\Rightarrow \frac{\hbar}{i}\left[a\frac{\partial \psi_1}{\partial t} + b\frac{\partial \psi_2}{\partial t}\right] = a\hat{H}\psi_1 + b\hat{H}\psi_2$$
$$= \hat{H}(a\psi_1 + b\psi_2)$$

$\therefore$ We can see $\psi = a\psi_1 + b\psi_2$ is a valid solution.

$\Rightarrow$ Now lets consider 2 slits;



When only 1 slit is open at a time



When Both slits are open.

$$\psi_{12} = a\psi_1 + b\psi_2$$

$\Rightarrow$ Now $P_1 = |\psi_1|^2$

$P_2 = |\psi_2|^2$, $P_{1,2}(x) = |\psi_{1,2}(x)|^2$

$$= |a\psi_1 + b\psi_2|^2$$

=

$$\rightarrow P_{1,2}(r) = |a|^2 \psi_1^2 + |b|^2 \psi_2^2 + ab^* \psi_1 \psi_2^* + a^* b \psi_1^* \psi_2$$

$$P_{1,2}(r) = \underbrace{a^2 |\psi_1|^2}_{\text{Related to } P_1} + \underbrace{b^2 |\psi_2|^2}_{\text{Related to } P_2} + \text{Terms related to interference of } \psi_1 \text{ \& } \psi_2$$

($b^*, a^* \rightarrow$ Complex conjugates)

$\therefore$ This is proof that they do interfere

———x———

$\cancel{\Rightarrow \hat{H} =}$

Potential / Kinetic energy Operators;

- $\hat{K} = \frac{-\hbar^2}{2m} \nabla^2$ } K.E operator where $\hat{H} = \frac{-\hbar^2}{2m} \nabla^2 + V(\bar{r})$
- P.E $\Rightarrow V(\bar{r})$

$\cancel{\Rightarrow \hat{p} = \hbar\nabla}$

$\Rightarrow \hat{p} = i\hbar\nabla$ } Momentum operator

$\Rightarrow$ Energy operator $\Rightarrow \hat{H}$ or $i\hbar \frac{\partial}{\partial t}$

$\Rightarrow \langle \hat{O} \rangle$ } Expected value of operator,

$$\Rightarrow \boxed{\langle \hat{O} \rangle = \frac{\int dx\, \psi^*(x)\, \hat{O}\, \psi(x)}{\int dx\, \psi^*(x)\, \psi(x)}}$$

———x———

ex: $\hat{p}(A\sin kx) = i\hbar \frac{d}{dx}(A\sin kx) = A i\hbar k\cos kx$

$\rightarrow$ We get $\langle \hat{p} \rangle = 0$

$$\langle \hat{p}^2 \rangle = \frac{\int dx \sin kx \cdot \hat{p}^2 (\sin kx) dx}{\int dx \cdot \sin kx \cdot \sin kx}$$

$$\Rightarrow \hat{p}^2(\sin kx) = \hbar^2 k^2 \sin kx$$

$$\therefore \langle \hat{p}^2 \rangle = \frac{\int dx \cdot \sin kx \cdot \hbar^2 k^2 \sin kx}{\int dx \cdot \sin^2 kx}$$

$$\boxed{\langle \hat{p}^2 \rangle = \hbar^2 k^2} \quad , \quad \boxed{\langle \hat{p} \rangle = 0}$$

$$\hbar^2 \times \left(\frac{n\pi}{L}\right)^2 = \frac{\hbar^2 \pi^2 n^2}{L^2}$$

—— x ——

Note: Momentum, position, $\hat{H}$, Energy operators are needed for this course.

—— x ——

((Chapter → 5 of arthur Beizer))

—— x ——

(1D oscillator)

Note: $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) \quad , \quad 0 \leq x \leq L$

$\psi_n(x) = 0 \quad , \quad x > L \text{ or } x < 0$

—— x ——

**

* Free Particle: (1-dimension)

- T.D.S.E

$$\hat{H} = -\frac{\hbar^2}{2m} \cdot \frac{d^2}{dx^2} + 0 \rightarrow \text{No Potential}$$

$$\Rightarrow \hat{H}\psi = \frac{\hbar^2}{2m} \frac{d^2}{dx^2} (\underbrace{\psi(x,t)}_{\text{or } \psi(x) \text{ in T.I.S.E}})$$

$$\Rightarrow \hat{H}\psi = i\frac{\partial \psi}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x} = i\frac{\partial \psi}{\partial t} \quad , \quad \text{Suppose } \boxed{\psi = \psi(x) \cdot \ldots}$$

$$\Rightarrow \frac{-\hbar^2}{2m}\psi''\cdot X = c\cdot\psi\cdot X'$$

$$\Rightarrow \frac{-\hbar^2}{2m}\cdot\frac{1}{\psi}\frac{d^2}{dx^2}\psi = c\cdot\frac{1}{X}\cdot\frac{\partial X}{\partial t}$$

So solutions are of the form,

$$* \quad \psi = Ae^{ikx} + Be^{-ikx}$$

$$\psi'' = A(ik)^2\cdot e^{ikx} + B(-ik)^2 e^{-ikx}$$

$$= -k^2\psi(x)$$

$$\Rightarrow k^2 = c_2 \times \frac{2m}{\hbar} = \frac{2mE}{\hbar^2}$$

$$(c_2 = E/\hbar)$$

$$* \quad \psi(x,t) = \left(Ae^{ikx} + Be^{-ikx}\right)\cdot \underbrace{e^{c_1 kt}}$$

For all stationary states where ψ can be split into space & time parts. $\leftarrow$ $e^{-\frac{iEt}{\hbar}}$

$$\Rightarrow \text{Now } \psi(x) = \underbrace{Ae^{ikx}}_{\psi_+} + \underbrace{Be^{-ikx}}_{\psi_-}$$

(Consider the time independent part)

$$\psi = \psi_+ + \psi_-$$

a) $$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi_+ = -\frac{\hbar^2}{2m}A(ik)^2 e^{ikx} = \frac{\hbar^2k^2}{2m}\psi_+$$

b) $-\frac{\hbar^2}{2m}\frac{d^2\psi_-}{dx^2} = -\frac{\hbar^2 k^2}{2m}\psi_-$

$\Rightarrow$ From ⓐ & ⓑ, we can see ψ_+ & ψ_- are stationary states.

complex conjugate

* $\Rightarrow \psi_+ = Ae^{ikx}$, $|\psi_+^*|^2 = A^2 = |\psi_+|^2$
* $\psi_- = Be^{-ikx}$, $|\psi_-^*|^2 = B^2 = |\psi_-|^2$

$\Rightarrow$ These are not normalizable, since we need $\int_{-\infty}^{\infty} dx\, |\psi_+|^2 = 1$

$\Rightarrow A^2(\infty - (-\infty)) = 1$

$\therefore$ A must be 0, similarly B.

$\Rightarrow$ So the wave functions can't be normalized.

Now lets calculate $\langle \hat{p} \rangle_{\psi_+}$

$$\Rightarrow \langle \hat{p} \rangle_{\psi_+} = \frac{\int_{-\infty}^{\infty} dx\, (Ae^{-ikx})\, \hat{p} A e^{ikx}}{\int_{-\infty}^{\infty} dx\, Ae^{ikx} \cdot Ae^{-ikx}} \rightarrow ①$$

Now,

$$\hat{p} e^{ikx} = -i\hbar (ik) e^{ikx}$$
$$= -i^2 \hbar k e^{ikx}$$
$$= \hbar k e^{ikx} \longrightarrow \text{Put in ①}$$

$$\Rightarrow \langle \hat{p} \rangle_{\psi_+} = \hbar k \quad \left(k = \sqrt{\frac{2mE}{\hbar}}\right)$$

$$\langle \hat{p} \rangle_{\psi_+} = \hbar k = \sqrt{2mE}$$ } we can relate with classical mechanics (So momentum is similar to classical mech)

Similarly,

$$\langle \hat{p} \rangle_{\psi_-} = -\hbar k = -\sqrt{2mE}$$

——x——

Note: We can see our original example,

$$\psi(x) = \sqrt{\frac{2}{L}} \sin(kx) = \sqrt{\frac{2}{L}} \left(\frac{e^{ikx} - e^{-ikx}}{2i} \right)$$

$$\psi(x) = \underbrace{\psi_+}_{\text{+ve momentum wave}} + \underbrace{\psi_-}_{\text{-ve momentum wave}}$$

$$\Rightarrow k = \frac{n\pi}{L},$$

$$E = \frac{n^2 h^2}{8mL^2} = \frac{\hbar^2 k^2}{2m},$$

——x——

*

* Note

Bound State → Energy < 0 } Wave function is not normalizable.

Unbound state → Energy ≥ 0

——x——

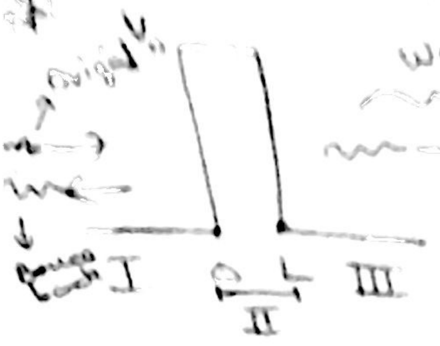
Note: Correspondence statement.

——x——

* Note: Axioms of schrodinger's equation/theory

——x——

Potential Boundary;



we cannot have any bounce backs in this region.

$\psi_I = \psi_{III} \rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow ①$

$\psi_{II} \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \rightarrow ②$

$\Rightarrow \frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E-V_0)\psi$ *

Consider,
$\Rightarrow E < V_0$

$\therefore E - V_0 < 0$

From ①, ②

$$\Rightarrow \frac{d^2\psi}{dx^2} = -\left(\frac{2mE}{\hbar^2}\right)\psi \quad \text{for region I \& III}$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = +\left(\frac{(V_0-E)\cdot 2m}{\hbar^2}\right)\psi \rightarrow V_0 - E > 0$$

$\rightarrow ③$

a) $\Rightarrow$ In region I & III,

$$\psi = A\psi_+ + B\psi_- \quad \text{\} Free particle}$$

Solution is of type. e^{ikx}, e^{-ikx}

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\Rightarrow \psi_I = A_1 e^{ikx} + B_1 e^{-ikx}$$

$$\psi_{III} = A_3 e^{ikx} + B_3 e^{-ikx}$$

b) $\Rightarrow$ In region II, lets try solutions of type

$$\psi = Ae^{kx} + Be^{-kx}$$

$$\rightarrow \psi'' = k^2(Ae^{kx} + Be^{-kx}) = k^2\psi$$

• We want this type becaus

as we showed in 3, $\frac{d^2\psi}{dx^2} \propto \psi$ {the proportional}

$\therefore$ ~~ψ_{III}~~

$$\psi_{II} = A_2 e^{k_2 x} + B_2 e^{-k_2 x},$$

$$k_2^2 = \left(\frac{(V_0 - E) 2m}{\hbar^2}\right)$$

*Finally,

- Now there ψ's must be continuous,

$$\therefore \psi_I(x=0) = \psi_{II}(x=0) \longrightarrow 1$$

$$\Rightarrow \frac{d}{dx}\psi_I(x=0) = \frac{d}{dx}\psi_{II}(x=0) \longrightarrow 2$$

$\Rightarrow$ Similarly,

$$\psi_{II}(x=L) = \psi_{III}(x=L) \longrightarrow 3$$

$$\Rightarrow \frac{d}{dx}\psi_{II}(x=L) = \frac{d}{dx}\psi_{III}(x=L) \longrightarrow 4$$

(We have 6 unknowns $A_1, B_1, A_2, B_2, A_3, B_3$)
but only 4 equations

- Now in region III we can only have a wave moving to the right but not to the left. (No $-ve$ momentum wave in 3rd region) (Since it cant be reflected back if $x > L$).

$\therefore B_3 = 0$

**$\Rightarrow$ Transmission coefficient $= \frac{|\psi_{III}|^2_{x=\infty}}{|\psi_I|^2_{x=-\infty}} = \left(|\psi_{III}|^2 / |\psi_I|^2\right)$

$$T = \frac{|A_3|^2}{|A_1|^2} = e^{-k_2 L}$$

** * ⇒ Transmission coefficient $T = \frac{|A_3|^2}{|A_1|^2}$

*

$$\boxed{T = e^{-2k_2 L}}$$

Similar to probability

(Barrier potential - V_0)

(Particle energy $= E$)

$$k_2 = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

- So we can see in quantum mech, even if energy of particle $< V_0$, it can tunnel through the barrier with probability T.

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* Tunneling effect;

- $T = e^{-k_2 L}$, where particle can cross barrier with a smaller energy than the barrier

⇒ $P \propto \frac{1}{k_2}$ ⇒ As energy ↓, probability drops exponentially

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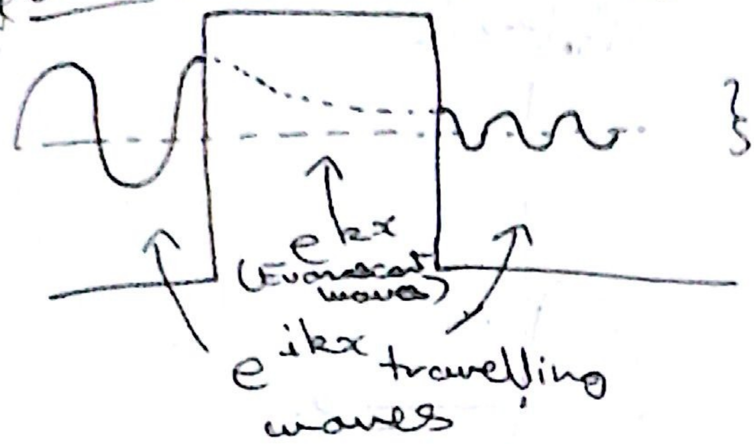
*Note: Due to this effect, electrons can jump & swap places across atoms, this is what chemical bonding is

—— x ——

Note: (Tunneling microscope uses this principle)

—— x ——

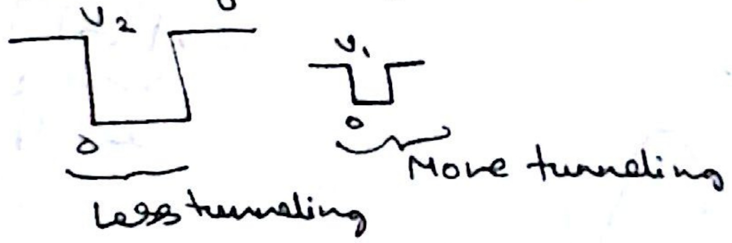
*Wave function diagram;



} Only a small probability (wave function amplitude) gets tunnels through.

—— x ——

Note: Potential well is simple oscillating waves. (If potential walls are not ∞, then tunneling may occur).



—— x ——

*Note: In Q.M a particle can have -ve K.E ⟶ evanescent wave. (So we can enter potentials > T.E).

—— x ——

*SHM in Q.M;

$$\boxed{E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega}, \quad \omega = \sqrt{\frac{k}{m}}$$

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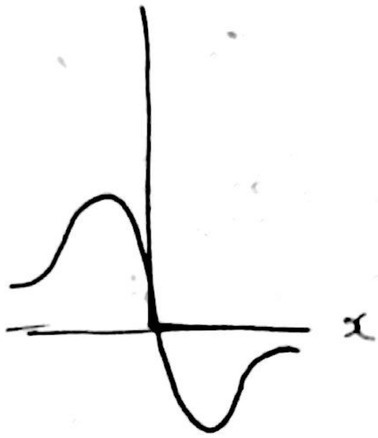
$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \underbrace{\frac{1}{2}kx^2}_{\substack{\text{Potential} \\ \text{of SHM oscillator}}}\psi(x) = E\psi(x)$$

$\Rightarrow$ From this we get

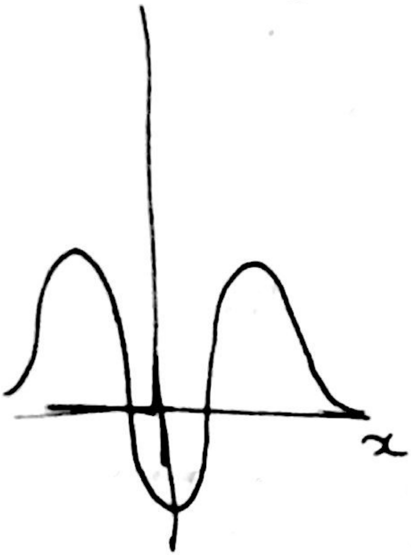
$$E = \left(\frac{1}{2} + n\right)\hbar\omega$$

$\psi_0(x)$ (E_0)

$\psi_1(x)$ (E_1)



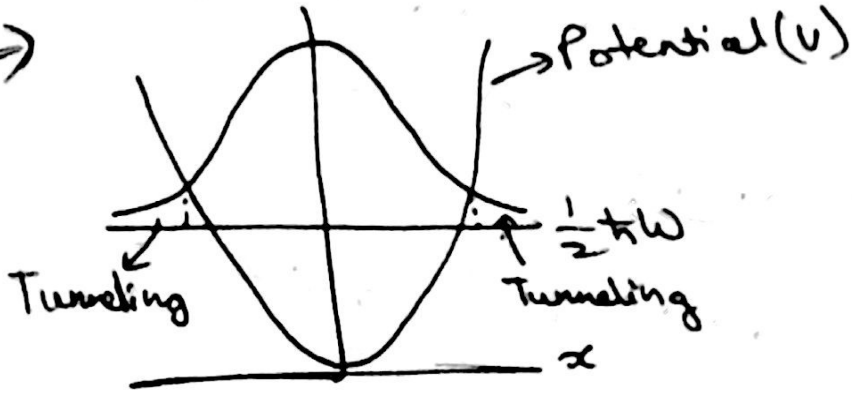
$\psi_2(x)$ (E_2)



$\Rightarrow P(x) = |\psi|^2$

$\Rightarrow P(x)$ has as many nodes as ψ does.

$\rightarrow$



- For $n=0$, No nodes in $P(x)$
  $n=1$, 1 node in $P(x)$
  $n=2$, 2 nodes in $P(x)$ and so on.

$\Rightarrow$ As $n\uparrow$, and $n\to\infty$, we start to get classical mechanical probabilities with no tunneling.

—— x ——

* Hydrogen Atom;

- Assume nucleus is stationary at the center of the atom at {origin}.

$$-\frac{\hbar^2}{2m_e}\nabla^2\psi(\bar{r}) + \underbrace{V(\bar{r})}_{-\frac{e^2}{4\pi\varepsilon_0 r}}\psi(\bar{r}) = E\psi(\bar{r})$$

- Consider the electron.

$\Rightarrow E_n = -\frac{\text{Const}}{n^2}$, $E_1 = -13.6\text{eV}$

- Angular momentum is also quantized

* $\boxed{|L| = \sqrt{l(l+1)}\,\hbar}$, $l = 0, 1, \ldots, (n-1)$

* $L_z = m_L \cdot \hbar$ s.t. $m_L = 0, \pm 1, \pm 2, \ldots, \pm l$

$\Rightarrow$ 3 Quantum numbers n, l, m_L.

$\Rightarrow$ Probability does not change on any rotations of wave functions.

—— x ——

Note:

* Pauli postulated that every orbital has atmost 2 electrons.

—— x ——

Note: Hund's rule of multiplicity:

- Fill up all empty orbitals with same l first.

——— x ———